Maximal ideals

Today: Rings are comm. w/ $1 \neq 0$

<u>Prop</u>: Let $I \subset R$ an ideal

① $I = R$ iff I contains a unit.

② R is a field iff the only ideals of R are 0 and R

<u>Pf</u>: ① If $I = R$, then $1 \in I$

Conversely, if $u \in I$ and $u \in R^\times$ say $u \cdot v = 1$

Then $u \cdot v = 1 \in I \implies$ if $r \in R$, then $r \cdot (u \cdot v) = r \in I$

$\implies R \subset I \implies R = I$

② If $I \subset R$ is an ideal in a field, and $\exists\, a \in I \setminus \{0\}$

Then $a \in R^\times$ (b/c it is a field) $\implies I = R$

Conversely, suppose 0 and R are the only ideals in R.

Let $a \in R \setminus \{0\}$

Consider $(a) \subset R$.

Then $(a) \neq 0 \implies (a) = R$

$\implies \exists\, u \in (a)$ s.t. $u \in R^\times$ say $u \cdot v = 1$

We may write $u = r \cdot a$, $r \in R \implies (r \cdot a) \cdot v = 1$

$a \cdot (r \cdot v) = 1$

$\implies a \in R^\times$

$\implies R$ is a field $\square$

Cor: If F is a field.

Then any nonzero ring homomorphism

$$f: F \to R$$

is an injective map.

Pf: $\operatorname{Ker} f = 0$ or F. Because f is nonzero, we conclude that $\operatorname{Ker} f = 0$ $\square$

Defn: An ideal $M \subset R$ is called a <u>maximal ideal</u> if

① $M \neq R$

② If $I \subset R$ is an ideal s.t. $M \subset I$

Then $I = M$ or $I = R$.

Not all rings admit maximal ideals

A given ring may admit multiple maximal ideals.

e.g. $2\mathbb{Z}, 3\mathbb{Z}$ are maximal ideals in $\mathbb{Z}$.

Prop: If R is a ring w/ $1 \neq 0$.

Then every proper ideal is contained in a maximal ideal.

<u>A digression on Zorn's Lemma:</u>

Defn: A <u>partial order</u> on a non-empty set A is a relation $\leq$

s.t. ① $x \leq x$ (Reflexive)

② $x \leq y, y \leq x \Longrightarrow x = y$ (Anti-Symmetric)

③ $x \leq y, y \leq z \Longrightarrow x \leq z$ (Transitive)

Example: If X is any set

$$\mathcal{P}(X) = \{\text{subsets } U \subset X\}$$

Then inclusion is a partial order on $\mathcal{P}(X)$

e.g.

$$\{a, b, c\}$$

$$\supset \qquad \cup \qquad \supset$$

$$\{a, b\} \qquad \{a, c\} \qquad \{b, c\}$$

$$\cup \quad \cup\subset \qquad \cup\subset \quad \cup$$

$$\{a\} \qquad \{b\} \qquad \{c\}$$

$$\cup \qquad \cup \qquad \subset$$

$$\emptyset$$

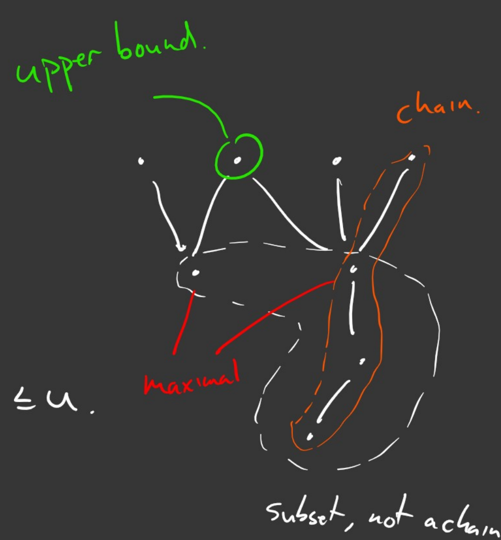
Defn: If $(A, \leq)$ is a partially ordered set (poset)

① A subset $B \subset A$ a chain if

$\forall\, x, y \in B \quad x \leq y$ or $y \leq x$

② An upper bound on a subset $B \subset A$ is an element $u \in A$ s.t. $\forall b \in B$, $b \leq u$.

③ A maximal element of a subset $B \subset A$ is an element $m \in B$ s.t. if $b \in B$ and $b \geq m$, then $b = m$.



