Lecture 8

More on Maximal Ideals

Recall: $(x) \subset \mathbb{Z}[x]$ is prime, but $(x) \subsetneq (2, x)$, so it not maximal.
$(x) \in \mathbb{R}[x]$ is maximal because $\mathbb{R}[x]/(x) \cong \mathbb{R}$ is a field

Example 8.1 Let $a \in \mathbb{R}$. We defined the evaluation map before:

$$\begin{aligned} \mathrm{Ev}_a \colon \mathbb{R}[x] &\to \mathbb{R} \\ p(x) &\mapsto p(a) \end{aligned}$$

Observe that Ev_a is infact surjective. Then

$$\mathbb{R}[x]/\operatorname{Ker}(\mathrm{Ev}_a) \cong \mathbb{R} \implies \operatorname{Ker}(\mathrm{Ev}_a) \text{ is a maximal ideal}$$

Denote $M_a \coloneqq \operatorname{Ker}(\mathrm{Ev}_a)$

Claim: $M_a = (x - a) \quad (\text{e.g } M_0 = (x))$

Proof. If $p(x) \in (x - a)$ then we may write $p(x) = q(x) \bullet (x - a)$, $q(x) \in \mathbb{R}[x]$, then

$$\mathrm{Ev}_a(p(x)) = p(a) = q(a) \bullet (a - a) = 0 \implies p(x) \in M_a \implies (x - a) \subset M_a$$

Conversely suppose $p(x) \in M_a = \operatorname{Ker}(\mathrm{Ev}_a)$. Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then you can check with polynomial division that $x - a$ divides $p(x)$ with no remainder i.e

$$\frac{p(x)}{x - a} = q(x)$$

therefore,

$$p(x) = q(x) \bullet (x - a) \implies p(x) \in (x - a) \implies M_a \subset (x - a) \implies M_a(x - a)$$

■

Q: Is every maximal ideal of $\mathbb{R}[x]$ of the form M_a?
For example, in $\mathbb{Z}$, the $\{\text{maximal ideals}\} = \{\text{prime ideals}\}$ but we saw above that in $\mathbb{Z}[x]$ there exist prime ideals that are not maximal.
Two standard questions:
(1) What are the primes?
(2) What are the maximal ideals?

Claim: Consider $I = (x^2 + 1)$, then $I \subset \mathbb{R}[x]$ is a maximal ideal.

Proof. We have that

$$\mathbb{R}[x] = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n | a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots n\}$$

What does $\overline{x^n}$ look like in $\mathbb{R}[x]/(x^2+1)$?

$$x^2+1 \in (x^2+1) \implies \overline{x^2} = \overline{-1} \in \mathbb{R}[x]/I$$
$$x^3 = x \cdot x^2 \implies \overline{x^3} = \overline{x} \cdot \overline{(-1)} \in \mathbb{R}[x]/I$$
$$x^4 = x^2 \cdot x^2 \implies \overline{x^4} = \overline{(-1)} \cdot \overline{(-1)} \in \mathbb{R}[x]/I$$

Therefore

$$\mathbb{R}[x]/I = \{\overline{a_0 + a_1 x} | a_0, a_1 \in R\}$$

There is a ring isomorphism

$$\mathbb{R}[x]/I \to \mathbb{C}$$
$$\overline{1} \mapsto 1$$
$$\overline{x} \mapsto i$$

therefore since the quotient ring is isomorphic to the field $\mathbb{C}$, I is maximal. ■

<u>Claim:</u> (x^2+1) is **not** maximal in $\mathbb{C}[x]$

Proof. We know that $x+i, x-i \in \mathbb{C}[x]$ and

$$(x+i)(x-i) = x^2+1 \in (x^2+1)$$

But $x+i, x-i \notin (x^2+1)$ therefore (x^2+1) is not prime in $\mathbb{C}[x]$ and consequently is not maximal. ■

Observe if $a \in R \subset S$ Then

$$(a)_R = \{r \cdot a | r \in R\}$$
$$\cap$$
$$(a)_s = \{s \cdot a | s \in S\}$$

can have different properties as ideals, e.g

$$\underbrace{(x) \subset \mathbb{Z}[x]}_{\text{prime}} \longrightarrow \underbrace{(x) \subset \mathbb{R}[x]}_{\text{maximal}}$$
$$\underbrace{(x^2+1) \subset \mathbb{R}[x]}_{\text{maximal}} \longrightarrow \underbrace{(x^2+1) \subset \mathbb{C}[x]}_{\text{not prime, not maximal}}$$

The Ring of Fractions

Q:How do we build $\mathbb{Q}$ out of $\mathbb{Z}$?
We want to add in multiplicative inverses like $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$
Consider

$$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{(m, n) | m, n \in \mathbb{Z}, n \neq 0\}$$

(think of the elements of this set as the fractions $\frac{m}{n}$)
There are some repeats if we care about multiplication like

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$$

We should define an equivalence relation

$$\frac{a}{b} \sim \frac{c}{d} \iff ad = bc$$

e.g $\frac{4}{6} \sim \frac{6}{9}$ because $4 \bullet 9 = 36 = 6 \bullet 6$.

