第十周作业答案

1. 我们用$|\varphi_n\rangle$表示厄米算符H的本征态（譬如，H可以是任何物理体系的哈密顿算符），假设全体$|\varphi_n\rangle$构成一个离散的正交归一基。算符$U(m,n)$定义是
$$U(m,n) = |\varphi_m\rangle\langle\varphi_n|,$$
   a. 计算$U(m,n)$的伴随算符$U^\dagger(m,n)$，
   b. 证明：
$$U(m,n)U^\dagger(p,q) = \delta_{n,q}U(m,p),$$
   c. 计算算符$U(m,n)$的迹$\mathrm{Tr}\{U(m,n)\}$,
   （参考迹的定义：$\mathrm{Tr}\{U(m,n)\} = \sum_i\langle\varphi_i|U(m,n)|\varphi_i\rangle$）
   d. 设A是一个算符，它的矩阵元是$A_{mn} = \langle\varphi_m|A|\varphi_n\rangle$；试证：
$$A = \sum_{m,n} A_{mn}U(m,n),$$
   e. 试证：$A_{pq} = \mathrm{Tr}\{AU^\dagger(p,q)\}$。

解：

a.
$$U^\dagger(m,n) = (|\varphi_m\rangle\langle\varphi_n|)^\dagger = |\varphi_n\rangle\langle\varphi_m|.$$

b.
$$\begin{aligned}U(m,n)U^\dagger(p,q) &= |\varphi_m\rangle\langle\varphi_n|\left(|\varphi_p\rangle\langle\varphi_q|\right)^\dagger \\ &= |\varphi_m\rangle\langle\varphi_n|\varphi_q\rangle\langle\varphi_p| \\ &= \delta_{n,q}|\varphi_m\rangle\langle\varphi_p| = \delta_{n,q}U(m,p).\end{aligned}$$

c.
$$\mathrm{Tr}\{U(m,n)\} = \sum_i\langle\varphi_i|U(m,n)|\varphi_i\rangle = \sum_i\langle\varphi_i|\varphi_m\rangle\langle\varphi_n|\varphi_i\rangle = \sum_i\delta_{im}\delta_{ni} = \delta_{mn}.$$
或参考公式：$\mathrm{Tr}(|\alpha\rangle\langle\beta|) = \langle\beta|\alpha\rangle$

d. 利用态空间中恒等算符的表达式
$$A = \sum_m|\varphi_m\rangle\langle\varphi_m|\,A\sum_n|\varphi_n\rangle\langle\varphi_n|$$
$$= \sum_{m,n}|\varphi_m\rangle\langle\varphi_m|A|\varphi_n\rangle\langle\varphi_n|$$

$$= \sum_{m,n} A_{mn} |\varphi_m\rangle\langle\varphi_n|$$

$$= \sum_{m,n} A_{mn} U(m,n).$$

e.

$$\mathrm{Tr}\{AU^\dagger(p,q)\} = \sum_n \langle\varphi_n|AU^\dagger(p,q)|\varphi_n\rangle$$

$$= \sum_n \langle\varphi_n|A|\varphi_q\rangle\langle\varphi_p|\varphi_n\rangle$$

$$= \sum_n \langle\varphi_n|A|\varphi_q\rangle\delta_{np} = A_{pq}.$$