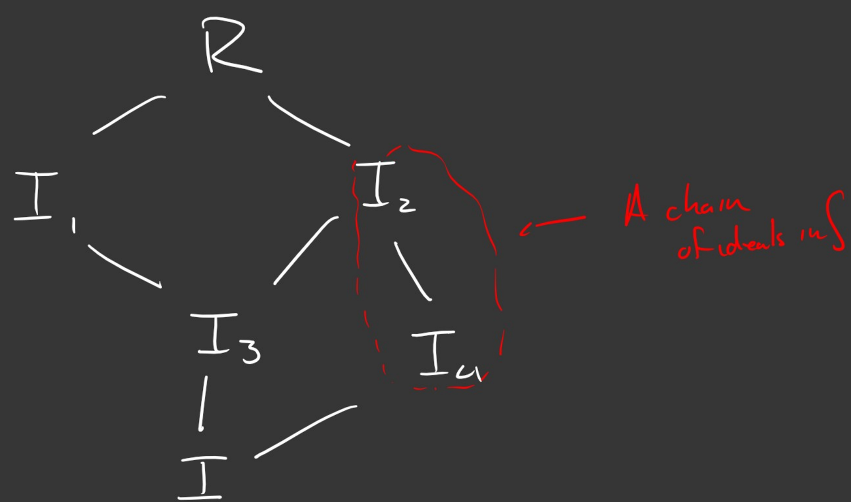
Zorn's Lemma: If A is a non-empty poset s.t. every chain admits an upper bound. Then A has a maximal element.

Prop: If R is a comm. ring w/ $1 \neq 0$. Then every proper ideal is contained in a maximal ideal.

Pf: Let $I \subsetneq R$ be a proper ideal.

Consider $S := \{\text{proper ideals of } R \text{ containing } I\}$

S is partially ordered by inclusion



A chain of ideals in S is a collection of ideals

$$C = \{ \cdots \subset I_{-1} \subset I_0 \subset I_1 \subset I_2 \subset \cdots \}$$

want to show C has an upper bound.

Let $J = \bigcup_{I_k \in \mathcal{C}} I_k$

Claim: J is an ideal containing I.

Pf: $I \subset J$ is clear

$0 \in J$ b/c $0 \in I_k$ for any k.

If $a, b \in J$, then $\exists\, I_{k_1}, I_{k_2}$ s.t. $a \in I_{k_1}$, $b \in I_{k_2}$

without loss of generality, $I_{k_1} \subset I_{k_2}$

$\implies a, b \in I_{k_2} \implies a - b \in I_{k_2} \subset J \implies a - b \in J$

If $r \in R$, then $r \cdot a \in I_{k_2} \subset J \implies r \cdot a \in J$ $\square$

$\implies J$ is an upper bound for $\mathcal{C}$

$\implies$ we can apply Zorn's Lemma.

i.e. S admits a maximal element.

i.e. a proper ideal $M \subset R$ s.t. $I \subset M$.

and if $M' \subset R$ is an ideal s.t. $M \subset M'$

Then $I \subset M' \implies$ either $M' \in S \implies M' = M$

or $M' \notin S \implies M' = R$ $\square$

Thm: If R is a comm. ring w/ $1 \neq 0$.

Then $M \subset R$ is maximal iff R/M a field.

Recall: $P \subset R$ is prime iff R/P an integral domain.

Cor: Maximal ideals are prime.

Pf: M maximal $\Longrightarrow$ R/M a field $\Longrightarrow$ R/M an int. dom.

$\Longrightarrow$ M is prime $\square$

Pf: of Thm.

Lattice isomorphism

$$\begin{array}{ccc} \{\text{Ideals of } R \text{ containing } M\} & \longleftrightarrow & \{\text{Ideals of } R/M\} \\ \| & & \| \\ \{M, R\} & \longleftrightarrow & \{0, R/M\} \end{array}$$

$\square$

Examples: ① $n\mathbb{Z} \subset \mathbb{Z}$ is maximal iff $\mathbb{Z}/n\mathbb{Z}$ is a field.

i.e. n is prime.

So in $\mathbb{Z}$ $\{\text{prime ideals}\} = \{\text{maximal ideals}\}$

② $(x) \subset \mathbb{Z}[x]$ is prime (check this)

However $(x) \subset (2, x)$, but $1 \notin (2, x) \Longrightarrow (2, x) \subsetneq \mathbb{Z}[x]$

③ $(x) \subset \mathbb{R}[x]$ is maximal.

$$\mathbb{R}[x]/(x) \cong \mathbb{R} \text{ — a field.}$$