



STATISTICAL THINKING IN PYTHON II

Welcome
to the course!



You will be able to...

- Estimate parameters
- Compute confidence intervals
- Perform linear regressions
- Test hypotheses

with real data!



Caltech



We use hacker statistics

- Literally simulate probability
- Broadly applicable with a few principles



Statistical analysis of the beak of the finch



Geospiza fortis



Geospiza scandens



STATISTICAL THINKING IN PYTHON II

**Let's start
thinking statistically!**

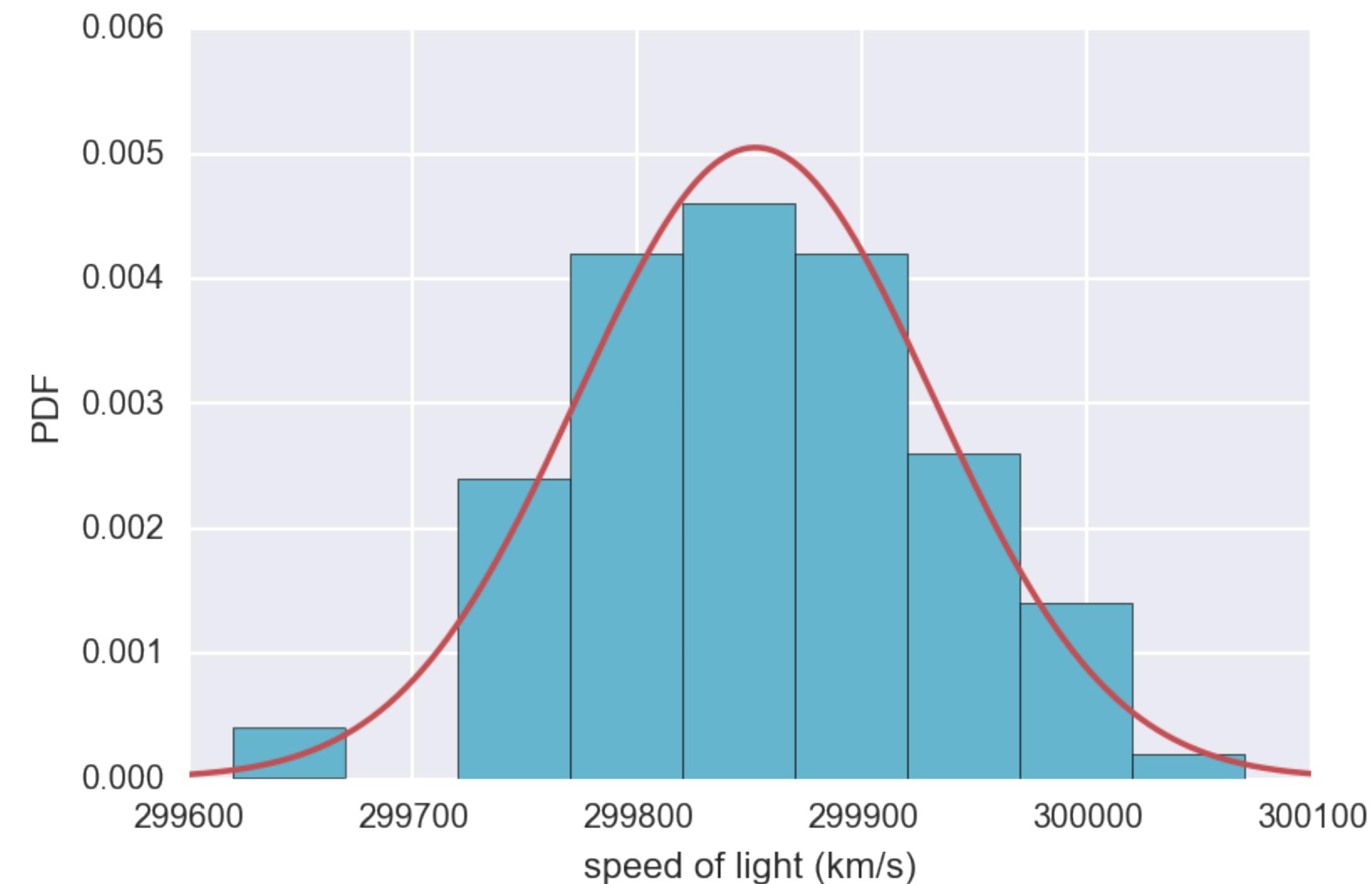


STATISTICAL THINKING IN PYTHON II

Optimal parameters

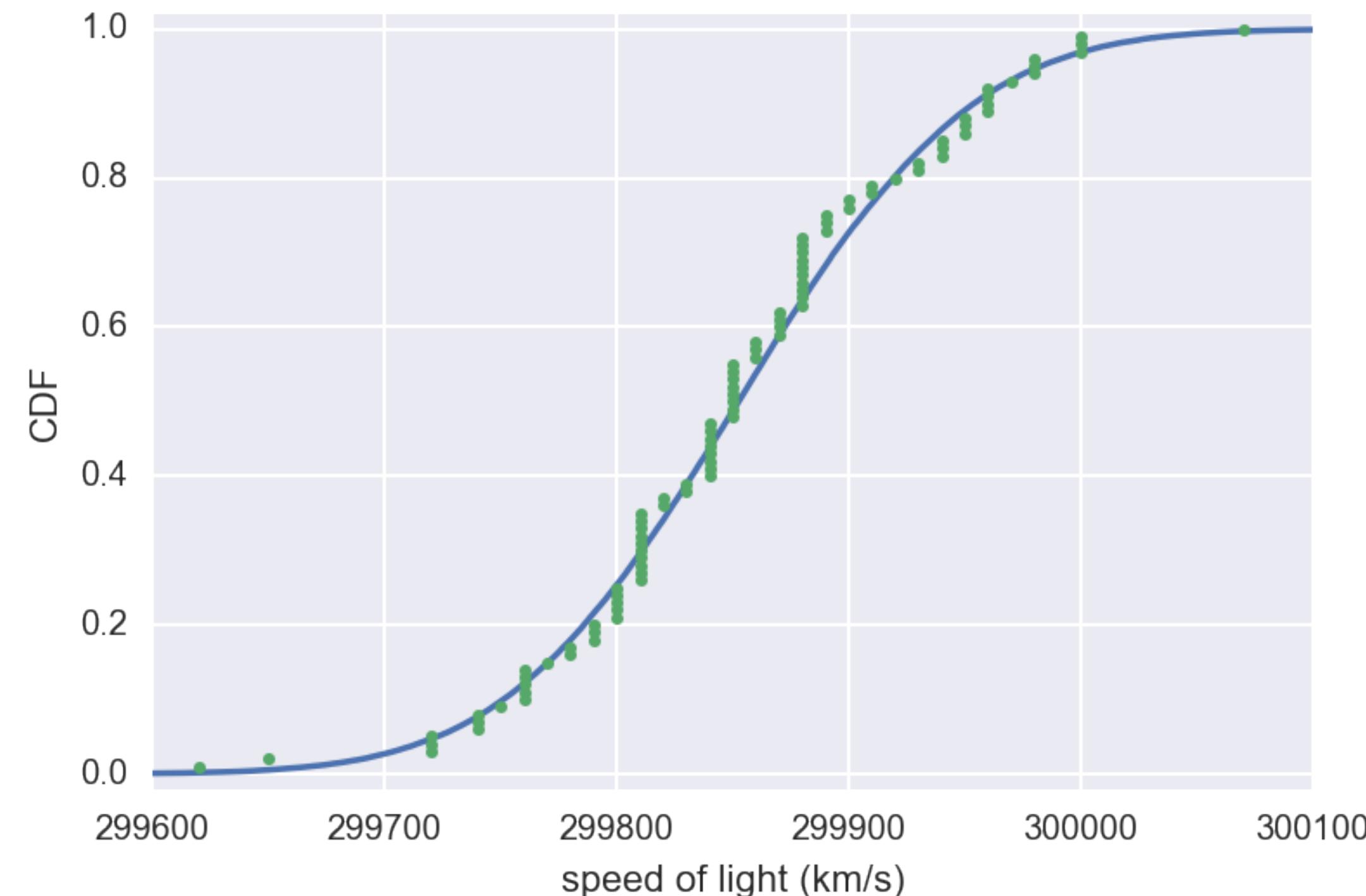


Histogram of Michelson's measurements





CDF of Michelson's measurements





Checking Normality of Michelson data

```
In [1]: import numpy as np
```

```
In [2]: import matplotlib.pyplot as plt
```

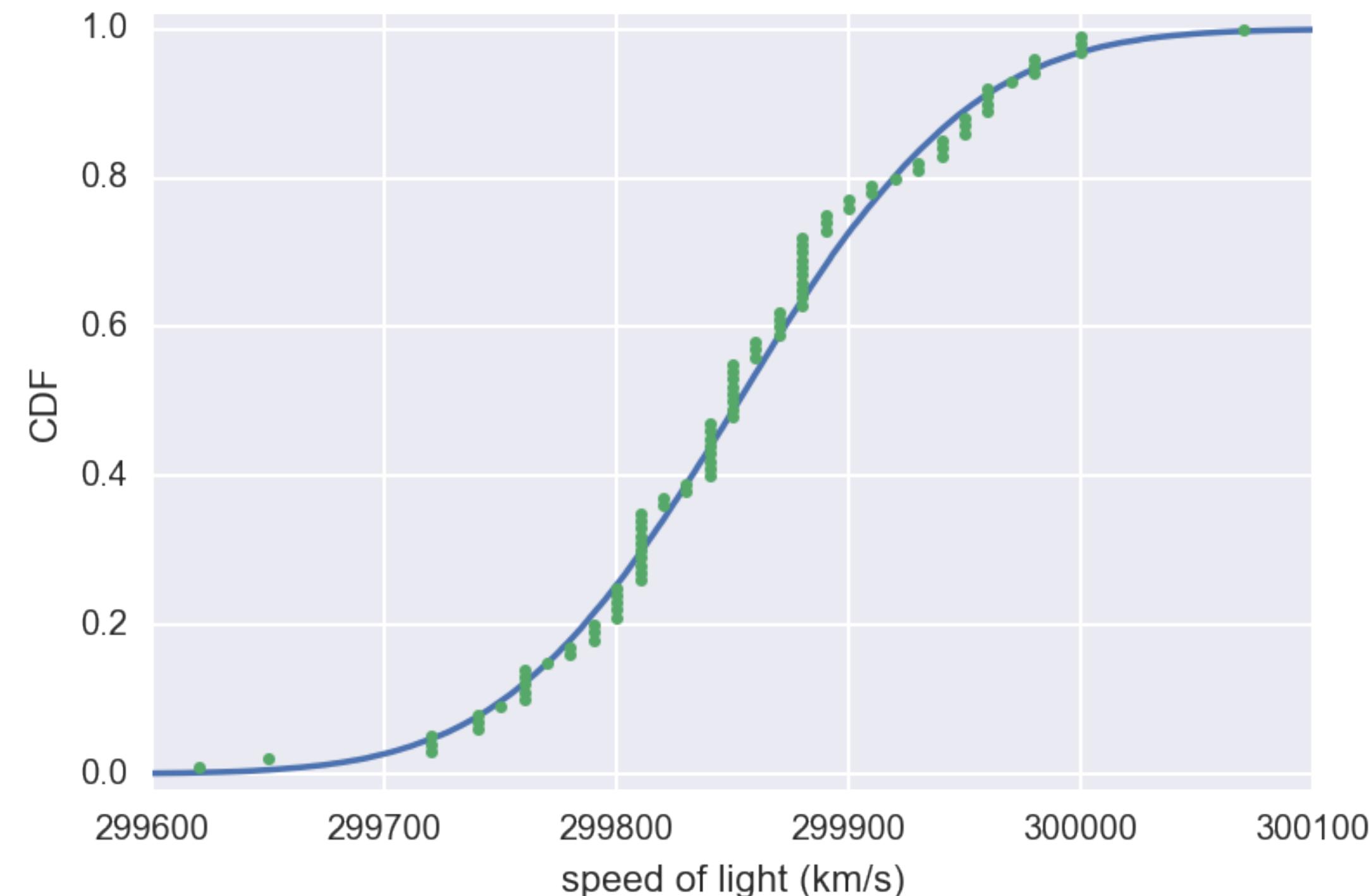
```
In [3]: mean = np.mean(michelson_speed_of_light)
```

```
In [4]: std = np.std(michelson_speed_of_light)
```

```
In [5]: samples = np.random.normal(mean, std, size=10000)
```

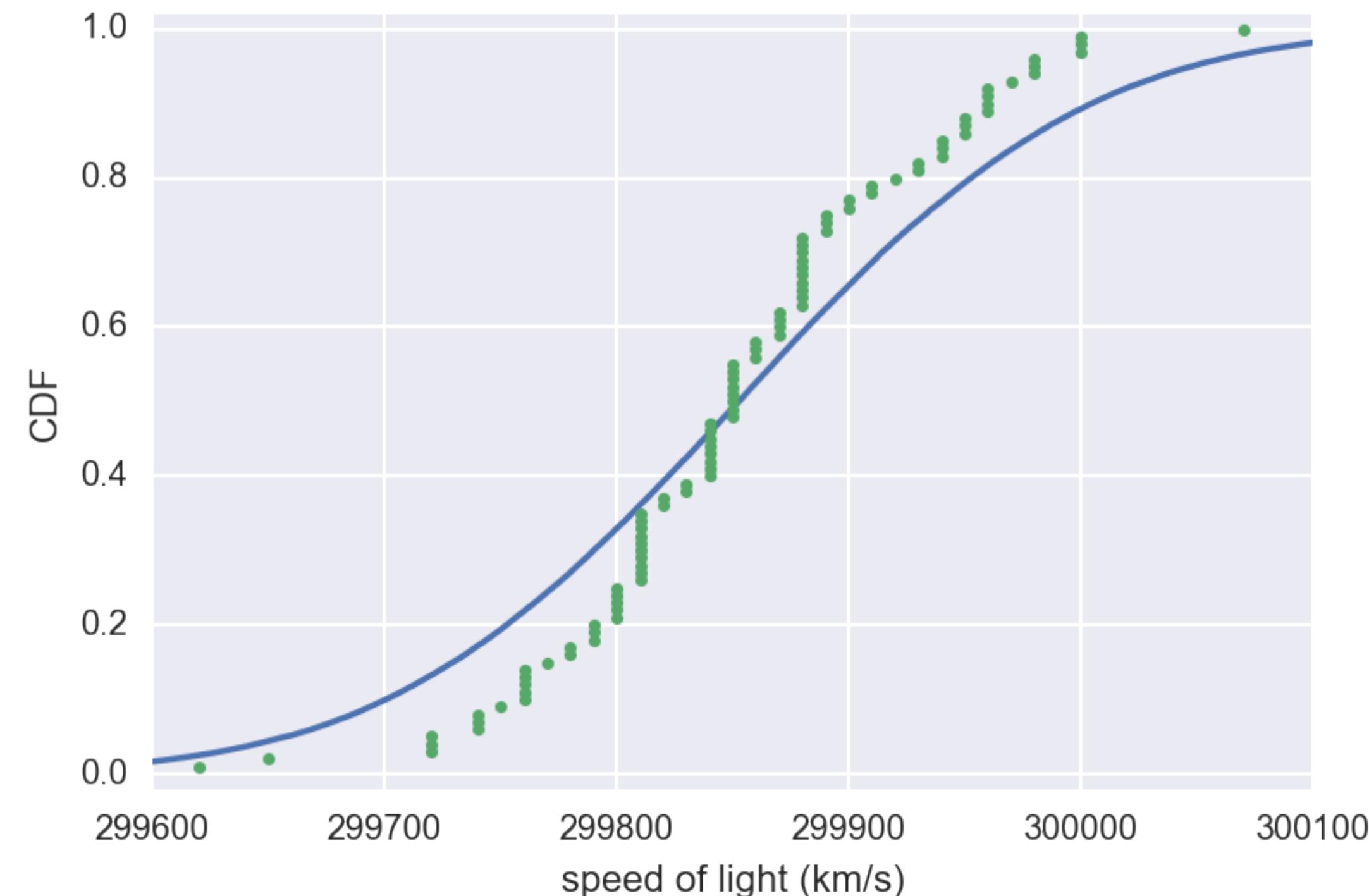


CDF of Michelson's measurements



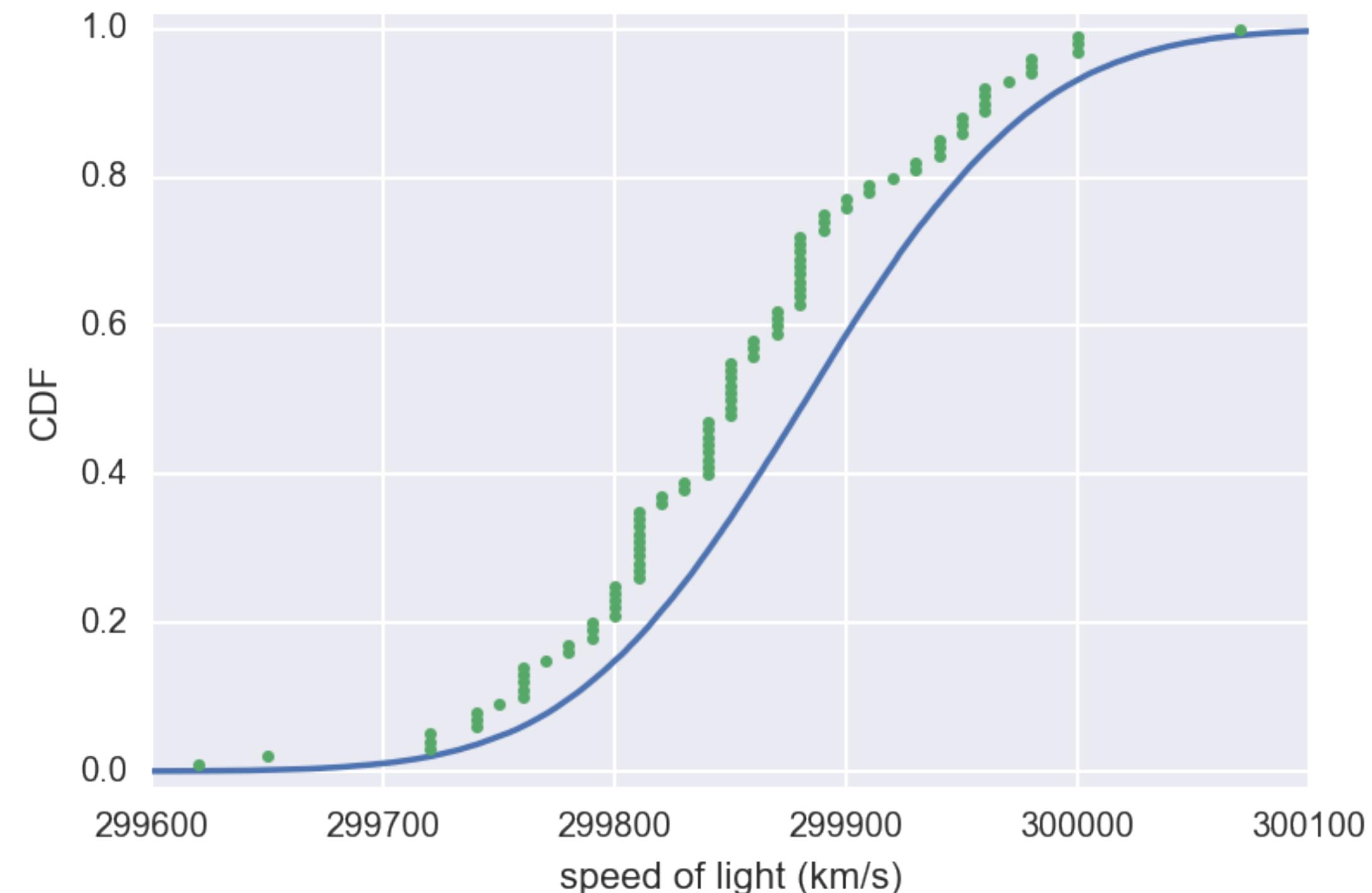


CDF with bad estimate of st. dev.





CDF with bad estimate of mean



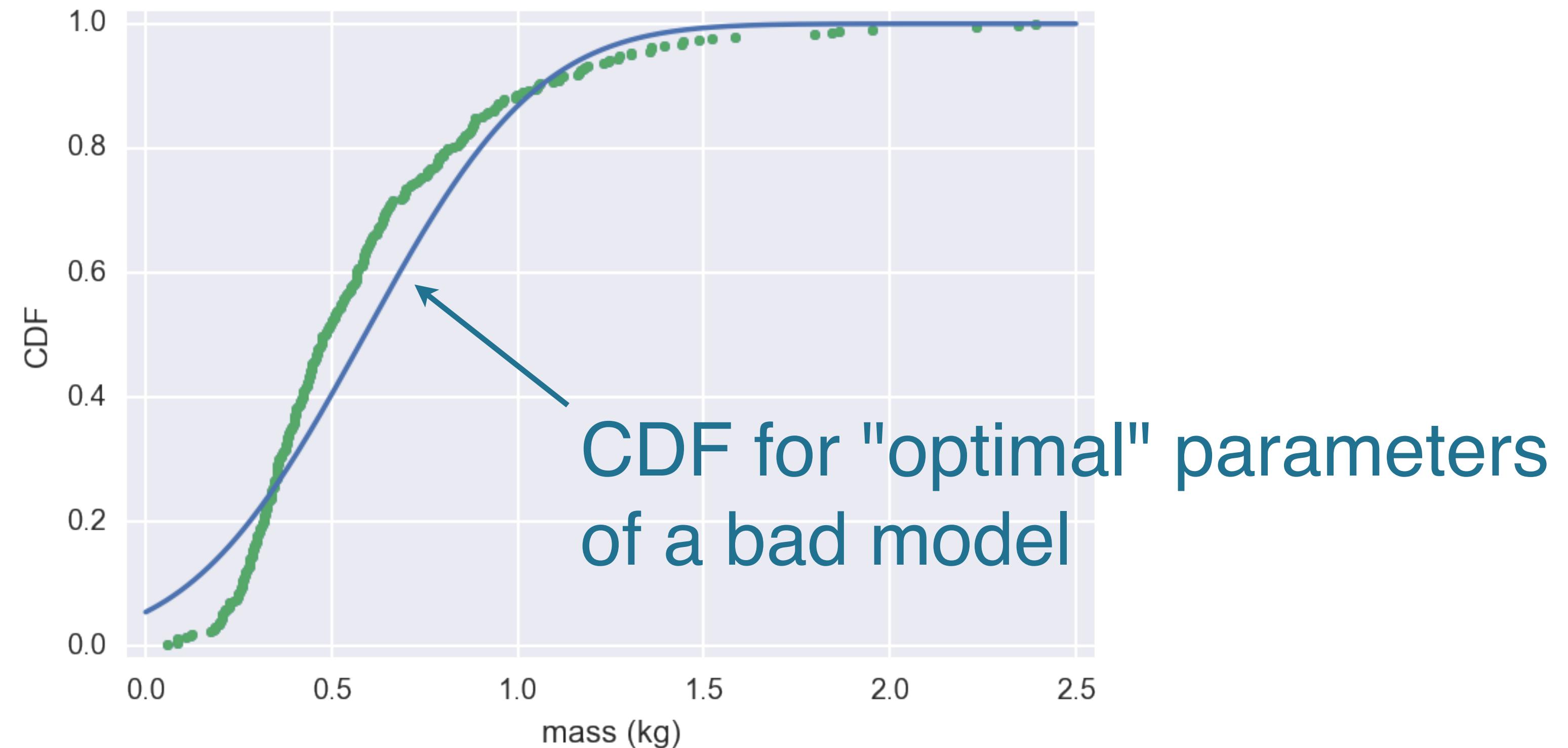


Optimal parameters

- Parameter values that bring the model in closest agreement with the data



Mass of MA large mouth bass





Packages to do statistical inference



`scipy.stats`



`statsmodels`



`hacker stats`
with numpy



STATISTICAL THINKING IN PYTHON II

Let's practice!

