Determinants

<u>Recap</u>: Fix a field F

V, W vector spaces

$f: V \to W$ a F-vector space homomorphism

$B = \{v_1, \ldots, v_n\}$ a basis for V

$D = \{w_1, \ldots, w_m\}$ a basis for W

(0) $\underline{\Phi}_B : V \longrightarrow F^n$ $\quad$ F-v.sp. iso

$$\sum_{i=1}^{n} \alpha_i v_i \longmapsto \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}_B$$

(1) $M_B^D : \mathrm{Hom}_F(V, W) \xrightarrow{\cong} M_{m,n}(F)$ $\quad$ F-v.sp. iso

$$(f: V \to W) \longmapsto M_B^D(f) = \left(f(v_1)_D \,\middle|\, f(v_2)_D \,\middle|\, \cdots \,\middle|\, f(v_n)_D \right)$$

(2) $\underset{B}{f}: V \longrightarrow \underset{D}{W}$ $\qquad$ $g: \underset{D}{W} \longrightarrow \underset{\mathcal{E}}{U}$

$$M_B^{\mathcal{E}}(g \circ f) = M_D^{\mathcal{E}}(g) \cdot M_B^D(f)$$

(3) $M_B^B : \mathrm{End}(V) \xrightarrow{\cong} M_n(F)$ $\quad$ is a <u>ring</u> iso

(4) $f \in \mathrm{End}(V)$ isomorphism $\longleftrightarrow$ $M_B^B(f) \in M_n(F)$ invertible/nonsingular

Defn: Let V an F-vector space

$$B = \{v_1, \_\_\_, v_n\}$$

$$D = \{w_1, \_\_\_, w_n\}$$

Suppose $\quad v_j = \sum_{i=1}^{n} \alpha_{ij} w_i$

Then $\quad M = (\alpha_{ij}) = \left((v_1)_D \,\middle|\, (v_2)_D \,\middle|\, \cdots \,\middle|\, (v_n)_D \right)$

is the <u>change-of-basis</u> matrix from B to D.

Obs: $v_i = Id_V(v_i)$

$$\Longrightarrow \quad M = M_B^D(Id_V)$$

Thm: If $f \in End(V)$

Then
$$M_B^B(f) = M_B^B(Id_V \circ f \circ Id_V)$$
$$= M_D^B(Id_V) \cdot M_D^D(f) \cdot M_B^D(Id_V)$$
$$= M^{-1} \cdot M_D^D(f) \cdot M$$

Defn: Matrices $A, B \in M_n(F)$ are said to be <u>similar</u> matrices if

$\exists$ invertible matrix $P \in M_n(F)$

s.t. $\quad A = P^{-1} B P$

Thm: Two matrices $M, N \in M_n(F)$ are similar

if and only if $\exists\, f \in End(F^n)$ and bases B, D for F^n

s.t.
$$M = M_B^B(f)$$
$$N = M_D^D(f)$$
$$P = M_B^D(Id_{F^n})$$

and
$$M = P^{-1} N P$$

Pf: Exercise $\square$

Recall: If $f: V \longrightarrow W$ an F-v.sp. homomorphism

Then there is a map (sometimes called the adjoint)

$$f^*: W^* \longrightarrow V^*$$
$$\varphi \longmapsto \left[f^*\varphi(v) = \varphi(f(v)) \right]$$

If we have bases

$$B = \{v_1, \ldots, v_n\} \quad \text{for } V$$
$$D = \{w_1, \ldots, w_m\} \quad \text{for } W$$

we have dual bases

$$B^* = \{v_1^*, v_2^*, \ldots, v_n^*\} \quad \text{for } V^*$$
$$D^* = \{w_1^*, w_2^*, \ldots, w_m^*\} \quad \text{for } W^*$$

Thm: If $f \in \mathrm{Hom}_F(V, W)$

and $M = M_B^D(f)$

Then $M_{D^*}^{B^*}(f^*) = M^T$

Pf: Exercise 12

Cor: Row rank = Column rank for any matrix

Determinants

Fix a field F

Consider a linear map

$$f : F \longrightarrow F \qquad (\text{1-dim'l v.sp.})$$

For any $v \in F$, $v = v \cdot 1 \implies f(v) = f(v \cdot 1) = v \cdot f(1)$

$\implies f$ is determined by $f(1)$.

