More on maximal ideals

Recall: $(x) \subset \mathbb{Z}[x]$ prime, but $(x) \subsetneq (2, x)$ so not maximal

$(x) \subset \mathbb{R}[x]$ maximal because $\mathbb{R}[x]/(x) \cong \underbrace{\mathbb{R}}_{\text{a field.}}$

Example: Let $a \in \mathbb{R}$.

$$\begin{aligned} Ev_a : \mathbb{R}[x] &\longrightarrow \mathbb{R} \\ p(x) &\longmapsto p(a). \end{aligned}$$

Observe: Ev_a is, in fact, surjective.

$\Longrightarrow \mathbb{R}[x]/Ker(Ev_a) \cong \mathbb{R} \Longrightarrow Ker(Ev_a)$ is a maximal ideal.

Denote $M_a := Ker(Ev_a)$

Claim: $M_a = (x-a)$ (e.g. $M_0 = (x)$)

Pf: If $p(x) \in (x-a)$

then we may write

$$p(x) = q(x) \cdot (x-a) \quad , \ q(x) \in \mathbb{R}[x]$$

$$\begin{aligned} \Longrightarrow Ev_a(p(x)) &= p(a) \\ &= q(a) \cdot (a-a) = 0 \end{aligned}$$

$\Longrightarrow p(x) \in M_a \Longrightarrow (x-a) \subset M_a$

Conversely, suppose $p(x) \in M_a = Ker(Ev_a)$

Say $p(x) = a_0 + a_1 x + a_2 x^2 + \text{—} + a_n x^n$

$$
\begin{array}{r l}
 & a_n x^{n-1} + (a_{n-1} + a_n a) x^{n-2} + \cdots\cdots + \\
\hline
x - a \,\big) & a_n x^n + a_{n-1} x^{n-1} + \_\_\_ + a_1 x + a_0 \\
 & -(a_n x^n - a_n \cdot a\, x^{n-1}) \\
\hline
 & (a_{n-1} + a_n a) x^{n-1} + a_{n-2} x^{n-2} \\
 & -(\_\_\_\_ \quad - a(a_{n-1} + a_n a) x^{n-2}) \\
\hline
 & (a_{n-2} + a_{n-1} \cdot a + a_n \cdot a^2) x^{n-2} \\
 & \qquad \ddots
\end{array}
$$

$$a_0 + a_1 a + a_2 a^2 + \cdots a_n a^n = p(a) = 0$$

$$\Longrightarrow p(x) = q(x) \cdot (x-a) \Longrightarrow p(x) \in (x-a)$$

$$\Longrightarrow M_a \subset (x-a)$$

$$\Longrightarrow M_a = (x-a) \qquad \square$$

Q: Is every maximal ideal of $\mathbb{R}[x]$ of the form M_a?

For example, in $\mathbb{Z}$, the $\{\text{max ideals}\} = \{\text{prime ideals}\}$

we saw above that in $\mathbb{Z}[x]$, $\exists$ prime ideals that are not maximal.

Two standard questions:

① what are the primes?

② what are the maximal ideals.

Consider $I = (x^2+1)$

Claim: $I \subset \mathbb{R}[x]$ is a maximal ideal.

Pf: $\mathbb{R}[x] = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \mid a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots n \}$

what does $\overline{x^n}$ look like in $\mathbb{R}[x]/(x^2+1)$?

$$x^2+1 \in (x^2+1) \implies \overline{x^2} = \overline{-1} \in \mathbb{R}[x]/I.$$

$$x^3 = x \cdot x^2 \implies \overline{x^3} = \overline{x} \cdot \overline{(-1)} \in \mathbb{R}[x]/I$$

$$x^4 = x^2 \cdot x^2 \implies \overline{x^4} = \overline{(-1)} \cdot \overline{(-1)} \in \mathbb{R}[x]/I$$

$$\mathbb{R}[x]/I = \{ \overline{a_0 + a_1 x} \mid a_0, a_1 \in \mathbb{R} \}$$

with $\overline{x} \cdot \overline{x} = \overline{-1}$.

There is a ring isomorphism

$$\begin{aligned} \mathbb{R}[x]/I &\longrightarrow \mathbb{C} \\ \overline{1} &\longmapsto 1 \\ \overline{x} &\longmapsto i \end{aligned}$$

$\implies I$ is maximal

$\square$

Claim: (x^2+1) is not maximal in $\mathbb{C}[x]$

Pf: $x+i,\ x-i \in \mathbb{C}[x]$

$$(x+i)(x-i) = x^2+1 \in (x^2+1)$$

But $x+i,\ x-i \notin (x^2+1)$

$\Longrightarrow$ (x^2+1) is not prime in $\mathbb{C}[x]$

$\Longrightarrow$ (x^2+1) is not maximal $\square$

Obs: If $a \in R \subset S$

Then $(a)_R = \{ r \cdot a \mid r \in R \}$

$\cap$

$(a)_S = \{ s \cdot a \mid s \in S \}$

can have different properties as ideals.

e.g. $(x) \subset \mathbb{Z}[x] \rightsquigarrow (x) \subset \mathbb{R}[x]$

prime $\qquad$ maximal

$(x^2+1) \subset \mathbb{R}[x] \rightsquigarrow (x^2+1) \subset \mathbb{C}[x]$

maximal $\qquad$ not prime, not maximal.