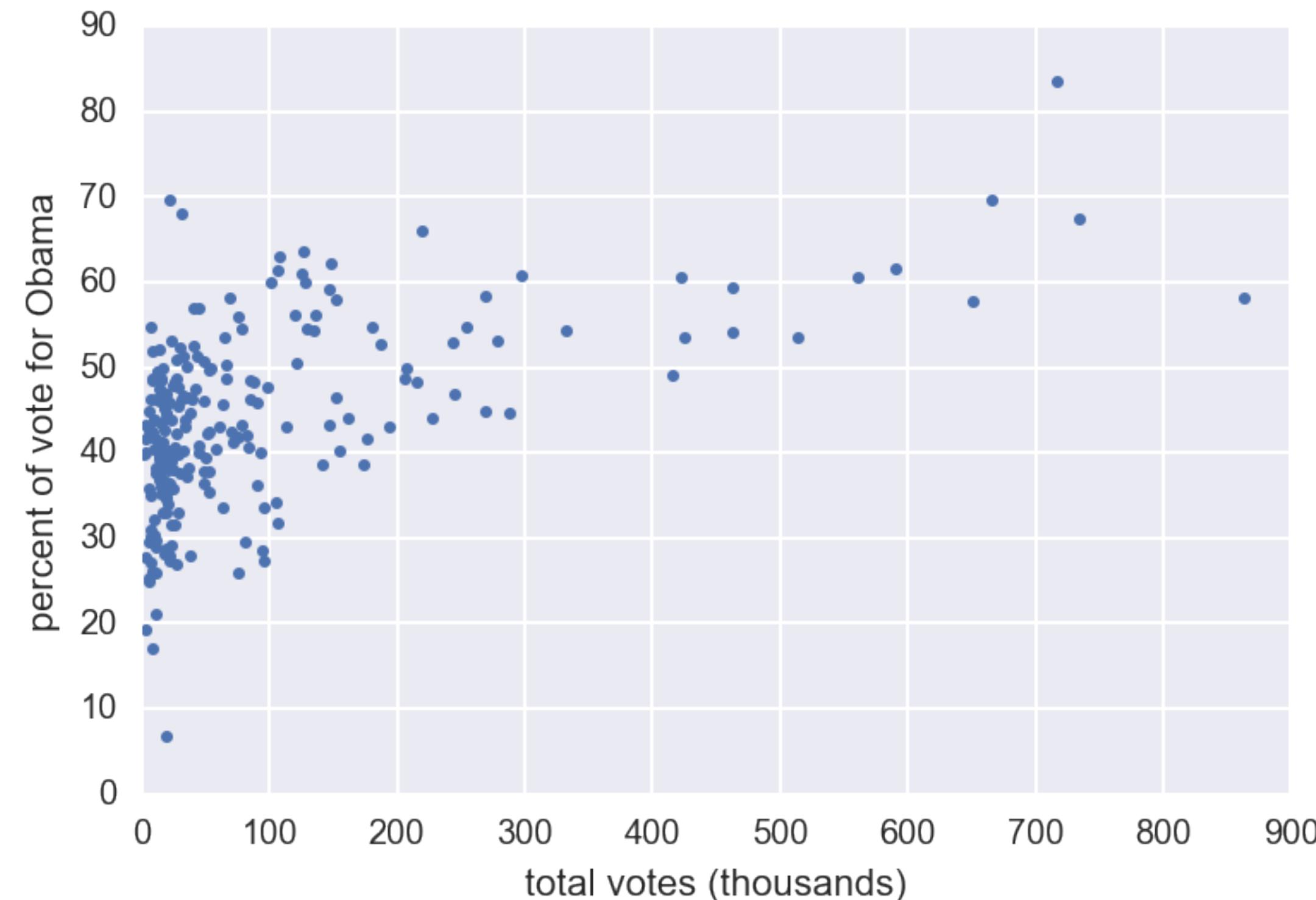


STATISTICAL THINKING IN PYTHON II

Linear regression by least squares



2008 US swing state election results



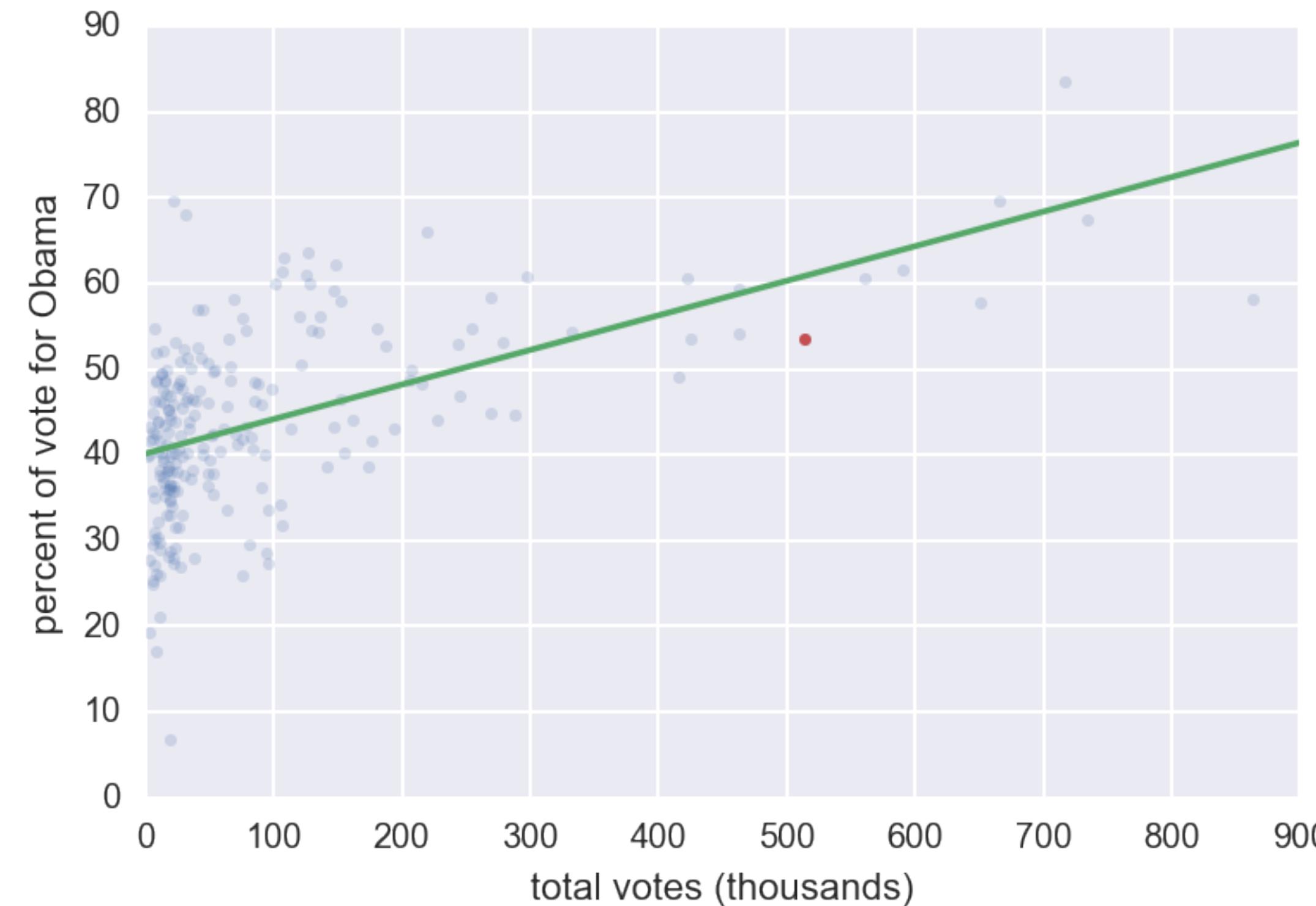


2008 US swing state election results



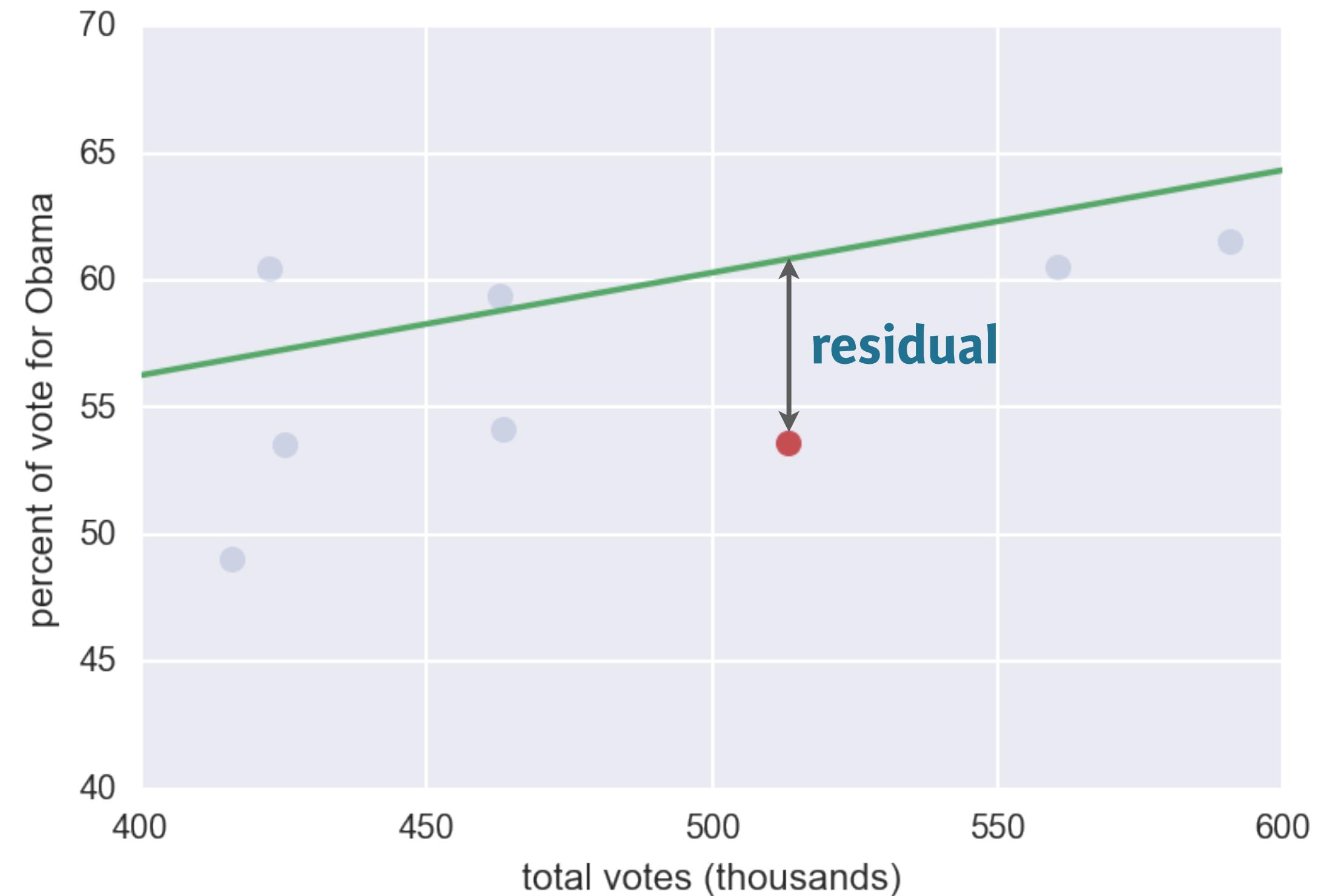


2008 US swing state election results





Residuals





Least squares

- The process of finding the parameters for which the sum of the squares of the residuals is minimal



Least squares with np.polyfit()

```
In [1]: slope, intercept = np.polyfit(total_votes,  
...:  
           dem_share, 1)
```

```
In [2]: slope  
Out[2]: 4.037071700946555e-05
```

```
In [3]: intercept  
Out[3]: 40.113911968641744
```



STATISTICAL THINKING IN PYTHON II

Let's practice!

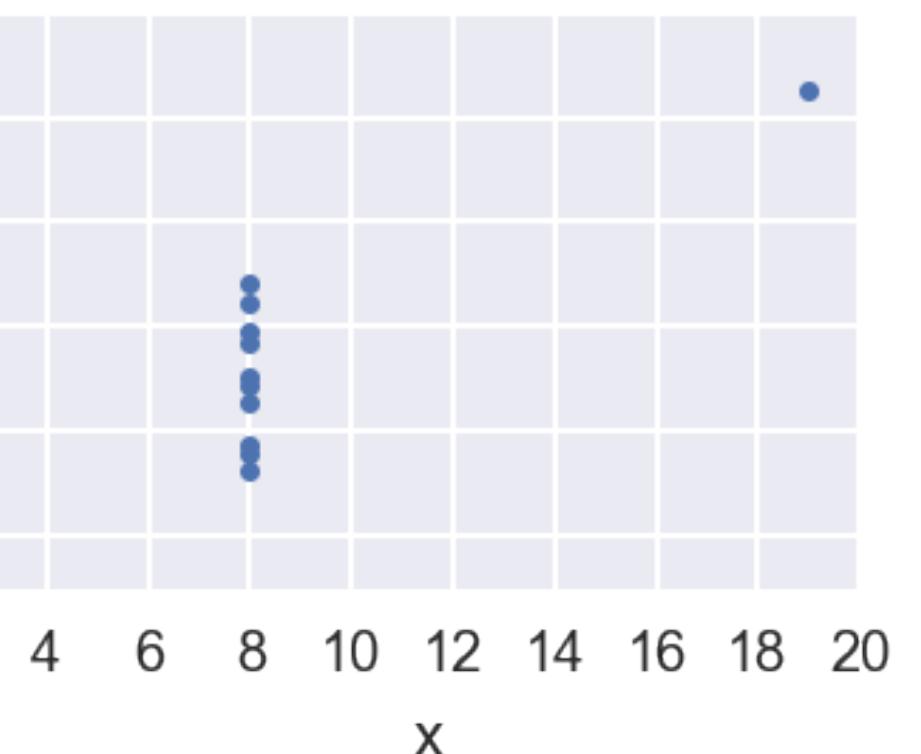
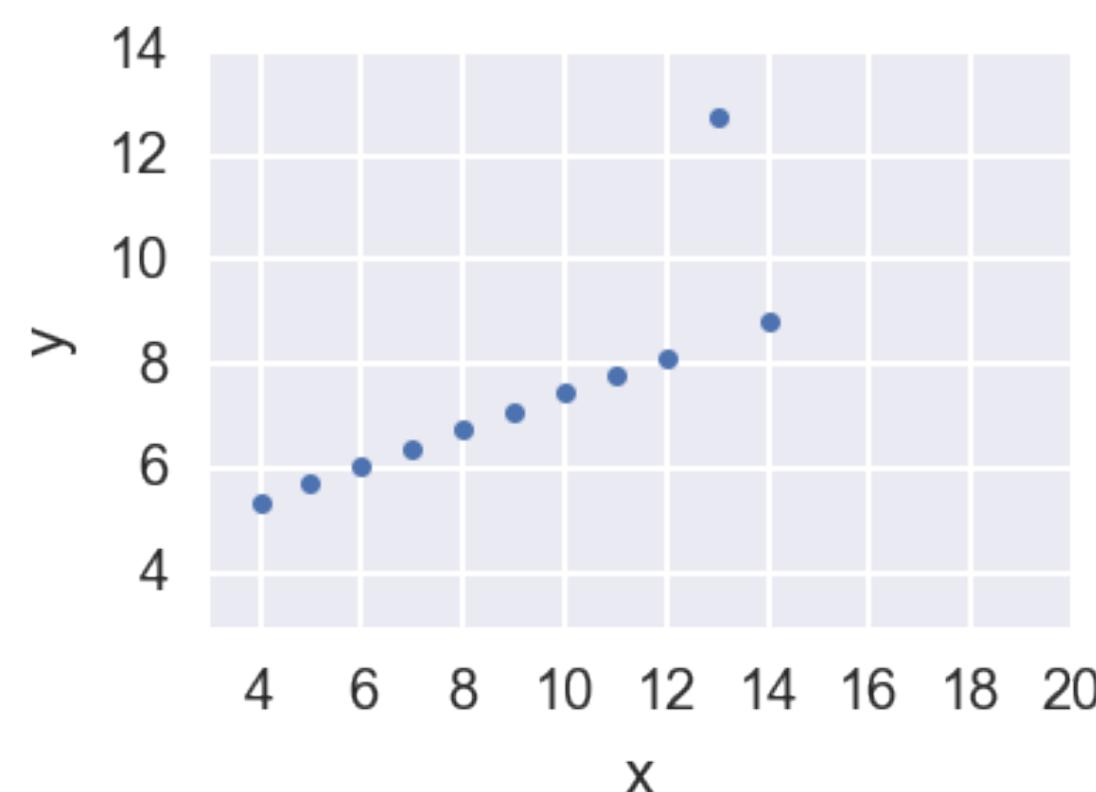
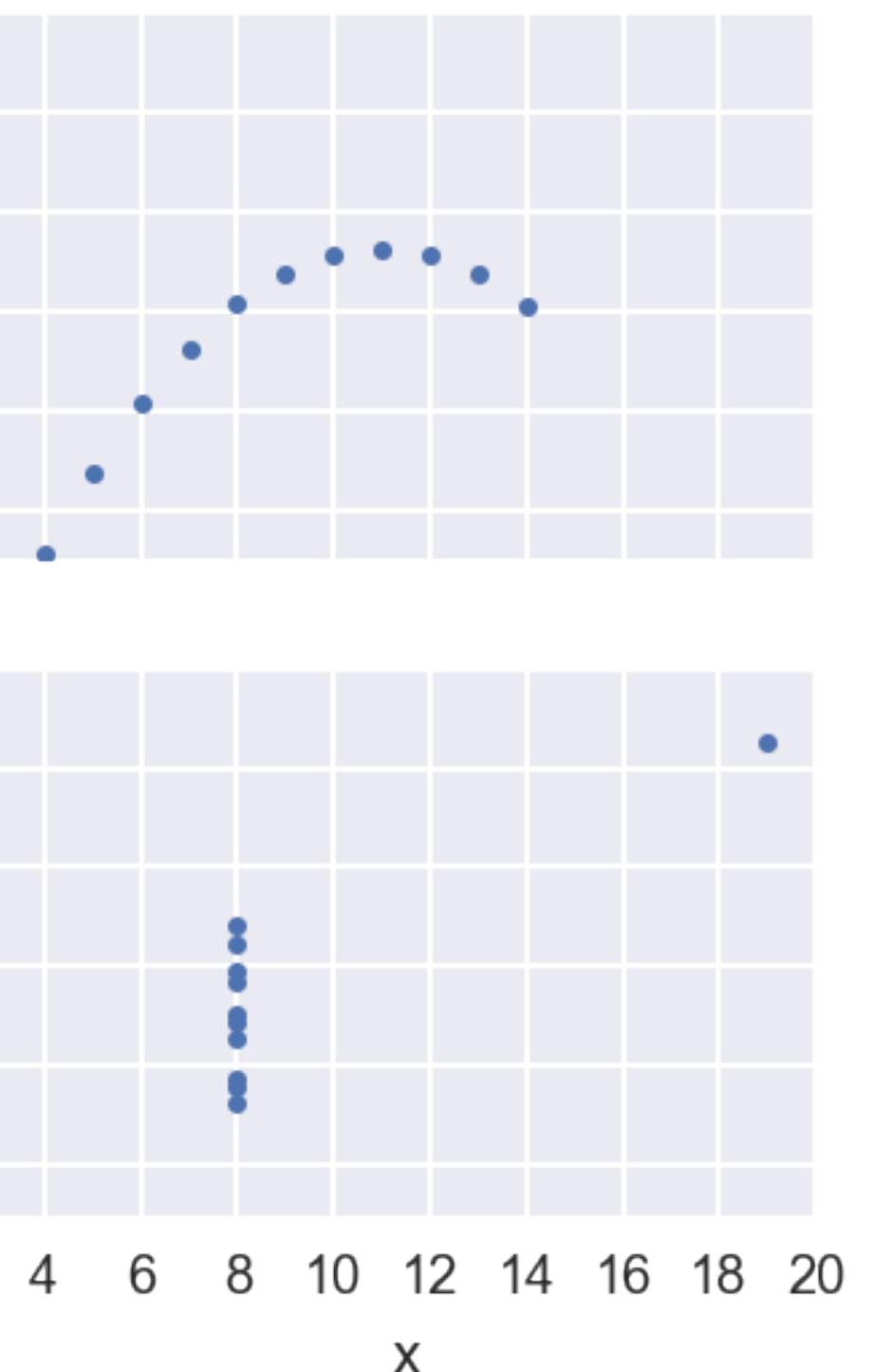
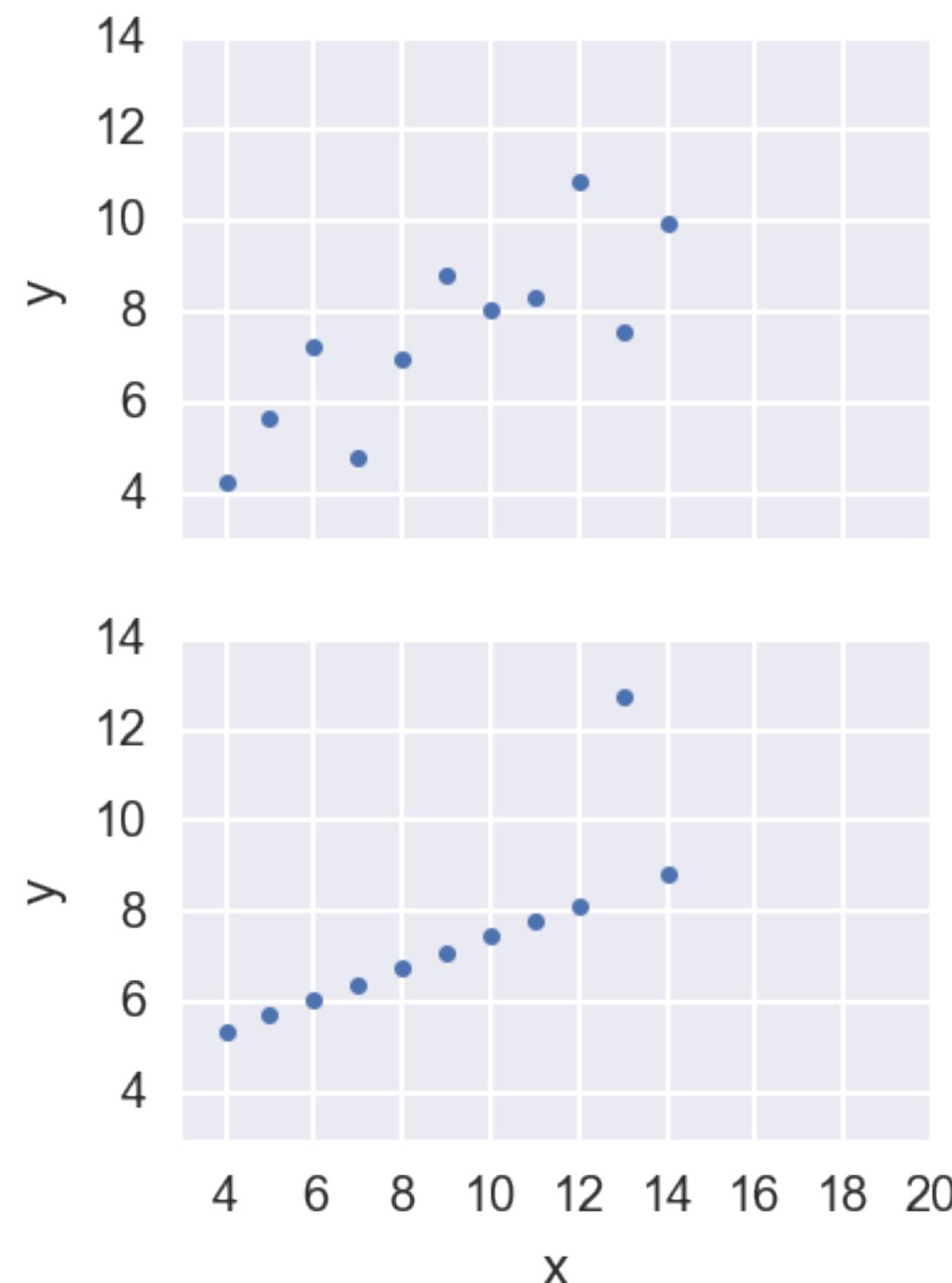


STATISTICAL THINKING IN PYTHON II

The importance of EDA: Anscombe's quartet

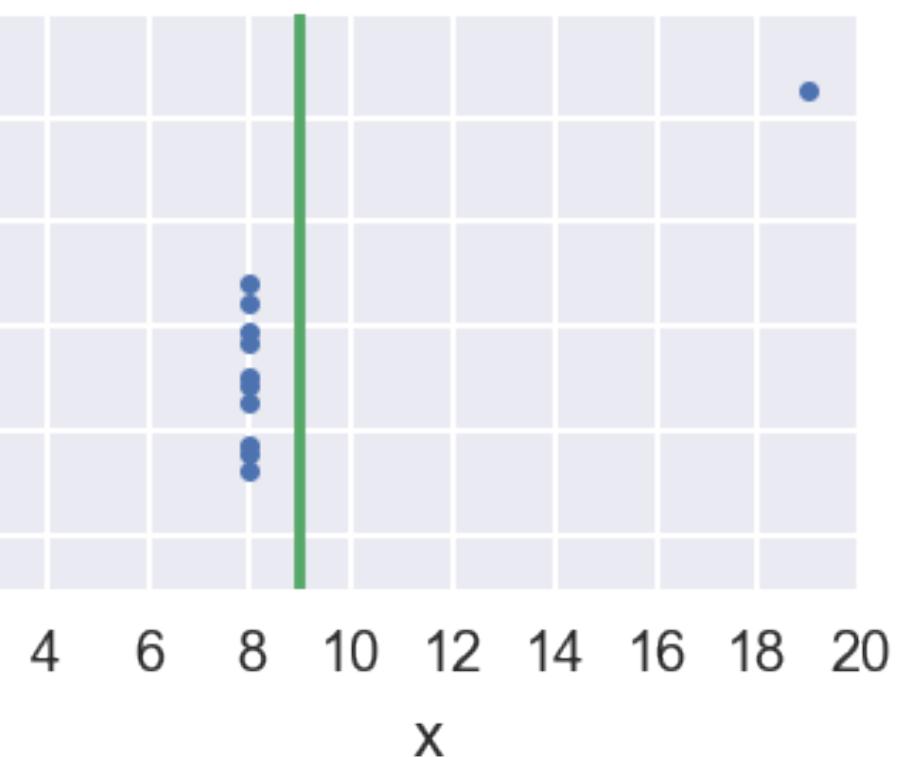
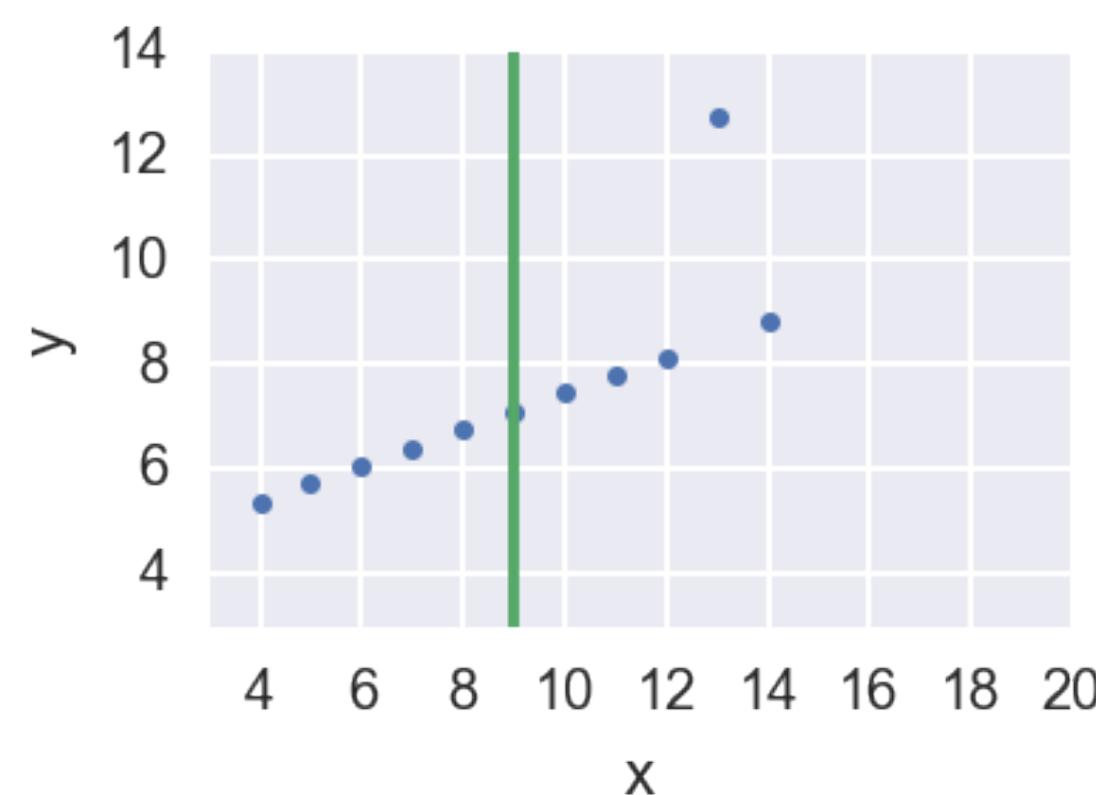
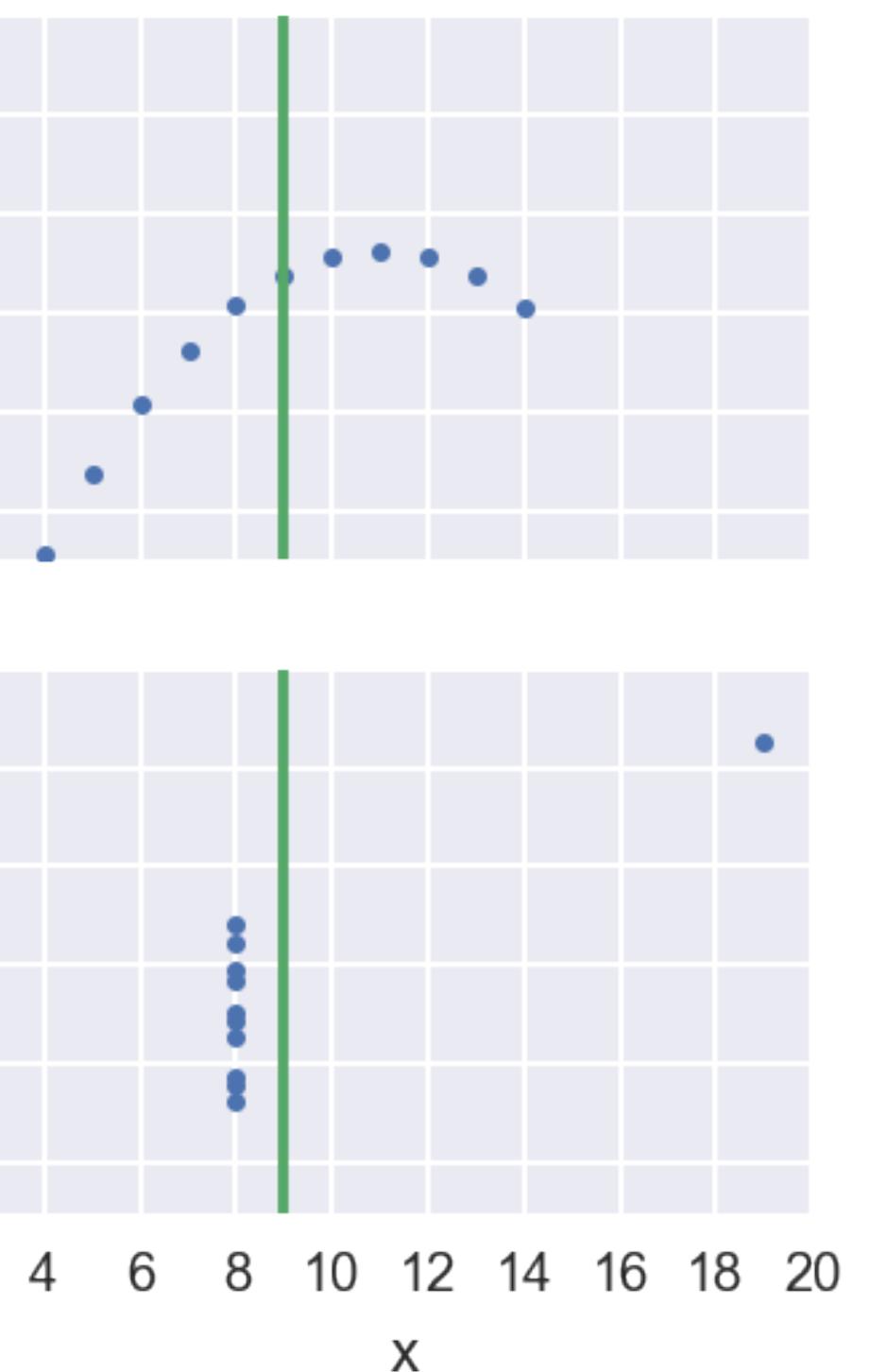
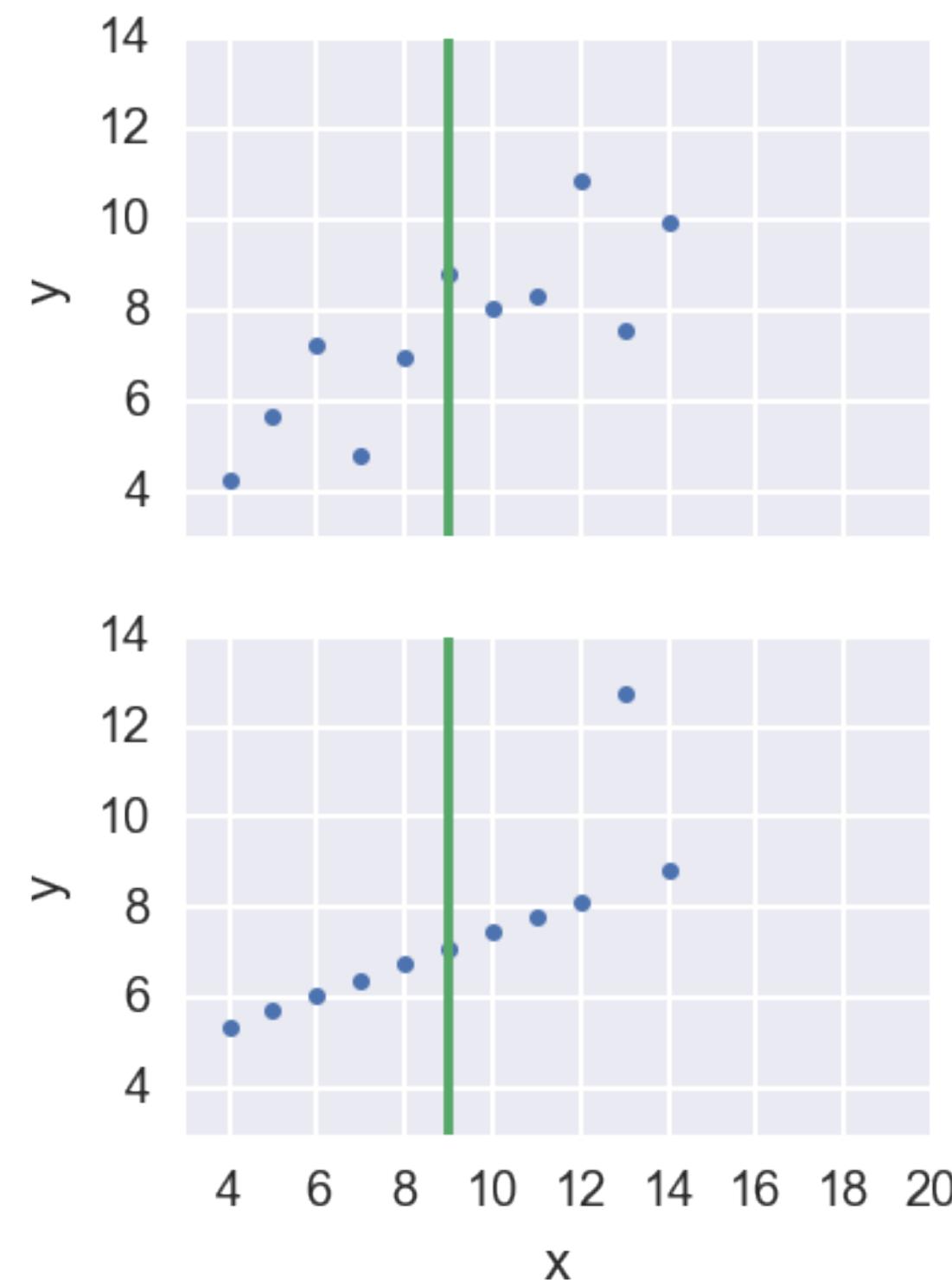


