The matrix of a linear transformation

Recall: If $B = \{v_1, \_\_, v_n\}$ is a basis for a vector space V over $\mathbb{F}$

Then $\Phi_B : V \longrightarrow \mathbb{F}^n$

$$\sum_{i=1}^{n} \alpha_i v_i \longmapsto \sum \alpha_i e_i = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}_B = (\alpha_1, \alpha_2, \_\_, \alpha_n)_B$$

Consider a linear transformation

$$\varphi : V \longrightarrow W$$

Suppose we have bases

$B = \{v_1, \_\_, v_n\}$ for V

$D = \{w_1, \_\_, w_m\}$ for W

Then $\varphi(v_j) = \alpha_{1j} \cdot w_1 + \alpha_{2j} \cdot w_2 + \dots + \alpha_{mj} \cdot w_m$

where $\alpha_{ij} \in \mathbb{F}$ for $i = 1, \_\_, m$
$j = 1, \_\_, n.$

i.e. $$\Phi_D[\varphi(v_j)] = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}_D$$

Recall: If $v \in V$, say $v = \sum_{j=1}^{n} \alpha_j v_j$

Then $\varphi(v) = \sum \alpha_j \cdot \big(\underbrace{\varphi(v_j)}\big)$

we only need to know these to define the lin. trans.

Defn: If $\varphi : V \to W$,

$B = \{v_1, \ldots, v_n\}$ for V

$D = \{w_1, \ldots, w_m\}$ for W

Then the matrix representing φ in the bases B, D is

$$M_B^D(\varphi) := \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ddots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \ddots & \alpha_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

Note: we sometimes write $M_B^D(\varphi) = (\alpha_{ij})$

Obs: If $v \in V$, say $\Phi_B(v) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}_B$

Then
$$\varphi(v) = \sum_{j=1}^{n} \alpha_j \varphi(v_j)$$
$$= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} \alpha_{ij} w_i \right).$$
$$= \underbrace{\sum_{i=1}^{m} \left[\sum_{j=1}^{n} \alpha_{ij} \alpha_j \right]}\, w_i$$

i.e. $$\Phi_D(\varphi(v)) = \begin{pmatrix} \sum_{j=1}^{n} \alpha_{1j} \alpha_j \\ \sum_{j=1}^{n} \alpha_{2j} \alpha_j \\ \vdots \\ \sum_{j=1}^{n} \alpha_{mj} \alpha_j \end{pmatrix}_D$$

Thm: With notation as above

$$\Phi_D(\varphi(v)) = M_B^D(\varphi) \cdot \Phi_B(v)$$

matrix multiplication

Example: $V = \mathbb{R}^2$ w/ basis $\mathcal{E}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$W = \mathbb{R}^3$ w/ basis $\mathcal{E}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$B_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x - y \\ x + y \\ y \end{pmatrix}$$

Compute $M_{\mathcal{E}_2}^{\mathcal{E}_3}(\varphi)$:

$$\varphi\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \quad M_{\mathcal{E}_2}^{\mathcal{E}_3}(\varphi) = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Compute $M^{\mathcal{B}_3}_{\mathcal{E}_2}(\varphi)$

$$\varphi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\varphi\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix} = -2\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Longrightarrow M^{\mathcal{B}_3}_{\mathcal{E}_2}(\varphi) = \begin{pmatrix}1 & -2\\1 & 0\\0 & 1\end{pmatrix}$$

compare. $M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi) = \begin{pmatrix}2 & -1\\1 & 1\\0 & 1\end{pmatrix}$

<u>Check</u>: $M^{\mathcal{B}_3}_{\mathcal{E}_2}(\varphi)\cdot\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix}x-2y\\x\\y\end{pmatrix}_{\mathcal{B}_3}$

These represent the same vector in $\mathbb{R}^3$

$M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi)\cdot\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix}2x-y\\x+y\\y\end{pmatrix}_{\mathcal{E}_3}$

