L12: PIDs are UFDs and Polynomial Rings

Definition 12.1: Ascending Chains, Noetherian Ring

Let R be a commutative ring with $1 \neq 0$.
An **ascending chain** of ideals in R is a sequence

$$I_1 \subset I_2 \subset I_3 \subset \dots. \subset R$$

We say an ascending chain **stabilizes** if there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $I_n = I_m$.
We say R satisfies the **ascending chain condition** (a.c.c.) if every ascending chain stabilizes.
If R satisfies the a.c.c., we say it is a **Noetherian ring**.

Theorem 12.2: PID is Noetherian

If R is a PID, then R is Noetherian.

Proof. Let

$$I_1 \subset I_2 \subset I_3 \subset \dots. \subset R$$

be an ascending chain in a PID.
Consider

$$I := \bigcup_{n \in \mathbb{N}} I_n$$

which is an ideal. Then since R is a PID, $I = (a)$ for some $a \in R$.
In particular,

$$a \in I = \bigcup_{n \in \mathbb{N}} I_n \implies a \in I_N$$

for some $N \in \mathbb{N}$.
But if $a \in I_N$ then we also know $(a) \subset I_N$ implying $I \subset I_N$.
But by the definition of I, we also have the containment in the other direction, i.e $I_N \subset I$, and hence we have $I = I_N$.
Furthermore the chain stops "growing" at a finite ideal I_N and so the ascending chain stabilizes

$$I = I_N = I_{N+1} = I_{N+2} = \dots$$

Therefore, R is a Noetherian ring. ■

Theorem 12.3: PID is UFD

Every PID is a UFD.

Let R be a PID.
We want to show if $r \in R \setminus \{0\}$, $r \notin R^{\times}$, then r admits a **unique** expression as a product of irreducibles.

Lemma 12.4: Existence of product of irreducibles in PID

A element r in a PID has **some** expression as a product of irreducibles

Proof.
If r is irreducible, then $r = r$ and we are done.
If not, then we can write $r = r_1 \cdot r_2$, where $r_1, r_2 \notin R^\times$. Then $r \in (r_1)$ but $(r) \neq (r_1)$ because in order for that to be the case, r_2 would have to a be a unit. Therefore, it is a proper subset i.e, $(r) \subsetneq (r_1)$.
If r_1, r_2 are irreducibles, then we are done.
If not,

$$r_1 = r_{11} \cdot r_{12}$$
$$r_2 = r_{21} \cdot r_{22}$$

where $r_{ij} \notin R^\times$, $i, j \in \{1, 2\}$. Again, $r_1 \in (r_{11})$ but (since r_{12} is not a unit) $(r_1) \neq (r_{11})$, and hence $(r) \subsetneq (r_1) \subsetneq (r_{11})$.
Since R is a PID, it is also Noetherian, and so this chain stabilizes eventually. This means we will reach a point where and $(r_{1111}) = (r_{11111})$ implying $r_{1111} = r_{11111} \cdot u$ for some unit u, and thus r_{1111} is irreducible. Hence in general r will be factored into something like

$$r = (r_{111\ldots 1} \cdot r_{111\ldots 2}) \cdot \ldots \cdot (r_{222\ldots 1} \cdot r_{222\ldots 2})$$

where each term on the right side of the inequality is irreducible. ■

Lemma 12.5: Uniqueness of product of irreducibles in PID

The factorization into irreducibles is **unique** (up to reordering and associates).

Proof.
Say the factorization into irreducibles is $r = p_1 \cdot p_2 \cdot \ldots \cdot p_n$. We proceed by induction on n.
Base Case: If $n = 1$, then $r = \underbrace{p_1}_{\text{irred.}}$ implies r is irreducible.
Suppose now r factors into a different product of irreducibles,

$$r = q_1 \cdot q_2 \cdot \ldots \cdot q_n,\ n \geq 2,\ q_i \text{ irreducible } \forall i \in \{1, \ldots, n\}$$

But then $q_1, (q_2 \cdot \ldots \cdot q_n) \notin R^\times$ (since by definition irreducibles are non-units) implying r is not irreducible, which is a contradiction.
Therefore, $r = p_1$ is the unique way to write r as the product of irreducibles when $n = 1$.
Induction Hypothesis: Now suppose if r admits a factorization into at most $n - 1$ irreducibles, then the factorization is unique.

