# Intro to modular forms.

(for Algebraic Geometry I Spring 2024)

## 0.1 REVIEW OF MEROMORPHIC FUNCTIONS

This is just for me to warm up, of course this is elementary and covered in 503 (Ch3 Stein's book).
(of course this will not be mentioned in class)

DEF. A **point singularity** of a function $f$ is some $z_0 \in \mathbb{C}$ such that $f$ is defined in a nbgh of $z_0$ but not defined at $z_0$ (and holomorphic)
or **isolated singularity**

- We say that $z_0 \in \mathbb{C}$ is a **zero** for $f$ if $f(z_0)=0$

Theorem (Holomorphic functions near a zero)
Suppose $\Omega$ is a region, $0 \not\equiv f : \Omega \to \mathbb{C}$ holomorphic with $f(z_0)=0$. Then

i) $\exists U$ open nbgh of $z_0$ st $f(z)\neq 0$ in $U - \{z_0\}$ (the zero is isolated)

ii) $\exists U$ open nbgh of $z_0$, $\exists g : U \to \mathbb{C}$ holomorphic on $U$, $\underset{z\in U}{g(z)\neq 0}$ and $\exists!\, n \in \mathbb{N}$ st

$$f(z) = (z-z_0)^{n} g(z) \quad \forall z \in U$$

order of

DEF Let $f$ have an isolated singularity at $z_0$, $\left(f : \overset{\text{open}}{\Omega} \setminus \{z_0\} \to \mathbb{C} \text{ holom}\right)$ $(z_0 \in \Omega)$ then

i) We say that $z_0$ is a **removable singularity** if $\exists\, g : U \to \mathbb{C}$ holom ($U \subseteq \Omega$ open nbgh of $z_0$) st $g \equiv f$ in $U - \{z_0\}$

ii) $z_0$ is a pole of $f$ if $1/f$ defined to be zero in $z_0$ is holomorphic in a nbgh of $z_0$
($1/f$ removable sing at $z_0$ and when removed is a zero)

iii) $z_0$ is an **essential singularity** if none of the above hold.

Theorem (Structure thm of poles) Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holom and suppose $z_0$ is a pole
Then $\exists\, U \subseteq \Omega$ nbgh of $z_0$, $\exists\, g : U \to \mathbb{C}\setminus\{0\}$ (holom) and $\exists!\, n \in \mathbb{N}$ / $\forall z \in U \setminus \{z_0\}$

$$f(z) = (z-z_0)^{-n} g(z)$$

**order or multiplicity of the pole** **($n=1$, simple)**

**Theorem** (Riemann's thm on removable sing) Suppose $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic where $\Omega \ni z_0$ is an open subset. If $f$ is bounded then $z_0$ is a removable singularity.

**Corollary** $f : \Omega \setminus \{z_0\} \to \mathbb{C}$, (where $\Omega$ open, $z_0 \in \Omega$) $z_0$ is a pole of $f$ $\leftrightarrow$ $\lim_{z \to z_0} |f(z)| = \infty$

**DEF**
Let $f : \Omega \setminus \{z_0, \dots\} \to \mathbb{C}$ holomorphic $z_0, \dots \in \Omega$ poles and the sequence has no limit points, then $f$ is said to be meromorphic in $\Omega$.
(Sometimes we say $f$ meromorphic at $z_0$, this means $z_0$ removable or a pole).

**DEF** $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the following topology
Open sets $:=$ (Open in $\mathbb{C}$) $\cup$ ($\{z \in \mathbb{C} : |z| > M, M > 0\} \cup \{\infty\}$)
is called the extended complex plane.

**DEF** Let $f : \mathbb{C} \to \mathbb{C}$, we say that

- $f$ has a pole at infinity / essential sing at infinity / removable sing (holomorphic) at infinity if $F(z) = f(1/z)$ has a pole at the origin / essential sing at the origin / removable sing at the origin

Suppose that $f$ is meromorphic on $\mathbb{C}$ and it is holomorphic at infinity or has a pole at infinity, then we say that $f$ is meromorphic in the extended complex plane.

**Rmk** · Say $f : \mathbb{C} \to \mathbb{C}$ (or $\Omega \setminus \{\dots\}$, $\setminus \{\dots\}$) meromorphic we can extend it to $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ ($\Omega \cup \{\infty\} \subseteq \hat{\mathbb{C}}$) via
$f(\text{pole}) = \infty$, $f(\infty) = \lim_{z \to 0} f(1/z)$ if it exists, $\infty$ if not (this is why we say meromorphic in $\mathbb{C}$)
And similarly one can reduce functions from $\hat{\mathbb{C}}$ to $\mathbb{C}$.
$f$ is meromorphic in the extended complex plane ($\Omega$) $\leftrightarrow$ $\hat{f}$ meromorphic in $\hat{\mathbb{C}}$ ($\Omega \cup \{\infty\}$) with adequate def.

**Theorem** (Classification of meromorphic) Suppose $f$ is meromorphic in the extended complex plane, then $f(z) = \frac{P(z)}{Q(z)}$ for $P, Q \in \mathbb{C}[z]$. (Rational function)

**Corollary** · Meromorphic in $\hat{\mathbb{C}}$ are determined by their zeros and poles with mult.
- $f$ entire and meromorphic on the extended complex plane $\to$ $f$ polynomial (no poles).

## 0.2 REVIEW OF LFTs

This is again for review and it is covered in 503 (Ch 8 Stein's book). However, it contains interesting ideas for what we will do.

**DEF** Let $f: U \to V$ be holomorphic ($U$ open and hence $V$ open by Open Mapping thm) and bijective then $f$ is said to be a conformal map or biholomorphism. If such $f$ exists between $U, V$ (hence they have to be open sets) we say that $U, V$ are biholomorphic or conformally equivalent, we write $U \sim V$.

**Theorem** i) If $f: U \to \mathbb{C}$ holomorphic and injective. Then $f'(z) \neq 0$ $\forall z \in U$.
ii) $f: U \to V$ conformal. Then $f^{-1}$ is holomorphic.

**Theorem** $f: \mathbb{C} \to \mathbb{C}$ entire and 1-1. Then $f(z) = az + b$ $a, b \in \mathbb{C}$. $(a \neq 0)$

**Notes** • Even though these functions should be called affine, they are sometimes called linear transformations. Characteristics

1) The comp of linear transf $\to$ linear transf
2) Fixes 1 or 0 points
3) $(z_1, z_2), (w_1, w_2) \in \mathbb{C}$ $\exists!$ such $f$ with $f(z_1) = w_1$, $f(z_2) = w_2$
4) lines $= \{ a + tb, t \in \mathbb{R}\ a, b \in \mathbb{C} \} \to$ lines
circles $\to$ circles.

The reason why we have talked about linear transformations is because they are the numerator and denominator of something called linear fractional transformations or Möebius transformations. Namely the map from example 1 is one such map. The maps are:

$$\mathbb{C} - \{\text{one point}\} \longmapsto \mathbb{C} \quad \equiv \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$
$$z \longmapsto \frac{az+b}{cz+d} \qquad \text{st}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

How can we understand this mappings on $\mathbb{C}$. A LFT can be naturally identified with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$. Note that

$$GL_2(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}$$
$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d}$$

is a group action. (However the action identification is not isomorphic and the action is not faithful)

Faithful action from $G$ in $\Omega$ means that if $g \cdot \alpha = \alpha$ $\forall \alpha \in \Omega$, $g = 1_G$.

However if we let $PGL_2(\mathbb{C}) = GL_2(\mathbb{C}) / Z(GL_2(\mathbb{C}))$ ; quotient group by the center (scalar matrices)

We get: Group of LFTs $\cong GL_2(\mathbb{C}) / \underset{\text{scalar mat}}{Z(GL_2(\mathbb{C}))} = PGL_2(\mathbb{C})$

and it acts on $\mathbb{C}$ (faithfully)

<u>Thm</u> (LFTs, meromorphic in $\hat{\mathbb{C}}$) Let $f : \mathbb{C} - \{-d/c\} \to \mathbb{C}$ , $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$

$f(z) = \dfrac{az+b}{cz+d}$ . Then

i) $f$ is 1-1, and meromorphic in the extended complex plane

ii) $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the natural extension is bijective with inverse given by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (an LFT)

and the comp of two LFTs is again an LFT.

($PGL_2(\mathbb{C})$ acts faithfully on $\hat{\mathbb{C}}$)

iii) An LFT fixes 1 or 2 points unless it is the identity.

iv) $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \hat{\mathbb{C}}^3$ $\exists !$ LFT ($g \in PGL_2(\mathbb{C})$) st $z_i \mapsto w_i$

v) Sends circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$

$\hookrightarrow \{C_r(A_0)\} \cup \{\text{Lines in } \mathbb{C} \cup \{\infty\}\}$

vi) An injective meromorphic function in the extended complex plane is of this form.

