

# More on Ideals

Let  $R$  be a ring with  $1 \neq 0$ .  
Recall that if  $A \subset R$ , then

$$(A) = \bigcap_{\substack{I \subset R \text{ ideals} \\ A \subset I}} I$$

## Definition 6.1: Ring Multiplication

For fixed sets  $A, B \subset R$ , we define **ring multiplication** as

$$A \cdot B := \{a_1 b_1 + \cdots + a_n b_n \mid a_1, \dots, a_n \in A, b_1, \dots, b_n \in B, n \in \mathbb{N}\}$$

## Proposition 6.1: Characterization of ideal generated by a set

If  $A \subset R$  is any subset, then:

- (i)  $R \cdot A$  is the left ideal generated by  $A$
- (ii)  $A \cdot R$  is the right ideal generated by  $A$
- (iii)  $R \cdot A \cdot R$  is the (two-sided) ideal generated by  $A$

*Note:* If

- $A = \emptyset$ , then we say  $RA = AR = RAR = \{0\}$
- $R$  is commutative, then  $RA = AR = RAR$ .

**Proof.** We will only check for the left ideal, the others follow similarly.

First the subring criterion for  $RA \subset R$

- (i)  $0 = 0 \cdot a \in RA \implies RA \neq \emptyset$
- (ii) Let  $x, y \in RA$ , then there exist

$$\begin{aligned} r_1, \dots, r_n \in R, a_1, \dots, a_n \in A \\ r'_1, \dots, r'_m \in R, a'_1, \dots, a'_m \in A \end{aligned}$$

such that

$$\begin{aligned} x &= r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \\ y &= r'_1 a'_1 + r'_2 a'_2 + \cdots + r'_m a'_m \end{aligned}$$

then

$$\begin{aligned} x - y &= (r_1 a_1 + \cdots + r_n a_n) - (r'_1 a'_1 + \cdots + r'_m a'_m) \\ &= r_1 a_1 + \cdots + r_n a_n + (-r'_1) a'_1 + \cdots + (-r'_m) a'_m \in RA \end{aligned}$$

and

$$\begin{aligned} xy &= (r_1 a_1 + \cdots + r_n a_n) \cdot (r'_1 a'_1 + \cdots + r'_m a'_m) \\ &= (r_1 a_1 r'_1) a'_1 + \cdots + (r_1 a_1 r'_m) a'_m \\ &\quad + \vdots \\ &\quad + (r_n a_n r'_1) a'_1 + \cdots + (r_n a_n r'_m) a'_m \in RA \end{aligned}$$

Then  $RA$  is a subring.

To see  $RA$  is an ideal: Let  $r \in R, x \in RA$  as above.

$$r \cdot x = r \cdot (r_1 a_1 + \cdots + r_n a_n) = (rr_1)a_1 + \cdots + (rr_n)a_n \in RA$$

Moreover

$$A \subset RA \quad (1 \in R \implies \forall a \in A, 1 \cdot a = a \in RA)$$

So  $RA$  is an ideal containing  $A$  i.e

$$(A) \subset RA$$

On the other hand, if  $I$  is a left ideal such that  $A \subset I$ , then  $a \in A, r \in R \implies r \cdot a \in I$  which implies for any finite list  $r_1, \dots, r_n \in R, a_1, \dots, a_n \in A$

$$r_1 a_1, \dots, r_n a_n \in I \implies r_1 a_1 + \cdots + r_n a_n \in I \implies RA \subset I$$

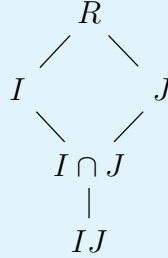
and since  $(A)$  is a left ideal, we have

$$RA = (A)$$

and specifically this is the smallest ideal needed to contain  $A$ . ■

**Proposition 6.2:**  $I \cdot J \subset I \cap J$

If  $I, J \subset R$  are ideals, then  $I \cdot J$  is an ideal,  $I \cdot J \subset I \cap J$ .



Note:  $I \cdot I = I^2, \dots, \underbrace{I \cdot I \cdot \dots \cdot I}_{n\text{-times}} = I^n$

**Example 6.1.** Consider  $2\mathbb{Z}, 3\mathbb{Z} \subset \mathbb{Z}$ , then

$$2\mathbb{Z} \cdot 3\mathbb{Z} = \left\{ \sum_{k=1}^n 2a_k \cdot 3b_k \mid a_k, b_k \in \mathbb{Z} \right\} = \left\{ 6 \left( \sum_{k=1}^n a_k \cdot b_k \right) \mid a_k, b_k \in \mathbb{Z} \right\} = 6\mathbb{Z}$$

and

$$2\mathbb{Z} \cap 3\mathbb{Z} = \underbrace{\{2n = 3m\}}_{2|m, 3|n} = 6\mathbb{Z}$$

In this case we have  $2\mathbb{Z} \cdot 3\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z}$ .

