POTENTIAL THEORY AND HARMONIC FUNCTIONS

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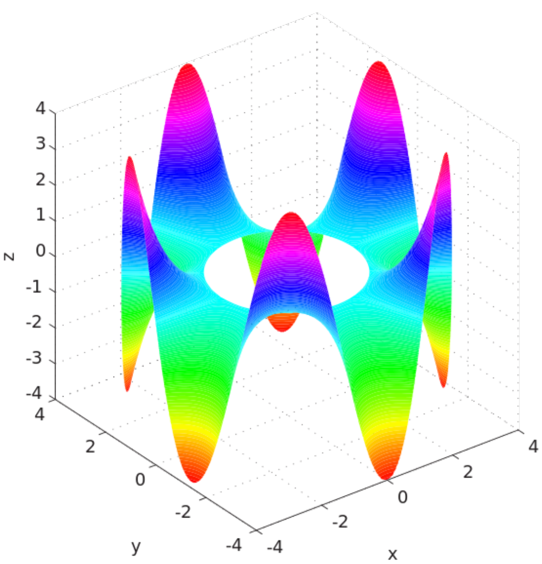

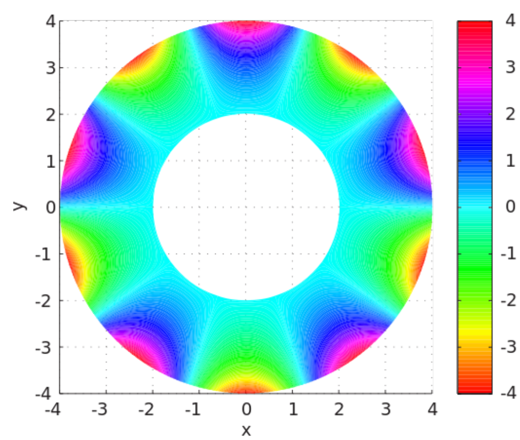

1 Introduction [5][8]

The theory of potentials as a concept has its origins in the mid 17th and 18th centuries, with it's basic notions being expressed by physicists in the search of symmetrical configurations. Such examples include Newton's Law of Gravitation and Coloumb's theory of electrostatic attraction/repulsion. Then it was Lagrange, Legendre and Laplace who studied further the work of their predecessors; the first expressing a *potential function* of a given force in a *field*. Shortly afterwards, it was the likes of Green and Gauss who coined the terms **potential function** and **potential** respectively

From then on, the study of potentials reached farther than the physical framework it was began on. In the mid to late 19th century, potential theory applied itself onto more complicated problems in mathematical physics and it transitioned from a question on physical masses suspended in space to arbitrary ones serving certain restrictions. What became of central importance was the partial differential equation named after Laplace and functions satisfying such symmetrical variational problems became a special class - *Harmonic functions*. From within this framework, the theory enriched further. Imposing certain boundary conditions and varying the domain leads to the Dirichlet, Neumann and Robin boundary value problems. Due to certain "harmonious" properties of harmonic functions we will see later, it leads itself very naturally to inverse problems and methods such as *Balayage*. Lyapunov and Steklov solved the fundamentals of such boundary problems, the point at which the study became more focused on the potentials themselves.

Throughout the 20th century, potentials gained several levels of abstraction, finding their place in the theory of measures and generalized functions. As more of this theory was studied, more of its fundamental connections arose; complex and functional analysis, variational calculus, topology, and more surprisingly, probability theory. Not only abstractly but also tangibly the idea of potentials were prevalent in Einstein's theory of general relativity and the Minkowski metric and more significantly in the Aharanov-Bohm effect, showing it's existence in fundamental natural phenomena.

2 Early Motivations[1]

We begin by understanding potential theory from a familiar physical perspective. Consider the concept of Coloumb's law: Suppose there are N charges, q_i, located at positions given by $\mathbf{r}_i \in \mathbb{R}^3$. Then we know that the electrical force acting on an external test charge q with position vector $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = q \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

Then it is canonical to write $\mathbf{F} = q\mathbf{E}$ where $\mathbf{E}$ is the electric field given by the summation. Generalizing this to a continuous charge distribution over a volume V given by charge density

ρ we see that the electric field at a position $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \int_V \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \, \mathrm{d}V$$

Now let $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$, then the x-component of the integrand is given by

$$\frac{x-x'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} = -\frac{\partial}{\partial x}\left(\frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}\right)$$

Since x was an arbitrary coordinate, we can in fact write

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

which leads us to the notion that the electric field is given by the gradient of some other field, namely

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$$

where

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}V$$

This function, φ, is what is given the name of a **potential**, that is, the function whose gradient the Electric field is defined by.

Consider the divergence of the electric field,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = -\nabla^2\varphi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \mathrm{d}V$$

We know from our previous explorations in the lectures that the solution to $-\nabla^2 u(\mathbf{r}) = \delta(\mathbf{r})$ is given by

$$u(\mathbf{r}) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

and hence

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \mathrm{d}V = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') \, \mathrm{d}V = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and we have remarkably derived Gauss' theorem by sufficiently knowing a solution to *Poisson equation*, i.e the Green's function to the 3D Poisson equation.