Now we can start to talk about the upper half plane. We might need more results after; this was not attempting to be fully self contained but we are good to start now.

## 1. THE MODULAR GROUP

Let $\mathbb{H}$ be the upper half plane $= \{x+iy \in \mathbb{C} : y > 0\}$. The next result follows from studying automorphisms of the disk.

<u>Theorem</u> Let $SL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \det A = 1\}$

$SL_2(\mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{H}) = \{f : \mathbb{H} \to \mathbb{H} \text{ bijective and holomorphic}\}$ ← hence $f^{-1}$ holom.

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto g : \mathbb{H} \to \mathbb{H}$, $z \longmapsto \dfrac{az+b}{cz+d}$ ← pole at the real line, outside $\mathbb{H}$.

Is a surj group hom (the group structure of $SL_2(\mathbb{R})$ is with product and in $\mathrm{Aut}(\mathbb{H})$ is with comp).

Note that the kernel of the previous map is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

$I$ (written above the first matrix)

Let $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{I, -I\}$ ; then $PSL_2(\mathbb{R}) \cong Aut(\mathbb{H})$ by 1st iso thm.

and it acts faithfully on $\mathbb{H}$. ($SL_2(\mathbb{R})$ acts on $\mathbb{H}$ ; $g \cdot z = g(z)$; so induces action of the quotient group)

Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\} \leq SL_2(\mathbb{R})$

DEF A topological group $G$ is a topological space with group structure $\cdot : G \times G \to G$, $(x,y) \mapsto xy$ such that this and the inversion maps are continuous functions.

Example Consider $\mathbb{R}^4$ with usual topology, $SL_2(\mathbb{R}) \subseteq \mathbb{R}^4$ with subspace topology and the product of matrices is a topological group.

A subgroup $H$ of a topological group $G$ is called a discrete (sub)group if $\forall g \in H$, $\exists U \subseteq G$ open such that $g \in U$ and $H \cap U = \{g\}$.

Example $SL_2(\mathbb{Z}) \leq SL_2(\mathbb{R})$ is discrete.

DEF $G = SL_2(\mathbb{Z}) / \{I, -I\}$ is called the modular group.

(Let $\pi : SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$ canonical proj ; $G = \pi(SL_2(\mathbb{Z}))$)

Convention: If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we abusively denote $\pi(g) \in G$ by $g$.

## Understanding $G$ and its action.

Our goal now is to understand $G$ and its action on $\mathbb{H}$. For this we will find a geometrically nice set of representatives of the orbits. (aka Fundamental domain of the action of $G$ on $\mathbb{H}$)

We will find a generating set of $G$ and we will also mention a presentation of $G$.

In general if $G$ acts on $\Omega$, $Y \subseteq \Omega$ is a fund domain for $G \circlearrowleft \Omega$ if $\forall x \in \Omega$, $|Y \cap \underbrace{x^G}_{G \text{ orbit}}| = 1$

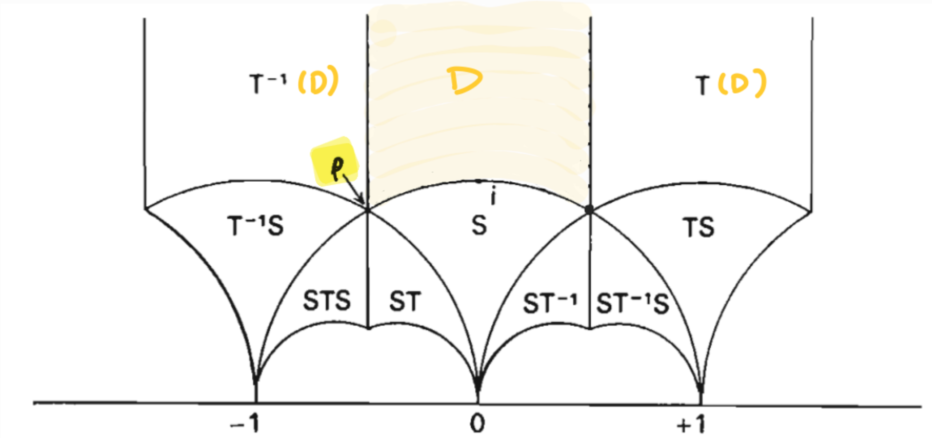
Let $S \in G$ defined by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (so technically speaking $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle \in G$ .... )

$T \in G$ defined by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Thus if $z \in \mathbb{H}$, $S \cdot z = \frac{-1}{z}$, $T \cdot z = z+1$.

Also $S^2 = 1_G = (ST)^3$ (ST has order 3)

Let $D = \{z \in \mathbb{H} : |z| \geq 1, |\mathrm{Re}(z)| \leq 1/2\}$. The next is an illustration of how $\{1, T, TS, ST^{-1}S, S, ST, STS, T^{-1}S, T^{-1}\} \subseteq G$ transform D.




D is a "fundamental domain" for the action of G in $\mathbb{H}$.

↳ we need to take out some boundary points to match the definition; who cares ...

<u>Theorems 1,2</u>

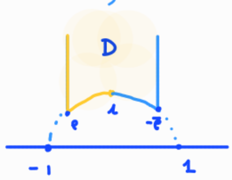
i) $\forall z \in \mathbb{H}, \exists g \in G : g \cdot z \in D$

ii) If $z \neq z' \in D$ and $\exists g \in G : gz = z'$ then
- $R(z) = \pm 1/2$ and $z = z' \pm 1$
- $|z| = 1$, $z' = -1/z$

(not)

- So technically speaking the fund domain is $\mathrm{Int}\, D \cup \{z \in \mathbb{H} : |z| \geq 1, \mathrm{Re}(z) = -1/2\} \cup \{z \in \mathbb{H} : |z| = 1, -\frac{1}{2} < \mathrm{Re}(z) \leq 0\}$

- Thus D maps surjectively to $\mathbb{H}/G$ (reps of G-orbits) and Int (D) injectively. (of course canonical maps point $\longmapsto$ its equivalence class)




iii) Let $z \in D$ and let $I(z) = \mathrm{Est}_G(z) = \{g \in G : g \cdot z = z\}$

Then $I(z) = \{1_G\}$ except if

•) $z = i$ in which case $I(z) = \langle S \rangle = \{1_G, S\}$ (subgroup generated by S)

••) $z = \rho = e^{i 2\pi/3}$ in which case $I(z) = \langle ST \rangle$ (order 3)

•••) $z = -\bar{\rho} = e^{\pi i/3}$ in which case $I(z) = \langle TS \rangle$ (order 3)

iv) $G = \langle S, T \rangle$

Proof/ Let $G' = \langle S, T \rangle$

• Claim Let $z \in \mathbb{H}$, $\exists g' \in G' : g' \cdot z \in D$ (of course this proves i) in a strong way).

Proof/ Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent an element of $G'$. Then

$\mathrm{Im}(g \cdot z) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$ note this is positive.

$\downarrow$ easy computational check.

• $\exists\, g \in G'$ such that $\mathrm{Im}(gz)$ is maximum. (among $\mathrm{Im}(gz)$, $g \in G'$)

• Need to minimize denominator. Pick $g \in G'$. Let $\alpha = \mathrm{Im}(g \cdot z) \in \mathbb{R}$. The number of $(c,d) \in \mathbb{Z}^2 : |cz+d| < \alpha$ is finite. Check among all of these if any pair is of coprime numbers (ie $\exists a, b \in \mathbb{Z} : ac - bd = 1$). If there are not $\alpha$ is maximum. If there is such, check $|cz+d|$ for all of those and consider the minimum. Any corresponding $g'$ gives $\mathrm{Im}(g \cdot z)$ maximum.

• Choose $n \in \mathbb{Z}$ so that

$$-1/2 \leq \mathrm{Re}\left(T^n (g \cdot z)\right) \leq 1/2$$

• $\underbrace{T^n (g \cdot z)}_{z'} \in D$ | NTS $|z'| \geq 1$. If $|z'| < 1$, then $\mathrm{Im}(S T^n g \cdot z) = \mathrm{Im}(S \cdot z') = \mathrm{Im}(-1/z') >$
$> \mathrm{Im}(z') = \mathrm{Im}(g \cdot z)$ ∮.

$\downarrow$ T doesn't change imaginary part.

Thus the element $g' = T^n g$ has the desired property.

i) Is proved. We work to prove the next two items, let $z \in D$

Suppose that $g \cdot z \in D$ where $g$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (not $1_G$).

We may assume $\mathrm{Im}(g \cdot z) \geq \mathrm{Im}(z)$ (if not take $gz$ as your original point $w$ and $g^{-1} w = z \in D$).