Inductive Step: If we can write into two different factorizations

$$\begin{aligned} r &= p_1 \bullet p_2 \bullet \ldots \bullet p_n, \quad p_i\text{'s irreducible} \\ &= q_1 \bullet q_2 \bullet \ldots \bullet q_m, \quad q_j\text{'s irreducible and } m \geq n \end{aligned}$$

Then $p_1 \mid q_1 \bullet (q_2 \bullet \ldots \bullet q_m)$ and recall irreducibles are prime in a PID. Since p_1 is irreducible, it is prime and so either $p_1 \mid q_1$ or $p_1 \mid (q_2 \bullet \ldots \bullet q_n)$. W.l.o.g. assume $p_1 \mid q_1$ i.e $q_1 = u \bullet p_1$, $u \in R$.

Since q_1 is irreducible, then $u \in R^\times$ or $p_1 \in R^\times$. But p_1 is irreducible, so it can be not an element of $R^\times$, therefore $u \in R^\times$ and so p_1 and q_1 associate.

So we write

$$\begin{aligned} r = p_1 \bullet p_2 \bullet \ldots \bullet p_n &= q_1 \bullet q_2 \bullet \ldots \bullet q_m \\ &= (u \bullet p_1) \bullet q_2 \bullet \ldots \bullet q_m \end{aligned}$$

Since R is an integral domain, we can cancel p_1 from both sides to get

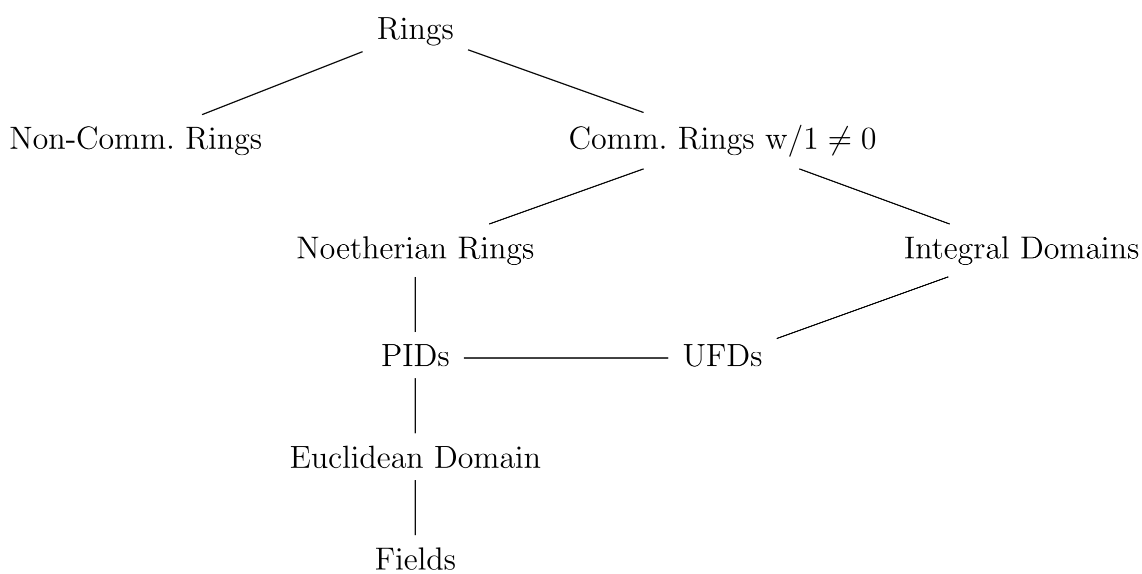
$$\underbrace{p_2 \bullet \ldots \bullet p_n}_{\text{product of (n-1) irred.}} = (u \bullet q_2) \bullet q_3 \bullet \ldots \bullet q_m$$

Now by our induction hypothesis, r admits a unique factorization into at most $(n-1)$ irreducibles, which implies the list of irreducibles on the left and right side of the equality are the same (up to associates) i.e.

