CS 2601 Linear and Convex Optimization

10. KKT conditions

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Active and inactive constraints

Let $\boldsymbol{x} \in \mathbb{R}^n$ and $n \geq k$. Consider

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & h_i(\boldsymbol{x}) = 0,\ i = 1, 2, \ldots, k \\ & g_j(\boldsymbol{x}) \leq 0,\ j = 1, 2, \ldots, m \end{aligned} \tag{ICP}$$

We do not assume it is a convex problem. Assume the domain is $\mathbb{R}^n$. The feasible set is

$$X = \{\boldsymbol{x} : h_i(\boldsymbol{x}) = 0,\ 1 \leq i \leq k;\ g_j(\boldsymbol{x}) \leq 0,\ 1 \leq j \leq m\}$$

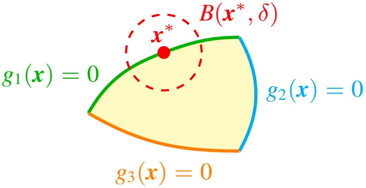
Let $\boldsymbol{x}_0 \in X$. The j-th inequality constraint $g_j(\boldsymbol{x}) \leq 0$ is called active at $\boldsymbol{x}_0$ if $g_j(\boldsymbol{x}_0) = 0$, and inactive at $\boldsymbol{x}_0$ if $g_j(\boldsymbol{x}_0) < 0$. Denote by $J(\boldsymbol{x}_0)$ the set of indices of the active inequality constraints at $\boldsymbol{x}_0$,

$$J(\boldsymbol{x}_0) = \{j : g_j(\boldsymbol{x}_0) = 0\}$$

By convention, equality constraints are considered active at all $\boldsymbol{x} \in X$.

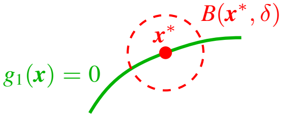
Reduction to equality constrained problem

Suppose $\boldsymbol{x}^*$ is a local minimum of (ICP). It is the solution to



$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\
\text{s.t.} \quad & h_i(\boldsymbol{x}) = 0,\ i = 1, 2, \dots, k \\
& g_j(\boldsymbol{x}) \le 0,\ j = 1, 2, \dots, m \\
& \boldsymbol{x} \in B(\boldsymbol{x}^*, \delta)
\end{aligned}
$$

for some small enough δ. It is equivalent to



$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\
\text{s.t.} \quad & h_i(\boldsymbol{x}) = 0,\ i = 1, 2, \dots, k \\
& g_j(\boldsymbol{x}) = 0,\ j \in J(\boldsymbol{x}^*) \\
& \boldsymbol{x} \in B(\boldsymbol{x}^*, \delta)
\end{aligned}
$$

If it is known a priori which constraints are active at $\boldsymbol{x}^*$, we can find $\boldsymbol{x}^*$ by solving the above equality constrained problem.

Reduction to equality constrained problem (cont'd)

A local minimum $\boldsymbol{x}^*$ of (ICP) is also a local minimum of the following

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & h_i(\boldsymbol{x}) = 0,\ i = 1, 2, \ldots, k \\ & g_j(\boldsymbol{x}) = 0,\ j \in J(\boldsymbol{x}^*) \end{aligned}$$

$\boldsymbol{x}^* \in X$ is a regular point if $\nabla h_i(\boldsymbol{x}^*), 1 \le i \le k$ and $\nabla g_j(\boldsymbol{x}^*), j \in J(\boldsymbol{x}^*)$ are linearly independent.

