Serre Duality

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Theorem 15.1

Suppose $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is exact sequence of sheaves on the topological space X such that $H^1(X, \mathcal{G}) = 0$. Then

$$H^1(X, \mathcal{F}) \cong \mathcal{H}(X)/\beta\mathcal{G}(X)$$

Proof. By the long exact sequence from before, we get the exact sequence

$$\mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to 0$$

Then by the first isomorphism theorem it's evident the surjection $\mathcal{H} \to H^1(X, \mathcal{F})$ becomes

$$H^1(X, \mathcal{F}) \cong \mathcal{H}(X)/\beta\mathcal{G}(X)$$

■

Theorem 16.2

Suppose X is compact RS and $D \in \mathrm{Div}(X)$ is a divisor with $\deg(D) < 0$. Then $H^0(X, \mathcal{O}_D) = 0$

Theorem 16.3

Let $D \leq D'$ be divisors on a compact RS X, then the inclusion map $\mathcal{O}_D \to \mathcal{O}_{D'}$ induces an epimorphism

$$H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0$$

16.1 Definition of a linear form $\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}$

Let X be a compact RS, then by (15.14), we have that the exact sequence

$$0 \to \Omega \to \mathcal{E}^{1,0} \to \mathcal{E}^{(2)} \to 0$$

induces the isomorphism

$$H^1(X, \Omega) \cong {\mathcal{E}^{(2)}}\big/{\mathrm{d}\mathcal{E}^{1,0}}$$

Let $\xi \in H^1(X, \Omega)$ and $\omega \in \mathcal{E}^{(2)}(X)$ be a representative of ξ through this isomorphism... Set

$$\mathrm{Res}(\xi) \equiv \frac{1}{2\pi i} \iint_X \omega$$

and by theorem (10.20) this is independent of choice of ω

17.2 Mittag-Leffler Distributions of Differential Forms

Now let X be a RS, $\mathcal{M}^{(1)}$ the sheaf of meromorphic 1-forms on X, $\mathfrak{U} = (U_i)_{i \in I}$ an open cover of X.

Definition 4

A cochain $\mu = (\omega_i) \in C^0(\mathfrak{U}, \mathcal{M}^{(1)})$ is called a **Mittag-Leffler distribution** if the differences $\omega_j - \omega_i$ are holomorphic on $U_i \cap U_j$ i.e $\delta\mu \in Z^1(\mathfrak{U}, \Omega)$as

$$\delta\mu = \omega_j - \omega_i$$
$$\delta\delta\mu = (\omega_j - \omega_k) + (\omega_k - \omega_i) + (\omega_i - \omega_j) = 0$$

Denote $[\delta\mu] \in H^1(X, \Omega)$ the cohomology class of $\delta\mu$.

Definition 5

Let $a \in X$ then the **residue** of the Mittag-Leffler distribution $\mu = (\omega_i)$ at a is defined by choosing $i \in I$ such that $a \in U_i$ and set

$$\operatorname{Res}_a(\mu) \coloneqq \operatorname{Res}_a(\omega_i)$$

where $\operatorname{Res}_a(\omega_i) = c_{-1}$ the principal part of the in the Laurent series of expansion of f given by $\omega_i = f_i \mathrm{d}z$

With this definition, if $a \in U_i \cap U_j$, the difference $\omega_i - \omega_j$ is holomorphic and $\operatorname{Res}_a(\omega_i) = \operatorname{Res}_a(\omega_j)$

Theorem 17.6

Given the definitions above,

$$\operatorname{Res}(\mu) = \operatorname{Res}([\delta\mu])$$

Proof. omitted for sake of time ■

17.4 The Sheaves Ω_D

Let X be a compact R, $D \in \operatorname{Div}(X)$ then

Definition 7

We define the sheaf of meromorphic 1-forms which are multiples of $-D$ by

$$\Omega_D \coloneqq \{\omega \in \mathcal{M}^{(1)} \mid \operatorname{ord}_x(\omega) + D(x) \geq 0 \, \forall x \in U\}$$

In particular, $\Omega_0 = \Omega$ is just the sheaf of holomorphic 1-forms.