$$\{(u \bullet q_2), q_3, q_4, \ldots, q_m\} = \{p_2, p_3, \ldots, p_n\}$$

and so $m = n$ and the p_i's are unique. ■

We can now see a hierarchy for the specific structures we have discussed thus far



Polynomial Rings (Again)

Let R be an commutative ring with $1 \neq 0$.
Now assume R is an integral domain and recall some facts we've already proven about :
(1) $R[X]$ is an integral domain.
(2) $R[X]^\times = R^\times$ e.g. $\mathbb{Z}[X]$, the only units are $\{\pm 1\}$.
(3) $\deg[p(X) \bullet q(X)] = \deg p(X) + \deg q(X)$
(4) The field of fractions of $R[X]$ is the field of rational functions

$$R(X) \coloneqq \left\{ \frac{p(X)}{q(X)} \middle| \, p, q \in R[X],\, q \neq 0 \right\}$$

(5) If F is a field, then $F[X]$ is a Euclidean Domain.

Corollary 12.6: $F[X]$ is PID, UFD, and Noetherian

If F is a field, $F[X]$ is a PID, UFD, and Noetherian.

(6) Let $I \subset R$ be an ideal and R a commutative ring (not necessarily integral) and consider the ideal generated by I in $R[X]$, i.e.

$$(I) \coloneqq I[X] \coloneqq \{p(X) \in R[X] \mid \text{coeffs. are in } I\}$$

Then

$$R[X]/(I) \cong (R/I)[X]$$

Proof.
Consider the map

$$\phi \colon R[X] \to (R/I)[X]$$
$$a_0 + a_1 X + \cdots + a_n X^n \mapsto \overline{a_0} + \overline{a_1} X + \overline{a_2} X^2 + \cdots + \overline{a_n} X^n$$

for example

$$\phi \colon \mathbb{Z}[X] \to (\mathbb{Z}/3\mathbb{Z})[X]$$
$$1 + 2X + 4X^3 \mapsto \overline{1} + \overline{2}X + \overline{4}X^3 = \overline{1} + \overline{2}X + X^3$$

"Clearly" ϕ is a surjective ring homomorphism, so

$$(R/I)[X] \cong R[X]/\operatorname{Ker}\phi$$

But the kernel is exactly the set of polynomials with coefficients that are zero (i.e. in the ideal), hence $\operatorname{Ker}\phi \coloneqq \{a_0 + a_1 X + \cdots + a_n X^n \mid a_i \in I\} = (I)$. ■

Example 12.1. Consider $3\mathbb{Z} \coloneqq \{0, 3, -3, 6, -6, \ldots\}$ and

$$(3\mathbb{Z}) \coloneqq \{a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n \mid a_i \in 3\mathbb{Z}\} \implies \mathbb{Z}[X]/(3\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})[X]$$

e.g $1 + 2X + 4X^3 = 1 + 2X + X^3 + \underbrace{3X^3}_{\in (3\mathbb{Z})}$; so we can think about the coefficients in either ring

$$\underbrace{1, 2, 4}_{\in \mathbb{Z}} \to \underbrace{\overline{1}, \overline{2}, \overline{1}}_{\in \mathbb{Z}/3\mathbb{Z}}$$

Corollary 12.7

If $I \subset R$ is prime, then $(I) \subset R[X]$ is prime.

Theorem 12.8: $F[X]$ satisfies unique euclidean condition

If $a(X), b(X) \in F[X]$ where F is a field. Then there exist **unique** $q(X), r(X) \in F[X]$ such that $\deg(r(X)) < \deg(b(X))$ (or $r(X) = 0$) for which

$$a(X) = q(X) \cdot b(X) + r(X)$$

Note: The point of the above theorem being that elements in $F[X]$ have unique quotients and remainders, which doesn't always happen in a general Euclidean domain as $\mathbb{Z}$ is a Euclidean Domain with $N(n) = |n|$, e.g

$$7 = 3 \cdot 2 + 1 \quad N(1) = 1 < N(2)$$
$$7 = 4 \cdot 2 - 1 \quad N(-1) = 1 < N(2)$$