Take the basis $B = \{1\}$ for F

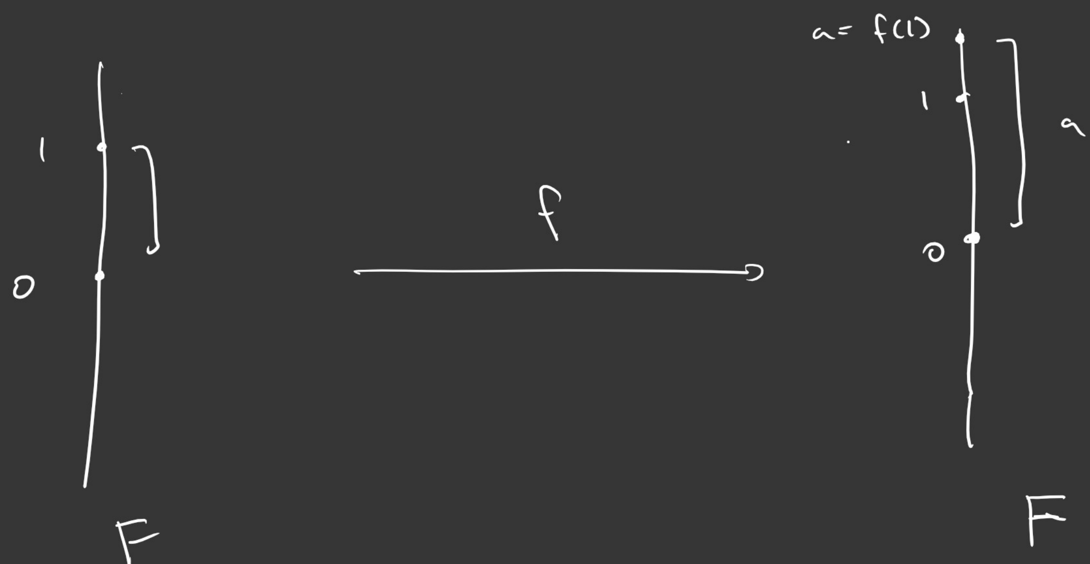
$$M_B^B(f) = (f(1))$$

Defn: The determinant of

a 1×1 matrix $A = (a)$

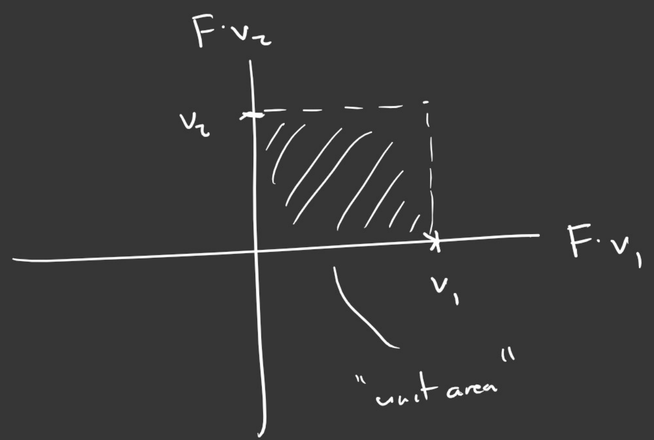
an endomorphism $f \in End(F)$

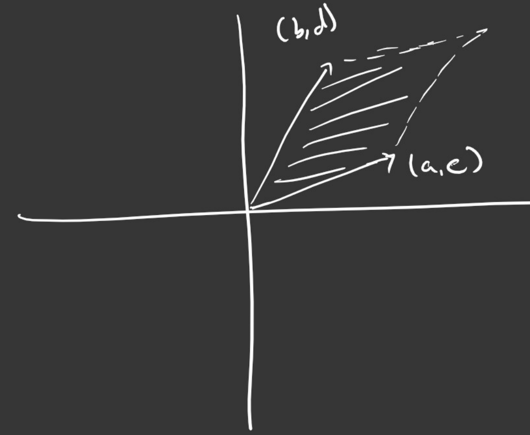
is $\det A := a \in F$

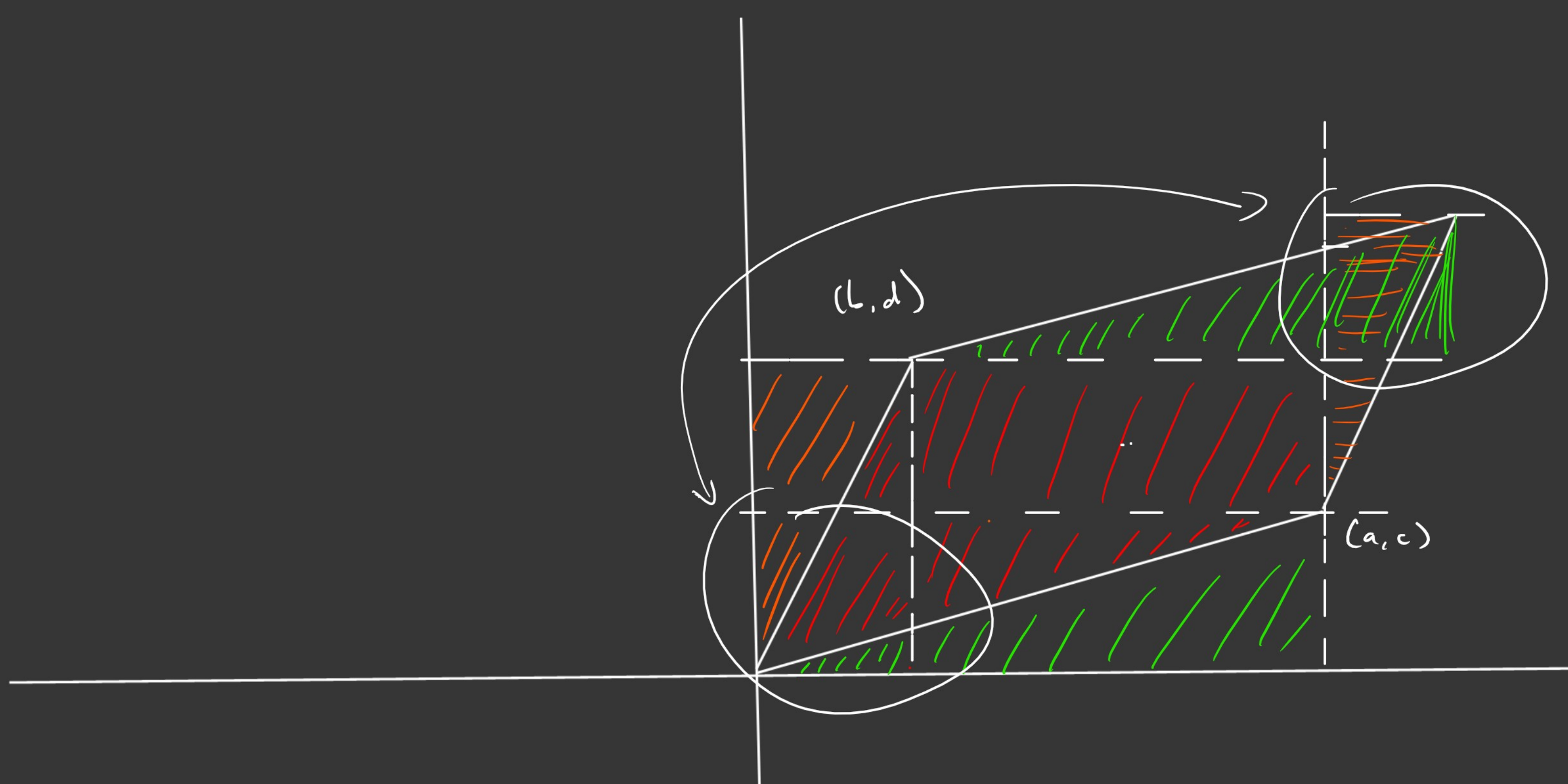


Consider $f : F^2 \longrightarrow F^2$

$$M_B^B(f) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \{v_1, v_2\}$$







$\Longrightarrow$ Area(▱) = $ad - bc$

<u>Defn</u>: The <u>determinant</u> of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$

is $\det A := ad - bc \in F$

Defn: If $A = (a_{ij}) \in M_n(F)$

Then the $(i,j)^{th}$ **minor** of A

is obtained by deleting the i^{th} row, j^{th} column from A

and denote it by A_{ij}.

The $(i,j)^{th}$ cofactor of A is

$$(-1)^{i+j} \cdot A_{ij}$$

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$,

$$A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

$$A_{31} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$$

Defn: The **determinant** of $A = (a_{ij}) \in M_n(F)$

is defined inductively

$$\det A := (-1)^{i+1} a_{i1} \cdot \det A_{i1} + (-1)^{i+2} \cdot a_{i2} \cdot \det A_{i2} + \dots + (-1)^{i+n} a_{in} \det A_{in}.$$

