Abstract linear algebra

<u>Defn</u>: A subset A of an R-module M is said to be <u>linearly independent</u> if

$$a_1, \ldots, a_n \in R, \quad m_1, \ldots, m_n \in A$$

s.t. $$a_1 \cdot m_1 + a_2 \cdot m_2 + \ldots + a_n \cdot m_n = 0$$

Then $$a_1 = a_2 = a_3 = \cdots = a_n = 0$$

If A is <u>not</u> lin. ind., we say it is <u>linearly dependent</u>

<u>Example</u>: A basis B for a free R-module is linearly independent

$B = \{1, X, X^2, X^3, \ldots\}$ is lin. ind. in $\mathbb{R}[X]$ (viewed as an $\mathbb{R}$-module)

<u>Redefinition</u>: A <u>basis</u> of a free R-module is a linearly independent spanning set.

<u>Non-examples</u>:

- $\{0\} \subset M$ is <u>not</u> lin. ind. (assuming $R \neq 0$)

  e.g. $1 \cdot 0 = 0 = 0 \cdot 0$

- $\mathbb{Z}/2\mathbb{Z}$ as a $(\mathbb{Z}/4\mathbb{Z})$-module

  The only possible linearly independent subset is $\{\bar{1}\}$

  $$\bar{2} \in \mathbb{Z}/4\mathbb{Z} \Rightarrow \bar{2}_4 \cdot \bar{1}_2 = \bar{0}_2 \in \mathbb{Z}/2\mathbb{Z}$$

  $\bar{2} \neq \bar{0} \in \mathbb{Z}/4\mathbb{Z}$

Thm: If V is a finitely generated vector space over a field F

Then V is a free F-vector space.

Pf: Let $A = \{v_1, \dots, v_n\}$ a finite spanning set of V

we may suppose no proper subset of A is spanning.

We show that A is linearly independent:

Suppose not

Let $\alpha_1, \dots, \alpha_n \in F$ s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \quad \text{s.t. } \alpha_1, \dots \alpha_n \text{ not all zero.}$$

After possibly rearranging, we may assume $\alpha_1 \neq 0$.

Because F is a field, $\frac{1}{\alpha_1} \in F$

$$\Longrightarrow \quad v_1 = \frac{1}{\alpha_1} \cdot \left(-\alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4 \cdots - \alpha_n v_n \right)$$

$$= \left(-\frac{\alpha_2}{\alpha_1}\right) \cdot v_2 + \left(-\frac{\alpha_3}{\alpha_1}\right) \cdot v_3 + \left(-\frac{\alpha_4}{\alpha_1}\right) \cdot v_4 + \dots + \left(-\frac{\alpha_n}{\alpha_1}\right) v_n$$

$$\Longrightarrow \quad v_1 \in \operatorname{Span}\{v_2, \dots, v_n\}$$

$$\Longrightarrow \quad \operatorname{Span}\{v_2, \dots, v_n\} = V \quad \rightarrow\leftarrow$$

$$\Longrightarrow \quad A = \{v_1, \dots, v_n\} \text{ is linearly independent}$$

Remains to show that V is a free F-vector space

Suppose $v \in V$

$$v = a_1 \cdot v_1 + a_2 \cdot v_2 + \_\_\_ + a_n \cdot v_n \qquad a_1, \_\_, a_n \in F$$
$$= b_1 \cdot v_1 + b_2 \cdot v_2 + \_\_\_ + b_n \cdot v_n \qquad b_1, \_\_, b_n$$

$$\Longrightarrow (a_1 - b_1) \cdot v_1 + (a_2 - b_2) \cdot v_2 + \_\_\_ + (a_n - b_n) \cdot v_n = 0$$

Since A is linearly independent

$$a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \quad \dots, \quad a_n - b_n = 0$$
$$a_1 = b_1, \quad a_2 = b_2, \quad \dots, \quad a_n = b_n$$

$\Longrightarrow$ V is free on A $\square$

Cor: If V is a finitely generated F-vector space and A is a minimal spanning set

Then V is a free F-vector space on A and A is a basis for V.

Cor: If V is an F-vector space w/ finite spanning set A

Then A contains a basis B for V.

Pf: Take a minimal spanning subset of A $\square$

Replacement Theorem

Suppose V is an F-vector space

w/ basis $A = \{a_1, \_\_, a_n\}$

and $B = \{b_1, \_\_, b_m\}$ is a linearly independent set.

After possibly rearranging A, the sets

$$C_k := \{b_1, \_\_, b_k, a_{k+1}, \_\_, a_n\} \quad \forall\, 0 \le k \le m$$

are bases for V.

In particular, $n \geq m$

Pf: Prove this by induction:

When $k=0$, $C_0 = A = \{a_1, \_\_, a_n\}$ this is already true.

Now suppose C_k is a basis for V

we will show C_{k+1} is a basis for V.

$$C_k = \{b_1, \_\_, b_k, a_{k+1}, \_\_, a_n\} \text{ spans } V$$

$$\Longrightarrow b_{k+1} = \alpha_1 \cdot b_1 + \alpha_2 \cdot b_2 + \_\_ + \alpha_k \cdot b_k + \alpha_{k+1} \cdot a_{k+1} + \_\_ + \alpha_n \cdot a_n$$

Now B is linearly independent $\Longrightarrow \exists\, \alpha_{k+i} \neq 0,\ i \geq 1$

After rearranging, we may assume $\alpha_{k+1} \neq 0$.

