

CS2601 Linear and Convex Optimization

Homework 8

Due: 2022.12.7

1. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{x_1, x_2} \quad & f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = 1 \end{aligned}$$

- (a). Find the optimal solution x_1^*, x_2^* by reduction to an unconstrained problem.
- (b). Find the optimal solution and the corresponding Lagrange multiplier λ^* using the Lagrangian multipliers method.

2. Solve the following problem,

$$\begin{aligned} \min_{x_1, x_2} \quad & f(x_1, x_2) = x_1x_2 + x_1^2 \\ \text{s.t.} \quad & x_1^2 + \frac{1}{8}x_2^2 = 1 \end{aligned}$$

3. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{g}^T \mathbf{x} + c \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} = \mathbf{b} \end{aligned}$$

where $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $\mathbf{Q} \succ \mathbf{O}$, $\mathbf{g} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $\mathbf{A} \in \mathbb{R}^{k \times n}$ with $\text{rank } \mathbf{A} = k$, and $\mathbf{b} \in \mathbb{R}^k$.

- (a). Write down the Lagrange condition for this problem.
- (b). Find a **closed form solution** for the optimal solution \mathbf{x}^* and the corresponding Lagrange multiplier λ^* .
Hint: Show $\mathbf{A} \mathbf{Q}^{-1} \mathbf{A}^T \succ \mathbf{O}$ and hence is invertible.
- (c). Use part (b) to find the projection $\text{Proj}_S(\mathbf{x}_0)$ of a point \mathbf{x}_0 onto the the affine space $S = \{\mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}\}$, i.e. solve

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2} \|\mathbf{x} - \mathbf{x}_0\|_2^2 \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} = \mathbf{b} \end{aligned}$$

When $\mathbf{x}_0 = \mathbf{0}$, you should recover the result on slide 11 of §9.

- (d). Consider a hyperplane $P = \{\mathbf{x} : \mathbf{w}^T \mathbf{x} = b\}$. Use the result in (c) to find the distance $d(\mathbf{x}_0, P)$ between \mathbf{x}_0 and P . You should recover the result on slide 13 of §1.