Homework 9

Question 1

First we construct the Lagrangian as below:

$$\mathcal{L}\left(x_1, x_2, \mu_1, \mu_2\right)=x_1^2+\left(x_2-1\right)^2+\mu_1\left(x_1-x_2-1\right)+\mu_2\left[\left(x_1-1\right)^2+x_2^2-1\right] \tag{1}$$

The KKT conditions are

$$\begin{cases}\partial_{x_1}\mathcal{L}=2x_1+\mu_1+2\mu_2\left(x_1-1\right)=0\\ \partial_{x_2}\mathcal{L}=2\left(x_2-1\right)-\mu_1+2\mu_2x_2=0\\ g_1(\boldsymbol{x})=x_1-x_2-1\leq 0\\ g_2(\boldsymbol{x})=\left(x_1-1\right)^2+x_2^2-1\leq 0\\ \mu_1\geq 0\\ \mu_2\geq 0\\ \mu_1g_1(\boldsymbol{x})=\mu_1\left(x_1-x_2-1\right)=0\\ \mu_2g_2(\boldsymbol{x})=\mu_2\left[\left(x_1-1\right)^2+x_2^2-1\right]=0\end{cases} \tag{2}$$

For $j=1,2$, we have the following four cases:

Case 1: $g_1(\boldsymbol{x}), g_2(\boldsymbol{x})$ are both inactive.

Then we have $\mu_1=\mu_2=0,\ x_1=0, x_2=1$

However, $x_1=0, x_2=1$ doesn't satisfy $g_2(\boldsymbol{x})=\left(x_1-1\right)^2+x_2^2-1\leq 0$

Thus we should exclude this case.

Case 2: $g_1(\boldsymbol{x}), g_2(\boldsymbol{x})$ are both active.

Then we have:

$$\begin{cases}g_1(\boldsymbol{x})=x_1-x_2-1=0\\ g_2(\boldsymbol{x})=\left(x_1-1\right)^2+x_2^2-1=0\end{cases} \tag{3}$$

And induce that $x_1=1\pm\frac{\sqrt{2}}{2}, x_2=\pm\frac{\sqrt{2}}{2}$, together with the equations below:

$$\begin{cases}\partial_{x_1}\mathcal{L}=2x_1+\mu_1+2\mu_2\left(x_1-1\right)=0\\ \partial_{x_2}\mathcal{L}=2\left(x_2-1\right)-\mu_1+2\mu_2x_2=0\end{cases} \tag{4}$$

we have $\mu_2=-1$, which contradicts the fact that μ_2 should be larger than zero, thus this case should be excluded.

Case 3: $g_1(\boldsymbol{x})$ is active, while $g_2(\boldsymbol{x})$ is inactive.

Then we have:

$$\begin{cases}g_1(\boldsymbol{x})=x_1-x_2-1=0\\ \mu_2=0\end{cases} \tag{5}$$

With

$$\begin{cases}\partial_{x_1}\mathcal{L}=2x_1+\mu_1+2\mu_2\left(x_1-1\right)=0\\ \partial_{x_2}\mathcal{L}=2\left(x_2-1\right)-\mu_1+2\mu_2x_2=0\end{cases} \tag{6}$$

We have $x_2=0, \mu_1=-2,$ which contradicts the fact that μ_1 should be larger than zero, thus this case should be excluded.

Case 4: $g_2(\boldsymbol{x})$ is active, while $g_1(\boldsymbol{x})$ is inactive.

Then we have:

$$\begin{cases}g_2(\boldsymbol{x})=\left(x_1-1\right)^2+x_2^2-1=0\\ \mu_1=0\end{cases} \tag{7}$$

With

$$\begin{cases}\partial_{x_1}\mathcal{L} = 2x_1 + \mu_1 + 2\mu_2\left(x_1 - 1\right) = 0 \\ \partial_{x_2}\mathcal{L} = 2\left(x_2 - 1\right) - \mu_1 + 2\mu_2 x_2 = 0\end{cases} \tag{8}$$

We have

$$(1)\begin{cases}x_1 = 1 - \frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \\ \mu_1 = 0 \\ \mu_2 = \sqrt{2} - 1\end{cases} \quad (2)\begin{cases}x_1 = 1 + \frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \\ \mu_1 = 0 \\ \mu_2 = -\sqrt{2} - 1\end{cases} \tag{9}$$

In (2), we have $g_1(\boldsymbol{x}) = x_1 - x_2 - 1 \geq 0$, thus get excluded.

