The other isomorphism theorems

Thm: The 1^{st} Isomorphism Thm.

$$f: R \to S \implies R/I \cong \operatorname{Im} f$$
$$\begin{array}{c} R \\ \cup \\ I = \operatorname{Ker} f \end{array}$$

Thm: The 2^{nd} Isomorphism Thm.

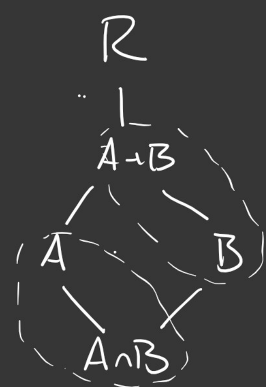
Let $A \subset R$ be a subring

$B \subset I$ be an ideal

Then $A+B := \{a+b \mid a \in A,\ b \in B\}$ is a subring of R

$A \cap B$ is an ideal in A.

and $(A+B)/B \cong A/(A\cap B)$

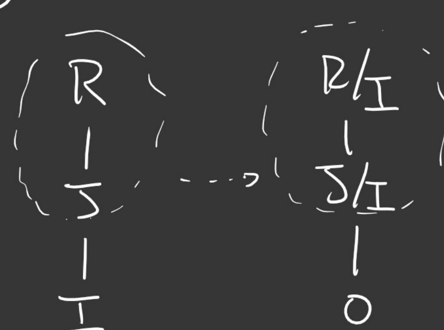


Thm: The 3^{rd} Isomorphism Thm.

Let $I, J \subset R$ be ideals $I \subset J$

Then $J/I := \{a + I \in R/I \mid a \in J\}$ is an ideal in R/I

and $(R/I)/(J/I) \cong R/J$



Thm: The 4^{th} Isomorphism Thm.

Let $I \subset R$ be an ideal

Then the correspondence

$$I \subset A \subset R \longleftrightarrow A/I \subset R/I$$

is a bijection between

$$\{\text{subrings of } R \text{ containing } I\} \longleftrightarrow \{\text{subrings of } R/I\}$$

Moreover, $A \subset R$ is an ideal iff A/I is an ideal in R/I.

$$\begin{tikzcd}
 & R \arrow[dl, dash] \arrow[dr, dash] & & & & R/I \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \arrow[rr, leftrightarrow] & & A/I \arrow[dr, dash] & & B/I \arrow[dl, dash] \\
 & C \arrow[d, dash] & & & & C/I \arrow[d, dash] & \\
 & I & & & & 0 &
\end{tikzcd}$$

Pf: (of 2^{nd})

$A \subset R$ subring, $B \subset R$ ideal.

Easy check: $A+B$ is a subring

$A \cap B$ is an ideal in A

want an isomorphism $A+B/_B \longrightarrow A/_{A\cap B}$

Idea: Use 1^{st} Isomorphism Thm.

i.e. we want to find a surjective homomorphism

$$f: A+B \longrightarrow A/A\cap B$$

s.t. $\operatorname{Ker} f = B$

Define a map $\phi: A+B \longrightarrow A/A\cap B$

$$a+b \longmapsto a + A\cap B$$

easy: ϕ is a homomorphism if it is well-defined

Generally, f $x \in A+B$, there are many ways to express $x \in A+B$.

i.e. there may exist, $a, a' \in A$, $b, b' \in B$

s.t. $x = a+b = a'+b'$

So is $\phi(x) = a + A\cap B$ or $\phi(x) = a' + A\cap B$?

