Parabolic Optimization

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***Abstract*—the parabolic optimization is another method that is used to find the root of a function given any 3 points it finds a local optima, its similar to the secant method in the recursiveness.**

***Keywords*—parabolic, recursive, root, function.**

I. Introduction

Parabolic optimization is a technique used to find an optima of a function and that's by continuously creating parabolas around 3 points and then replacing the point with the highest/lowest value in the function (depending on if we are finding a minima or a maxima), there are other methods for finding optima of a function such as the golden search which is used to find an extrema in between a specified interval. The parabolic optimization is a similar method to the secant method, as it finds a number of roots of secant lines to find the root of a function.

II. Parabolic optimization

When given any 3 points we can calculate a local optima of this parabola using the parabolic optimization and that's by calculating ***a*** in the formula

$$f(x) = ax^2 + bx + c \qquad (1)$$

we can calculate a using the following equation:

$$a = \frac{(x_2 - x_3)f(x_1) + (x_3 - x_1)f(x_2) + (x_1 - x_2)f(x_3)}{(x_1 - x_2)(x_1x_2 - x_1x_3 - x_2x_3 - x_3{}^2)} (2)$$

And according to the equation (2) we can calculate a so if ***a*** is larger than 0 it will be a local minimum and if ***a*** is smaller than 0 then it will be a local maximum. Also, if a=0 and the slope = 0 then that means that we are extremely close to an extremum for the function.

Now that we decided whether it's a minimum or a maximum, we can start calculating the root, if it was a minimum we need to get rid of the point that give us the highest value from the given function and replace it with x_e which is calculated using the following equation:

$$x_e = \frac{(x_2{}^2 - x_3{}^2)f(x_1) + (x_3{}^2 - x_1{}^2)f(x_2) + (x_1{}^2 - x_2{}^2)f(x_3)}{2(x_2 - x_3)f(x_1) + (x_3 - x_1)f(x_2) + (x_1 - x_2)f(x_3)} (3)$$

We keep iterating for x_e until we get a satisfying answer depending on E_a (or until we reach the end of the iterations) which is:

$$E_a = abs(\frac{x_1 - x_e}{x_e})(4)$$

And if we wanted to get the maximum of a function, we get rid of the point that gives us the lowest value from the given function and replace it with x_e which was calculated the same as before, and we keep iterating for x_e until we get a satisfying answer depending on E_s or until we reach the end of the number of iterations.

There are a lot of advantages for the parabolic optimization over other methods and algorithms, for example we only use function values for this algorithm so when it converges to an extreme it does with a rate of convergence (super-linear convergence) of O($h^{1.3247}$), also, the parabolic optimization does not used derivative of functions unlike other algorithms (gradient descent and Newton's method). On the other hand there are some disadvantages for the parabolic optimization, for example the convergance to an extremum is not always guaranteed because sometime the 3 points may be colinear and that results in a degentrate parabola which does not generate a new point. Also if the derivatives of a function are available we just use Newton's method because it exhibits convergence.

III. Algorithm implementation

The function accepts 4 inputs, the function handle and the 3 points we want to apply the parabolic optimization on and returns the point at which we have reached an optima. We initialized A at the start of the function, then we enter a while loop where it calculates X_e and then checks if a is larger or smaller than 0 so it can discard the point with the lowest value in the function, we also keep checking to see if ea is smaller than es, if it was then we break the loop and we would've found the point we want if its not then we continue taking points and calculating X_e until we reach
ea < es.

IV. Solving an example

Now we will use the parabolic optimization to calculate a local optima of the function:

$$g(x) = x^4 + 3x^3 - 2x^2$$

We will take the points:

$x_1 = 0$
$x_2 = 0.55$

$x_3 = 0.225$

First of all we calculate a using the equation (2), which gives us ***a = 0.0043*** which means that a is larger than 0 and that means it's a local minimum, so we apply the algorithm on the function:

1- We calculate x_1, x_2, x_3, $f(x_1)$, $f(x_2)$, $f(x_3)$, and we add the values to the table.

2- For each iteration we look for the highest value in between $f(x_1)$, $f(x_2)$, $f(x_3)$

3- We get rid of the point that gives us the highest value

4- We will assume E_a as 0.01

5- First we got rid of x_1 as it gave us the highest value from the function and we replaced it with x_e which was calculated using the equation (3).

