Optimising distribution of power during a cycling time trial

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Abstract

A simple mathematical model is used to find the optimal distribution of a cyclist's effort during a time trial. It is shown that maintaining a constant velocity is optimal if the goal is to minimise the time taken to complete the course while fixing amount of work done. However, this is usually impractical on a non-flat course because the cyclist would be unable to maintain the power output required on the climbs. A model for exertion is introduced and used to identify the distribution of power that minimises time while restricting the cyclist's exertion. It is shown that, for a course with a climb followed by a descent, limits on exertion prevent the cyclist from improving performance by shifting effort towards the climb and away from the descent. It is also shown, however, that significant improvement is possible on a course with several climbs and descents. An analogous problem with climbs and descents replaced by headwinds and tailwinds is considered and it is shown that there is no significant advantage to be gained by varying power output. Lagrange multipliers are used solve the minimisation problems.

Keywords: power output, exertion, time trial, critical power.

List of symbols

P = power output (W)
P_c = climbing power (W)
P_d = descending power (W)
P_m = maximum power (W)
P_0 = critical power (W)
W = total work (J)
v = velocity (m s^{-1})
v_c = climbing velocity (m s^{-1})
v_d = descending velocity (m s^{-1})
w = headwind velocity (m s^{-1})
t = time (s)
T = time to complete course (s)
D = duration of effort at a fixed power output (s)
x = distance travelled (m)
x_c = climb length (m)
x_d = descent length (m)
L = course length (m)
R = rate of exertion
E = exertion
m = mass of cyclist with bicycle (kg)
g = 9.8 m s^{-2}
c = drag coefficient (kg m^{-1})
μ = friction coefficient
s = road grade
s_c = climbing grade
s_d = descending grade

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Introduction

In a cycling time trial, a cyclist's objective is to cover a fixed distance as quickly as possible without the benefit of drafting other cyclists. If external forces on the cyclist remain constant (i.e. a course with constant grade with no wind), conventional wisdom dictates that the cyclist's effort should be distributed evenly throughout the race (Atkinson *et al.*, 2003). The mathematical model developed here is used to show that, in the absence of such ideal conditions, the cyclist may benefit from distributing effort unevenly during the race.

A similar modelling study has been carried out (Swain, 1997) which also shows that significant improvements in performance can result from variations in power that preserve mean power. This was followed up by experimental work by Liedl *et al.* (1999) which showed that some physiological variables are not adversely affected by relatively small (5%) power variations about a fixed mean. The natural extension of such an investigation is to determine the variation of power that produces the fastest time. It will be shown that such power variations are typically quite large and not practical because of physiological limitations. This clearly suggests that a model should include a physiological component if it is to be useful in optimising performance.

The physiological component of the model used here is simple in that exertion is regarded as one dimensional and that it increases at a rate dependent purely on power output. In fact, exertion has many aspects that interact in a complex way during prolonged effort and depend not only on power output but on how the effort has been applied. For that reason, it may be premature to suppose that the results presented here have direct implications for the racing cyclist. Still, the idea started in Swain (1997) of improving cycling performance with power variation clearly deserves further investigation, and the model introduced here seems like a natural starting point for incorporating physiology into such an investigation.

The model for power output

The model used here for a cyclist's power output P (in watts) is

$$P = cv^3 + mgv(\mu + s) \tag{1}$$

where c is the drag coefficient (in kg m^{-1}), v is velocity (in m s^{-1}), m is the mass (in kg) of the cyclist (and bicycle), μ is the friction coefficient, s is the uphill grade of the road (sine of the road angle), and g = 9.8 m s^{-2}. (This is a simplified version of the model given in Broker (2003).) The validity of this model rests on following assumptions:

1. variations in a cyclist's power output due to acceleration and deceleration can be ignored
2. the cyclist maintains the same aerodynamic position throughout the time trial
3. there is no wind (this assumption will be removed in the last section).

The first assumption is justified only if the course lacks rapidly varying terrain resulting in frequent and significant changes in velocity. This will certainly be the case for the first example considered here; the grade of the course will change abruptly at only one point and the cyclist's velocity will be constant otherwise. A course with several grade changes will also be considered but, even in this case, the power associated with accelerating will be small in comparison to the cyclist's total power output.

Inspection of eqn. 1 reveals why work done by the cyclist may be decreased by shifting effort more towards climbs. Notice that, when velocity is high (during flats and descents), dP/dv is large and decreasing velocity causes a relatively large decrease in power output but, when velocity is low (during climbs), dP/dv is small and a similar increase in velocity causes a small increase in power output.