At a regular local minimum, Lagrange condition yields

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{k} \lambda_i^* \nabla h_i(\boldsymbol{x}^*) + \sum_{j \in J(\boldsymbol{x}^*)} \mu_j^* \nabla g_j(\boldsymbol{x}^*) = \boldsymbol{0}$$

Setting $\mu_j^* = 0$ for inactive constraints, i.e. $j \notin J(\boldsymbol{x}^*)$,

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{k} \lambda_i^* \nabla h_i(\boldsymbol{x}^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(\boldsymbol{x}^*) = \boldsymbol{0}$$

Karush-Kuhn-Tucker (KKT) conditions

Theorem. If $\boldsymbol{x}^*$ is a local minimum of (ICP) and also a regular point, then there exist Lagrange multipliers[1] $\lambda_1^*, \ldots, \lambda_k^*, \mu_1^*, \ldots, \mu_m^* \in \mathbb{R}$ s.t. the following KKT conditions hold,

1. $\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{k} \lambda_i^* \nabla h_i(\boldsymbol{x}^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(\boldsymbol{x}^*) = \mathbf{0}$ (stationarity)
2. $\mu_j^* \geq 0, j = 1, 2, \ldots, m$
3. $\mu_j^* g_j(\boldsymbol{x}^*) = 0, j = 1, 2, \ldots, m$ (complementary slackness)

Note. Condition 1 says $\nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}$ for the Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{\mu}^T \boldsymbol{g}(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{i=1}^{k} \lambda_i h_i(\boldsymbol{x}) + \sum_{j=1}^{m} \mu_j g_j(\boldsymbol{x})$$

Note. Condition 3 is called complementary slackness condition, as it implies either $\mu_j^* = 0$ or $g_j(\boldsymbol{x}^*) = 0$.

[1] Sometimes also called KKT multipliers. Sometimes λ_i are called Lagrange multipliers while μ_j are called KKT multipliers.

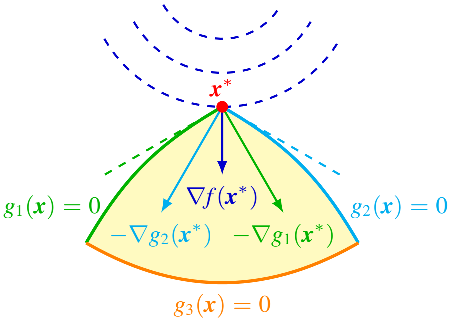
Geometric interpretation

Let $\boldsymbol{x} \in \mathbb{R}^2$. Consider

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad g_j(\boldsymbol{x}) \le 0,\ j = 1, 2, 3$$

Suppose $\boldsymbol{x}^*$ is a local minimum and only g_1 and g_2 are active at $\boldsymbol{x}^*$. The KKT condition says $\mu_1^* \ge 0$, $\mu_2^* \ge 0$, $\mu_3^* = 0$ and

$$\nabla f(\boldsymbol{x}^*) = -\mu_1^* \nabla g_1(\boldsymbol{x}^*) - \mu_2^* \nabla g_2(\boldsymbol{x}^*)$$



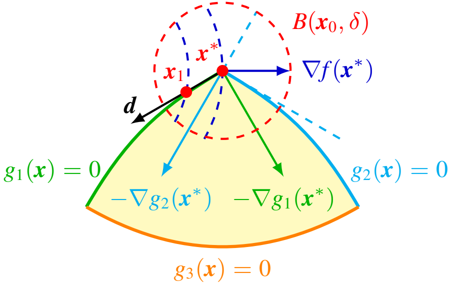
Geometric interpretation (cont'd)

Why $\mu_j^* \geq 0$? Assume $\mu_2^* < 0$ and we show a contradiction.