This means $|cz+d| \leq 1$. If $|c| \geq 2$ this is impossible; so we have the following cases

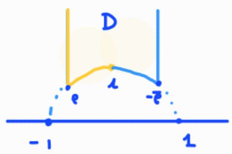
• $c = 0$ Then $d = \pm 1$. So $g \cdot z = \dfrac{\pm z + b}{\pm 1} = z \pm b$. Since $g \cdot z \in D$,

we have that

$$-1/2 \leq \mathrm{Re}(z),\ \mathrm{Re}(gz) \leq 1/2$$

This of course implies that $b = 0$ or $\mathrm{Re}(z) = -1/2$, $b = 1$ or $\mathrm{Re}(z) = 1/2$, $b = -1$.
so $g = 1$

• $c = 1$ $|cz+d| \leq 1$ means $|z+d| \leq 1$ so this is $d = 0$ except $z = \rho$ (or $-\bar{\rho}$)




In which case $d = 0$ or $1$ (or $0, -1$).

If $d = 0$, $|z| \leq 1$ so $|z| = 1$ since we are in $D$. But $ad - bc = 1$ implies $b = -1$

So $gz = \dfrac{az+b}{cz+d} = a - 1/z \in D$ so as above $a = 0$ or $\mathrm{Re}(-1/z) = \pm 1/2$ so $-1/z \in D$, $|-1/z| = 1$

thus this means $-1/z = \rho$ or $-\bar\rho$ so $z = \rho$ or $-\bar\rho$. And in this case $a = 0, -1$ for $z = \rho$
$a = 0, 1$ for $z = -\bar\rho$

. If $d = 1$, $z = \rho$ gives $a - b = 1$ so $g \cdot \rho = \frac{a-1}{1+\rho} = a + \rho$ so $a = 0, 1$
. If $d = -1$, $z = -\bar\rho$ then $g \cdot (-\bar\rho) \underset{\text{similarly}}{=} a + (-\bar\rho)$ $a = 0, -1$.

If $c = -1$ then this leads to case $c = 1$ since $g \in G$ is also represented by $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

ii, iii Follow from this. Why? For ii read the underlined parts. For iii) it is obvious but formally:
Suppose $g \in G$, $z \in D$ and $g \cdot z = z$. In particular $g \cdot z \in D$ so we can use what we did for ii). Thus, we have 2 options

. $c = 0$, then $g$ given by $\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ but $gz = z$ means $b = 0$ so $g = 1_G$.

. $c = \pm 1$ — $d = 0$ then $g$ given by $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$. We know $g \cdot z \in D$ iff $a = 0 \wedge |z| = 1$ or $z = \rho$, $a = 0, -1$ ; $z = -\bar\rho$ $a = 0, 1$. But out of this we only have $g \cdot z = z$ if $z = i$ and in this case $a = 0$. Or $z = \rho$ and $a = -1$ Or $z = -\bar\rho$ and $a = -1$

$d = 1$ then $z = \rho$ $a = 0$ (to have $g \cdot z = z$)

$d = -1$ then $z = -\bar\rho$ $a = 0$

This shows that except $z = i, \rho, -\bar\rho$, $I(z) = \{1_G\}$.
In case $z = i$, $I(z) = \{1_G, S\}$. In case $z = \rho$, $I(z) = \{1_G, \underset{z \mapsto -1 - \frac{1}{z}}{\overset{ST}{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}}}, \underset{z \mapsto \frac{-1}{z+1}}{\overset{(ST)^2}{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}}\}$
In case $z = -\bar\rho$, $I(z) = \langle TS \rangle$ similarly.

Now we prove iv). We show $G' \supseteq G$. Let $g \in G$, let $z_0 \in \operatorname{int}(D)$ let $z = g \cdot z_0$.
We know that $\exists g' \in G'$: $g' \cdot z \in D$. Thus $z_0$, $g' \cdot z = g' \cdot g \cdot z_0$ are both in $D$ and $z_0 \in \operatorname{int}(D)$. By ii, $z_0 = g' \cdot z$ now $gg' \cdot z_0 = z_0$ so by iii) since $z_0 \in \operatorname{int}(D)$ $gg' = 1$ so $g \in G'$. $\square$

Fact: A presentation of this group $G$ is $\langle S, T : S^2 = 1 = (ST)^3 \rangle$

# 2. MODULAR FUNCTIONS

<u>DEF</u> Let $k\in\mathbb{Z}$, we say that $f:\mathbb{H}\setminus\{\dots\}\to\mathbb{C}$ is **weakly modular of weight $2k$** if it is meromorphic in $\mathbb{H}$ and verifies $f(z)=(cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ $\forall \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$.

<u>Stupid Rmk</u> i) With the (equivalent) defn we have we could also write $f:\mathbb{H}\longrightarrow\hat{\mathbb{C}}$ it would make sense

ii) Some authors say weight $-2k$, $k$ ....

<u>Observation</u> Let $g\in G$ be represented by $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$; then $g:\mathbb{H}\longrightarrow\mathbb{H}$, $z\longmapsto\frac{az+b}{cz+d}$ $\in Aut(\mathbb{H})$

and the derivative which we denote by $\frac{d(g\cdot z)}{dz}=\frac{a(cz+d)-c(az+b)}{(cz+d)^2}=\frac{1}{(cz+d)^{-2}}$

So we can rewrite the relation as $f(gz)\left(\frac{d(g\cdot z)}{dz}\right)^k=f(z)$.

<u>Proposition</u> Let $f$ be meromorphic on $\mathbb{H}$. Then $f$ is a weakly modular form of weight $2k$ iff

i) $f(z+1)=f(z)$

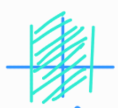
ii) $f(-1/z)=z^{2k}f(z)$.

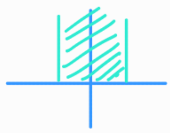
<u>Proof</u>: $S,T$ generate $G$ (essentially) □

Suppose $f:\mathbb{H}\to\mathbb{C}$ satisfies $f(z)=f(z+1)$. Then $\exists\,\tilde f$: Unit disk $\setminus\{0\}\to\mathbb{C}$ such that $\tilde f(e^{2\pi i z})=f(z)$ for every $z\in\mathbb{H}$. <u>Proof</u>: Define in $\mathbb{H}$ the following relation, $z_1\sim z_2$ if $z_1-z_2\in\mathbb{Z}$. Let $\mathbb{H}/\mathbb{Z}$ abusively denote the quotient set. Let $g:\mathbb{C}/\mathbb{Z}\to\mathbb{C}\setminus\{0\}$ bijective and well defined

$z+\mathbb{Z}\longmapsto e^{i2\pi z}$

legit quotient group.





Note $\mathbb{H}/\mathbb{Z}$ canonically identified with a sub group of $\mathbb{C}/\mathbb{Z}$. The image of $\mathbb{H}/\mathbb{Z}$ under $g$ is $\{e^{-2\pi y}e^{2\pi i x} : 0<y<\infty,\ -0.5<x<0.5\}=$ Unit disk $\setminus\{0\}$.

Let $h$: Unit disk $\setminus\{0\}\longrightarrow\mathbb{H}/\mathbb{Z}$ the inverse. Let $\bar f:\mathbb{H}/\mathbb{Z}\longrightarrow\mathbb{C}$ well defined

$z+\mathbb{Z}\longmapsto f(z)$

(seen as a subset of $\mathbb{C}/\mathbb{Z}$)

Let $\tilde f=\bar f\circ h$ : Unit disk $\setminus\{0\}\longmapsto\mathbb{C}$ and if $z\in\mathbb{H}$, $e^{2\pi i z}\in$ Unit disk $\setminus\{0\}$ so $\tilde f(e^{2\pi i z})=\bar f(h(e^{2\pi i z}))=f(z)$.

$q\longmapsto\tilde f(q)$

It is an exercise in complex analysis to see that if $f$ holom/meromorphic in $\mathbb{H}$ then $\tilde f$ holom/meromorphic in Unit disk $\setminus\{0\}$.