This is not a problem so long as $a + A\cap B = a' + A\cap B$

In other words, if $a - a' \in A\cap B$

BUT $a+b = a'+b' \implies \underbrace{a-a'}_{\in A} = b'-b \in B$

$\implies a-a' \in A\cap B$

we also need to check that

$$\phi : A+B \longrightarrow A/_{A\cap B}$$
$$a+b \longmapsto a + A\cap B$$

is surjective

Clearly, if $a + A\cap B \in A/_{A\cap B}$, then say $a \in A$ and is a rep for $a + A\cap B$

Then $a+0 \in A+B$ and $\phi(a) = a + A\cap B$

Finally, we must check that

$$\operatorname{Ker}\phi = B$$

If $\phi(a+b) = 0 + A\cap B$

Then $\implies a \in A\cap B \implies a \in B$

$\implies \operatorname{Ker}\phi \subset B.$

On the other hand, if $b \in B \subset A+B.$

Then we can write it as $b = 0+b.$

$\implies \phi(b) = 0 + A\cap B \implies b \in \operatorname{Ker}\phi$

$\implies B \subset \operatorname{Ker}\phi$

$\implies \operatorname{Ker}\phi = B$

$\square$

Pf: of 3^{rd} Iso Thm.

$I \subset J \subset R$ are ideals

we want to show: $J/I \subset R/I$ is an ideal

and $\left(R/I\right) / \left(J/I\right) \cong R/J$.

$$\begin{array}{c} a \in R \\ \cup \\ J \\ \cup \\ I \end{array}$$

Check: J/I is an ideal in R/I

Define a map. $\phi : R/I \longrightarrow R/J$

$$a + I \longmapsto a + J$$

Obs: • If $a \in J$, then $\phi(a+I) = a + J = J = "0"$

• ϕ is clearly surjective: Pick any rep. $a \in R$ for $a + J$.

Then $\phi(a+I) = a + J$

Remains to show that $\operatorname{Ker} \phi = J/I$.

$\ulcorner$ If $a + I \in \operatorname{Ker}\phi$, then $\phi(a+I) = a + J = J$

$\Longrightarrow a \in J \quad \Longrightarrow a + I \in J/I$

$\Longrightarrow \operatorname{Ker}\phi \subset J/I$.

If $a \in J$, then $\phi(a+I) = a + J = J$

$\Longrightarrow a + I \in \operatorname{Ker}\phi$

$\Longrightarrow \operatorname{Ker}\phi \supset J/I \Longrightarrow \operatorname{Ker}\phi = J/I \lrcorner$

$\square$

Defn: Let R be a ring, w/ $1 \neq 0$

$A \subset R$ any subset

The ideal generated by A is

$$A \subset (A) \subset R$$

the smallest ideal of R containing A.

If an ideal I is generated by a single element set,

then we say I is a principal ideal

If I is generated by a finite set

then we say I is a finitely generated ideal

Note: Instead of writing $I = (\{a\})$, we often omit the set

and just write $I = (a)$.

Similarly, we'll write $I = (a_1, \ldots, a_n)$

Prop: For any $A \subset R$

$$(A) = \bigcap_{\substack{I \subset R \text{ ideals} \\ A \subset I}} I.$$

Pf: **Obs:** $R \subset R$ and is always an ideal of itself.

$\Longrightarrow \{A \subset I \subset R\} \neq \emptyset$.

Check: $(A) \subset \bigcap_{\substack{I \subset R \text{ ideals} \\ A \subset I}} I$.

$\ulcorner$ Suppose $A \subset I$ and $(A) \not\subset I$

- $(A) \cap I \subsetneq (A)$
- $A \subset (A),\ A \subset I \Longrightarrow A \subset (A) \cap I$.
- $(A) \cap I$ is an ideal.

  $\ulcorner$ use the isomorphism thms. $\lrcorner$

$\Longrightarrow$ There is an ideal containing A (i.e. $(A) \cap I$) that is smaller than (A) $\rightarrow \leftarrow$

$\lrcorner$

Check: $\bigcap_{A \subset I} I \subset (A)$.

$\ulcorner$ • $\bigcap_{A \subset I} I$ is an ideal, • $A \subset \bigcap I$

$\Longrightarrow \bigcap_{A \subset I} I \subset (A)$ because (A) is an ideal $\lrcorner$ $\square$