HEAT KERNELS AND ASYMPTOTICS WITH A VIEW TOWARDS INDEX THEORY

Hussein Hijazi

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Project Supervisor: Prof. Simon Scott

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For my family, of course…

Abstract

In the following paper, we build on the ideas developed from the study of the heat equation. Let Δ_g be the Laplacian associated to a manifold, then there is a unique heat operator $e^{-t\Delta_g}$ which describes the manner in which a distribution is propagated. To such a heat operator exist asymptotics that describe the short-time behavior and local geometry of the manifold. Such asymptotics can be constructed from the theory of pseudodifferntial operators and traces of the asymptotics reveal topological and geometric information related to Index theory. Furthermore, explicit calculations are completed at the end to demonstrate the feasibility of such expansions.

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Introduction

Perhaps one of the most remarkable theorems of the 21st century with far reaching implications and generalization and the main motivation for our study of the heat kernel is the Atiyah-Singer Index theorem

Theorem 1.1

Let D be a Dirac operator and M a Riemannian manifold

$$\underbrace{\operatorname{ind}(D)}_{\text{Analytical index}} = \underbrace{\frac{1}{(2\pi i)^n}\int_M \hat{A}(M)\operatorname{ch}(V,F)}_{\text{Topological index}}$$

On the lefthand side we have the **Fredholm index** assigned to an elliptic operator and is a purely analytical object and important to note is an integer. It is defined as

$$\operatorname{ind}(D) = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D$$

and which were originally studied on systems of differential equations. In some sense, the index measures the defect of an operator The right-side may look confusing but we remark that it is a purely topological object. The **A-hat genus**, $\hat{A}$, and the **Chern character**, ch, are related to the cohomology of the manifold. A priori, the topological index is known to be rational. However, if one is able to realize a certain topological datum in the form of the topological index, then one can learn more information by knowing that the value is an integer.

The theorem itself is built upon many others and is a generalization of results which link analytical, topological and geometric concepts including but not limited to the Gauss-Bonnet theorem, the Hirzebruch–Riemann–Roch theorem, the Hirzebruch signature theorem and more. Index theory or Invariance theory are names given to the field of understanding such concepts and draw on ideas from Differential/Algebraic Geometry/Topology, Hodge Theory, Analysis, K-Theory, operator theory, number theory and even mathematical physics.

Of particular interest to us is the heat kernel method and we will set out to understand its pillars. The heat kernel approach began in the late 1960s due to Mckean and Singer[HS67] after inspiration from Minakshisundaram and Pleijel [MP49]. From there, Patodi [Pat71] and Gilkey [Gil18]developed the theory heavily and since then Index theory has been an amazing amalgamation of many different faces. Inherently, the propagation of heat is not a global property; indeed, a probabilistic interpretation is that the heat kernel tells you the probability that heat from a starting point x on a manifold will reach a point y at time t. This is governed by stochastic Brownian motion and a particularity of it is that a random walk can run arbitrarily far and still have nonzero return probability. Hence, the propagation of heat depends on the global topology of the manifold, rather than the local.

However, for short times, the heat distribution is relatively the same and has not smeared out, and hence can be determined by local data. This local data turns out to reveal remarkable and simple identities relating the spectrum of the Laplacian to the manifold.

Our discussion is organic and does not require too many preliminaries. Indeed, as the topic draws on from various areas, it is the author's choice not to spend time discussing preliminaries but rather invoking them as necessary in order to maintain fluidity of prose. In Chapter 1, we begin by understanding the familiar heat kernel to us a standard course on PDEs and Fourier Analysis. From there we build up to the construction of the more general heat operator and associated heat kernel on the manifold using [Ros10][Can13]. Chapter 2 discusses the asymptotic expansions using Riemannian geometry to reveal geometric invariants associated to the manifold and from there the discussion pivots to what is known as the symbol calculus of pseudo differential operators and follow [Sco18][Gri15][Gil18]. Finally, in Chapter 3 we give examples of how the heat kernel method finds its place in Index theory and conclude with an asymptotic calculation of an important ODE.

The Heat Kernel

2.1 The basics

The following constructions result mainly from [Ros10].

We begin with a discussion of the Laplacian on the unit circle, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Consider the Hilbert space $L^2(S^1, \mathbb{C})$ of L^2 functions taking $S^1 \to \mathbb{C}$. The Laplacian operator in this situation is taken to be

$$\Delta = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$$

Here the minus sign is taken for conventional reasons. Recall from standard Fourier Analysis that this space admits an orthonormal basis of eigenfunctions as the complex trigonometric exponentials $e^{in\theta}$ where $n \in \mathbb{Z}$ with eigenvalues $\{n^2\}$ as $\Delta e^{in\theta} = n^2 e^{in\theta}$. Then any function $f \in L^2$ can be represented by the decomposition

$$f = \sum_n a_n e^{in\theta}, \quad a_n = \frac{1}{2\pi}\int_{S^1} f(\theta)e^{-in\psi}\mathrm{d}\psi$$

We notice that because the spectrum of Δ, denoted $\sigma(\Delta)$, on S^1 is discrete, that such a formula is possible. The case is not similar on $L^2(\mathbb{R})$ as $\sigma(\Delta) = [0, \infty]$ and hence is no longer discrete. The calculation of $\sigma(\Delta)$ outlined as an exercise in [Ros10, p.4] through operator theory. However, as is remarked in [Ros10] the peculiar similarity of the Fourier series decomposition and the Fourier inversion formula.

$$f(\theta) = \sum_n \left(\frac{1}{2\pi}\int_{S^1} f(\psi)e^{-in\psi}\mathrm{d}\psi\right)e^{in\theta}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-i\xi y}\mathrm{d}y\right)e^{i\xi x}\mathrm{d}\xi$$

Turning our attention to the discussion at hand, consider the standard diffusion of heat on a perfectly insulated circle of an initial distribution of heat given by $f(0, \theta) = f(\theta)$. The evolution of this system, according to thermodynamics, is demosntrated by the equation

$$(\partial_t + \Delta)f(t, \theta) = 0$$

whose solution can be derived, given $f(t, \theta) = \sum_n a_n(t)e^{in\theta}$

$$f(t, \theta) = \sum_n a_n e^{-n^2 t}e^{in\theta} \tag{2.1}$$

where a_n are once more the Fourier coefficients of f. By physical intuition, one predicts the long time behavior of the temperature distribution to be a constant. This is true as simply seeing that $t\infty$ results all terms of the sum, except for a_0, to dissipate and hence the constant reached is the average value of the initial distribution. To understand the heat flow on $\mathbb{R}$ requires a bit more effort, employing the Fourier transform whose presence will be consistent in our discussion, hence we formally define:

Definition 2.1

The **Fourier transform** on $\mathbb{R}^n$ is an isometry on $L^2(\mathbb{R}^n)$ given by the equation (and its inversion)