Minimising time with work held constant

This section finds the power distribution that minimises the time to complete the course for a given amount of work for a course with a climb and descent of constant grade. Specifically, it is assumed that the course consists of a climb x_c metres long of constant grade s_c followed by a descent x_d metres long of constant grade s_d. It follows from eqn. 1 that

$$P_c = cv_c^3 + mgv_c(\mu + s_c) \tag{2}$$

$$P_d = cv_d^3 + mgv_d(\mu - s_d) \tag{3}$$

where P_c and P_d are the climbing and descending power, and v_c and v_d are the climbing and descending velocity, all assumed to be constant. (The fact that constant velocity is optimal on sections of constant grade follows from the discussion in Appendix 1.) The time T required to complete the course is given by

$$T = \frac{x_c}{v_c} + \frac{x_d}{v_d} \tag{4}$$

The work W done by the cyclist in completing the course is given by

$$W = \frac{x_c P_c}{v_c} + \frac{x_d P_d}{v_d}$$

$$= cx_c v_c^2 + cx_d v_d^2 + mgx_c(\mu + s_c) + mgx_d(\mu - s_d) \tag{5}$$

Suppose that the goal is to minimise the time T in which the course is completed, subject to the restriction that a fixed amount of work W is done. (Notice that this is equivalent to fixing mean power, as is done in Swain (1997).) This problem can be solved using Lagrange multipliers: assuming that W and T are functions of v_c and v_d, their gradients must be parallel at extreme values of T; i.e.

$$\nabla W = \lambda \nabla T$$

λ a constant. Combining this with eqns. 4 and 5 yields

$$2cx_c v_c = -\frac{\lambda x_c}{v_c^2}$$

$$2cx_d v_d = -\frac{\lambda x_d}{v_d^2}$$

Eliminating λ yields $2cv_c^3 = 2cv_d^3$ which implies $v_c = v_d$. It is not hard to verify that this happens when T is at a minimum. Appendix 1 generalises this calculation by showing that constant velocity minimises time for fixed work on any course of varying grade (still ignoring acceleration).

Example

In order to understand the implications of this calculation, it is worthwhile to consider the following example. Suppose that course is 40 km long with uphill and downhill grades of 2.5%; i.e. $x_c = x_d = 2 \times 10^4$ m and $s_c = s_d = 0.025$, and that the cyclist and bicycle have a mass $m = 84$ kg. Typical values for the drag and friction coefficients for this cyclist on an aero bicycle are $c = 0.17\ \mathrm{kg\ m^{-1}}$ and $\mu = 0.003$ (Kyle, 2003). Suppose also that the cyclist is capable of completing the course in 3100 s (51 min 40 s) at a constant power output. Setting eqns. 2 and 3 equal and solving simultaneously with eqn. 4 with $T = 3100$; one obtains $v_c = 10.64\ \mathrm{m\ s^{-1}}$ and $v_d = 16.38\ \mathrm{m\ s^{-1}}$. Plugging these into eqn. 5 yields $W = 1.396 \times 10^6$ J.

For comparison, now assume that the cyclist does the same amount of work, but minimises time by maintaining a constant velocity over the entire course. Solving eqn. 5 for velocity yields $v_c = v_d = 13.81\ \mathrm{m\ s^{-1}}$. Plugging this into eqn. 4 yields $T = 2\,896$ s, a savings of 204 s (3 min 24 s) and 6.6% increase in average velocity (from maintaining a constant power output). The impossibility of achieving this average velocity, however, is revealed by calculating the climbing power required: $P_c = 766$ W. Such a power output would exhaust the cyclist in a matter of seconds. This illustrates the fact that there is not a direct relationship between level of exertion and work done. This is not surprising, however, since the relationship between rate of exertion and power output is nonlinear; rate of exertion increases much more rapidly as the cyclist approaches their maximum power output, as is clearly illustrated by experiments comparing power output to duration of exercise (see, for example, Whitt & Wilson (1983)). Still, these calculations suggest that there is a benefit to working harder while climbing than descending. To quantify such possible benefits, it is necessary to determine a more appropriate measure for the cyclist's level of exertion during the race.

Minimising time with exertion held constant

Defining exertion

As mentioned in the introduction, exertion E will be modelled as a one-dimensional quantity regardless of the fact that there are several variables associated with exertion level that evolve in complex ways during exercise. Incorporating a more realistic model for exertion is left as a goal for future research.