A physical corollary of Gauss' Law interesting to us is known as Earnshaw's Theorem, which states that the classical theory of electrostatics is not enough to explain the stability of charges. Suppose there exists a positive charge in an arbitrary space. If this positive charge was stable, then any pertubation of the charge will result in it moving back to its equilibrium point. However, this would mean that the electric field in the neighborhood of the equilibrium point points radially inwards. This would imply the charge is negative, but it isn't, its positive. Hence some field line in the neighborhood of the charge must point outwards, and thus the particle can always escape by moving along this line. But this implies the electric field is divergence-free (no sources or sinks) and hence

$$\nabla \cdot \mathbf{E} = -\nabla^2\varphi = 0$$

i.e Laplace's equation is true for electric potentials. This also shows that there are no local minima or maxima of electric field potentials in free space, only saddle points; a **key** defining property we will see for harmonic functions. One might imagine that the field lines could curl into the charge, but by the same logic this also can't happen - electric potentials are also irrotational.

Aside: If you're curious, the stability of atoms and molecules is explained by quantum mechanics

3 Harmonic Functions

We begin by defining the most familiar notion of harmonic functions:

Definition 3.1: Harmonic Function

A twice continuously differentiable function $u\colon \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is an open subset is called **harmonic** if it satisfies Laplace's equation, that is

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

From this simple notion, we can think of a variety of harmonic functions straight off the bat. Consider homogeneous polynomials (all terms have the same total degree):

- Degree 0: All constants C are harmonic
- Degree 1: All linear polynomials $ax + by$ are harmonic
- Degree 2: $x^2 - y^2$ and xy are harmonic and moreover any linear combination of the two is also harmonic
- Degree n: We will return to this later

One can easily check that there are several harmonic functions exhibiting radial symmetry, that is $f(x_1, x_2, \ldots, x_n) = f(r)$ where r is the radius,

- The function $f(x, y) = a\ln\left(\sqrt{x^2 + y^2}\right) = a\ln(r)$
- In dimensions greater than 2, $f(r) = 1/r$

Finally, exponentially oscillations are also harmonic, that is

$$f(x, y) = e^{kx}\sin ky, \quad f(x, y) = e^{kx}\cos ky$$

One important idea to notice is that by definition, harmonic functions are the kernel of the Laplacian operator. Since Δ is a linear operator, and the kernel is a subspace, the any linear combination of harmonic functions will also be harmonic. This leads to the idea, that if one can find a *fundamental* set of harmonic functions, or equivalently a *basis*, then any harmonic function can be constructed from them.

Theorem 3.2: Mean Value Property[3]

Let σ be the $(n-1)$-dimensional surface measure and $u\colon \Omega \to \mathbb{R}$ a harmonic function. Let $B(x,r)$ be the ball with center x and radius r completely contained in Ω, the the value of u at the center x is given by the average value of u on the surface of the ball and also by the average value in the interior of the ball, that is:

$$u(x) = \frac{1}{S(B(0,1))r^{n-1}} \int_{\partial B(x,r)} u \,\mathrm{d}\sigma = \frac{1}{V(B(0,1))r^{n}} \int_{B(x,r)} u \,\mathrm{d}V$$

where S is the surface area operator and V is the volume operator. If we denote a symbol for the integration implying the integral divided by the measure of the domain of integration.

$$u(x) = \fint_{B(x,r)} u \,\mathrm{d}V = \fint_{\partial B(x,r)} u \,\mathrm{d}S$$

Proof. The proof will be omitted as it is not so important, usually it is proved by an application of Green's identities or the divergence theorem. ■

This theorem holds remarkable value on the nature of harmonic functions. Since the radius of the ball is arbitrary, this means the value at centered at x is the average of all its neighbors regardless on how large the domain of neighbors is specified. Thinking for a moment, if u is not trivially a constant, then the values of points in the ball and on the surface of the ball must have both positive and negative values, otherwise the average will surely be greater or smaller than the center. This further enforces the idea that there can not be any local minima or maxima, because the average of points around a minimum (maximum) must be greater (smaller) than the minimum (maximum) itself. Furthermore, this means the "flow" of values on the function are inherently connected, because *every* point has to be the average of its neighborhood.

Corollary 3.3

A real-valued harmonic function can not have isolated zeros. [3]

Proof. Suppose u is harmonic and real valued on Ω and x^* is a zero in Ω. By the mean value property, the average of u on $B(x^*, r)$ for any $r > 0$ has to also be 0. This leads to two possibilities: either u is identically 0 on the domain or u takes values greater and smaller than 0 on the $\partial B(x^*, r)$. But since the ball is connected, by the concept of the intermediate value theorem, u must 0 somewhere on the boundary. ■

This furthers the notion of inherent links between the values of the function u; the zeros come in groups and the function adapts to satisfy the average values.

Theorem 3.4: Maximum principle

Suppose Ω is a connected and u is real and harmonic on it. If u attains a maximum or minimum in Ω, then u is constant. [3]

Proof. If u has a maximum at some point a, then by the mean value property, the average in $B(a, r) = u(a)$. However, the ball around a maximum only takes values less than or equal to it (by continuity), and if even a single point was less than the maximum the average would be less. Hence, it must be that u is constant in the ball. This same logic applies to $\partial B(a, r)$ and hence the whole ball is constant. Therfore as $r \to \infty$, u is constant on its whole domain.
If u attains a minimum, then just apply the argument to $-u$. ■

Corollary 3.5

Suppose Ω is bounded and u is continuous and real on $\overline{\Omega}$ and harmonic on Ω. Then u attains its maximum and minimum values on $\partial\Omega$. [3]

Proof. Directly follows from the Maximum principle. ■

This corollary plays in important role in the uniqueness of harmonic functions. It precisely shows that a harmonic function on a bounded domain is determined by its values on the boundary. Indeed suppose $u, v \in C^2(\Omega)$ are harmonic, then $w = u - v$ is also harmonic. But then $w = 0$ on $\partial\omega$ and by the maximum principles this implies that $w = 0$ on the interior. Hence $u = v$ on Ω. We have seen this notion before, as the solutions to the Dirchlet problem are unique.