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathrm{d}x \qquad \check{f}(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi$$

where $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$ are vectors in $\mathbb{R}^n$ and $\mathrm{d}x$ is the modified Lebesgue measure $(2\pi)^{n/2}\mathrm{d}x_1 \dots \mathrm{d}x_n$ to reduce the notation. Furthermore, for $f, g \in L^2(\mathbb{R}^n)$, we define their **convolution** as

$$f * g(x) := \int_{R^n} f(x-y)g(y)\mathrm{d}y$$

Now given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote the complex derivative

$$D^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$$

we recall the following properties of the Fourier transform

Lemma 2.2

(i) $\widehat{f * g} = \hat{f} \cdot \hat{g}$

(ii) $(\widehat{fg}) = \hat{f} * \hat{g}$

(iii) $(\widehat{D^\alpha f})(\xi) = \xi^\alpha \hat{f}(\xi)$

Using the above, it is a standard result from undergraduate courses on PDE that on $\mathbb{R}$, by transforming the heat equation, solving and inverting one retrieves the solution to be

$$f(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4t}} f(y)dy \tag{2.2}$$

for which one can verify that $(\partial_t + \Delta)f(t,x) = 0$ and $\lim_{t\to 0} f(t,x) = f(x)$. Indeed, one directly recognizes that the function being integrated against is commonly referred to as the heat kernel. We note it is a smooth function

$$e(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}} \in C^\infty((0,\infty) \times \mathbb{R} \times \mathbb{R})$$

Once more, the physical intution tells us that the long time behavior should be a constant. However, given $\mathbb{R}$ is infinitely long and non-compact, the constant should be 0 (and it is easily verified taking $t \to \infty$ in $f(t,x)$). Looking back we can find a similar expression on S^1 simply by substitution the defintion of a_n into 2.1 to get

$$f(t,\theta) = \frac{1}{2\pi} \int_{S^1} \underbrace{\sum_n e^{-n^2 t} e^{in\theta} e^{-in\psi}}_{e(t,\theta,\psi)} f(\psi)\mathrm{d}\psi$$

which is also an integral kernel and smooth on its domain.

2.2 The setting on manifolds

Much of this behavior and intuition can be extended to a general compact Riemannian manifold (M, g) but we will only focus on functions rather than differential forms. We will construct our heat kernel in this general setting but will not dwell on the elements of geometry required now. Instead, to facilitate a continuous construction, we will invoke geometric preliminaries when necessary, starting with the following definition

Definition 2.3

The **Laplace-Beltrami** operator on a Riemannian manifold (M, g) is the operator defined as

$$\Delta_g : C^\infty(M) \to C^\infty(M)$$
$$\Delta_g : -\mathrm{div}_g \circ \nabla_g$$

where div and ∇ are the respective divergence and gradient operators. In local coordinates, this is given by

$$\Delta_g = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i (g^{ij} \sqrt{|g|} \partial_j)$$

Now let (M, g) be compact, Riemannian manifold with $\partial M = \emptyset$ and define the **heat operator** as $L := \partial_t + \Delta$ acting on function in $C((0, \infty) \times M)$ such that it is C^1 in time and C^2 in space. Then, on this manifold, we have two heat equations

$$\begin{cases} Lu(x,t) = F(x,t) \\ u(x,0) = f(x) \end{cases} \qquad \begin{cases} Lu(x,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

the former being the more general inhomogenous equation with dispersive force $F(x, t)$ and initial distribution $f(x)$ while the latter is homogeneous. As one does when dealing with partial differential equations, we wonder if this solution is unique. Indeed it is

Proposition 2.4

The solution to the inhomogeneous heat equation is unique.

Proof. Let u_1 and u_2 be two solutions to the homogeneous equation, then their difference $u = u_1 - u_2$ solves

$$\begin{cases} Lu(x,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Furthemore, note that energy decreases with time under the L^2 norm (here ω_g is the volume form on M)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} |u(\cdot, t)|^2_{L^2} &= 2 \int_M \partial_t u(x,t) u(x,t) \omega_g \\ &= -2 \int_M \Delta_g u(x,t) u(x,t) \omega_g \\ &= -2 |\nabla_g u(\cdot, t)|^2_{L^2} \leq 0 \end{aligned}$$

Hence, as the energy of the difference is decreasing but the initial condition is 0, hence $u(x,t) = 0$ for all $x \in M$ and the result follows. ■

We now state without proof Duhamel's principle in the context of the manifold which will be used in forthcoming proofs. The statement itself may look convoluted, but the principle is the same in classical case: it associates perturbations applied at two different times, that is u perturbing v at α and β, to the propagation of heat at each time t in between α and β.

Proposition 2.5: Duhamel's Formula

[Can13]Let $u(x,t)$ and $v(x,t)$ be two functions on M which are C^2 in space and C^1 in time. Then, for any $t > 0$ and α, β such that $[\alpha, \beta] \subset (0,t)$, we have the equality

$$\begin{aligned} &\int_M u(y, t-\alpha)v(y,\alpha) - u(y,t-\beta)v(y,\beta)\omega_g \\ &= \int_\alpha^\beta \int_M Lu(y,t-s)v(y,s) - u(y,t-s)Lv(y,s)\omega_g \mathrm{d}s \end{aligned}$$

Now we are ready to state the first main definition

Definition 2.6

A **fundamental solution** or **heat kernel** is a continuous function $e(t,x,y)$: $(0,\infty) \times M \times M \to \mathbb{R}$, C^2 in space and C^1 in time, satisfying

$$L_y e(t,x,y) = 0 \quad \text{and} \quad \lim_{t \to 0} e(t,\cdot,y) = \delta_y = \delta(\cdot - y)$$

where δ_y is the Dirac distribution in the y variable.

In this sense, it is a nascent delta function satisfying the heat equation and is a pointwise distributor of the heat flow, hence being called fundamental. Using Duhamel's principle we can easily show why the heat kernel is an interesting object.

Theorem 2.7

The heat kernel on M is unique and is symmetric in its space variables.

Proof. Let $e_1(t,x,y)$ and $e_2(t,x,y)$ be two heat kernels on the same manifold. Let $u(y,t) = e_1(t,x,y)$ and $v(y,t) = e_2(t,z,y)$ with x and z fixed. Since they are both heat kernels, the right hand side of Duhamel's formula is 0 and hence

$$\int_M e_1(t-\alpha,x,y)e_2(\alpha,z,y) - e_1(t-\beta,x,y)e_2(\beta,z,y)\omega_g = 0$$

Then by sending $\alpha \to 0$ and $\beta \to t$

$$\int_M e_1(t,x,y) \lim_{\alpha \to 0} e_2(\alpha,z,y) - \lim_{\beta \to t} e_1(t-\beta,x,y)e_2(t,z,y)\omega_g = 0$$

By integrating and invoking the second property of the heat kernel, this simply becomes $e_1(t,x,z) = e_2(t,z,x)$ and so by now instead considering $e_1 = e_2$ we first find that the heat kernel is indeed symmetric on $M \times M$. Furthermore, since it is symmetric we also have $e_1(t,x,z) = e_2(t,x,z)$ and hence the heat kernel is unique. ■

From this we understand that for each compact Riemannian manifold there is only one way to diffuse an initial heat distribution. This ties into the concept that heat kernels can detect the topology of a manifold. Indeed, consider 2.2 and understand that this says the heat at a point x is affected by all points y, arbitrarily far, i.e it has infinite propagation speed. As such, the global structure of the manifold is critical to the behavior. As before, one can use the heat kernel to find a general solution to the heat equation