The exertion model will be based on the three-parameter critical power model (Morton, 1996):

$$D = \frac{A(P_m - P)}{(P_m - P_0)(P - P_0)} \tag{6}$$

where D is the duration that exercise can continue at a constant power output P, P_m is maximum power output, P_0 is critical power, and A is anaerobic work capacity (Moritani *et al.*, 1981). Notice that this model puts a limiting value (P_m) on the power that the cyclist can sustain for any appreciable amount of time and implies that the rate at which the cyclist exerts themselves grows without bound as that limiting value is approached. It also sets a lower limit (P_0) on the range of power outputs which increase the cyclist's exertion. This should not be interpreted to mean that the cyclist can generate a power output of P_0 indefinitely, just much longer than the time frame of interest here. With that in mind, the parameter values in eqn. 6 will be chosen based on data for a cyclist performing in this time frame. Data in Broker *et al.* (1999) indicates that the British cyclist Chris Boardman maintained a power output of 520 W for 251 s in a record-setting 4000 m time trial and, based on data for team pursuits in the same article, it will be assumed that Boardman is capable of generating 700 W for 60 s. Boardman's power output in a record-setting hour ride (3600 s) was measured by Keen (1994) at 442 W. These values represent three pairs of coordinates (P, D) that, if assumed to satisfy eqn. 6, yield three equations for the three unknown parameters. The following values result from solving these equations:

$$P_m = 1234\text{ W}, \qquad P_0 = 435\text{ W},$$

$$A = 1.243 \times 10^4\text{ J} \tag{7}$$

In defining exertion, it is assumed (i) that exertion is initially zero and (ii) that, for a fixed power output above P_0, exertion increases linearly to the point of exhaustion. This means that the rate of change of exertion at a given power output is inversely proportional to the corresponding duration D. Specifically, rate of exertion $R = dE/dt$ will be defined as the reciprocal of D:

$$R(P) = \frac{(P_m - P_0)(P - P_0)}{K(P_m - P)} \tag{8}$$

(This function is plotted in Fig. 1.) Exertion E is then given by

$$E(x) = \int_0^t R\, d\tau = \int_0^x \frac{R}{v}\, d\xi \tag{9}$$

at a point x on the course, and finishing exertion is

$$E(L) = \int_0^L \frac{R}{v}\, dx \tag{10}$$

where L is the length of the course. Notice that this definition of exertion implies that exertion at the point of exhaustion is 1.

It should also be noted that eqn. 6 is intended to apply when $P > P_0$ but eqn. 8 implies that exertion decreases when $P < P_0$. This is consistent with experimental work (e.g. Coats *et al.*, 2003) but it is not clear whether the rate of decrease implied by eqn. 8 is appropriate. An argument in favour of this is the fact that this rate makes exertion rate a smooth function of power near P_0 and many aspects of exertion (e.g. heart rate, oxygen uptake) vary smoothly, if not linearly, as functions of power output throughout the three phases of exercise as defined in Skinner & McLellan (1980). It is also supported by the experimental work in Liedl *et al.* (1999): variations of the order of 5%

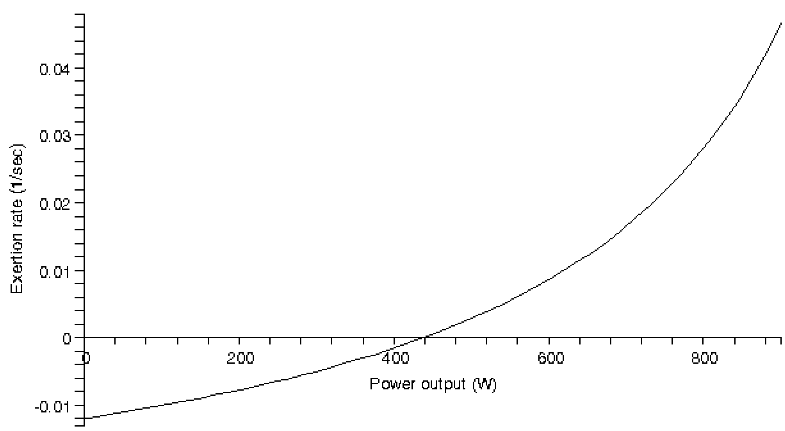

Figure 1 Exertion rate R versus power output P.

about critical power were shown to have little effect on physiological variables as compared with constant power, suggesting exertion is nearly linear in the neighbourhood of P_0.