In (1), we have $g_1(\boldsymbol{x}) = x_1 - x_2 - 1 \leq 0$, satisfy the KKT conditions.

Thus the optimal point $\boldsymbol{x}^*$ is $\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and the Lagrange multipliers are $\mu_1 = 0, \mu_2 = \sqrt{2} - 1$.

Question 2

We know the fact that KKT conditions for convex problems are both sufficient and necessary.

First, we transform the problem into the following convex form:

$$\begin{aligned}\min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & g_1(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 + 1 \leq 0 \\ & g_2(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + x_2^2 + 1 \leq 0\end{aligned} \tag{10}$$

We construct the Lagrangian as below:

$$\mathcal{L}\left(x_1, x_2, \mu_1, \mu_2\right) = x_1^2 + x_2^2 + \mu_1\left[\left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1\right] + \mu_2\left[\left(x_1 - 1\right)^2 + x_2^2 - 1\right] \tag{11}$$

The KKT conditions are:

$$\begin{cases}\partial_{x_1}\mathcal{L} = 2x_1 + 2\mu_1\left(x_1 - 1\right) + 2\mu_2\left(x_1 - 1\right) = 0 \\ \partial_{x_2}\mathcal{L} = 2x_2 + 2\mu_1\left(x_2 - 1\right) + 2\mu_2 x_2 = 0 \\ g_1(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1 \leq 0 \\ g_2(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + x_2^2 - 1 \leq 0 \\ \mu_1\left[\left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1\right] = 0 \\ \mu_2\left[\left(x_1 - 1\right)^2 + x_2^2 - 1\right] = 0 \\ \mu_1 \geq 0 \\ \mu_2 \geq 0\end{cases} \tag{12}$$

Then we check the following points:

(1) $\left(x_1, x_2\right) = (1, 1)$

Since $g_1(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1 < 0$

$\therefore \mu_1 = 0$

Then $\partial_{x_1}\mathcal{L} = 2x_1 + 2\mu_1\left(x_1 - 1\right) + 2\mu_2\left(x_1 - 1\right) = 0$ not holds for $\left(x_1, x_2, \mu_1\right) = (1, 1, 0)$

Thus this point is not an optimal point.

(2) $\left(x_1, x_2\right) = (0, 1)$

This is not an optimal point since $g_2(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + x_2^2 - 1 = 1 > 0$

(3) $\left(x_1, x_2\right) = (1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2})$

Since $g_2(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + x_2^2 - 1 = 1 - \sqrt{2} < 0$

$\therefore\ \mu_2 = 0$

By solving the KKT conditions, we derive that $\mu_1 = \sqrt{2} - 1 > 0$

And it satisfies:

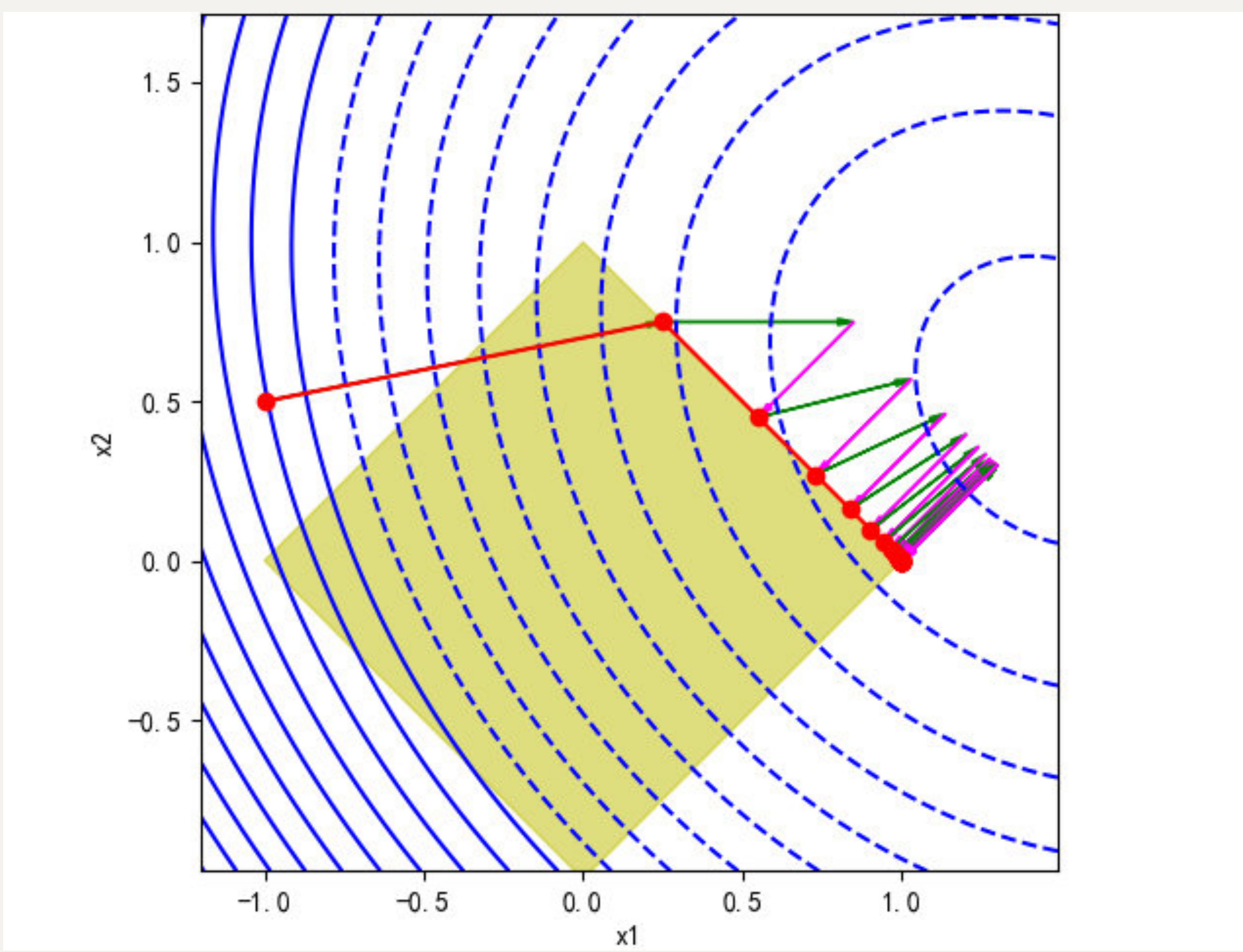
$$\begin{cases} \partial_{x_1}\mathcal{L} = 2x_1 + 2\mu_1\left(x_1 - 1\right) + 2\mu_2\left(x_1 - 1\right) = 0 \\ \partial_{x_2}\mathcal{L} = 2x_2 + 2\mu_1\left(x_2 - 1\right) + 2\mu_2 x_2 = 0 \\ g_1(\boldsymbol{x}) = \left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1 \le 0 \\ \mu_1\left[\left(x_1 - 1\right)^2 + \left(x_2 - 1\right)^2 - 1\right] = 0 \\ \mu_2\left[\left(x_1 - 1\right)^2 + x_2^2 - 1\right] = 0 \\ \mu_1 \ge 0 \end{cases} \tag{13}$$

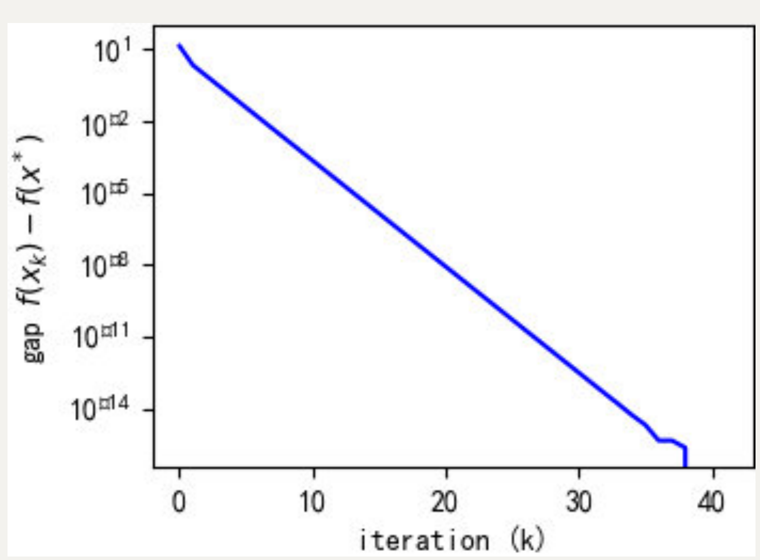
$\therefore$ the point $(x_1, x_2) = (1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2})$ is the optimal point.