Theorem 2.8

Let $f \in C(M)$ and $F \in C(M \times (0,\infty))$, then the solution to

$$\begin{cases} Lu(x,t) = F(x,t) \\ u(x,0) = f(x) \end{cases}$$

is given by

$$u(x,t) = \int_M e(t,x,y)f(y)\omega_g + \int_0^t \int_M e(s,x,y)F(y,t-s)\omega_g \mathrm{d}s$$

Proof. Let u be as it is and $v(y,t) = e(t,x,y)$, then by fixing x and once again using Duhamel's formula, applying the properties of the heat kernel

$$\int_M u(y,t-\alpha)e(\alpha,x,y) - u(y,t-\beta)e(\beta,x,y)\omega_g = \int_\alpha^\beta \int_M F(y,t-s)e(s,x,y)\omega_g \mathrm{d}s$$

Sending $\alpha \to 0$ and $\beta \to t$ the first term on the left hand side becomes $u(x,t)$, the second moves to the rightside and the result is proven. ■

One interpretation is as follows: the first term in the solution is the homogeneous natural effect of the heat distribution onto the initial condition without any external forces. Then the force of the inhomogeneity is propagated at all times up to t and at each time t sending the effect of the external force by heat diffusion across the solution. Their sum is then the resulting heat distribution on the manifold.

2.3 The heat operator

With what we have constructed, knowing that to each (M,g) we have a unique integration kernel describing the behavior of heat flow on the manifold, we come to another core concept:

Definition 2.9

On a compact Riemannian manifold (M, g), the **heat operator, heat propagator** and sometimes, by abuse, the **heat kernel** is the operator, for $t > 0$, given by

$$e^{-t\Delta_g} : L^2(M, g) \to L^2(M, g)$$
$$e^{-t\Delta_g} f(x) := \int_M e(t, x, y) f(y) \omega_g$$

We look to understand what sort of operator we are dealing with and use this result from [Can13]

Proposition 2.10

(i) $e^{-t\Delta_g} e^{-t'\Delta_g} = e^{-(t+t')\Delta_g}$.

(ii) The heat operator is self-adjoint and positive.

(iii) The heat operator is compact.

(iv) As $t \to 0$, $e^{-t\Delta_g} f$ converges to f in $L^2(M, g)$.

Proof. For (i) refer to [Ros10, p.26]. For (ii) simply compute for $f, g \in L^2(M, g)$ the inner product with the operator applied to f and under reasonable conditions one can move things around to show $\langle e^{-t\Delta_g} f, g\rangle = \langle f, e^{-t\Delta_g} g\rangle$. For positivity

$$\langle e^{-t\Delta_g} f, f\rangle = \langle e^{-t/2\Delta_g} e^{-t/2\Delta_g} f, f\rangle = \langle e^{-t/2\Delta_g} f, e^{-t/2\Delta_g} f\rangle = |e^{-t/2\Delta_g} f|^2_{L^2} \geq 0$$

For (iii) it can be shown in fact the codomain is $H_1(M, g)$, which is the $1st$ Sobolev space , i.e the complection of $C_c^\infty(M)$ with respect to the norm

$$|f|_{H_1} = \left(\sum_{|\alpha| \leq 1} |D^\alpha f|^2_{L^2} \right)$$

Furthermore, the mapping is continuous and there is a wrap around compact inclusion of $H_1(M) \subset L^2(M)$ and hence the composition

$$L^2(M) \xrightarrow{e^{-t\Delta_g}} H_1 \overset{L}{\hookrightarrow}{}^2 (M)$$

is compact. For (iv) see [Can13, p. 67] ■

The first property satisfies our intuition of the physical world: the heat flow at the sum of two times t and t' should be the same as completing a heat flow up to t and starting with that as the initial condition flowing to time t'. Two and three make it a compact, self-adjoint operator, which is key in the upcoming discussion. Finally, the last point tells us the operator, too, behaves like a nascent delta distribution and returns the initial condition as $t \to 0$. Indeed, we see that $e^{-t\Delta_g}$ is compact and self-adjoint and so recalling the spectral theorem for compact self-adjoint operators, we expect there to be an eigendecomposition just as in the case of S^1. This is true:

Theorem 2.11: Hodge theorem for functions

Let (M, g) be a compact connected oriented Riemannian manifold. There exists an orthonormal basis $\{\varphi_0, \varphi_1, \dots\}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g with φ_i having eigenvalue λ_i satisfying

$$\lambda_0 \leq \lambda_1 \leq \dots \to \infty$$

Furthemore for all i the φ_i are smooth and

$$e(t, x, y) = \sum_{j=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$$

The proof for this is quite technical and can be found in [Can13, p.68-69] and is also known as the Sturm-Lioville decomposition. Hence, similar as to the heat flow on S^1 earlier, we have a decomposition of the heat kernel into eigenspaces. This tells us that the heat kernel is *audible*: it is dependent on the eigenbasis of the manifold but more specifically to its eigenvalues, hence a critical link to Spectral geometry. Certainly it is beautiful that one can concretely deduce that heat flow describes the geometry of a manifold by 'hearing its frequencies'. It is important to remark however that throughout all of this, we have not justified the existence of the heat kernel. Thankfully, it does exist, but the proof is technical and can be found in section 3.2.2 of [Ros10]. We complete this section with a final corollary leading into our next discussion

Corollary 2.12

For all $t > 0$

$$\operatorname{Tr} e^{-t\Delta_g} := \int_M e(t, x, x) \omega_g = \sum_{k=0}^{\infty} e^{-\lambda_k t}$$

where Tr represents the trace of an operator.

Proof. Integrate the decomposition and simply use the fact that the φ's are orthonormal giving $\int_M \varphi_i^2 = 1$ and the result is proven. ■

Asymptotics of the Heat Kernel

We now turn to understanding how one can determine the heat kernel of a manifold and a elliptic differential operator on it. First, given functions $f(t)$ and $g(t)$ we write $f(t) \sim g(t)$ if

$$\lim_{t \to 0} \frac{f(t) - g(t)}{t^m} = 0, \quad \forall m \in \mathbb{R}^+$$

and say that f is asymptotic to g and so

Definition 3.1

Let f be a function on $\mathbb{R}^+$ with values in a Banach space E. A formal series

$$f(t) \sim \sum_{k=0}^{\infty} a_k(t)$$

where the a_k are functions $\mathbb{R}^+ \to E$ is called an **asymptotic expansion** for f near 0 if for all $n \in \mathbb{Z}^+$ there exists a cutoff ℓ_n such that for all $\ell \geq \ell n$ there is a constant $C_{l,n}$ such that

$$\left| f(t) - \sum_{k=0}^{\ell} a_k(t) \right| \leq C_{l,n} |t|^n$$

Equivalently, $f(t)$ is asymptotic to its expansion

Though we did not cover it, the proof of the heat kernel's existence has one important construction of an approximate solution to the heat equation otherwise known as a **parametrix**. It is not an exact solution, that is it does not directly satisfy the heat equation, but does so up to terms t^k with an error term decaying faster than polynomials for x and y close, that is in short time. This then leads showing the heat kernel also takes the form

$$e(t, x, y) = (H_k - Q_k * H_k)(t, x, y)$$

where H_k and Q_k are some types of parametricies being convolved. We will not focus on this but it is important to understand that these approximations work only for t near 0. Using similar methods of parametricies one can prove the following, though we will omit it (see [Roe01] for details).