The fact that exertion can decrease means that, in addition to the assumption that the cyclist is exhausted upon finishing the course, i.e. $E(L) = 1$, it is also necessary to put internal restrictions on exertion. Specifically,

$$0 \le E(x) \le 1 \quad \text{for} \quad 0 \le x \le L \tag{11}$$

Piecewise constant grade course

As was the case when work was held constant, constant velocity is optimal when the course grade is constant (this is verified in Appendix 2), so velocity will be assumed constant on each constant-grade section of the course. To begin with, the restriction in eqn. 11 will be ignored and T will be minimised subject to the constraint $E(L) = 1$. Applying eqn. 9 to the course discussed in the previous section yields

$$E(L) = \frac{x_c R(P_c)}{v_c} + \frac{x_d R(P_d)}{v_d} \tag{12}$$

Lagrange multipliers can again be used to minimise T:

$$\nabla E = \lambda \nabla T$$

the derivatives again being with respect to v_c and v_d. Combining this with eqns. 4 and 12 implies

$$\frac{x_c R'(P_c)}{v_c}\frac{\mathrm{d}P_c}{\mathrm{d}v_c} - \frac{x_c R\,(P_c)}{v_c^2} = -\frac{\lambda x_c}{v_c^2}$$

$$\frac{x_d R'(P_d)}{v_d}\frac{\mathrm{d}P_d}{\mathrm{d}v_d} - \frac{x_d R\,(P_d)}{v_d^2} = -\frac{\lambda x_d}{v_d^2}$$

Eliminating λ yields

$$v_c R'(P_c)\frac{\mathrm{d}P_c}{\mathrm{d}v_c} - R(P_c) = v_d R'(P_d)\frac{\mathrm{d}P_d}{\mathrm{d}v_d} - R(P_d) \tag{13}$$

This equation must be solved simultaneously with $E(L) = 1$ to obtain the values of v_c and v_d that minimise T.

Peak-centred course

Consider the course in the previous example, and suppose that the cyclist whose exertion rate is modeled by eqn. 8 completes the course first with constant power output to exhaustion ($E = 1$). Setting $P_c = P_d$ and $E = 1$ in eqn. 12, solving for v_c and v_d, and plugging the result into eqns. 2 and 4 gives $T = 3121$ s and $P_c = P_d = 443$ W. Solving eqns. 12 and 13 with $E = 1$ for v_c and v_d, and plugging the result into eqns. 2–4 gives $P_c = 522$ W, $P_d = 309$ W and $T = 3069.2$ s, a time savings of 51.8 s from maintaining a constant power output. As for the earlier result with fixed work, this result is also not practical. The climbing power is great enough that cyclist's exertion will exceed its limiting value of $E = 1$ (in approximately 4 min) well before the climbs ends (in approximately 29 min). The final value for exertion is 1 only because of the negative rate of exertion during the descent. This illustrates the need for the internal restriction in eqn. 11, which can be satisfied by specifying that the cyclist's exertion reach a value of 1 on the climb and then be kept at a value of 1 by having the cyclist maintain critical power ($P_d = 435$ W) on the descent. If this is done, the climbing power is $P_c = 448$ W and $T = 3115$, a saving of only 6 s. This essentially means that no significant improvement in performance can result from varying power on the course in this example.

Multiple climbs

One way that improved performance could be realised with this strategy is if the course, instead of having one long climb and descent, was composed of several smaller climbs and descents. Suppose, for example, that the 40 km course consisted of 20 climbs 1000 m in length, each followed by a descent of the same length (all 2.5% grades). The cyclist could still maintain a high power output on the climbs but use the descents to recover and avoid exceeding the limiting exertion. The optimal distribution of power for this course is determined in a similar way as for the peak-centred course. First, T is minimised subject to $E(L) = 1$ (in fact, the solution is the same as for the peak-centred course since T and $E(L)$ are not affected by the ordering of the climbs and descents). Then a new constraint is added at the last local extremum of exertion where eqn. 11 is violated and T is minimised again with this added constraint. For a general piecewise constant grade course, this procedure may require repetition if eqn. 11 is still not satisfied. In this case, however, only one additional constraint is needed. Specifically, since exertion reaches a maximum of 1.31 at $x = 39\,000$ (the peak of the last climb), T is minimised again subject to $E(39\,000) = E(40\,000) = 1$. For $39\,000 < x < 40\,000$, grade is constant so power and exertion rate must also be constant, which means $E = 1$ and $P = P_0$. The time required for this section is 61.5 s. The rest of the solution can be found using eqns. 12 and 13, letting $L = 39\,000$, $x_c = 20\,000$, and $x_d = 19\,000$ (since ordering of the climbs and descents does not matter) and the time required for this section is 3009 s for a total of $T = 3070.5$ s. The time necessary for the cyclist to complete this course with constant power output is the same as for the peak-centred course (as is the power output), so the time saved by varying power is 50.5 s. The corresponding exertion is plotted as a function of x in Fig. 2. (Note that the solution satisfies eqn. 11.) Power output is 520 W for the climbs and 306 W for the descents except the last one, which is done at critical power (435 W). It is interesting to note that the cyclist's average power output is 430 W, 13 W lower than if constant power is used. This is due to the fact that exertion is a nonlinear function of power output.