Suppose $\omega \in \mathcal{M}^{(1)}(X)$ is non trivial on X i.e $\omega = \mathrm{d}f$ where $f \in \mathcal{M}^{(1)}$ is a non-constant meromorphic function. Let K be the divisor of ω i.e

$$K : x \mapsto \mathrm{ord}_x \omega = \mathrm{ord}_x f$$

when $\omega = f\mathrm{d}z$ locally on a neighborhood U of a. Then for arbitrary $D \in \mathrm{Div}(X)$, mulitplication by ω induces a sheaf isomorphism

$$\begin{aligned} \mathcal{O}_{D+K} &\tilde{\to} \Omega_D \\ f &\mapsto f\omega \end{aligned}$$

Lemma 8

There is a constant $k_0 \in \mathbb{Z}$ such that

$$\dim H^0(X, \Omega_D) \geq \mathrm{deg} D + k_0$$

for every $D \in \mathrm{Div}(X)$

Proof. Suppose $w \in \mathcal{M}^{(1)}(X)$ is non-trivial, K is a divisor of ω, $g$is the genus. Set $k_0 := 1 - g + \mathrm{deg} K$, then by Riemann-Roch

$$\begin{aligned} \dim H^0(X, \Omega_D) &= \dim H^0(X, \mathcal{O}_{D+K}) \\ &= \dim H^1(X, \mathcal{O}_{D+K}) + 1 - g + \mathrm{deg}(D+K) \\ &= \dim H^1(X, \mathcal{O}_{D+K}) + 1 - g + \mathrm{deg} D + \mathrm{deg} K \\ &\geq \mathrm{deg} D + k_0 \end{aligned}$$

■

17.5 Definition of Dual Pairing

Let X be a compact RS,$D \in \mathrm{Div}(X)$, then the product

$$\Omega_{-D} \times \mathcal{O}_D \to \Omega, \quad (\omega, f) \mapsto \omega f$$

induces a mapping in cohomology

$$H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \to H^1(X, \Omega)$$

By composing this map with $\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}$ we get

$$\begin{aligned} \langle , \rangle : H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) &\to \mathbb{C} \\ \langle \omega, \xi \rangle &:= \mathrm{Res}(\omega\xi) \end{aligned}$$

And so by considering it as operator which acts on the second factor, we get the induced map

$$\begin{aligned} s_D : H^0(X, \Omega_{-D}) &\to \times H^1(X, \mathcal{O}_D)^* \\ \omega &\mapsto \langle \omega, \rangle = \mathrm{Res}(\omega \_) \end{aligned}$$

and Serre Duality says that s_D is an isomorphism

Theorem 17.9

The mapping s_D is injective.

Proof. We must show for $\omega \in H^0(X, \Omega_{-D})$, there exists $\xi \in H^1(X, \mathcal{O}_D)$ such that $\langle \omega, \xi \rangle \neq 0$ (otherwise we would have a nontrivial kernel).
Let $a \in X$ such that $D(a) = 0$ and U_0, z is a coordinate neighborhood of a with $z(a) = 0$ and $D \mid_{U_0} = 0$. On U_0, $\omega = f\mathrm{d}z$ where $f \in \mathcal{O}_{U_0}$.
Assume U_0 is small enough so f has no zeroes in $U_0\{a\}$. Set $U_1 = X \setminus \{a\}$ and $\mathfrak{U} = (U_0, U_1)$.
Let $\eta = (f_0, f_1) \in C^0(\mathfrak{U}, \mathcal{M}^{(1)})$ where

$$f_0 = \frac{1}{zf}, \quad f_1 = 0$$

Then

$$\omega\eta = \left(\frac{dz}{z}, 0\right) \in C^0(\mathfrak{U}, \mathcal{M}^{(1)})$$

is a Mittag-Leffler distribution. By the definition of residue and noting $0 \in U_0$ means the residue is applied to $\frac{dz}{z}$

$$\operatorname{Res}(\omega\eta) = \sum_{a \in X} \operatorname{Res}_a \omega\eta = \operatorname{Res}_0 \omega_i = \operatorname{Res}_0 \frac{dz}{z} = 1$$

