Spanning sets and free modules Defn: let M be an R-module An R-linear combination of elements m,, _, m, EM is an element of the form a, m, + az · mz+ - + an· m, a,, az, - , an eR We say a subset ACM spans or generales the module if every element of M is an R-linear combination of elements in A. More generally, if BCM, the submodule spanned/generated by B 13 RB := { a, m, t a, m, t - + a, m, | n, Zt, a, , -, a, eR, m, m, eB} Exercise: RB is an R-module Example: For any ring R w/ 1+0 every element is a "linear combination" of {13 i.e. if rol, then r=r. So R=RE133 is spenned by a single element as an R-module.

So $R = R \{17\}$ is spanned by a single element as an R-module. Example: The polynomial ring REX has a natural R-module structure!

If a_0R , $p(x) = a_0 + a_1 X + \dots + a_n X \cap CRCX$ then $a_1 \cdot (a_0 + a_1 X + \dots + a_n X^n)$ $i = (a_0a_0) \cdot (a_0a_1) \cdot X + \dots + (a_0a_n) \cdot X^n$ REX is spanned by $\{1, X, X^7, X^3, X^4, \dots \}$

Obs: REXT has no finite spanning set! To see this, suppose RCXI is spanned by P.(X), Pr(X), ___, Pn(X) arex] let d= max { deg p, (x), ____, deg pn(x) } Then d< 00 => Ya,, __,an oR deg [a, p,(x) + az. pz(x) + _____ ranpa(x)] < d. => X del & Span & pi(x), ---, pa(x)} Defn: We say an R-module Mis finitely generated if it has a finite spanning cet. We say M is cyclic if it is spanned by a single element. Example: If Risaring, ACR

Example! If R is a ring, ACR

Then RA = (A)

(the module generated by A is the ideal generated by A)

A cyclic submodule of R is just a principal ideal.

Example: Raring, F=R is the free R-module of rank n. F has a natural spanning set: $\begin{cases}
e_1 = (1,0,0,-..,0) \\
e_2 = (0,1,0,-..,0)
\end{cases}$ $e_3 = (0,0,1,...,0)$ $e_4 = (0,0,0,-..,0)$ Any element (a,, az, _ an) & R can be written as (a,, az, ____, a) = a,.(1,0,0, ___,0) + az.(0,1,0, __,0) t.... + aa(0,0,0,-,0,1) = 9, ·e, · 92 · ez - - · en · en Re contextualizing the free R-module of rank n. Consider the set &1,2,3,__,n3 A function a: {1, 2,3, __, n3 _> } -----> a(z) = 9z $n \longmapsto a(n) = a_n$

we can think of an ordered n-tuple of elements in R

as a function

a: \(\xi_{1,2}, -n \, \cdots - \cdot \)

1.e. we can think of 2 as Dn= & a: &1,2, __,n3 __> R } The obvious addition is a +6: {1,2, -, n} -> R 1 ----> a(1) 16(1) 21-3 a(2)-16(2) и 1------ a(u) -b(u) obvious scalar multiplication is The r.a: {1,2, __,a} ___ } 1 1-s r.a(1) Z ----> r.a(z) " r.a(n) Defn: Fix a ring R An R-module I is free on a set A if YmeF are unique elements m, mz, __, m, & A a,, az, ___, a, e \ \ s.t. m= a, m, +9, m, + = +9, m, we call A sot at free generators of F or a basis of F

Note: usually, we ask that the basis is ordered in some way.

Example: The set $\Sigma_n = \{e_1, e_2, \dots, e_n\}$ is a basis for the free module of rank n.

Non-example: Z/2Z 1s a non-free Z-module.

Non-example: Is every submodule of a fee module free?

2/47 1) a fee module over 2/47

(Chak: 2/42 = 2/42 {] 3 , free)

7. 2/42 = 80, 23 c 2/42 15 a submodule

But: $\overline{2} \cdot \overline{2} = \overline{0}$ There is no unique way of writing $\overline{0}$ as a $\overline{2/42}$ -linear combination of $\overline{23}$

 \Rightarrow 2. $\mathbb{Z}/4\mathbb{Z} = (\overline{2})$ is not free.

Example: Fix a rmy R. Let A be any set FCA) := { \$\phi: A -> R \ | \phi(a) = 0 for all but finishly many acA} is a free module over R on the set A. 6,4: A → R PE: Addition: 6+7: A -> 12 a >>> \$(a) + 2(a) Scalars: b.A -> R, reR r. ø: A --> R a -> r. p(n) Consider the inclusion map

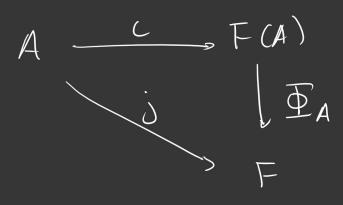
Consider the inclusion map $C: A \longrightarrow F_{\mathcal{P}}(A)$ $a \longmapsto \begin{cases} \phi_a: A \longrightarrow \Gamma \\ \times \longmapsto \begin{cases} 1 & \text{xen} \\ 0 & \text{xtn} \end{cases} \end{cases}$

Obviously this map is injective: If $\phi_a = \phi_b$ then $\phi_a(a) = 1 = f_b(a)$ $\Rightarrow a = b$.

we call $c(A) = \mathcal{E}_A$ and we see that

() EA Spans FR(A) PE; (&: A -> R) & FR(A) Ea,, _ an 3 CA st. \$(a;) ≠0 $\phi(a_i) = \phi(a_i) \cdot | = \phi(a_i) \cdot \phi_{a_i}(a_i)$ $\Rightarrow \phi = \phi(a) \cdot \phi_{a_1} + \phi(a_2) \cdot \phi_{a_2} + \dots + \phi(a_n) \cdot \phi_{a_n}$ R R \$ & Span EA (E) FR(A) is fre on EA φ= r.· pa, t rz· paz t - tr.· pan = 5, . pa, - 5 z · pa, - + 5 n dan ([-5]). da, - (1-52). daz + --- - (1-52) daz = 0 => (r,-s,) pa, (a) + (r,-s,) pa, (a) + __+ (r,-s) pa, (a) =0 $(r_1-s_1)\cdot l=(r_1-s_1)=0$ Similarly, r;=s; \f;

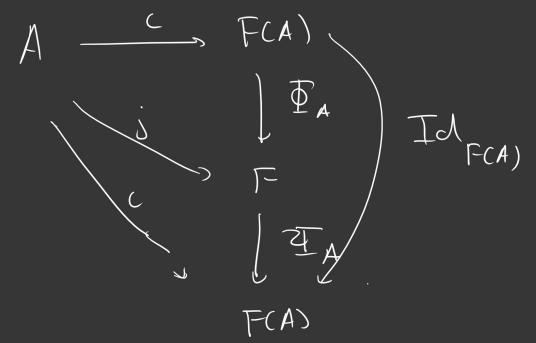
Than: (The universal property of free R-modules) Ra ring, A is any set Mis an R-module s.t. 3 f. A -> M. unique R-module homomorphism $\Phi_A: F(A) \longrightarrow M$ A _ C _ F(A) A paid 3! PA PE: DA: F(A) -> MER OM (p:A-R) - Splas. flas Cor. If Ris aring, F is any free module on aset A Then FSFCA) PF: ACF that governdes F freely over R j: A --- F



Thee is an obvious map T:F -> F(A)

rait _ + ran L _ ripait rapay

Clearly this map



By uniqueness
$$\overline{I}_A \circ \overline{\Phi}_A = \overline{I}_A \circ \overline{F}_{(A)}$$