6- We kept iterating until we reached $E_s - E_a < 0.01$

7- We finally got the output of 0.3789, and that is a satisfying answer looking at the graph drawn on desmos.

8- We could have had a more accurate answer if we lowered E_s

Ps. Some points had to be rounded to 2 decimal places as the table wouldn't have fit inside the column

X1	X2	X3	F(X1)	F(X2)	F(X3)	Xe	Es-ea	replaced
0	0.55	0.22	0	-0.01	-0.06	0.29	-	X1
0.29	0.55	0.22	-0.08	-0.01	-0.06	0.35	0.15	X2
0.29	0.35	0.22	-0.08	-0.1	-0.06	0.42	0.3	X3
0.29	0.35	0.42	-0.08	-0.1	-0.09	0.37	-0.01	X1
0.37	0.35	0.42	-0.1	-0.1	-0.09	0.38	0.002	X2
0.37	0.38	0.42	-0.1	-0.103	-0.09	-	-	

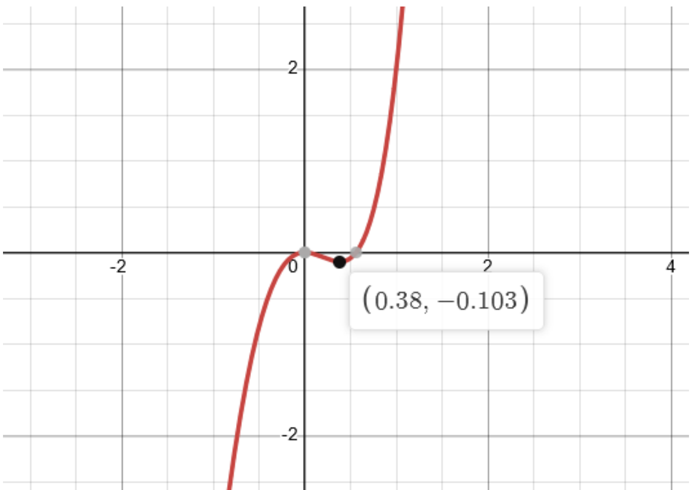

Figure 1

Graph showing a minima point for g(x)

Now we will use the algorithm to find the extreme minimum of the same function *g(x)*

We will take the points:

$x_1 = -2.7$
$x_2 = -2.5$
$x_3 = -2.6$

First of all we calculate a using the equation (2), which gives us ***a = 0.0124*** which means that a is larger than 0 and that means it's a minimum, so we apply the algorithm on the function to get the following table:

As we did in the previous example we did the same here and we got x_e = -2.62 which is f(xe) = -20.56 and as we can see on the graph it gives us a very close result.

X1	X2	X3	F(X1)	F(X2)	F(X3)	Xe	replaced
-2.7	-2.5	-2.6	-20.5	-20.3	-19.8	-2.62	X3
-2.7	-2.5	-2.62	-20.5	-20.3	-20.56	-2.62	X2

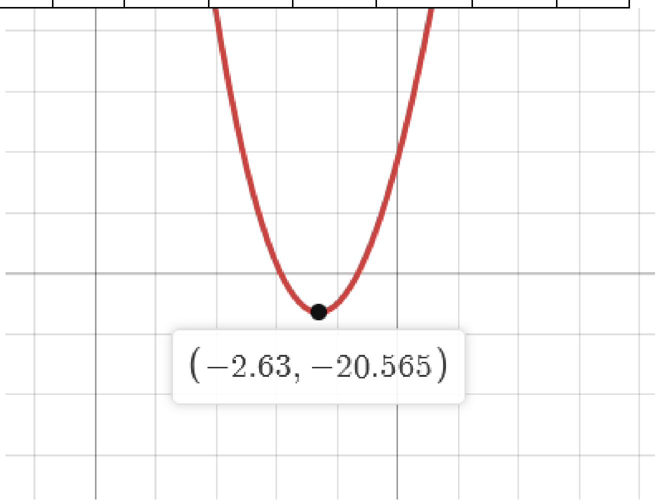

Figure 2 Graph showing the minimum point of g(x)

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