Now $\delta\eta = f_1 - f_0 = \frac{1}{zf} \in Z^1(\mathfrak{U}, \mathcal{O}_D)$ as it is holomorphic on $U_0 \cap U_1 = U_0 \setminus \{a\}$ and every coboundary is a cocycle. Let $\xi = [\delta\eta] \in H^1(X, \mathcal{O}_D)$ be the class of $\delta\eta$, then finally

$$\langle \omega, \xi \rangle = \operatorname{Res}(\omega\xi) = \operatorname{Res}([\delta(\omega\eta)]) = \operatorname{Res}(\omega\eta) = 1$$

where the first equality is by definition, the second because $\omega\xi = \omega \cdot [\delta\eta] = [\delta(\omega\eta)]$ and the third from (17.3) ∎

Suppose $D, D' \in \operatorname{Div}(X)$ are two divisors on compact RS X such that $D' \leq D$. Then by theorem (16.8) implies that the inclusion $0 \to \mathcal{O}_{D'} \to \mathcal{O}_D$ induces an epimorphism

$$H^1(X, \mathcal{O}_{D'}) \to H^1(X, \mathcal{O}_D) \to 0$$

By dualizing this sequence, we get a monomorphism

$$0 \to H^1(X, \mathcal{O}_D)^* \xrightarrow{i^D_{D'}} H^1(X, \mathcal{O}_{D'})^*$$

where this is just the restriction of a functional to a subdomain.

Then one can check that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{O}_D)^* \arrow[r, "i^D_{D'}"] & H^1(X, \mathcal{O}_{D'})^* \\
0 \arrow[r] & H^0(X, \Omega_{-D}) \arrow[r] \arrow[u, "s_D"] & H^0(X, \Omega_{-D'}) \arrow[u, "s_{D'}"']
\end{tikzcd}$$

Proof. Take $\omega \in H^0(X, \Omega_{-D})$, then $s_D(\omega) = \text{Res}(\omega\_)$ and $i^D_{D'}(s_D(\omega)) = \text{Res}(\omega\_) \mid_{\mathcal{O}'_D}$. On the otherhand, viewing ω as a cocycle consisting of 1-forms in the included sheaf $\Omega_{-D'}$, it is immediate that $s_{D'} = \text{Res}(\omega\_)\colon H^1(X, \mathcal{O}_{D'}) \to \mathbb{C}$. ■

Lemma 10

Suppose $\lambda \in H^1(X, \mathcal{O})^*$ and $\omega \in H^0(X, \Omega_{-D'})$ satsify

$$i^D_{D'} = s_{D'}(\omega)$$

Then $\omega \in H^0(X, \Omega_{-D})$ and $\lambda = s_D(\omega)$

Proof. Suppose for a contradiction $\omega \notin H^0(X, \Omega_{-D})$, i.e $\exists a \in X$ such that $\text{ord}_a(\omega) < D(a)$. Once again choose (U_0, z), a coordinate neighborhood of a with $z(a) = 0$ and $D \mid_{U_0} = 0$ (but now also $D' \mid_{U_0} = 0$). On U_0, $\omega = f\mathrm{d}z$ where $f \in \mathcal{M}(U_0)$. Assume U_0 is small enough so f has no zeroes or poles in $U_0\{a\}$.
Set $U_1 = X \setminus \{a\}$ and $\mathfrak{U} = (U_0, U_1)$. Let $\eta = (f_0, f_1) = (1/(fz), 0) \in C^0(\mathfrak{U}, \mathcal{M})$ and because $\text{ord}_a(\omega) < D(a)$ then more specifically $\eta \in C^0(\mathfrak{U}, \mathcal{O}_D)$. On the intersection $U_0 \cap U_1$, $\delta\eta = 1/(fz)$ is holomorphic and a multiple of both D, D' so that

$$\delta\eta \in Z^1(\mathfrak{U}, \mathcal{O}) = Z^1(\mathfrak{U}, \mathcal{O}_{\mathcal{D}}) = Z^1(\mathfrak{U}, \mathcal{O}_{\mathcal{D}'})$$