Proof.
Suppose $a(X) = q(X) \cdot b(X) + r(X) = q'(X) \cdot b(X) + r'(X)$, then

$$r(X) = a(X) - q(X) \cdot b(X)$$
$$r'(X) = a(X) - q'(X) \cdot b(X)$$

and $\deg(r), \deg(r') < \deg(b)$ (or they're both 0 but then obviously they are unique). Consider

$$r(X) - r'(X) = q'(x) \cdot b(X) - q(X) \cdot b(X) = [q'(X) - q(X)] \cdot b(X)$$

Assume $q' - q \neq 0$ and $b \neq 0$, then since Euclidean domains are integral we have

$$\deg[(q' - q) \cdot b] = \deg(q' - q) + \deg(b)$$

but also $(q' - q) \cdot b = r - r'$ for which we know

$$\deg[r - r'] < \deg b$$

and hence a contradiction arises and it must be that $q' - q = 0$ and so

$$q' - q = 0 \implies q' = q \implies r = r' \qquad \blacksquare$$

The idea here behind this theorem and proof being that in regular Euclidean domains adding or subtracting alters the value of a norm while in polynomial rings, which has norm as the degree of the polynomial, it doesn't. This can be seen by considering that in the integers, if you start with 8 and subtract off 1, the norm is now 7, but in the polynomial ring if you start with a polynomial of degree 8 and subtract off a polynomial of strictly less degree (possibly even the same degree) then the norm (degree) does not change.

Corollary 12.9

Suppose F, K are fields with $F \subset K$ and $a(X), b(X) \in F[X]$.
Then the quotient and remainder polynomials of a by b are independent of field.

Proof. There exist $q(X), r(X) \in F[X]$ and $Q(X), R(X) \in K[X]$ with $\deg r < \deg b$ and $\deg R < \deg b$, such that

$$a(X) = q(X) \cdot b(X) + r(X) \quad a(X) = Q(X) \cdot b(X) + R(X)$$

But by the previous theorem, there is uniqueness since $q, r \in K[X]$ it must mean that

$$q(X) = Q(X) \quad r(X) = R(X)$$

■

Corollary 12.10

For fields F, K with $F \subset K$, $b(X) \mid a(X)$ in $K[X]$ *iff* $b(X) \mid a(X)$ in $F[X]$

Example 12.2.

$$(X-1) \mid X^2 - 1 \text{ in } \mathbb{R}[X] \text{ and so also in } \mathbb{C}[X]$$

However note the case where $b(X) \mid a(X)$ in $K[X]$ but not in $F[X]$ e.g.

$$(X-i) \mid X^2 + 1 \text{ in } \mathbb{C}[X] \text{ but not } \mathbb{R}[X]$$

Since $X^2 + 1$ has no non-trivial factors in $\mathbb{R}[X]$.

Definition 12.11: Multivariable Polynoimal Ring

Let R be a commutative ring with $1 \neq 0$.
The **polynomial ring in the variables** $X_1, \ldots, X_n$ **with coefficients in R** is defined inductively as

$$R[X_1, X_2, \ldots, X_n] \coloneqq R[X_1, X_2, \ldots, X_{n-1}][X_n]$$

Concretely, think of $R[X_1, \ldots, X_n]$ as finite sums of **monomials**, i.e

$$aX_1^{d_1} X_2^{d_1} \ldots X_n^{d_n}, \quad d_i \in \mathbb{Z},\ d_i \geq 0$$

e.g

$$1 + 2XY + Y^2, 2X - 7X^3 y + 2XY^4 + 1 \in \mathbb{Z}[X, Y]$$

Definition 12.12: Multi-Degree

The **degree** of a monomial

$$aX_1^{d_1} X_2^{d_1} \ldots X_n^{d_n}$$

is $d = d_1 + d_2 + \cdots + d_n$.
The **multi-degree** is $(d_1, d_2, d_3, \ldots, d_n)$.
The **degree** of a polynomial is the highest degree of any monomial in it.

Proposition 12.13

Let R be an integral domain and

$$p(X_1, \ldots, X_n), q(X_1, \ldots, X_n) \in R[X_1, X_2, \ldots, X_n] \setminus \{0\}$$

then

(1) $R[X_1, X_2, \ldots, X_n]$ is an integral domain.
(2) $R[X_1, X_2, \ldots, X_n]^\times = R^\times$
(3) $\deg[p \bullet q] = \deg p + \deg q$