Statistical Techniques in Robotics (16-831, S22) Lecture #25 (Wednesday, Apr 20)

Maximum Entropy IRL and Matrix Game IRL

Lecturer: Kris Kitani *Scribes: Siqi Chai, Jiajie Xu (Group A)*

1 Review

In the previous lecture, we discussed Max Margin IRL. It relates the active imitation learning problem to max margin classifier problem. We also talked about Max-Margin Planning, but didn't finish this topic.

1.1 Policy Value and Expected Policy Features

We define policy value $V(\pi)$ (or equivalently $\mathbb{E}[V^{\pi}(s)]$) as:

$$\begin{aligned} V(\pi) &= \mathbb{E}_p\left[\sum_{t=0}^{\infty} \gamma^t r(s_t)\right] \\ &= \boldsymbol{\theta} \cdot E\left[\sum_{t=0}^{\infty} \gamma^t \phi(s_t)\right] && \triangleright\ \phi(s_t) \text{ is features at each state} \\ &= \boldsymbol{\theta} \cdot \boldsymbol{\mu}(\pi) \end{aligned}$$

We name $\boldsymbol{\mu}(\pi)$ expected feature counts. From the above equation, we know that policy value is linear in expected features. $\boldsymbol{\mu}(\pi)$ is an expected value over all possible trajectories and all possible states, it doesn't depend on a single state.

1.2 Max Margin IRL

Using the linearity above, we rewrite the policy value comparison between expert and other policy:

$$\boldsymbol{\theta} \cdot \boldsymbol{\mu}(\pi^*) \geq \boldsymbol{\theta} \cdot \boldsymbol{\mu}(\pi)$$

Transfer this inequality into a max margin optimization problem with L2 regularization:

$$\begin{aligned} \min_{\theta, t} \quad & \lambda \|\theta\|_2 - t \\ \text{s.t.} \quad & \theta^{\intercal}(\mu(\pi^*) - \mu(\pi_n)) \geq t \quad \forall \pi_n \in \Pi \end{aligned}$$

where Π is a finite set of competitive policies other than expert. This format of optimization is exactly the same as a max-margin classifier (SVM). Substitute $t = \sum_i 1 - \xi_i$ we get:

$$\min_{\theta, \xi} \quad \lambda \|\theta\|_2 + \sum_i \xi_i \tag{1}$$

$$\text{s.t.} \quad \theta^{\intercal}(\mu(\pi^*) - \mu(\pi_i)) \geq 1 - \xi_i \quad \forall i \tag{2}$$

We call the 1 margin, and ξ_i slack as in the soft SVM setting. Then we can use quadratic programming to solve this optimization.

Algorithm 1 MaxMargin IRL

```
1: function MaxMargin IRL($\mu(\pi^*)$)
2:     $\bar{\boldsymbol{\mu}}_D$ = CountFeatures($\mathcal{E}, \pi^*$)
3:     for $i = 1 \cdots$ do
4:         $\hat{\pi} = argmax_\pi\, \theta^\intercal \mu(\pi)$        ▷ Solve RL to get competitive policy
5:         $\mu(\hat{\pi_i}) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t \phi(s_t)\right]$        ▷ Pre-compute feature count using MC roll out
6:         $\theta$ = QuadProg($\lambda, \mu(\pi^*), \{\mu(\hat{\pi}_n)\}_{n=1}^{i}$)        ▷ update $\theta$ using quadratic programming, gradient descent, etc.
7:     end for
8: return $\theta$
9: end function
```

1.3 Maximum Margin Planning

Structured Output Max-Margin IRL (aka Maximum Margin Planning) is an algorithm built up on Max Margin IRL, and it can be used in different real robotics tasks instead of toy examples.

Define new quantities:

- Cumulative Feature count from policy and trajectory: $\mu(\pi) \quad \mu(\zeta)$
- Feature map, features for each state-action pair: $F \in \mathbb{R}^{N\times|S||A|}$
- Occupancy measure: visitation count of each state-action pair: $\eta(a, s) \in \mathbb{R}^{|S||A|}$
- Relation: $\mu = F\eta$

Starting from equation (1), we make the margin depend on the "distance" between policies:

$$\begin{aligned} \min_{\theta,\xi} \quad & \lambda\|\theta\|_2 + \sum_i \xi_i \\ \text{s.t.} \quad & \theta^\intercal(\mu(\pi^*) - \mu(\pi_i)) \geq \ell(\pi^*, \pi_i) - \xi_i \quad \forall i \end{aligned}$$

Through a lot of math, we can derive the final object function for Max-Margin planning:

$$\min_\theta \frac{1}{2}||\theta||^2 + \lambda \sum_d \left\{(\theta^\intercal F_d + l_d^\intercal)\eta_d^* - \theta^\intercal F_d \eta_d\right\}$$

which is the form of SVM with hinge loss, and we can use online sub-gradient descent to solve it.