(Note $\widetilde{fg}=\overline{fg}\circ h=\bar f h\,\bar g h=\tilde f\tilde g$)

DEF Let $f : \mathbb{H} \to \mathbb{C}$ meromorphic such that $f(z+1) = f(z)$, then let $\tilde{f}$ : unit disk $\setminus \{0\} \to \mathbb{C}$ as above; if $\tilde{f}$ has a removable singularity at $0$ we say $f$ is holomorphic at $i\infty$.
If $\tilde{f}$ has a pole or removable sing at $0$ then we say $f$ is meromorphic at $i\infty$. This means that $\tilde{f}$ admits a Laurent expansion in a nbgh of the origin $\tilde{f}(q) = \sum_{-\infty}^{\infty} a_n q^n$ with $a_n = 0$ $\forall n < -N$ for some $N \in \mathbb{N}$.
($f$ has a pole, rem sing, ... at $i\infty$ when $\tilde{f}$ has such thing at $0$)

(idea, meromorphic at $\infty$ we look at $f(1/z)$ so we care

about all directions going to infinity;

here we just care about

)

DEF A weakly modular function is called modular function if it is meromorphic at $i\infty$. When $f$ is holomorphic at $i\infty$ we set $f(i\infty) = \tilde{f}(0)$. "the value of $f$ at $i\infty$"

A modular function holomorphic in $\mathbb{H}$ and $i\infty$ is called a modular form. If $f(i\infty) = 0$ is called a cusp form.

Observation A modular form of weight $2k$ is given by a series $f(z) = \sum_{n=0}^{\infty} a_n q^n =$ (here we're using the thm holom implies analytic; thm 9 ch 2)
$= \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ converging $\forall q : |q| < 1 \;\; \vdots \;\; \mathrm{Im} z > 0$.
(recall Hadamard's thm; it converges absolutely there) and uniformly on compact sets)
verifying $f(-1/z) = z^{2k} f(z)$ $\forall z \in \mathbb{H}$; it is cusp if $a_0 = 0$.

Examples: i) $f, g$ are modular forms of weight $2k, 2m$. Then $fg$ is a modular form of weight $2k+2m$
ii) We'll see that $q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$ is a cusp form of weight 12

## Lattice functions and modular forms. Identifications.

DEF Let $V$ be a finite dimensional real vector space. (It is isomorphic to $\mathbb{R}^n$ by fixing a basis; with this isom we can give a topology on $V$ by declaring open sets to be preimages of open sets in $\mathbb{R}^n$ with usual topology; the topology is independent of the chosen basis. We now have easily that $(V,+)$ is a topological group). A subgroup $\Gamma$ of $V$ is said to be a lattice if it verifies one of the following equivalent conditions

i) $\Gamma$ is discrete and $V/\Gamma$ compact topological space (quotient topology)

ii) $\Gamma$ discrete and generates the $\mathbb{R}$ vs $V$.

iii) $\exists \{e_1, \dots, e_n\}$ $\mathbb{R}$-basis which is a $\mathbb{Z}$-basis of $\Gamma$ (ie $\Gamma = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$ internal)

Consider $\mathbb{C}$ as a $\mathbb{R}$ vector space, let $\mathcal{R} = \{\text{lattices of } \mathbb{C}\}$, $M = \{(w_1,w_2)\in \mathbb{C}^*\times\mathbb{C}^* : \mathrm{Im}(w_1/w_2)>0\}$

makes sure that $w_1, w_2$ are $\mathbb{R}$-l.i.

We define the map $\Gamma : M \longrightarrow \mathcal{R}$ surjective

kind of abuse

$$(w_1,w_2) \longmapsto \Gamma((w_1,w_2)) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$$

It is a lattice by cond iii.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $g\in SL(2,\mathbb{Z})$, $(w_1,w_2)\in M$. Let $g\cdot(w_1,w_2) := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^t = (\overbrace{aw_1+bw_2}^{"w_1'"}, \overbrace{cw_1+dw_2}^{"w_2'"})$

Clearly $hg\cdot(w_1,w_2) = h\cdot(g\cdot(w_1,w_2))$. It is easy to check $\{w_1',w_2'\}$ $\mathbb{Z}$ basis of $\Gamma((w_1,w_2))$ and also if we let

$z = w_1/w_2$, $z' = w_1'/w_2'$ then $z' = g\cdot z \in \mathbb{H}$, therefore $(w_1',w_2')\in M$. Thus $SL(2,\mathbb{Z})$ acts on $M$.

action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$

From what we've said two elements in the same orbit have equal images in our map $\Gamma$

Conversely if $(w_1,w_2),(w_1',w_2')\in M$ and $\Gamma((w_1,w_2)) = \Gamma((w_1',w_2'))$ then by linear algebra

$\exists\, g\in SL_2(\mathbb{Z}) : g\cdot(w_1,w_2) = (w_1',w_2')$. (both change of basis matrix have integer coefficients this means that the matrix change of basis $g\cdot(w_1,w_2) = (w_1',w_2')$ has det $= \pm 1$ if det $g = -1$ one sees by computation $\mathrm{Im}(w_1/w_2)$ has opposite sign to $\mathrm{Im}(w_1'/w_2')$ ↯ )

We have proved

<u>Corollary</u> Two elements of $M$ have the same image via $\Gamma$ (define same lattice) iff they are in the same orbit under the action of $SL_2(\mathbb{Z})$. So we identify using this map the set of orbits of $M \circlearrowleft SL_2(\mathbb{Z})$ with $\mathcal{R}$. (meaning we have a natural bijection and we will move from one to another; if we have some $m\in M$ we think of it as a lattice... )

Let $\mathbb{C}^*$ act on $\mathcal{R}$ (resp on $M$) by $\lambda\cdot\Gamma = \lambda\Gamma$ (resp $\lambda\cdot(w_1,w_2) = (\lambda w_1, \lambda w_2)$) $\forall \lambda\in\mathbb{C}^*$

(it acts in $M/SL_2(\mathbb{Z})$, the quotient set (the action induces an equivalence relation...) and all of these are compatible ... obvious meaning)

$M/\mathbb{C}^* \xleftrightarrow{\Psi} \mathbb{H}$ well defined bijection, now $SL_2(\mathbb{Z})$ acts on $M/\mathbb{C}^*$

$(w_1,w_2) \longmapsto z = w_1/w_2$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g}\cdot(w_1,w_2) = (aw_1+bw_2,\ cw_1+dw_2)$$

$$\cdot\ \lambda(w_1,w_2) = \lambda(aw_1+bw_2,\ cw_1+dw_2).$$

Now let $g\in SL_2(\mathbb{Z})$, $\Psi(g\cdot(w_1,w_2)) = g\cdot\Psi(w_1,w_2)$ with $g$ seen as a member of $SL_2(\mathbb{Z})$.

$$\frac{aw_1+bw_2}{cw_1+dw_2} \qquad\qquad \frac{a\,w_1/w_2+b}{c\,w_1/w_2+d}$$

"So $\Psi$ is translating the action of $SL_2(\mathbb{Z})$ on $M/\mathbb{C}^*$ to the action of $G$ on $\mathbb{H}$."

Thus $\dfrac{M/\mathbb{C}^*}{SL_2(\mathbb{Z})} \underset{\text{natural bij}}{\longleftrightarrow} \mathbb{H}/G$

But $\dfrac{M/\mathbb{C}^*}{SL_2(\mathbb{Z})} \underset{\text{bijection}}{\longleftrightarrow} \mathcal{R}/\mathbb{C}^*$ (this is intuitive and natural via the following; a bit abusive)

$$\underbrace{\lambda(w_1, w_2)}_{\substack{\cdot \text{ denote an elmt of } M/\mathbb{C}^* \\ \cdot \text{ taken as a rep for the further ident}}} \longrightarrow \lambda\Gamma(w_1, w_2) \quad \text{(seen as an orbit of the action of } \mathbb{C}^* \text{ on } \mathcal{R}\text{)}$$

Therefore we've obtained an explicit bijection $\mathcal{R}/\mathbb{C}^* \longleftrightarrow H/G$; so an elmt of $H/G$ can be identified with a lattice in $\mathbb{C}$ defined up to multiplying it by a nonzero scalar. (this is just reading the first sentence)

We now explore a way to get modular functions.

Let $F : \mathcal{R} \to \mathbb{C}$ a function, $k \in \mathbb{Z}$. We say that **F is of weight 2k** if $F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma)$ $\forall \Gamma \in \mathcal{R}$, $\lambda \in \mathbb{C}^*$.

Let $F : \mathcal{R} \to \mathbb{C}$ of weight $2k$, let $(w_1, w_2) \in M$. We define
$$F : M \to \mathbb{C}, \quad (w_1, w_2) \longmapsto F((w_1, w_2)) := F(\Gamma((w_1, w_2)))$$

Since $F$ is of weight $2k$ this translates to our second $F$ as

$$F((\lambda w_1, \lambda w_2)) = \lambda^{-2k} F((w_1, w_2)) \quad (a)$$

Suppose now $w_1/w_2 = w_1'/w_2'$. Then $w_2^{2k} F((w_1, w_2)) \overset{\text{by (a)}}{=} F((w_1/w_2, 1)) = F((w_1'/w_2', 1)) =$

$\underset{\text{by (a)}}{=} (w_2')^{2k} F((w_1', w_2'))$. Therefore $w_2^{2k} F((w_1, w_2))$ is a function of $w_1/w_2$ (which is in $\mathbb{H}$)

Thus $\exists f : \mathbb{H} \to \mathbb{C}$ : $F((w_1, w_2)) = w_2^{-2k} f(w_1/w_2)$ (b)

Note that $F((w_1, w_2)) = F(g \cdot (w_1, w_2))$ $\forall g \in SL_2(\mathbb{Z})$) ie invariant under the action.

