Rank-nullity, dual spaces

Recall: If $A = \{v_1, \ldots, v_k\}$ is lin. ind. in a fin-dim'l vector space V

and $B = \{b_1, \ldots, b_n\}$ is a basis

Then after possibly reordering

$$C_i = \{v_1, \ldots, v_i, b_{i+1}, \ldots, b_n\}$$

is a basis for all $0 \le i \le k$

In particular, $k \le n$.

Cor: (Building-up Lemma)

If $A = \{a_1, \ldots, a_k\}$ is a lin. ind. set in a f.d. F-vector space V

Then there is a basis $B \supset A$.

Pf: Take any basis D for V and apply replacement to A and D □

Thm: Let V be an F-vector space, $W \subset V$ a subspace

In particular, V/W is an F-vector space

Then $$\dim V/W + \dim W = \dim V$$

(If either side is infinite, then both are)

Pf: Suppose V is finite dimensional, $\dim V = n$

$$\begin{array}{c} V \\ \cup \\ W \end{array}, \quad \dim W = m.$$

Let $\mathcal{B} = \{v_1, \ldots, v_m\} \subset W$ a basis for W

$\mathcal{B} \subset V$ is a lin. ind.

Building-up $\Rightarrow \exists\ \mathcal{B}' = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ a basis for V.

Consider the quotient map

$$\phi : V \longrightarrow V/W$$

① $\phi(v_i) = 0 \qquad i \in \{1, \ldots, m\}$

② $\phi(v_i) \neq 0 \qquad i \in \{m+1, \ldots, n\}$

$\ulcorner$ If $\phi(v_i) = 0$, then $v_i \in W$

$\Rightarrow v_i = \sum_{j=1}^{m} a_j v_j$

$\Rightarrow \{v_1, v_2, \ldots, v_m, v_i\}$ is linearly dependent $\to \Leftarrow$ $\lrcorner$

$\Rightarrow \{\phi(v_{m+1}), \ldots, \phi(v_n)\} \subset V/W$ is lin. ind.

$\ulcorner$ Suppose $a_{m+1}, \ldots, a_n \in F$ s.t.

$$a_{m+1}\phi(v_{m+1}) + \cdots + a_n \phi(v_n) = 0$$

$$\phi(\underbrace{a_{m+1} v_{m+1} + \cdots + a_n v_n}_{\in W}) = 0$$

$\Rightarrow a_{m+1} v_{m+1} + \cdots + a_n v_n = a_1 v_1 + \cdots + a_m v_m$

$\mathcal{B}'$ lin. ind. $\Rightarrow a_{m+1} = \cdots = a_n = 0$ $\lrcorner$

$B' \subset B$

$$\{ \underbrace{v_1, \text{———}, v_m}_{\substack{\text{Span the kernel} \\ \text{of } \phi}}, \underbrace{v_{m+1}, \text{———}, v_n}_{\substack{\phi(v_{m+1}), \text{———}, \phi(v_n) \\ \text{lin. ind. in } V/W}} \}$$

ϕ is surjective $\Rightarrow \{ \underset{0}{\underset{\|}{\phi(v_1)}}, \text{———}, \underset{0}{\underset{\|}{\phi(v_m)}}, \phi(v_{m+1}), \text{———}, \phi(v_n) \}$ spanning for V/W

$\Rightarrow \{ \phi(v_{m+1}), \text{———}, \phi(v_n) \}$ is spanning.

$\Rightarrow \{ \phi(v_{m+1}), \text{———}, \phi(v_n) \}$ is a basis for V/W.