Anscombe's quartet



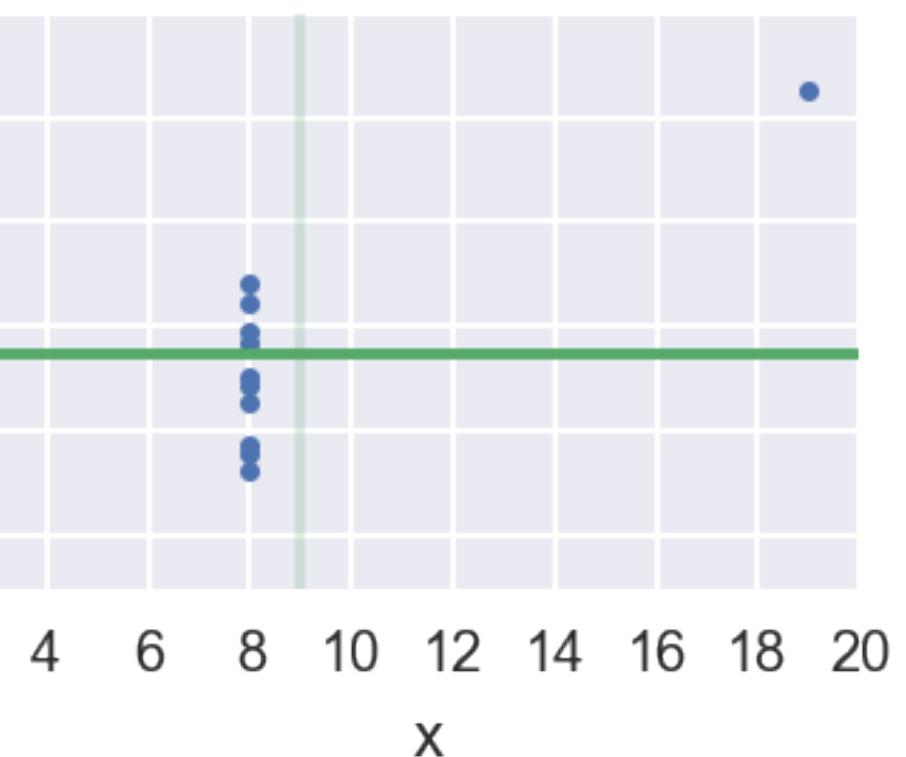
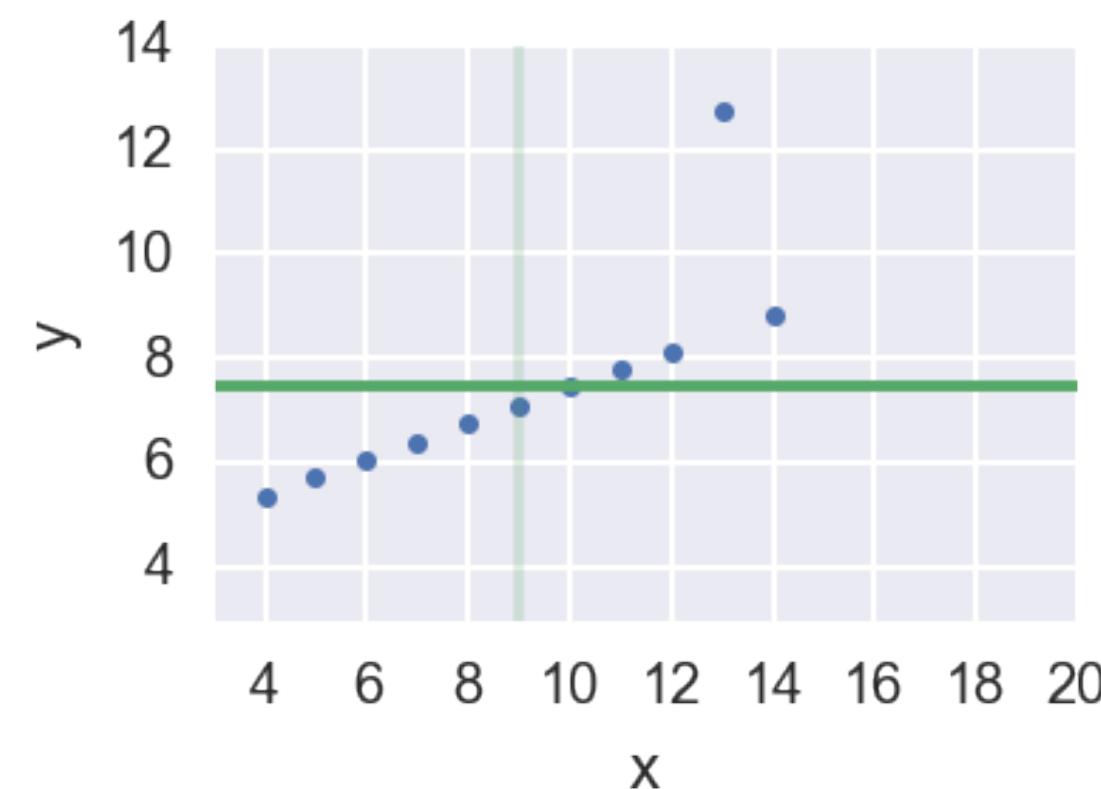
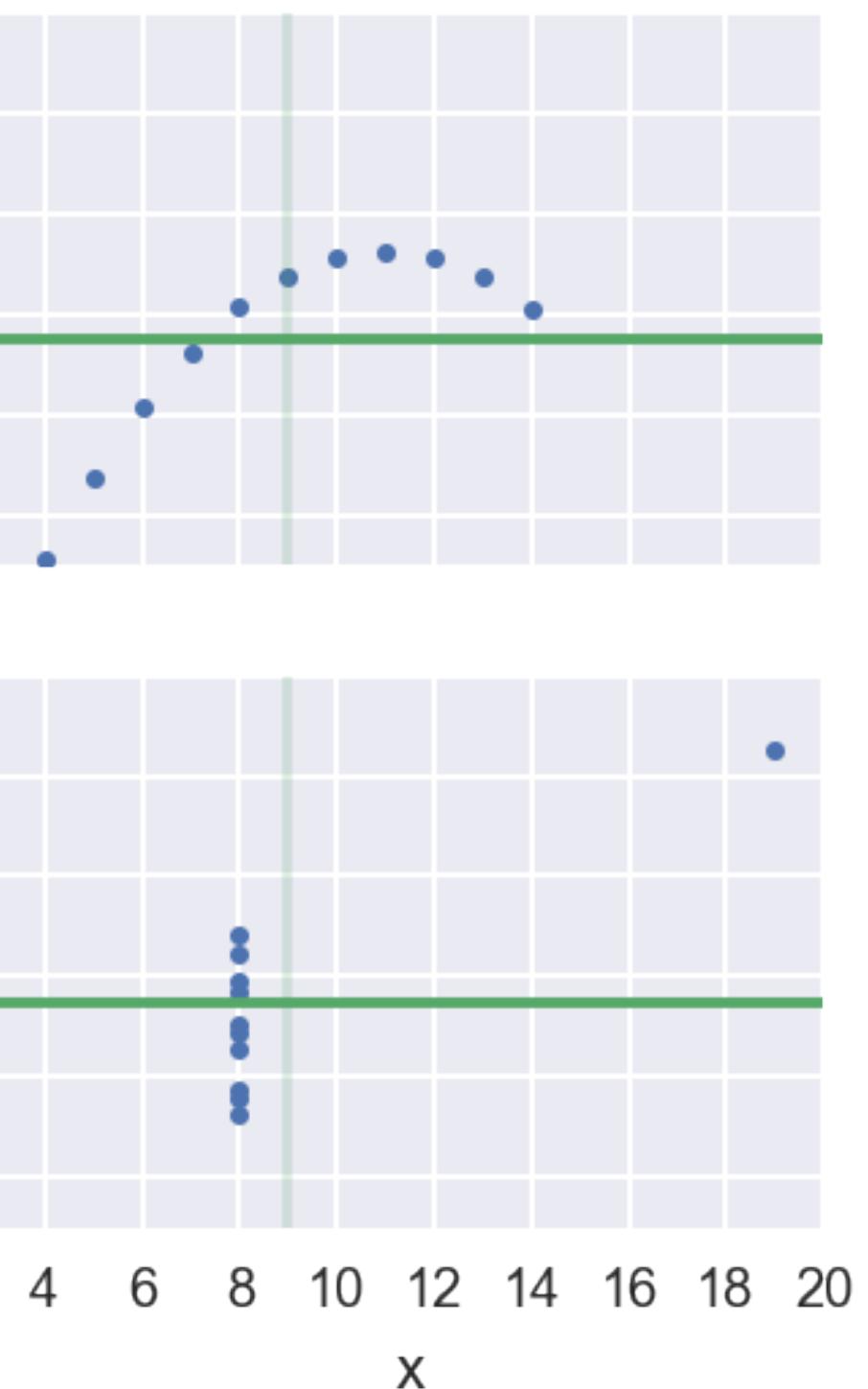
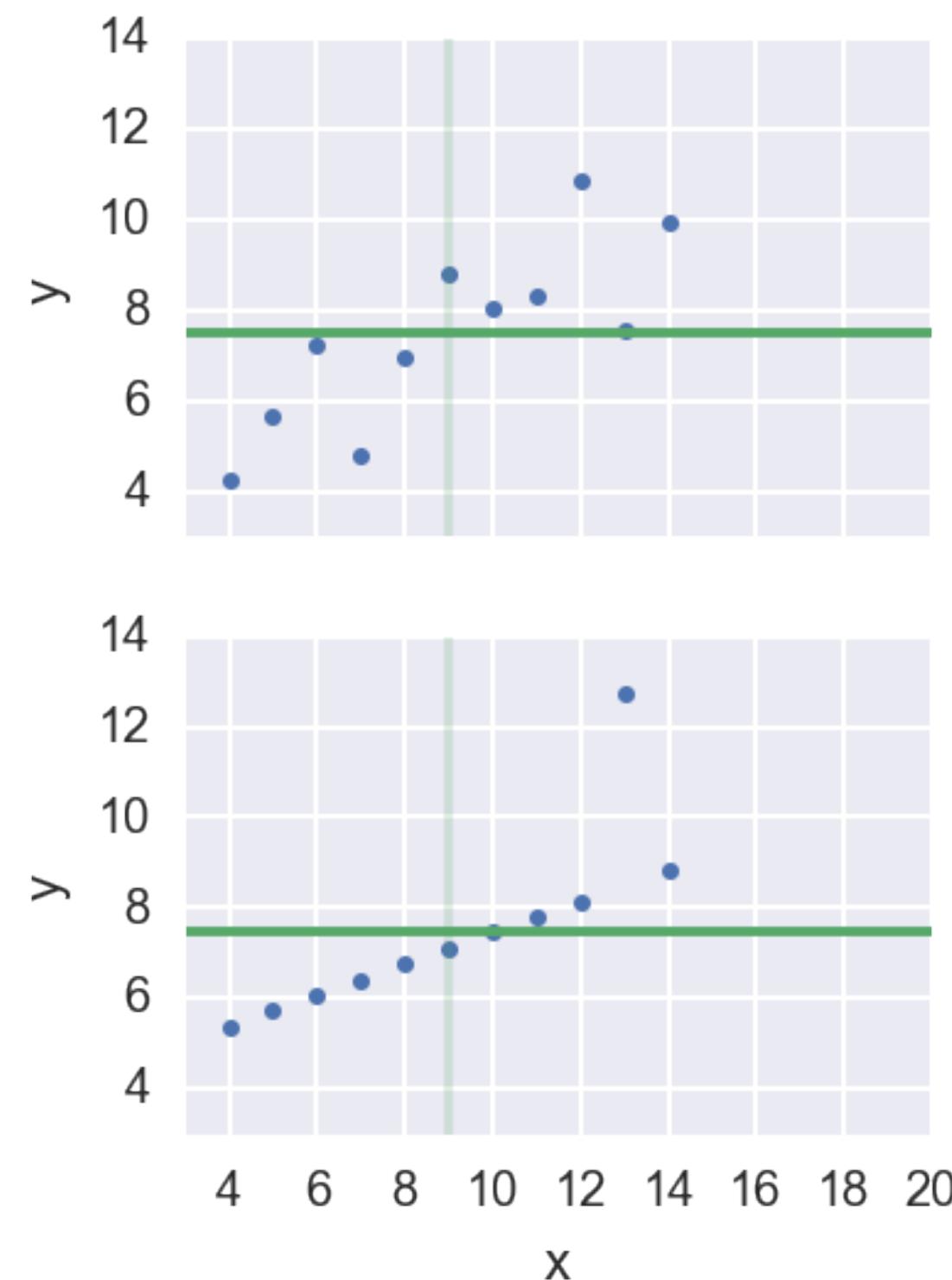


Anscombe's quartet



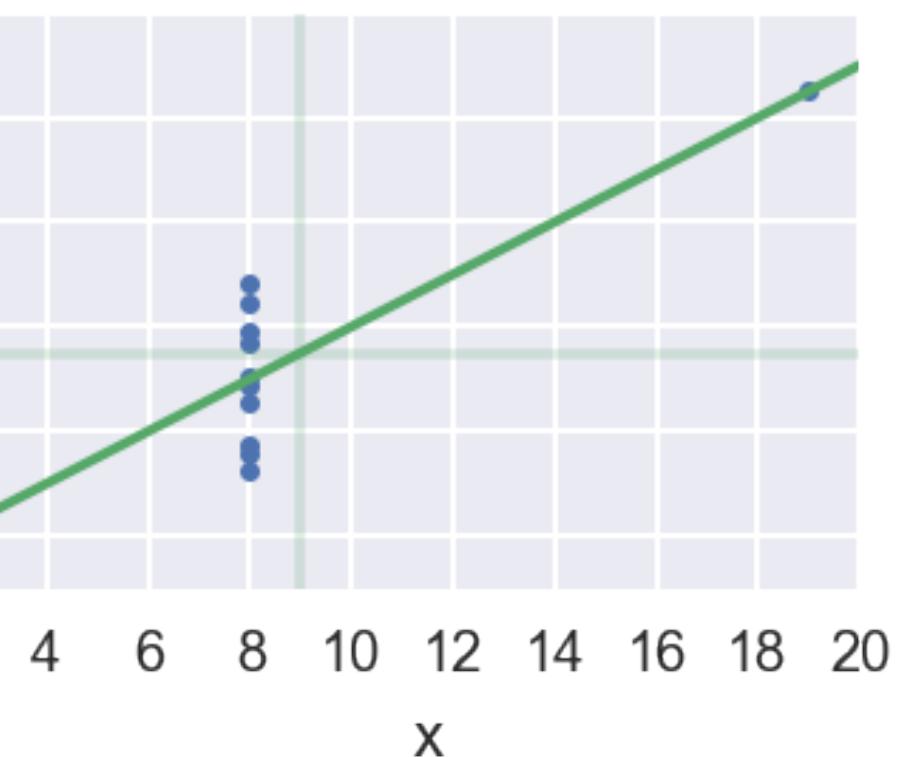
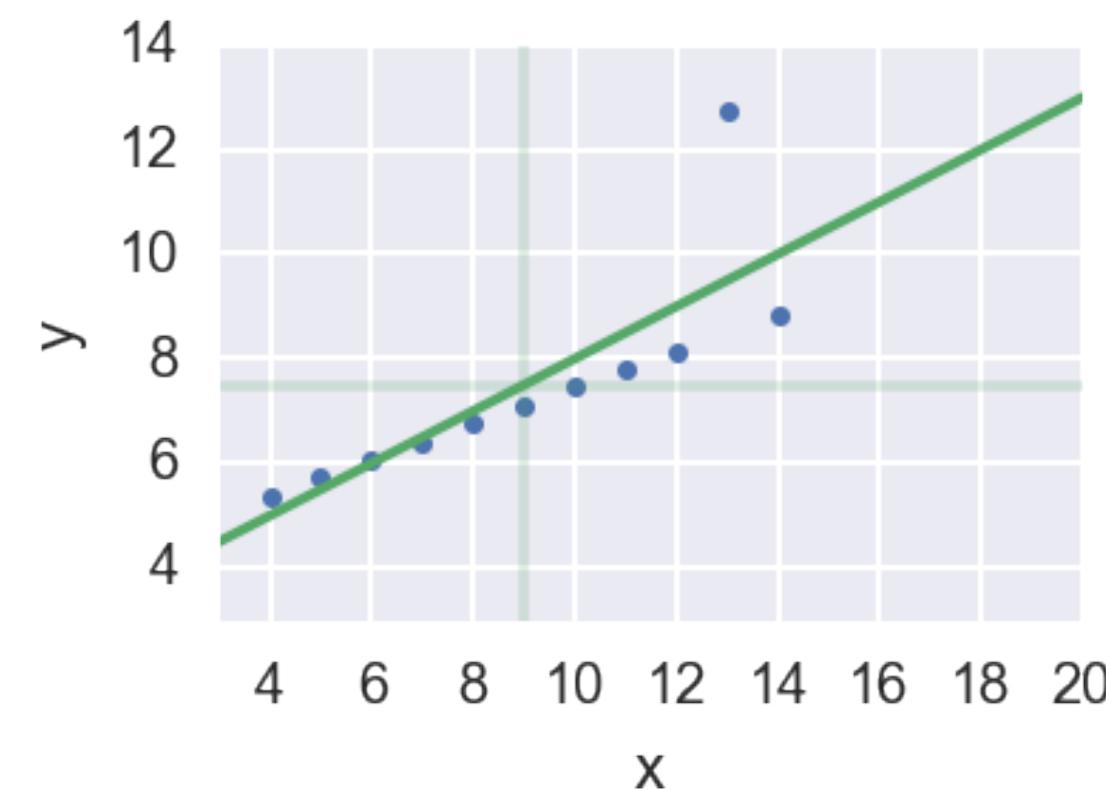
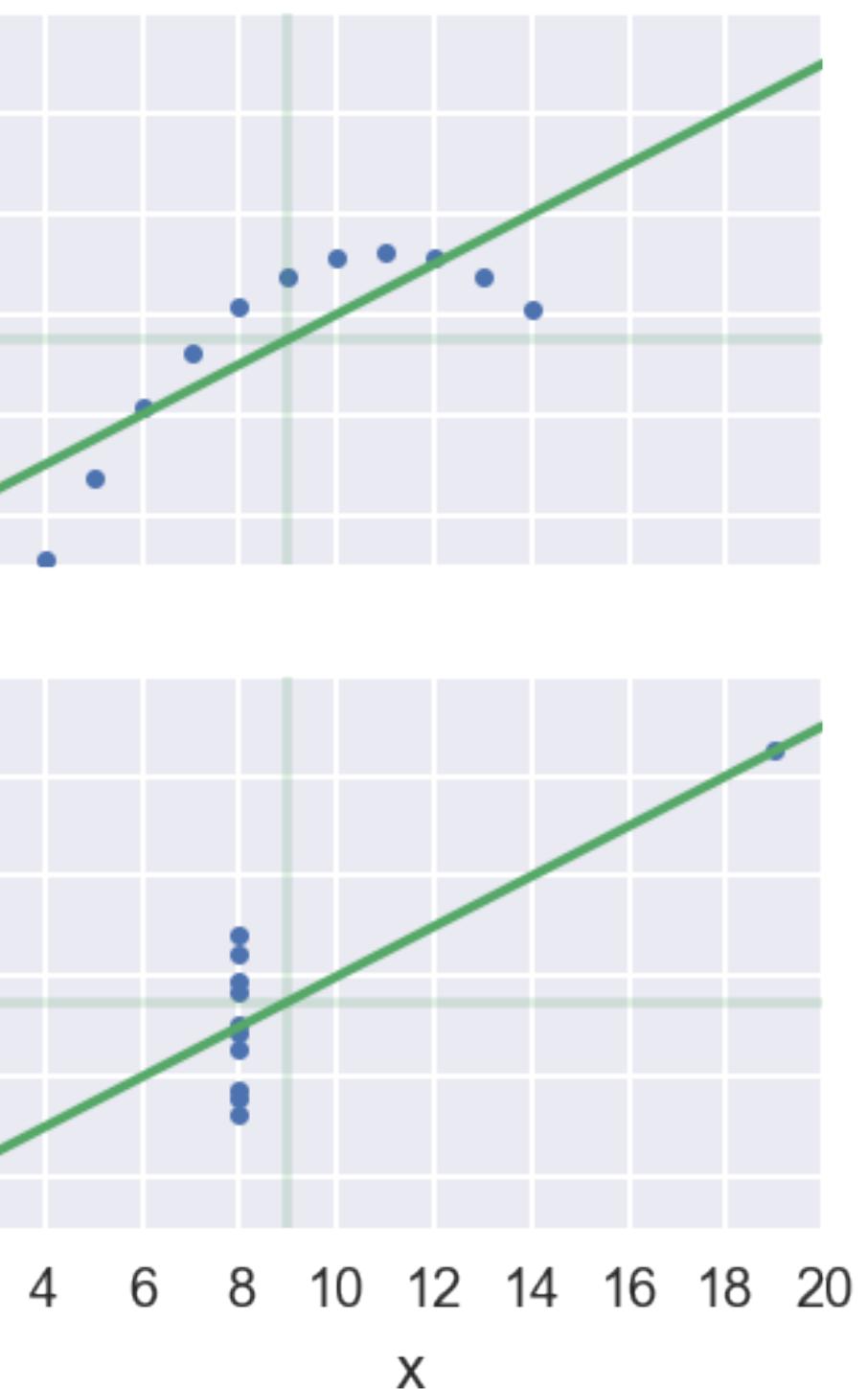
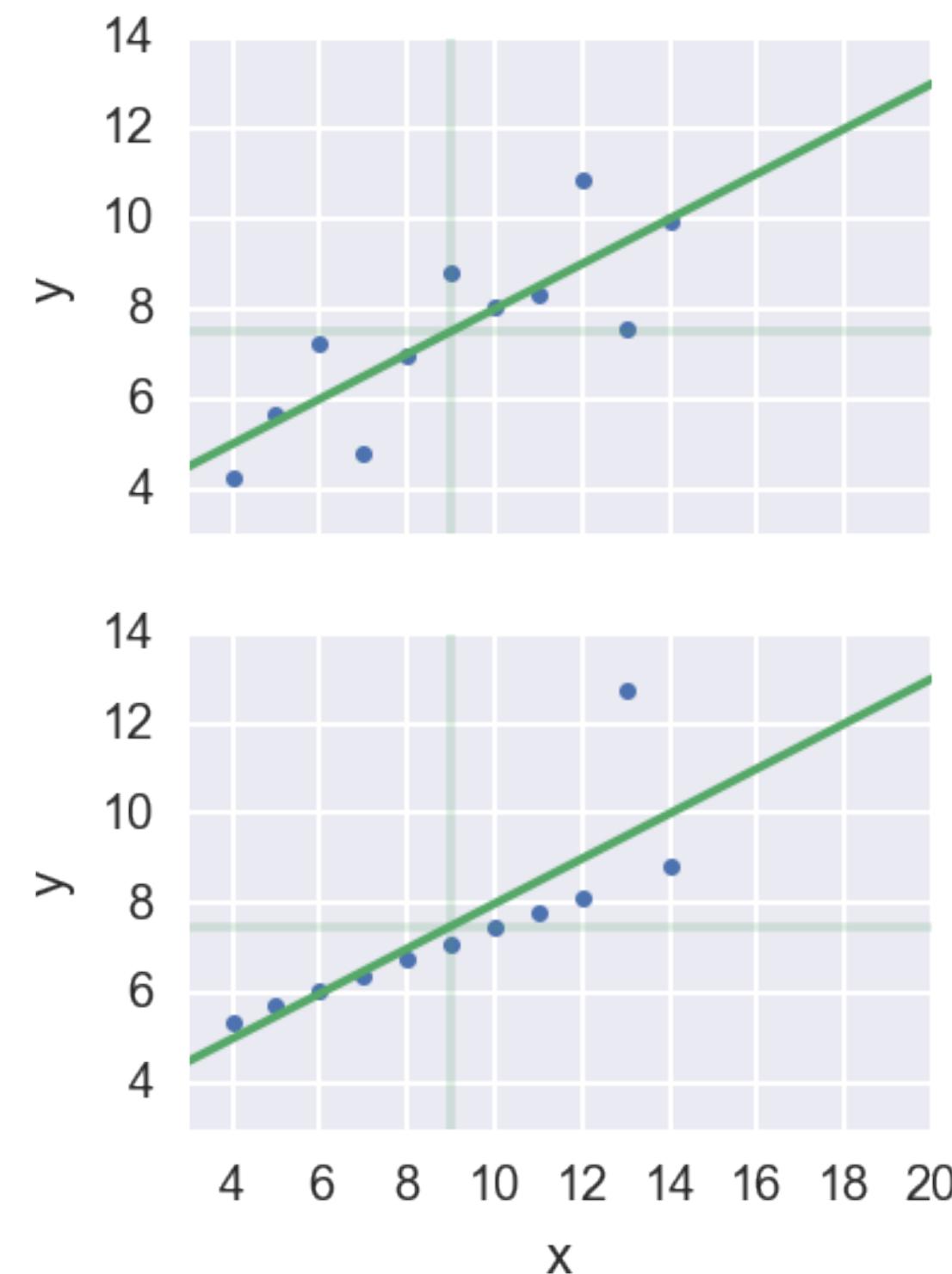


