

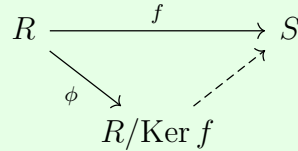
L5: Isomorphism Theorems

Theorem 5.1: The First Isomorphism Theorem

If $f : R \rightarrow S$ is a ring homomorphism and $I = \text{Ker } f$. Then

$$R/I \cong \text{Im } f$$

as rings.



Theorem 5.2: The Second Isomorphism Theorem

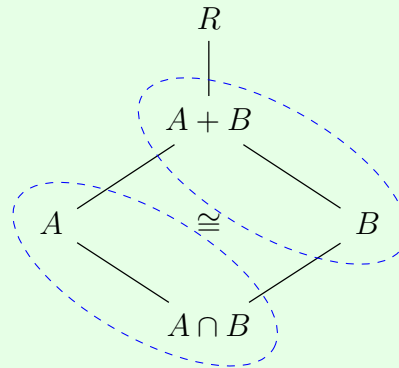
Let $A \subset R$ be a subring and $B \subset I$ an ideal.

Then

$$A + B := \{a + b \mid a \in A, b \in B\}$$

is a subring of R and $A \cap B$ is an ideal of A and

$$(A + B)/B \cong A/(A \cap B)$$



Proof of 5.2.

Let $A \subset R$ be a subring and $B \subset I$ an ideal.

It is **Easy to check** that $A + B$ is a subring and $A \cap B$ is an ideal in A .

Now we want to find an isomorphism

$$(A + B)/B \longrightarrow A/(A \cap B)$$

Idea: Use the First Isomorphism Theorem, i.e we want to find a surjective homomorphism

$$f: A + B \rightarrow A/(A \cap B)$$

such that $\text{Ker } f = B$.

Define a map

$$\begin{aligned}\phi: A + B &\rightarrow A/(A \cap B) \\ a + b &\mapsto a + A \cap B\end{aligned}$$

which can be shown to be homomorphism if it is well defined. Generally, if $x \in A + B$, there are many ways to express $x \in A + B$, i.e there may exist, $a, a' \in A$ and $b, b' \in B$ such that

$$x = a + b = a' + b'$$

So is $\phi(x) = a + A \cap B$ or $\phi(x) = a' + A \cap B$?

This is not a problem so long as $a + A \cap B = a' + A \cap B$. In other words, if $a - a' \in A \cap B$ BUT

$$a + b = a' + b' \implies \underbrace{a - a'}_{\in A} = b' - b \in B \implies a - a' \in A \cap B$$

We also need to check that ϕ is surjective.

Clearly, if $a + A \cap B \in A/(A \cap B)$, then say $a \in A$ and is a representative for $a + A \cap B$.

Then, $a + 0 \in A + B$ and $\phi(a) = a + A \cap B$.

Finally, we must check that

$$\text{Ker } \phi = B$$

If $a + b \in \text{Ker } \phi$ then $\phi(a + b) = 0 + A \cap B$ and so

$$a \in A \cap B \implies a \in B \implies \text{Ker } \phi \subset B$$

On the other hand, if $b \in B \subset A + B$, then we can write it as $b = 0 + b$ and so

$$\phi(b) = 0 + A \cap B \implies b \in \text{Ker } \phi \implies B \subset \text{Ker } \phi$$

Therefore, $\text{Ker } \phi = B$. ■

Theorem 5.3: The Third Isomorphism Theorem

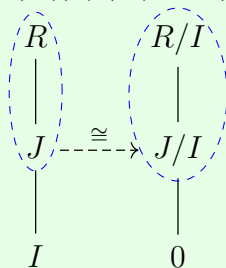
Let $I, J \subset R$ be ideals $I \subset J$.

Then

$$J/I := \{a + I \in R/I \mid a \in J\}$$

(the cosets of R/I whose representatives are in J or similarly the restriction of the quotient map from R to R/I to the domain J) is an ideal in R/I and

$$(R/I)/(J/I) \cong R/J$$



Proof of 5.3.