$\Rightarrow \dim V/W + \dim W = \dim V$ $\square$

<u>Defn</u>: If $\varphi : V \to W$ is an F-linear transformation.

we sometimes refer to the kernel of φ as the <u>null space</u> of φ

The <u>nullity</u> of φ is the $\dim(\operatorname{Ker} \varphi)$

The <u>rank</u> of φ is the $\dim(\operatorname{Im} \varphi)$

If $\operatorname{Ker} \varphi = 0$, then we say φ is <u>non-singular</u>

otherwise we say φ is <u>singular</u>

The <u>cokernel</u> of φ is

$$\operatorname{Coker} \varphi := W / \operatorname{Im} \varphi$$

Cor: If $\varphi: V \to W$ is an F-linear transformation

① $\operatorname{Ker}\varphi \subset V$, $\operatorname{Im}\varphi \subset W$ are subspaces

② (Rank-nullity) $\dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$

Pf: Isomorphism Thm $\Longrightarrow \operatorname{Im}\varphi \cong V/\operatorname{Ker}\varphi$

$\Longrightarrow \dim V = \dim \operatorname{Ker}\varphi + \dim \operatorname{Im}\varphi$ $\square$

Cor: If $\varphi: V \to W$ is an F-lin. trans.

$\dim V = \dim W$

Then the following are equivalent:

① φ is an isomorphism

② $\operatorname{Ker}\varphi = 0$ (i.e. φ is injective)

③ $\operatorname{Im}\varphi = W$ (i.e. φ is surjective)

④ If $B \subset V$ is a basis

Then $\phi(B) := \{\phi(v_1), \dots, \phi(v_n) \mid v_1, \dots, v_n \in B\}$

is a basis for W.

The dual of a vector space

Defn: Let V be an F-vector space

The dual space is

$$V^* := \mathrm{Hom}_F(V, F)$$

Elements of V^* are called linear functionals

e.g. $V := \{\text{continuous functions } f : [0,1] \to \mathbb{R}\}$

$$\int : V \longrightarrow \mathbb{R} \qquad \text{is a linear functional on } V.$$
$$f \longmapsto \int_0^1 f\,dx$$

Lemma: If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V

the any linear functional $f \in V^*$ is determined

by its values on $\mathcal{B}$.

Pf: If $v \in V$, then $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$$\Longrightarrow \quad f(a_1 v_1 + \cdots + a_n v_n) = a_1 \underbrace{f(v_1)} + a_2 \underbrace{f(v_2)} + \cdots + a_n \underbrace{f(v_n)}$$

$\Longrightarrow$ Given $\alpha_1 = f(v_1)$, $\alpha_2 = f(v_2)$, $\ldots$ $\alpha_n = f(v_n)$

For any vector $v = a_1 v_1 + \cdots + a_n v_n$

$$f(v) = a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n$$

$\square$

Defn: Let $B = \{v_1, \_\_\_, v_n\}$ be a basis for V

Denote by $v_i^* \in V^*$ the linear functional

$$v_i^*(v_j) := \begin{cases} 1 & i=j \\ 0 & j \neq i \end{cases}$$

Thm: $B^* = \{v_1^*, \_\_\_, v_n^*\}$ is a basis for V^*

In particular, if $\dim V = n$, then $\dim V^* = n$.

Pf: $f \in V^*$, $v \in V$ $\quad v = a_1 v_1 + \_\_\_ + a_n v_n$

So $f(v) = f(a_1 v_1 + \_\_\_ + a_n v_n)$

$= a_1 f(v_1) + \_\_\_ + a_n f(v_n)$

On the other hand

$v_1^*(v) = v_1^*(a_1 v_1 + \_\_\_ + a_n v_n)$

$= a_1 \underbrace{v_1^*(v_1)}_{=1} + a_2 \cdot v_1^*\cancel{(v_2)}^{0} + \_\_\_ + a_n v_1^*\cancel{(v_n)}^{0}$

$= a_1$

$v_2^*(v) = a_2$

$\vdots$

$v_n^*(v) = a_n$

$\Longrightarrow f(v) = a_1 \cdot f(v_1) + \_\_\_ + a_n \cdot f(v_n)$

$= v_1^*(v) \cdot f(v_1) + \_\_\_ + v_n^*(v) \cdot f(v_n)$

$= \left(f(v_1) \cdot v_1^* + f(v_2) \cdot v_2^* + \_\_\_ + f(v_n) \cdot v_n^*\right)(v)$

$\Longrightarrow f = \sum_{j=1}^{n} f(v_j) \cdot v_j^* \Longrightarrow B^*$ is spanning.