2 Summary

2.1 Maximum Entropy IRL

In this class, we introduce a new algorithm under inverse reinforcement learning, called Maximum Entropy (MaxEnt) IRL. We use a different method related to entropy to set up objective function, instead of comparing value functions between expert and other competitive policy. In all previous methods, we assume linearization of reward function, now we don't make this assumption, but it will emerge as part of the derivation.

We want to estimate the distribution over all possible state trajectories generated by Markov decision process, $p(\zeta)$, which is huge and often infinity in size. As we have no data/evidence from this distribution, we consider every trajectory to be equally likely (a uniform distribution). This comes from the idea of Occam's razor, we look for the simplest explanation, which is most likely to be true.

2.1.1 Information Entropy and Objective Function

The definition of information entropy is central to the derivation of MaxEnt IRL. Recall, the entropy of a discrete distribution is:

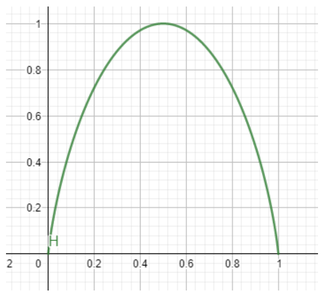
$$\mathcal{H}(x) = -\sum_{x} p(x)\log(p(x))$$

Entropy for a continuous distribution:

$$\mathcal{H}(x) = -\int_{x} p(x)\log(p(x))dx$$

Entropy measures the uncertainty of the distribution. The more information we get about x from distribution, the lower entropy it has. A simple example to understand this is the entropy of binomial random variable:

$$\mathcal{H}(x) = -(x\log_2 x) - (1-x)\log_2(1-x)$$



This entropy takes maximum when probability of $x = 1$ is 0.5. Intuitively for our trajectory distribution, when it is a uniform distribution, the entropy is maximized.

Maximizing entropy is the same as minimizing negative entropy, therefore our objective function is:

$$\underset{p(\zeta)}{min} \sum_{\zeta} p(\zeta)\log p(\zeta)$$

2.1.2 Constraints on Trajectory Distribution

We want to take into account the sampled trajectories, and make our estimated distribution $\tilde{p}(\zeta)$ as close to distribution of expert demonstration $\hat{p}(\zeta)$ as possible:

$$\tilde{p}(\zeta) \approx \hat{p}(\zeta)$$

However, matching a full distribution is hard, so we match their statistics instead. In this case we match the expected feature counts. LHS is the average feature count of our model (all trajectories), RHS is that of expert trajectories:

$$\sum_{\zeta} p(\zeta)\mu(\zeta) = \sum_{d} p(\zeta_d)\mu(\zeta_d)$$

This is the first constraint we have on the optimization problem. Note that matching the average feature count doesn't guarantee that the distributions are the same.

Another constraint we have is that the sum of all probabilities should be one:

$$\sum_{\zeta} p(\zeta) = 1$$

2.1.3 MaxEnt IRL Optimization

Combining 2.2.1 and 2.1.2, the complete optimization problem with constraints is the following:

$$\begin{aligned} \min_{p(\zeta)} \quad & \sum_{\zeta} p(\zeta) \log p(\zeta) \\ \text{s.t.} \quad & \sum_{\zeta} p(\zeta)\mu(\zeta) = \sum_{d} \tilde{p}(\zeta_d)\mu(\zeta_d) \\ & \sum_{\zeta} p(\zeta) = 1 \end{aligned}$$

We again use Lagrangian to optimize these equations, where $\boldsymbol{\theta}$ and V are Lagrangian multipliers.