Denote the class of $[\delta\eta]$ as $\xi' \in H^1(X, \mathcal{O}_{D'})$ and $\xi \in H^1(X, \mathcal{O}_D)$.
Actually, $\xi = 0$ because $\delta\eta$ is a coboundary and so by hypothesis

$$\langle \omega, \xi' \rangle = i^D_{D'}(\lambda)(\xi') = \lambda(\xi) = 0$$

But then

$$1 = \text{Res}(\omega\eta) = \langle \omega, \xi' \rangle = \lambda(x') = 0$$

a contradiction and so in fact $\omega \in H^0(X, \Omega_{-D})$.
Finally, $i^D_{D'}(\lambda) = s_{D'}(\omega) = i^D_{D'}(\lambda)(s_D(\omega))$ where the second equality is due to commutativity of the diagram and the injectivity of the natural inclusion by restriction. ■

Now suppose $D, B \in \text{Div}(X)$ for X compact. Given a meromorphic function $\psi \in H^0(X, \mathcal{O}_B)$, the sheaf morphism

$$\mathcal{O}_{D-B} \xrightarrow{\psi} \mathcal{O}_D,\ f \mapsto \psi f$$

induces a linear mapping $H^1(X, \mathcal{O}_{D-B} \to H^1(X, \mathcal{O}_D))$ and thus a linear mapping

$$H^1(X, \mathcal{O}_D)^* \xrightarrow{\psi} H^1(X, \mathcal{O}_{D-B})^*$$

By definition, $(\psi\lambda)(\xi) = \lambda(\psi\xi)$ for $\xi \in H^1(X, \mathcal{O}_{D-B})$ and $\lambda \in H^1(X, \mathcal{O}_D)^*$, i.e. the pullback of λ by ψ. Then the following diagram commutes

$$\begin{array}{ccc} H^1(X, \mathcal{O}_D)^* & \xrightarrow{\psi} & H^1(X, \mathcal{O}_{D-B})^* \\ {\scriptstyle s_D}\uparrow & & \uparrow{\scriptstyle s_{D-B}} \\ H^0(X, \Omega_{-D}) & \xrightarrow{\psi} & H^0(X, \Omega_{-D+B}) \end{array}$$

since

$$\langle \psi\omega, \xi \rangle = \operatorname{Res}((\psi\omega)\xi) = \operatorname{Res}(\omega(\psi\xi)) = \langle \omega, \psi\xi \rangle$$

Lemma 11

If $\psi \in H^0(X, \mathcal{O}_B)$ is not the zero element, then the mapping

$$\psi : H^1(X, \mathcal{O}_D)^* \to H^1(X, \mathcal{O}_{D-B})^*$$

is injective

Proof. Let $A \coloneqq (\psi) \geq -B$ be the divisor of ψ.The mapping $\mathcal{O}_{D-B} \xrightarrow{\psi} \mathcal{O}_D$ factors

$$\begin{tikzcd} & \mathcal{O}_{D+A} \arrow[dr, "\psi\cdot"] & \\ \mathcal{O}_{D-B} \arrow[ur, "\text{incl}"] \arrow[rr, "\psi\cdot"'] & & \mathcal{O}_D \end{tikzcd}$$

where the top right map is an isomorphism because if $f \in \mathcal{O}_{D+A}(X)$ then

$$\begin{aligned} &\operatorname{ord}_x \psi f + D \\ \operatorname{ord}_x \psi + &\operatorname{ord}_x f + D \\ &\geq \operatorname{ord}_x \psi - A = 0 \end{aligned}$$