Hence $w_2^{-2k} f(w_1/w_2) = (cw_1 + dw_2)^{-2k} f\left(\dfrac{aw_1 + bw_2}{cw_1 + dw_2}\right)$ $\forall (w_1, w_2) \in M$. Let $z \in \mathbb{H}$ take $(w_1, w_2) = (z, 1)$

So we get $f(z) = (cz + d)^{-2k} f\left(\dfrac{az + b}{cz + d}\right)$ $\forall z \in \mathbb{H}$. $\forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Conversely if we start with $f : \mathbb{H} \to \mathbb{C}$ satisfying ↑ let
$$F : M \to \mathbb{C}, \quad (w_1, w_2) \longmapsto w_2^{-2k} f(w_1/w_2)$$

invariant under the action of $SL_2(\mathbb{Z})$ and $F((\lambda w_1, \lambda w_2)) = \lambda^{-2k} F((w_1, w_2))$

so let $F : \mathcal{R} \to \mathbb{C}$
$$\Gamma_0 \longmapsto F(\Gamma_0) := F((w_1, w_2))$$
with $(w_1, w_2)$ any elmt in $M$ such that $\Gamma((w_1, w_2)) = \Gamma_0$
($\Gamma$ as defined in the discussion).

So we can identify (bijection) lattice functions of weight $2k$ and $\{f : \mathbb{H} \to \mathbb{C} : f(z) = (cz+d)^{-2k} f(g \cdot z) \ \forall g \in SL_2(\mathbb{Z}), \forall z \in \mathbb{H}\}$

In particular modular functions of weight $2k$ are identified with SOME lattice functions of weight $2k$

The ones such that the $f(w_1/w_2)$ is meromorphic and meromorphic at infinity.

## Important example ; Eisenstein series.

(I will not check every analysis detail but I'll give idea of proof and be very clear.)

SOME TRIVIALITIES

In a Banach space if a series is absolutely convergent then is unconditionally convergent ; this means that it converges (to the same element) up to any reordering of the terms. This is essentially: Banach → (abs cv → cv) & same results in analysis 1.

Thus if we are in a Banach space $X$ and we have $\Gamma \subseteq X$ any countable subset then if we choose an ordering $\{\gamma_i\}_{i=1}^{\infty} = \Gamma$ of the elements and prove $\sum_{i=1}^{\infty} \|\gamma_i\| < \infty$ ; the quantity $\sum_{\gamma \in \Gamma} \gamma$ is well defined element of $X$ ; to express it as a limit one just picks an arbitrary ordering.

It is also clear from this that whenever one says $\sum_{\gamma \in \Gamma} \|\gamma\| < \infty$ it means that for any ordering this is $< \infty$ but enough to check for one precise ordering.

**Lemma** Let $\Gamma$ be a lattice in $\mathbb{C}$. $\sum'_{\gamma \in \Gamma} \frac{1}{|\gamma|^{\sigma}} < \infty$ for $\sigma > 2$ ($\sum'$ means summation over the $\gamma \in \Gamma \setminus \{0\}$)

About proof i) Majorize by some constant times $\iint_{\mathbb{R}^2 \setminus B_1(0)} \frac{1}{(x^2+y^2)^{\sigma/2}}$ ; this integral is easily computed with change to polar coordinates. ii) There is a more elementary approach L 1.5 Ch 9 Stein Shakarchi. □

. Let $k \in \mathbb{Z}_{>1}$. $G_k : \mathcal{R} = \{\text{lattices in } \mathbb{C}\} \longrightarrow \mathbb{C}$

$$\Gamma \longmapsto G_k(\Gamma) = \sum'_{\gamma \in \Gamma} \frac{1}{\gamma^{2k}}$$

i) Well defined by trivial remark and lemma.

ii) $G_k(\lambda\Gamma) = \sum'_{\lambda\gamma \in \lambda\Gamma} \frac{1}{(\lambda\gamma)^{2k}} = \frac{1}{\lambda^{2k}} \sum_{\gamma \in \Gamma} \frac{1}{\gamma^{2k}} = \lambda^{-2k} G_k(\Gamma)$ so $G_k$ is a lattice function of weight $2k$

We call it the Einsenstein series of index $k$ (sometimes called $2k$)

We can thus define $G_k : M \to \mathbb{C}$

$$(w_1, w_2) \longmapsto G_k((w_1, w_2)) := G_k((\Gamma((w_1, w_2))) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \frac{1}{(mw_1 + nw_2)^{2k}}$$

The function $f$ from the last discussion satisfies $f : \mathbb{H} \to \mathbb{C}$

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq\bar{0}}} \frac{1}{(mw_1+nw_2)^{2k}} = w_2^{2k} f(w_1/w_2) \qquad \forall (w_1,w_2)\in M$$

Hence $\underset{G_k}{f(z)} = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \frac{1}{(mz+n)^{2k}} \in \mathbb{C}$ (absolutely converged). We also call it $G_k$;

Proposition Let $k\in\mathbb{Z}_{>1}$. $G_k: \mathbb{H}\to\mathbb{C}$ is a modular form of weight $2k$. We have $G_k(i\infty) = 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$

ABOUT PROOF We have that $G_k(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$ $\forall z\in\mathbb{H}$, $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$

from above discussion so we only need to check that it is holomorphic in $\mathbb{H}$, at infinity and find its value

- Holomorphic in $\mathbb{H}$: Stein's proof shows that $\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \frac{1}{|mz+n|^{2k}}$ converges uniformly in $Im(z)\geq\delta>0$ $\forall\delta>0$.

(this is uniformly absolutely cv). This (in Banach spaces) easily implies $\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \frac{1}{(mz+n)^{2k}}$ unif cv.

Now by complex analysis $G_k: \mathbb{H}\to\mathbb{C}$ is holomorphic

↳ thm 16 in my ch2 notes ("sequence of holomorphic functions in $\Omega$ converging uniformly on compact subsets imply that the limit is holomorphic")

(from the meaning of $i\infty$)

- At $i\infty$? It should be intuitively clear that we want to see that $\lim_{Im(z)\to\infty} G_k(z)$ exists and compute its value. Since we have a weakly modular function we have periodicity so we just look at $D$, the fundamental domain. From the previous paragraph $\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \frac{1}{(mz+n)^{2k}}$ converges uniformly in $D$ so we can write

$$\lim_{Im(z)\to\infty} G_k(z) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \left(\lim_{Im(z)\to\infty}\frac{1}{(mz+n)^{2k}}\right) \underset{\downarrow}{=} \sum_{n\in\mathbb{Z}-\{0\}} 1/n^{2k} \underset{\downarrow}{=} 2\sum_{n=1}^{\infty} 1/n^{2k}.$$

- $m\neq 0$ we get $0$
- $m=0$ we get $1/n^{2k}$

we can sum in whatever order

Rmk: The $\widetilde{G_{2k}}(q)$ that we should have formally used to compute $G_k(i\infty)$ is given in Serre Ch7.4.2.

Examples: Let $G_2, G_3$ are modular forms of weights 4 and 6. The first prop of the section shows that $g_2 = 60\,G_2$, $g_3 = 140\,G_3$ are also modular forms of weight 4, 6.

Also $g_2(i\infty) \underset{\substack{\downarrow\\ \text{easy}}}{=} 60\,G_2(i\infty) = 120\sum_{n=1}^{\infty}\frac{1}{n^4} = 4/3\,\pi^4$

Known to be that but also proved in Harmonic analysis notes. (in very similar)

$$g_3(i\infty) = 140\,G_3(i\infty) = 240\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{8}{27}\pi^6.$$

If we let $\Delta = g_2^3 - 27 g_3^2$ holomorphic, $\Delta(z+1) = \Delta(z)$ and $\Delta(i\infty) = 0$ so cusp form of weight 12.

$\Delta(-1/z) = z^{12}\Delta(z)$

$\Delta(-1/z) = g_2(-1/z)^3 - 27 g_3(-1/z)^2 = z^{12} g_2(z)^3 - 27 z^{12} g_3(z)^2$

Remark The reason why we define $g_2, g_3$ has to do with theory of elliptic curves which will potentially be covered. (Rmk after prop 3 in Serre and last part of sec 7.2).

## 3. THE SPACE OF MODULAR FORMS.

We first study the zeros and poles of modular functions.

