# Nonstandard finite difference methods for solving models of population dynamics

Cite as: AIP Conference Proceedings **2202**, 020042 (2019); https://doi.org/10.1063/1.5141655
Published Online: 27 December 2019

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# Nonstandard Finite Difference Methods for Solving Models of Population Dynamics

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**Abstract.** For years, mathematical modeling has been involved in solving problems occurred in ecology relating to population dynamics. In this paper, we solve several models of population dynamic using nonstandard finite difference methods. We compare our results of nonstandard finite difference methods with those of Euler's and Heun's methods. Our research shows that the nonstandard finite difference method produces most accurate solution in comparison with the results of Euler's and Heun's methods.

## INTRODUCTION

Population dynamics is a science that designs or describes growth and changes of the population of species which may involves birth, death, migration, etc. One example is the growth rate of one or more populations. In this paper we will discuss the development of a population of one species. To solve the corresponding equation models, we will use Euler's method, Heun's method, and the nonstandard finite difference method. The results of these methods will be compared to its exact solution.

The next sections, we provide mathematical models and methods to solve the considered problems, numerical results and discussions, and finally conclusions.

## MODELS AND METHODS

Suppose that we have a population of one species. We assume that the variable of time $t$ is free and the number of population is denoted by $N(t)$. We shall discuss two types of population dynamics models. The first is that if the population growth rate is a constant $R_0$ then we have the exponential Malthusian model [1]:

$$\frac{dN}{dt} = R_0 N\,, \qquad N(0) = N_0. \tag{1}$$

The second model to be discussed in this paper is the logistic model. This logistic model is

$$\frac{dN}{dt} = N(a - bN), \qquad N(0) = N_0 \tag{2}$$

where $a$ and $b$ are positive constants.

*International Conference on Science and Applied Science (ICSAS) 2019*
AIP Conf. Proc. 2202, 020042-1–020042-7; https://doi.org/10.1063/1.5141655
Published by AIP Publishing. 978-0-7354-1953-7/$30.00

There are three methods used in this paper to solve the two models above, namely Euler's method, Heun's method and the nonstandard finite difference method. Obtaining numerical solutions to the two models above using these three methods, we will see which method produces the closest results to the exact solution.

## Euler's Method

The standard step of Euler's method for $y'(x) = f(t, y)$ is [2]

$$y_{k+1} = y_k + \Delta t\, f(t_k, y_k), \tag{3}$$

for $k = 0, 1, 2, \ldots . n - 1$ where $\Delta t$ is the step-size of $t$.

Based on equation (3), we can construct Euler's scheme for equation (1) as follows

$$N_{k+1} = N_k + \Delta t\, R_0 N_k, \tag{4}$$

while Euler's scheme for equation (2) is

$$N_{k+1} = N_k + \Delta t(aN_k - b(N_k)^2). \tag{5}$$

## Heun's Method

The standard step of Heun's method for $y'(x) = f(t, y)$ is [2]

$$\begin{aligned} \tilde{y}_{k+1} &= y_k + \Delta t\, f(t_k, y_k), \\ y_{k+1} &= y_k + \frac{\Delta t}{2}[f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1})], \end{aligned} \tag{6}$$

where $\Delta t$ is step-size of $t$.

Based on equation (6), we can construct Heun's scheme for equation (1) as follows

$$\begin{aligned} \widetilde{N}_{k+1} &= N_k + \Delta t\, R_0 N_k, \\ N_{k+1} &= N_k + \frac{\Delta t}{2}\left[R_0 N_k + f(t_{k+1}, \widetilde{N}_{k+1})\right], \end{aligned} \tag{7}$$

while Heun's scheme for equation (2) is

$$\begin{aligned} \widetilde{N}_{k+1} &= N_k + \Delta t(aN_k - b(N_k)^2), \\ N_{k+1} &= N_k + \frac{\Delta t}{2}\left[(aN_k - b(N_k)^2) + f(t_{k+1}, \widetilde{N}_{k+1})\right]. \end{aligned} \tag{8}$$