- By regularity of $\boldsymbol{x}^*$, $\nabla g_2(\boldsymbol{x}^*) \notin S \triangleq$ span $\{\nabla g_1(\boldsymbol{x}^*)\}$.
- Let $\boldsymbol{d} = -\nabla g_2(\boldsymbol{x}^*) - \mathcal{P}_S(-\nabla g_2(\boldsymbol{x}^*))$. Then $\boldsymbol{d} \perp S$, $\boldsymbol{d}^T \nabla g_2(\boldsymbol{x}^*) < 0$
- Move along the curve $g_1(\boldsymbol{x}) = 0$ in the direction of $\boldsymbol{d}$ from $\boldsymbol{x}^*$ to $\boldsymbol{x}_1$.

$$\boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) = \boldsymbol{d}^T[-\mu_1 \nabla g_1(\boldsymbol{x}^*) - \mu_2 \nabla g_2(\boldsymbol{x}^*)] = -\mu_2^* \boldsymbol{d}^T \nabla g_2(\boldsymbol{x}^*) < 0.$$

For a small move, $f(\boldsymbol{x}_1) < f(\boldsymbol{x}^*)$, contradicting minimality of $f(\boldsymbol{x}^*)$.



Appendix: Proof of $\boldsymbol{\mu} \geq \mathbf{0}$

Suppose $\mu_{j_0}^* < 0$ for some j_0. We will show we can move away from $\boldsymbol{x}^*$ so that feasibility is maintained but f decreases, contradicting the minimality of $\boldsymbol{x}^*$. Let $J'(\boldsymbol{x}^*) = J(\boldsymbol{x}^*) \setminus \{j_0\}$, and

$$S = \text{span}\left\{\nabla h_i(\boldsymbol{x}), i = 1, 2, \ldots, k;\ \nabla g_j(\boldsymbol{x}),\ j \in J'(\boldsymbol{x}^*)\right\}$$

1. $\nabla g_{j_0}(\boldsymbol{x}^*) \notin S$ by regularity of $\boldsymbol{x}^*$.
2. Let $\boldsymbol{d} = -\nabla g_{j_0}(\boldsymbol{x}^*) - \mathcal{P}_S(-\nabla g_{j_0}(\boldsymbol{x}^*))$. Then $\boldsymbol{d} \perp S$, $\boldsymbol{d}^T \nabla g_{j_0}(\boldsymbol{x}^*) < 0$
3. KKT then implies $\boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) < 0$
4. By the lemma on slide 23 of §9, there exists a local path $\boldsymbol{x}(t)$ s.t. $\boldsymbol{x}(0) = \boldsymbol{x}^*$, $\boldsymbol{x}'(0) = \boldsymbol{d}$, $h_i(\boldsymbol{x}(t)) = 0, \forall i$, and $g_j(\boldsymbol{x}(t)) = 0$ for $j \in J'(\boldsymbol{x}^*)$.
5. For $j \notin J(\boldsymbol{x}^*)$, $g_j(\boldsymbol{x}^*) < 0$. By continuity, $g_j(\boldsymbol{x}(t)) < 0$ for small t.
6. By the chain rule,

$$\left.\frac{d}{dt} g_{j_0}(\boldsymbol{x}(t))\right|_{t=0} = \nabla g_{j_0}(\boldsymbol{x}^*)^T \boldsymbol{x}'(0) = \nabla g_{j_0}(\boldsymbol{x}^*)^T \boldsymbol{d} < 0$$

   For small $t > 0$, $g_{j_0}(\boldsymbol{x}(t)) < g_{j_0}(\boldsymbol{x}^*) = 0$. Similarly, $f(\boldsymbol{x}(t)) < f(\boldsymbol{x}^*)$.

Proof of $\boldsymbol{\mu} \geq \mathbf{0}$ (cont'd)

Proof of step 3.

- By KKT Conditions 2 and 3

$$\nabla f(\boldsymbol{x}^*) = -\sum_{i=1}^{k} \lambda_i^* \nabla h_i(\boldsymbol{x}^*) - \sum_{j \in J(\boldsymbol{x}^*)} \mu_j^* \nabla g_j(\boldsymbol{x}^*) - \sum_{j \notin J(\boldsymbol{x}^*)} \underbrace{\mu_j^* \nabla g_j(\boldsymbol{x}^*)}_{=\mathbf{0}}$$