<u>Obs</u>: (1) In general, $M^{D}_{B}(\varphi)$ depends on B and D

(2) $M^{D}_{B}(\varphi) = \left(\phi(v_1)_D \mid \phi(v_2)_D \mid \cdots \mid \phi(v_n)_D \right)$

e.g. $\varphi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}_{\mathcal{B}_3} = \begin{pmatrix}2\\1\\0\end{pmatrix}_{\mathcal{E}_3}$

$\varphi\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix}_{\mathcal{B}_3} = \begin{pmatrix}-1\\1\\1\end{pmatrix}_{\mathcal{E}_3}$

<u>Thm:</u> Let V, W F-vector spaces

$\mathcal{B} = \{v_1, \ldots, v_n\}$ basis for V

$\mathcal{D} = \{w_1, \ldots, w_m\}$ basis for W

Then the map

$$M_{\mathcal{B}}^{\mathcal{D}} : \mathrm{Hom}_F(V, W) \longrightarrow M_{m,n}(F)$$
$$\varphi \longmapsto M_{\mathcal{B}}^{\mathcal{D}}(\varphi)$$

is a vector space isomorphism.

<u>Pf:</u> If $\varphi, \psi \in \mathrm{Hom}_F(V,W)$, $\alpha \in F$

$$(\alpha\varphi + \psi)(v_j) = \alpha \cdot \varphi(v_j) + \psi(v_j)$$

Consider $\Phi_{\mathcal{D}}(\varphi(v_j)) = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}_{\mathcal{D}}$, $\Phi_{\mathcal{D}}(\psi(v_j)) = \begin{pmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{mj} \end{pmatrix}_{\mathcal{D}}$

Then $M_{\mathcal{B}}^{\mathcal{D}}(\alpha\varphi + \psi) = \left(\begin{array}{c|c|c|c} \alpha\alpha_{11} + \beta_{11} & \alpha \cdot \alpha_{12} + \beta_{12} & & \alpha \cdot \alpha_{1n} + \beta_{1n} \\ \alpha\alpha_{21} + \beta_{21} & \alpha \cdot \alpha_{22} + \beta_{22} & \cdots & \alpha \cdot \alpha_{2n} + \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha \cdot \alpha_{m1} + \beta_{m1} & \alpha \cdot \alpha_{m2} + \beta_{m2} & & \alpha \cdot \alpha_{mn} + \beta_{mn} \end{array}\right)$

$$= \alpha \cdot M_{\mathcal{B}}^{\mathcal{D}}(\varphi) + M_{\mathcal{B}}^{\mathcal{D}}(\psi)$$

$\Rightarrow M_{\mathcal{B}}^{\mathcal{D}}$ is a homomorphism.

Surjective: Let $M = (\alpha_{ij}) \in M_{m \times n}(F)$

Define $\varphi_M : V \longrightarrow W$

$$\begin{aligned} v_1 &\longmapsto \alpha_{11} w_1 + \alpha_{21} w_2 + \dots + \alpha_{m1} w_m \\ &\vdots \\ v_n &\longmapsto \alpha_{1n} w_1 + \alpha_{2n} w_2 + \dots + \alpha_{mn} w_m \end{aligned}$$

extend linearly.

Clearly $M_B^D(\varphi_M) = M$.

Injectivity: Suppose $M_B^D(\varphi) = 0$

Then $\Phi_D(\varphi(v_1)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_D$

$\vdots$

$\Phi_D(\varphi(v_n)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_D$

$$\Longrightarrow \Phi_D(\varphi(v)) = \Phi_D\left[\sum_{j=1}^{n} \alpha_j \varphi(v_j)\right]$$

$$= \sum_{j=1}^{n} \alpha_j \cdot \Phi_D[\varphi(v_j)]$$

$$= \sum_{j=1}^{n} \alpha_j \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_D = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_D$$

$\Longrightarrow \Phi_D$ is an isomorphism $\Longrightarrow \varphi \equiv 0 \Rightarrow M_B^D$ injective

$\square$

Cor: $\dim_F \mathrm{Hom}(V,W) = m\cdot n.$

Pf: $\dim_F M_{m,n}(F) = m\cdot n$ $\square$

Defn: An $m\times n$ matrix $M \in M_{m,n}(F)$ is said to be non singular if

$$M\cdot v = 0 \implies v = 0 \in F^n$$

Thm: If $\varphi : V \to W$, $\psi : W \to U$ lin. trans.