$$\mathcal{L}(p(\zeta), \boldsymbol{\theta}, V) = \sum_{\zeta} p(\zeta) \log p(\zeta) + \boldsymbol{\theta}^{\intercal} \left\{ \sum_{d} p(\zeta_d)\mu(\zeta_d) - \sum_{\zeta} p(\zeta)\mu(\zeta) \right\} + V \left\{ \sum_{\zeta} p(\zeta) - 1 \right\}$$

Take partial derivative w.r.t. each $p(\zeta)$:

$$\frac{\partial \mathcal{L}(p(\zeta), \boldsymbol{\theta}, V)}{\partial p(\zeta)} = (\log p(\zeta) + 1) - \boldsymbol{\theta}^{\intercal}\mu(\zeta) + V$$

Set it to 0:

$$\log p(\zeta) = \boldsymbol{\theta}^{\intercal}\mu(\zeta) - V - 1$$

$$\begin{aligned} p(\zeta) &= \frac{exp(\boldsymbol{\theta}^{\intercal}\mu(\zeta))}{exp(V+1)} \\ &\propto exp(\boldsymbol{\theta}^{\intercal}\mu(\zeta)) \end{aligned}$$

This shows that the optimal form of trajectory likelihood is a Boltzmann distribution [1] with energy $\boldsymbol{\theta}^\intercal \mu(\zeta)$ and partition function $exp(V+1)$.

From what we learned in Lec 23, we can interpret $\boldsymbol{\theta}^\intercal \mu(\zeta)$ as the value function of a trajectory, which is a linear function of expected state features, then it means the trajectories are generated by a reward function of the form:

$$r(s) = \boldsymbol{\theta}^\intercal \phi(s)$$

which is a linear function of step-wise state features.

2.1.4 Gradient Descent Update and Algorithm

Our goal is to find optimal weight vector θ that defines the reward function $R(s)$, using gradient descent. In the Lagrangian, only the second term is a function of θ:

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}) = \sum_d p(\zeta_d)\mu(\zeta_d) - \sum_\zeta p(\zeta)\mu(\zeta)$$

The two terms on RHS are both expected feature count under learned policy, we use short-hand notation:

$$\nabla \mathcal{L}(\boldsymbol{\theta}) = \bar{\boldsymbol{\mu}}_D - \bar{\boldsymbol{\mu}}$$

Then the gradient descent update equation is:

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \lambda \cdot (\bar{\boldsymbol{\mu}}_D - \bar{\boldsymbol{\mu}})$$

On RHS, $\boldsymbol{\theta}, \lambda$ and $\bar{\boldsymbol{\mu}}_D$ are known or can be easily calculated, but to get $\bar{\boldsymbol{\mu}}$, we need to:

1. compute optimal policy π in $\mathcal{E}$ using RL
2. compute expected feature count $\bar{\boldsymbol{\mu}}$ by rolling out Monte Carlo estimate, etc.

Algorithm pseudocode:

Algorithm 2 MaxEnt IRL

```
1: function MaxEnt IRL($\lambda, \pi^*$)
2:     $\bar{\boldsymbol{\mu}}_D$ = CountFeatures($\mathcal{E}, \pi^*$)
3:     while $\nabla \mathcal{L} > \epsilon$ do
4:         $\pi$ = SoftValueIteration($\mathcal{E}, \boldsymbol{\theta}$)        ▷ Solve RL
5:         $\bar{\boldsymbol{\mu}}$ = CountFeatures($\mathcal{E}, \pi$)        ▷ Pre-compute feature count
6:         $\boldsymbol{\theta} = \boldsymbol{\theta} - \lambda \cdot (\bar{\boldsymbol{\mu}}_D - \bar{\boldsymbol{\mu}})$        ▷ GD
7:     end while
8: return $\boldsymbol{\theta}$
9: end function
```

2.1.5 Key Concepts

- Entropy over trajectories $\mathcal{H}(p(\zeta))$ should be maximized (as objective function) when we have no demonstrations
- Statistics of demonstrations (from experts) can be used to setup constraints in the optimization problem
- Optimal form of the trajectory likelihood $p(\zeta)$ is a Boltzmann distribution
- Lagrangian multiplier $\boldsymbol{\theta}$ and V can be interpreted as the linear parameters of the reward function and the partition function, respectively

2.2 Activity Forecasting

In an activity forecasting task, we aim to predict the activity of an agents given a scene. Specifically, given a novel static scene, for example a view of a parking lot with some cars and an agent, we want to predict the path that the agent will take in order to move to its goal. When we apply the MaxEnt IRL algorithm to this type of problem, we get back a distribution over possible future trajectories. Note that this distribution not only captures the shortest path, but also all possible paths and interactions between scene and the agent, parameterize by path probabilities.