Proposition 3.2

The heat kernel $e(t, x, x)$ has a short-time asymptotic expansion

$$e(t, x, x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} u_k(x, x) t^k$$

where u_k are polynomials to be determined and $u_0(x, x) = 1$. Furthermore, $e(t, x, y) \sim 0$ for $x \neq y$.

We now make the relation to the conclusion of the previous section

Theorem 3.3

Let $\{\lambda_i\}$ be the spectrum of the Laplacian on functions on (M, g). Then

$$\sum_i e^{-\lambda_i t} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k, \qquad a_k = \int_M u_k(x,x)\omega_g$$

Proof. Simply by combining the above theorem and 2.12 we see

$$\begin{aligned}\operatorname{Tr} e^{-t\Delta_g} = \sum_i e^{-\lambda_i t} = \int_M e(t,x,x)\omega_g \sim \int_M (4\pi t)^{-n/2} \sum_k u_k(x,x) t^k \omega_g \\ \sim (4\pi t)^{-n/2} \sum_k \left(\int_M u_k(x,x)\omega_g \right) t^k \quad \blacksquare\end{aligned}$$

With the last two results we now have asymptotics for both the kernel and it's trace and can finally demonstrate how one may use these expansions to determine the geometry of the manifold

Corollary 3.4

Let M and N be compact isospectral[1] Riemannian manifolds. Then M and N have the same dimension and the same volume.

Proof. Let $\{\lambda_i\}$ denote the spectrum of M and N and m, n their respective dimensions. Since they are isospectral, we have a chain of similarities

$$(4\pi t)^{-m/2} \sum_k \left(\int_M u_k(x,x)\omega_g \right) t^k \sim \sum_i e^{-\lambda_i t} \sim (4\pi t)^{-n/2} \sum_k \left(\int_N u_k(x,x)\omega_g \right) t^k$$

Therefore, m and n are equal and hence by splitting the sum at $k = 0$ and taking the differences we see (using the superscript to differ the polynomials)

$$\begin{aligned}&(4\pi t)^{-m/2} \left(\int_M u_0^M(x,x)\omega_g - \int_N u_0^N(x,x)\omega_g \right) \\ &\sim (4\pi t)^{-m/2} \sum_k \left(\int_M u_k^M(x,x)\omega_g - \int_N u_k^N(x,x)\omega_g \right) t^k\end{aligned}$$

This directly implies, since these differences are asymptotic to 0 by isospectrality, that

$$\int_M u_0^M(x,x) = \int_N u_0^N(x,x)$$

and by iterating one arrives that such a relation holds for all k. From Prop 3.2, we know that $u_0 = 1$ and so we are integrating each manifold over their own volume measure and therefore $\mathrm{Vol}(M) = \mathrm{Vol}(N)$. $\blacksquare$

This isn't the only theorem of it's kind as one can further show that $u_1(x,x) = 1/6s(x)$ where $s(x)$ is the scalar curvature and even more obtusely as shown by Mckean

[1] The eigenvalues of Δ on both manifolds conicide, including mulitplicities

and Singer

$$u_2(x,x,) = \frac{1}{72}s(x)^2 - \frac{1}{180}|\mathrm{Ric}|^2 + \frac{1}{180}|R_x|^2 + \frac{1}{30}\Delta_g s(x)$$

where Ric is the Ricci tensor and R_x the Riemann curvature tensor. Even more surprising is that two compact isospectral manifolds are *diffeomorphic*!
We can see from the previous proof that in order for two manifolds to be isospectral, they need to agree on all of the integrals. We have just seen that these integrals result in geometric information and so it begs the question, how does one actually determine the functions $u_k(x,x)$. In fact as it turns, the u_k are all polynomials in the components of R_x and its covariants derivatives, that is to say

$$\begin{aligned} u_1(x,x) &= P(R_x) \\ u_k(x,x) &= P(R_x, \nabla R_x, \nabla^2 R_x, \ldots, \nabla^{2i-4}R_x), \quad k \geq 2 \end{aligned}$$

for universal polynomials. Much of this discussion and further can be found in [Ros10, p. 3.3] and beyond.

3.1 Pseudodifferential operators and symbols

Now that we have seen the power of the heat kernel and its trace, it would be good to see how one can actually construct the kernel. For this we follow a different theory, namely that of pseudodifferential operators (PDOs) and their symbol calculus. This will mainly follow [Sco18] supplemented by other resources. Once again, let (M,g) be a smooth compact Riemannian manifold with dimension n.
When trying to solve a differential equation on a manifold, what we are essentially doing is finding an operator that inverts Δ_G, but itself may not be a differential operator. Thus it is in our interest to enlarge the class of differential operators so that their 'inverses' are also of the same class. The key connection be made is that, given a differential operator $P = \sum_{|\alpha|\leq d} p_\alpha(x)D_x^\alpha$ of degree d with p_α smooth on x, then for an appropriately behaving f, we have

$$Pf(x) = \sum_{|\alpha|\leq d} p_\alpha(x)D_x^\alpha f = \sum_{|\alpha|\leq d} p_\alpha(x)(D_x^\alpha \widehat{f})^* = \sum_{|\alpha|\leq d} p_\alpha(x)(\widehat{\xi^\alpha f})^* = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\mathrm{d}\xi$$

where $*$ is denoting the Fourier inversion. Hence a differential operator can be expressed by converting differentiation into multiplication using the Fourier transform and hence direct calculations can be used to determine information about the operator.

Definition 3.5

A **pseudodifferential operator** (PDO) P of order k on a compact Riemannian manifold acting on an appropriate function f is given by

$$Pf := \int_M e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\mathrm{d}\xi$$

where $\hat{f}$ is the Fourier transform of f.
The function $\sigma(x,\xi) = \sum_{|\alpha|\leq k} p_\alpha(x)\xi^\alpha$ is the **(full) symbol** associated to P. The highest order term is the **principal symbol** $\sigma_k(x,\xi) = \sum_{|\alpha|=k} p_\alpha(x)\xi^\alpha$.