## Nonstandard Finite Difference Method

The exact solution to equation (1) is

$$N(t) = ke^{R_0 t}, \tag{9}$$

if we take the initial condition $N_0 = 1$, so we have $k = 1$, then equation (9) becomes

$$N(t) = e^{R_0 t}. \tag{10}$$

Using the theory in [3-5], we take

$$y_k^{(i)} \equiv y^{(i)}(t_k), \qquad t_k = (\Delta t)k; \tag{11}$$

so a difference equation can be constructed by calculating the following determinant [3-5]

$$\begin{vmatrix} y_k & y_k^1 & y_{k+1}^2 & \cdots & y_k^n \\ y_{k+1} & y_{k+1}^1 & y_{k+1}^2 & \cdots & y_{k+1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{k+n} & y_{k+n}^1 & y_{k+n}^2 & \cdots & y_{k+n}^n \end{vmatrix} = 0.$$

Let us consider the first model. Equation (10) can be written [3-5]

$$N^{(1)}(t) = e^{R_0 t}, \tag{12}$$

so the corresponding difference equation is

$$\det \begin{vmatrix} N_k & N_k^{(1)} \\ N_{k+1} & N_{k+1}^{(1)} \end{vmatrix} = \begin{vmatrix} N_k & e^{R_0 \Delta t k} \\ N_{k+1} & e^{R_0 \Delta t (k+1)} \end{vmatrix} = e^{R_0 \Delta t k} \begin{vmatrix} N_k & 1 \\ N_{k+1} & e^{R_0 \Delta t} \end{vmatrix}$$

$$= e^{R_0 \Delta t k}\left[e^{R_0 \Delta t} N_k - N_{k+1}\right] = 0$$

or can be written as

$$N_{k+1} = e^{R_0 \Delta t} N_k, \tag{13}$$

or

$$\frac{N_{k+1} - N_k}{\left(\frac{1 - e^{R_0 \Delta t}}{\lambda}\right)} = R_0 N_k. \tag{14}$$

Equation (14) is a nonstandard finite difference scheme for equation (1).

Now let us consider the second model. It has the exact analytical solution as given by

$$N(t) = \frac{a N_0}{(a - N_0 b) e^{-a(t - t_0)} + b N_0}. \tag{15}$$

Equation (2) is a nonlinear differential equation, so that the method (11)-(14) cannot be used to obtain a nonstandard difference scheme. To obtain a nonstandard difference scheme, substitutions will be made for equation (15) as follows [3-6]

$$t_0 \to t_k, \qquad t \to t_{k+1}, \qquad N_0 \to N_k, \;\; N(t) \to N_{k+1}. \tag{16}$$

These lead to

$$\frac{N_{k+1} - N_k}{\left(\frac{e^{a\Delta t} - 1}{a}\right)} = a N_k - b N_{k+1} N_k. \tag{17}$$

Equation (17) is a nonstandard finite difference scheme for equation (2).

## RESULTS AND DISCUSSION

In this section the results of calculations and simulations will be addressed from the three methods.

### Simulation Results and Discussion of Equations (1)

We do calculations using the MATLAB software. For equation (1), we take the value of $t$ for the interval $[0,1]$ with $\Delta t = 0.1$ and $R_0 = 1$, where the solution graph is shown in Fig. 1. This Fig. 1 shows that the graph of solution from the nonstandard method, Heun's method and the exact solution coincide, so that it looks like just one graph. However, when we magnify the figure, as shown in Fig. 2, the solution graph of Heun's method does not coincide at all with the exact solution, but the graph of the exact solution coincide with the nonstandard finite difference solution.