- Since $\boldsymbol{d} \perp S$, $\mu_{j_0}^* < 0$,

$$\boldsymbol{d}^T \nabla f(\boldsymbol{x}^*) = -\sum_{i=1}^{k} \lambda_i^* \underbrace{\boldsymbol{d}^T \nabla h_i(\boldsymbol{x}^*)}_{=0} - \sum_{j \in J'(\boldsymbol{x}^*)} \mu_j^* \underbrace{\boldsymbol{d}^T \nabla g_j(\boldsymbol{x}^*)}_{=0} - \mu_{j_0}^* \underbrace{\boldsymbol{d}^T \nabla g_{j_0}(\boldsymbol{x}^*)}_{<0}$$
$$< 0$$

Sufficiency of KKT conditions for convex problems

Theorem. For a convex problem, i.e. f and g_j are convex, and h_i are affine, if there exist $\lambda_1^*, \ldots, \lambda_k^*$ and $\mu_1^*, \ldots, \mu_m^*$ s.t. the KKT conditions are satisfied at a feasible $\boldsymbol{x}^* \in X$, then $\boldsymbol{x}^*$ is a global minimum.

Note. The previous necessary conditions assume $\boldsymbol{x}^*$ is regular point. The sufficient conditions here assume convexity but not regularity.

Proof. We show $\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) \geq 0, \forall \boldsymbol{x} \in X$.

1. By the KKT conditions,

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) = -\sum_i \lambda^* \nabla h_i(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) - \sum_{j \in J(\boldsymbol{x}^*)} \mu_j^* \nabla g_j(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*)$$

   It suffices to show $\nabla h_i(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) = 0$ and $\nabla g_j(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) \leq 0$.

2. Since $h_i(\boldsymbol{x}) = \boldsymbol{a}_i^T\boldsymbol{x} - b_i$ is affine, and $h_i(\boldsymbol{x}) = h(\boldsymbol{x}^*) = 0$ by feasibility,

$$\nabla h_i(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) = \boldsymbol{a}_i^T(\boldsymbol{x}-\boldsymbol{x}^*) = h_i(\boldsymbol{x}) - h(\boldsymbol{x}^*) = 0$$

3. For $j \in J(\boldsymbol{x}^*)$, $g_j(\boldsymbol{x}^*) = 0$ and $g_j(\boldsymbol{x}) \leq 0$. By the convexity of g_j,

$$\nabla g_j(\boldsymbol{x}^*)^T(\boldsymbol{x}-\boldsymbol{x}^*) \leq g_j(\boldsymbol{x}) - g_j(\boldsymbol{x}^*) \leq 0$$

Example

$$\min_{\boldsymbol{x}\in\mathbb{R}^3} \quad f(\boldsymbol{x}) = x_1 + 2x_2 + x_3$$
$$\text{s.t.} \quad h(\boldsymbol{x}) = x_1 + x_2 + 2x_3 = 0$$
$$g(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 - 1 \leq 0$$

All feasible points are regular. The Lagrangian is

$$\mathcal{L}(\boldsymbol{x}, \lambda, \mu) = x_1 + 2x_2 + x_3 + \lambda(x_1 + x_2 + 2x_3) + \mu(x_1^2 + x_2^2 + x_3^2 - 1)$$

The KKT conditions (including the constraints) are

$$\begin{cases} \mu \geq 0 \\ \partial_{x_1}\mathcal{L} = 1 + \lambda + 2\mu x_1 = 0 \\ \partial_{x_2}\mathcal{L} = 2 + \lambda + 2\mu x_2 = 0 \\ \partial_{x_3}\mathcal{L} = 1 + 2\lambda + 2\mu x_3 = 0 \\ \mu(x_1^2 + x_2^2 + x_3^2 - 1) = 0 \\ x_1 + x_2 + 2x_3 = 0 \\ x_1^2 + x_2^2 + x_3^2 - 1 \leq 0 \end{cases}$$

Example (cont'd)

Case I. g is inactive. Thus $\mu = 0$. But this leads to a contradiction.