Notation We use the structure theorem of poles and the theorem "holomorphic near a zero" to establish a cavar notation: Let $f$ be meromorphic on $U \subseteq \mathbb{C}$ open, $p \in U$. $v_p(f)$ is by definition the $n \in \mathbb{Z}$: $f/(z-p)^n$ is holomorphic and nonzero at a nbgh of $p$. We call it order of $f$ at $p$. (mainly interested in the case $f: \mathbb{H} \to \mathbb{C}$.)

Suppose also that $f$ is a modular function we define the order of $f$ at $i\infty$ $v_\infty(f) := v_0(\tilde{f})$ where $\tilde{f}$ is the associated function to $f$ (defined in unit disk $-2\infty$).

(the two mentioned thms allow us to say it is well defined)

Observation When $f$ is a modular function of weight $2k$ we have that

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \text{so} \quad v_p(f) = v_{g\cdot p}(f) \quad \text{for } g \in G.$$

Proof/ Suppose $v_p(f) = n$. This means that $\frac{f(z)}{(z-p)^n} = \tilde{f}(z)$ in a nbgh of $p$ with $\tilde{f}$ holomorphic non vanishing at $p$.

Consider $\lim_{z\to g\cdot p} \frac{f(z)}{(z-g\cdot p)^n} = \lim_{z\to p} \frac{f(g\cdot z)}{(g\cdot z - g\cdot p)^n} = \lim_{z\to p} \frac{(cz+d)^{2k} f(z)}{(g\cdot z - g\cdot p)^n} = \lim_{z\to p} (cz+d)^{2k} \frac{f(z)}{(z-p)^n} (cz+d)^n (cp+d)^n =$

$$g\cdot z - g\cdot p = \frac{az+b}{cz+d} - \frac{ap+b}{cp+d} = \frac{(az+b)(cp+d) - (ap+b)(cz+d)}{(cz+d)(cp+d)} = \frac{(z-p)(ad-bc)}{(cz+d)(cp+d)} = \frac{z-p}{(cz+d)(cp+d)}$$

$= (cp+d)^{2(k+n)} \lim_{z\to p} \frac{f(z)}{(z-p)^n} \neq 0$.

$(cp+d)^{2(k+n)} \neq 0$ since $p \in \mathbb{H}$, $c,d \in \mathbb{Z}$

It now follows (with some easy arguments; maybe a bit annoying to write) that $v_{g\cdot p}(f) = n$.

recall that meromorphic functions have isolated poles / zeros.

Thus $v_p(f)$ only depends on the orbit of $p$ in $\mathbb{H}/G$. Finally if $p \in \mathbb{H}$ recall that $I(p) = \mathrm{Est}_G(p) = \{1, -1\}$ except $\begin{cases} p = i \\ p = e^{2\pi i/3} \\ p = e^{\pi i/3} \end{cases}$ in which case the estabilizer has $\begin{cases} 2 \\ 3 \\ 3 \end{cases}$ elements. We denote $e_p$ the order (# elements) of $I(p)$.

**Theorem 3** Let $f$ be a modular function of weight $2k$ not identically zero. Then

$$v_\infty(f) + \sum_{p \in H/G} \frac{1}{e_p} v_p(f) = k/6 \qquad \equiv v_{\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum^{*}_{p \in H/G} v_p(f) = \frac{k}{6}$$

$\hookrightarrow e^{2\pi i/3}$

where $\Sigma^*$ means summation over the points in $H/G$ distinct from the classes of $i$, $\rho$.

Proof/ Step 1 The sum $\sum_{p \in H/G} \frac{1}{e_p} v_p(f)$ has only finite number of nonzero terms.

Why? In $D$ $f$ has only finitely many poles or zeros. Why? $\exists \tilde{f}$ meromorphic at unit disk such that $\tilde{f}(e^{2\pi i z}) = f(z)$

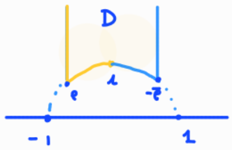
So $\tilde{f}$ has a removable singularity or pole at $0$; since zeros and poles of meromorphic functions are isolated $\exists\, r > 0$ : $\tilde{f}$ holomorphic at $0 < |q| < r$. This easily implies that $\exists\, c > 0$ : $f$ has no zeros or poles in $\operatorname{Im}(z) > c$. For $\operatorname{Im}(z) \leq c$, $z \in D$ we have a compact set so (since zeros and poles are isolated) we have finitely many

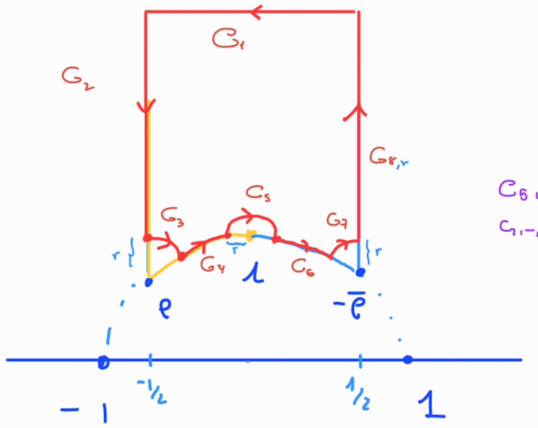
We now explain the equality.

CASE 1 : Suppose $f$ has not a zero or pole on the boundary of $D$ except possibly at $\rho, -\bar{\rho}$ or $i$. For a sufficiently large length we can get

($\rho$ and $-\bar{\rho}$ are congruent mod $G$)

a contour (toy contour) such that its interior contains a representative of each zero or pole of $f$ not congruent to $i$ or $\rho$




$C_5$ is a semicircle around $i$

$C_1, \dots, C_8$ carry a parameter $r$.

Let $\mathcal{C}_r$ denote this toy contour with the param. given by the picture by argument principle + def of $v_p(f)$

$$\frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{f'(z)}{f(z)} dz = \sum^{*}_{p \in H/G} v_p(f)$$

• We have $\displaystyle\int_{\mathcal{C}_1} \frac{f'(z)\, dz}{f(z)} = \int_{\mathcal{C}_{1,r}} \frac{\tilde{f}'(e^{2\pi i z})\, 2\pi i\, e^{2\pi i z}}{\tilde{f}(e^{2\pi i z})} dz = \int_{\mathcal{C}_1} \frac{\tilde{f}'(q(z))\, q'(z)}{\tilde{f}(q(z))} dz =$

$f(z) = \tilde{f}(e^{2\pi i z})$

call $q(z) = e^{i 2\pi z}$

$$= \int_{g(C_1)} \frac{\tilde{f}'(q)}{\tilde{f}(q)}\, dq = -2\pi i\, v_0(\tilde{f}) = -2\pi i\, v_{i\infty}(f) \quad ; \text{ so } \quad \frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)}\, dz = -v_{i\infty}(f).$$

def of integral

meaning that we integrate over the set $g(C_1)$ with param $= g \circ$ old param


compose old (clockwise) param with $g$ and get a circle with clockwise param.

Now we apply argument principle (note that for sufficiently large height we get sufficiently small circle so that $\tilde{f}$ has no zeros or poles besides the origin.

• Since $f(z) = f(z+1)$ we have $f'(z) = f'(z+1)$ so clearly $\int_{C_2} \frac{f'(z)}{f(z)}\, dz = \int_{C_2} \frac{f'(z+1)}{f(z+1)}\, dz = -\int_{C_8} \frac{f'(z)}{f(z)}\, dq$

---

Rmk If we denote by $C_{p,r,\alpha}$ an arc of a circle of radius $r$, angle $\alpha$ centered at $p \in \mathbb{C}$ if $f$ has a zero or pole at $P$ (meromorphic in a nbgh of $P$) we have


$$\lim_{r \to 0} \int_{C_{p,r,\alpha}} \frac{f'(z)}{f(z)}\, dz = \alpha i\, v_p(f)$$

(obvious param counterclockwise)

This is essentially taking this and doing an analogous residue thm for angle $\alpha$ and then get an argument principle from there (so quite intuitive)

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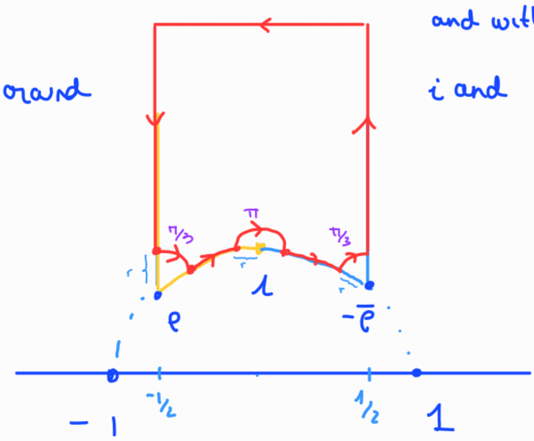
• From this remark and with the easy observation about angles (note $C_5$ has always been a semicircle around $i$ and $\rho = e^{i2\pi/3}$, $-\bar{\rho} = e^{i\pi/3}$