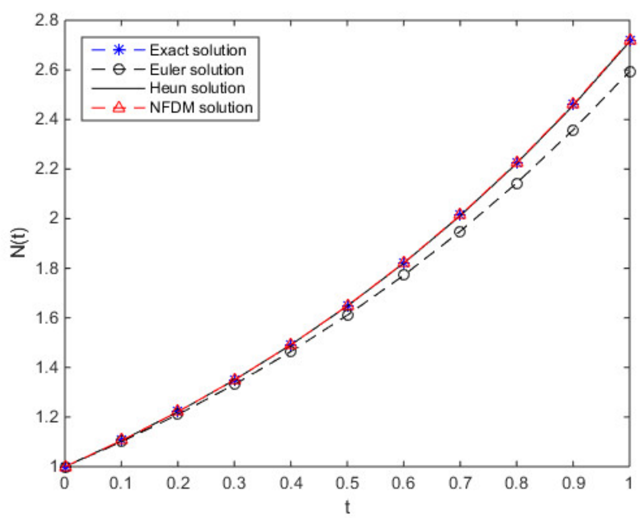


**FIGURE 1.** A numerical simulation graph of equation (1).

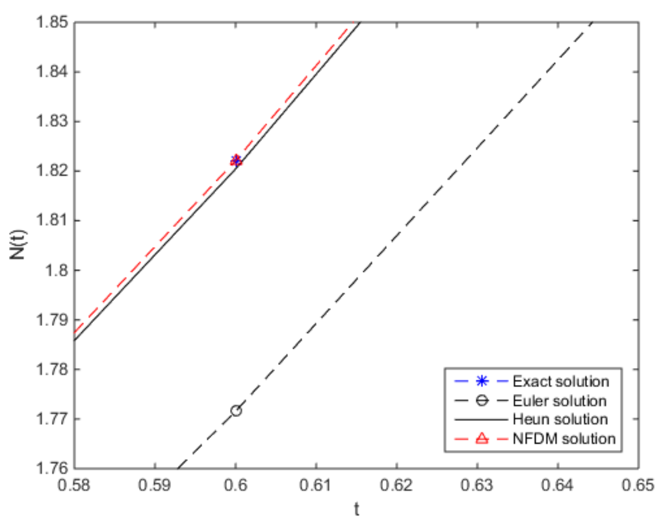


**FIGURE 2.** A magnification of Fig. 1 of the results for equation (1).

Absolute errors are recorded in Table 1. The difference from the value of the exact solution and the solution of the nonstandard method is zero. This shows that from Euler's method, Heun's method and the nonstandard finite difference method, the later has solution closest to the exact solution to equation (1). In fact, its error is zero.

**TABLE 1.** Errors of calculation of equations (1).
NFDM is Nonstandard Finite Difference Method.

| $t$ | Error of Euler's | Error of Heun's | Error of NFDM |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 0.005171 | 0.000171 | 0.000000 |
| 0.2 | 0.011403 | 0.000378 | 0.000000 |
| 0.3 | 0.018859 | 0.000626 | 0.000000 |
| 0.4 | 0.027725 | 0.000923 | 0.000000 |
| 0.5 | 0.038211 | 0.001275 | 0.000000 |
| 0.6 | 0.050558 | 0.001690 | 0.000000 |
| 0.7 | 0.065036 | 0.002179 | 0.000000 |
| 0.8 | 0.081952 | 0.002752 | 0.000000 |
| 0.9 | 0.101655 | 0.003421 | 0.000000 |
| 1.0 | 0.124539 | 0.004201 | 0.000000 |

## Simulation Results and Discussion of Equations (2)

Now let us consider equation (2). We take $t$ in the interval $[0,1]$ with $\Delta t = 0.1$, $a = 1$, $b = 2$, and $N\ (0) = 1$ where the solution graph is shown in Fig. 3 and Fig. 4. In these figures, the solution graph which is closest to the exact solution is the graph of the nonstandard solution. This is also shown from the results of errors calculations in Table 2 that the error value generated from the nonstandard difference method is smaller (in fact, its error is zero) than Heun's method and Euler's method.