Notice that the time saved by varying power on the multiple-climb course is close to the time savings predicted for the peak-centred course when internal restrictions on exertion were ignored. Although the latter prediction is not useful for the peak-centred course, it could be regarded as an upper limit for time saved by varying power on a course with the same amount of vertical climbing and grades but with many alternating climbs and descents. Fig. 3 shows a plot of

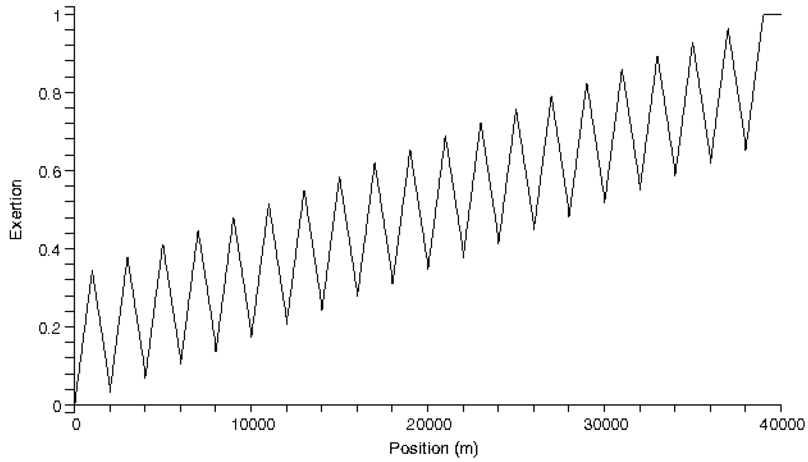

Figure 2 Exertion E versus position x for multiple-climb course.

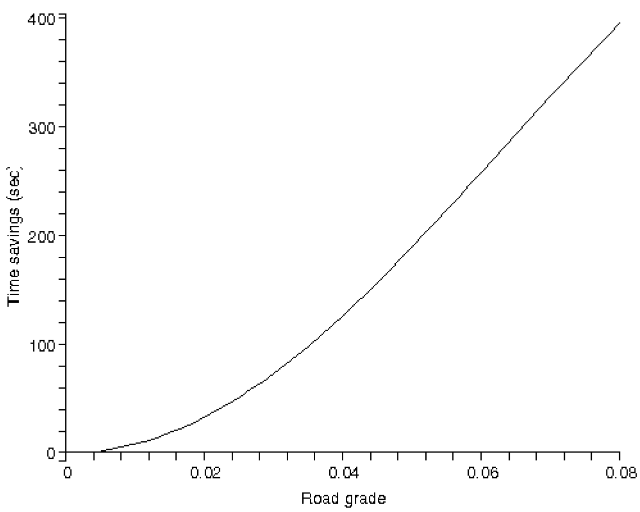

Figure 3 Time savings versus road grade s for peak-centred course.

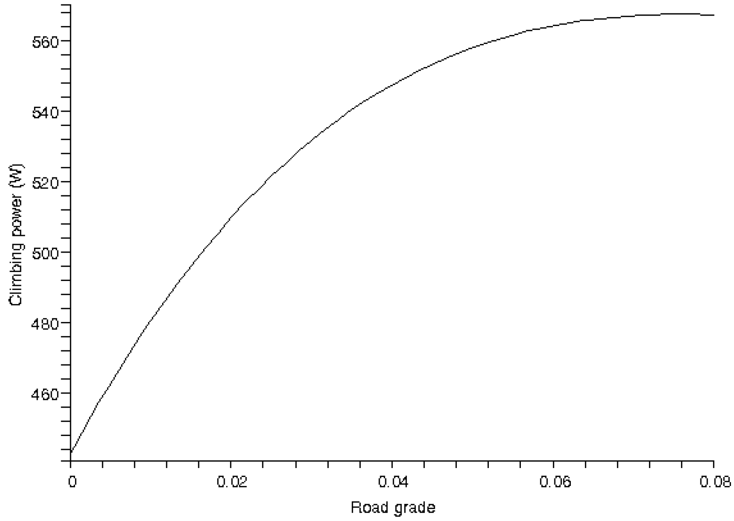

Figure 4 Climbing power P_c versus road grade s for peak-centred course.