which is injective as $\psi f = \psi g \Leftrightarrow f = g$ and surjective becasuse if $g \in \mathcal{O}_D$ then it comes from $(1/\psi)g$. By 16.8, the inclusion $\mathcal{O}_{D-B} \to \mathcal{O}_{D+A}$ induces an epimorphism $H^1(X, \mathcal{O}_{D-B}) \to H^1(X, \mathcal{O}_{D+A})$ and the composition of an isomorphism and epimorphism is still an epimorphism it follows that

$$H^1(X, \mathcal{O}_{D-B}) \xrightarrow{\psi\cdot} H^1(X, \mathcal{O}_D)$$

is an epimorphism and dualizing gives the monomorphism. ■

Theorem 17.12: Serre Duality

For any divisor D on a compact RS X the mapping

$$s_D : H^0(X, \Omega_{-D}) \to H^1(X, \mathcal{O}_D)^*$$

is an isomorphism.

Proof. Injectivity is already done, so only need to show surjectivity. Suppose $0 \neq \lambda \in H^1(X, \mathcal{O}_D)^*$, we want to show that λ lies in the image of s_D.
Let P be a divisor such that $\deg(P) = 1$. For any $n \in \mathbb{N}$, define

$$D_n \coloneqq D - nP$$

We want this divisor because by taking n large enough, $H^0(X, \mathcal{O}_{D_n})$ will be trivial.
Denote $\Lambda \subset H^1(X, \mathcal{O}_{D_n})^*$ the vector subspace of all linear forms of the form $\psi\lambda$, where $\psi \in H^0(X, \mathcal{O}_{nP})$.

Lemma (17.8) says that the map

$$\psi : H^1(X, \mathcal{O}_D)^* \to H^1(X, \mathcal{O}_{D_n})^*$$

is injective. So, $\Lambda \cong H^0(X, \mathcal{O}_{nP})$.
By the R-R theorem,

$$\dim \Lambda \geq 1 - g + \deg(nP) = 1 - g + n\deg(P) \geq 1 - g + n$$

What we do is intersect Λ with the image of s_{D_n}, whence it will result this intersection is in the image of s_D.
Set $J = \mathrm{Im}(s_{D_n})$, then by 17.4 we have

$$\begin{aligned}\dim J = \dim H^0(X, \Omega_{-D_n}) &\geq k_0 + \deg(-D_n)\\ &= k_0 + n - \deg(D)\end{aligned}$$

The first equality follows since s_{D_n} is injective.
If $n > \deg(D)$ then $\deg(D_n) = \deg(D) - n < 0$ and thus by (16.5) $H^1(X, \mathcal{O}_{D_n}) = 0$.
Now, by the R-R

$$\dim H^1(X, \mathcal{O}_{D_n})^* = \dim H^1(X, \mathcal{O}_{D_n}) = g - 1 + n - \deg(D) = n + (g - 1 - \deg(D))$$

If one chooses n large enough then

$$\begin{aligned}\dim \Lambda + \dim J \geq (1 - g + n) + (k_0 + n - \deg(D)) &\geq 2 - 2g + 2n - \deg(D)\\ &> n + g - 1 - \deg(D)\\ &= \dim H^1(X, \mathcal{O}_{D_n})^*\end{aligned}$$

Thus $\Lambda \cap J \neq 0$ and we conclude that there exists $\psi \in H^0(X, O_{nP}), \omega \in H^0(X, \Omega_{-D_n})$ such that $\psi\lambda = s_{D_n}(\omega)$.

