L5: Isomorphism Theorems

Theorem 5.1: The First Isomorphism Theorem

If $f : R \to S$ is a ring homomorphism and $I = \operatorname{Ker} f$. Then

$$R/I \cong \operatorname{Im} f$$

as rings.

$$\begin{tikzcd}
R \arrow[rr, "f"] \arrow[dr, "\phi"'] & & S \\
& R/\operatorname{Ker} f \arrow[ur, dashed] &
\end{tikzcd}$$

Theorem 5.2: The Second Isomorphism Theorem

Let $A \subset R$ be a subring and $B \subset I$ an ideal.
Then

$$A + B \coloneqq \{a + b \mid a \in A, b \in B\}$$

is a subring of R and $A \cap B$ is an ideal of A and

$$(A + B)/B \cong A/(A \cap B)$$

$$\begin{tikzcd}
& R \arrow[d, dash] & \\
& A + B \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] \arrow[rr, phantom, "\cong"] & & B \arrow[dl, dash] \\
& A \cap B &
\end{tikzcd}$$

Proof of 5.2.
Let $A \subset R$ be a subring and $B \subset I$ an ideal.
It is **Easy to check** that $A + B$ is a subring and $A \cap B$ is an ideal in A.
Now we want to find an isomorphism

$$(A + B)/B \longrightarrow A/(A \cap B)$$

Idea: Use the First Isomorphism Theorem, i.e we want to find a surjective homomorphism

$$f \colon A + B \to A/(A \cap B)$$

such that $\operatorname{Ker} f = B$.

Define a map

$$\begin{aligned}\phi\colon A+B &\to A/(A\cap B)\\ a+b &\mapsto a+A\cap B\end{aligned}$$

which can be shown to be homomorphism if it is well defined. Generally, if $x \in A+B$, there are many ways to express $x \in A+B$, i.e there may exist, $a, a' \in A$ and $b, b' \in B$ such that

$$x = a+b = a'+b'$$

So is $\phi(x) = a + A\cap B$ or $\phi(x) = a' + A\cap B$?
This is not a problem so long as $a+A\cap B = a'+A\cap B$. In other words, if $a-a' \in A\cap B$ BUT

$$a+b = a'+b' \implies \underbrace{a-a'}_{\in A} = b'-b \in B \implies a-a' \in A\cap B$$

We also need to check that ϕ is surjective.
Clearly, if $a+A\cap B \in A/(A\cap B)$, then say $a \in A$ and is a representative for $a+A\cap B$.
Then, $a+0 \in A+B$ and $\phi(a) = a+A\cap B$.
Finally, we must check that

$$\operatorname{Ker}\phi = B$$

If $a+b \in \operatorname{Ker}\phi$ then $\phi(a+b) = 0+A\cap B$ and so

$$a \in A\cap B \implies a \in B \implies \operatorname{Ker}\phi \subset B$$

On the other hand, if $b \in B \subset A+B$, then we can write it as $b = 0+b$ and so

$$\phi(b) = 0+A\cap B \implies b \in \operatorname{Ker}\phi \implies B \subset \operatorname{Ker}\phi$$

Therefore, $\operatorname{Ker}\phi = B$. ■

Theorem 5.3: The Third Isomorphism Theorem

Let $I, J \subset R$ be ideals $I \subset J$.
Then

$$J/I := \{a+I \in R/I \mid a \in J\}$$

(the cosets of R/I whose representatives are in J or similarly the restriction of the quotient map from R to R/I to the domain J) is an ideal in R/I and

$$(R/I)/(J/I) \cong R/J$$

$$\begin{tikzcd}
R \arrow[d, dash] & R/I \arrow[d, dash] \\
J \arrow[r, dashed, "\cong"] \arrow[d, dash] & J/I \arrow[d, dash] \\
I & 0
\end{tikzcd}$$

Proof of 5.3.
Let $I \subset J \subset R$ be ideals.
Then we want to show, $J/I \subset R/I$ is an ideal and

$$(R/I)/(J/I) \cong R/J$$

Check: J/I is an ideal in R/I.
Then define a map

$$\begin{aligned} \phi \colon R/I &\to R/J \\ a + I &\mapsto a + J \end{aligned}$$

Observe that if $a \in J$, then $\phi(a + I) = a + J = J = \overline{0}$
ϕ is also clearly surjective: Pick any representative $a \in R$ for $a + J$, then

$$\phi(a + I) = a + J$$

It remains to be shown that $\operatorname{Ker} \phi = J/I$ as follows:
If $a + I \in \operatorname{Ker} \phi$ then $\phi(a + I) = a + J = 0 + J = J$ which implies

$$a \in J \implies a + I \in J/I \implies \operatorname{Ker} \phi \subset J/I$$

If $a \in J$, then $\phi(a + I) = a + J = J$ which implies

$$a + I \in \operatorname{Ker} \phi \implies \operatorname{Ker} \phi \supset J/I$$

and therefore $\operatorname{Ker} \phi = J/I$. ■

Theorem 5.4: The Fourth Isomorphism Theorem

Let $I \subset R$ be an ideal.
Then the correspondence

$$I \subset A \subset R \longleftrightarrow A/I \subset R/I$$

is a bijection between

$$\{\text{subrings of } R \text{ containing } I\} \longleftrightarrow \{\text{subrings of } R/I\}$$

Moreover, $A \subset R$ is an ideal *iff* A/I is an ideal in R/I.

$$\begin{tikzcd}
& R \arrow[dl, dash] \arrow[dr, dash] & & & & R/I \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \arrow[rr, leftrightarrow] & & A/I \arrow[dr, dash] & & B/I \arrow[dl, dash] \\
& C \arrow[d, dash] & & & & C/I \arrow[d, dash] & \\
& I & & & & 0 &
\end{tikzcd}$$

Definition 5.5: Ideal Generation, Principal and Finitely Generated Ideal

Let R be a ring, with $1 \neq 0$ and let $A \subset R$ be any subset.
The **ideal generated by** A is

$$A \subset (A) \subset R$$

i.e, the smallest ideal of R containing A.
If an ideal I is generated by a single element set, then we say I is a **principal ideal**.
If I is generated by a finite set then we say I is a **finitely generated ideal**.

Note: Instead of writing $I = (\{a\})$ for a principal ideal, we often omit the set notation and just write

$$I = (a)$$

Similarly, we will write $I = (a_1, \ldots, a_n)$ for finitely generated ideals.

Proposition 5.6: Minimality of ideal generated by a set

For any subset $A \subset R$ and ideals $I \subset R$ such that $A \subset I$, we have

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

Proof.
Observe that $R \subset R$ and is always an ideal of itself which implies that there always exists an ideal containing A (at least R)

$$\{A \subset I \subset R\} \neq \emptyset$$

First check that $(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$
Suppose, for a contradiction, $A \subset I$ and $(A) \not\subset I$, then

(i) $(A) \cap I \subsetneq (A)$ (proper subset otherwise $(A) \subset I$)
(ii) $A \subset (A)$ and $A \subset I \implies A \subset (A) \cap I$
(iii) $(A) \cap I$ is an ideal (second isomorphism theorem).

Therefore there is an ideal containing A (i.e $(A) \cap I$) that is smaller than (A), which is contradictory the definition of (A). Hence

$$(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

Now check that $\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$
We have that

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I$$

is an ideal and therefore $A \subset \bigcap I$ which implies

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$$

because (A) is an ideal. Therefore,

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I \qquad \blacksquare$$