Anscombe's quartet



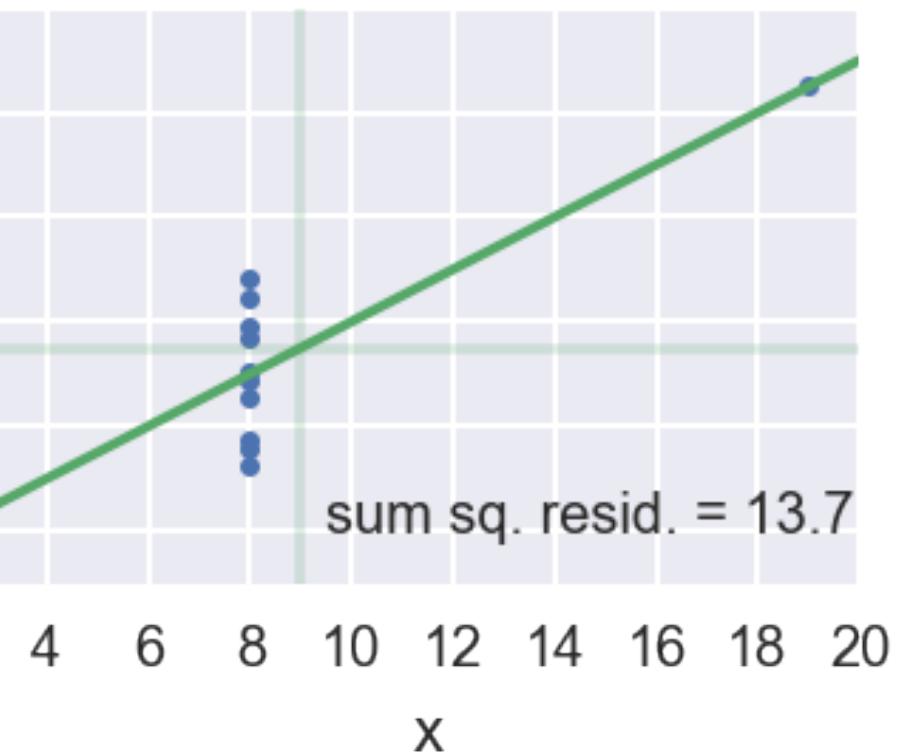
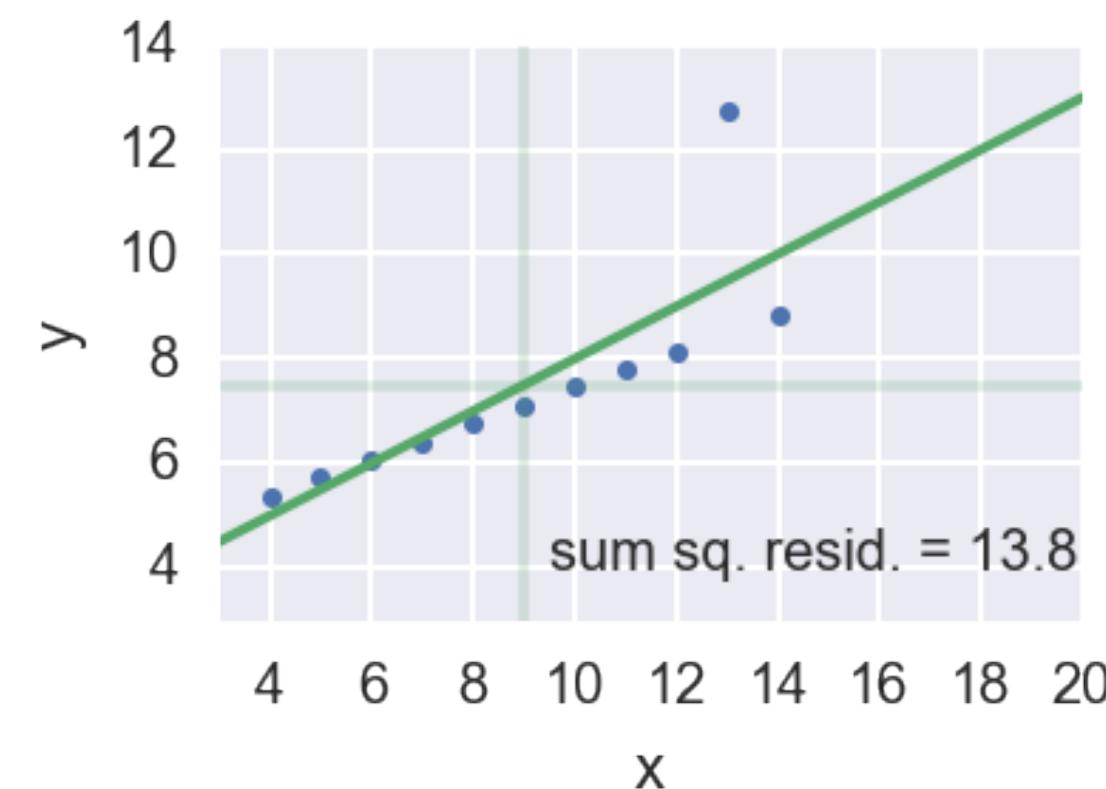
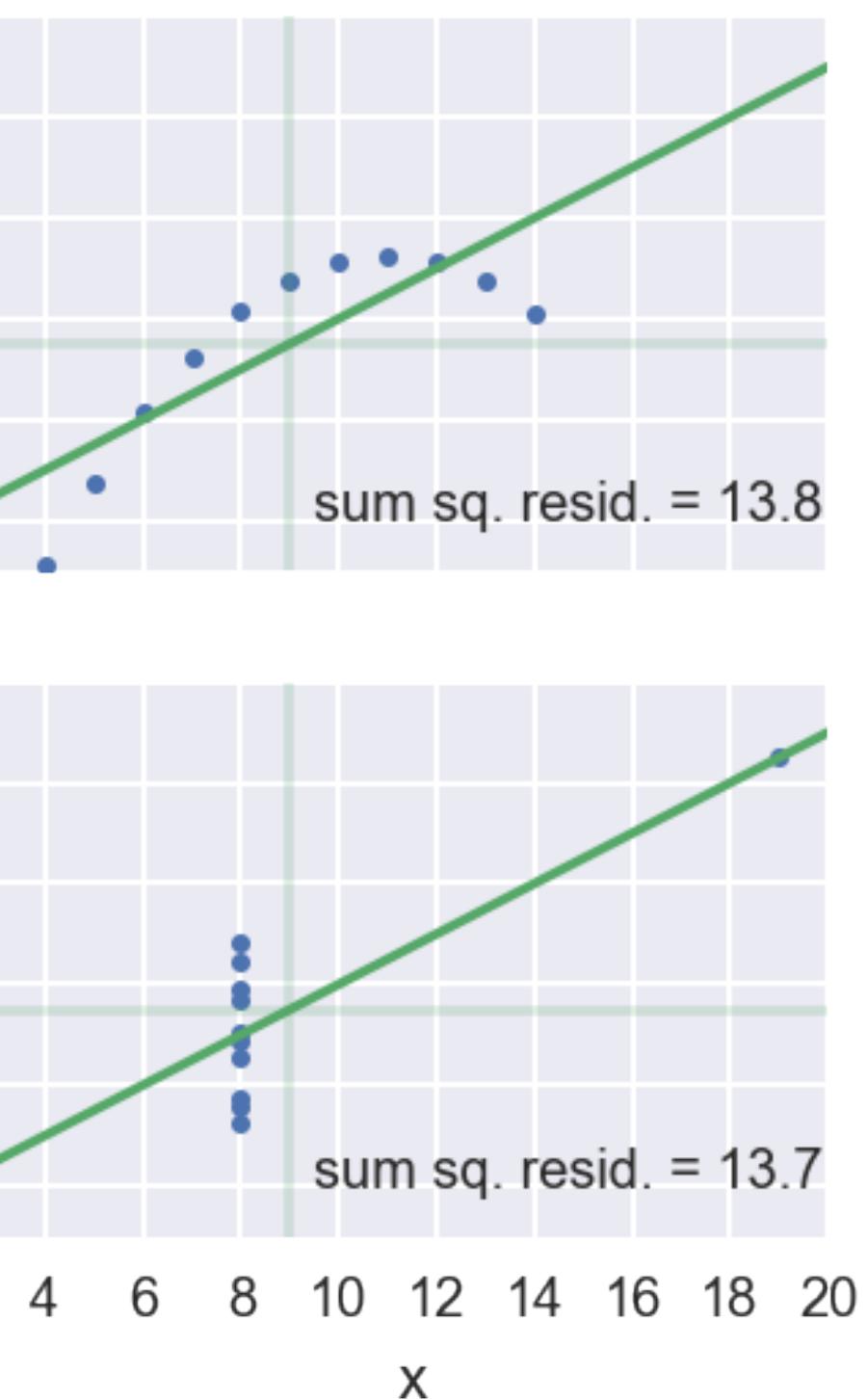
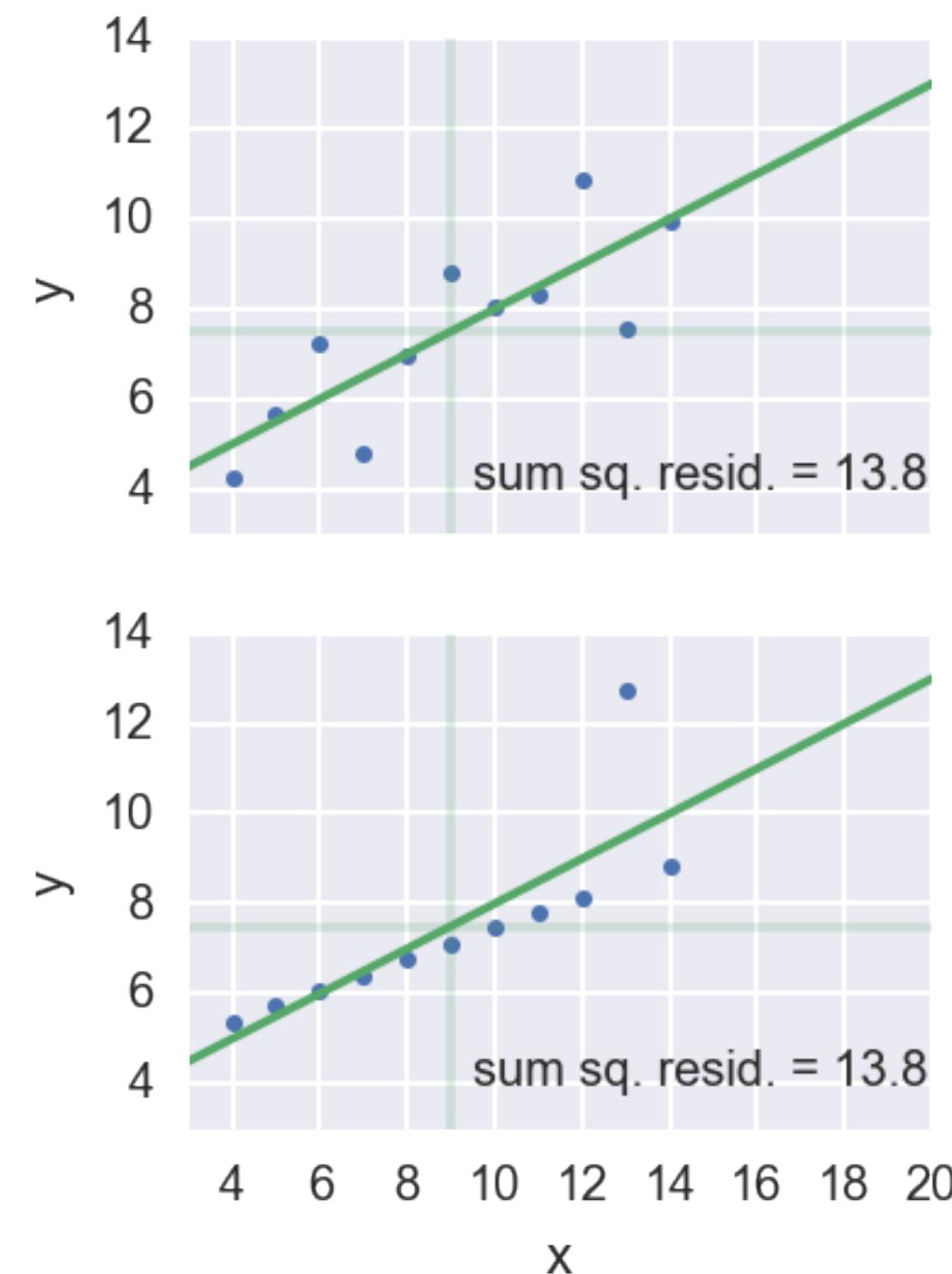


Anscombe's quartet





Anscombe's quartet



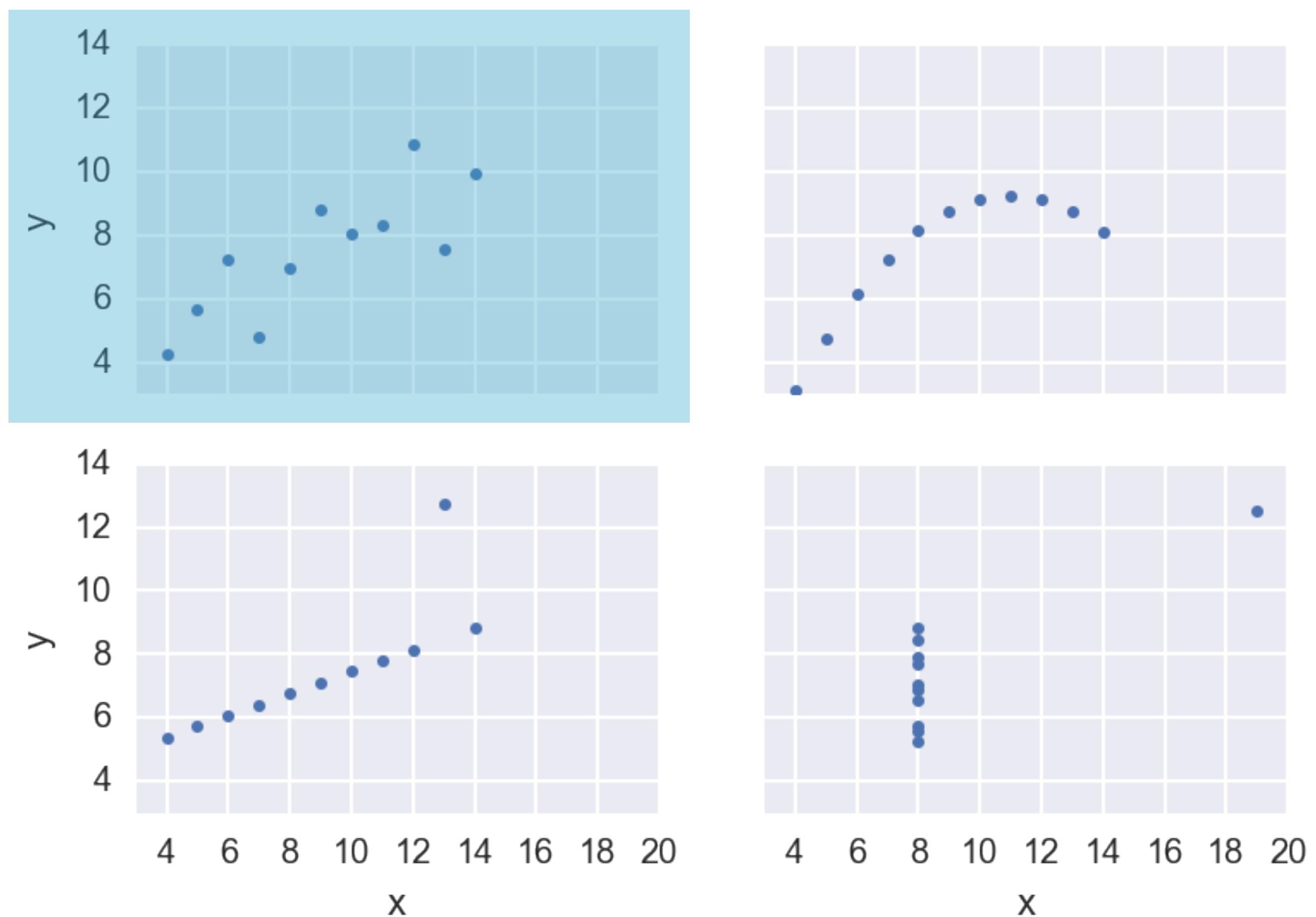


Look before you leap!

- Do graphical EDA first

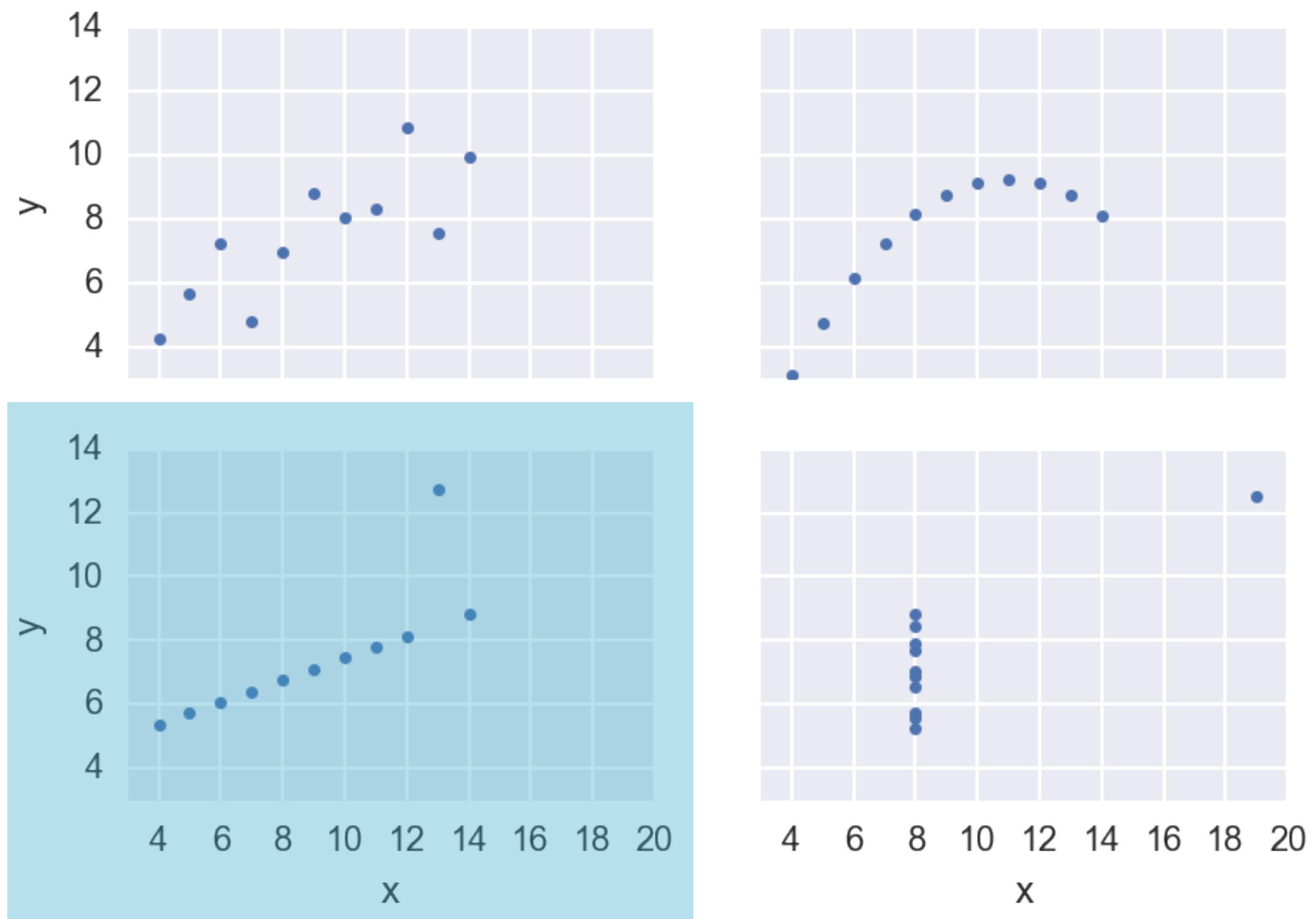


Anscombe's quartet



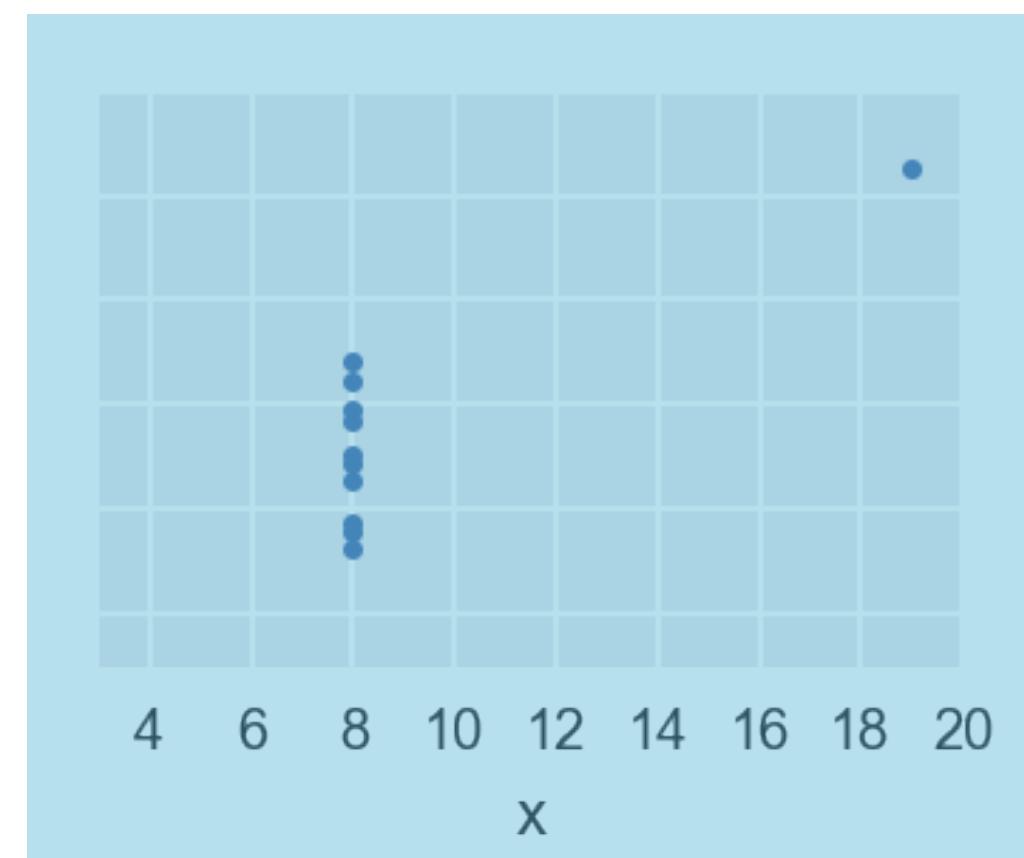
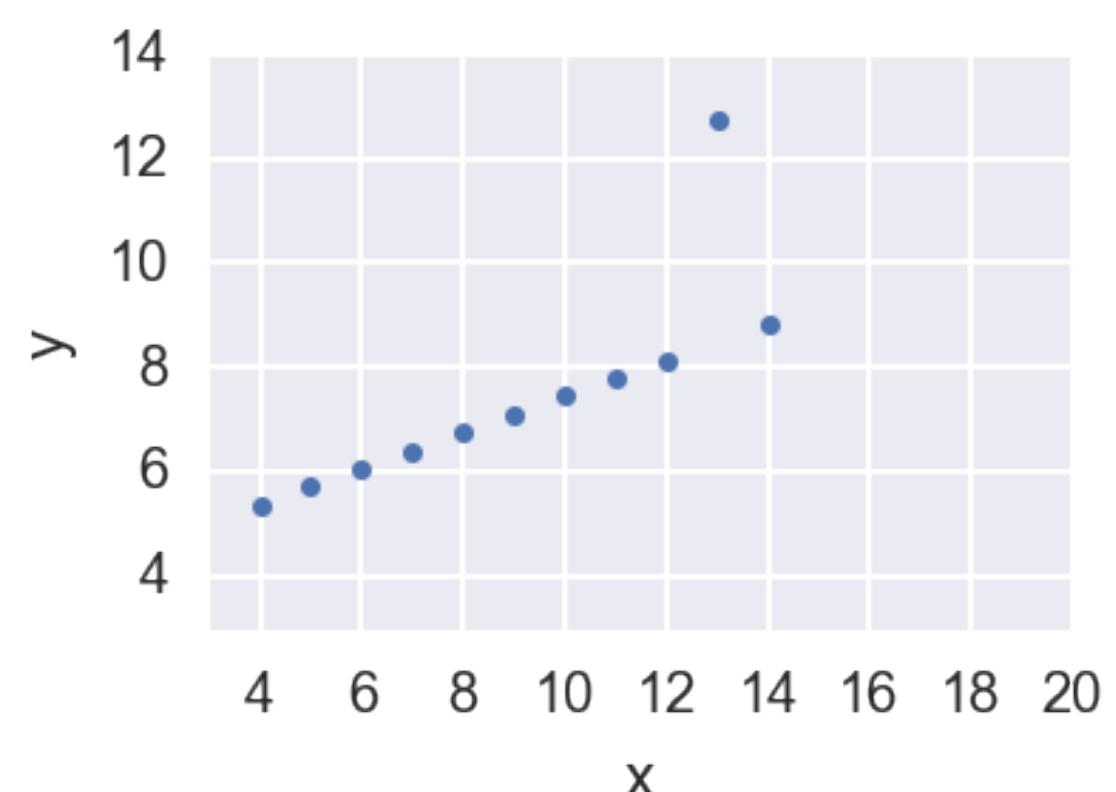
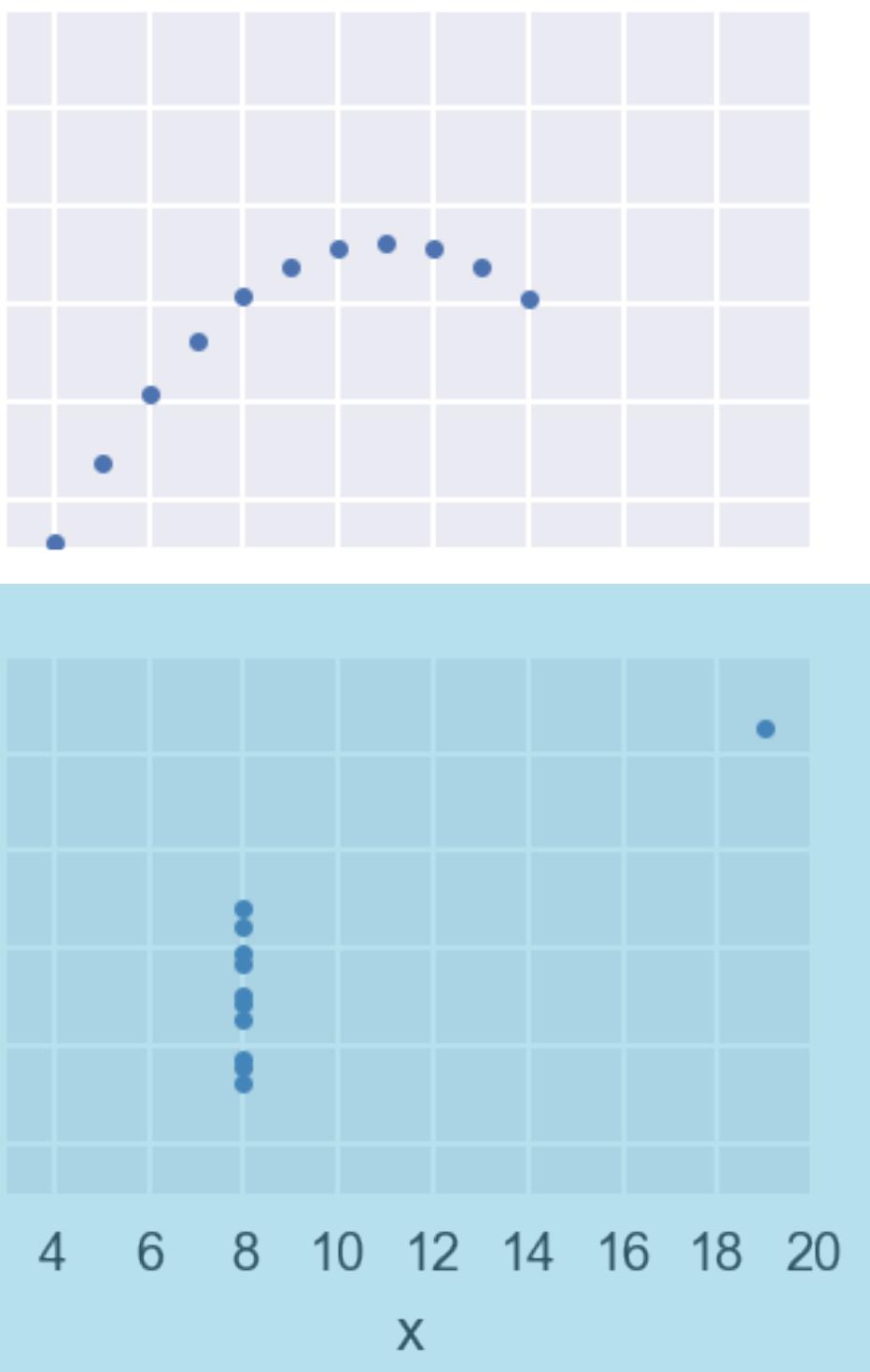
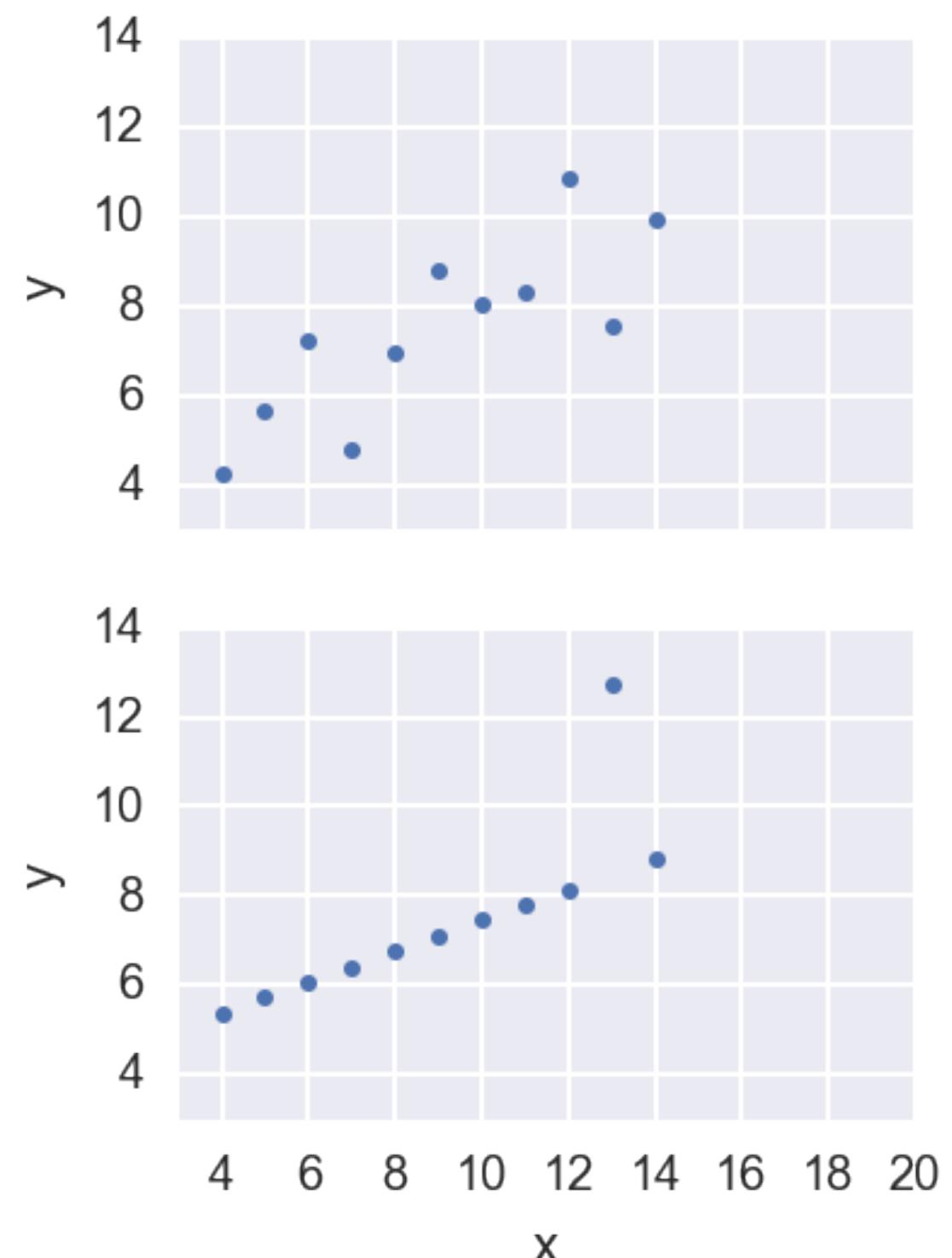