this potential time savings as a function of s, obtained by varying the height of the peak-centred course; Fig. 4 shows the corresponding plot of climbing power.

Non-centred peak course

Fig. 3 shows that potential time saving increases with amount of vertical climbing. The potential time saving is also increased when the climbs are steeper than the descents, even if vertical climbing is the same. To see this, consider a course with a vertical climb of 500 m (the same as the peak-centred course above) occupying the first 8000 m of the course (for a climbing grade of 6.25%) followed by a 32 000 m descent (at 1.6%). The predicted time for the cyclist (whose exertion rate is modelled by eqn. 8) is T = 3229 s using constant power and T = 3125 s if time is minimised without taking eqn. 11 into account, a difference of 104 s. Notice this is twice the corresponding time difference calculated for the peak-centred course with the same vertical climb. Fig. 5 shows a plot of this time difference as a function of the length x_c of the climb (with the vertical climb held constant). Again, actual time savings could approach these differences only on a course with the same grades and a large number of climbs and descents. (A time saving of 103 s was calculated for a course with 20 climbs at 6.25%.) Fig. 6 shows the corresponding plot of climbing power. Notice that climbing power increases much more rapidly as the climbing grade increases than for the peak-centred course. This is because, on the non-centred peak course, the climbs are shorter and increased exertion on the climbs can be compensated for by a much smaller decrease in exertion on the descents.

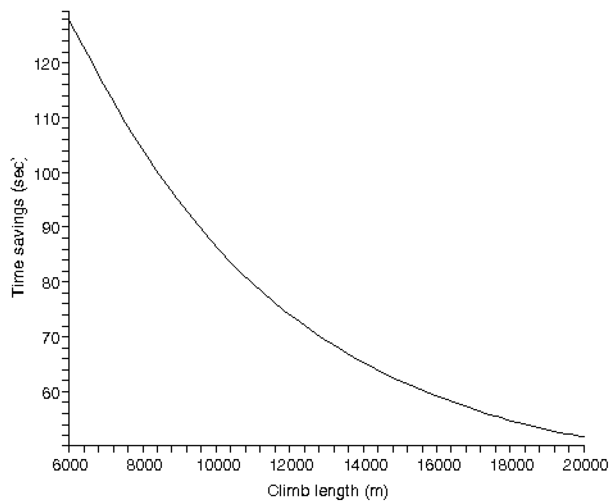

Figure 5 Time savings versus climb length x_c for course with 500 m vertical climb and descent.

Effects of wind

This section considers the effect on the results for the peak-centred course if the climb and descent are replaced by a headwind and tailwind. The inclusion of wind parallel to the cyclist's direction yields the following modification of eqn. 1:

$$P = cv\,(v + w)^2 + mgv\,(\mu + s) \tag{14}$$

where w is headwind velocity in m s^{-1} (negative for a tailwind). (This is also simplified from the model in Broker (2003).) Consider a flat course ($s_c = s_d = 0$) of 20 000 m which the cyclist covers in both directions. Assume that there is a 4.5 m s^{-1} wind parallel to the course so that the cyclist has a 4.5 m s^{-1} headwind on the way out and a 4.5 m s^{-1} tailwind on the way back. Thus eqns. 2 and 3 are replaced by

