第三周作业参考答案

1. 根据谐振子波函数的形式:

$$\psi_n(x) = \left(\frac{\alpha}{2^n\sqrt{\pi}n!}\right)^{\frac{1}{2}} H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}, \alpha = \sqrt{\frac{m\omega}{\hbar}},$$

$$\psi_{n+1}(x) = \sqrt{\frac{1}{2(n+1)}}\left(\frac{\alpha}{2^n\sqrt{\pi}n!}\right)^{\frac{1}{2}} H_{n+1}(\alpha x)e^{-\frac{1}{2}\alpha^2x^2},$$

$$\psi_{n-1}(x) = \sqrt{2n}\left(\frac{\alpha}{2^n\sqrt{\pi}n!}\right)^{\frac{1}{2}} H_{n-1}(\alpha x)e^{-\frac{1}{2}\alpha^2x^2},$$

由$[H_{n+1}(\alpha x) - 2\alpha x H_n(\alpha x) + 2nH_{n-1}(\alpha x)]\left(\frac{\alpha}{2^n\sqrt{\pi}n!}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\alpha^2x^2} = 0,$

$$\Rightarrow \sqrt{2(n+1)}\psi_{n+1}(x) - 2\alpha x\psi_n(x) + \sqrt{2n}\psi_{n-1}(x) = 0,$$

即

$$x\psi_n(x) = \frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right].$$

2. 由题 1 中所得的递推关系有:

$$x\psi_{n-1}(x) = \frac{1}{\alpha}\left[\sqrt{\frac{n-1}{2}}\psi_{n-2}(x) + \sqrt{\frac{n}{2}}\psi_n(x)\right],$$

$$x\psi_{n+1}(x) = \frac{1}{\alpha}\left[\sqrt{\frac{n+1}{2}}\psi_n(x) + \sqrt{\frac{n+2}{2}}\psi_{n+2}(x)\right],$$

$$\Rightarrow x^2\psi_n(x) = \frac{1}{2\alpha^2}\Big[\sqrt{n(n-1)}\psi_{n-2}(x) + (2n+1)\psi_n(x) + \sqrt{(n+1)(n+2)}\psi_{n+2}(x)\Big].$$

3. 由题 1 中所得结果知:

$$\bar{x} = \int \psi_n^*(x)x\psi_n(x)\,dx = \int \psi_n^*(x)\frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right]dx = 0.$$

$$\bar{V} = \int \psi_n^*(x)\,\frac{1}{2}m\omega^2x^2\,\psi_n(x)\,dx = \frac{1}{2}m\omega^2\frac{2n+1}{2\alpha^2} = \frac{(n+1/2)\hbar\omega}{2} = \frac{E_n}{2}.$$

4. 入射动量为$\hbar\boldsymbol{k}_0$，入射能量和入射角为

$$E=\frac{\hbar^2k_0^2}{2m}=\frac{\hbar^2}{2m}(k_1^2+k_2^2), \tag{1}$$

$$\theta=\arctan\frac{k_2}{k_1}=\arcsin\frac{k_2}{k_0}. \tag{2}$$

由于$\partial V/\partial y=\partial V/\partial z=0$，反射和折射时粒子动量的$y$分量和$z$分量不变。以$\boldsymbol{k}_R$表示反射波波矢量，由于反射波出现于$x<0$区域，$V=0$，所以$|\boldsymbol{k}_R|$必须等于$|\boldsymbol{k}_0|$，因此

$$\boldsymbol{k}_R=(-k_1,k_2,0), \tag{3}$$

所以反射角必然等于入射角。这就是反射定律。折射波出现于$x>0$区域，设波矢量为$\boldsymbol{k}_D$，即折射粒子动量为$\hbar\boldsymbol{k}_D$。$\boldsymbol{k}_D$应满足能量关系

$$\frac{\hbar k_D^2}{2m}=E+V_0. \tag{4}$$

$\boldsymbol{k}_D$的直角坐标分量可以写成

$$\boldsymbol{k}_D=(k,k_2,0). \tag{5}$$

由式(1)(4)(5)易见，k和k_1间有下列关系

$$k^2-k_1^2=\frac{2mV_0}{\hbar^2}. \tag{6}$$

设折射角为φ，

$$\sin\theta=\frac{k_2}{k_0},\qquad \sin\varphi=\frac{k_2}{k_D},$$

$$\frac{\sin\theta}{\sin\varphi}=\frac{k_D}{k_0}=\left(1+\frac{V_0}{E}\right)^{\frac{1}{2}}=n. \tag{7}$$

相当于光学中的折射定律。

下面求入射、反射、折射波的振幅及相位关系，即系数R,D。在分界面两侧，ψ及$\partial\psi/\partial x$应该连续，即

$$\begin{gathered}(\psi_0+\psi_R)|_{x=0}=\psi_D|_{x=0},\\ \left(\frac{\partial\psi_0}{\partial x}+\frac{\partial\psi_R}{\partial x}\right)\bigg|_{x=0}=\frac{\partial\psi_D}{\partial x}\bigg|_{x=0},\end{gathered} \tag{8}$$

即得

$$
\begin{aligned}
&1+R=D,\\
&(1-R)k_1=Dk,\\
&R=\frac{k_1-k}{k_1+k},\qquad D=\frac{2k_1}{k_1+k}.
\end{aligned}\tag{9}
$$

由于$k_1<k$，所以R是负实数，D是正实数。即折射波和入射波同相，反射波和入射波反相。由于

$$
\begin{aligned}
k_1&=k_0\cos\theta=k_0\sqrt{1-\sin^2\theta}\,,\\
k&=k_D\cos\varphi=k_0\sqrt{n^2-\sin^2\theta}\,,
\end{aligned}\tag{10}
$$

式(9)可以写成

$$
R=\frac{\cos\theta-\sqrt{n^2-\sin^2\theta}}{\cos\theta+\sqrt{n^2-\sin^2\theta}},\qquad D=\frac{2\cos\theta}{\cos\theta+\sqrt{n^2-\sin^2\theta}}.\tag{11}
$$

根据公式

$$
j_x=-\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right),
$$

容易得出入射粒子流量为$\hbar k_1/m$，反射粒子流量为$\hbar k_1|R|^2/m$，折射粒子流量为$\hbar k|D|^2/m$。所以

$$
\begin{aligned}
\frac{\text{反射流量}}{\text{入射流量}}&=|R|^2=\left(\frac{k_1-k}{k_1+k}\right)^2,\\
\frac{\text{折射流量}}{\text{入射流量}}&=\frac{k}{k_1}|D|^2=\frac{4kk_1}{(k_1+k)^2},
\end{aligned}\tag{12}
$$

正入射时，$\theta=\varphi=0, k_0=k_1, k_D=k, k/k_1=n$,

$$
\frac{\text{反射流量}}{\text{入射流量}}=\left(\frac{n-1}{n+1}\right)^2,\qquad \frac{\text{折射流量}}{\text{入射流量}}=\frac{4n}{(n+1)^2}.\tag{13}
$$

本题结果和电磁波的反射、折射规律相似。