# Unique Factorization Domains

**Definition 11.1: Irreducible/Reducible, Prime, Associate Elements**

Let $R$ be an integral domain

(i) Suppose $r \in R \setminus \{0\}$, $r \notin R^\times$.
We say $r$ is **irreducible** if whenever $r = a \cdot b$, either $a \in R^\times$ or $b \in R^\times$.
We say $r$ is **reducible** if it is not irreducible.

(ii) Suppose $r \in R \setminus \{0\}$, $r \notin R^\times$
We say $r$ is **prime** if $(r)$ is a prime ideal.
In other words, if $r \mid a \cdot b$, then either $r \mid a$ or $r \mid b$.

(iii) We say $a, b \in R$ are **associates** if there exists $u \in R^\times$ such that $a = u \cdot b$.

**Proposition 11.1: Prime elements in integral domain are irreducible**

Any prime element in an integral domain is irreducible.

***Proof.*** Suppose $p = a \cdot b \in R$ and $(p)$ is a prime ideal.
Then $p \in (p)$ implies $a \in (p)$ or $b \in (p)$. W.l.o.g let $a \in (p)$.
So $\exists r \in R$ such that $a = p \cdot r$ and hence

$$p = (p \cdot r) \cdot b = p \cdot (r \cdot b)$$

Since $R$ is an integral domain then, $1 = r \cdot b$, so $b \in R^\times$. ■

**Example 11.1.** Irreducible but not prime.
Consider the ring

$$\mathbb{Z}[\sqrt{-5}] :- \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

Then

- $N(a + b\sqrt{-5}) := a^2 + 5b^2$
- $N(x \cdot y) = N(x) \cdot N(y)$
- $N(x) = \pm 1$ if an only if $x \in \mathbb{Z}[\sqrt{-5}]^\times$

<u>**Claim:**</u> $2 + \sqrt{-5}$ is irreducible

***Proof.*** Suppose

$$2 + \sqrt{-5} = (a + b\sqrt{-5}) \cdot (c + d\sqrt{-5})$$

Then

$$N(2 + \sqrt{-5}) = 4 + 5 = 9 \implies N(a + b\sqrt{-5}) \mid 9 \implies N(a + b\sqrt{-5}) = \pm 1 \text{ or } \pm 3$$

*Observe* that if $b \neq 0$, then

$$N(a + b\sqrt{-5}) = a^2 + 5b^2 \geq 5$$

Therefore
$b = 0 \implies N(a+b\sqrt{-5}) = N(a) = a^2 \implies N(a+b\sqrt{-5}) = 1 \implies a+b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]^\times$
$\blacksquare$

**Claim:** $2+\sqrt{-5}$ is **not** prime.

***Proof.*** We know

$$3^2 = 9 = (2+\sqrt{-5}) \cdot (2-\sqrt{-5}) \in (2+\sqrt{-5})$$

However, $3 \notin (2+\sqrt{-5})$.
If $3 = (a+b\sqrt{-5}) \cdot (2+\sqrt{-5})$, then
$9 = N(3) = N(a+b\sqrt{-5}) \cdot N(2+\sqrt{-5}) = N(a+b\sqrt{-5}) \cdot 9 \implies N(a+b\sqrt{-5}) = 1$
which immediately tells us $b = 0$ and $a = \pm 1$.
But $3 \notin \pm(N(a+b\sqrt{-5}) \cdot N(2+\sqrt{-5}))$ $\blacksquare$

**Proposition 11.2: Element in PID is prime iff it is irreducible**

In a PID an element is prime *iff* it is irreducible.

***Proof.*** It suffices to show that irreducible $\implies$ prime.
Suppose $r \in R$ is irreducible and recall that maximal ideals are prime. Hence we will show that $(r)$ is maximal.
Suppose $(r) \subset (m) \subsetneq R$, then

$$r \in (m) \implies \exists s \in R,\, r = s \cdot m \implies r \text{ irreducible} \implies s = R^\times \text{ or } m \in R^\times$$

By assumption $(m) \subsetneq R$ and this implies

$$m \notin R^\times \implies s \in R^\times \implies (r) = (m)$$

$\blacksquare$

**Example 11.2.** In $\mathbb{Z}$, the irredcibles are the primes (and their negatives)

*Observe* that the factorization of any integer into primes is unique!