Definition 8.1

The **field of rational numbers** is

$$\mathbb{Q} := \{\frac{m}{n} |\, m, n \in \mathbb{Q},\, n \neq 0\} / \sim$$

and this is a field with operations given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$$

There is an injective ring homomorphism

$$\mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto \frac{n}{1}$$

Moreover,

<u>**Claim:**</u> If F is a field and there is an injective ring homomorphism

$$f : \mathbb{Z} \to F$$

Then it factors through $\mathbb{Q}$, i.e there is a ring homomorphism

$$\overline{f} : \mathbb{Q} \to F \text{ such that } f(n) = \overline{f}\left(\frac{n}{1}\right)$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathbb{Q} \arrow[dl, dashed, "\overline{f}"] \\
& F &
\end{tikzcd}$$

Suppose R is any commutative ring with $1 \neq 0$.

Q: Can we do something similar? i.e

$$R \times (R \setminus \{0\}) = \{(r,s) | r,s \in R, s \neq 0\}$$

(again we will write (r,s) as $\frac{r}{s}$). We want to define $r^{-1} = \frac{1}{r}$, $r \neq 0$
However, if r is a zero divisor, $r \bullet s = 0$ then we want to exclude

$$\frac{1}{r} \bullet \frac{1}{s} = \frac{1}{r \bullet s} = \frac{1}{0}$$

Definition 8.2

Let R be an integral domain with $1 \neq 0$. Consider

$$R \times (R \setminus \{0\}) = \{(r,s) | r,s \in R, s \neq 0\}$$

Define an equivalence relation (**exercise to show it is**) by

$$\frac{a}{r} \sim \frac{b}{s} \iff a \bullet d = b \bullet r$$

There is no ambiguity in the equality of products since R is integral there are no zero zero divisors, $s, r \neq 0$.

Definition 8.3

The **field of fractions** of R is

$$Q(R) \coloneqq R \times (R \setminus \{0\}) / \sim = \left\{ \left[\frac{a}{b}\right] \middle| a, b \in R,\, b \neq 0 \right\}$$

Theorem 8.1

$Q(R)$ is a field with operations

$$\frac{a}{r} + \frac{b}{s} = \frac{as + br}{rs}, \qquad \frac{a}{r} \bullet \frac{b}{s} = \frac{ab}{rs}$$

The map

$$i \colon R \to Q(R)$$
$$r \mapsto \frac{r}{1}$$

is an injective ring homomorphism (we say R is a subring of its field of fractions). Moreover, if F is any field such that $R \subset F$ is a subring (i.e there exists an injective ring homomorphism $f \colon R \to F$), then there is a ring homomorphism

$$\overline{f} \colon Q(R) \to F \text{ such that } f(x) = \overline{f} \bullet i(x)$$

$$\begin{tikzcd} R \arrow[rr, "i"] \arrow[dr, "f"'] & & Q(R) \arrow[dl, dashed, "\overline{f}"] \\ & F & \end{tikzcd}$$

Proof. Think about it...... ■

Example 8.2 $Q(\mathbb{Z}) = \mathbb{Q}$

Example 8.3 $R = \mathbb{R}[x]$ is an integral domain. The fractional field of R is the field of rational functions

$$Q(R) = \mathbb{R}(x) := \left\{ \frac{p(x)}{q(x)} \middle| \, p, q \in \mathbb{R}[x], \; q \neq 0 \right\}$$

Example 8.4 If R is any integral domain with field of fractions $Q(R) = F$. Consider the integral domain $R[x]$. Then $R \subset R[x]$, $R[x] \subset Q(R[x])$ implies

$$\begin{tikzcd} R \arrow[rr, "\text{inclusion}"] \arrow[dr] & & Q(R[x]) \\ & F \arrow[ur] & \end{tikzcd}$$

e.g $\mathbb{Z} \subset \mathbb{Z}[x]$, so $\mathbb{Q} \subset Q(\mathbb{Z}[x])$.
In fact

$$Q(Z[x]) = \mathbb{R}[x]$$

Furthermore, this is generally true, i.e

$$Q(R[x]) = F(x)$$