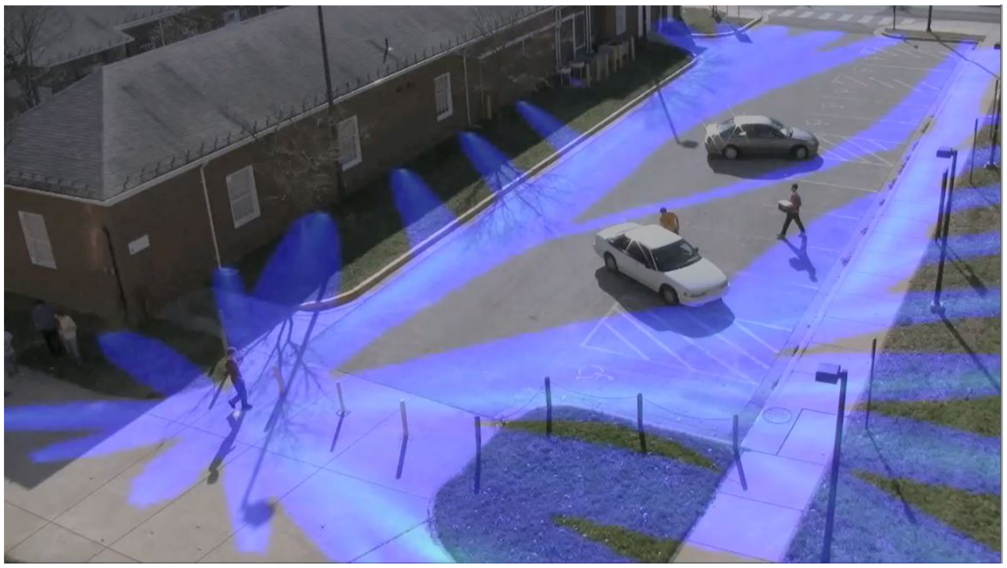


Figure 1: Activity Forecasting trajectories distribution (Kitani et al, 2012)

2.2.1 Research

In the research "Activity Forecasting" by Kitani et al (2012), both an expert demonstration and a physical representation of the scene are used to learn a person's decision process, or equivalently, the person's reward function that is not scene specific. The crucial part is that the learned reward function can be generalized to novel scenes. In contrast, if we only learn the trajectory distribution for the parking lot scene, we can only predict trajectory in that one scene, which is not very useful.

2.2.2 Approach

In this activity forecasting task, we assume an activity sequence is generated by an Markov Decision Process (MDP) as follow:

$$\zeta = \{x_0, a_0, r(x_0), x_1, a_1, r(x_1), \ldots, x_g, a_g, r(x_g)\}$$

$$x_i \in \mathbb{R}^2, \quad a_i \in \mathbb{R}^2, \quad r \in \mathbb{R}$$

where at each time stamp we have a state-action-reward tuple. Each x's represent a 2D coordinate, each a's represent a 2D velocity vector, and each r's is a scalar reward. We model the sequence by a policy:

$$p(a|x) \propto \exp\{r(x) + \sum_{x} p(x'|x,a)V(x')\}$$

The policy is furthered determined by the reward function:

$$V(x) = softmax_a Q(x,a)$$

$$Q(x,a) = r(x) + \sum_{x'} p(x'|x,a)V(x')$$

Put it together, given a reward function, we can compute the value function under MDP. With the value function, we can compute the policy. With the policy, we can sample trajectories from the environment. Although we have this generative process, we have a problem. We only have observations of trajectories, but not the reward function. That is, given a collection of sequences, how do we infer the reward function?