We will now see further notions of comparisons on harmonic functions. From the previous paragraph, we can further deduce that $u \geq v$ on $\partial\Omega$ implies $u \geq h$ in Ω.

Definition 3.6: Subharmonic and Superharmonic

Suppose Ω is an open set. A function $u \in C^2(\Omega)$ is **subharmonic** if $\Delta u \geq 0$ in Ω and **superharmonic** if $\Delta u \leq 0$ in Ω. The mean value property also adjusts for the change in equality respectively.

The subharmonic and superharmonic are aptly named, because the above definition is enough to conclude that if u is harmonic, v is subharmonic and $u = v$ on $\partial\Omega$, then the graph of v is below the graph of u (the same logic shows a superharmonic will lie above u). And so in this sense, the functions are *sub*- and *super*- when relating to harmonics. We can see from this that there is some sense of an ordering in the class of harmonic functions, that is to say for every harmonic function one will always be able to find corresponding subharmonic and superharmonic functions to bound it. As such, if u, v are harmonic and $u \leq v$ then one should be able to find u_1, v_1 subharmonic such that $u_1 \leq v_1$ as long as the values agree on the boundary on $\partial\Omega$.

Theorem 3.7: Harnack's Inequality [4][6]

Suppose that $\Omega' \subset \Omega$ is a connected open set completely contained in an open set Ω such that $\overline{\Omega'}$ is compact. There exists a constant C, depending *only* on Ω and Ω' such that if $u \in C^2(\Omega)$ is a non-negative harmonic function then,

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

and in particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

forall $x, y \in \Omega'$.

Proof: Evans.
Let $r = \frac{1}{4}\text{dist}(\Omega', \partial\Omega)$. Choose $x, y \in \Omega'$, $|x - y| \leq r$. Then

$$u(x) = \fint_{B(x,2r)} u \, \mathrm{d}V \geq \frac{1}{w_n 2^n r^n} \int_{B(y,r)} u \, \mathrm{d}V = \frac{1}{2n} \fint_{B(y,r)} u \, \mathrm{d}V = \frac{1}{2^n} u(y)$$

Hence $2^n u(y) \geq u(x) \geq \frac{1}{2^n} u(y)$. Since Ω' is connected and $\overline{\Omega'}$ is compact, by definition there must be a finite subcover of Ω' by disjoint balls $\{B_i\}_{i=1}^N$ with radius r, then

$$u(x) \geq \frac{1}{2^{nN}} u(y)$$

for all $x, y \in \Omega'$. ■

Harnack's inequality is a powerful theorem which gives bounds on the growth of harmonic functions. It controls the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function. The fact that the constant C depends only on the domain shows that this is a universal property for harmonic functions on Ω. This shows the values of harmonic functions on Ω' are comparable, that is they "flow" together and only are small when every value in Ω' is small. Since C depends only on Ω' then Ω' hosts only harmonic functions that staisfy certain comparability requirements. If $\inf u$ was 0 on Ω' then $u \equiv 0$ on all of Ω'. Harnack's inequality has strong consequences, and in fact is used in Perelman's proof of the Poincare Conjecture. One of the consequences is the following principle:

Theorem 3.8: Harnack's Principle [3]

Suppose Ω is connected and (u_m) is a pointwise increasing sequence of harmonic functions on Ω. Then either (u_m) converges uniformly on compact subsets of Ω to a harmonic function on Ω or $u_m(x) \to \infty$ for every $x \in \Omega$.

In general real analysis the pointwise limit of a monotonic sequence of continuous functions need not converge nicely. Even in the finite case, there is no reason one should expect a sequence to converge uniformly on every compact subset of Ω. In particular, the convergence of an arbitrary sequence of smooth functions at a single point does not imply its convergence nor smoothness at any other point. In contrast, we have the complete opposite for harmonic functions: their convergence is uniform not only to a point but to full harmonic function on the same domain. This is a very strong condition that demonstrates the beauty of harmonic functions.

4 Relationship to Complex Analysis

When learning the theory of functions of a complex variable, one often realizes that the "nice" properties of working over a complex domain leads to beauty and symmetry. In fact, we will see that harmonic functions are surrogates for complex functions in the domain of real numbers. The first concept is that differentiation in complex numbers (holomorphy) is a very strong property, implying that if a function is once differentiable then it is C^∞. In contrast to real analysis, differentiability is a weak condition. In no way does differentiability imply smoothness and one can construct counter-intuitive examples like the Weierstrass function. Now we see that harmonic functions are extend their beauty to behave like functions of complex variables and are in fact realizations of holomorphic functions in the real domain:

Theorem 4.1: Smoothness [7]

Let Ω be an open set in $\mathbb{R}^n$ and let u be harmonic on Ω, then $u \in C^\infty(\Omega)$

To prove this theorem, we define functions which approximate the Dirac Delta distribution

Definition 4.2: Approximation to the Identity

Let $T\colon \mathbb{R}^n \to \mathbb{R}$ be a function such that

(i) $T \in C^\infty(\mathbb{R}^n)$
(ii) $T \geq 0$
(iii) $\operatorname{supp}(T) \subset B(0,1)$
(iv) $T(x) = T(|x|)$
(v) $\int_{\mathbb{R}^n} T \, \mathrm{d}\sigma = 1$

T is called an **approximate identity** or **nascent delta function**.