Note: Independent of i.

Example:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (-1)^{1+1} \cdot a_{11} \cdot \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

$$= (-1)^2 \cdot (1) \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + (-1)^3 \cdot (2) \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (-1)^4 \cdot (3) \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7$$

Properties of determinants

① $\det A = 0$ iff A is singular

iff the columns are linearly dependent

iff the rows are linearly dependent.

② $\det(A \cdot B) = \det A \cdot \det B$

Cor: If $P \in M_n(F)$ is inventible

Then $\det P^{-1} = (\det P)^{-1}$

Pf: $\det(Id_n) = 1 \implies \det(P \cdot P^{-1}) = \det P \cdot \det(P^{-1})$

$\implies \det(P^{-1}) = \dfrac{1}{\det P}$ $\square$

Cor: If $f \in End(V)$

Then $\det f := \det(M_B^B(f))$ is well-defined

(i.e. independent of B)

Pf: $M_B^B(f) = P^{-1} M_D^D(f) \cdot P$

$$\det(M_B^B(f)) = \det P^{-1} \cdot \det(M_D^D(f)) \cdot \det P$$
$$= \det(M_D^D(f)) \cdot (\det P)^{-1} \cdot \det P$$
$$= \det(M_D^D(f))$$

$\square$

③ $\det(A^T) = \det A$

Cor: If $f: V \longrightarrow V$

$f^*: V^* \longrightarrow V^*$

Then $\det(f^*) = \det(f)$.