**Definition 11.2: Unique Factorization Domain**

A **unique factorization domain** (UFD) is an integral domain $R$ such that for all $r \in r \setminus \{0\}$, $r \notin R^\times$

(i) $r = p_1 \cdot p_2 \cdot \ldots \cdot p_k$ for $p_i$ irreducible.

(ii) This decomposition is unique up to associates and reordering, i.e if

$$r = q_1 \cdot \ldots \cdot q_m, \quad q_j \text{ irreducible}$$

Then after reordering, $q_i = u_i p_i, u_i \in R^\times$ and $n = m$.

**Example 11.3.** Fields are vacuously UFDs

**Example 11.4.** $\mathbb{Z}$ are a UFD

**Example 11.5.** $\mathbb{Z}[\sqrt{-5}]$ is **not** a UFD as

$$3^2 = (2+\sqrt{-5}) \cdot (2-\sqrt{-5})$$

and $3, 2 \pm \sqrt{-5}$ are irreducibles.

**Proposition 11.3: Element in UFD is prime iff it is irreducible**

In a UFD, an element is prime *iff* it is irreducible.

***Proof.*** It suffices to show once more that irreducible $\implies$ prime.
Suppose $r \in R$ is irreducible and $a \cdot b \in (r)$ i.e there exists $c \in R$ such that $a \cdot b = r \cdot c$
By unique factorization

$$\begin{aligned} a =& p_1 \cdot p_2 \cdot \ldots p_n, \quad p_i \text{ irreducible, unique} \\ b =& q_1 \cdot q_2 \cdot \ldots q_n, \quad q_j \text{ irreducible, unique} \\ c =& r_1 \cdot r_2 \cdot \ldots r_l, \quad r_k \text{ irreducible, unique} \end{aligned}$$

Hence

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n \cdot q_n \cdot \ldots \cdot q_m = r \cdot r_1 \cdot r_2 \cdot \ldots \cdot r_l$$

so by unique factorization, w.l.og

$$r = u \cdot p_1,\ u \in R^\times \implies r|a$$

■

**Proposition 11.4: Nonzero elements in UFD have GCD**

Let $a, b \in R \setminus \{0\}$ in a UFD. Then there is a greatest common divisor of $a, b$ in $R$.

***Proof.*** We write for $u, v \in R^\times$ and $p_i$'s irreducible

$$a = u \cdot p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_n^{e_n} b = v \cdot p_1^{f_1} \cdot p_2^{f_2} \cdot \ldots \cdot p_n^{f_n}$$

We allow some exponents to be 0 ($p_i^0 = 1$) and we require $p_i \neq p_j$ if $i \neq j$ for example

$$\begin{pmatrix} 12 = 2^2 \cdot 3 \to 12 = 2^2 \cdot 3^1 \cdot 5^0 \\ 20 = 2^2 \cdot 5 \to 20 = 2^2 \cdot 3^0 \cdot 5^1 \end{pmatrix}$$

<u>**Claim:**</u>

$$d = p_1^{\min\{e_1, d_1\}} \cdot p_2^{\min\{e_2, d_2\}} \cdot \ldots \cdot p_n^{\min\{e_n, d_n\}}$$

is the $gcd(a, b)$.

***Proof.*** Clearly $d \mid a$, $d \mid b$.
If $c \mid a$, $c \mid b$, then we want to see that $c \mid d$.
Unique factorization says for $q_i$ irreducible, $q_i \neq q_j$ and $g_i > 0$, we have

$$c = q_1^{g_1} \bullet \ldots \bullet q_m^{g_m}$$

Since $c \mid a$, $c \mid b$, then after changing associates

$$\{q_1, \ldots, q_n\} \subset \{p_1, \ldots, p_n\},\ g_i \leq \min\{e_i, f_i\} \implies c \mid d$$

■

And so there exists a greatest common divisor of $a, b$ in $R$. ■