Here $|\cdot|$ denotes the standard euclidean norm and so property (iv) says that the function should be radial. An example of such a function

$$T(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}} & x \in B(0,1) \\ 0 & \text{otherwise} \end{cases}$$

The point of interest in the function is for the following theorem

Theorem 4.3

Let Ω be an open subset of $\mathbb{R}^n$. Then if f is a Lebesgue measurable and integrable function on any compact subset $\Omega' \subset \Omega$ and $g \in C^k(\mathbb{R})$ with $\operatorname{supp}(g) \subset B(0,r)$ then the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}\sigma$$

is in $C^k(\Omega_r)$ where $\Omega_r = \{x \in \Omega \mid d(x, \partial\Omega) > r\}$. [7]

With this theorem, one can directly prove Theorem 4.1.

Proof. Informally, choosing f to be harmonic and g to be an approximate identity, after direct calculation and invocation of the Mean Value Property one arrives to

$$(T * f)(x) = f(x) \int_{B(0,1)} T(y)\,\mathrm{d}\sigma(y) = f(x)$$

And so f which is harmonic is equivalent to its convolution with the approximate identity. By the previous theorem, the convolution is C^∞ since the approximate identity is and hence f is also C^∞. ■

We see that harmonic functions employ the strong property of differentiability found in the complex domain. It turns out, that this is not the only similarity found with complex analysis. The following three results are mentioned without proof to preserve space and continue exposition, their proofs can be found in

Theorem 4.4: Liouville's Theorem[3]

A bounded harmonic function on $\mathbb{R}^n$ is constant.

Corollary 4.5

Suppose u is a continuous bounded function on $\overline{\Omega}$ that is harmonic on Ω. If $u = 0$ then $u \equiv 0$ on $\overline{\Omega}$ [3]

Theorem 4.6: Singular behavior[3]

If u is a harmonic function defined on an open subset $\Omega \setminus \{x_0\}$ of $\mathbb{R}^n$ which is less singular at x_0 than the fundamental solution of the Laplacian, that is for $n > 2$,

$$u(x) = o(|x - x_0|^{2-n})$$

then u extends to a harmonic function on Ω

From this we see that many natural consequences of complex functions extend naturally to harmonic functions. Of particular interest is the last. We see that harmonic functions with isolated singularities can be reconciled by allowing the behavior of the function to mimic the fundamental solution centered at the singularity. But through all of this, we ask what is *the fundamental solution*? It turns out that fundamental solutions are precisely the radial solutions defined before, and they are basis from which harmonic functions can be constructed.

5 Fundamental Solutions

We begin by stating the following theorem whose proof is not too difficult, but will not be written

Theorem 5.1

Among all second-order homogeneous PDEs in two dimensions with constant coefficients, show that the only ones that do not change under a rotation of the coordinate system (i.e., are rotationally invariant), have the form

$$a\Delta u = bu$$

By choice of $b = 0$ we retrieve Lapalce's equation. The point of understanding this theorem is that solutions to Laplace's equation can be rotated by an orthogonal group O and still satsify the equation. But not all solutions will look the same when rotated, and is in our best interest to understand the solutions which are radial (invariant under rotation). Let u be a solution to $\Delta u = 0$ on $\mathbb{R}^n$ and assume the solution takes the form $u = u(r)$ where $r = |x|$ is the euclidean norm in n-dimensions. After substitution into the equation, one derives that the solutions are

$$u(r) = \begin{cases} b\ln r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3 \end{cases}$$

You also asked what the fundamental solutions of $\Delta u + u = 0$ might look like. In fact this is a hard question with no general closed form. There have been specific cases where it has been solved[9], but it seems not to be as simple as the other possible choices for a and b. One should notice that this function is not defined for $r = 0$ and is singular at that point. This motivates the search for our fundamental solutions.

Definition 5.2: Fundamental solution [4]

For a linear partial differential operator L, a fundamental solution F is a solution to the inhomogeneous equation

$$LG = \delta(x)$$

where δ is the Dirac delta function.

The function G is commonly known as the Green's function for the partial differential operator. It is the function which satisfies an impulse response of the the operator in question. It turns out that one can choose specific constants for the derived radial solutions to find the fundamental solution. In particular, the fundamental solutions to $\Delta u = \delta$ are given by

$$G(x) = \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2 \\ \frac{1}{n(2-n)V(B(0,1))} \frac{1}{|x|^{2-n}} & n > 2 \end{cases}$$

It can be seen that the fundamental solutions serves as the kernel of the integral operator which inverts the Laplacian. This allows one to solve Poisson's inhomogenous equation $\Delta u = f$ for some compactly supported function f. Since G is harmonic on $x \neq 0$(by construction of the radial solution), then G shifted to a point y given by $G(x-y)$ is also harmonic and furthermore $G(x-y)f(y)$ is also harmonic when $x \neq y$. This leads to the following theorem whose proof is also omitted

Theorem 5.3: Solution to Poisson's Equation [4]

Consider the Poisson equation in $\mathbb{R}^n$

$$\Delta u = f$$

where f is compactly supported. Then the solution to the Poisson problem is given by the convolution

$$u = (G * f)(x) = \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}^n} \ln |x-y| f(y) \, \mathrm{d}y & n = 2 \\ \frac{1}{n(2-n)V(B(0,1))} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} \, \mathrm{d}y & n > 2 \end{cases}$$

As a final point, I'd just like to bring up some final theorems.