Question 3

Log

```
t =  1
number of iterations: 41
solution: [9.99999999e-01 1.00256220e-09]
value: 1.5
```





Question 4

(1.)

$$\mathcal{L}\left(x_1, x_2, x_3, \lambda\right) = e^{2x_1} + e^{x_2} + e^{x_3} + \lambda\left(x_1 + x_2 + x_3 - 1\right) \tag{14}$$

$$
\begin{cases}
\partial_{x_1}\mathcal{L} = 2e^{2x_1} + \lambda = 0 \\
\partial_{x_2}\mathcal{L} = e^{x_2} + \lambda = 0 \\
\partial_{x_3}\mathcal{L} = e^{x_3} + \lambda = 0 \\
x_1 + x_2 + x_3 - 1 = 0
\end{cases} \tag{15}
$$

By solving the equations above we could get:

$$
\begin{cases}
x_1^* = \frac{1-2ln2}{5} \\
x_2^* = \frac{ln2+2}{5} \\
x_3^* = \frac{ln2+2}{5}
\end{cases} \tag{16}
$$

With Lagrange multiplier and optimal value:

$$
\begin{aligned}
\lambda^* &= -\sqrt[5]{2e^2} \\
f^* &= \frac{5}{2}\sqrt[5]{2e^2}
\end{aligned} \tag{17}
$$

(2.)

KKT system for this problem is:

$$
\begin{bmatrix} \nabla^2 f(\boldsymbol{x}) & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\nabla f(\boldsymbol{x}) \\ \boldsymbol{0} \end{bmatrix} \tag{18}
$$

Where:

$$
\nabla f(\boldsymbol{x}) = \begin{pmatrix} 2e^{2x_1} \\ e^{x_2} \\ e^{x_3} \end{pmatrix} \tag{19}
$$

$$
\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 4e^{2x_1} & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 0 & 0 & e^{x_3} \end{pmatrix} \tag{20}
$$

With $\boldsymbol{A} = (1, \quad 1, \quad 1)$.

By Solving equation (18), we get:

$$
\begin{cases}
d_1 = \frac{5}{2}\frac{e^{x_2+x_3}}{e^{x_2+x_3}+4e^{2x_1+x_3}+4e^{2x_1+x_2}} - \frac{1}{2} \\
d_2 = \frac{10e^{2x_1+x_3}}{e^{x_2+x_3}+4e^{2x_1+x_3}+4e^{2x_1+x_2}} - 1 \\
d_3 = \frac{10e^{2x_1+x_2}}{e^{x_2+x_3}+4e^{2x_1+x_3}+4e^{2x_1+x_2}} - 1
\end{cases} \tag{21}
$$

(3.)

```
iteration 0: [0. 1. 0.]
iteration 1: [-0.11369186  0.56845929  0.54523257]
iteration 2: [-0.09499115  0.5534111   0.54158005]
iteration 3: [-0.08601059  0.54598966  0.54002094]
iteration 4: [-0.0816069   0.5423022   0.53930469]
iteration 5: [-0.07942602  0.54046402  0.538962  ]
iteration 6: [-0.07834074  0.53954628  0.53879446]
iteration 7: [-0.07779938  0.53908775  0.53871163]
iteration 8: [-0.07752902  0.53885857  0.53867046]
iteration 9: [-0.07739392  0.53874399  0.53864993]
iteration 10: [-0.07732639  0.53868671  0.53863968]
iteration 11: [-0.07729263  0.53865807  0.53863455]
iteration 12: [-0.07727575  0.53864376  0.538632  ]
iteration 13: [-0.07726731  0.5386366   0.53863072]
iteration 14: [-0.07726309  0.53863302  0.53863008]
iteration 15: [-0.07725887  0.53862944  0.53862944]
iteration 16: [-0.07725887  0.53862944  0.53862944]
optimal value: 4.284141440311191
```