$$\begin{cases} \partial_{x_1}\mathcal{L} = 1 + \lambda + 2\mu x_1 = 0 \implies \lambda = -1 \\ \partial_{x_2}\mathcal{L} = 2 + \lambda + 2\mu x_2 = 0 \implies \lambda = -2 \end{cases}$$

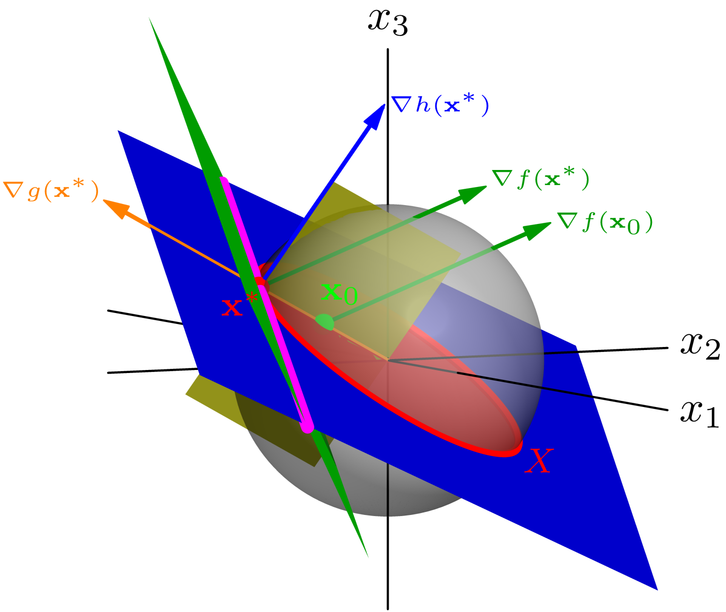
Case II. g is active. This essentially reduces to the example on slide 31 of §9, but we only take the solution with $\mu \geq 0$,

$$\begin{cases} x_1 = -\frac{1}{\sqrt{66}} \\ x_2 = -\frac{7}{\sqrt{66}} \\ x_3 = \frac{4}{\sqrt{66}} \\ \lambda = -\frac{5}{6} \\ \mu = \sqrt{\frac{33}{72}} \end{cases}$$

Since the problem is convex, the above gives a global minimum.

Note. By minimizing $-f$, one can verify the other solution for the example on slide 31 of §9 is a global maximum.

Example (cont'd)



Example

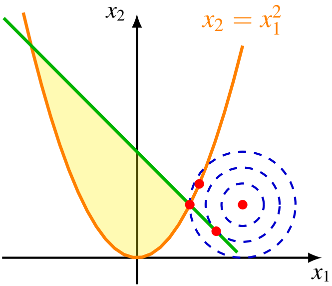
$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \quad f(\boldsymbol{x}) = (x_1-2)^2 + (x_2-1)^2$$
$$\text{s.t.} \quad g_1(\boldsymbol{x}) = x_1^2 - x_2 \leq 0$$
$$g_2(\boldsymbol{x}) = x_1 + x_2 - 2 \leq 0$$

All feasible points are regular. The Lagrangian is

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{\mu}) = (x_1-2)^2 + (x_2-1)^2 + \mu_1(x_1^2 - x_2) + \mu_2(x_1+x_2-2)$$

The KKT conditions (including the constraints) are

$$\begin{cases} \mu_1 \geq 0 \\ \mu_2 \geq 0 \\ \partial_{x_1}\mathcal{L} = 2(x_1-2) + 2\mu_1 x_1 + \mu_2 = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2-1) - \mu_1 + \mu_2 = 0 \\ \mu_1(x_1^2 - x_2) = 0 \\ \mu_2(x_1+x_2-2) = 0 \\ x_1^2 - x_2 \leq 0 \\ x_1 + x_2 - 2 \leq 0 \end{cases}$$



Example (cont'd)

Case I. Both g_1 and g_2 are inactive, so $\mu_1 = \mu_2 = 0$.

