The Chinese Remainder Theorem

Definition 9.1: Direct Product

Let R, S be rings.
The **direct product** of R and S is the ring

$$R \times S \coloneqq \{(r,s) | r \in R, s \in S\}$$

with ring operations

$$(r_1, s_1) + (r_2, s_2) \coloneqq (r_1 + r_2, s_1 + s_2)$$
$$(r_1, s_1) \cdot (r_2, s_2) \coloneqq (r_1 \cdot r_2, s_1 \cdot s_2)$$

More generally, if $\{R_\alpha \mid \alpha \in A\}$ is any collection of rings, then the **direct product** of the collection is the ring

$$\bigtimes_{\alpha \in A} R_\alpha \coloneqq \{(r_\alpha)_{\alpha \in A} \mid r_\alpha \in R_\alpha\}$$

with ring operations

$$(r_\alpha)_{\alpha \in A} + (s_\alpha)_{\alpha \in A} \coloneqq (r_\alpha + s_\alpha)_{\alpha \in A}$$
$$(r_\alpha)_{\alpha \in A} \cdot (s_\alpha)_{\alpha \in A} \coloneqq (r_\alpha \cdot s_\alpha)_{\alpha \in A}$$

Given $a, b \in \mathbb{Z}$, we say they are **relatively prime** if the greatest common divisor is 1. Equivalently (Bezout's Identity), we say a, b are relatively prime if there exists $m, n \in \mathbb{Z}$ such that

$$am + bn = 1$$

Definition 9.2: Comaximal Ideals

In a commutative ring R with $1 \neq 0$, we say two ideals $A, B \subset R$ are **comaximal** (i.e relatively prime) if $A + B = R$. This implies there exists a sum $a + b$ such that $a + b = 1$.

Theorem 9.1: Product of pairwise comaximals is intersection

Let $A_1, \ldots, A_k \subset R$ be ideals in a commutative ring with $1 \neq 0$.
If they are pairwise comaximal then

$$A_1 \cdot A_2 \cdot \ldots \cdot A_k = A_1 \cap A_2 \cap \ldots \cap A_k$$

Proof.
We already know that

$$A_1 \cdot A_2 \cdot \ldots \cdot A_k \subset A_1 \cap A_2 \cap \ldots \cap A_k$$

It suffices to show

$$A_1 \cap A_2 \cap \ldots \cap A_k \subset A_1 \cdot A_2 \cdot \ldots \cdot A_k$$

Let's prove this for two ideals and then generalize. First, consider comaximal ideals A, B.

Let $x \in A \cap B$, then we want to show $x \in A \cdot B$
By comaximality,

$$\exists a \in A, b \in B,\ a + b = 1 \in A + B$$

In particular,

$$x = x \cdot 1 = x \cdot (a + b) = x \cdot a + x \cdot b$$

and so

$$x \in A \cap B \implies \left.\begin{matrix} x \in A \implies x \cdot b \in A \cdot B \\ x \in B \implies x \cdot a \in A \cdot B \end{matrix}\right\} \implies x \cdot a + x \cdot b \in A \cdot B$$

Hence $x \in A \cdot B \implies A \cap B \subset A \cdot B$, and we can conclude

$$A \cdot B = A \cap B$$

The general case follows if we can show

$$A = A_1,\ B = A_2 \cdot A_3 \cdot \ldots \cdot A_k$$

are comaximal; we can do this with induction.
By assumption of comaximality A_1, A_i are comaximal for all $i \in \{2, \ldots, k\}$ therefore

$$\begin{aligned} \exists x_2 \in A_1,\ y_2 \in A_2, &\quad \text{s.t.} \quad 1 = x_2 + y_2 \\ \exists x_3 \in A_1,\ y_3 \in A_3, &\quad \text{s.t.} \quad 1 = x_3 + y_3 \\ &\vdots \\ \exists x_k \in A_1,\ y_k \in A_k, &\quad \text{s.t.} \quad 1 = x_k + y_k \end{aligned}$$

and this implies

$$1 = (x_2 + y_2) \cdot (x_3 + y_3) \cdot \ldots \cdot (x_k + y_k) \in A_1 + (A_2 \cdot \ldots \cdot A_k)$$

since all x's are in A_1 and all y's are in the product of the other ideals, the expanded product will have some mix of x's and some mixes of the y's. Hence, we conclude $A_1, A_2 \cdot \ldots \cdot A_k$ are comaximal. ■

Theorem 9.2: Chinese Remainder Theorem

Let $A_1, \ldots A_k \subset R$ ideals in a commutative ring with $1 \neq 0$.
The map

$$\begin{aligned} \phi \colon R &\to (R/A_1) \times (R/A_2) \times (R/A_3) \times \ldots \times (R/A_k) \\ r &\mapsto (r + A_1, r + A_2, r + A_3, \ldots, r + A_k) \end{aligned}$$

is a ring homomorphism with $\operatorname{Ker} \phi = A_1 \cap A_2 \cap \ldots \cap A_k$.
Moreover, if they are pairwise comaximal, then ϕ is surjective.

