

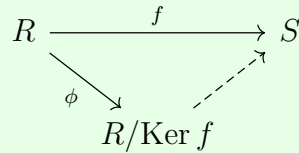
# L5: Isomorphism Theorems

## Theorem 5.1: The First Isomorphism Theorem

If  $f : R \rightarrow S$  is a ring homomorphism and  $I = \text{Ker } f$ . Then

$$R/I \cong \text{Im } f$$

as rings.



## Theorem 5.2: The Second Isomorphism Theorem

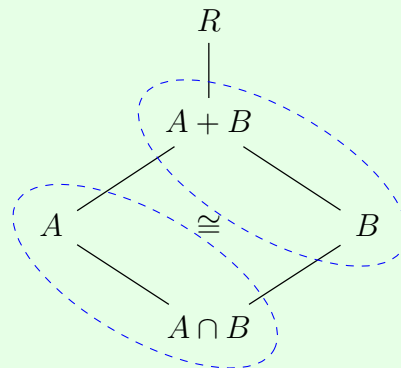
Let  $A \subset R$  be a subring and  $B \subset I$  an ideal.

Then

$$A + B := \{a + b \mid a \in A, b \in B\}$$

is a subring of  $R$  and  $A \cap B$  is an ideal of  $A$  and

$$(A + B)/B \cong A/(A \cap B)$$



### **Proof of 5.2.**

Let  $A \subset R$  be a subring and  $B \subset I$  an ideal.

It is **Easy to check** that  $A + B$  is a subring and  $A \cap B$  is an ideal in  $A$ .

Now we want to find an isomorphism

$$(A + B)/B \longrightarrow A/(A \cap B)$$

Idea: Use the First Isomorphism Theorem, i.e we want to find a surjective homomorphism

$$f: A + B \rightarrow A/(A \cap B)$$

such that  $\text{Ker } f = B$ .

Define a map

$$\begin{aligned}\phi: A + B &\rightarrow A/(A \cap B) \\ a + b &\mapsto a + A \cap B\end{aligned}$$

which can be shown to be homomorphism if it is well defined. Generally, if  $x \in A + B$ , there are many ways to express  $x \in A + B$ , i.e there may exist,  $a, a' \in A$  and  $b, b' \in B$  such that

$$x = a + b = a' + b'$$

So is  $\phi(x) = a + A \cap B$  or  $\phi(x) = a' + A \cap B$ ?

This is not a problem so long as  $a + A \cap B = a' + A \cap B$ . In other words, if  $a - a' \in A \cap B$  BUT

$$a + b = a' + b' \implies \underbrace{a - a'}_{\in A} = b' - b \in B \implies a - a' \in A \cap B$$

We also need to check that  $\phi$  is surjective.

Clearly, if  $a + A \cap B \in A/(A \cap B)$ , then say  $a \in A$  and is a representative for  $a + A \cap B$ .

Then,  $a + 0 \in A + B$  and  $\phi(a) = a + A \cap B$ .

Finally, we must check that

$$\text{Ker } \phi = B$$

If  $a + b \in \text{Ker } \phi$  then  $\phi(a + b) = 0 + A \cap B$  and so

$$a \in A \cap B \implies a \in B \implies \text{Ker } \phi \subset B$$

On the other hand, if  $b \in B \subset A + B$ , then we can write it as  $b = 0 + b$  and so

$$\phi(b) = 0 + A \cap B \implies b \in \text{Ker } \phi \implies B \subset \text{Ker } \phi$$

Therefore,  $\text{Ker } \phi = B$ . ■

### Theorem 5.3: The Third Isomorphism Theorem

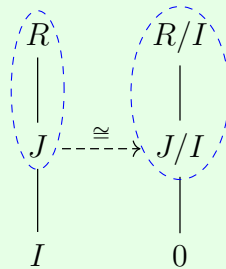
Let  $I, J \subset R$  be ideals  $I \subset J$ .

Then

$$J/I := \{a + I \in R/I \mid a \in J\}$$

(the cosets of  $R/I$  whose representatives are in  $J$  or similarly the restriction of the quotient map from  $R$  to  $R/I$  to the domain  $J$ ) is an ideal in  $R/I$  and

$$(R/I)/(J/I) \cong R/J$$



**Proof of 5.3.**

Let  $I \subset J \subset R$  be ideals.