We get that the radius $r$ goes to $0$ we get $\lim_{r\to 0} \int_{C_5} \frac{f'(z)}{f(z)}\, dz = -\pi i\, v_i(f)$ ; $\lim_{r\to 0} \int_{C_3} \frac{f'(z)}{f(z)} = -\frac{\pi i}{3} v_\rho(f)$

; $\lim_{r\to 0} \int_{C_7} \frac{f'(z)}{f(z)} = -\frac{\pi i}{3} v_{-\bar{\rho}}(f) = -\frac{\pi i}{3} v_\rho(f)$ ($\rho, -\bar{\rho}$ are congruent mod $G$ )

• Finally for $C_4, C_6$. Consider $\overset{S(z)}{S\cdot z} = -1/z$ note it maps $C_6$ to $-C_4$ and also

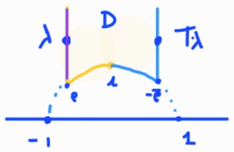
$f(z) = (z)^{-2k} f(S\cdot z)$. So $f'(z) = -2k\,(z)^{-2k-1} f(-1/z) + z^{-2k} f'(-1/z)\,(-1/z^2)$

So $\displaystyle\int_{C_4} \frac{f'(z)}{f(z)}\,dz = \int_{C_4} \frac{-2k}{z}\,dz + \int_{C_4} \frac{f'(S(z))\,S'(z)}{f(S(z))}\,dz \underset{\text{easy}}{=} \frac{2\pi i k}{6} - \int_{C_6} \frac{f'(u)}{f(u)}\,du$

So $\displaystyle\int_{C_4} \frac{f'(z)}{f(z)}\,dz + \int_{C_6} \frac{f'(z)}{f(z)}\,dz = 2\pi i \frac{k}{6}$

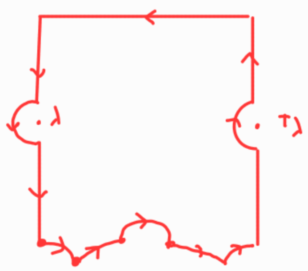
Thus $\displaystyle\sum^{*}_{p\in H/G} v_p(f) = \lim_{r\to 0} \frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i} \lim_{r\to 0} \sum_{i=1}^{8} \int_{C_i} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Big(-2\pi i\, v_\infty(f) - \pi i\, v_i(f) - \frac{2\pi i}{3} v_\rho(f) + 2\pi i \frac{k}{6}\Big) = -v_\infty(f) - \frac{1}{2} v_i(f) - \frac{1}{3} v_\rho(f) + k/6$ as wanted.

CASE 2




If $f$ has a zero or pole on the half line $\{z : \mathrm{Re}(z) = -1/2,\ \mathrm{Im}(z) > \sqrt{3}/2\}$

then $v_{T\lambda}(f) = v_\lambda(f)$ so we also have the same so we consider




and with a bit more detail we have the same result (repeating the argument)

CASE 3 We proceed in an analogous way if we have more zeros or poles at $\partial D$. (finitely many always)

$\square$

Remark One can define a structure of complex analytic manifold on $\widehat{H/G}$ (the "compactification") and the proof is easier.

## The algebra of modular forms.

It is clear that weakly modular functions of weight $2k$ $(k\in\mathbb{Z})$ form a vector space over $\mathbb{C}$ (from the 1st prop of sec. 2)
modular forms and cusp forms (of weight $2k$) are $\mathbb{C}$-subspaces that we will denote by $M_k, M_k^0$ respectively.
$(M_k^0 \leq M_k)$

Our goal is to understand this vector spaces better; for example know dimension and basis.

Notes i) $M_k \xrightarrow{\Psi} \mathbb{C}$, $f \longmapsto f(i\infty)$ then $M_k^0 = \ker \Psi$ ($\Psi$ linear form)

ii) $\dim_{\mathbb{C}}(M_k/M_k^0) \leq 1$. ✓

iii) For $k \geq 2$ $G_k \in M_k$ and $G_k(i\infty) \neq 0$ (by last prop of sec 2) thus for $k\geq 2$

$$M_k = M_k^0 \oplus \underbrace{\mathbb{C}\cdot G_k}_{\mathbb{C}\text{-span of } G_k}.$$

**Theorem 4** i) $M_k = 0$ for $k<0$, $k=1$

ii) For $k=0,2,3,4,5$, $\dim M_k = 1$ with basis $1, G_2, G_3, G_4, G_5$ respectively; In these cases $M_k^0 = 0$

iii) $M_{k-6} \longrightarrow M_k^0$, $f \longmapsto f\cdot\Delta$ isomorphism. (recall $\Delta = g_2^3 - 27g_3^2 \in M_6^0$ ; $g_2 = 60G_2$, $g_3 = 140G_3$)

Proof/ i) Let $f \in M_k \setminus \{0\}$. We have $v_{i\infty}(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum^{*}_{p\in H/G} v_p(f) = k/6$

Since $f$ modular form all the terms on LHS are $\geq 0$. So $k \geq 0$. Also $k\neq 1$ because $1/6$ can't be
(holomorphic so no poles)

written as $n + n'/2 + n''/3 + n'''$ with $n, n', n'', n''' \geq 0$ integers

iii) Some previous observations

Claim 1 $v_\rho(G_2) = 1$, $v_p(G_2) = 0$ for $p \neq \rho \bmod G$.
Applying the expression from thm 3 to $G_2$ we get $\frac{2}{6} = n + n'/2 + n''/3 + n'''$ with $n, n', n'', n''' \in \mathbb{Z}_{\geq 0}$
But the only solution to this is $n = 0 = n' = n'''$, $n'' = 1$.

Claim 2 $v_i(G_3) = 1$, $v_p(G_3) = 0$ $p \neq i \bmod G$.
Same argument.

Claim 3 $\Delta$ does not vanish on $\mathbb{H}$ and has a simple zero at $i\infty$.

$\Delta(i) = g_2^3(i) - 27g_3^2(i) \neq 0$. So $\Delta$ non identically $0$.
$g_3^2(i)$: $0$; $v_i(G_2) = 0$.

Note that $\Delta$ has weight 12, and it is a cusp form so has a 0 at $i\infty$ thus $v_{i\infty}(\Delta) \geq 1$
If we apply the formula from thm 3 we easily see $v_{i\infty}(\Delta)$ has to be 1 and $v_p(\Delta) = 0$ $p \in H/G$.

From the 1st prop of sec 2 it is clear that if $f$ weakly modular of weight $2k$, $1/f$ is weakly modular of weight $-2k$. This combined with the fact that the product of weakly modular of weights $2k, 2m$ is weakly modular of weight $2k+2m$ (by prop in sec 2) yields that if $f \in M_k^0$ and we set $g = \frac{f}{\Delta}$, $g$ is of weight $2k - 12 = 2(k-6)$. Moreover

$$v_p(g) \underset{\underset{\text{easy}}{\downarrow}}{=} v_p(f) - v_p(\Delta) = \begin{cases} v_p(f) & p \neq i\infty \\ v_p(f) - 1 & p = i\infty \end{cases}$$

Shows $v_p(g) \geq 0$ $\forall p$ so $g \in M_{k-6}$ (it holomorphic everywhere). We now prove iii

$$\begin{aligned} M_{k-6} &\xrightarrow{\psi} M_k^0 \\ g &\longmapsto g\Delta \end{aligned}$$

- $g\Delta$ is clearly a cusp form and the weight is $2(k-6)+12 = 2k$ so $g\Delta \in M_k^0$
- linear ✓
- 1-1 ✓
- Surj? If $f \in M_k^0$ were seen $f/\Delta \in M_{k-6}$ so surjective.

ii) For $k = 0,2,3,4,5$, $k-6 < 0$. Thus $M_{k-6} = 0$ by i) and by 3 $M_k^0 = 0$. This shows by note ii, $\dim M_k \leq 1$. But $\underset{\in M_0}{1}, \underset{\in M_2}{G_2}, \underset{\in M_3}{G_3}, \underset{\in M_4}{G_4}, \underset{\in M_5}{G_5} \neq 0$ so $\dim M_k = 1$ and ii is clear. □

**Corollary** i) We have $\dim M_k = \begin{cases} [k/6] & \text{if } k \equiv 1 \bmod 6,\ k \geq 0 \\ [k/6] + 1 & \text{if } k \not\equiv 1 \bmod 6,\ k \geq 0 \end{cases}$ ($[x]$ is the largest $n \in \mathbb{Z}$ : $n \leq x$)

ii) $\{G_2^\alpha G_3^\beta : \alpha, \beta \in \mathbb{Z}_{\geq 0},\ 2\alpha + 3\beta = k\}$ is a $\mathbb{C}$-basis for $M_k$.