Theorem 5.4: Bôcher's Theorem [3][2]

Let $a \in \Omega$. If u is harmonic on $\Omega \setminus \{a\}$ and positive near a, then there is a harmonic function v on Ω and a constant $b \in \mathbb{R}$ such that

$$u(x) = \begin{cases} v(x) + b \ln |x-a| & n = 2 \\ v(x) + b|x-a|^{2-n} & n > 2 \end{cases}$$

This shows that harmonic functions with singularities that behave positively near the point of singularity can be reconciled by allowing the singularity to behave as an impulse response at that point. It implies that the fundamental solutions play a large role in defining the behavior of harmonics at undefined points.

Theorem 5.5: Laplace in Spherical [4]

The general solution, f, to Laplace's equation on a sphere is given by

$$f(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{l} F_\ell(r) Y_\ell^m(\theta, \varphi)$$

where $F(r)$ is a radial function and $Y(\theta, \varphi)$ is a spherical harmonic. This is analogous to Fourier series on the unit circle.

From this we see that there extends a natural analogue to the Fourier series to Laplace's equation on the sphere. The spherical harmonics are an orthonormal basis of functions on the sphere, just as sin and cos are orthonormal on the unit circle. The spherical harmonics are given by

$$Y_\ell^m(\theta,\varphi) = Ne^{im\varphi}P_\ell^m(\cos\theta)$$

where N is a normalization constant and P_l^m are the associated Legendre polynomials

$$P_\ell^m(x) = (-1)^m 2^\ell (1-x^2)^{m/2} \sum_{k=m}^{\ell} \frac{k!}{(k-m)!} x^{k-m} \binom{\ell}{k} \binom{\frac{\ell+k-1}{2}}{\ell}$$

In fact, you asked me in class what the radial solutions look like. It turns out my intuition was incorrect in thinking they would be linear combinations of the radial solutions to Laplace's equation in $\mathbb{R}^n$. As it turns out they are

$$F(r) = Ar^\ell + Br^{-\ell-1}$$

This methodology of separation of variables can be used to solve the Schrodinger equation for the hydrogen atom, whose normalized position wavefunctions are given by

$$\psi_{n\ell m}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na_0^*}\right)^3 \frac{(n-\ell+1)!}{2n(n+\ell)!}} e^{-\rho/2}\rho^\ell L_{n-\ell-1}^{2\ell+1}(\rho) Y_\ell^m(\theta,\varphi)$$

where $\rho = \frac{2r}{na_0^*}$, a_0^* is the Bohr radius, L are generalized Laguerre polynomials, and Y are spherical harmonics. Furthemore, the parameters n, ℓ, m turn out to be, in this context, quantum numbers, specifically n is the principal, ℓ is the azimuthal, m is the magnetic.

6 Concluding Remarks

In conclusion, Harmonic functions are no ordinary class of functions. They exhibit deep connections between their values, causing their behaviour on certain subsets of the domain to exhibit collective group properties. When one value of the function is very small or large, the values in the neighborhood need to also follow along. Additionally, they are analogues to holomorphic functions but in $\mathbb{R}^n$, and with them, they bring very beautiful properties that allow one to analyze their behaviour using very strong conditioned theorems. I hope in the future I would be able to continue my study of Harmonic Functions, as their presence in physics is quintessential, and any revelation on harmonic functions is surely useful in physical contexts. Shoutout to Sheldon Axler et al. [3] for the only self-contained book about Harmonic Function Theory rather than the more general Potential Theory.

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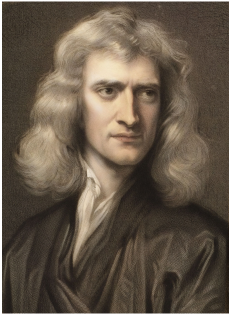
Overview

Historical Introduction



Newton

Coulomb

Lagrange

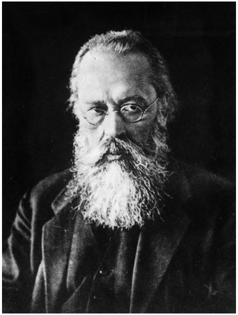
Laplace

Legendre

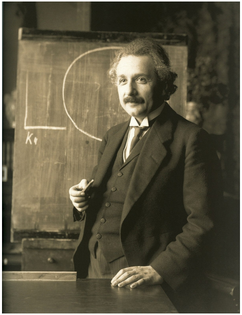
Gauss

Green

Lyapunov



Steklov



Einstein

Aharonov

Bohm

Coulomb's Law

Suppose there are N charges, q_i, located at positions given by $\mathbf{r}_i \in \mathbb{R}^3$. Then we know that the electrical force acting on an external test charge q with position vector $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = q \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}$$

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Then it is canonical to write $\mathbf{F} = q\mathbf{E}$ where $\mathbf{E}$ is the electric field given by the summation. Generalizing this to a continuous charge distribution over a volume V given by charge density ρ we see that the electric field at a position $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V$$

Notice that in the integrand, we can in fact write

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which leads us to the notion that the electric field is given by the gradient of some other field, namely

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$$

where

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V$$

This function, φ, is what is given the name of a **potential**, that is, the function whose gradient the Electric field is defined by.

Consider the divergence of the electric field,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = -\nabla^2 \varphi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V$$

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We know from our previous explorations in the lectures that the solution to $-\nabla^2 u(\mathbf{r}) = \delta(\mathbf{r})$ is given by

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and hence

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V &= \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') \, \mathrm{d}V \\ &= \frac{\rho(\mathbf{r})}{\varepsilon_0} \end{aligned}$$

So we have

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}$$

and hence have derived Gauss's law just by knowing the solution to $-\nabla^2 u = \delta$, i.e the **fundamental solution** for the Laplacian in three dimensions.