The space of pseduodifferntial operators of order k is denoted Ψ^k and is in fact an algebra. The algebra homomorphism is given by symbol composition in the following way. If A and B are PDOs with symbols $a(x,\xi)$ and $\underline{(}x,\xi)$, then the composition AB has symbol

$$a \circ b(x,\xi) = \sum_{\mu} \frac{1}{\mu!} \partial_{\xi}^{\mu} a(x,\xi) D_{x}^{\mu} b(x,\xi)$$

where

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n, \quad \frac{1}{\mu!} = \frac{1}{\mu_1! \dots \mu_2!}, \quad \partial_{\xi}^{\mu} = \partial_{\xi_1}^{\mu_1} \dots \partial_{\xi_n}^{\mu_n}, \quad D_{x_j} = -i\partial_{x_j}$$

Furthermore we define $\Psi^{-\infty} := \bigcap_k \Psi^k$ and call them smoothing operators. It remains to understand what spaces are being invoked under such transformations.
Recall $C_c^{\infty}(M)$ are smooth, compactly supported functions on the manifold and are called **test functions**. The dual to this space, $C_c^{\infty}(M)' = \mathcal{D}'(M)$, is called the space of **distributions**, that is, linear functionals T on $C_c^{\infty}(M)$ which are continuous with respect to the topology. Distributions which extend to a functional on the Schwartz space $\mathcal{S}(M)$ of rapidly decaying functions are called **tempered** and this space is denoted by $\mathcal{S}'(M)$. Lastly, distributions extending continuously and linearly to $C^{\infty}(M)$ are known as **compactly supported** denoted as $\mathcal{E}'$ and we have $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$ as vector spaces. It is enough to know that $f \in C_c^{\infty}(M)$ for the above definition to be reasonably behaved and we write $P : C_c^{\infty}(M) \to \mathcal{D}'(M)$, as smooth compactly supported f is sent into $\mathcal{D}'(M)$ by using the inner product $(T_f, \varphi) \mapsto \int_M f(x)\overline{\varphi(x)}$ for $\varphi \in C_c^{\infty}(M)$. The final result we will use from the theory of distributions is the following result

Theorem 3.6

Let X, Y be open sets of $\mathbb{R}^n$. To every continuous operator $K : \mathcal{S} \to \mathcal{S}'$ there exists a distribution $k \in \mathcal{S}'(X \times Y)$, called its Schwartz kernel, such that for $u \in C_c^{\infty}(X)$ and $v \in C_c^{\infty}(Y)$, the following holds

$$Kv = \int_Y k(\cdot, y)v(y)\mathrm{d}y$$

and

$$\langle Kv, u \rangle = \int_X \int_Y k(x,y)v(y)u(x)\mathrm{d}y\mathrm{d}x$$

With this, to any $P \in \Psi^k$, there exists an integral kernel to represent it. If this kernel is smooth and the norm of its derivatives are smoothly bounded, i.e $|D^{\alpha} f(x)| \leq C_{\alpha}$, then we say P is **smoothing**. For further discussion, see [Tre80, p. C.1][Sin17][Shu01] [Pat]
With that preliminary covered, we proceed onto the construction from [Sco18]. Consider operators of the form $P = \Delta_g + \sum_{k=0}^{n} a_k \partial_k$ where $a_k \in C_c^{\infty}(M)$. We will assume that the lower order terms do not affect the overarching behavior and that $\sigma(P)$ exhibits the same properties/structure as $\sigma(\Delta_g)$. Specifically, the spectrum must be discrete with a complete orthonormal eigenbasis and the eigenvalues must be non-negative and accumulate at infinity. First, we redefine the heat operator for P in a spectral context

3.2 The resolvent construction

Definition 3.7

The **heat operator** e^{-tP} can be alternatively defined using the Cauchy integral formula

$$e^{-tP} := \int_\Gamma e^{-\lambda}(tP-\lambda)^{-1}\overline{\mathrm{d}\lambda} \overset{\lambda\mapsto t\lambda}{=} \int_\Gamma e^{-t\lambda}(P-\lambda)^{-1}\overline{\mathrm{d}\lambda}$$

where $\overline{\mathrm{d}\lambda} = \frac{i}{2}\pi \mathrm{d}\lambda$ and Γ is a contour enclosing the spectrum. A common contour, given small $c \in \mathbb{R}$, consists of two rays $\{re^{\pm i\pi/4} | c \leq r\}$ and the arc $\{ce^{i\theta} | \theta \in (\pi/4, 3\pi/4)\}$

This equivalent definition of the heat operator is based solely on a 'functional calculus' by treating the operator expression as a complex function one uses the Cauchy integral formula. Note that only if $\lambda \notin \sigma(P)$ then $P-\lambda$ is invertible. Indeed, $(P-\lambda)^{-1}$, called the **resolvent** satisfies 3.6 and as such is associated to a kernel which also depends on λ, denoted $R(x,y,\lambda)$ and acts as such

$$(P-\lambda)^{-1}\varphi(x) = \int_M R(x,y,\lambda)\varphi(y)|\mathrm{d}y|$$

where ω_g is the volume form and $\varphi \in C_c^\infty$. Applying this to the definition of the heat operator

$$\begin{aligned}\left(e^{-tP}\varphi\right)(x) &= \int_\Gamma e^{-t\lambda}(P-\lambda)^{-1}\varphi(x)\overline{\mathrm{d}\lambda} \\ &= \int_\Gamma e^{-t\lambda}\int_M R(x,y,\lambda)\varphi(y)\omega_g\overline{\mathrm{d}\lambda} \\ &= \int_M \underbrace{\int_\Gamma e^{-t\lambda}R(x,y,\lambda)\overline{\mathrm{d}\lambda}}_{e(t,x,y)}\varphi(y)\omega_g \end{aligned} \tag{3.1}$$

Thus, we have retrieved the first definition with a new perspective on the expression of the kernel. The goal is to use this formulation to find an asymptotic expansion for the heat kernel which is computable and so it remains to asymptotically understand $R(x,y,\lambda)$. It turns out that the resolvent symbol admits an asymptotic expansion

$$R(x,y,\lambda) \sim \sum_{j\geq 0} r_{-2-j}(x,y,\lambda)$$

where r_{-2-j} become smoother as $j \to \infty$ and are **quasi-homogeneous**, i.e.

$$r_{-2-j}(x,ty,t^2\lambda) = t^{-2-j}r_{-2-j}(x,y,\lambda)$$

Following the discussion on [Sco18, L.2 p.6][Gri15, p. C.3], we can write the kernel as Fourier transform

$$R(x,y,\lambda) = \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}r(x,y,\xi,\lambda)\mathrm{d}\xi = \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}\sum_{j\geq 0} r_{-2-j}(x,y,\lambda)\mathrm{d}\xi + E_{-\infty}(x,y)$$

where $E_{-\infty}$ is a smoothing operator whose trace is null and $r(x,y,\xi,\lambda)$ is the symbol amplitude associated to $(P-\lambda)^{-1}$. In this sense, the two sides are asymptotic because their difference vanishes locally and thus

$$R(x,y,\lambda) \sim \sum_{j\geq 0} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} r_{-2-j}(x,\xi,\lambda)\mathrm{d}\xi \tag{3.2}$$

To compute the terms recursively, we need to localize the situation. By choosing suitable coordinate charts between M and $\mathbb{R}^n$, locally we can write P as

$$\begin{aligned} P = \Delta_g + \sum_{k=0}^{n} a_k \partial_k &= -\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i (g^{ij}\sqrt{|g|}\partial_j) + \text{lower order terms} \\ &= \sum_{k,l} g^{kl}(-i\partial_k)(-i\partial_l) + \text{lower order terms} \end{aligned}$$

since $|g|= 1$. Replacing $(-i\partial_k)$ by ξ_k we find the local symbol of $\sigma(P)(x,\xi)$ as

$$\sigma(P)(x,\xi) = a(x,\xi) = \underbrace{|\xi|_g^2}_{a_2(x,\xi)} + \underbrace{\sum_k b_k(x)\xi_k}_{a_1(x,\xi)} + \underbrace{c(x)}_{a_0(x,\xi)}$$

