HOMEWORK2

QUESTION 1

(a).

$$f_{x_1} = 4x_1 + 2x_2 + 2$$

$$f_{x_2} = 5x_2 + 2x_1 - 2x_3 - 3$$

$$f_{x_3} = 6x_3 - 2x_2 + 2$$

Since the stationary point holds that

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0}$$

Then we can derive that the station point $\mathbf{x} = (x_1, x_2, x_3)^T = (-1, 1, 0)^T$

Check Hessian matrix:

$$H = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

Its leading principal minor is:

$$H_1 = 4, H_2 = 16, H_3 = 80$$

According to Theorem (Sylvester), H is is positive definite, so $\mathbf{x}$ is local minima of f.

(b).

$$f_{x_1} = x_1 + 2x_2$$

$$f_{x_2} = 2x_2 + 2x_1 - x_3 + 1$$

$$f_{x_3} = -3x_3 - x_2 - 3$$

Since the stationary point holds that

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0}$$

Then we can derive that the station point $\mathbf{x} = (x_1, x_2, x_3)^T = (-\frac{12}{5}, \frac{6}{5}, -\frac{7}{5})^T$

Check Hessian matrix:

$$H = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & -3 \end{pmatrix}$$

check Its leading principal minor:

$$H_1 = 1, H_2 = -2$$

Then we know that H is indefinite matrix, so $\mathbf{x} = (x_1, x_2, x_3)^T = (-\frac{12}{5}, \frac{6}{5}, -\frac{7}{5})^T$ is neither a local minima, nor a local maxima

QUESTION 2

According to Theorem (Sylvester), α should satisfy:

$$A = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 1 & \alpha \\ 2 & \alpha & 2 \end{vmatrix} \geq 0$$

$$A_2 = \begin{vmatrix} 1 & \alpha \\ \alpha & 2 \end{vmatrix} \geq 0$$

Then we have:

$$\begin{cases} -4\alpha - 3\alpha^2 \geq 0 \\ 2 - \alpha^2 \geq 0 \end{cases}$$

$$-\frac{4}{3} \leq \alpha \leq 0$$

QUESTION 3

If C is convex, according to the definition, $\forall \mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n$, which correspond to $f(\mathbf{x_1}), f(\mathbf{x_2})$, we have:

$$\theta f(\mathbf{x_1}) + \bar{\theta} f(\mathbf{x_2}) \in C, \theta \in [0, 1]$$

Since

$$\theta f(\mathbf{x_1}) + \bar{\theta} f(\mathbf{x_2}) = \mathbf{A}(\theta \mathbf{x_1} + \bar{\theta} \mathbf{x_2}) + \mathbf{b}$$

We can derive that

$$\theta \mathbf{x_1} + \bar{\theta} \mathbf{x_2} \in \mathbb{R}^n, \forall \mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n$$

Therefore, $f^{-1}(C)$ is also convex.

QUESTION 4

For $\forall X, Y \in C$, we consider:

$$\theta X + \bar{\theta} Y = \theta a_1 + \theta a_2 + \bar{\theta} b_1 + \bar{\theta} b_2, \theta + \bar{\theta} = 1$$

here $a_1, b_1 \in C_1, a_2, b_2 \in C_2$ .Since C_1, C_2 are convex sets, we could infer that

$$\theta a_1 + \theta b_1 \in C_1, \theta a_2 + \theta b_2 \in C_2$$

Therefore,

$$\theta X + \bar{\theta} Y \in C$$

As a result, we have proved that C is a convex set.

QUESTION 5

(a.)

$\forall \mathbf{x}, \mathbf{y} \in intC$, we could find a small enough number r, s.t. $B(\mathbf{x}, r), B(\mathbf{y}, r) \in C$.

Then we can find a number $\delta < r$. $\forall \mathbf{x}, \mathbf{y} \in intC$ (mentioned above), consider:

$$\begin{aligned} B(\theta\mathbf{x} + \bar{\theta}\mathbf{y}, \delta) &= \Big\{\theta\mathbf{x} + \bar{\theta}\mathbf{y} + \delta\mathbf{u} : \mathbf{u} \in B(\mathbf{0}, 1)\Big\} \\ &= \Big\{\theta(\mathbf{x} + \delta\mathbf{u}) + \bar{\theta}(\mathbf{y} + \delta\mathbf{u}) : \mathbf{u} \in B(\mathbf{0}, 1)\Big\} \end{aligned}$$

$\because \mathbf{x} + \delta\mathbf{u}, \mathbf{y} + \delta\mathbf{u} \in C$

$\therefore \theta(\mathbf{x} + \delta\mathbf{u}) + \bar{\theta}(\mathbf{y} + \delta\mathbf{u}) \in C$

$\therefore \theta\mathbf{x} + \bar{\theta}\mathbf{y}$ is the interior point of C

$\therefore intC$ is a convex set

(b).

$\forall \mathbf{x}, \mathbf{y} \in \bar{C}$, we can find arrays $\{\mathbf{x_i}\}, \{\mathbf{y_i}\}$ $s.t.$ $\lim\limits_{i \to +\infty} \mathbf{x_i} = \mathbf{x}, \lim\limits_{i \to +\infty} \mathbf{y_i} = \mathbf{y}$,

where $\mathbf{x_i}, \mathbf{y_i} \in C$

Note that we could acutally consider all the points in $\bar{C}$ as cluster point. if not, just take $\mathbf{x_i} \equiv \mathbf{x}, \mathbf{y_i} \equiv \mathbf{y}$

$\because C$ is convex set

$\therefore \theta \mathbf{x_i} + \bar{\theta}\mathbf{y_i} \in C$

$\because \lim\limits_{i \to +\infty} \theta \mathbf{x_i} + \bar{\theta}\mathbf{y_i} = \theta \mathbf{x} + \bar{\theta}\mathbf{y}$,

$\therefore \theta \mathbf{x} + \bar{\theta}\mathbf{y} \in \bar{C}$ (according to the definition of closure)

$\therefore \bar{C}$ is a convex set