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2(x_1 - 2) = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2 - 1) = 0 \end{cases} \implies \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$$

But

$$x_1^2 - x_2 = 3 > 0$$

violating $g_1 \leq 0$.

Case II. g_2 is active, but g_1 is inactive, so $\mu_1 = 0$.

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2(x_1 - 2) + \mu_2 = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2 - 1) + \mu_2 = 0 \\ x_1 + x_2 - 2 = 0 \end{cases} \implies \begin{cases} x_1 = \frac{3}{2} \\ x_2 = \frac{1}{2} \\ \mu_2 = 1 \end{cases}$$

But

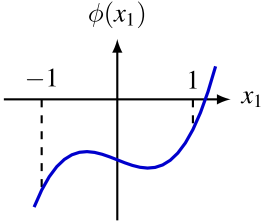
$$x_1^2 - x_2 = \frac{7}{4} > 0$$

violating $g_1 \leq 0$.

Example (cont'd)

Case III. g_1 is active, but g_2 is inactive, so $\mu_2 = 0$.

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2(x_1 - 2) + 2\mu_1 x_1 = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2 - 1) - \mu_1 = 0 \\ x_1^2 - x_2 = 0 \end{cases}$$



From the last two equations,

$$x_2 = x_1^2, \quad \mu_1 = 2(x_2 - 1) = 2(x_1^2 - 1)$$

Plugging into the first equation,

$$2(x_1 - 2) + 4x_1(x_1^2 - 1) = 0 \implies \phi(x_1) \triangleq 2x_1^3 - x_1 - 2 = 0$$

Note $\mu_1 \geq 0 \implies x_1^2 \geq 1 \implies x_1 \geq 1$ or $x_1 \leq -1$.

If $x_1 \geq 1$, then $x_2 = x_1^2 \geq 1$, contradicting $x_1 + x_2 < 2$ (g_2 is inactive).
If $x_1 \leq -1$, $\phi(x_1) = 0$ has no solution since $\phi'(x_1) = 6x_1^2 - 1 > 0$ for $x_1 \leq -1$ and $\phi(-1) = -3 < 0$.

Example (cont'd)

Case IV. Both g_1 and g_2 are active.

$$\begin{cases} x_1^2 - x_2 = 0 \\ x_1 + x_2 - 2 = 0 \end{cases} \implies \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} x_1 = -2 \\ x_2 = -2 \end{cases}$$

Plugging into

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2(x_1 - 2) + 2\mu_1 x_1 + \mu_2 = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2 - 1) - \mu_1 + \mu_2 = 0 \end{cases}$$

yields

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ \mu_1 = \frac{2}{3} \\ \mu_2 = \frac{2}{3} \end{cases} \quad \text{or} \quad \begin{cases} x_1 = -2 \\ x_2 = -2 \\ \mu_1 = -\frac{2}{3} \quad \text{(violating } \mu_1 \geq 0\text{)} \\ \mu_2 = \frac{16}{3} \end{cases}$$

This is a convex problem, so $\boldsymbol{x}^* = (1, 1)^T$ is the global minimum.

Example: Power allocation

Recall the power allocation problem on slide 6 of §1,

$$\begin{aligned}
\max_{P_1,\ldots,P_n} \quad & \sum_{i=1}^{n} W_i \log_2(1+\frac{P_i}{N_i}) \\
\text{s.t.} \quad & \sum_{i=1}^{n} P_i \le P \\
& P_i \ge 0, \quad i = 1,2,\ldots,n
\end{aligned}$$

The optimal solution should satisfy $\sum_{i=1}^{n} P_i = P$ (why?). Equivalent to

$$\begin{aligned}
\min_{\boldsymbol{P}} \quad & f(\boldsymbol{P}) = -\sum_{i=1}^{n} W_i \log(1+\frac{P_i}{N_i}) \\
\text{s.t.} \quad & h(\boldsymbol{P}) = \sum_{i=1}^{n} P_i - P = 0 \\
& g_i(\boldsymbol{P}) = -P_i \le 0, \quad i = 1,2,\ldots,n
\end{aligned}$$