$$a_{k+1} = \frac{1}{\alpha_{k+1}}\left(b_{k+1} - \alpha_1 b_1 - \alpha_2 b_2 \cdots - \alpha_k b_k - \alpha_{k+2} a_{k+2} - \cdots - \alpha_n a_n\right)$$

$$= \left(\frac{1}{\alpha_{k+1}}\right) \cdot b_{k+1} + \left(\frac{-\alpha_1}{\alpha_{k+1}}\right) \cdot b_1 + \cdots + \left(\frac{-\alpha_k}{\alpha_{k+1}}\right) b_k + \left(\frac{-\alpha_{k+2}}{\alpha_{k+1}}\right) \cdot a_{k+2} + \_\_ + \left(\frac{-\alpha_n}{\alpha_{k+1}}\right) a_n$$

$$\Longrightarrow \quad a_{k+1} \in \operatorname{Span}\{b_1, \ldots, b_{k+1}, a_{k+2}, \ldots, a_n\} = \operatorname{Span} C_{k+1}$$

$$\Longrightarrow \quad \operatorname{Span} C_{k+1} \supset \operatorname{Span}\{b_1, \ldots, b_k, a_{k+1}, \ldots, a_n\} = \operatorname{Span} C_k = V$$

$$\Longrightarrow \quad \operatorname{Span} C_{k+1} = V$$

It remains to show C_{k+1} is linearly independent.

Suppose $\beta_1 b_1 + \ldots + \beta_k b_k + \beta_{k+1} \underbrace{b_{k+1}} + \gamma_{k+2} a_{k+2} + \ldots + \gamma_n a_n = 0$

$$= \left(\sum_{i=1}^{k} \beta_i \cdot b_i\right) + \beta_{k+1} \cdot \left(\sum_{i=1}^{k} \alpha_i \cdot b_i + \sum_{j=k+1}^{n} \alpha_j \cdot a_j\right) + \left(\sum_{j=k+2}^{n} \gamma_j \cdot a_j\right)$$

$$= \left[\sum_{i=1}^{k} \left(\beta_i + \beta_{k+1} \alpha_i\right) \cdot b_i\right] + \left(\beta_{k+1} \cdot \alpha_{k+1}\right) \cdot a_{k+1} + \left[\sum_{j=k+2}^{n} \left(\beta_{k+1} \alpha_j + \gamma_j\right) \cdot a_j\right]$$

Because C_k is linearly independent

$$\Longrightarrow \quad \beta_i + \beta_{k+1} \alpha_i = 0 \qquad i = 1, \ldots, k$$

$$\boxed{\beta_{k+1} \cdot \alpha_{k+1} = 0}$$

$$\beta_{k+1} \cdot \alpha_j + \gamma_j = 0 \qquad j = k+2, \ldots, n$$

By assumption $\alpha_{k+1} \neq 0 \xrightarrow[\text{F a field}]{} \beta_{k+1} = 0$

$$\Longrightarrow \quad \beta_i = 0 \qquad i = 1, \ldots, k$$

$$\gamma_j = 0 \qquad j = k+2, \ldots, n$$

$\Longrightarrow$ C_{k+1} is linearly independent $\square$

Cor: (1) If V is an F-vector space w/ basis $B = \{b_1, \ldots, b_n\}$

Then any lin. ind. set A has at most n elements

any spanning set C has at least n elements

(2) Any two bases B, B' of a finitely generated F-vectorspace have the same cardinality.

Defn: If V is a f.g F-v.sp.

Then the <u>dimension</u> of V is

$$\dim_F V := \dim V := \text{cardinality of any basis of } V$$

we say V is <u>finite dimensional</u>

If V is not f.g., then we say it is <u>infinite dimensional</u>

$(\dim V = \infty)$

Example: . $\dim \mathbb{R}^2 = 2$

. $\dim \{\text{real polynomials of degree at most } 3\} = 4$

. $\dim \mathbb{R}[x] = \infty$

Cor: If V is a fin.-dim'l F-vector space with a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$

Then $\mathcal{B}$ defines an F-vector space isomorphism

$$\Phi_{\mathcal{B}} : V \xrightarrow{\cong} F^n$$

Pf:

$$\begin{aligned}
\Phi_{\mathcal{B}} : V &\longrightarrow F^n \\
b_1 &\longmapsto e_1 = (1, 0, 0, \ldots, 0) \\
b_2 &\longmapsto e_2 = (0, 1, 0, \ldots, 0) \\
&\vdots \\
b_n &\longmapsto e_n = (0, 0, \ldots, 0, 1)
\end{aligned}$$

"extend linearly"

i.e.
$$\begin{aligned}
\Phi_{\mathcal{B}}(\alpha_1 \cdot b_1 + \alpha_2 \cdot b_2 + \ldots + \alpha_n \cdot b_n) &= \alpha_1 \cdot \Phi_{\mathcal{B}}(b_1) + \alpha_2 \cdot \Phi_{\mathcal{B}}(b_2) + \ldots + \alpha_n \cdot \Phi_{\mathcal{B}}(b_n) \\
&= \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \ldots + \alpha_n \cdot e_n
\end{aligned}$$

Check: Injectivity

$$\operatorname{Ker} \Phi_{\mathcal{B}} = \{ \alpha_1 \cdot b_1 + \ldots + \alpha_n b_n \mid \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \ldots \alpha_n \cdot e_n = 0 \}$$
$$= \{0\}$$

$\mathcal{E} = \{e_1, \ldots, e_n\}$ a basis for F^n

Check: Surjectivity

$v = \alpha_1 \cdot e_1 + \ldots + \alpha_n \cdot e_n \in F^n$, then $\Phi_{\mathcal{B}}(\alpha_1 \cdot b_1 + \ldots + \alpha_n b_n) = v$

$\square$