L7: Maximal Ideals

Let R be a commutative ring with $1 \neq 0$.

Proposition 7.1: Ideals containing units

Let $I \subset R$ an ideal
(i) $I = R$ if and only if I contains a unit.
(ii) R is a field if and only if the only ideals of R and 0 and R

Proof.
(i) If $I = R$, then $1 \in I$
Conversely, if $u \in I$ and $u \in R^\times$ say $u \cdot v = 1$, then $u \cdot v = 1 \in I$ implies, if $r \in R$, then

$$r \cdot (u \cdot v) = r \in I \implies R \subset I \implies R = I$$

(ii) If $I \subset R$ is an ideal in a field, and $\exists a \in I \setminus \{0\}$ (non-zero element of the field), then $a \in R^\times$ (since it is a field) implies $I = R$ (by part (i)).
Conversely, suppose 0 and R are the only ideals in R. Let $a \in R \setminus \{0\}$ and consider $(a) \subset R$, then

$$(a) \neq 0 \implies (a) = R \underbrace{\implies}_{\text{by part (i)}} \exists u \in (a),\, u \in R^\times (\text{say } u \cdot v = 1)$$

Since $u \in (a)$, we may write $u = r \cdot a, r \in R$, then

$$(r \cdot a) \cdot v = u \cdot v = 1 = a \cdot (r \cdot v) \implies a \in R^\times \implies R \text{ is a field}$$ ■

Corollary 7.2: Homomorphism from field to ring is injective

If F is a field, then any nonzero ring homomorphism

$$f : F \to R$$

is an injective map

Proof. $\operatorname{Ker} f = 0$ or F. Because f is nonzero, we conclude that $\operatorname{Ker} f = 0$, which means f is injective since the only element that maps to 0 is 0. ■

Definition 7.3: Maximal Ideal

An ideal $M \subset R$ is called a **maximal ideal** if
(i) $M \neq R$
(ii) If $I \subset R$ is an ideal such that $M \subset I$, then $I = M$ or $I = R$

Not all rings admit maximal ideals and a given ring may admit multiple maximal ideals, e.g $2\mathbb{Z}, 3\mathbb{Z}$ are maximal ideals in $\mathbb{Z}$.

A Digression on Zorn's Lemma

Definition 7.4: Partial Order

A **partial order** on a non-empty set A is a relation $\leq$ such that

(i) $x \leq x$ (Reflexive)

(ii) $x \leq y, y \leq x \implies x = y$ (Anti-symmetric)

(iii) $x \leq y, y \leq z \implies x \leq z$ (Transitive)

Example 7.1.
If X is any set then the power set (the set of all subsets) is written

$$\wp(X) = \{\text{subsets } U \subset X\}$$

Then inclusion is a partial order on $\wp(X)$, e.g

$$
\begin{array}{ccccc}
 & & \{a,b,c\} & & \\
 & \rotatebox{45}{$\subset$} & \cup & \rotatebox{135}{$\subset$} & \\
\{a,b\} & & \{a,c\} & & \{b,c\} \\
\cup & \rotatebox{45}{$\subset$}\,\rotatebox{135}{$\subset$} & & \rotatebox{45}{$\subset$}\,\rotatebox{135}{$\subset$} & \cup \\
\{a\} & & \{b\} & & \{c\} \\
 & \rotatebox{135}{$\subset$} & \cup & \rotatebox{45}{$\subset$} & \\
 & & \varnothing & &
\end{array}
$$

Definition 7.5: Poset, Chain, Upper Bound, Maximal Element

If $A, \leq$ is a **partially ordered set** (**poset**), then

(i) A subset $B \subset A$ is a **chain** if $\forall x, y \in B \implies x \leq y$ or $y \leq x$ (everything can be compared).

(ii) An **upper bound** on a subset $B \subset A$ is an element $u \in A$ such that

$$\forall b \in B, b \leq u$$

(iii) A **maximal element** of a subset $B \subset A$ is an element of $m \in B$ such that if $b \in B$ and $b \geq m$, then $b = m$.

Lemma 7.6: Zorn's Lemma

If A is a non-empty poset such that every chain admits an upper bound, then A has a maximal element.