$\mathcal{B} = \{v_1, \_\_, v_n\}$ a basis for V

$\mathcal{D} = \{w_1, \_\_, w_m\}$ a basis for W

$\mathcal{E} = \{u_1, \_\_, u_\ell\}$ a basis for U.

Then $\psi\circ\varphi : V \to U$

and $M_{\mathcal{B}}^{\mathcal{E}}(\psi\circ\varphi) = M_{\mathcal{D}}^{\mathcal{E}}(\psi)\cdot M_{\mathcal{B}}^{\mathcal{D}}(\varphi)$

Pf: Check: $$\psi\circ\varphi(v_j) = \psi\left[\sum_{i=1}^{m} \alpha_{ij} w_i\right]$$
$$= \sum_{i=1}^{m} \alpha_{ij}\left[\sum_{k=1}^{\ell} \beta_{ki} u_k\right]$$
$$= \sum_{k=1}^{\ell} \underbrace{\left[\sum_{i=1}^{m} \beta_{ki}\cdot\alpha_{ij}\right]} u_k$$

$\square$

Defn: An $n \times n$ matrix $M \in M_n(F)$ is <u>invertible</u>

if $\exists\, M^{-1} \in M_n(F)$

s.t. $M \cdot M^{-1} = M^{-1} \cdot M = Id_n$

The <u>row rank</u> (<u>column rank</u>) of a $m \times n$ matrix

$M \in M_{m,n}(F)$

is the number lin. ind. rows (or columns) of M

viewed as vectors in F^m (or F^n).

Thm: ① $M \in M_n(F)$ is nonsingular

iff M is invertible

② If B is a basis for an n-dim'l v. sp. V

Then $M_B^B : \mathrm{Hom}_F(V,V) \longrightarrow M_n(F)$

$$\mathrm{Hom}_F(V,V) = \mathrm{End}(V)$$

is a <u>ring</u> isomorphism

Pf: (1) If M is invertible

Suppose $\exists\, x \in F^n$ s.t. $Mx = 0$

Then $M^{-1} \cdot Mx = (Id_n) \cdot x = x$

$$\begin{array}{c} \shortparallel \\ M^{-1}(0) = 0 \end{array}$$

$\Longrightarrow M$ is non singular

If M is nonsingular, fix bases B, D for V

$\exists\, \phi \in End(V)$ s.t. $M_B^D(\phi) = M$

M is nonsingular $\Longrightarrow Ker\, \phi = 0$

$\Longrightarrow \phi$ is an isomorphism

$\Longrightarrow \exists\, \phi^{-1} : V \longrightarrow V$

and $M_B^D(\phi) \cdot M_D^B(\phi^{-1}) = M_D^D(\phi \cdot \phi^{-1}) = M_D^D(Id_V) = Id_n$

$\Longrightarrow M_D^B(\phi^{-1}) = \left[M_B^D(\phi)\right]^{-1} = M^{-1}$

(2) $\phi, \psi \in End(V)$

$$M_B^B(\phi \circ \psi) = M_B^B(\phi) \cdot M_B^B(\psi)$$

$\square$

Obs: $M_B^D(\phi \circ \psi) = M_\mathcal{E}^D(\phi) \cdot M_B^\mathcal{E}(\psi)$ we need $B = \mathcal{E}$, $D = \mathcal{E}$ for this to make sense.

<u>Defn</u>: Two $n \times n$ matrices $M, N \in M_n(F)$ are said to be <u>similar</u> if

$\exists$ non-singular (i.e. invertible) $P \in M_n(F)$

s.t. $N = P^{-1} M P$.

Two endomorphisms $\varphi, \psi \in End(V)$ are <u>similar</u> if there is an isomorphism $\xi \in End(V)$

s.t. $\psi = \xi^{-1} \circ \varphi \circ \xi$