# L19: Rank-nullity and spaces

Recall: If $A = \{v_1, \dots, v_n\}$ is linearly independent in a finite dimensional vector space $V$ and $B = \{b_1, \dots, b_n\}$ is a basis.
Then after possibly reordering

$$C_i = \{v_1, \dots, v_i, b_{i+1}, \dots, b_n\}$$

is a basis for all $0 \leq i \leq k$ and in particular, $k \leq n$.

**Corollary 19.1**

If $A = \{a_1, \dots, a_n\}$ is a linearly independent set in a finite dimensional $F$-vector space $V$, then there is a basis $B \supset A$.

***Proof.*** Take any basis $D$ for $V$ and apply replacement to $A$ and $D$. ∎

**Theorem 19.2**

Let $V$ be an $F$-vector space, $W \subset V$ a subspace. Then, in particular, $V/W$ is an $F$-vector space and

$$\dim V/W + \dim W = \dim V$$

(if either side is infinite, then both are)

***Proof.*** Suppose $V$ is finite dimensional and $\dim V = n$ and $\dim W = m$.
Let $B = \{v_1, \dots, v_m\} \subset W$ be a basis for $W$. Then $B \subset V$ is linearly independent and by the building up lemma there exists

$$B' = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$$

which is a basis for $V$.
Consider the quotient map

$$\phi \colon V \to V/W$$

∎

**Definition 19.3**

If $\varphi \colon V \to W$ is an $F$-linear transformation, we sometimes refer to the kernel of $\varphi$ as the **null space** of $\varphi$.
The **nullity** of $\varphi$ is the $\dim \operatorname{Ker} \varphi$.
The **rank** of $\varphi$ is the $\dim \operatorname{Im} \varphi$.
If $\operatorname{Ker} \varphi = 0$, then we say $\varphi$ is **non-singular**, otherwise we say $\varphi$ is **singular**.
The **cokernel** of $\varphi$ is

$$\operatorname{Coker} \varphi := W/\operatorname{Im} \varphi$$

**Corollary 19.4**

If $\varphi\colon V \to W$ is an $F$ linear transformation, then:

(1) $\operatorname{Ker}\varphi \subset V$ and $\operatorname{Im}\varphi \subset W$ are subspaces.

(2) (Rank-nullity) $\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$.

***Proof.*** First isomorphism theorem implies $\operatorname{Im}\varphi \cong V/\operatorname{Ker}\varphi$ and hence

$$\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$$

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**Corollary 19.5**

If $\varphi\colon V \to W$ is an $F$-linear transformation and $\dim V = \dim W$, then the following are equivalent:

(1) $\varphi$ is an isomorphism

(2) $\operatorname{Ker}\phi = 0$ (i.e. $\varphi$ is injective)

(3) $\operatorname{Im}\varphi = W$ (i.e. $\varphi$ is surjective)

(4) If $B \subset V$ is a basis, then

$$\phi(B) \coloneqq \{\phi(v_1), \dots, \phi(v_n) \mid v_1, \dots, v_n \in B\}$$

is a basis for $W$.

## The dual of a vector space

**Definition 19.6**

Let $V$ be an $F$-vector space. The **dual space** is

$$V^* \coloneqq \operatorname{Hom}_F(V, F)$$

Elements of $V^*$ are called **linear functionals**

**Example 19.1.** Let $V$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$, then the integral operator is a linear functional on $V$

$$\int\colon V \to \mathbb{R}$$

$$f \mapsto \int_0^1 f \, \mathrm{d}\mathrm{d}x$$

**Lemma 19.7**

If $B = \{v_1, \dots, v_n\}$ is a basis for $V$, then any linear functional $f \in V^*$ is determined by its values on $B$.

***Proof.*** If $v \in V$, then

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \\ \implies & f(a_1 + \cdots + a_n v_n) = a_1 f(v_1) + \cdots + a_n f(v_n) \\ \implies & a_1 \alpha_1 + \cdots + a_n \alpha_n \end{aligned}$$

given $\alpha_1 = f(v_1), \ldots, \alpha_n = f(v_n)$. ■

**Definition 19.8**

Let $B = \{v_1, \ldots, v_n\}$ be a basis for $V$. Denote by $v_i^* \in V^*$ the linear functional

$$v_i^*(v_j) \coloneqq \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

**Theorem 19.9**

$B^* = \{v_1^*, \ldots, v_n^*\}$ is a basis for $V^*$. In particular, if $\dim V = n$, then $\dim V^* = n$.