Now, set $A \coloneqq (\psi)$ i.e. $1/\psi \in H^0(X, \mathcal{O}_A)$ because $\mathrm{ord}_x\psi + \mathrm{ord}_x 1/\psi = 0$. Let $D' \coloneqq D_n - A$, then

$$i^D_{D'}(\lambda) = \frac{1}{\psi}(\psi\lambda) = \frac{1}{\psi} s_{D_n}(\omega) = s_{D'}\left(\frac{1}{\psi}\omega\right)$$

and by Lemma 17.7, we get

$$\omega_0 \coloneqq \frac{1}{\psi}\omega \in H^0(X, \Omega_{-D}), \quad \lambda = s_D(\omega_0)$$

■

This theorem is usually used to obtain equality $\dim H^1(X, \mathcal{O}_D) = \dim H^0(X, \Omega_{-D})$ (as any finite dimensional space and its dual have the same dimension). In particular, for $D = 0$ one headers $g = \dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega)$ and so the genus of a compact RS can be realized as the maximum number of linearly independent holomorphic 1-forms on X. Thus the R-R theorem becomes

$$H^0(X, \mathcal{O}_{-D}) - \dim H^0(X, \Omega_{-D}) = 1 - g - \deg(D)$$

Theorem 17.13

Suppose $D \in \mathrm{Div}(X)$ for compact X. Then
$$H^0(X, \mathcal{O}_{-D}) \cong H^1(X, \Omega_D)^*$$

Proof. Let $\omega_0 \neq 0 \in \mathcal{M}^{(1)}$ and let K be its divisor. Then
$$H^0(X, \mathcal{O}_{-D}) \cong H^0(X, \Omega_{-D-K}) \cong H^1(X, \mathcal{O}_{D+K})^* \cong H^1(X, \Omega_D)^*$$
where the middle inequality is Serre Duality. ■

In particular, $D = 0$ implies $\dim H^1(X, \Omega) = \dim H^0(X, \mathcal{O}) = 1$ because
$$\dim H^0(X, \mathcal{O}) - \dim H^0(X, \Omega) = 1 - g = 1 - H^1(X, \mathcal{O}) = 1 - H^0(X, \Omega)$$
and thus $\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}$ is an isomorphism because it is not identically 0.

Theorem 17.14

The divisor of a non-vanishing meromorphic 1-form ω on a compact RS of genus g satisfies
$$\deg(\omega) = 2g - 2$$

Proof. Let K be the divisor of ω. By R-R,
$$\dim H^0(X, \mathcal{O}_K) - \dim H^1(X, \mathcal{O}_K) = 1 - g + \deg(K)$$
By (17.4), $\Omega \cong \mathcal{O}_K$ and thus
$$1-g+\deg(K) = \dim H^0(X, \Omega) - \dim H^1(X, \Omega) = \dim H^1(X, \mathcal{O}) - \dim H^0(X, \mathcal{O}) = g-1$$
and hence $\deg(K) = 2(g-1)$ ■

Then because of this, by considering
$$\dim H^1(X, \mathcal{O}_D) \cong \dim H^1(X, \mathcal{O}_D)^* \cong \dim H^0(X, \Omega_{-D}) \cong \dim H^0(X, \mathcal{O}_{K-D})$$
and $\deg(K - D) = \deg(K) - \deg(D)$ the R-R equation becomes:
$$\begin{aligned}
\dim H^0(X, \mathcal{O}_D) - H^0(X, \mathcal{O}_{K-D}) &= 1 - g + deg(D) \\
\dim H^0(X, \mathcal{O}_D) - H^0(X, \mathcal{O}_{K-D}) &= 1 - \left(\frac{\deg(K)}{2} + 1\right) + deg(D) \\
&= -\frac{\deg(K)}{2} + \deg(D) \\
&= -\frac{\deg(K-D)}{2} + \frac{\deg(D)}{2}
\end{aligned}$$

Corollary 15

For any lattice $\Gamma \subset \mathbb{C}$ the torus ${}^{\mathbb{C}}/_{\Gamma}$ has genus 1.

Proof. $\mathrm{d}z \in \mathcal{M}^{(1)}(\mathbb{C})$ induces a 1-form $\omega \in \mathcal{M}^{(1)}({}^{\mathbb{C}}/_{\Gamma})$ having no zeroes or poles (10.14), thus $\deg(\omega) = 2g - 2 = 0$ and hence $g = 1$. ∎

Suppose X, Y are compact Riemann Surfaces and $f : X \to Y$is a non-constant holomorphic mapping. For $x \in X$, let $v(x, f)$ be the multipicity of $f(x)$ at x. In other words, f locally looks like z^k where $k = v(f, x)$ is the valency.