Example: Power allocation (cont'd)

The Lagrangian is

$$\mathcal{L}(\boldsymbol{P}, \lambda, \boldsymbol{\mu}) = -\sum_{i=1}^{n} W_i \log(1 + \frac{P_i}{N_i}) + \lambda(\sum_{i=1}^{n} P_i - P) - \sum_{i=1}^{n} \mu_i P_i$$

All feasible points are regular. By the stationarity condition,

$$\partial_{P_i}\mathcal{L} = -\frac{W_i}{P_i + N_i} + \lambda - \mu_i = 0, \quad i = 1, 2, \ldots, n.$$

For each i,

1. If g_i is inactive, then $\mu_i = 0$ and
$$P_i = \frac{W_i}{\lambda} - N_i > 0$$
2. if g_i is active, then $P_i = 0$ and
$$-\frac{W_i}{N_i} + \lambda = \mu_i \geq 0 \implies \frac{W_i}{\lambda} - N_i \leq 0$$

Example: Power allocation (cont'd)

In both cases,

$$P_i = \left(\frac{W_i}{\lambda} - N_i\right)^+ = W_i\left(r - \frac{N_i}{W_i}\right)^+, \quad \text{where } r = \frac{1}{\lambda}$$

It remains to solve for r from

$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} W_i\left(r - \frac{N_i}{W_i}\right)^+ = P$$

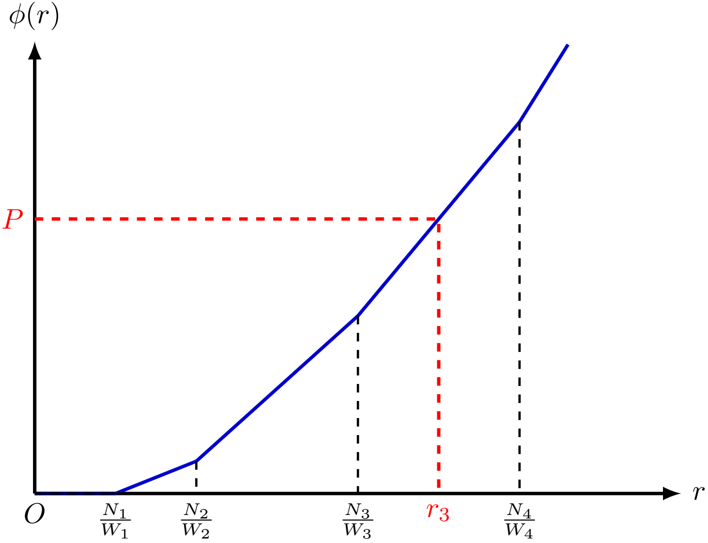
WLOG, assume $\frac{N_1}{W_1} \leq \frac{N_2}{W_2} \leq \ldots \frac{N_n}{W_n}$. Note $\phi(r) = \sum_{i=1}^{n} W_i\left(r - \frac{N_i}{W_i}\right)^+$ is continuous, piecewise linear, strictly increasing on $[\frac{N_1}{W_1}, +\infty)$ with $\phi(\frac{N_1}{W_i}) = 0$. Thus $\phi(r) = P$ has a unique solution $r = r_{k_0}$, where

$$r_k = \frac{P + \sum_{i=1}^{k} N_i}{\sum_{i=1}^{k} W_i}$$

and

$$k_0 = \max\left\{k : \phi(\frac{N_k}{W_k}) \leq P\right\} = \max\left\{k : r_k \geq \frac{N_k}{W_k}\right\}$$

Example: Power allocation (cont'd)



Example: Power allocation (cont'd)

Famous water filling solution.