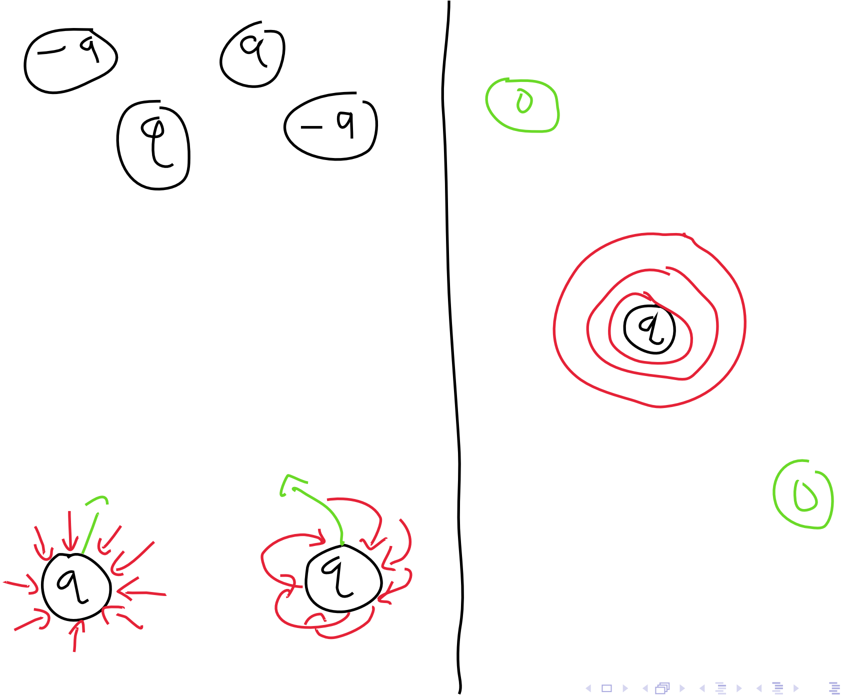
Earnshaw's Theorem

Theorem 1 (Earnshaw's Theorem)

A collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges.

- Why?
- Gauss's law implies $\nabla \cdot \mathbf{E} = -\nabla^2 \varphi = 0$, divergence-free.
- No sources or sinks, only saddles (i.e no local minima or maxima).
- Also, curl-free.

Aside: If you're curious, the stability of atoms and molecules is explained by quantum mechanics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Harmonic Functions

Definition 2 (Harmonic Function)

A twice continuously differentiable function $u\colon \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is an open subset is called **harmonic** if it satisfies Laplace's equation, that is

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

- Homogeneous Polynomials
  - Degree 0: constants C.
  - Degree 1: $ax + by$
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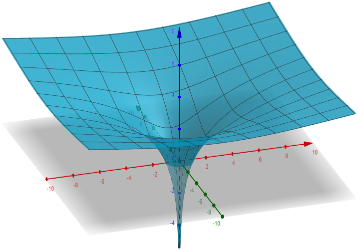
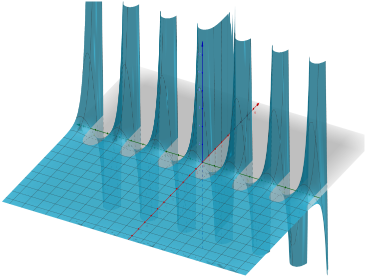
One important idea to notice is that by definition, harmonic functions are the kernel of the Laplacian operator. Since Δ is a linear operator, and the kernel is a subspace, the any linear combination of harmonic functions will also be harmonic.



$\ln\left(x^2+y^2\right)$



$e^x \sin y$

Credits: https://www.geogebra.org/3d

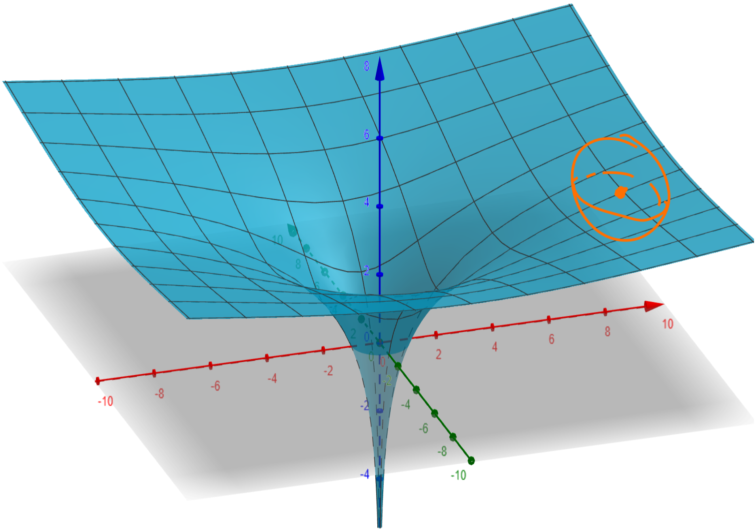
Theorem 3 (Mean Value Property)

Let σ be the $(n-1)$-dimensional surface measure and $u \colon \Omega \to \mathbb{R}$ a harmonic function. Let $B(x,r)$ be the ball with center x and radius r completely contained in Ω, the the value of u at the center x is given by the average value of u on the surface of the ball and also by the average value in the interior of the ball, that is:

$$u(x) = \frac{1}{S(B(0,1))r^{n-1}} \int_{\partial B(x,r)} u \, \mathrm{d}S = \frac{1}{V(B(0,1))r^{n}} \int_{B(x,r)} u \, \mathrm{d}V$$

