Algorithm Design and Analysis (Fall 2022)

Assignment 1

Deadline: Mar 19, 2022

1. (20 Points) Prove the following generalization of the master theorem. Given constants $a \geq 1, b > 1, d \geq 0$, and $w \geq 0$, if $T(n) = 1$ for $n < b$ and $T(n) = aT(n/b) + n^d \log^w n$, we have

$$T(n) = \begin{cases} O(n^d \log^w n) & \text{if } a < b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \\ O(n^d \log^{w+1} n) & \text{if } a = b^d \end{cases}.$$

2. (20 points) Recall the median-of-the-medians algorithm we learned in the lecture. It groups the numbers by 5. What happens if we group them by $3, 7, 9 \ldots$? Please analyze those different choices and discuss which one is the best. Note that in this problem, we may drop the big-O notation and discuss the constants.

3. (20 points) Given n integers, where n is even. Can you find both the maximum and minimum within $3n/2 - 2$ comparisons? Note that we also do not use big-O notations here.

4. (20 points) Recall the Word RAM model we learn in the lecture. Let w be the number of bits in a machine word. We know that those basic operations on each word (like addition, right shift, and/or, ...) can be viewed as unit operations. We will design algorithms to compute the transpose A^T of a binary matrix $A \in \{0, 1\}^{h \times h}$.

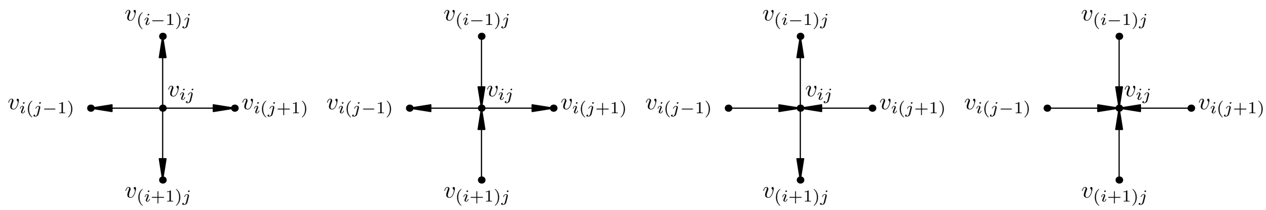
   (a) (5 points) If we want to store the matrix in one word, how large h can be? How to store it in one word?

   (b) (15 points) Consider an one-word size matrix A. Design an $O(\log w)$ algorithm to compute its transpose A^T.

5. (20 points) Let $G = (V, E)$ be a $n \times n$ *grid* which is an undirected graph with n^2 vertices labeled $V = \{v_{ij}\}_{i=0,1,\dots,n-1;j=0,1,\dots,n-1}$. Vertex v_{ij} is adjacent to $v_{(i-1)j}$ (unless $i = 0$), $v_{(i+1)j}$ (unless $i = n-1$), $v_{i(j-1)}$ (unless $j = 0$), and $v_{i(j+1)}$ (unless $j = n-1$).

   Let H be a directed graph sharing the same vertex set V as G. The edge set of H is the same as the edge set of G, except that a direction of each edge in G is specified for the corresponding edge in H.

   (a) (10 points) Suppose the directions of the edges in H satisfy the followings. The directions of the edges between v_{ij} and the "left" and the "right" neighbors of v_{ij} are the same, i.e., either both edges point towards v_{ij} or both edges point outwards v_{ij}, and the directions of the edges between v_{ij} and the "upper" and the "lower" neighbors of v_{ij} are the same. Specifically, the four edges incident to v_{ij} (where $0 < i < n-1$ and $0 < j < n-1$) must be in one of the following four configurations.

   Design an algorithm that finds *one vertex with out-degree* 0 <u>or</u> *one directed cycle*. Your algorithm must run in $O(\log n)$ time.

   (b) (10 points) Suppose the directions of the edges in H are defined as follows. There is a function (given as input) $f : V \to \mathbb{Z}$ that assigns an integer *value* to each vertex, and assume $f(u) \neq f(v)$ for any $u \neq v \in V$. The direction of the edge between the two vertices u, v is from the vertex with a higher value to the vertex with a lower value.

   Design an algorithm that finds *one vertex with out-degree* 0 <u>or</u> *one directed cycle*. Your algorithm must run in $O(n)$ time.

   (c) (Bonus 5 points) If no further assumptions were made for directions of the edges in H, prove that any algorithm that finds *one vertex with out-degree* 0 <u>or</u> *one directed cycle* requires $\Omega(n^2)$ time.

6. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.