Anscombe's quartet



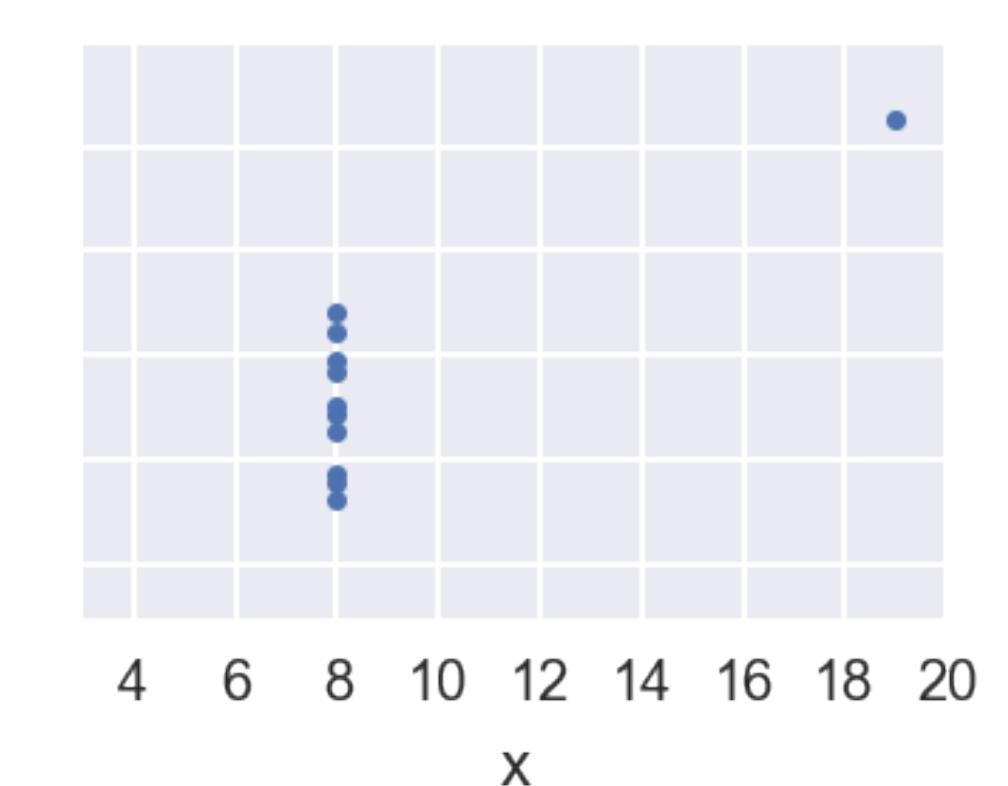
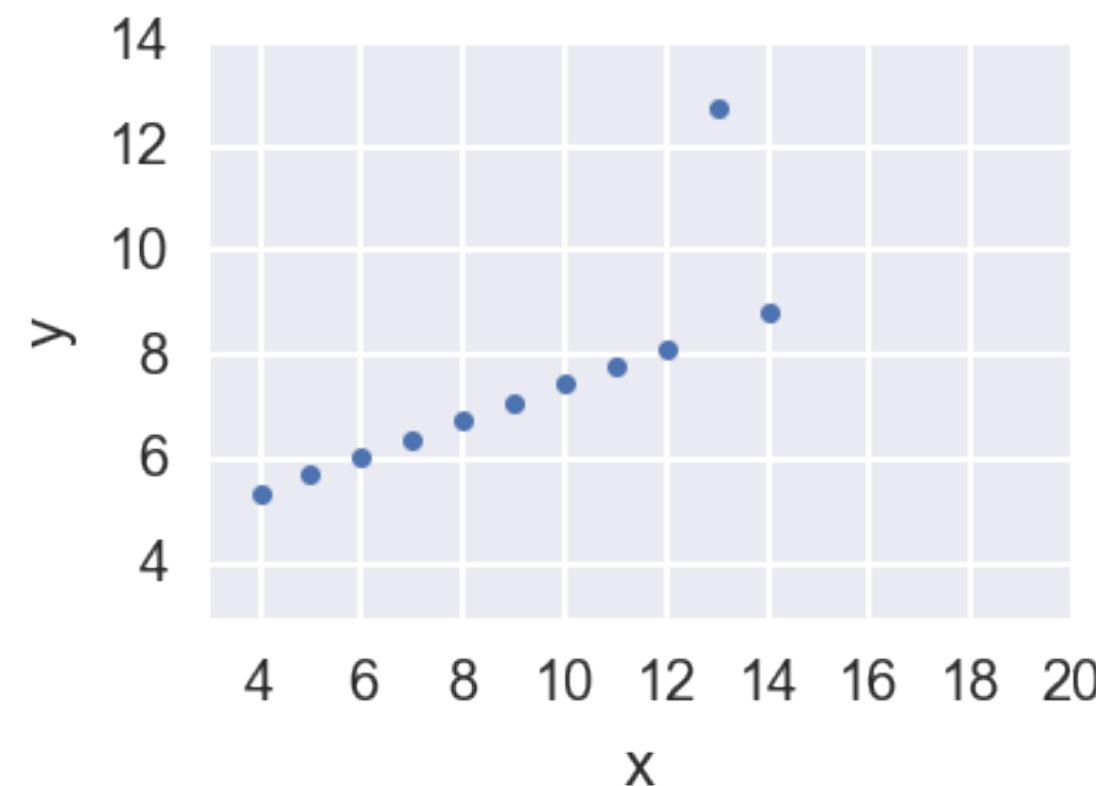
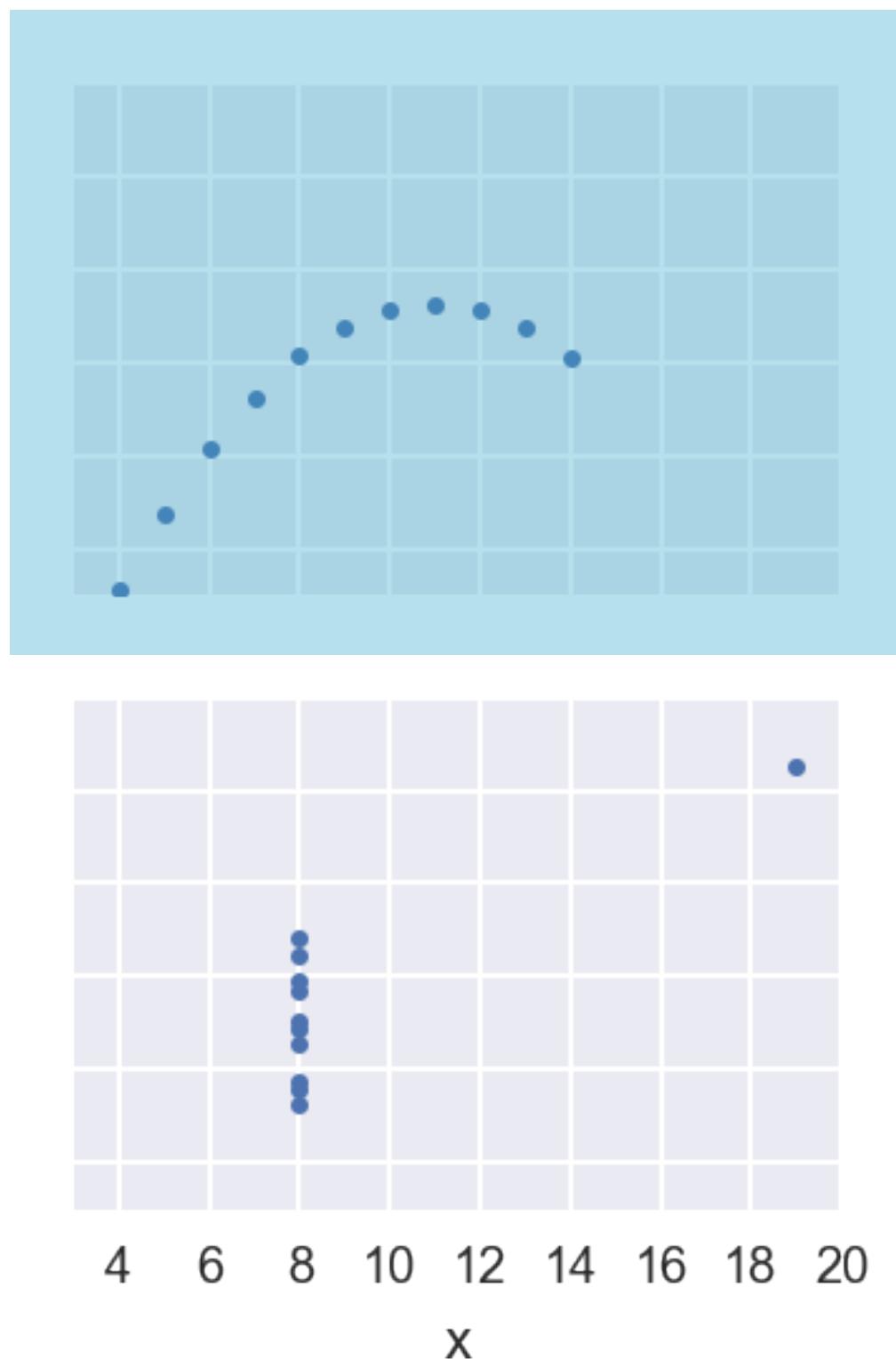
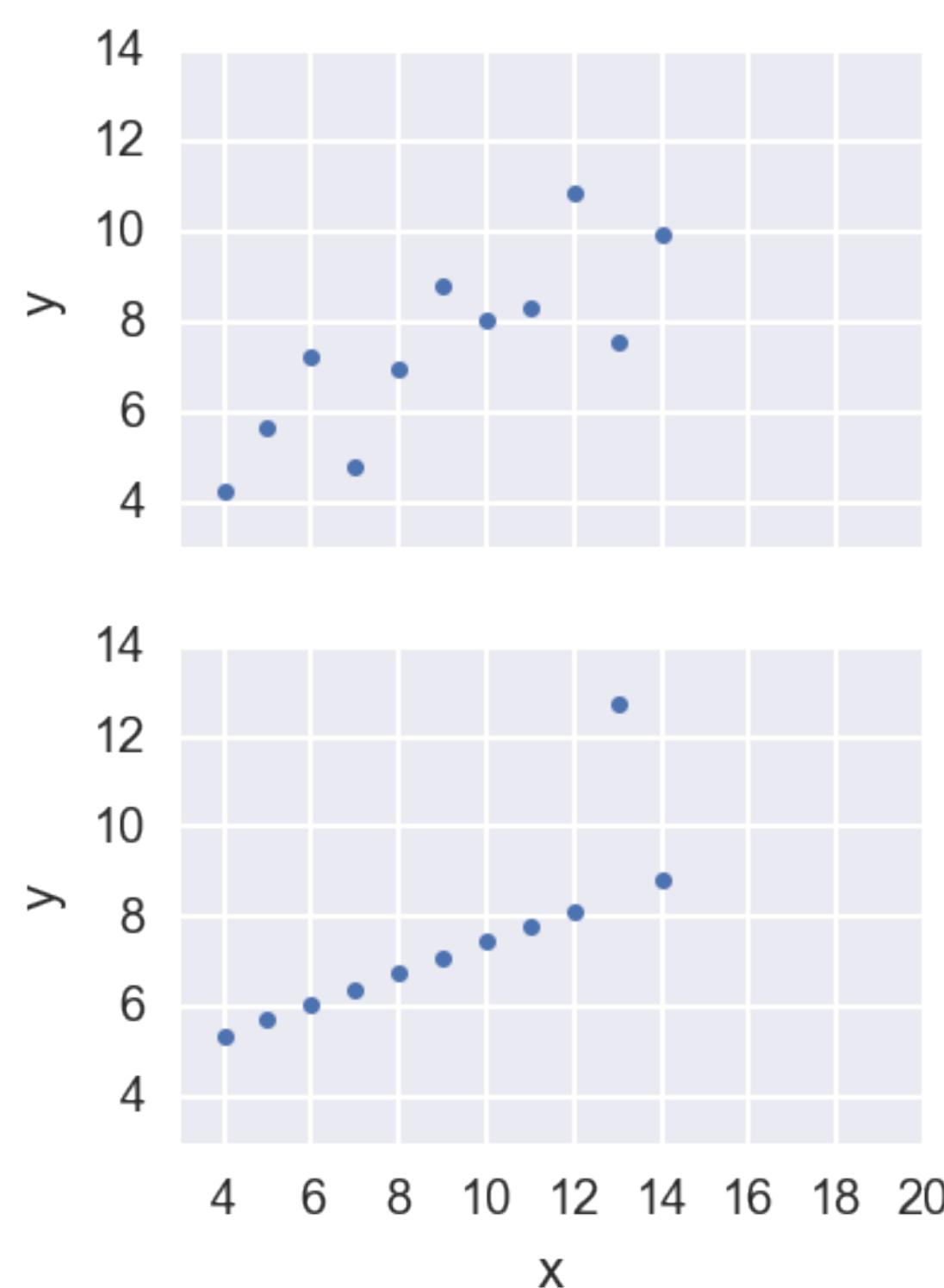


Anscombe's quartet





Anscombe's quartet





STATISTICAL THINKING IN PYTHON II

Let's practice!

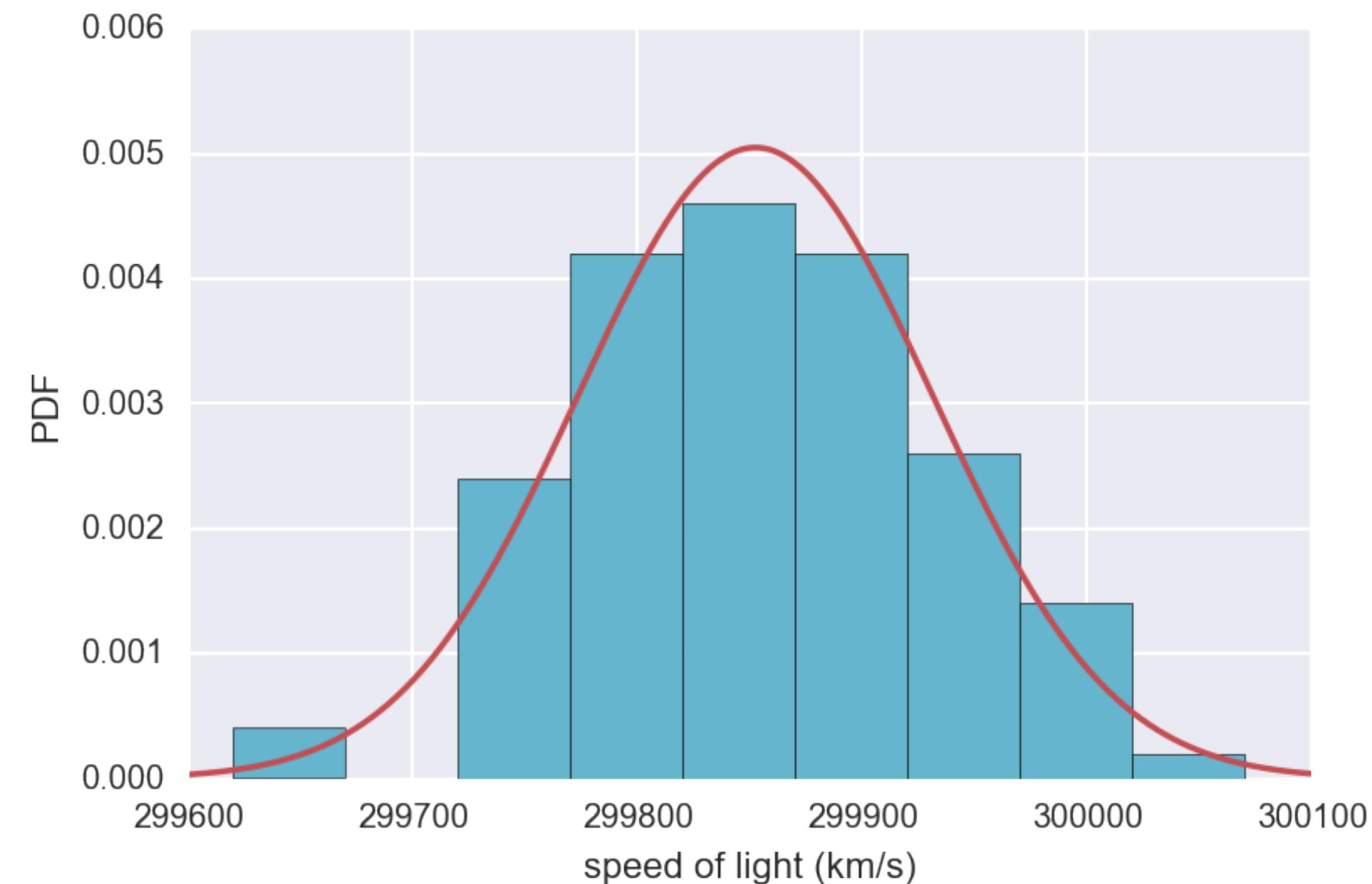


STATISTICAL THINKING IN PYTHON II

Generating bootstrap replicates



Michelson's speed of light measurements





Resampling an array

Data:

[23.3, 27.1, 24.3, 25.3, 26.0]

Mean = 25.2

Resampled data:

[, , , ,]



Resampling an array

Data:

[23.3, , 24.3, 25.3, 26.0]

Mean = 25.2

Resampled data:

[27.1, , , ,]



Resampling an array

Data:

[23.3, 27.1, 24.3, 25.3, 26.0]

Mean = 25.2

Resampled data:

[27.1, , , ,]



Resampling an array

Data:

[23.3, 27.1, 24.3, 25.3, 26.0]

Mean = 25.2

Resampled data:

[27.1, 26.0, , ,]



Resampling an array

Data:

[23.3, 27.1, 24.3, 25.7, 26.0]

Mean = 25.2

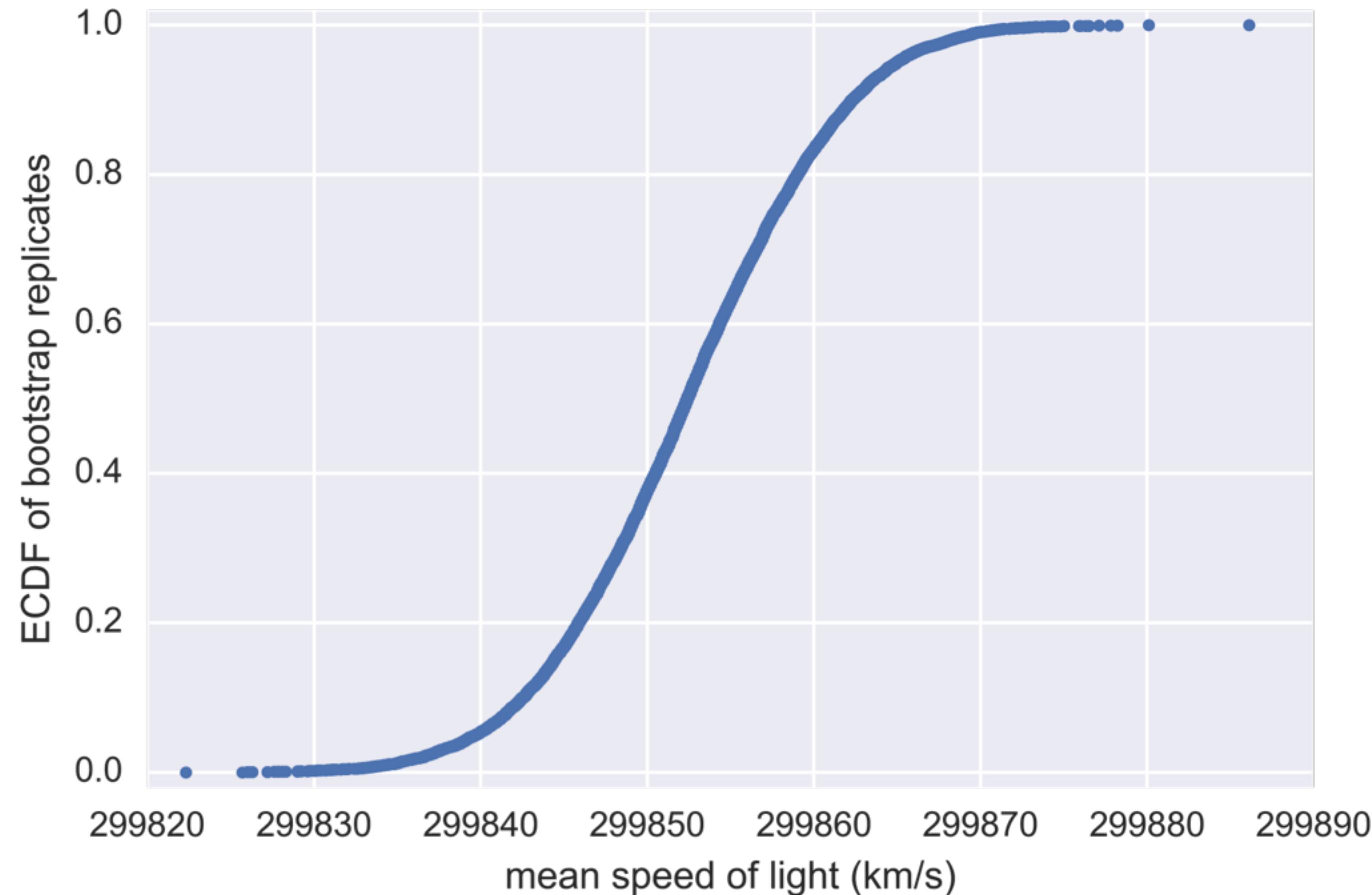
Resampled data:

[27.1, 26.0, 23.3, 25.7, 23.3]

Mean = 25.08



Mean of resampled Michelson measurements





Bootstrapping

- The use of resampled data to perform statistical inference



Bootstrap sample

- A resampled array of the data



Bootstrap replicate

- A statistic computed from a resampled array



Resampling engine: np.random.choice()

```
In [1]: import numpy as np
```

```
In [2]: np.random.choice([1,2,3,4,5], size=5)  
Out[2]: array([5, 3, 5, 5, 2])
```



Computing a bootstrap replicate

```
In [1]: bs_sample = np.random.choice(michelson_speed_of_light,  
...:  
           size=100)
```

```
In [2]: np.mean(bs_sample)  
Out[2]: 299847.7999999999
```

```
In [3]: np.median(bs_sample)  
Out[3]: 299845.0
```

```
In [4]: np.std(bs_sample)  
Out[4]: 83.56428602572931
```



STATISTICAL THINKING IN PYTHON II

Let's practice!



STATISTICAL THINKING WITH PYTHON II

Bootstrap confidence intervals



Bootstrap replicate function

```
In [1]: def bootstrap_replicate_1d(data, func):  
...:     """Generate bootstrap replicate of 1D data."""  
...:     bs_sample = np.random.choice(data, len(data))  
...:     return func(bs_sample)  
...:
```

```
In [2]: bootstrap_replicate_1d(michelson_speed_of_light, np.mean)  
Out[2]: 299859.2000000001
```

```
In [3]: bootstrap_replicate_1d(michelson_speed_of_light, np.mean)  
Out[3]: 299855.7000000001
```

```
In [4]: bootstrap_replicate_1d(michelson_speed_of_light, np.mean)  
Out[4]: 299850.2999999999
```



Many bootstrap replicates

```
In [1]: bs_replicates = np.empty(10000)

In [2]: for i in range(10000):
....:     bs_replicates[i] = bootstrap_replicate_1d(
....:         michelson_speed_of_light, np.mean)
....:
```



Plotting a histogram of bootstrap replicates

```
In [1]: _ = plt.hist(bs_replicates, bins=30, normed=True)

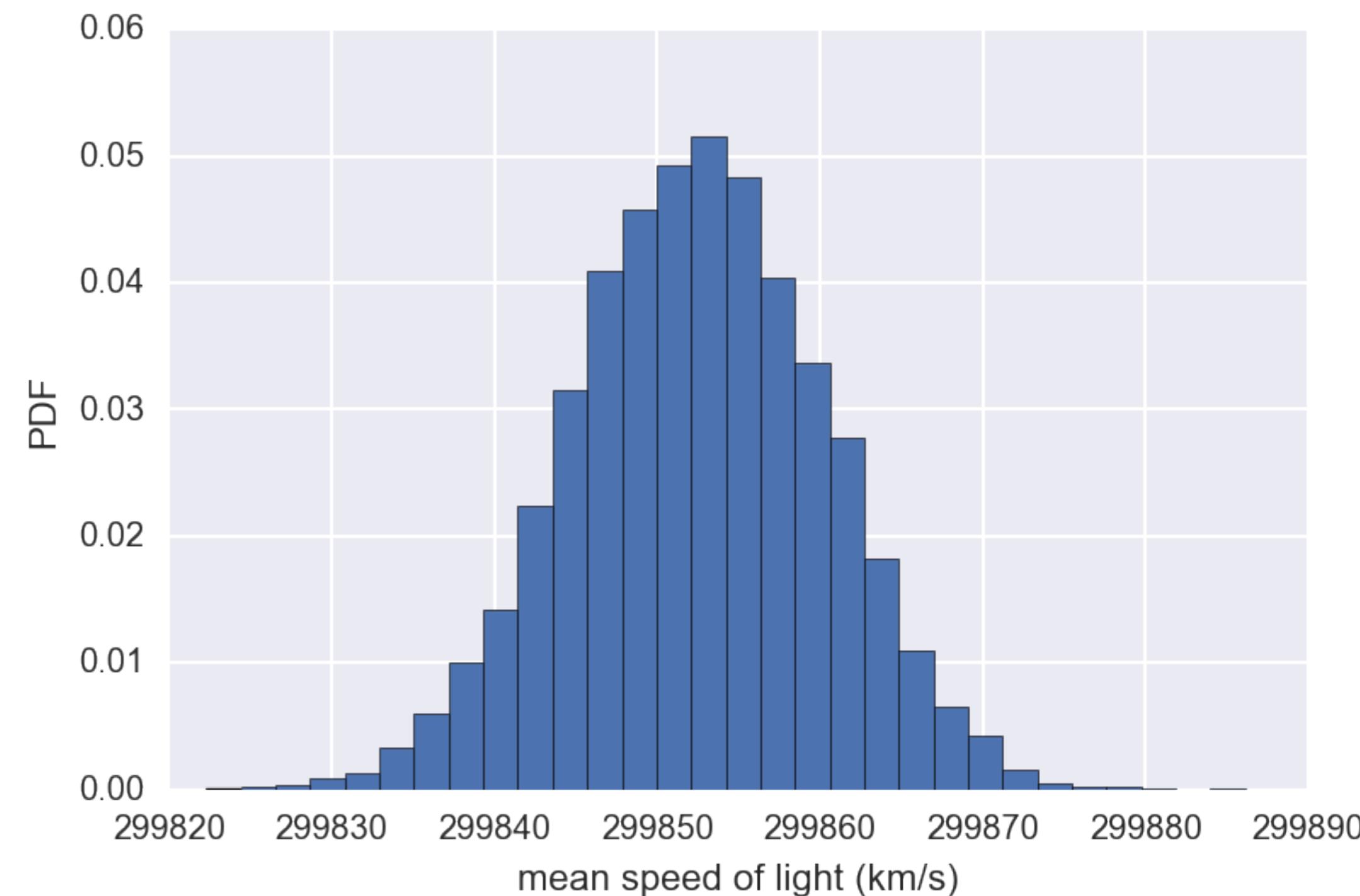
In [2]: _ = plt.xlabel('mean speed of light (km/s)')

In [3]: _ = plt.ylabel('PDF')

In [4]: plt.show()
```



Bootstrap estimate of the mean





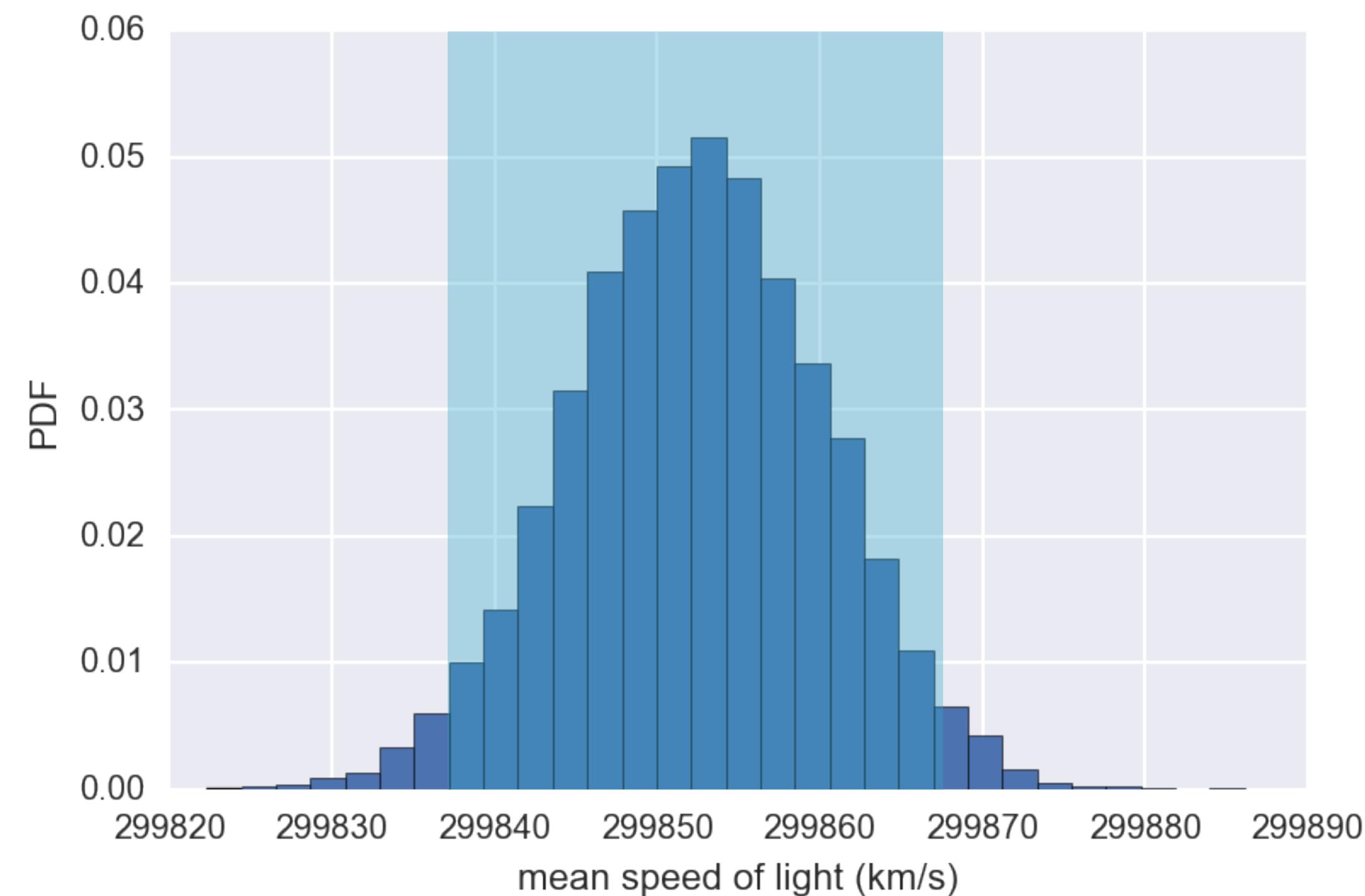
Confidence interval of a statistic

- If we repeated measurements over and over again, $p\%$ of the observed values would lie within the $p\%$ confidence interval.