If we denote a symbol for the integration implying the integral divided by the measure of the domain of integration.

$$u(x) = \fint_{B(x,r)} u \, \mathrm{d}V = \fint_{\partial B(x,r)} u \, \mathrm{d}S$$



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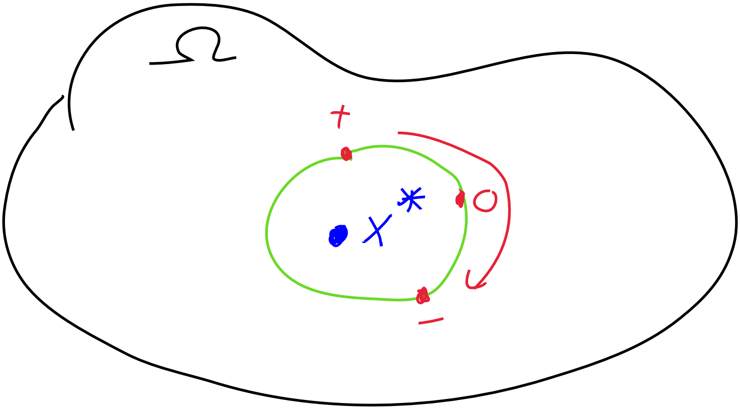
Corollary 4

A real-valued harmonic function can not have isolated zeros.

Proof.

Suppose u is harmonic and real valued on Ω and x^* is a zero in Ω. By the mean value property, the average of u on $B(x^*, r)$ for any $r > 0$ has to also be 0. This leads to two possibilities: either u is identically 0 on ∂B or u takes values greater and smaller than 0 on the $\partial B(x^*, r)$. But since the ball is connected, by the concept of the intermediate value theorem, u must 0 somewhere on the boundary. ■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Theorem 5 (Maximum Principle)

Suppose Ω is a connected and u is real and harmonic on it. If u attains a maximum or minimum in Ω, then u is constant.

Proof.

If u has a maximum at some point a, then by the mean value property, the average in $B(a, r) = u(a)$. However, the ball around a maximum only takes values less than or equal to it (by continuity), and if even a single point was less than the maximum the average would be less. Hence, it must be that u is constant in the ball. This same logic applies to $\partial B(a, r)$ and hence the whole ball is constant. Therfore as $r \to \infty$, u is constant on its whole domain.

If u attains a minimum, then just apply the argument to $-u$. ∎

Corollary 6

Suppose Ω is bounded and u is continuous and real on $\overline{\Omega}$ and harmonic on Ω. Then u attains its maximum and minimum values on $\partial\Omega$.

It precisely shows that a harmonic function on a bounded domain is determined by its values on the boundary. Indeed suppose $u, v \in C^2(\Omega)$ are harmonic, and $u = v$ on $\partial\Omega$ then $w = u - v$ is also harmonic. But then $w = 0$ on $\partial\omega$ and by the maximum principles this implies that $w = 0$ on the interior. Hence $u = v$ on Ω. Soon we will see this notion arising in the Dirichlet problem. We can further deduce that $u \geq v$ on $\partial\Omega$ implies $u \geq h$ in Ω.

Definition 7 (Subharmonic and Superharmonic)

Suppose Ω is an open set. A function $u \in C^2(\Omega)$ is **subharmonic** if $\Delta u \geq 0$ in Ω and **superharmonic** if $\Delta u \leq 0$ in Ω. The mean value property also adjusts for the change in equality respectively.

Let u be harmonic and v subharmonic which agree on $\partial\Omega$, then $v \leq u$ on Ω, the graph of v lies below the the graph of u at all points inside the domain. We can see from this that there is some sense of an ordering in the class of harmonic functions.

Theorem 8 (Harnack's Inequality)

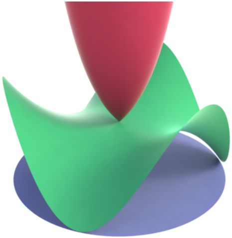
Suppose that $\Omega' \subset \Omega$ is a connected open set completely contained in an open set Ω such that $\overline{\Omega'}$ is compact. There exists a constant C, depending only on Ω and Ω' such that if $u \in C^2(\Omega)$ is a non-negative harmonic function then,

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

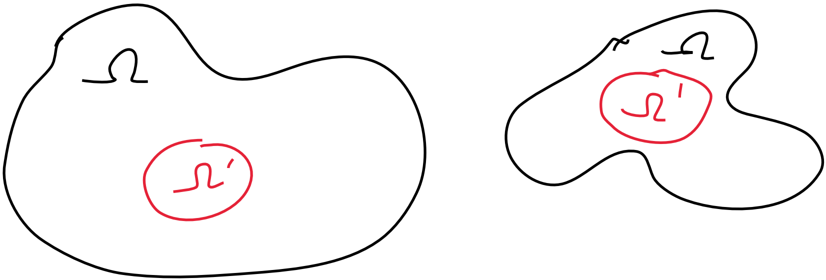
and in particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

forall $x, y \in \Omega'$.



It controls the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function. The fact that the constant C depends only on the domain shows that this is a universal property for harmonic functions on Ω. Since C depends only on Ω' then Ω' hosts only harmonic functions that satisfy certain comparability requirements. If $\inf u$ was 0 on Ω' then $u \equiv 0$ on all of Ω'. Harnack's inequality has strong consequences, and in fact is used in Perelman's proof of the Poincare Conjecture.