Notes i) This gives a complete classification of modular forms (we have a ring which is a $\mathbb{C}$ vspace; and graded bearing)

ii) Let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ direct sum note it has a structure of a graded algebra $f \in M_{k_1}, g \in M_{k_2}$ then $fg \in M_{k_1+k_2}$ by the last example before we talked about lattices. The assertion ii) implies that

$M \cong \mathbb{C}[G_1, G_2]$ — $\cong$: as a graded algebra if we wait; $\mathbb{C}[G_1, G_2]$: polynomial algebra

Proof / i) By i) ii) thm 4 it holds for $0 \leq k < 6$. Now recall that for $k \geq 6$; $M_k = M_k^0 \oplus \mathbb{C} \cdot G_k$ (this was for $k \geq 2$)

So $\dim M_k = 1 + \dim M_k^0 = 1 + \dim M_{k-6} \underset{\underset{\text{induction}}{\downarrow}}{=} \begin{cases} 1 + [k/6 - 1] = [k/6] & \text{if } k \equiv 1 \bmod 6 \\ 1 + [\frac{k-6}{6}] + 1 = 1 + [k/6] & \text{if } k \not\equiv 1 \bmod 6 \end{cases}$

ii) $\{G_2^\alpha G_3^\beta : \beta, \alpha \in \mathbb{Z}_{\geq 0}\ 2\alpha + 3\beta = k\}$ generate $M_k$.

For $k \leq 3$ follows from i) ii) thm 4. For $k \geq 4$ we argue by induction

Write $2\gamma + 3\delta = k$ , $(\gamma, \delta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ ( for $k \geq 2$ we can do it ; if $k$ even ✓ if odd $k-3$ even so ✓ )

Consider the modular form $g = G_2^{\gamma} G_3^{\delta} \in M_{2\gamma + 3\delta} = M_k$ ; of course is nonzero at infinity

Let $f \in M_k$ , of course $\exists \lambda \in \mathbb{C}$ : $f - \lambda g \in M_k^0$ cusp form. By thm 4 iii , $f - \lambda g = \Delta h$ for $h \in M_{k-6}$

Apply inductive hypothesis to $h$ and we are done (easy check) □

$\{ G_2^{\alpha} G_3^{\beta} : \beta, \alpha \in \mathbb{Z}_{\geq 0} \; 2\alpha + 3\beta = k \} \overset{=\mathcal{A}}{}$ li. (of course this means li ; generating set with appropriate dim )

By induction on $k$ we show that there are exactly $\dim M_k$ monomials in $\mathcal{A}$ . For $k < 6$ we are fine.

Now if $k \geq 6$ a pair $(\tilde{\alpha}, \tilde{\beta}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ st $2\tilde{\alpha} + 3\tilde{\beta} = k - 6$ gives $(\alpha, \beta)$ : $2\alpha + 3\beta = 2\tilde{\alpha} + 3\tilde{\beta} + 2 \cdot 3 = k$
$\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta} + 2$

so we get $\dim M_{k-6}$ many monomials $G_2^{\alpha} G_3^{\beta}$ in $\mathcal{A}$

($\dim M_{k-6} = \dim M_k - 1$.) All the elements in $\mathcal{A}$ come from such pairs except if we can write $k = 2\alpha$ or $k = 2\alpha + 3$ ; Exactly one of these always happens so we get that $|\mathcal{A}| = \dim M_{k-6} + 1 = \dim M_k$ and thus it follows that it is li □

□

## The modular invariant

Let $j = \dfrac{1728\, g_2^3}{\Delta}$

<u>Proposition</u> i) $j$ is a modular function of weight 0

ii) $j$ is holomorphic in $\mathbb{H}$ and has a simple pole at $i\infty$

iii) Induces a bijection $\mathbb{H}/G \xrightarrow{j} \mathbb{C}$ .

Proof/ i) $g_2^3$ , $\Delta$ are modular functions of weight 12 (now argue as in the proof of thm 4.iii )

ii) $\Delta$ is $\neq 0$ in $\mathbb{H}$ and has a simple zero at $i\infty$ by the proof of thm 4 ; $g_2$ holomorphic everywhere

iii) The map is bijective iff $\forall \lambda \in \mathbb{C}$ , $f_\lambda = 1728\, g_2^3 - \lambda \Delta$ has a unique 0 mod $G$ .

$f_\lambda$ is a modular form of weight 12 thus $\underset{\geq 0}{v_{i\infty}(f_\lambda)} + \frac{1}{2} \underset{\geq 0}{v_i(f_\lambda)} + \frac{1}{3} \underset{\geq 0}{v_\rho(f_\lambda)} + n = 1$

The only possibilities correspond to $(v_{i\infty}(f_\lambda), v_i(f_\lambda), v_\rho(f_\lambda)) = (1,0,0)$ or $(0,2,0)$ or $(0,0,3)$ or $(0,0,0)$ and $n = 1$. This shows $f_\lambda$ has a unique 0 in $\mathbb{H}/G$ . □

<u>Proposition</u> (Characterization of modular functions of weight 0) Let $f$ be meromorphic on $\mathbb{H}$ . TFAE

i) $f$ is a modular function of weight 0

ii) $f$ is a quotient of two modular forms of the same weight

iii) $f$ is a rational function of $j$ .

Proof / iii) → ii) → i) unclear

We of course need to remind ourselves that:
- product of two different weights gives weight = sum of weights
- $\tilde{f}\tilde{g} = \widetilde{fg}$ so that we can make assertions about blw at $i\infty$. (sometimes)
- $1/f$ has weight - weight $f$

this holds for weakly modular functions. Then you argue that they are whatever you need.

We show i → iii).

WMA: $f$ holomorphic on $\mathbb{H}$.

If not multiply $f$ by a suitable pol in $j$; now new $f$ is holomorphic on $\mathbb{H}$ and by the following proof rational function of $j$, so $f$ too.

Since $\Delta$ is 0 at $\infty$, $\exists n \in \mathbb{N}$: $g = \Delta^n f$ is holomorphic at $\infty$. $g$ is of course a modular form and has weight $12n$. By the second part of the last corollary $g$ is a $\mathbb{C}$-lin combo of $G_2^\alpha G_3^\beta$ $2\alpha + 3\beta = 6n$. Of course we may assume $g = G_2^\alpha G_3^\beta$ (if not apply what we are about to do for each summand) Recall $6n = 2\alpha + 3\beta$, thus $p = \alpha/3, q = \beta/2 \in \mathbb{Z}$ and

$\mathbb{Z} \ni \alpha = \frac{6n - 3\beta}{2}$ but this is further div. by 3. (same for $q$).

$$f = \frac{G_2^\alpha G_3^\beta}{\Delta^n} = \frac{G_2^{3p} G_3^{2q}}{\Delta^{p+q}}$$

$$n = \frac{2\alpha + 3\beta}{6} = \frac{\alpha}{3} + \frac{\beta}{2} = p + q$$

So NTS $\frac{G_2^3}{\Delta}$, $\frac{G_3^2}{\Delta}$ are rational functions of $j$, this is obvious.

$$G_2^3 = \frac{g_2^3}{60^3} \quad \text{so} \quad \frac{G_2^3}{\Delta} = \frac{1}{60^3} \frac{j}{1728} \quad ✓$$

$$G_3^2 = \frac{g_3^2}{140^2} = \frac{\Delta - g_2^3}{(-27)140^2} = \frac{\Delta - \frac{j\Delta}{1728}}{(-27)140^2} = \frac{1728\Delta - j\Delta}{(-27)(1728)(140)^2} \; ; \text{ so } \frac{G_3^2}{\Delta} = \frac{1728 - j}{(-27)(140)^2(1728)} \quad ✓$$

□

<u>Remark</u> The coefficient 1728 has been introduced so that the Laurent expansion of $\tilde{j}(q)$ where $j(z) = \tilde{j}(e^{2\pi i z})$ has residue 1 at 0, more precisely

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n \;, \quad z \in \mathbb{H}, \; q = e^{2\pi i z}$$

Moreover $c(1) = 196884$, $c(2) = 21493760$, $c(3) = 864299970$

Now if we consider the monster group and we look at the list of degrees of irred $\mathbb{C}$-rep (194 irreps) we get $1, 196883, 21296876, 842609326$. The crazy part is that

$196884 = 1 + 196883$
$21493760 = 1 + 196883 + 21296876$
$864299970 = 2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326$

("Monstrous Moonshine conjecture")