Bootstrap confidence interval

```
In [1]: conf_int = np.percentile(bs_replicates, [2.5, 97.5])  
Out[1]: array([ 299837.,  299868.])
```





STATISTICAL THINKING WITH PYTHON II

Let's practice!



STATISTICAL THINKING IN PYTHON II

Pairs bootstrap

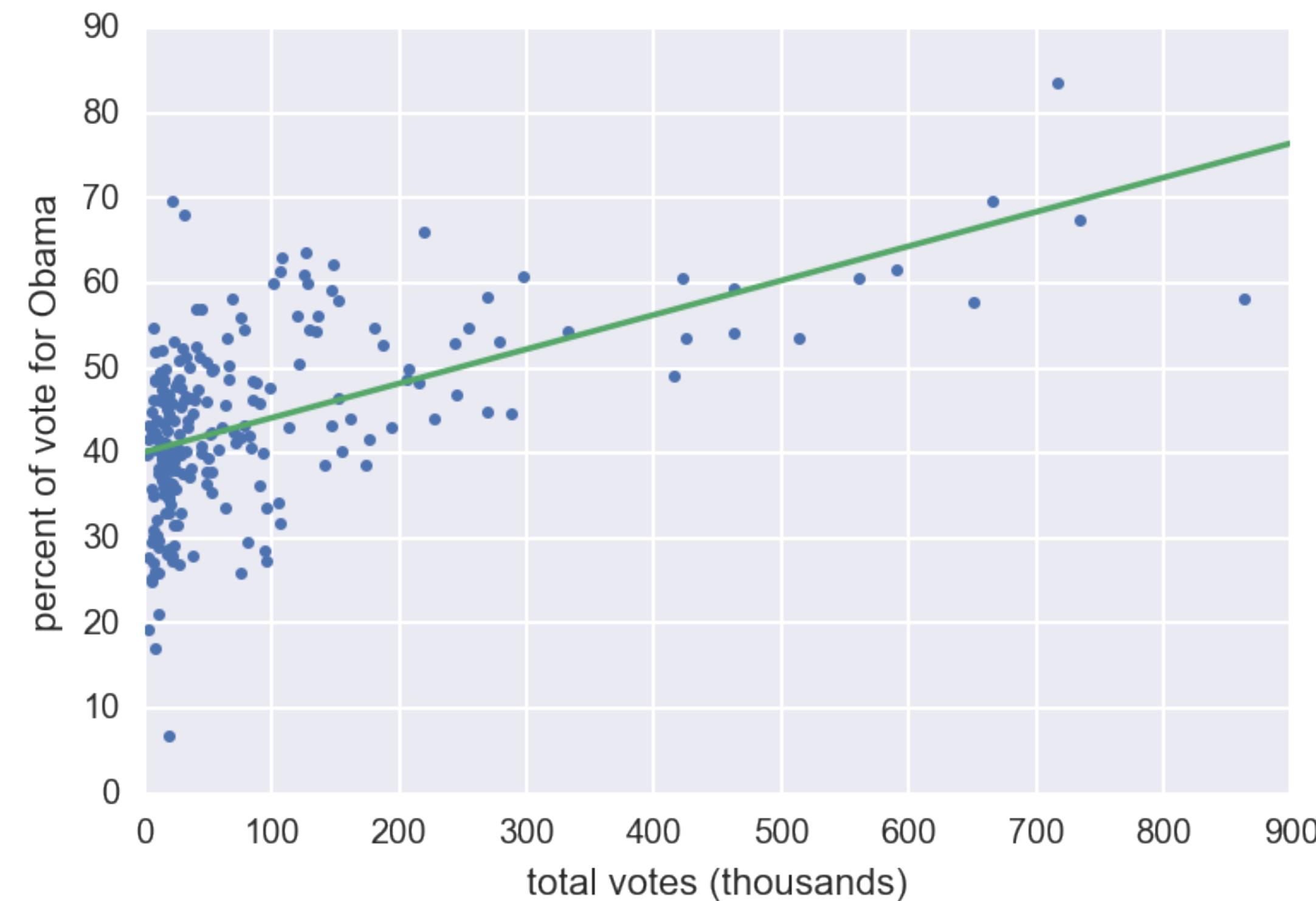


Nonparametric inference

- Make no assumptions about the model or probability distribution underlying the data



2008 US swing state election results





Pairs bootstrap for linear regression

- Resample data *in pairs*
- Compute slope and intercept from resampled data
- Each slope and intercept is a bootstrap replicate
- Compute confidence intervals from percentiles of bootstrap replicates



Generating a pairs bootstrap sample

```
In [1]: np.arange(7)
Out[1]: array([0, 1, 2, 3, 4, 5, 6])

In [1]: inds = np.arange(len(total_votes))

In [2]: bs_inds = np.random.choice(inds, len(inds))

In [3]: bs_total_votes = total_votes[bs_inds]

In [4]: bs_dem_share = dem_share[bs_inds]
```



Computing a pairs bootstrap replicate

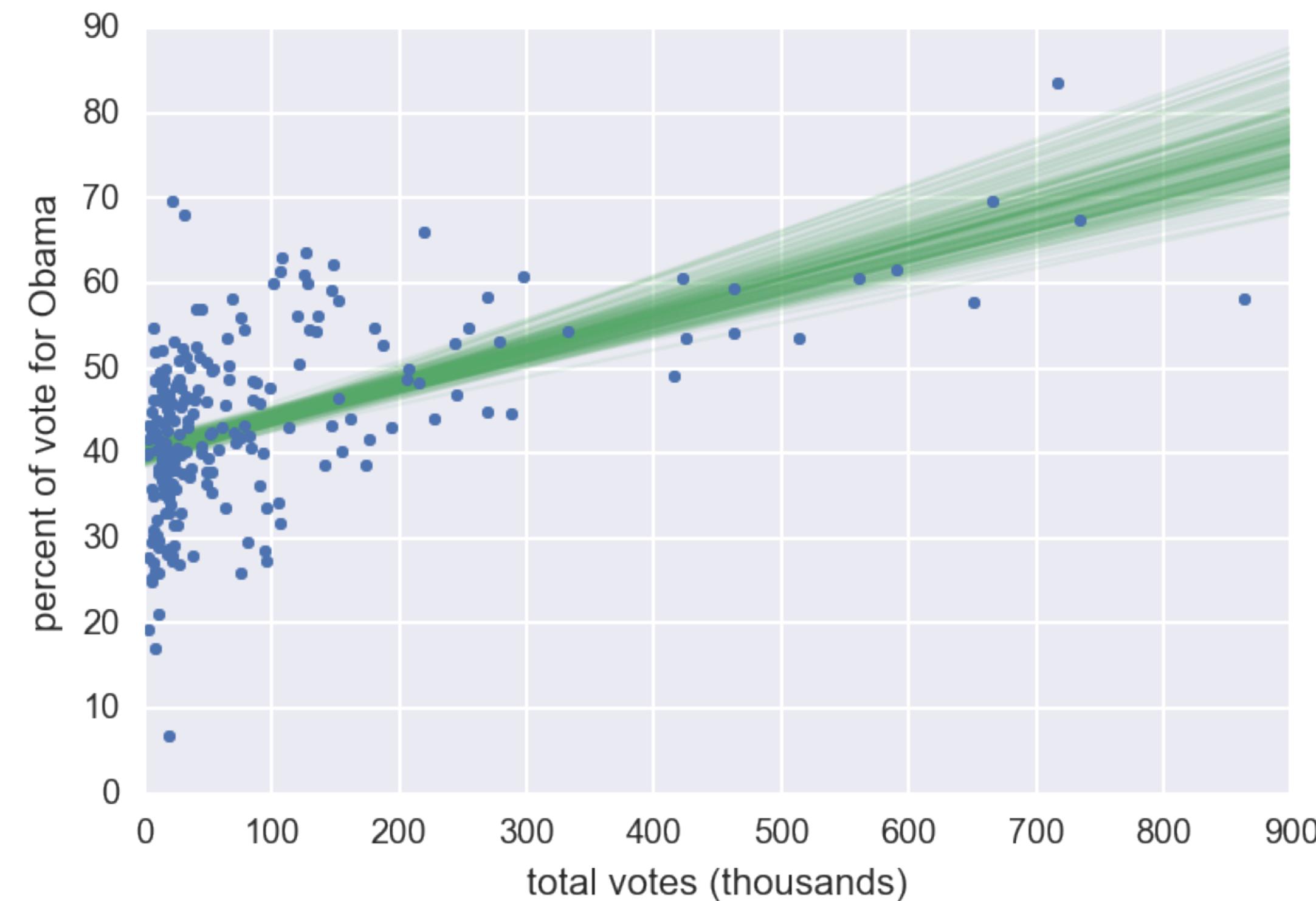
```
In [1]: bs_slope, bs_intercept = np.polyfit(bs_total_votes,  
...:                                     bs_dem_share, 1)
```

```
In [2]: bs_slope, bs_intercept  
Out[2]: (3.9053605692223672e-05, 40.387910131803025)
```

```
In [3]: np.polyfit(total_votes, dem_share, 1) # fit of original  
Out[3]: array([-4.03707170e-05, 4.01139120e+01])
```



2008 US swing state election results





STATISTICAL THINKING IN PYTHON II

Let's practice!

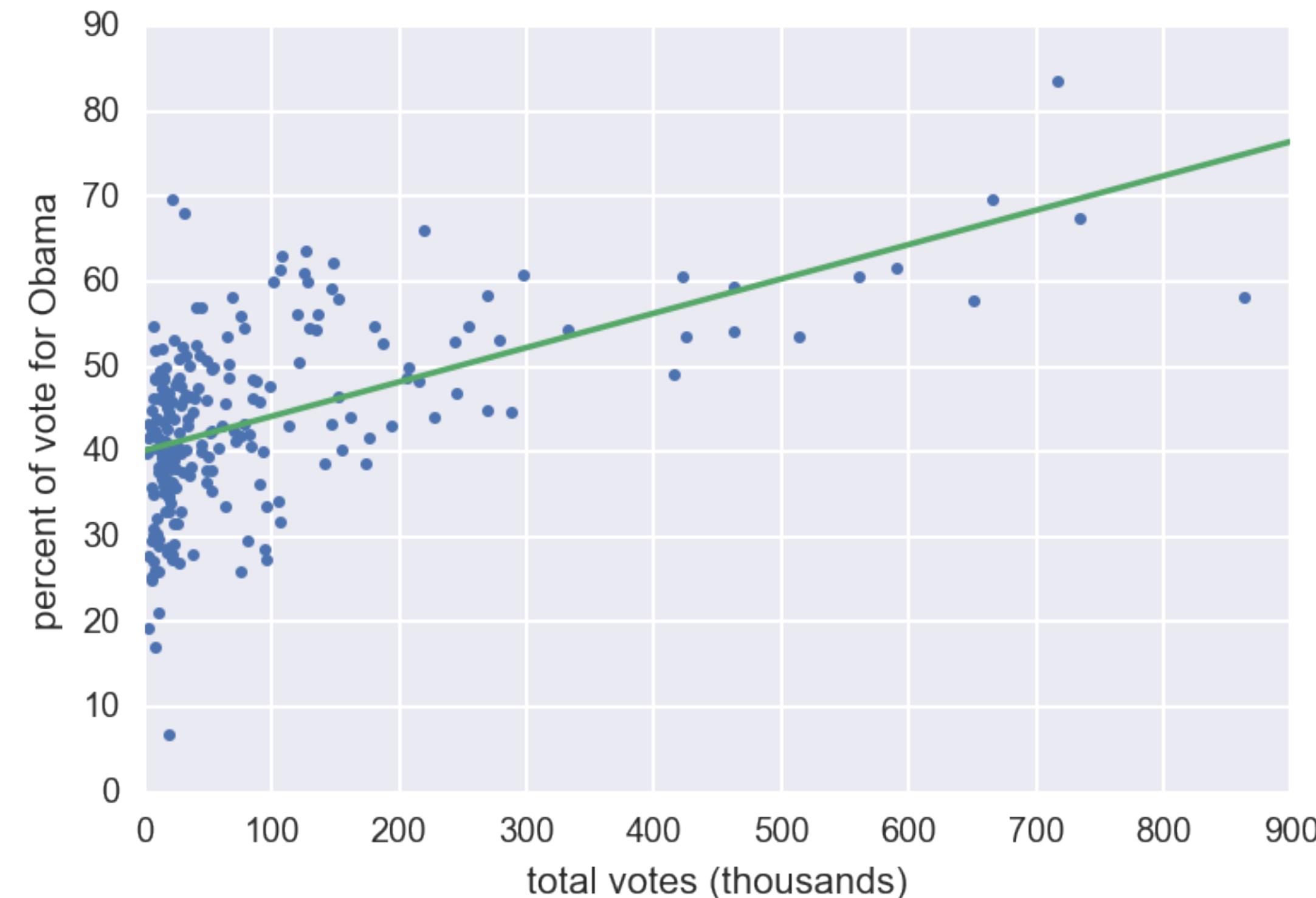


STATISTICAL THINKING IN PYTHON II

Formulating and simulating hypotheses



2008 US swing state election results







Hypothesis testing

- Assessment of how reasonable the observed data are assuming a hypothesis is true

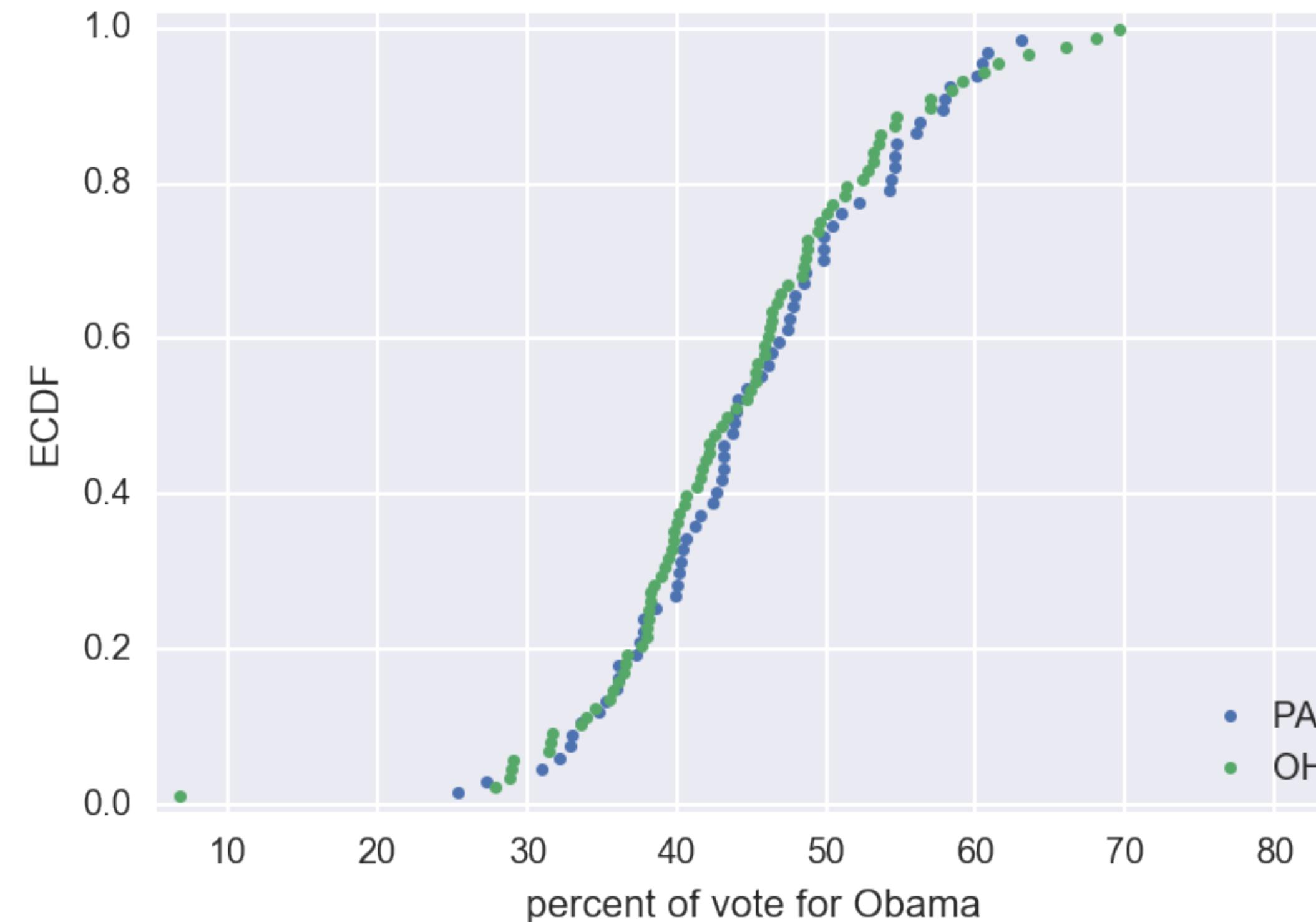


Null hypothesis

- Another name for the hypothesis you are testing



ECDFs of swing state election results





Percent vote for Obama

	PA	OH	PA — OH difference
mean	45.5%	44.3%	1.2%
median	44.0%	43.7%	0.4%
standard deviation	9.8%	9.9%	-0.1%



Simulating the hypothesis

60.08,	40.64,	36.07,	41.21,	31.04,	43.78,	44.08,	46.85,
44.71,	46.15,	63.10,	52.20,	43.18,	40.24,	39.92,	47.87,
37.77,	40.11,	49.85,	48.61,	38.62,	54.25,	34.84,	47.75,
43.82,	55.97,	58.23,	42.97,	42.38,	36.11,	37.53,	42.65,
50.96,	47.43,	56.24,	45.60,	46.39,	35.22,	48.56,	32.97,
57.88,	36.05,	37.72,	50.36,	32.12,	41.55,	54.66,	57.81,
54.58,	32.88,	54.37,	40.45,	47.61,	60.49,	43.11,	27.32,
44.03,	33.56,	37.26,	54.64,	43.12,	25.34,	49.79,	83.56,
40.09,	60.81,	49.81,	56.94,	50.46,	65.99,	45.88,	42.23,
45.26,	57.01,	53.61,	59.10,	61.48,	43.43,	44.69,	54.59,
48.36,	45.89,	48.62,	43.92,	38.23,	28.79,	63.57,	38.07,
40.18,	43.05,	41.56,	42.49,	36.06,	52.76,	46.07,	39.43,
39.26,	47.47,	27.92,	38.01,	45.45,	29.07,	28.94,	51.28,
50.10,	39.84,	36.43,	35.71,	31.47,	47.01,	40.10,	48.76,
31.56,	39.86,	45.31,	35.47,	51.38,	46.33,	48.73,	41.77,
41.32,	48.46,	53.14,	34.01,	54.74,	40.67,	38.96,	46.29,
38.25,	6.80,	31.75,	46.33,	44.90,	33.57,	38.10,	39.67,
40.47,	49.44,	37.62,	36.71,	46.73,	42.20,	53.16,	52.40,
58.36,	68.02,	38.53,	34.58,	69.64,	60.50,	53.53,	36.54,
49.58,	41.97,	38.11					

Pennsylvania

Ohio



Simulating the hypothesis

```
60.08, 40.64, 36.07, 41.21, 31.04, 43.78, 44.08, 46.85,  
44.71, 46.15, 63.10, 52.20, 43.18, 40.24, 39.92, 47.87,  
37.77, 40.11, 49.85, 48.61, 38.62, 54.25, 34.84, 47.75,  
43.82, 55.97, 58.23, 42.97, 42.38, 36.11, 37.53, 42.65,  
50.96, 47.43, 56.24, 45.60, 46.39, 35.22, 48.56, 32.97,  
57.88, 36.05, 37.72, 50.36, 32.12, 41.55, 54.66, 57.81,  
54.58, 32.88, 54.37, 40.45, 47.61, 60.49, 43.11, 27.32,  
44.03, 33.56, 37.26, 54.64, 43.12, 25.34, 49.79, 83.56,  
40.09, 60.81, 49.81, 56.94, 50.46, 65.99, 45.88, 42.23,  
45.26, 57.01, 53.61, 59.10, 61.48, 43.43, 44.69, 54.59,  
48.36, 45.89, 48.62, 43.92, 38.23, 28.79, 63.57, 38.07,  
40.18, 43.05, 41.56, 42.49, 36.06, 52.76, 46.07, 39.43,  
39.26, 47.47, 27.92, 38.01, 45.45, 29.07, 28.94, 51.28,  
50.10, 39.84, 36.43, 35.71, 31.47, 47.01, 40.10, 48.76,  
31.56, 39.86, 45.31, 35.47, 51.38, 46.33, 48.73, 41.77,  
41.32, 48.46, 53.14, 34.01, 54.74, 40.67, 38.96, 46.29,  
38.25, 6.80, 31.75, 46.33, 44.90, 33.57, 38.10, 39.67,  
40.47, 49.44, 37.62, 36.71, 46.73, 42.20, 53.16, 52.40,  
58.36, 68.02, 38.53, 34.58, 69.64, 60.50, 53.53, 36.54,  
49.58, 41.97, 38.11
```



Simulating the hypothesis

```
59.10, 38.62, 51.38, 60.49, 6.80, 41.97, 48.56, 37.77,  
48.36, 54.59, 40.11, 57.81, 45.89, 83.56, 40.64, 46.07,  
28.79, 55.97, 33.57, 42.23, 48.61, 44.69, 39.67, 57.88,  
48.62, 54.66, 54.74, 48.46, 36.07, 43.92, 49.85, 53.53,  
48.76, 41.77, 36.54, 47.01, 52.76, 49.44, 34.58, 40.24,  
44.08, 46.29, 49.81, 69.64, 60.50, 27.32, 45.60, 63.10,  
35.71, 39.86, 40.67, 65.99, 50.46, 37.72, 50.96, 42.49,  
31.56, 38.23, 37.26, 41.21, 37.53, 46.85, 44.03, 41.32,  
45.88, 40.45, 32.12, 35.22, 49.79, 43.12, 43.18, 45.45,  
25.34, 46.73, 44.90, 56.94, 58.23, 39.84, 36.05, 43.05,  
38.25, 40.47, 31.04, 54.25, 46.15, 57.01, 52.20, 47.75,  
36.06, 47.61, 51.28, 43.43, 42.97, 38.01, 54.64, 45.26,  
47.47, 34.84, 49.58, 48.73, 29.07, 54.58, 27.92, 34.01,  
38.07, 31.47, 36.11, 39.26, 41.56, 52.40, 40.18, 47.87,  
46.33, 46.39, 43.11, 38.53, 33.56, 42.65, 68.02, 35.47,  
40.09, 36.43, 36.71, 60.08, 50.36, 39.43, 28.94, 58.36,  
42.20, 47.43, 44.71, 43.78, 39.92, 37.62, 63.57, 53.61,  
40.10, 46.33, 53.16, 32.88, 38.96, 41.55, 56.24, 38.11,  
42.38, 38.10, 43.82, 45.31, 60.81, 54.37, 53.14, 32.97,  
61.48, 50.10, 31.75
```



Simulating the hypothesis

59.10,	38.62,	51.38,	60.49,	6.80,	41.97,	48.56,	37.77,
48.36,	54.59,	40.11,	57.81,	45.89,	83.56,	40.64,	46.07,
28.79,	55.97,	33.57,	42.23,	48.61,	44.69,	39.67,	57.88,
48.62,	54.66,	54.74,	48.46,	36.07,	43.92,	49.85,	53.53,
48.76,	41.77,	36.54,	47.01,	52.76,	49.44,	34.58,	40.24,
44.08,	46.29,	49.81,	69.64,	60.50,	27.32,	45.60,	63.10,
35.71,	39.86,	40.67,	65.99,	50.46,	37.72,	50.96,	42.49,
31.56,	38.23,	37.26,	41.21,	37.53,	46.85,	44.03,	41.32,
45.88,	40.45,	32.12,	35.22,	49.79,	43.12,	43.18,	45.45,
25.34,	46.73,	44.90,	56.94,	58.23,	39.84,	36.05,	43.05,
38.25,	40.47,	31.04,	54.25,	46.15,	57.01,	52.20,	47.75,
36.06,	47.61,	51.28,	43.43,	42.97,	38.01,	54.64,	45.26,
47.47,	34.84,	49.58,	48.73,	29.07,	54.58,	27.92,	34.01,
38.07,	31.47,	36.11,	39.26,	41.56,	52.40,	40.18,	47.87,
46.33,	46.39,	43.11,	38.53,	33.56,	42.65,	68.02,	35.47,
40.09,	36.43,	36.71,	60.08,	50.36,	39.43,	28.94,	58.36,
42.20,	47.43,	44.71,	43.78,	39.92,	37.62,	63.57,	53.61,
40.10,	46.33,	53.16,	32.88,	38.96,	41.55,	56.24,	38.11,
42.38,	38.10,	43.82,	45.31,	60.81,	54.37,	53.14,	32.97,
61.48,	50.10,	31.75					

"Pennsylvania"

"Ohio"



Permutation

- Random reordering of entries in an array



Generating a permutation sample

```
In [1]: import numpy as np
```

```
In [2]: dem_share_both = np.concatenate(  
...:  
        (dem_share_PA, dem_share_OH))
```

```
In [3]: dem_share_perm = np.random.permutation(dem_share_both)
```

```
In [4]: perm_sample_PA = dem_share_perm[:len(dem_share_PA)]
```

```
In [5]: perm_sample_OH = dem_share_perm[len(dem_share_PA):]
```



STATISTICAL THINKING IN PYTHON II

Let's practice!