$$P_c = cv_c(v_c + 4.5)^2 + mgv_c\mu \tag{15}$$

$$P_d = cv_d(v_d - 4.5)^2 + mgv_d\mu \tag{16}$$

Suppose that the cyclist whose exertion rate is modelled by eqn. 8 completes the course first with constant power output to exhaustion ($E = 1$). Setting $P_c = P_d$ and $E = 1$ in eqn. 12 (with eqns. 15 and 16 in place of eqns. 2 and 3), solving for v_c and v_d, and plugging the result into eqns. 4 and 15 gives T = 3089 s and $P_c = P_d = 443$ W. Solving eqns. 12 and 13 with $E = 1$ for v_c and v_d, and plugging the result into eqns. 4 and 15 gives $P_c = 481.7$ W and $T = 3077$ s, a time saving of 12 s from maintaining a constant power output. This is far less promising than the result for the peak-centred course. As with that course, the cyclist will exceed the maximum exertion of 1 before reaching the midpoint of the course. (Using the first term in eqn. 12, one finds $E = 3.74$ at $x = 20\,000$ m.) However, even the unlikely situation of having the wind repeatedly change direction to mimic the multiple-climb example would not produce significant time savings. The reason for this is that, unlike for climbing, power is still a very nonlinear function of velocity when cycling into a headwind. Thus increasing effort into the wind causes a much less significant increase in velocity than increasing effort on a climb.

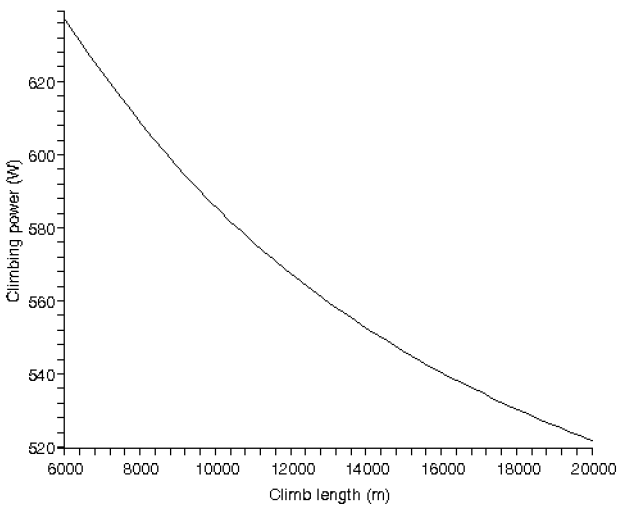

Figure 6 Climbing power *Pc* versus climb length *xc* for course with 500 m vertical climb and descent.

Conclusion

The model developed here suggests that, on a time trial course with varying grades, exertion can be decreased significantly by applying more effort on the climbs than on the descents, but only if there are several climbs and descents. The model also suggests that, in the absence of varying grades, there is no benefit to varying power in response to changes in wind speed.

Future research in this area could be directed at improving on the model for exertion. The assumption that rate of exertion is constant for fixed power output is clearly an oversimplification. In fact, measures of exertion evolve nonlinearly at rates dependent on many factors. The dynamical system model considered in Morton (1985) (derived from the three-component hydraulic model of Margaria (1976)) could be a useful starting point for further investigation. The benefit of this model is that it takes into account the interaction between aerobic and anaerobic energy sources. Further improvements to the model may also need to account for other factors affecting the exertion of a cyclist, e.g. differences in the physiological demands of climbing versus descending (Lucía *et al.*, 2003) or changes in pedaling cadence (Gaesser & Brooks, 1975).

Appendix 1: Variable grade course

The purpose of this section is to generalise the results of the first section to a course with variable road grade $s(x)$ where x is the distance the cyclist has travelled. Specifically, it is shown that, if work W is set at a fixed value, time T is minimised by maintaining a constant velocity. If $v(x)$ is the cyclist's velocity function, T is given by

$$T = \int_0^L \frac{\mathrm{d}x}{v} \tag{17}$$

where L is the length of the course, and the work W by

$$W = \int_0^T P\,\mathrm{d}t = \int_0^L \frac{P(v, x)}{v}\,\mathrm{d}x \tag{18}$$

The desired goal can also be accomplished by showing that work W is minimised for fixed T by constant velocity. First, suppose a constant velocity $\bar{v}$ is maintained and the associated time and work are $\bar{T}$ and $\bar{W}$. This means

$$\bar{T} = \int_0^L \frac{\mathrm{d}x}{\bar{v}} = \frac{L}{\bar{v}}$$

$$\bar{W} = \int_0^L \frac{P(\bar{v}, x)}{\bar{v}}\,\mathrm{d}x$$

If velocity is then varied by a function $\delta(x)$ so that time remains the same, then

$$\bar{T} = \int_0^L \frac{\mathrm{d}x}{\bar{v}+\delta} = \int_0^L \frac{\mathrm{d}x}{\bar{v}}$$