2.2.3 Reward Function Parameterization

We solve the problem by parameterizing the reward function with *theta*:

$$r(x;\theta) = \sum_{n} \theta_n f_n(x)$$

In the parking lot scene, for example, if we have three features:

$$r(x;\theta) = \theta_{car} f_{car}(x) + \theta_{lot} f_{lot}(x) + \theta_{grass} f_{grass}(x)$$

We can rewrite the reward function with a combination of linear functions over all features. Now the problem is reduced to estimating the θs for each of the features. We can use MaxEnt IRL to do this. The above process is called Inverse Optimal Control

2.2.4 Key Concepts: IRL under active imitation learning

- IRL can be formulated as a maximum entropy optimization problem
- Distribution over trajectories factorizes into a Boltzmann distribution
- We can use online mirror descent to solve for reward parameters

2.3 Matrix Game IRL

Matrix Game IRL demonstrates that a two player game can be solved via online learning. We begin by reviewing the objective function given expert trajectories, assuming trajectories are sample from policy and environment:

$$\zeta^* \sim \pi^*, \quad \zeta \sim \pi$$

When we compare a policy against the optimal policy, we use the policy value function. The relationship below must hold:

$$V(\pi^*) \geq V(\pi)$$

With the assumption that the reward function can be written as a linear function parameterized by some θ, we have:

$$\theta \cdot \mu(\pi^*) \geq \theta \cdot \mu(\pi)$$

By moving things around, we have this optimization setup:

$$\min_{\theta}\{\max_{\pi} \theta \cdot \mu(\pi) - \theta \cdot \mu(\pi_E)\}$$

where π_E is the expert policy. With more algebra, the objective function is now:

$$\min_{\theta}\{\max_{\pi} \theta^T(\mu(\pi) - \mu(\pi_E))$$

$$\theta \in \mathbb{R}^{1\times N}, \quad \mu(\pi) \in \mathbb{R}^{N\times 1}$$

The difference of $\mu(\pi)$ is the difference in feature counts. We can rewrite this difference with $G \in \mathbb{R}^{N\times M}$ and $\psi \in \mathbb{R}^{M\times 1}$:

$$\min_{\theta}\{\max_{\pi} \theta^T G\psi\}$$

With $\theta \in \mathbb{R}^{1\times N}$, $G \in \mathbb{R}^{N\times M}$, and $\psi \in \mathbb{R}^{M\times 1}$ we can now interpret the objective function under the setup of a two players game. We take θ as the row player, ψ as the column player (adversarial), and G as the matrix game. Note that, we can think ψ as a mixture of weights over many deterministic policies.

2.3.1 Online algorithm to solve Matrix Game

We can solve the matrix game using Multiplicative Weights Algorithm (MWA) Note that inside the for loop at line 5, we are sampling the row player's strategy using a multinomial distribution over the weights, which is similar to the weighted majority algorithm (WMA).

2.3.2 IRL algorithm to solve Matrix Game

We can use IRL to solve the matrix game. In the algorithm Multiplicative Weights Inverse Reinforcement Learning Algorithm (MW-IRL) on line 4-5, we make the prediction based on the current estimated reward function θ. On line 6, we make update to the reward function estimate.

Algorithm 3 MW-Matrix Game

```
1: $\beta$: learning rate
2: procedure MW-MATRIX GAME($\beta$)
3:     $w^{(0)} \leftarrow \{w_r^{(0)} = 1\}$
4:     for $t = 1, \cdots, T$ do
5:         $R^{(t)} = \frac{w^{(t-1)}}{\sum_r w_r^{t-1}}$
6:         $Receive(l^{(t)} = G(\cdot, c^{(t)}))$
7:         $w_r^{(t)} = w_r^{(t-1)} \beta^{l_r^{(t)}} \quad \forall r$
8:     end for
9: end procedure
```

Algorithm 4 MW-IRL

```
1: procedure MW-IRL($\hat{\pi_E}, \hat{\mu_E}, \beta$)
2:     for $t = 1, \cdots, T$ do
3:         $\theta_i = \frac{\theta_i}{\sum_{i'} \theta_{i'}} \quad \forall i$
4:         $\hat{\pi_\theta} = RL(r(\cdot; \theta))$
5:         $\hat{\mu} = Estimate(\pi_\theta)$
6:         $\theta_i = \theta_i \cdot \exp\{\ln \beta \cdot G_i(\hat{\mu_i})\} \quad \forall i$
7:     end for
8:     return $\theta$
9: end procedure
```

2.3.3 Key Concepts: Matrix Game

- Inverse Reinforcement Learning is a two player game between the reward function and a competitive policy
- Two player game can be solved with online mirror descent with entropic regularization

References

[1] Boltzmann distribution. Boltzmann distribution — Wikipedia, the free encyclopedia. [Online; accessed 23-April-2022].