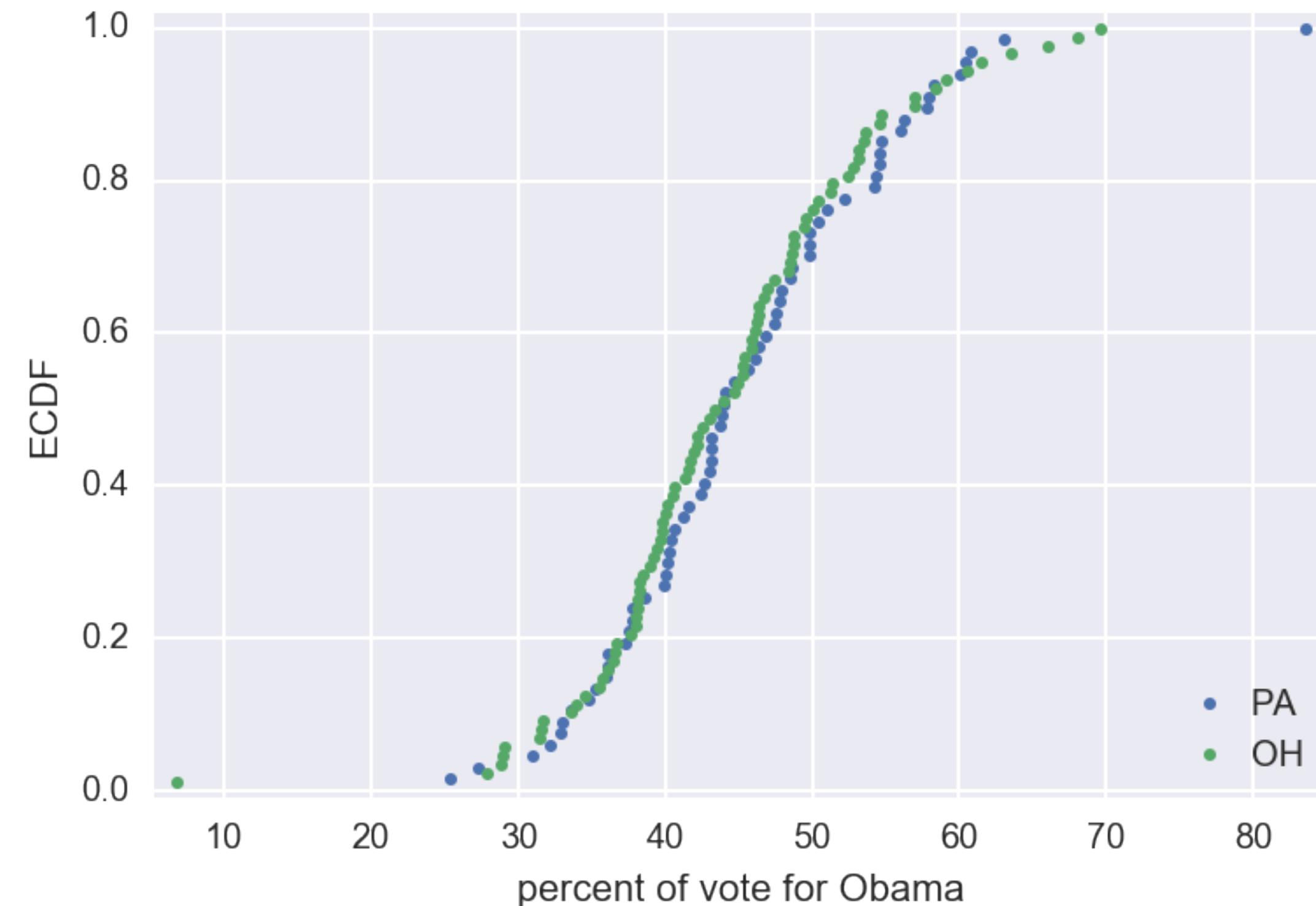


STATISTICAL THINKING IN PYTHON II

Test statistics and p-values



Are OH and PA different?





Hypothesis testing

- Assessment of how reasonable the observed data are assuming a hypothesis is true



Test statistic

- A single number that can be computed from observed data and from data you simulate under the null hypothesis
- It serves as a basis of comparison between the two



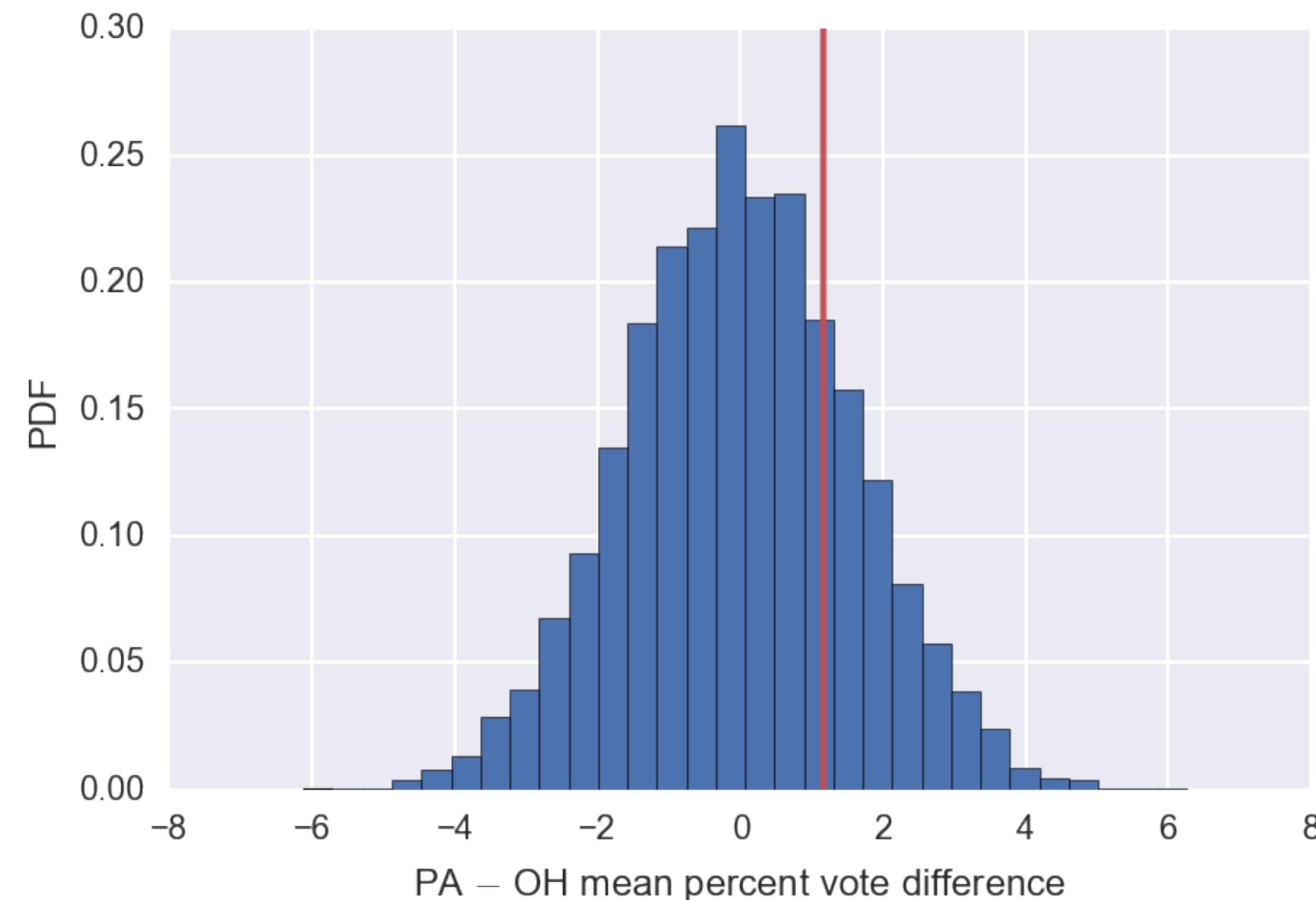
Permutation replicate

```
In [1]: np.mean(perm_sample_PA) - np.mean(perm_sample_OH)  
Out[1]: 1.122220149253728
```

```
In [2]: np.mean(dem_share_PA) - np.mean(dem_share_OH) # orig. data  
Out[2]: 1.1582360922659518
```

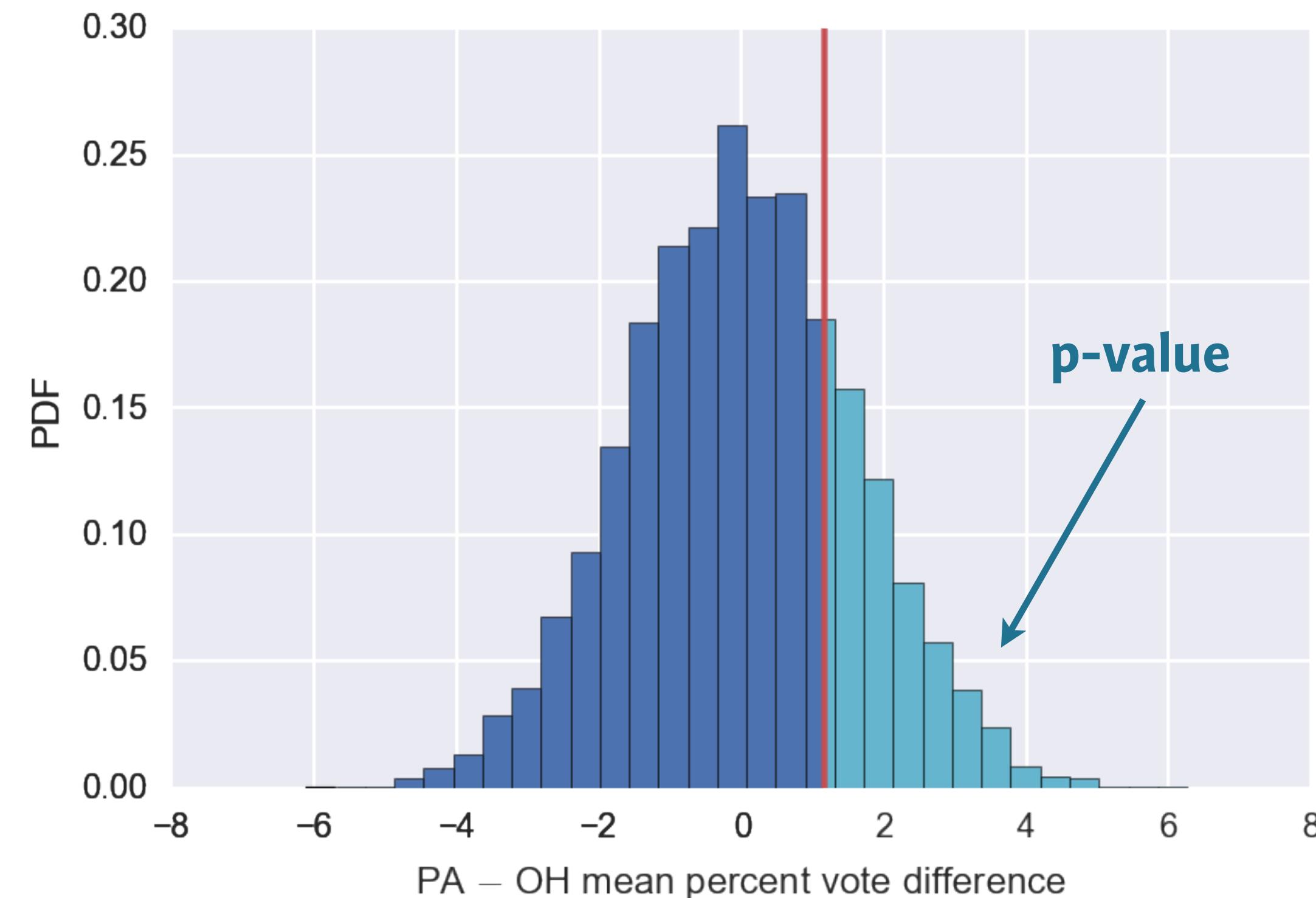


Mean vote difference under null hypothesis





Mean vote difference under null hypothesis





p-value

- The probability of obtaining a value of your test statistic that is at least as extreme as what was observed, under the assumption the null hypothesis is true
- NOT the probability that the null hypothesis is true



Statistical significance

- Determined by the smallness of a p-value



Null hypothesis significance testing (NHST)

- Another name for what we are doing in this chapter



statistical significance \neq practical significance



STATISTICAL THINKING IN PYTHON II

Let's practice!



STATISTICAL THINKING IN PYTHON II

Bootstrap hypothesis tests



Pipeline for hypothesis testing

- Clearly state the null hypothesis
- Define your test statistic
- Generate many sets of simulated data assuming the null hypothesis is true
- Compute the test statistic for each simulated data set
- The p-value is the fraction of your simulated data sets for which the test statistic is at least as extreme as for the real data

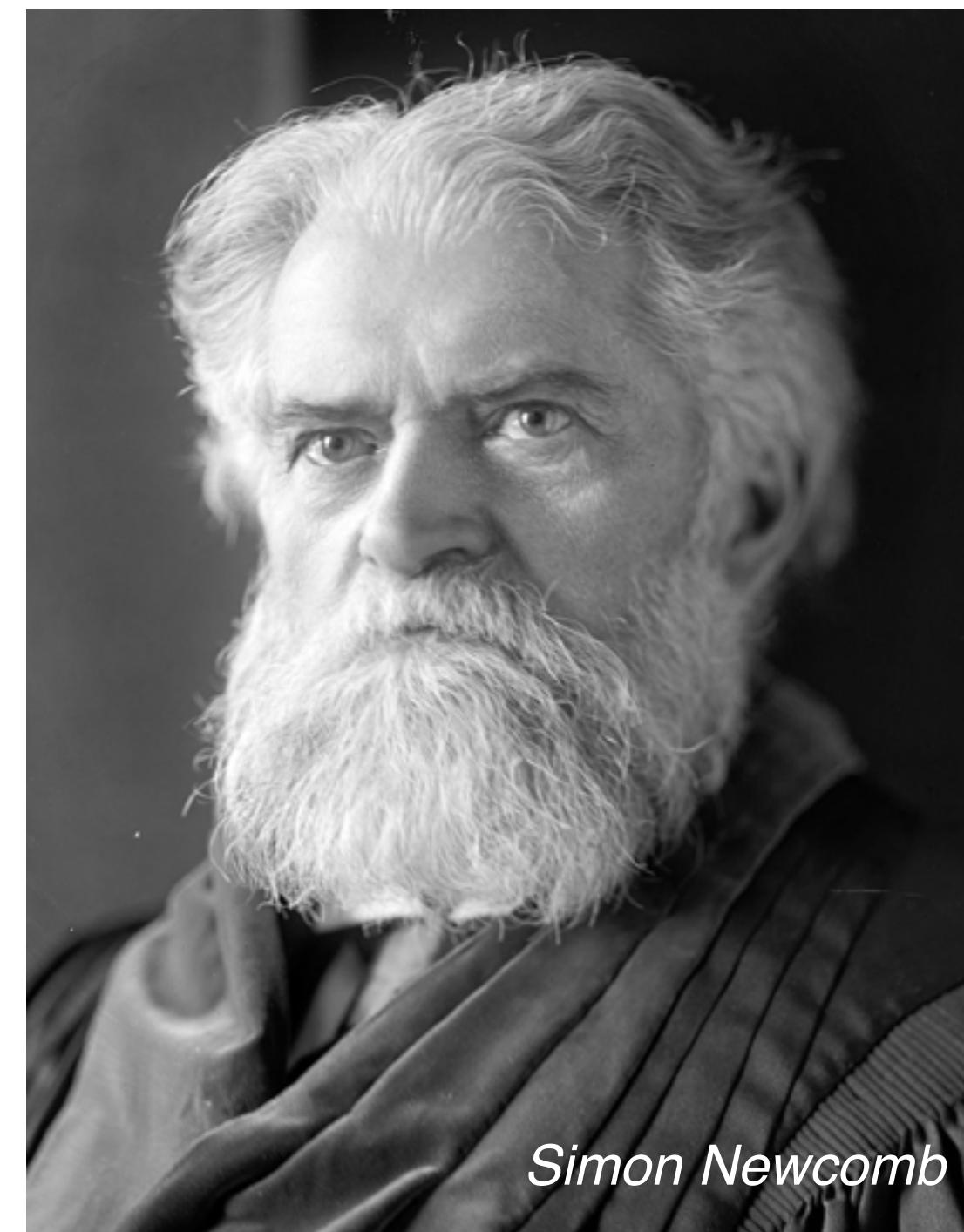


Michelson and Newcomb: speed of light pioneers



Albert Michelson

299,852 km/s



Simon Newcomb

299,860 km/s

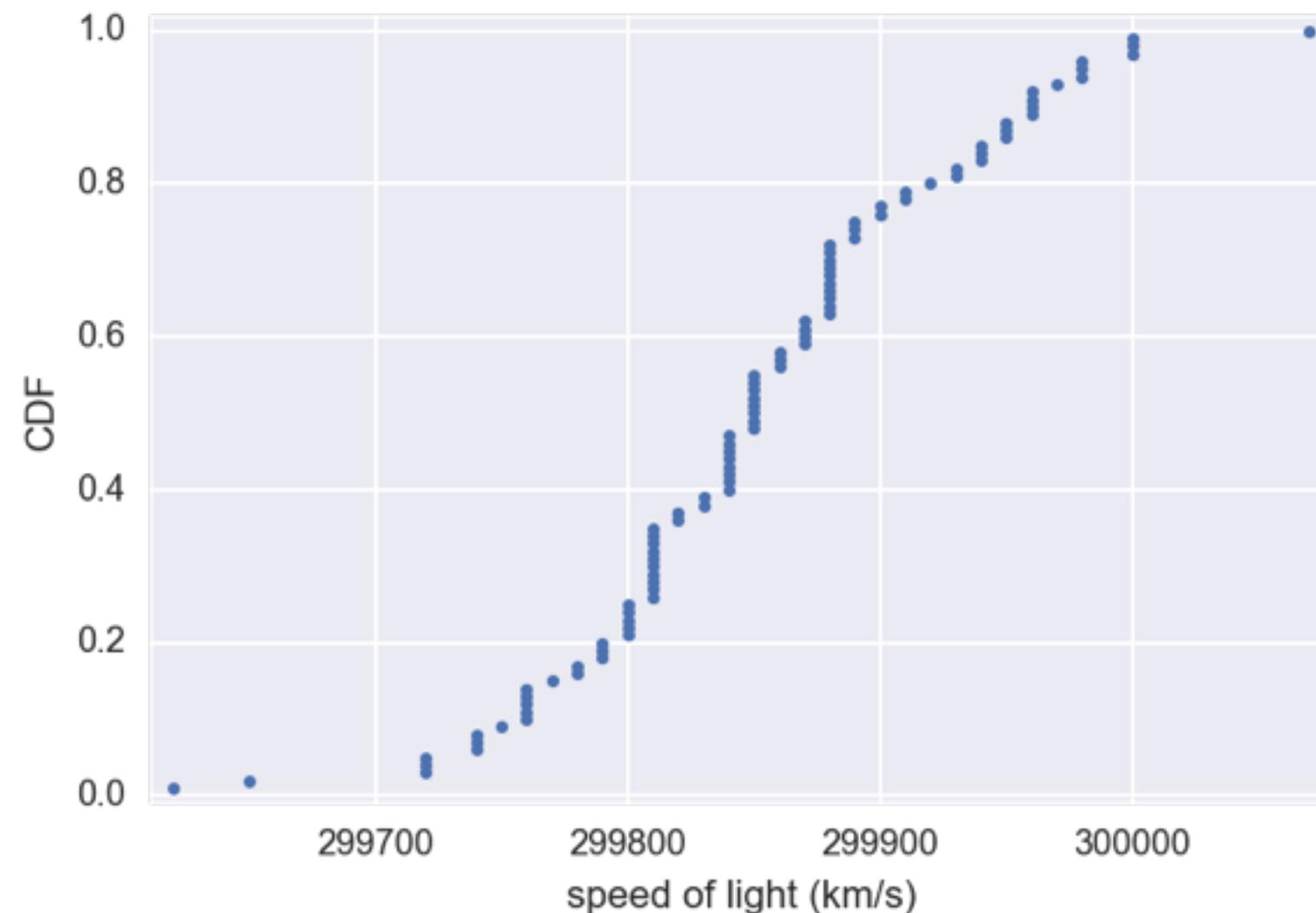
Michelson image: public domain, Smithsonian

Newcomb image: US Library of Congress



The data we have

Michelson:



Newcomb:

mean = 299,860 km/s



Null hypothesis

- The true mean speed of light in Michelson's experiments was actually Newcomb's reported value



Shifting the Michelson data

```
In [1]: newcomb_value = 299860 # km/s
```

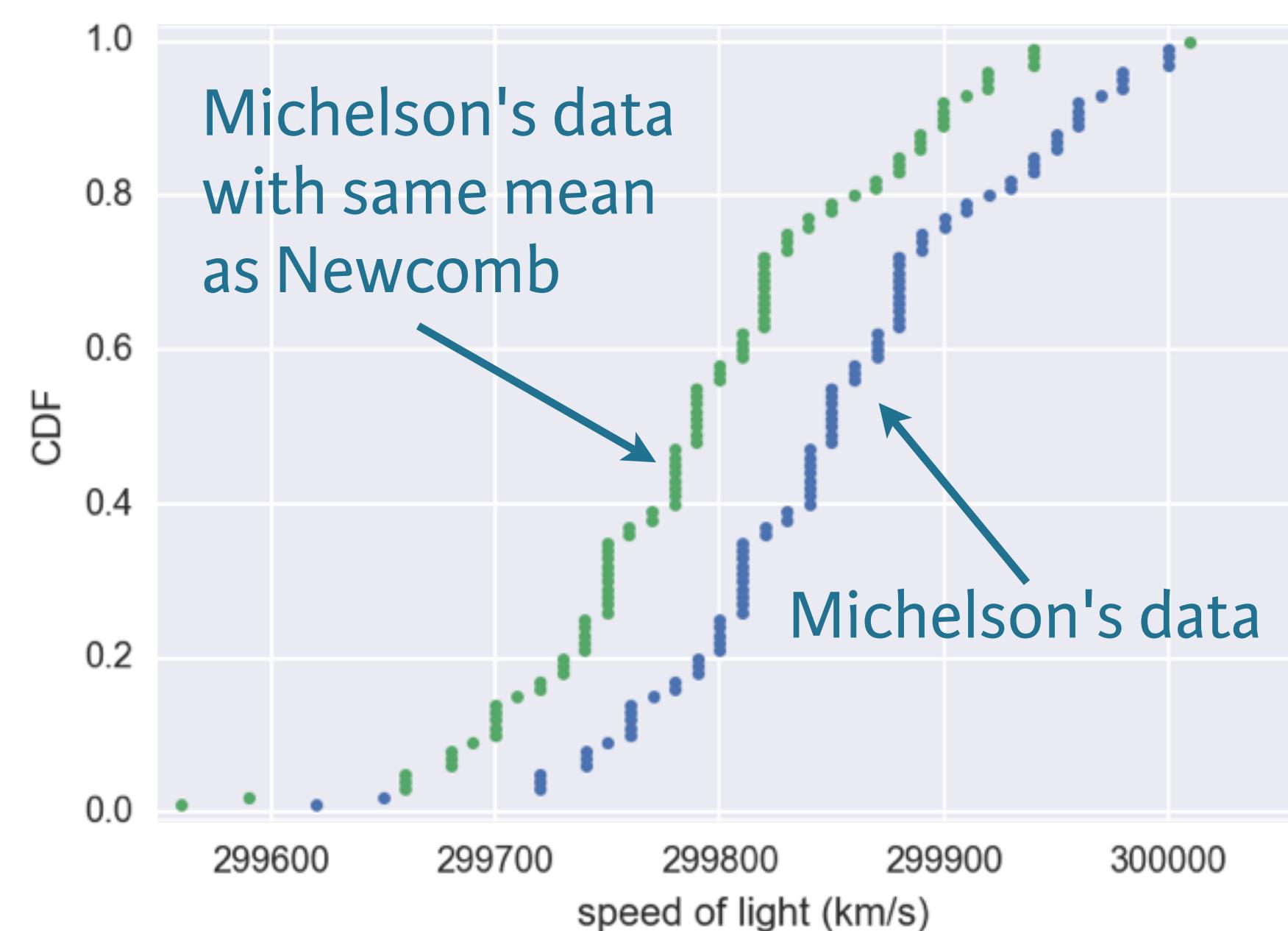
```
In [2]: michelson_shifted = michelson_speed_of_light \
...:         - np.mean(michelson_speed_of_light) + newcomb_value
```



Shifting the Michelson data

```
In [1]: newcomb_value = 299860 # km/s
```

```
In [2]: michelson_shifted = michelson_speed_of_light \
...: - np.mean(michelson_speed_of_light) + newcomb_value
```





Calculating the test statistic

```
In [1]: def diff_from_newcomb(data, newcomb_value=299860):  
...:     return np.mean(data) - newcomb_value  
...:
```

```
In [2]: diff_obs = diff_from_newcomb(michelson_speed_of_light)
```

```
In [3]: diff_obs  
Out[3]: -7.599999999767169
```



Computing the p-value

```
In [1]: bs_replicates = draw_bs_reps(michelson_shifted,  
...:                                diff_from_newcomb, 10000)  
  
In [2]: p_value = np.sum(bs_replicates <= diff_observed) / 10000  
  
In [3]: p_value  
Out[3]: 0.16039999999999999
```



One sample test

- Compare one set of data to a single number

Two sample test

- Compare two sets of data



STATISTICAL THINKING IN PYTHON II

Let's practice!

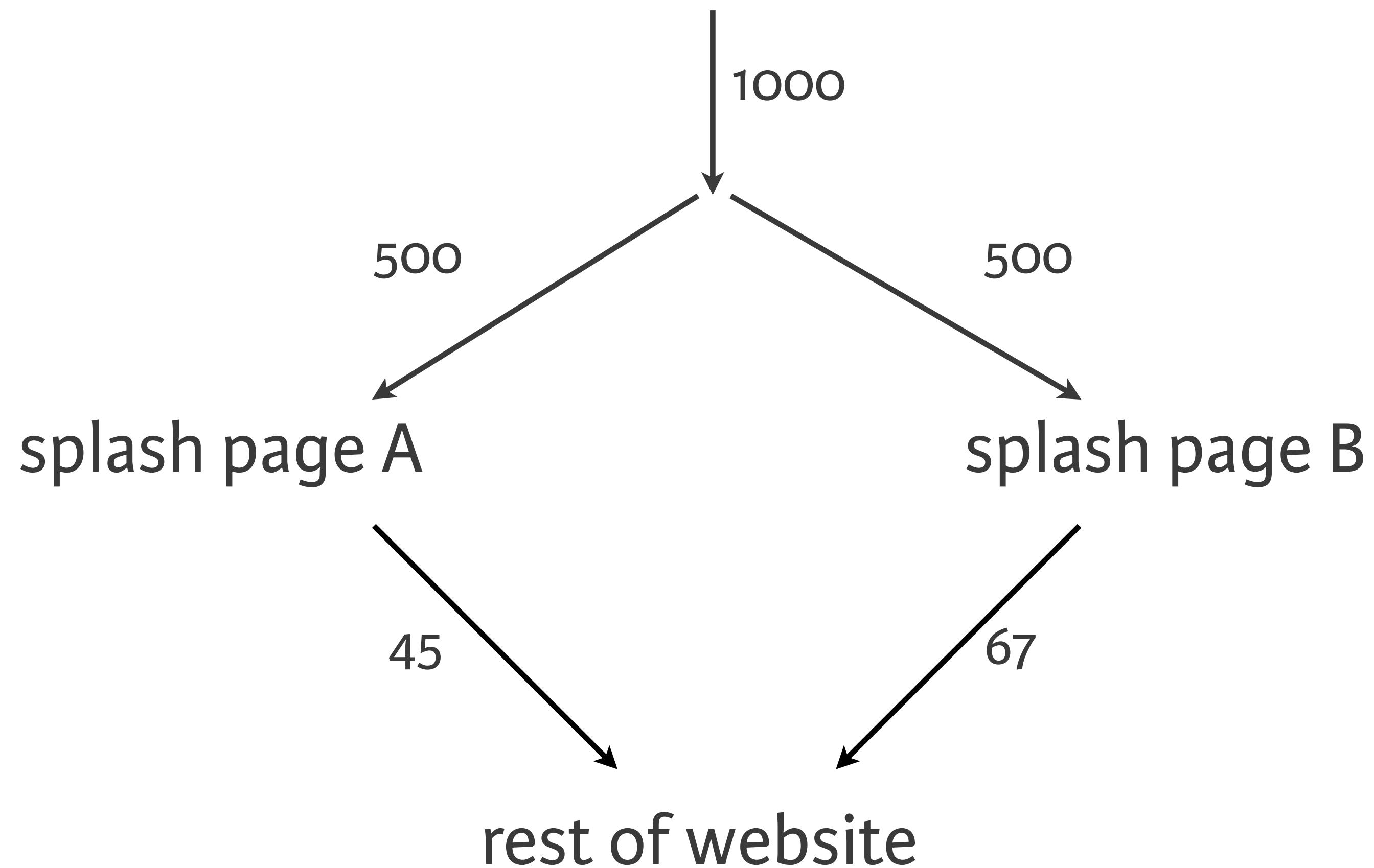


STATISTICAL THINKING IN PYTHON II

A/B testing



Is your redesign effective?