***Proof.*** Let $f \in V^*$, $v \in V$ with $v = a_1 v_1 + \cdots + a_n v_n$.
Then

$$f(v) = f(a_1 v_1 + \cdots + a_n v_n) = a_1 f(v_1) + \cdots + a_n f(v_n)$$

On the other hand,

$$v_1^*(v) = v_1^*(a_1 v_1 + \cdots + a_n v_n) = a_1 \underbrace{v_1^*(v_1)}_{=1} + a_2 \cancelto{0}{v_1^*(v_2)} + \cdots + a_n \cancelto{0}{v_1^*(v_n)} = a_1$$

Through this same logic it shown

$$v_i^*(v) = a_i \quad i = \{1, \ldots, n\}$$

Returning to the first equation

$$\begin{aligned} f(v) &= a_1 f(v_1) + \cdots + a_n f(v_n) \\ &= v_1^*(v) f(v_1) + \cdots + v_n^*(v) f(v_n) \\ &= (f(v_1) v_1^* + \cdots + f(v_n) v_n^*)(v) \end{aligned}$$

Hence $f = \sum_{i=1}^n f(v_i) v_i^*$ and $B^*$ is spanning.
On the other hand, if $\alpha_1, \ldots, \alpha_n \in F$ such that

$$\alpha_1 v_1^* + \cdots + \alpha_n v_n^*$$

Then

$$(\alpha_1 v_1^* + \cdots + \alpha_n v_n^*)(v_i) = \alpha_i = 0 \, \forall i$$

Therefore, $B^*$ is also linearly independent and we conclude $B^*$ is a basis for $V^*$. ■

***Note:*** If $\varphi\colon V \to W$ is a linear transformation, then there is an induced map

$$\varphi^*\colon W^* \to V^*$$
$$(f\colon W \to F) \mapsto (f \circ \varphi\colon V \to W \to F)$$

**Theorem 19.10**

If $\varphi\colon V \to W$ is a linear transformation of finite dimensional vector spaces inducing $\varphi^*\colon W^* \to V^*$. Then,

$$\operatorname{Ker}\varphi^* \cong \operatorname{Coker}\varphi$$
$$\operatorname{Coker}\varphi^* \cong \operatorname{Ker}\varphi$$

as $F$-vector spaces.

***Proof.*** Let $B = \{v_1, \dots, v_n\}$ a basis for $\operatorname{Ker}\varphi$, $B' = \{v_1, \dots, v_n, v_{n+1}, \dots, v_m\}$ a basis for $V$ and $\varphi(B') = \{\varphi(v_{n+1}), \dots, \varphi(v_m)\}$ a basis for $\operatorname{Im}\varphi$.
Since $\operatorname{Im}\varphi \subset W$ is a subspace then

$$C = \{\varphi(v_{n+1}), \dots, \varphi(v_m), w_1, \dots, w_k\}$$

is a basis for $W$.
Dualizing, we get the dual basis

$$C^* = \{\varphi(v_{n+1})^*, \dots, \varphi(v_m)^*, w_1^*, \dots, w_k^*\}$$

a basis for $W^*$.
Let $v \in V$ and consider

$$\varphi^*\colon W^* \to V^*$$
$$\varphi^*[\varphi(v_{n+i})^*](v) = \varphi(v_{n+i})^*(\varphi(v))$$

Since we can write $v = \sum_{j=1}^m a_j v_j$ then

$$\varphi^*[\varphi(v_{n+i})^*](v) = \varphi(v_{n+i})^* \left( \sum_{j=n+1}^m a_j \varphi(v_j) \right) = a_{n+i}$$

and hence

$$\varphi^*(w_j^*)(v) = w_j^*(\varphi(v)) = w_j^* \left( \sum_{j=n+1}^m a_j \varphi(v_j) \right) = 0$$

implying

$$\operatorname{Ker}\varphi^* = \operatorname{Span}\{w_1^*, \dots, w_k^*\}$$
$$\operatorname{Im}\varphi^* = \operatorname{Span}\{v_{n+1}^*, \dots, v_m^*\}$$

Therefore

$$\operatorname{Coker}\varphi = W/\operatorname{Im}\varphi = \frac{\operatorname{Span}\{\varphi(v_{n+1}), \dots, \varphi(v_m), w_1, \dots, w_k\}}{\operatorname{Span}\{\varphi(v_{n+1}), \dots, \varphi(v_m)\}} = \operatorname{Span}\{\overline{w}_1, \overline{w}_2, \dots, \overline{w}_k\}$$

and $\operatorname{Ker}\varphi = \operatorname{Span}\{v_1, \ldots, v_n\}$ to give

$$\operatorname{Coker}\varphi^* = V^*/\operatorname{Im}\varphi^* = \frac{\operatorname{Span}\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}}{\operatorname{Span}\{v_{n+1}^*, \ldots, v_m^*\}} = \operatorname{Span}\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n\}$$

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