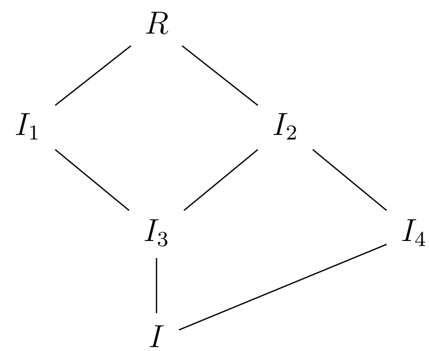
Proposition 7.7: All proper ideals contained in maximal ideal

If R is a commutative ring with $1 \neq 0$, then every proper ideal is contained in a maximal ideal

Proof.
Let $I \subsetneq R$ be a proper ideal.
Consider

$$\mathcal{S} := \{\text{proper ideals of } R \text{ containing } I\}$$

$\mathcal{S}$ is partially ordered by inclusion



A chain of ideals in $\mathcal{S}$ is a collection of ideals

$$\mathcal{C} = \{\ldots \subset I_{-1} \subset I_0 \subset I_1 \subset I_2 \subset \ldots\}$$

and to apply Zorn's Lemma, we need to show $\mathcal{C}$ has an upper bound.
Let

$$J = \bigcup_{I_k \in C} I_k$$

<u>**Claim:**</u> J is an ideal containing I.

> ***Proof.***
> $I \subset J$ is clear, since I is contained in all the ideals $I_k \in S$. It remains to show J itself is an ideal.
> $0 \in J$ because $0 \in I_k$ for any k.
> If $a, b \in J$, then $\exists I_{k_1}, I_{k_2}$ such that $a \in I_{k_1}, b \in I_{k_2}$, so w.l.o.g say $I_{k_1} \subset I_{k_2}$, then
>
> $$a, b \in I_{k_2} \implies a - b \in I_{k_2} \subset J \implies a - b \in J$$
>
> If $r \in R$, then $r \bullet a \in I_{k_2} \subset J \implies r \bullet a \in J$.
> Hence, J is an ideal containing I. ■

Therefore J is an upper bound for $\mathcal{C}$ and we can apply Zorn's lemma.
Therefore, $\mathcal{S}$ admits a maximal element, i.e a proper ideal $M \subset R$ such that $I \subset M$.
If $M' \subset R$ is an ideal such that $M \subset M'$, then $I \subset M'$ and so

$$\underbrace{M' \in \mathcal{S}}_{M' \text{ is proper}} \implies M' = M \quad \text{or} \quad \underbrace{M' \notin \mathcal{S}}_{M' \text{ is not proper}} \implies M' = R \qquad ■$$

Theorem 7.8: M maximal in comm. $R \iff R/M$ is field

If R is a commutative ring with $1 \neq 0$, then $M \subset R$ is maximal if and only if R/M is a field.

Proof.
Using the Lattice (fourth) Isomorphism Theorem we have

$$\{\text{Ideals of } R \text{ containing } M\} \longleftrightarrow \{\text{Ideals of } R/M\}$$
$$\{M, R\} \longleftrightarrow \{0, R/M\}$$

Since, the only ideals of R/M are 0 and itself, R/M is a field by Prop 7.1 (ii). ∎

Recall: $P \subset R$ is prime if and only if R/P is an integral domain.

Corollary 7.9: Maximal ideals are prime

Maximal ideals are prime.

Proof.
If M is maximal then R/M is a field. Therefore, R/M is an integral domain and hence M is prime. ∎

Example 7.2.
$n\mathbb{Z} \subset \mathbb{Z}$ is maximal if and only if $\mathbb{Z}/n\mathbb{Z}$ is a field, i.e n is prime.
So in $\mathbb{Z}$ we have

$$\{\text{prime ideals}\} = \{\text{maximal ideals}\}$$

Example 7.3.
The ideal generated by x, $(x) \subset \mathbb{Z}[x]$ is prime (check).
However, it is not maximal as $(x) \subset (2, x)$, but $1 \notin (2, x)$ and therefore $(2, x) \subsetneq \mathbb{Z}[x]$. So, in this case prime ideals are not necessarily maximal.

Example 7.4.
$(x) \subset \mathbb{R}[x]$ is maximal.

$$\mathbb{R}[x]/(x) \cong \mathbb{R}$$

and recall $\mathbb{R}$ is a field.