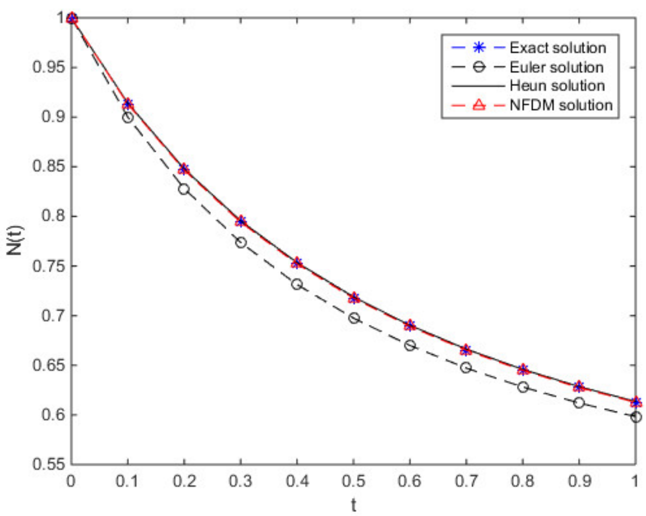


**FIGURE 3.** A numerical simulation graph of equation (2).

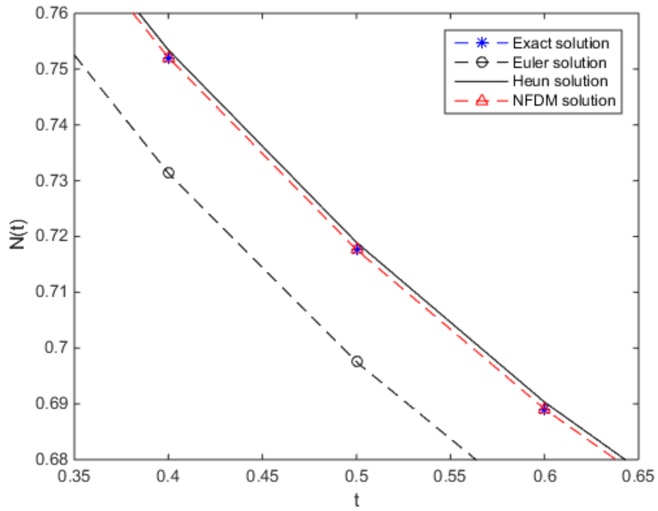


**FIGURE 4.** A magnification of Fig. 3 of the results for equation (2).

**TABLE 2.** Errors of calculation of equations (2).
NFDM is Nonstandard Finite Difference Method.

| *t* | Error of Euler's | Error of Heun's | Error of NFDM |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 0.013106 | 0.000894 | 0.000000 |
| 0.2 | 0.018547 | 0.001251 | 0.000000 |
| 0.3 | 0.020483 | 0.001362 | 0.000000 |
| 0.4 | 0.020726 | 0.001357 | 0.000000 |
| 0.5 | 0.020135 | 0.001297 | 0.000000 |
| 0.6 | 0.019143 | 0.001214 | 0.000000 |
| 0.7 | 0.017977 | 0.001123 | 0.000000 |
| 0.8 | 0.016756 | 0.001032 | 0.000000 |
| 0.9 | 0.015547 | 0.000945 | 0.000000 |
| 1.0 | 0.014383 | 0.000863 | 0.000000 |

As a final remark before we conclude this paper, we observe that the nonstandard finite difference methods produce exact solutions to the considered problems. This has been confirmed by observing that the error generated by the nonstandard finite difference methods is zero. Based on this fact, our nonstandard finite difference methods are actually exact finite difference methods for cases presented in this paper. This gives potential that nonstandard finite difference methods can be implemented to solve other interesting problems, such as those listed in some references [7-12].

## CONCLUSION

Based on our research results, among the solutions of Euler's method, Heun's method and the nonstandard finite difference method, we obtain that the nonstandard finite difference provides solution which is the most accurate in solving the considered models of population dynamics.

## ACKNOWLEDGMENTS

Some of this work was a part of the first author's thesis [13] under the supervision of the second author. The first author thank Sanata Dharma University for the financial support. The work of the second author was financially supported by *Direktorat Riset dan Pengabdian Masyarakat, Direktorat Jenderal Penguatan Riset dan Pengembangan, Kementerian Riset, Teknologi, dan Pendidikan Tinggi* of The Republic of Indonesia with contract number DIPA-042.06.1.401516/2019.


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