Let $I \subset J \subset R$ be ideals.

Then we want to show, $J/I \subset R/I$ is an ideal and

$$(R/I)/(J/I) \cong R/J$$

Check: J/I is an ideal in R/I .

Then define a map

$$\begin{aligned} \phi: R/I &\rightarrow R/J \\ a + I &\mapsto a + J \end{aligned}$$

Observe that if $a \in J$, then $\phi(a + I) = a + J = J = \bar{0}$

ϕ is also clearly surjective: Pick any representative $a \in R$ for $a + J$, then

$$\phi(a + I) = a + J$$

It remains to be shown that $\text{Ker } \phi = J/I$ as follows:

If $a + I \in \text{Ker } \phi$ then $\phi(a + I) = a + J = 0 + J = J$ which implies

$$a \in J \implies a + I \in J/I \implies \text{Ker } \phi \subset J/I$$

If $a \in J$, then $\phi(a + I) = a + J = J$ which implies

$$a + I \in \text{Ker } \phi \implies \text{Ker } \phi \supset J/I$$

and therefore $\text{Ker } \phi = J/I$. ■

Theorem 5.4: The Fourth Isomorphism Theorem

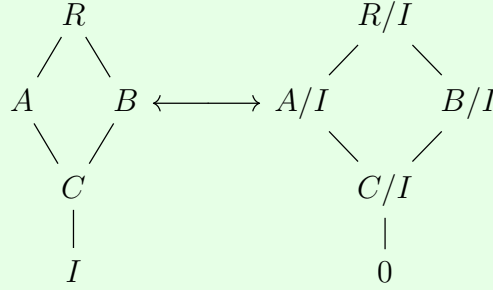
Let $I \subset R$ be an ideal.
Then the correspondence

$$I \subset A \subset R \longleftrightarrow A/I \subset R/I$$

is a bijection between

$$\{\text{subrings of } R \text{ containing } I\} \longleftrightarrow \{\text{subrings of } R/I\}$$

Moreover, $A \subset R$ is an ideal *iff* A/I is an ideal in R/I .



Definition 5.5: Ideal Generation, Principal and Finitely Generated Ideal

Let R be a ring, with $1 \neq 0$ and let $A \subset R$ be any subset.
The **ideal generated by** A is

$$A \subset (A) \subset R$$

i.e, the smallest ideal of R containing A .

If an ideal I is generated by a single element set, then we say I is a **principal ideal**.

If I is generated by a finite set then we say I is a **finitely generated ideal**.

Note: Instead of writing $I = (\{a\})$ for a principal ideal, we often omit the set notation and just write

$$I = (a)$$

Similarly, we will write $I = (a_1, \dots, a_n)$ for finitely generated ideals.

Proposition 5.6: Minimality of ideal generated by a set

For any subset $A \subset R$ and ideals $I \subset R$ such that $A \subset I$, we have

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

Proof.

Observe that $R \subset R$ and is always an ideal of itself which implies that there always exists an ideal containing A (at least R)

$$\{A \subset I \subset R\} \neq \emptyset$$

First check that $(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$

Suppose, for a contradiction, $A \subset I$ and $(A) \not\subset I$, then

(i) $(A) \cap I \subsetneq (A)$ (proper subset otherwise $(A) \subset I$)

(ii) $A \subset (A)$ and $A \subset I \implies A \subset (A) \cap I$

(iii) $(A) \cap I$ is an ideal (second isomorphism theorem).

Therefore there is an ideal containing A (i.e. $(A) \cap I$) that is smaller than (A) , which is contradictory the definition of (A) . Hence

$$(A) \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

Now check that $\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$

We have that

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I$$

is an ideal and therefore $A \subset \bigcap_{\substack{I \subset R \\ A \subset I}} I$ which implies

$$\bigcap_{\substack{I \subset R \\ A \subset I}} I \subset (A)$$

because (A) is an ideal. Therefore,

$$(A) = \bigcap_{\substack{I \subset R \\ A \subset I}} I$$

■