**Example 6.2.** Consider the ring  $R = \mathbb{Z}[X]$  with

$$(X) := \{p(X) \cdot x \mid p(X) \in R\}$$

$$(X^2) := \{q(X) \cdot x^2 \mid q(X) \in R\}$$

Then

$$(X) \cdot (X^2) = \{(p_1(X) \cdot X) \cdot (q_1(X) \cdot X^2) + \cdots + (p_n(X) \cdot X) \cdot (q_n(X) \cdot X^2)\}$$

$$= \{(p_1 \cdot q_1(X) + \cdots + p_n \cdot q_n(X)) \cdot X^3\} = (X^3)$$

On the other hand, since multiples of  $X^2$  are also multiples of  $X$ , we get

$$(X) \cap (X^2) = (X^2)$$

and so

$$(X) \cdot (X^2) = (X^3) \subsetneq (X) \cap (X^2) = (X^2)$$

Since a multiple of  $X^3$  is a multiple of  $X^2$  but there is no multiple of  $X^3$  which is equal to  $aX^2$  for nonzero  $a \in R$ .

## Large Ideals in $R$ and Arithmetic in $R$

Assume  $R$  is a commutative ring w/  $1 \neq 0$ .

If  $a \in R$ , then

$$(a) = \{ra \mid a \in R\} \quad (\text{the "multiples" of } a)$$

e.g.  $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\} = (2)$

*Note:* We sometimes write

$$(a) = R \cdot a = a \cdot R$$

We also say that if  $b \in (a)$ , that  $a$  **divides**  $b$ , i.e.  $a \mid b$ .

**Claim:**  $b \in (a)$  iff  $(b) \subset (a)$

**Proof.** Let  $b \in (a)$  then there exists  $r \in R$  such that  $b = r \cdot a$ . In particular,

$$c \in (b), \exists s \in R, c = s \cdot b = s \cdot (r \cdot a) = (s \cdot r) \cdot a \in (a) \implies (b) \subset (a)$$

On the other hand, if  $(b) \subset (a)$ , then  $b \in (b) \subset (a)$ . ■

### Definition 6.2: Prime Ideal

Let  $R$  be a commutative ring.

An ideal  $P \neq R$  is called a **prime ideal** if for all  $a, b \in R$  such that  $a \cdot b \in P$ , then either  $a \in P$  or  $b \in P$ .

**Example 6.3.**

- $2\mathbb{Z}$  is prime
- $6\mathbb{Z}$  is **not** prime e.g.  $2 \cdot 3 = 6 \in 6\mathbb{Z}$  but  $2, 3 \notin 6\mathbb{Z}$
- $\{0\} \subset \mathbb{Z}$  is prime. If  $a \cdot b = 0, a, b \in \mathbb{Z}$  then either  $a = 0$  or  $b = 0$  (integral domain).
- $(x) \subset \mathbb{R}[x]$  is prime
- $(x^2)$  is **not**, e.g.  $x \cdot x = x^2 \in (x^2)$  but  $x \notin (x^2)$ .

**Proposition 6.3:**  $R$  integral if  $\{0\}$  prime

$R$  is an integral domain iff  $\{0\}$  is prime

**Theorem 6.1:** Prime Ideal  $\iff R/P$  integral domain

Assume  $R$  is commutative.

An ideal  $P \subset R$  is prime iff  $R/P$  is an integral domain.

**Proof.**

$\Rightarrow$

Suppose  $P$  is prime and  $\bar{a}, \bar{b} \in R/P$  such that  $\bar{a} \cdot \bar{b} = \bar{0}$ .

We want  $\bar{a} = \bar{0}$  or  $\bar{b} = \bar{0}$ .

Pick representatives  $a \in \bar{a}, b \in \bar{b}$ . This implies  $\overline{a \cdot b} = \bar{0}$ , i.e.  $a \cdot b \in P$ .

But  $P$  is prime, so either  $a \in P$  or  $b \in P$ , i.e.  $\bar{a} = \bar{0}, \bar{b} = \bar{0}$ .

$\Leftarrow$

If  $R/P$  is integral and  $a \cdot b \in P$ , then

$$\overline{a \cdot b} = \bar{0} \implies \underbrace{\bar{a} = \bar{0} \text{ or } \bar{b} = \bar{0}}_{R/P \text{ integral}} \implies a \in P \text{ or } b \in P$$

■