Definition 16

The number $b(f, x) \coloneqq v(f, x) - 1$ is called the **branching order** of f at x (note that $b(f, x) = 0$ when f is unbranched). Since X is compact, there are finitely many x for which the branching order is non-zero and thus

$$b(f) \coloneqq \sum_{x \in X} b(f, x)$$

is called the **total branching order** of f.

Theorem 17.17: Riemann-Hurwitz

Suppose $f : X \to Y$ is an n-sheeted holomorphic covering mapping between compact Riemann surfaces X and Y with total branching order $b = b(f)$. Let g be the genus of X and g' the genus of Y Then

$$g = \frac{b}{2} + n(g' - 1) + 1$$

Proof. Let ω be a non-vanishing meromorphic 1-form on Y . Then we can pull back the form to X and we have

$$\deg((\omega)) = 2g' - 2, \quad \deg((f^*\omega)) = 2g - 2$$

Let $x \in X$ and $f(x) = y$. By (2.1) there is a coordinate neighborhood (U, z) and (V, w) of x and y such that with respect to these choices, f can be written $w = z^k$ where $k = v(f, x)$. Then on V, let $\omega = \phi(w)\mathrm{d}w$ so that on U

$$f^*\omega = \phi(z^k)\mathrm{d}z^k = kz^{k-1}\phi(z^k)\mathrm{d}z$$

We see that transfomring $w$to z^k we pickup a factor of $k = v(f, x)$ for the order and the z^{k-1} term adds another $k - 1 = b(f, x)$ to get

$$\mathrm{ord}_x(f^*\omega) = b(f, x) + v(f, x)\mathrm{ord}_y(\omega)$$

The total valence at any point in the preimage of $y \in Y$ must total to the rank of the covering, i.e

$$\sum_{x \in f^{-1}(y)} v(f, x) = n$$

Then

$$\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega) = \sum_{x\in f^{-1}(y)} b(f,x) + n\operatorname{ord}_y(\omega)$$

Summing over all y, we get

$$\deg((f^*\omega)) = \sum_{x\in X} \operatorname{ord}_x(f^*\omega) = \sum_{x\in X} b(f,x) + n\sum_{y\in Y} \operatorname{ord}_y(\omega) = b(f) + n\deg(\omega)$$

resulting in

$$2g-2 = b + n\deg(\omega) = b + n(2g'-2)$$

and the result follows. ■

So, in the case of an n-sheeted covering $\pi : X \to \mathbb{P}^1$ with total branching order b we get that

$$g = \frac{b}{2} + n(0-1) + 1 = \frac{b}{2} - n + 1$$

Thus, for a double covering (2-sheeted) we have $g = \frac{b}{2} - 1$.

For example, consider the RS generated by the equation $\pi : X \to \mathbb{P}^1$ defined by $\sqrt[n]{(1-z^n)}$. We can rewrite this as

$$\sqrt[n]{(1-z^n)} = \prod_{i=1}^{n} (z-\zeta_i)^{\frac{1}{n}}$$

which clearly has n branch points, one for each root of unity, and each one has $v(f,\zeta) = n$ and so $b(f,\zeta) = n-1$. This would imply $b(f) = n*(n-1)$ and since $\deg f = n$ we have

$$g_X = 1 - n + \frac{b}{2} = 1 - n + \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$$

Compact Riemann surfaces of $g > 1$ which admit a double covering of $\mathbb{P}^1$ are called **hyperelliptic**.

Theorem 17.18

Any genus two RS is hyperelliptic.