It is a known result in the construction of the paramterix of $(P-\lambda)^{-1}$ that if A and B are PDOs with corresponding symbols a and b and $AB = I + R$ where I is the identity operator and R is smoothing, then $a \circ b(x,\xi) = \iota$, the identity symbol. Let $a(x,\xi)$ be as defined in the local symbol and $b(x,\xi)$ be the asymptotic expansion for the resolvent kernel. Then by a result (c.f. [Gil18]) of comparing the homogeneites in ξ and equating coefficents when composing a and b the following holds

Proposition 3.8

The homogeneity comparison results in the following recursive relations:

$$\begin{aligned} r_{-2} &= (a_2(x,\xi)-\lambda)^{-1} \\ r_{-2-j} &= -r_{-2} \sum_{\substack{|\mu|+k+l=j \\ l<j}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-2-l} \end{aligned}$$

Hence, we can show the heat kernel $e(t,x,y)$ has an asymptotic expansion into integrals expressed in the resolvent asymptotics by substituting (3.2) into (3.1)

$$\begin{aligned} e(t,x,x) &= \int_\Gamma e^{-t\lambda} R(x,y,\lambda)\overline{\mathrm{d}\lambda} \\ &\sim \int_\Gamma e^{-t\lambda} \sum_{j\geq 0} \int_{\mathbb{R}^n} r_{-2-j}(x,\xi,\lambda) đ\xi \overline{\mathrm{d}\lambda} \\ &= \sum_{j\geq 0} \int_{\mathbb{R}^n} \int_\Gamma e^{-t\lambda} r_{-2-j}(x,\xi,\lambda)\overline{\mathrm{d}\lambda} đ\xi \end{aligned}$$

where $đ\xi = (2\pi)^{-n}\mathrm{d}\xi$. We want to write this as an asymptotic in t, so there is change variables in the following manner. Set $\mu = t\lambda$ then $\overline{\mathrm{d}\lambda} = \frac{\mathrm{d}\mu}{t}$ and so the expression becomes

$$\sum_{j\geq 0} \int_{\mathbb{R}^n} \int_\Gamma e^{-\mu} r_{-2-j}\left(x,\xi,\frac{\mu}{t}\right) \frac{\mathrm{d}\mu}{t} đ\xi$$

Next, $\xi = t^{-\frac{1}{2}}\eta$ with $\xi_i = t^{-\frac{1}{2}}\eta_i$ and $t^{-\frac{n}{2}}đ\eta = đ\xi$ yielding

$$\sum_{j\geq 0}\int_{\mathbb{R}^n}\int_\Gamma e^{-\mu} r_{-2-j}\left(x, \frac{\eta}{t^{1/2}}, \frac{\mu}{t}\right)\frac{\mathrm{d}\mu}{t}\frac{đ\eta}{t^{n/2}}$$

Finally, we can apply quasi-homogeneity by noticing that in the second parameter we have $s = t^{-\frac{1}{2}}$ and $s^2 = \frac{1}{t}$ in the third, giving

$$\begin{aligned} r_{-2-j}(x, sy, s^2\lambda) = s^{-2-j} r_{-2-j}(x, y, \lambda) &= \left(t^{-\frac{1}{2}}\right)^{-2-j} r_{-2-j}(x, y, \lambda) \\ &= t^{1+\frac{j}{2}} r_{-2-j}(x, y, \lambda) \end{aligned}$$

and hence we can conclude (by renaming dummy variables)

Theorem 3.9

The heat kernel for an elliptic differential operator on a manifold has an asymptotic expansion for short times t near 0 given by

$$e(t, x, x) \sim \sum_{j\geq 0}\left(\int_{\mathbb{R}^n}\int_\Gamma e^{-\lambda} r_{-2-j}(x, \xi, \lambda)\mathrm{d}\lambda \mathrm{d}\xi\right) t^{\frac{-n+j}{2}}$$

and its trace

$$\operatorname{Tr} e^{-tP} \overset{t\to 0+}{\sim} \sum_{j\geq 0}\int_M\int_{\mathbb{R}^n}\int_\Gamma e^{-\lambda} r_{-2-j}(x, \xi, \lambda)\mathrm{d}\lambda\, \mathrm{d}\xi\, \omega_g\, t^{\frac{-n+j}{2}}$$

Indeed comparing to he original asymptotic,

$$e(t, x, x) \sim (4\pi t)^{-n/2}\sum_{k=0}^{\infty} u_k(x, x) t^k$$

that this is the same as our theorem and one identifies

$$u_k(x, x) = \int_{\mathbb{R}^n}\int_\Gamma e^{-\lambda} r_{-2-j}(x, \xi, \lambda)\mathrm{d}\lambda \mathrm{d}\xi =: h_{\frac{-n+j}{2}}(x)$$

and calls $h_{\frac{-n+j}{2}}(x)$ the **heat trace coefficients** and their trace is denoted without $h_{\frac{-n+j}{2}}$. An interesting result arrives immediately

Corollary 3.10

$h_{\frac{-n+j}{2}}(x) = 0$ for odd j.

Proof. Take advantage that r_{-2-j} is quasi-homogeneous and apply the change of variables $\xi \mapsto -\xi$, then

$$r_{-2-j}(x, -\xi, \lambda) = r_{-2-j}(x, -\xi, -(1)^2\lambda) = (-1)^{-2-j} r_{-2-j}(x, \xi, \lambda)$$

Since j is odd then $-2-j = -2-(2k+1) = -2k-3$ is also odd and $(-1)^{-2-j} = -1$. Therefore, since $r_{-2-j}(x, \xi, \lambda) = -r_{-2-j}(x, -\xi, \lambda)$, the integration of ξ over $\mathbb{R}^n$ is anti-symmetric and everything cancels. ∎

Applications and calculations

4.1 Mckean-Singer Formula

The first result we discuss is the quintessential link between index theory and heat kernel asymptotics. A **Dirac Operator** is a first order differential operator D acting on a manifold such that $D^2 = \Delta$. It is, in some sense,a formal square root of the Laplacian on a manifold.

Theorem 4.1

Let (M, g) be a compact Riemannian manifold. Let e^{-tD^2} be the heat kernel of the operator D^2. Then for any $t > 0$ the following holds

$$\operatorname{ind}(D) = \operatorname{Str}(e^{-tD^2}) = \int_M \operatorname{Str} e(t, x, y) \mathrm{d}\omega$$

where ind is the index and Str is known as the **supertrace** given by

$$\begin{aligned}\operatorname{Str} e^{-tD^2} &:= \operatorname{Tr} e^{-tD^*D} - \operatorname{Tr} e^{-tDD^*} \\ &= \int_M \int_{\mathbb{R}^n} \int_\Gamma e^{-\lambda} r_{-2-j}(x, \xi, \lambda) \mathrm{d}\lambda \, \mathrm{d}\xi \, \omega_g \\ &- \int_M \int_{\mathbb{R}^n} \int_\Gamma e^{-\lambda} \widetilde{r}_{-2-j}(x, \xi, \lambda) \mathrm{d}\lambda \, \mathrm{d}\xi \, \omega_g\end{aligned}$$

where D^* is the adjoint to D.