On the other hand, if $\alpha_1, \_\_\_, \alpha_n \in F$

s.t. $\alpha_1 v_1^* + \_\_\_\_ + \alpha_n v_n^* = 0$

Then $(\alpha_1 v_1^* + \_\_\_\_ + \alpha_n v_n^*)(v_i) = \alpha_i = 0 \quad \forall i$

$\Longrightarrow$ B^* is lin. ind.

$\Longrightarrow$ B^* is a basis for V^*

$\square$

<u>Note</u>: If $\varphi : V \to W$ is a linear trans.

Then there is an induced map

$$\varphi^* : W^* \longrightarrow V^*$$
$$(f : W \to F) \longmapsto (f \circ \varphi : V \longrightarrow W \longrightarrow F)$$

<u>Thm</u>: If $\varphi : V \to W$ is a linear transformation (of f.d. v. spaces).

inducing $\varphi^* : W^* \longrightarrow V^*$

Then $\operatorname{Ker} \varphi^* \cong \operatorname{Coker} \varphi$ as F-vector spaces

$\operatorname{Coker} \varphi^* \cong \operatorname{Ker} \varphi$

<u>Pf</u>: $B = \{v_1, \_\_\_, v_n\}$ a basis for $\operatorname{Ker} \varphi$

$B' = \{v_1, \_\_\_ v_n, v_{n+1}, \_\_\_, v_m\}$ a basis for V.

$\varphi(B') = \{\varphi(v_{n+1}), \_\_\_\_, \varphi(v_m)\}$ a basis for $\operatorname{Im} \varphi$

(by rank-nullity)

$\mathrm{Im}\,\varphi \subset W \implies$

$$C = \{\varphi(v_{n+1}), \dots, \varphi(v_m), w_1, \dots, w_k\} \text{ a basis for } W$$

Dualizing, we get the dual basis

$$C^* = \{\varphi(v_{n+1})^*, \dots, \varphi(v_m)^*, w_1^*, \dots, w_k^*\} \text{ a basis for } W^*$$

Let $v \in V$ and consider

$$\varphi^* : W^* \longrightarrow V^*$$

$$\varphi^*[\varphi(v_{n+i})^*](v) = \varphi(v_{n+i})^*(\varphi(v))$$

Since we can write $v = \sum_{j=1}^{m} a_j v_j$

$$\implies \varphi^*[\varphi(v_{n+i})^*](v) = \varphi(v_{n+i})^*\left(\sum_{j=n+1}^{m} a_j \varphi(v_j)\right)$$
$$= a_{n+i}$$

$$\varphi^*(w_j^*)(v) = w_j^*(\varphi(v)) = w_j^*\left(\sum_{j=n+1}^{m} a_j \varphi(v_j)\right) = 0$$

$$\implies \mathrm{Ker}\,\varphi^* = \mathrm{Span}\{w_1^*, \dots, w_k^*\}$$

$$\mathrm{Im}\,\varphi^* = \mathrm{Span}\{v_{n+1}^*, \dots, v_m^*\}$$

$$\mathrm{Coker}\,\varphi = W/\mathrm{Im}\,\varphi = \frac{\mathrm{Span}\{\varphi(v_{n+1}), \dots, \varphi(v_m), w_1, \dots, w_k\}}{\mathrm{Span}\{\varphi(v_{n+1}), \dots, \varphi(v_m)\}}$$

$$\implies \mathrm{Coker}\,\varphi = \mathrm{Span}\{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_k\}$$

$$\mathrm{Ker}\,\varphi = \mathrm{Span}\{v_1, \dots, v_n\}$$

$$\operatorname{Coker}\varphi^* = V^*/\operatorname{Im}\varphi^* = \frac{\operatorname{Span}\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}}{\operatorname{Span}\{v_{n+1}^*, \ldots, v_m^*\}}$$

$$= \operatorname{Span}\{\bar{v}_1, \ldots, \bar{v}_n\}$$

$\square$

The four subspaces