Null hypothesis

- The click-through rate is not affected by the redesign



Permutation test of clicks through

```
In [1]: import numpy as np
```

```
In [2]: # clickthrough_A, clickthrough_B: arr. of 1s and 0s
```

```
In [3]: def diff_frac(data_A, data_B):
...:     frac_A = np.sum(data_A) / len(data_A)
...:     frac_B = np.sum(data_B) / len(data_B)
...:     return frac_B - frac_A
...:
```

```
In [4]: diff_frac_obs = diff_frac(clickthrough_A,
...:                               clickthrough_B)
```



Permutation test of clicks through

```
In [1]: perm_replicates = np.empty(10000)

In [2]: for i in range(10000):
...:     perm_replicates[i] = permutation_replicate(
...:         clickthrough_A, clickthrough_B, diff_frac)
...:

In [3]: p_value = np.sum(perm_replicates >= diff_frac_obs) / 10000

In [4]: p_value
Out[4]: 0.016
```



A/B test

- Used by organizations to see if a strategy change gives a better result



Null hypothesis of an A/B test

- The test statistic is impervious to the change



STATISTICAL THINKING IN PYTHON II

Let's practice!

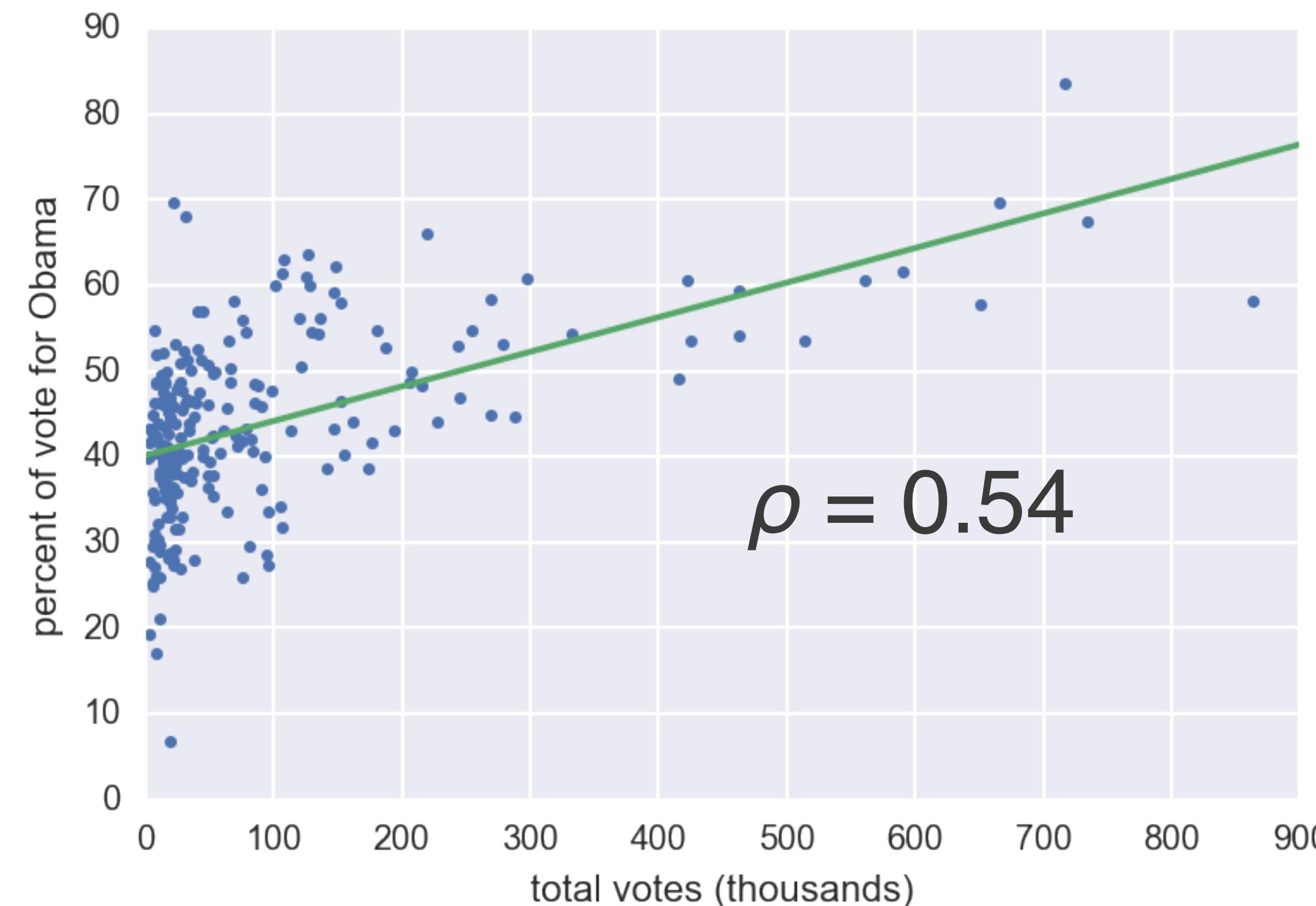


STATISTICAL THINKING IN PYTHON II

Test of correlation



2008 US swing state election results



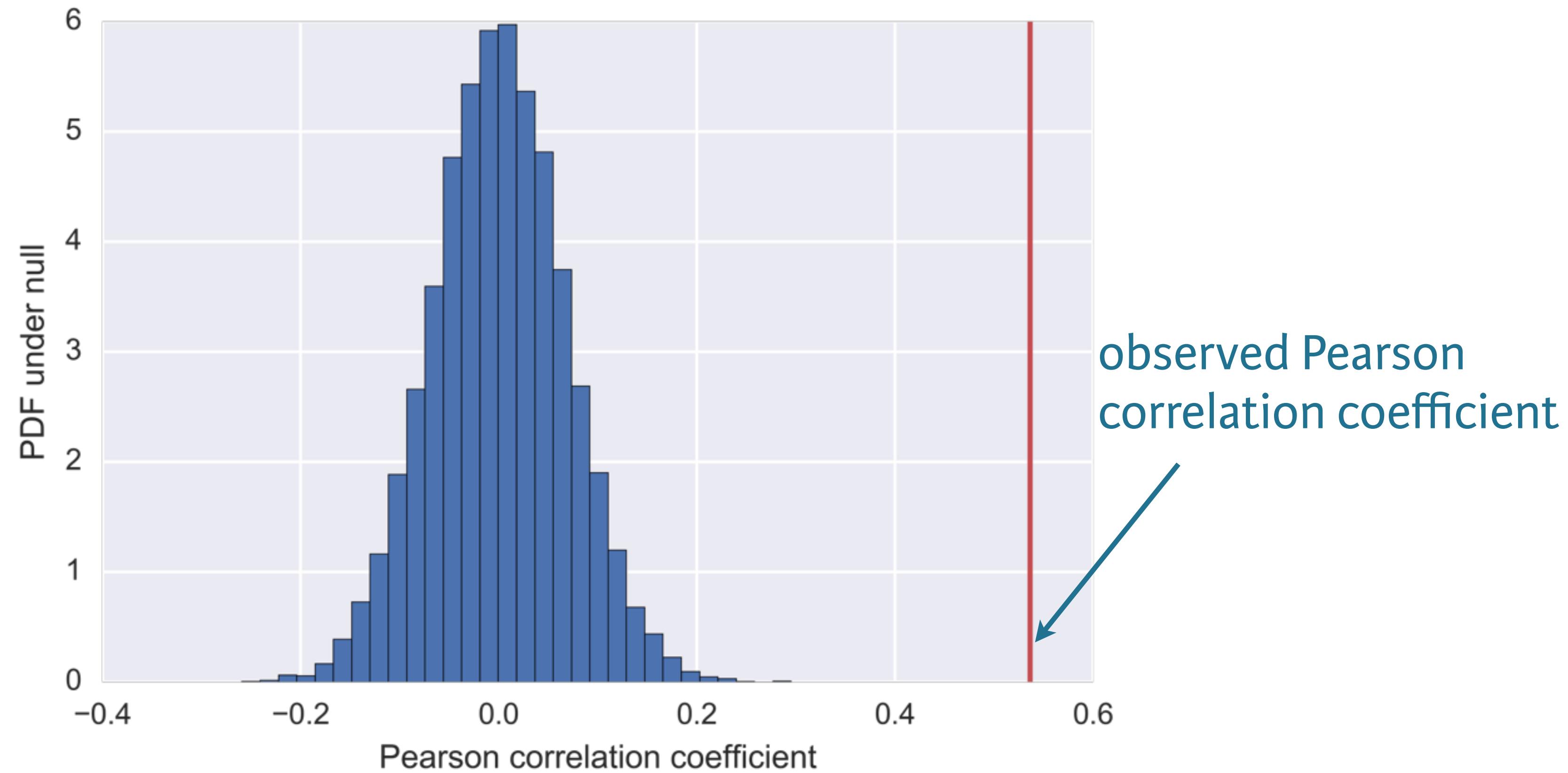


Hypothesis test of correlation

- Posit null hypothesis: the two variables are completely uncorrelated
- Simulate data assuming null hypothesis is true
- Use Pearson correlation, ρ , as test statistic
- Compute p-value as fraction of replicates that have ρ at least as large as observed.



More populous counties voted for Obama



p-value is very very small



STATISTICAL THINKING IN PYTHON II

Let's practice!



STATISTICAL THINKING IN PYTHON II

Darwin's finches: A full-blown statistical analysis



Your well-equipped toolbox

- Graphical and quantitative EDA
- Parameter estimation
- Confidence interval calculation
- Hypothesis testing

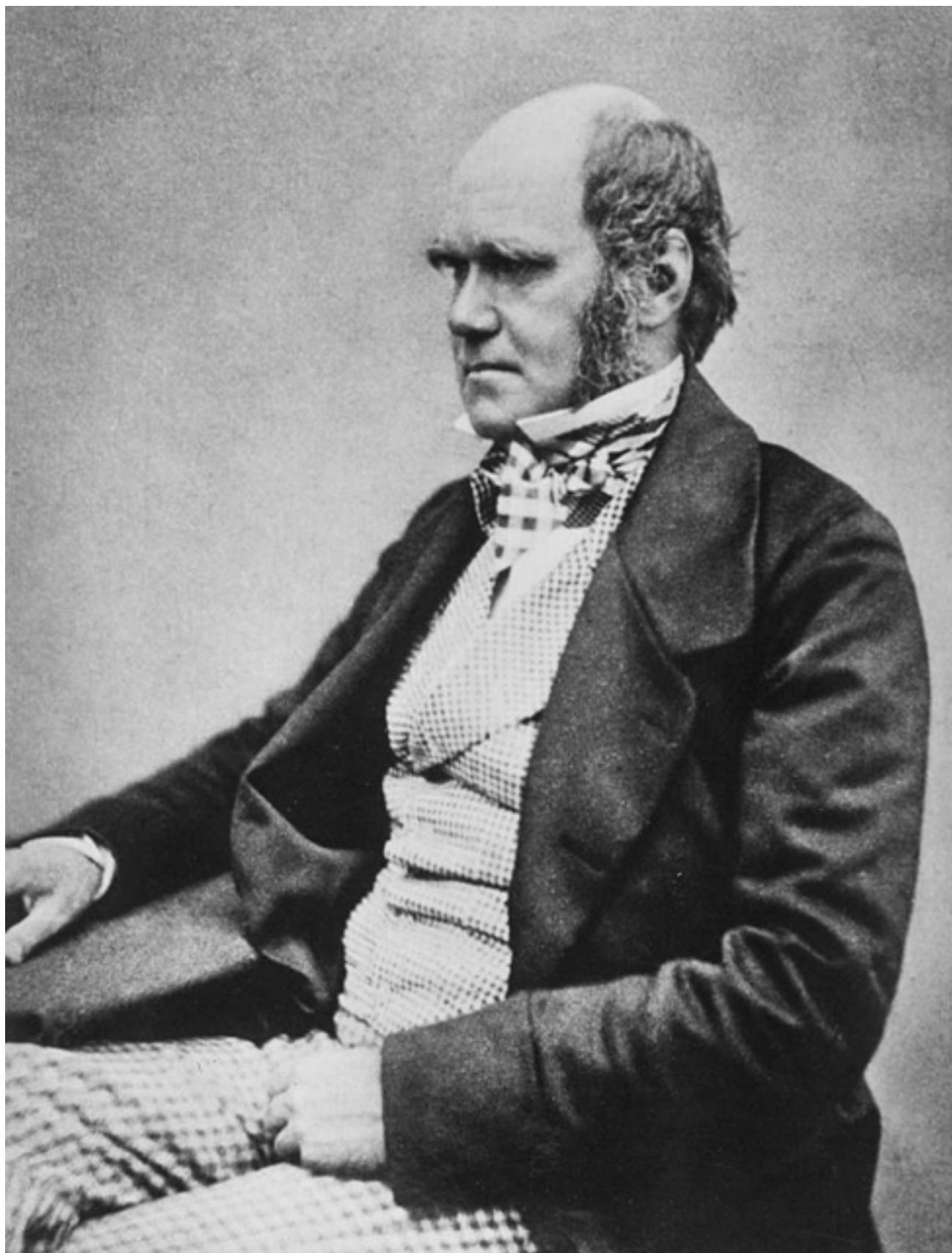




Image: NASA



The island of Daphne Major





The finches of Daphne Major



Geospiza fortis



Geospiza scandens



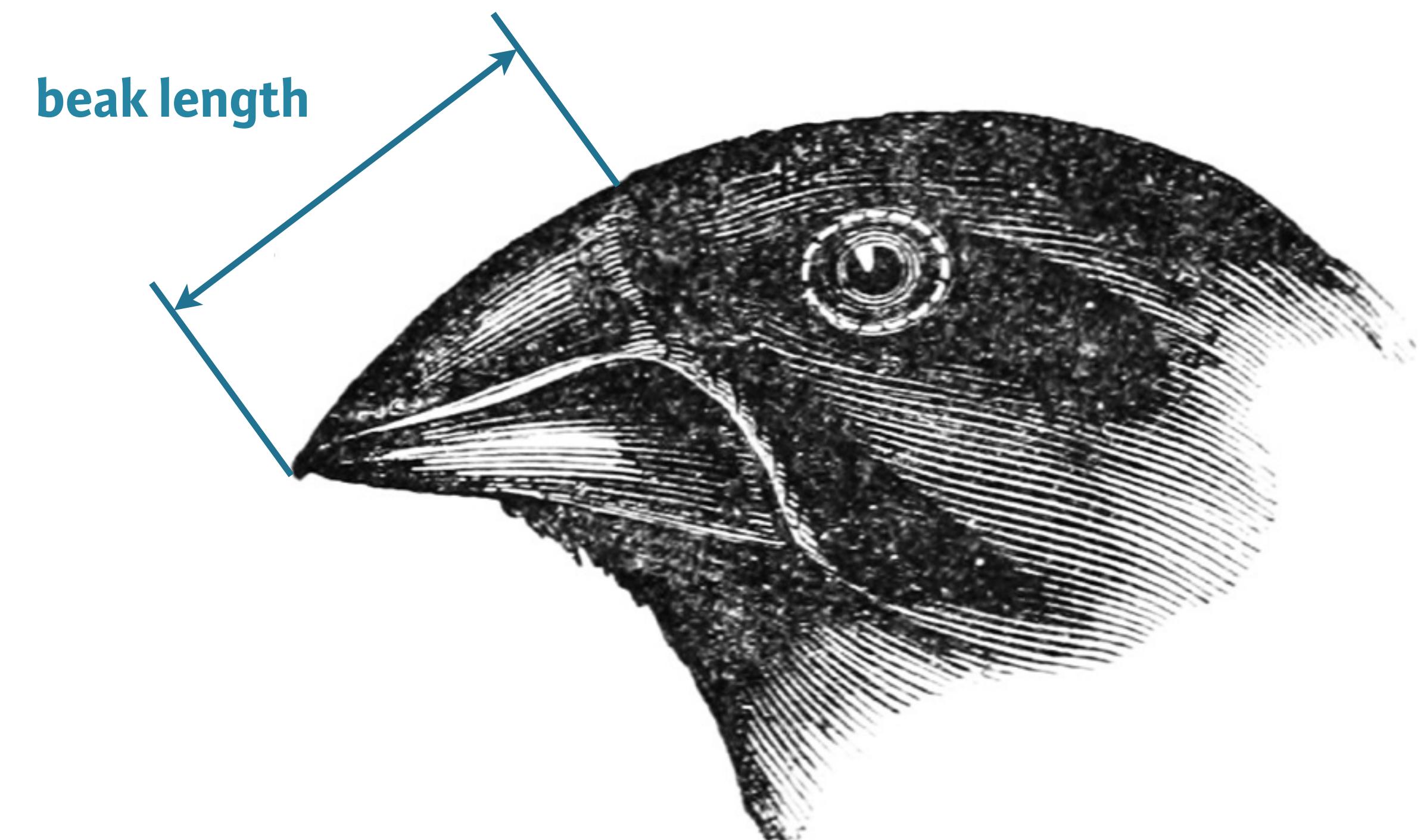
Our data source

- Peter and Rosemary Grant
40 Years of Evolution: Darwin's Finches on Daphne Major Island
Princeton University Press, 2014
- Data acquired from Dryad Digital Repository
<http://dx.doi.org/10.5061/dryad.g6g3h>



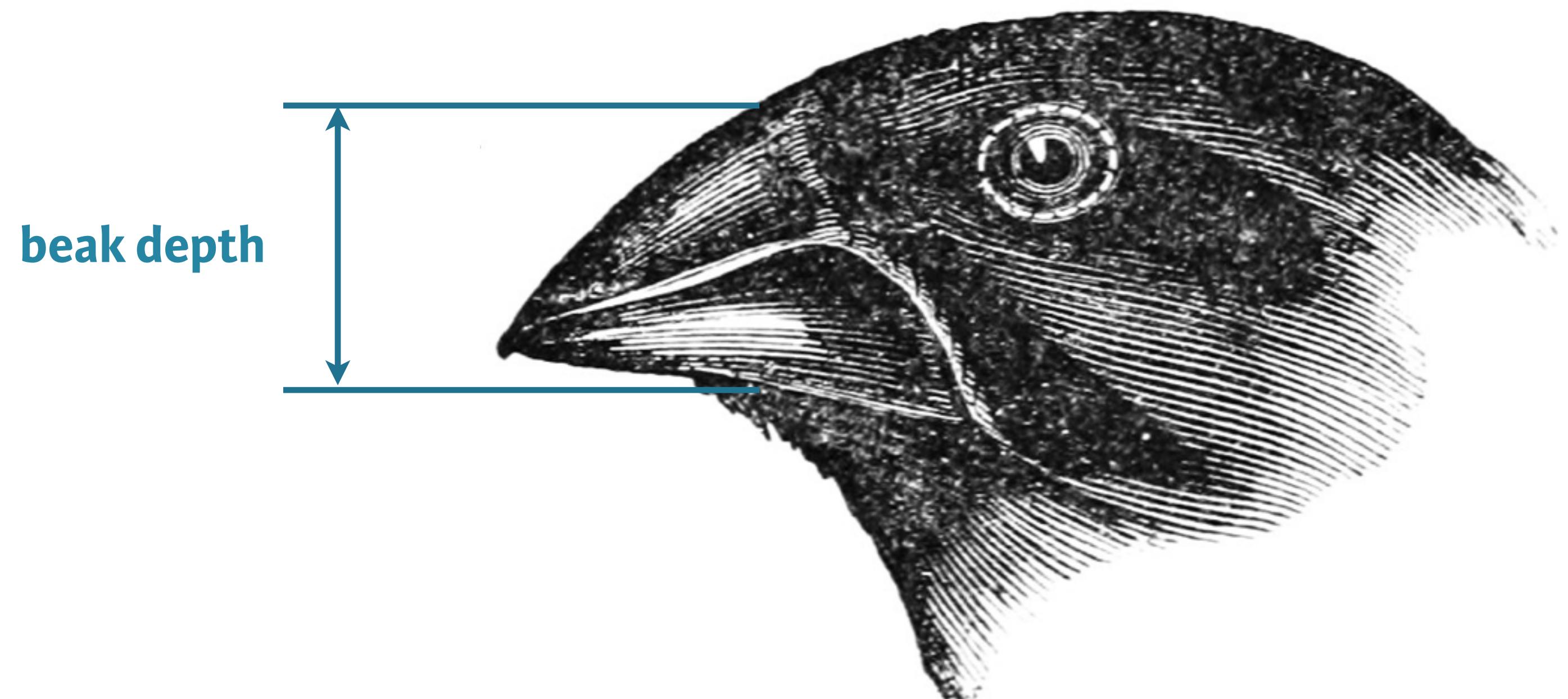


The dimensions of the finch beak





The dimensions of the finch beak





Investigation of *G. scandens* beak depth

- EDA of beak depths in 1975 and 2012
- Parameter estimates of mean beak depth
- Hypothesis test: did the beaks get deeper?



STATISTICAL THINKING IN PYTHON II

Let's do it!

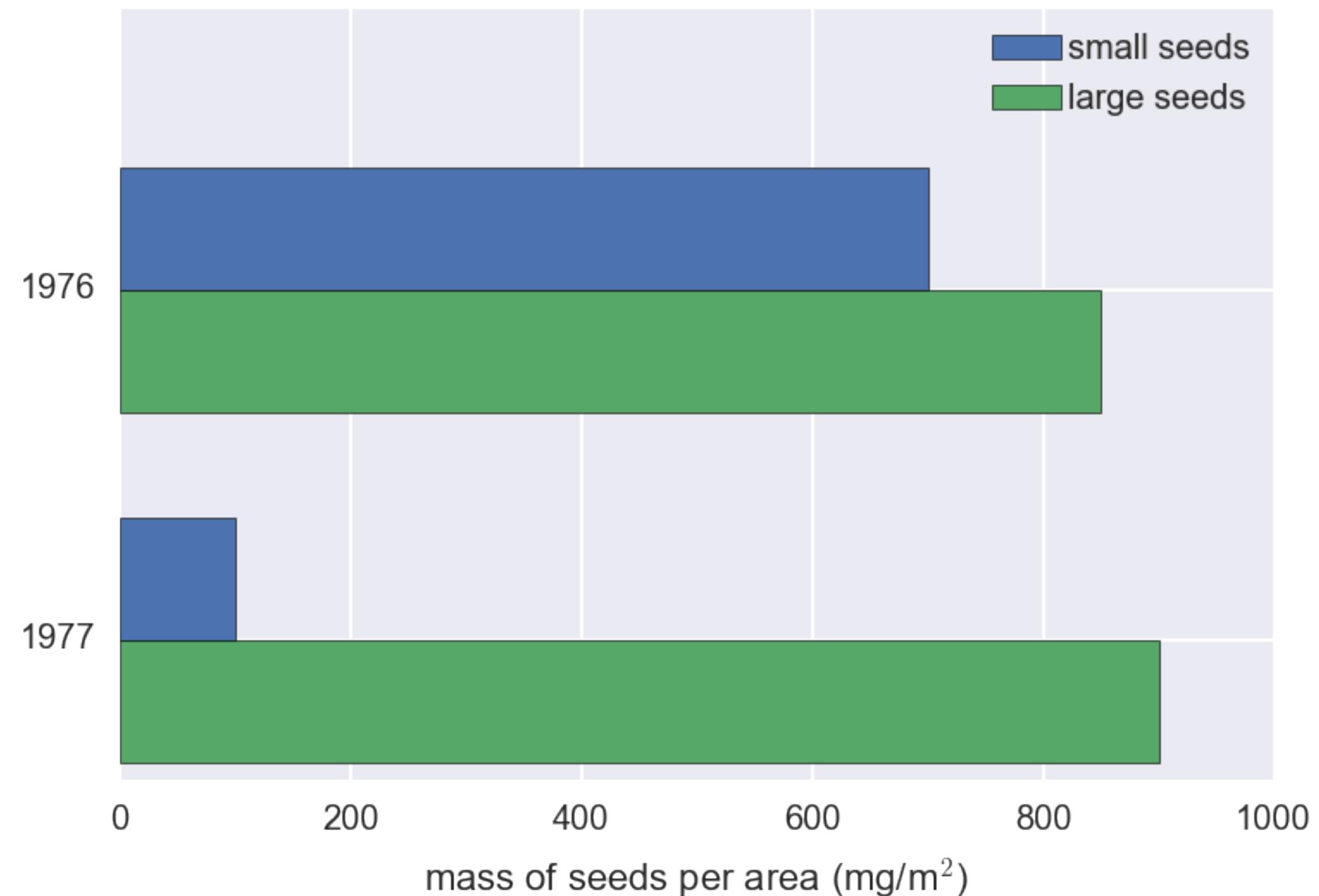


STATISTICAL THINKING IN PYTHON II

Variation in beak shapes

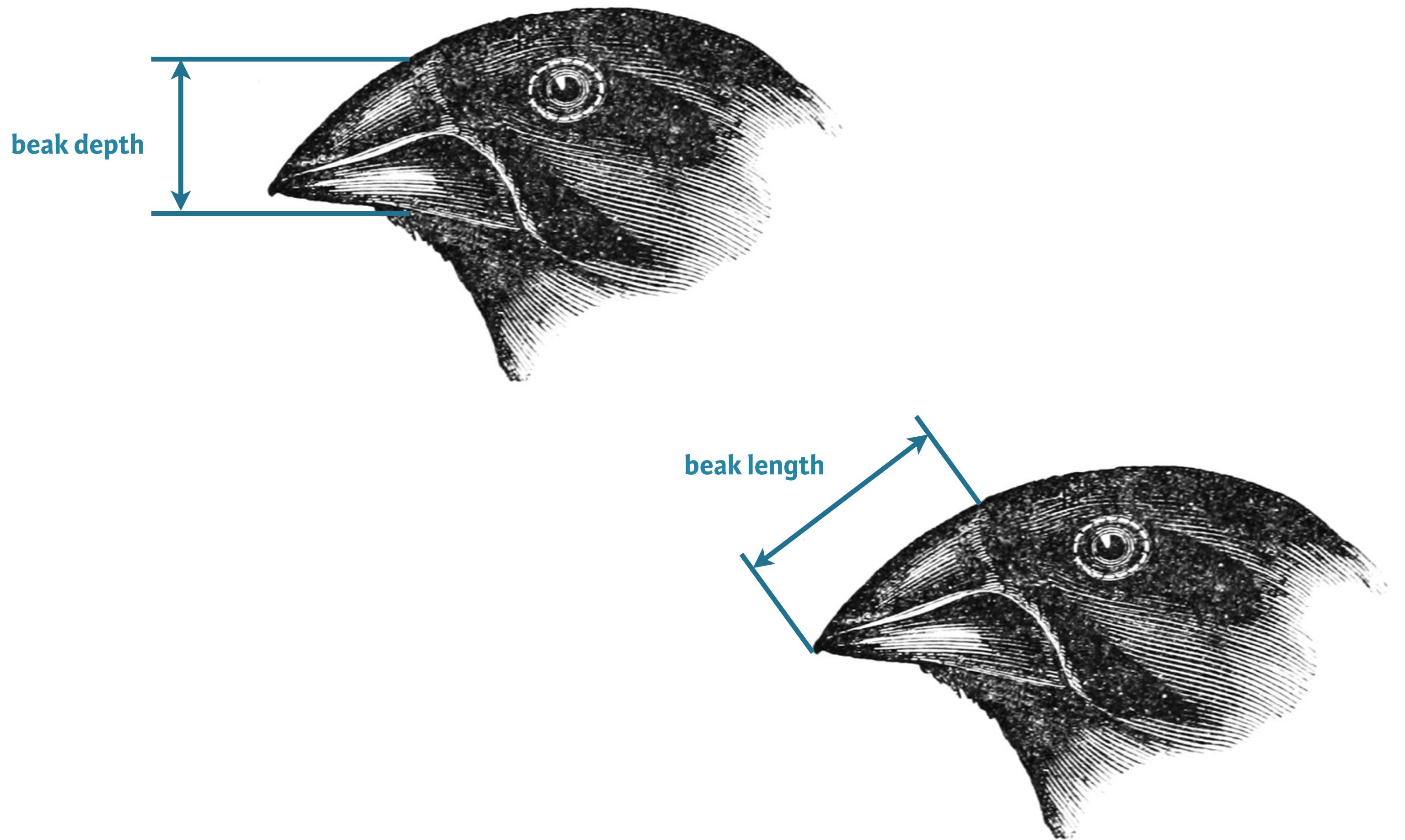


The drought of winter 1976/1977





Beak geometry





Hint

- `draw_bs_pairs_linreg()` will come in handy



STATISTICAL THINKING IN PYTHON II

Let's do it!



STATISTICAL THINKING IN PYTHON II

Calculation of heredity



The finches of Daphne Major



Geospiza fortis



Geospiza scandens



Heredity

- The tendency for parental traits to be inherited by offspring



STATISTICAL THINKING IN PYTHON II

Let's do it!



STATISTICAL THINKING IN PYTHON II

Final thoughts



Your statistical thinking skills

- Perform EDA
 - Generate effective plots like ECDFs
 - Compute summary statistics
- Estimate parameters
 - By optimization, including linear regression
 - Determine confidence intervals
- Formulate and test hypotheses



STATISTICAL THINKING IN PYTHON II

Bon voyage!