Perhaps the most striking part of the result is that, since the index of an operator is a constant integer, the trace difference is invariant to t. This motivates the interest to use short-time asymptotics we have constructed to determine the index, as knowing that it is an integer can reveal useful reductions in computation. The theorem plays a critical role in the heat equation proof for the Atiyah-Singer Index Theorem.

4.2 The Riemann Roch formula

Another wonderful result is that the Riemann Roch formula on a closed Riemann surface can be recovered by the calculation of resolvent symbols as outlined in the previous chapter. Though we will not prove it completely, we give a brief outline based on [Sco18]. Some familiarity with complex manifolds is required but we will not divulge into the details too much. Let M be a smooth compact manifold of dimension $2k$ with Hermitian metric h and let $D : C^\infty(M, V) \to C^\infty(M, V)$ be a Dirac operator acting on smooth sections and V a holomorphic vector bundle of rank n whose typical fiber is V itself. A Hermitian structure can be assigned to V defined by a partition of unity through the mapping of coordinate patches to positive definite Hermitian matrices through a collection of maps $\{E : U \to \mathrm{GL}(n, \mathbb{C})\}$. These maps allow one to define the Laplacian and its dual as

$$\Delta = -(hE^T)^{-1} \frac{\mathrm{d}}{\mathrm{d}\overline{z}} \left(E^T \frac{\mathrm{d}}{\mathrm{d}\overline{z}} \right), \quad \widetilde{\Delta} = -E \frac{\partial}{\partial z} \left((hE)^{-1} \frac{\partial}{\partial \overline{z}} \right)$$

which can be used to demonstrate their ellipticity.

Lemma 4.2

The operator has a non-negative discrete spectrum $\{\lambda\}$going to infinity and there exists a complete orthogonal basis for sections on $L^2(V)$. Each eigenspace $E_\Delta(\lambda)$ has finite dimension and the eigenspace associated to 0, otherwise known as the kernel $\operatorname{Ker}\Delta$, is equal to the space of holomorphic sections on the vector bundle V

$$E_\Delta(0) = \operatorname{Ker}\Delta = H^0(V)$$

A similar decomposition and result holds for $\widetilde{\Delta}$ and finally the two operators have the same eigenvalues

$$\dim(E_\Delta(\lambda_i)) = \dim(E_{\widetilde{\Delta}}(\lambda_i))$$

Now we can recall the Riemann-Roch formula as

Theorem 4.3

$$\dim H^0(V) - \dim H^0(\widetilde{V}) = \frac{1}{2\pi i}\int_M \partial\overline{\partial}\log(\det E) + \frac{n}{4\pi i}\int_M \partial\overline{\partial}\log h$$

where g_m is the genus and $\partial\overline{\partial}\log(\det E)$ is a form related to the De-Rham Cohomology and is the representative for differential form class known as the first **Chern class**

By calculating the resolvent symbols and the first three heat trace coefficients for Δ using the symbol calculus, one can arrive to the result

Proposition 4.4

$$h_0(x) = -\frac{1}{6\pi}\partial_z\partial_{\overline{z}}\log hI_n - \frac{1}{2\pi}\partial_{\overline{z}}\left((E^T)^{-1}\frac{\mathrm{d}E^T}{\mathrm{d}z}\right)$$

and by calculating the trace of this coefficient one arrives to

$$h_0 = \frac{1}{4\pi i}\int_M \partial\overline{\partial}\log\det E + \frac{n}{12\pi i}\int_M \partial\overline{\partial}\log h$$

and doing similarly for $\widetilde{\Delta}$ to find

$$\widetilde{h_0} = -\frac{1}{4\pi i}\int_M \partial\overline{\partial}\log\det E - \frac{n}{6\pi i}\int_M \partial\overline{\partial}\log h$$

and finally using the Mckean-Singer formula, since there is no dependence on t, we can send $t \to 0$ in the supertrace one finds that Δ and $\widetilde{\Delta}$ have the same heat traces except for $j = 2$, namely h_0 and $\widetilde{h_0}$, and hence

$$\operatorname{ind}(\Delta) = h_0 - \widetilde{h_0} = \frac{1}{2\pi i}\int_M \partial\overline{\partial}\log(\det E) + \frac{n}{4\pi i}\int_M \partial\overline{\partial}\log h$$

4.3 Mehler's Formula

We restrict ourselves to the Quantum Harmonic Oscillator on $\mathbb{R}$ and ξ and x are one dimensional and there are no multi-indices. This phenomena is defined by the equation

$$\partial_t - \Delta + a^2 x^2 = 0$$

The importance of this in relation to the index theorem was noticed by Getzler [BGM96]. Vaguely, one can rescale the parameters of a Dirac operator to show that in short time asymptotics, the operator likes like the harmonic oscillator with coefficients in $\Lambda^{\text{even}} T^* M$. By regular methods of decoupling ODEs and separation of variables to show that the heat kernel for this operator is in fact

$$e_H(t,x,x) = \sqrt{\frac{a}{2\pi \sinh(2at)}} \exp\left(\frac{-ax^2 \coth(2at)}{2}\right)$$

Since the context is fairly simple, we exhibit an actual calculation of the asymptotic to conclude the discussion. In regards to the last chapter, take the operator $H = -\Delta + a^2 x^2$ where Δ is the Laplacian on $\mathbb{R}$. Its symbol is

$$a(x,\xi) = \xi^2 + a^2 x^2$$

and we identify $a_2 = \xi^2$, $a_1 = 0$, $a_0 = a^2 x^2$. By proposition 3.8, we want to iterate the recurrence relation given through

$$r_{-2} = (a_2(x,\xi) - \lambda)^{-1}$$
$$r_{-2-j} = -r_{-2} \sum_{\substack{\mu+k+\ell=j \\ \ell<j}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} \, D_x^\mu r_{-2-\ell}$$

We would like to calculate four terms of the asymptotic expansion and hence will use $j = 0, 2, 4, 6$. Immediately we find $r_{-2} = \frac{1}{\xi^2 - \lambda}$. We now go through each situation evaluating case by case.