Corollary 9.1: Isomorphisms of quotient rings by product of ideals

If $A_1, \ldots, A_k \subset R$ are pairwise comaximal ideals in a commutative ring with $1 \neq 0$, then there is an isomorphism of rings (by the First Isomorphism Theorem)

$R/(A_1 \bullet \ldots \bullet A_k) \cong R/(A_1 \cap A_2 \cap \ldots \cap A_k) \cong (R/A_1) \times (R/A_2) \times (R/A_3) \times \ldots \times (R/A_k)$

So you can think of your quotient ring over the one ideal or over the separate components of the ideal.

Corollary 9.2: $\mathbb{Z}/n\mathbb{Z}$ isomorphic to quotients by prime factors

Let n be a positive integer with factorization into unique primes

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

then

$$\mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z}) \times \ldots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})$$

Example 9.1. Here are factorizations of two integer modulo rings:

$$\mathbb{Z}/30\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$$
$$\mathbb{Z}/168\mathbb{Z} \cong (\mathbb{Z}/8\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z})$$

Proof of CRT.
We want to see

$$\phi \colon R \to (R/A_1) \times (R/A_2) \times (R/A_3) \times \ldots \times (R/A_k)$$
$$r \mapsto (r + A_1, r + A_2, r + A_3, \ldots, r + A_k)$$

(1) $\operatorname{Ker}\phi = A_2 \cap \ldots \cap A_k$
(2) If $A_1, \ldots, A_k$ are pairwise comaximal then ϕ is surjective.
We will prove these for $k = 2$ and then generalize:
(1) Let $A, B \subset R$ be ideals and

$$\phi \colon R \to (R/A) \times (R/B)$$
$$r \mapsto (r + A, r + B)$$

Let $r \in \operatorname{Ker}\phi$, then

$$\left.\begin{aligned} r + A = 0 + A &\implies r \in A \\ r + B = 0 + B &\implies r \in B \end{aligned}\right\} \implies r \in A \cap B$$

If $r \in A \cap B$ then

$$\left.\begin{aligned} r \in A &\implies r + A = 0 + A \\ r \in B &\implies r + B = 0 + B \end{aligned}\right\} \implies r \in \operatorname{Ker}\phi$$

(2) If A, B are comaximal then there exists $x \in A, y \in B$ such that $1 = x + y$, then

$$1 - x = y \in B \implies 1 + A = y + A$$
$$1 - y = x \in A \implies 1 + B = x + B$$

and hence

$$\phi(x) = (x + A, x + B) = (0 + A, 1 + B)$$
$$\phi(y) = (y + A, y + B) = (1 + A, 0 + B)$$

So if we have any element $(r + A, s + B) \in R/A \times R/B$ then

$$\begin{aligned}(r + A, s + B) =& (r + A, 0 + B) + (0 + A, s + B)\\ =& (r + A, r + B) \cdot (1 + A, 0 + B) + (s + A, s + B) \cdot (0 + A, 1 + B)\\ =& \phi(r) \cdot \phi(y) + \phi(s) \cdot \phi(x)\\ =& \phi(ry + sx) \implies \phi \text{ surjective}\end{aligned}$$

More generally if $A_1, \ldots, A_k \subset R$ are ideals.
Let $A = A_1$, $B = A_2 \cdot A_3 \ldots \cdot A_k$, then we have a homomorphism

$$\phi_1 \colon R \to R/A \times R/B, \quad \operatorname{Ker} \phi_1 = A_1 \cap B$$

Now by the Lattice Isomorphism Theorem $A_2/B, A_3/B, \ldots, A_k/B \subset R/B$ are ideals. Take

$$A' = A_2/B, \quad B' = (A_3/B) \cdot (A_4/B) \cdot \ldots \cdot (A_k/B) = (A_3 \cdot A_4 \cdot \ldots A_k)/B$$

Then we get a homomorphism

$$\phi_2 \colon R/B \to (R/B)/A' \times (R/B)/B', \quad \operatorname{Ker} \phi_2 = A' \cap B'$$

By the third isomorphism theorem

$$(R/B)/A' = (R/B)/(A_2/B) \cong R/A_2$$

and similarly,

$$(R/B)/B' = (R/B)/(A_3 \cdot A_4 \cdot \ldots \cdot A_k)/B \cong R/(A_3 \cdot A_4 \cdot \ldots \cdot A_k)$$

Therfore, we have

$$\hat{\phi}_2 = (\mathrm{Id}, \phi_2) \circ \phi_1 \colon R \to R/A_1 \times R/A_2 \times R/(A_3 \cdot \ldots \cdot A_k)$$

Proceeding inductively on k, we end up with

$$\phi \colon R \to R/A_1 \times R/A_2 \times \ldots \times R/A_k$$

and the surjectivity when $A_1, \ldots, A_k$ are pairwise comaximal follow essentially because $A_1, A_2, \ldots A_k$ are comaximal. $\blacksquare$