The ring of fractions

Q: How do we build $\mathbb{Q}$ out of $\mathbb{Z}$?

We want to add in multiplicative inverses.

e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Consider $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{ (m,n) \mid m,n \in \mathbb{Z}, n \neq 0 \}$

(Think of these as fractions $\frac{m}{n}$)

Thee are some repeats if we care about multiplication

e.g. $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}, \dots$

We should define an equivalence relation.

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{iff} \quad ad = bc.$$

e.g. $\frac{4}{6} \sim \frac{6}{9}$ b/c $4 \cdot 9 = 36 = 6 \cdot 6$

Defn: The field of rational numbers is

$$\mathbb{Q} := \left\{ \tfrac{m}{n} \mid m,n \in \mathbb{Q}, n \neq 0 \right\} / \sim$$

and this is a field w/ operations given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

There is an injective ring homomorphism

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Q} \\ n &\longmapsto \frac{n}{1} \end{aligned}$$

Moreover:

<u>Claim</u>: If $\mathbb{F}$ is a field and there is an injective ring homomorphism

$$f : \mathbb{Z} \longrightarrow \mathbb{F}$$

Then it factors through $\mathbb{Q}$.

i.e. there is a ring homomorphism

$$\bar{f} : \mathbb{Q} \longrightarrow \mathbb{F}$$

s.t. $f(n) = \bar{f}\left(\frac{n}{1}\right)$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{Q} \arrow[d, "\bar{f}"] \\ & \mathbb{F} \end{tikzcd}$$

Suppose R is any comm. ring w/ $1 \neq 0$.

<u>Q</u>: Can we do something similar?

i.e. $R \times (R \setminus \{0\}) = \{ (r,s) \mid r,s \in R,\ s \neq 0 \}$

(again, we will write (r,s) as $\frac{r}{s}$)

we want to define $r^{-1} = \frac{1}{r}$, $r \neq 0$.

However, if r is a zero divisor, $r \cdot s = 0$

then we want $\frac{1}{r} \cdot \frac{1}{s} = \frac{1}{r \cdot s} = \frac{1}{0}$ ← we want to exclude this.

<u>Defn</u>: Let R be an integral domain w/ $1 \neq 0$

Consider
$$R \times (R \setminus \{0\}) = \{ (r,s) \mid r, s \in R, s \neq 0 \}$$

Define an equivalence relation by
$$\frac{a}{r} \sim \frac{b}{s} \quad \text{iff} \quad \underbrace{a \cdot s = b \cdot r}$$
no ambiguity b/c no zero divisors, $s, r \neq 0$

<u>Exercise</u>: This is an equivalence relation.

The <u>field of fractions</u> of R is
$$Q(R) := R \times (R \setminus \{0\}) / \sim$$
$$= \{ [\tfrac{a}{b}] \mid a, b \in R, b \neq 0 \}.$$

<u>Thm</u>: $Q(R)$ is a field with operations
$$\frac{a}{r} + \frac{b}{s} = \frac{as + br}{rs}$$
$$\frac{a}{r} \cdot \frac{b}{s} = \frac{ab}{rs}$$

The map
$$i : R \longrightarrow Q(R) \quad \text{is an injective ring homomorphism}$$
$$r \longmapsto \frac{r}{1}$$

(we say R is a subring of its field of fractions)

Moreover, if F is any field s.t. $R \subset F$ is a subring

(i.e. $\exists$ injective ring homomorphism $f : R \longrightarrow F$)

Then there is a ring homomorphism

$$\bar{f} : Q(R) \longrightarrow F$$

s.t. $f(x) = \bar{f} \circ i(x)$

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "f"'] & Q(R) \arrow[dl, dashed, "\bar{f}"] \\ F & \end{tikzcd}$$

Pf: Think about it $\square$

Example 0: $Q(\mathbb{Z}) = \mathbb{Q}$

Example 1: $R = \mathbb{R}[x]$ is an integral domain.

The fractional field of R is the field of rational functions

$$Q(R) = \mathbb{R}(x) := \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \in \mathbb{R}[x],\ q \neq 0 \right\}$$

Example 2: If R is any integral domain w/ field of fractions $Q(R) = F$.

Consider the integral domain $R[x]$

Then $R \subset R[x]$, $R[x] \subset Q(R[x])$

$$\Longrightarrow \begin{tikzcd} R \arrow[rr, "\text{inclusion}"] \arrow[dr] & & Q(R[x]) \\ & F \arrow[ur] & \end{tikzcd}$$

e.g. $\mathbb{Z} \subset \mathbb{Z}[x]$, so $\mathbb{Q} \subset Q(\mathbb{Z}[x])$

In fact. $Q(\mathbb{Z}[x]) = \mathbb{R}(x)$

This is generally true:

i.e. $Q(R[x]) = F(x)$