($j = 2$): The equation now becomes

$$r_{-4} = -r_{-2} \sum_{\substack{\mu+k+\ell=2 \\ \ell<2}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} \, D_x^\mu r_{-2-\ell}$$

If $\ell = 1$, then any term vanishes because of $r_{-3} = 0$. On the other hand, if $\ell = 0$ then the situation becomes

$$r_{-4} = -r_{-2} \sum_{\mu+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} \, D_x^\mu r_{-2}$$

If $\mu = 2, k = 0$ then the term vanishes because r_{-2} has no x derivative. If $\mu = k = 1$, the term also vanishes because $a_1 = 0$. However, if $\mu = 0, k = 2$ then the corresponding term is equal to $a_0 r_{-2} = \frac{a^2 x^2}{\xi^2 - \lambda}$ and we conclude by combining with the leading coefficient of the sum to get

$$\boxed{r_{-4} = -\frac{a^2 x^2}{(\xi^2 - \lambda)^2}}$$

($j = 4$): Now we have

$$r_{-6} = -r_{-2} \sum_{\substack{\mu+k+\ell=4 \\ \ell<4}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k} \, D_x^\mu r_{-2-\ell}$$

If $\ell = 0$ then $\mu + k = 4$ implying that $k \in \{0,1,2\}$ and $\mu \in \{4,3,2\}$ and hence $D_x^\mu r_{-2} = 0$, so there is no contribution. Once more, when $\ell = 1$ the terms contain r_{-3} which is 0. Furthermore when $\ell = 3$ is also 0 since $r_{-5} = 0$. What's left to consider is $\ell = 2$, whence the series becomes

$$r_{-6} = -r_{-2} \sum_{\mu+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-4}$$

When $\mu = k = 1$ the term vanishes because $a_1 = 0$. If $\mu = 2, k = 0$ then $\mu! = 2$, $\partial_\xi^2 a_2 = 2$ and $D_x^2 r_{-4} = -\frac{2a^2}{(\xi^2-\lambda)^2}$. If $\mu = 0, k = 2$ then the full term is $a_0 r_{-4} = -\frac{a^4x^4}{(\xi^2-\lambda)^2}$ and all together we combine to get

$$\boxed{r_{-6} = \left(\frac{-1}{\xi^2 - \lambda}\right)\left[\frac{-a^4x^4 - 2a^2}{(\xi^2-\lambda)^2}\right] = \frac{a^4x^4 + 2a^2}{(\xi^2-\lambda)^3}}$$

$(j = 6)$ Finally, we look at

$$r_{-8} = -r_{-2} \sum_{\substack{\mu+k+\ell=6 \\ \ell<6}} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-2-\ell}$$

Clearly we need only to consider $\ell = 0, 2, 4$ because otherwise the terms will have involving $r_1 = r_3 = r_5 = 0$. First consider $\ell = 0$ and by the same argument as before, $D_x^\mu r_{-2} = 0$ for all terms since $\mu \neq 0$ and r_{-2} does not depend on x. If instead $\ell = 2$ then we have

$$r_{-8} = -r_{-2} \sum_{\mu+k=4} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-4}$$

If $k = 1$ the term vanishes because of $a_1 = 0$. If $k = 2$ then $\mu = 2$ and $\partial_\xi^2 a_0 = 0$ and for $k = 0, \mu = 4$ the partial derivative vanishes again but this time because the degree of differentiation $\mu > 2$.
If $\ell = 4$ we have

$$r_{-8} = -r_{-2} \sum_{\mu+k=2} \frac{1}{\mu!} \partial_\xi^\mu a_{2-k}\, D_x^\mu r_{-6}$$

Again $\mu = k = 1$ is null as $a_1 = 0$. If $\mu = 0, k = 2$ then the term is $a_0 r_{-6}$ and $\mu = 2, k = 0$ is null as a_0 vanishes under the ξ-derivative. Collecting the terms we have

$$\boxed{r_{-8} = \left(\frac{-1}{\xi^2 - \lambda}\right)\left[\frac{a^6x^6 + 2a^4x^2}{(\xi^2-\lambda)^3}\right] = -\frac{a^6x^6 + 2a^4x^2}{(\xi^2-\lambda)^4}}$$

This completes the calculation for $j = 0, 2, 4, 6$. Since we are on $\mathbb{R}$, $n = 1$ and the heat trace coefficients are written $h_{\frac{j-1}{2}}(x)$ and in order to calculate them we mention two identites that will be useful to us without proof

$$\int_\Gamma e^{-\lambda}(\beta|\xi|^2 - \lambda I)^{-k} đ\lambda = \frac{e^{-\beta|\xi|^2}}{(k-1)!} \tag{4.1}$$

where once more $đ\lambda = i/2\pi \mathrm{d}\lambda$. Starting with $j = 0$ and directly applying (4.1)

$$h_{-\frac{1}{2}}(x) = \int_{\mathbb{R}} \int_\Gamma e^{-\lambda} \frac{1}{\xi^2 - \lambda} đ\lambda \mathrm{d}\xi = \int_{\mathbb{R}} e^{-\xi^2} \mathrm{d}\xi = \sqrt{\pi}$$

Next, for $j = 2$

$$h_{\frac{1}{2}}(x) = \int_{\mathbb{R}} \int_\Gamma e^{-\lambda} \frac{a^2x^2}{(\xi^2-\lambda)^2} đ\lambda \mathrm{d}\xi = a^2x^2 \int_{\mathbb{R}} e^{-\xi^2} \mathrm{d}\xi = -\sqrt{\pi} a^2 x^2$$

Next, for $j = 4$

$$h_{\frac{1}{2}}(x) = \int_{\mathbb{R}} \int_{\Gamma} e^{-\lambda} \frac{a^4x^4 + 2a^2}{(\xi^2 - \lambda)^3} đ\lambda \mathrm{d}\xi = (a^4x^4 + 2a^2) \int_{\mathbb{R}} \frac{e^{-\xi^2}}{2} \mathrm{d}\xi = \frac{\sqrt{\pi}}{2}(a^4x^4 + 2a^2)$$

and finally for $j = 6$

$$h_{\frac{5}{2}}(x) = -\int_{\mathbb{R}} \int_{\Gamma} e^{-\lambda} \frac{a^6x^6 + 2a^4x^2}{(\xi^2 - \lambda)^4} đ\lambda \mathrm{d}\xi = -(a^6x^6 + 2a^4x^2) \int_{\mathbb{R}} \frac{e^{-\xi^2}}{6} \mathrm{d}\xi = -\frac{\sqrt{\pi}}{6}(a^6x^6 + 2a^4x^2)$$

Ultimately leading us to our asymptotic

$$\boxed{e_H(t, x, x) \sim \sqrt{\frac{\pi}{t}} - \sqrt{\pi} a^2x^2t^{\frac{1}{2}} + \frac{\sqrt{\pi}}{2}(a^4x^4 + 2a^2)t^{\frac{3}{2}} - \frac{\sqrt{\pi}}{6}(a^6x^6 + 2a^4x^2)t^{\frac{5}{2}} + \mathcal{O}(t^{\frac{7}{2}})}$$

4.4 Conclusion

Indeed, throughout this paper we have seen the remarkable properties of the heat kernel both at the basic level and in the more general setting. The construction of the heat operator reveals to each manifold an associated unique heat kernel and hence one way to satisfy the heat equation. However, the story does not end here. There have been further and further generalizations and extensions of this theory to different settings. A good example is the discussion of the Zeta function in [Ros10, p. C.5] and the link between Number Theory and the heat kernel. The author looks towards the future with hopes of understanding how and whether such a theory can be extended to non-compact manifolds and also a better understanding and interpretation of the values resulting from the heat traces. For further comprehensive reading, see [Gil18][Pat][BB13].

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