Then we want to show,  $J/I \subset R/I$  is an ideal and

$$(R/I)/(J/I) \cong R/J$$

*Check:*  $J/I$  is an ideal in  $R/I$ .

Then define a map

$$\begin{aligned}\phi: R/I &\rightarrow R/J \\ a + I &\mapsto a + J\end{aligned}$$

**Observe** that if  $a \in J$ , then  $\phi(a + I) = a + J = J = \bar{0}$

$\phi$  is also clearly surjective: Pick any representative  $a \in R$  for  $a + J$ , then

$$\phi(a + I) = a + J$$

It remains to be shown that  $\text{Ker } \phi = J/I$  as follows:

If  $a + I \in \text{Ker } \phi$  then  $\phi(a + I) = a + J = 0 + J = J$  which implies

$$a \in J \implies a + I \in J/I \implies \text{Ker } \phi \subset J/I$$

If  $a \in J$ , then  $\phi(a + I) = a + J = J$  which implies

$$a + I \in \text{Ker } \phi \implies \text{Ker } \phi \supset J/I$$

and therefore  $\text{Ker } \phi = J/I$ . ■

**Theorem 5.4: The Fourth Isomorphism Theorem**

Let  $I \subset R$  be an ideal.

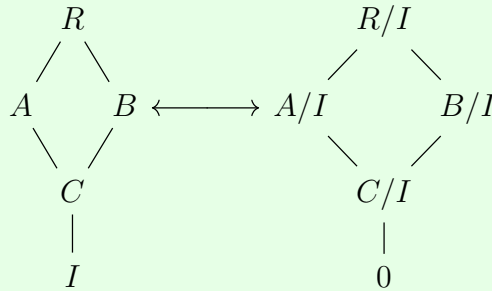
Then the correspondence

$$I \subset A \subset R \longleftrightarrow A/I \subset R/I$$

is a bijection between

$$\{\text{subrings of } R \text{ containing } I\} \longleftrightarrow \{\text{subrings of } R/I\}$$

Moreover,  $A \subset R$  is an ideal iff  $A/I$  is an ideal in  $R/I$ .



### Definition 5.5: Ideal Generation, Principal and Finitely Generated Ideal

Let  $R$  be a ring, with  $1 \neq 0$  and let  $A \subset R$  be any subset.

The **ideal generated by**  $A$  is

$$A \subset (A) \subset R$$

i.e, the smallest ideal of  $R$  containing  $A$ .

If an ideal  $I$  is generated by a single element set, then we say  $I$  is a **principal ideal**.

If  $I$  is generated by a finite set then we say  $I$  is a **finitely generated ideal**.

**Note:** Instead of writing  $I = (\{a\})$  for a principal ideal, we often omit the set notation and just write

$$I = (a)$$

Similarly, we will write  $I = (a_1, \dots, a_n)$  for finitely generated ideals.

### Proposition 5.6: Minimality of ideal generated by a set

For any subset  $A \subset R$  and ideals  $I \subset R$  such that  $A \subset I$ , we have

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

**Proof.**

**Observe** that  $R \subset R$  and is always an ideal of itself which implies that there always exists an ideal containing  $A$  (at least  $R$ )

$$\{A \subset I \subset R\} \neq \emptyset$$

First check that  $(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$

Suppose, for a contradiction,  $A \subset I$  and  $(A) \not\subset I$ , then

(i)  $(A) \cap I \subsetneq (A)$  (proper subset otherwise  $(A) \subset I$ )

(ii)  $A \subset (A)$  and  $A \subset I \implies A \subset (A) \cap I$

(iii)  $(A) \cap I$  is an ideal (second isomorphism theorem).

Therefore there is an ideal containing  $A$  (i.e  $(A) \cap I$ ) that is smaller than  $(A)$ , which is contradictory the definition of  $(A)$ . Hence

$$(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

Now check that  $\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$

We have that

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I$$

is an ideal and therefore  $A \subset \bigcap I$  which implies

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$$

because  $(A)$  is an ideal. Therefore,

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

■