Proof. It needs to admit a degree two meromorphic function. By the R-R we have

$$\begin{aligned}\dim H^0(X,\Omega) &= \dim H^0(X,\mathcal{O}_K)\\ &= 1 - g + \deg(K) + \dim H^0(X,\mathcal{O}) = 1 - 2 + 2 + 1 = 2\end{aligned}$$

so there exists a non-const holomorphic 1-form ω. Let $K = (\omega)$ then $K \geq 0$ and from above we know $\dim H^0(X,\mathcal{O}_K) = 2$ there is a nonconstant meromorphic function $f \in H^0(X,\mathcal{O}_K)$. f can have a pole of order at most two and cannot have only poles of degree one (otherwise it would be a sphere) and so f provides the double cover. ■

Theorem 17.19

If X is compact RS of genus g and $D \in \mathrm{Div}(X)$ then $H^1(X, \mathcal{O}_D) = 0$ whenever $\deg(D) > 2g - 2$

Proof. Let $\omega \in \mathcal{M}^{(1)}(X)$ be non-trivial and K its divisor. By (17.4), $\Omega_{-D} \cong \mathcal{O}_{K-D}$ and hence

$$H^1(X, \mathcal{O}_D) \cong H^1(X, \mathcal{O}_D)^* \cong H^0(X, \Omega_{-D}) \cong H^0(X, \mathcal{O}_{K-D}).$$

If $\deg(D) > 2g - 2$ then

$$\deg(K - D) = \deg(K) - \deg(D) < \deg(K) + 2 - 2g < 0$$

and thus $H^0(X, \mathcal{O}_{K-D}) = 0$ by theorem (16.5). ■

Corollary 20

If $\deg(D) > 2g - 2$, then $H^0(X, \mathcal{O}_D) = 1 - g + \deg(D)$

Proof. Apply result of previous theorem to R-R formula. ■

Corollary 21

If X is compact and $\mathcal{M}$ is the sheaf of meromorphic functions on X Then $H^1(X, \mathcal{M}) = H^1(X, \mathcal{M}^{(1)}) = 0$

Proof. First the sheaves are isomorphic by $f \mapsto f\omega$ where $\omega \neq 0$ is a fixed element in $\mathcal{M}^{(1)}$ Let $\xi \in H^1(X, \mathcal{M})$ with representative cocycle $(f_{ij}) \in Z^1(X, \mathcal{M})$. Possibly passing to a refinement of $\mathfrak{U}$, w.l.o.g assume that the total number of poles of all the f_{ij} is finite. Hence, there is a divisor D of large enough degree so that $\deg(D) > 2g - 2$ such that $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_D)$. By the previous theorem, (f_{ij}) is cohomologous to zero relative to the sheaf $\mathcal{O}_D$ and thus also relative to $\mathcal{M}$. ■

Definition 22

Let D be a divsisor on X. We say $\mathcal{O}_D$is **globally generated** if for every $x \in X$ there exists $f \in H^0(X, \mathcal{O}_D)$ such that

$$\mathcal{O}_{D,x} = \mathcal{O}_x f$$

i.e. every germ $\phi \in \mathcal{O}_{D,x}$ may be written $\phi = \psi f$ with $\psi \in \mathcal{O}_X$. The condition is equivalent to

$$\mathrm{ord}_x(f) = -D(X)$$

Theorem 17.23

Let X be a compact RS of genus g and D be a divisor on X with $\deg D \geq 2g$, then $\mathcal{O}_D$ is globally generated.

Theorem 17.24

On a compact RS X of genus g let D be a divisor of $\deg(D) \geq 2g+1$. Let $f_0, \ldots, f_n$ be a basis of $H^0(X, \mathcal{O}_D)$. Then

$$F = (f_0 : \cdots : f_N) : X \to \mathbb{CP}^N$$

is an embedding

One can also show $\deg(D) \geq 2g+1$ there exist $\phi_0, \ldots, \phi_3 \in H^0(X, \mathcal{O}_D)$ such that

$$(\phi_0 : \cdots : \phi_3) : X \to \mathbb{P}^3$$

is an embedding so that every CRS embeds into $\mathbb{P}^3$