which implies

$$\int_0^L \frac{\mathrm{d}x}{\bar{v}} - \int_0^L \frac{\mathrm{d}x}{\bar{v}+\delta} = \int_0^L \left(\frac{1}{\bar{v}} - \frac{1}{\bar{v}+\delta}\right)\mathrm{d}x$$

$$= \int_0^L \frac{\delta\,\mathrm{d}x}{\bar{v}(\bar{v}+\delta)} = 0$$

and, since $\bar{v}$ is constant,

$$\int_0^L \frac{\delta\,\mathrm{d}x}{\bar{v}+\delta} = 0 \tag{19}$$

The associated work is given by

$$W = \int_0^L \frac{P(\bar{v}+\delta, x)}{\bar{v}+\delta}\,\mathrm{d}x$$

$$= \int_0^L [(c\bar{v}+\delta)^2 + mg(\mu + s)]\,\mathrm{d}x$$

$$= \int_0^L [c\bar{v}^2 + + 2c\bar{v}\delta + c\delta^2 + mg(\mu + s)]\,\mathrm{d}x$$

$$= \int_0^L \left(\frac{P(\bar{v}, x)}{\bar{v}} + 2c\bar{v}\delta + c\delta^2\right)\mathrm{d}x$$

$$= \bar{W} + \int_0^L (2c\bar{v}\delta + c\delta^2)\,\mathrm{d}x$$

$$= \bar{W} + \int_0^L \left(\frac{2c\bar{v}\delta\,(\bar{v}+\delta)}{(\bar{v}+\delta)} + c\delta^2\right)\mathrm{d}x$$

$$= \bar{W} + \int_0^L \left(\frac{2c\bar{v}^2\delta}{(\bar{v}+\delta)} + \frac{2c\bar{v}\delta^2}{(\bar{v}+\delta)} + c\delta^2\right)\mathrm{d}x$$

$$= \bar{W} + 2c\bar{v}^2 \int_0^L \frac{\delta\,\mathrm{d}x}{(\bar{v}+\delta)} + \int_0^L \left(\frac{2c\bar{v}\delta^2}{(\bar{v}+\delta)} + c\delta^2\right)\mathrm{d}x$$

The first integral is zero by eqn. 19 and the second is clearly non-negative so $W > \bar{W}$, proving that constant velocity minimises work. As demonstrated in the first section, this is not practical on a course with significant uphill grades.

Appendix 2: Constant grade

The purpose of this section is to show that constant velocity is optimal when external forces are constant, i.e. road grade s is constant. Specifically, it is shown that exertion E is minimised for a fixed average velocity if v is constant. First, suppose a constant velocity $\bar{v}$ is maintained and the associated exertion is $\bar{E}$. Then

$$\bar{E} = \int_0^T \Gamma(\bar{v})\,\mathrm{d}t$$

where $\Gamma(v) = R[P(v)]$. Notice that $\Gamma'(v) = R'(P)\,P'(v)$ is an increasing function of v because R' is an increasing function of P and P' is an increasing function of v. If velocity is then varied by a function $\delta(t)$ so that average velocity remains the same, then

$$\int_0^T \delta\,\mathrm{d}t = 0 \tag{20}$$

The associated exertion is

$$E = \int_0^T \Gamma(\bar{v}+\delta)\,\mathrm{d}t$$

$$= \int_0^T [\Gamma(\bar{v}) + \Gamma'(c)\,\delta]\,\mathrm{d}t$$

$$= \bar{E} + \int_0^T \Gamma'(c)\,\delta\,\mathrm{d}t$$

where c is between $\overline{v}$ and $\overline{v} + \delta$ by the mean value theorem. This implies

$$\begin{aligned} E - \overline{E} &= \int_0^T \Gamma'(c)\,\delta\,\mathrm{d}t \\ &= \int_0^T \Gamma'(\overline{v})\,\delta\,\mathrm{d}t + \int_0^T [\Gamma'(c) - \Gamma'(\overline{v})]\,\delta\,\mathrm{d}t \\ &= \int_0^T [\Gamma'(c) - \Gamma'(\overline{v})]\,\delta\,\mathrm{d}t \qquad \text{by (20)} \qquad (21) \end{aligned}$$

If $\delta \geq 0$ then $c \leq \overline{v}$ so Γ' implies $\Gamma'(c) \geq \Gamma'(\overline{v})$; if $\delta \leq 0$ then $c \leq \overline{v}$ so $\Gamma'(c) \leq \Gamma'(\overline{v})$. Either way, the integrand in eqn. 21 is non-negative so $E - \overline{E} \geq 0$, proving the result.

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