Theorem 9 (Harnacks' Principle)

Suppose Ω is connected and (u_m) is a pointwise increasing sequence of harmonic functions on Ω. Then either (u_m) converges uniformly on compact subsets of Ω to a harmonic function on Ω or $u_m(x) \to \infty$ for every $x \in \Omega$.

In general real analysis the pointwise limit of a monotonic sequence of continuous functions need not converge nicely. Even in the finite case, there is no reason one should expect a sequence to converge uniformly on every compact subset of Ω. In particular, the convergence of an arbitrary sequence of smooth functions at a single point does not imply its convergence nor smoothness at any other point.

Relationship to $\mathbb{C}$

Theorem 10 (Smoothness)

Let Ω be an open set in $\mathbb{R}^n$ and let u be harmonic on Ω, then $u \in C^\infty(\Omega)$

Definition 11 (Approximation to the Identity)

Let $T\colon \mathbb{R}^n \to \mathbb{R}$ be a function such that

- $T \in C^\infty(\mathbb{R}^n)$
- $T \geq 0$
- $\operatorname{supp}(T) \subset B(0,1)$
- $T(x) = T(|x|)$
- $\int_{\mathbb{R}^n} T \, \mathrm{d}\sigma = 1$

T is called an **approximate identity** or **nascent delta function**.

$$T(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}} & x \in B(0,1) \\ 0 & \text{otherwise} \end{cases}$$

Theorem 12

Let Ω be an open subset of $\mathbb{R}^n$. Then if f is a Lebesgue measurable and integrable function on any compact subset $\Omega' \subset \Omega$ and $g \in C^k(\mathbb{R})$ with $\mathrm{supp}(g) \subset B(0,r)$ then the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}\sigma$$

is in $C^k(\Omega_r)$ where $\Omega_r = \{x \in \Omega \mid d(x, \partial\Omega) > r\}$.

Informally, choosing f to be harmonic and g to be an approximate identity, after direct calculation and invocation of the Mean Value Property one arrives to

$$(T * f)(x) = f(x) \int_{B(0,1)} T(y)\,\mathrm{d}\sigma(y) = f(x)$$

And so f which is harmonic is equivalent to its convolution with the approximate identity. By the previous theorem, the convolution is C^∞ since the approximate identity is and hence f is also C^∞.

Theorem 13 (Liouville's Theorem)

A bounded harmonic function on $\mathbb{R}^n$ is constant.

Corollary 14

Suppose u is a continuous bounded function on $\overline{\Omega}$ that is harmonic on Ω. If $u = 0$ on $\partial\Omega$ then $u \equiv 0$ on $\overline{\Omega}$

Theorem 15

If u is a harmonic function defined on an open subset $\Omega \setminus \{x_0\}$ of $\mathbb{R}^n$ which is less singular at x_0 than the fundamental solution of the Laplacian, that is for $n > 2$,

$$u(x) = o(|x - x_0|^{2-n})$$

then u extends to a harmonic function on Ω

Fundamental Solutions

Theorem 16

Among all second-order homogeneous PDEs in two dimensions with constant coefficients, show that the only ones that do not change under a rotation of the coordinate system (i.e., are rotationally invariant), have the form

$$a\Delta u = bu$$

Let u be a solution to $\Delta u = 0$ on $\mathbb{R}^n$ and assume the solution takes the form $u = u(r)$ where $r = |x|$ is the euclidean norm in n-dimensions. After substitution into the equation, one derives that the solutions are

$$u(r) = \begin{cases} b \ln r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3 \end{cases}$$

Definition 17 (Fundamental solution)

For a linear partial differential operator L, a fundamental solution G is a solution to the inhomogeneous equation

$$LG = \delta(x)$$

where δ is the Dirac delta function.

The function G is commonly known as the Green's function for the partial differential operator. It is the function which satisfies an impulse response of the the operator in question. It turns out that one can choose specific constants for the derived radial solutions to find the fundamental solution. In particular, the fundamental solutions to $\Delta u = \delta$ are given by

$$G(x) = \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2 \\ \frac{1}{n(2-n)V(B(0,1))} \frac{1}{|x|^{n-2}} & n > 2 \end{cases}$$

This allows one to solve Poisson's inhomogenous equation $\delta u = f$ for some compactly supported function f. Since G is harmonic on $x \neq 0$(by construction of the radial solution), then G shifted to a point y given by $G(x-y)$ is also harmonic and furthermore $G(x-y)f(y)$ is also harmonic when $x \neq y$. This leads to the following theorem whose proof is also omitted

Theorem 18

Consider the Poisson equation in $\mathbb{R}^n$

$$\Delta u = f$$

where f is compactly supported. Then the solution to the Poisson problem is given by the convolution

$$u = (G * f)(x) = \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}^n} \ln |x-y| f(y) \, \mathrm{d}y & n = 2 \\ \frac{1}{n(2-n)V(B(0,1))} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} \, \mathrm{d}y & n > 2 \end{cases}$$

Decomposition Theorems

Theorem 19 (Bôcher's Theorem)

Let $a \in \Omega$. If u is harmonic on $\Omega \setminus \{a\}$ and positive near a, then there is a harmonic function v on Ω and a constant $b \in \mathbb{R}$ such that

$$u(x) = \begin{cases} v(x) + b \ln |x - a| & n = 2 \\ v(x) + b|x - a|^{2-n} & n > 2 \end{cases}$$

Theorem 20

The general solution, f, to Laplace's equation on a sphere is given by

$$f(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F_l(r) Y_l^m(\theta, \varphi)$$

where $F(r)$ is a radial function and $Y(\theta, \varphi)$ is a spherical harmonic. This is analogous to Fourier series on the unit circle.

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Questions?