CS2601 Linear and Convex Optimization
Homework 10 Solution

Due: 2022.12.29

For this assignment, you should submit a **single** pdf file as well as your source code (.py or .ipynb files). The pdf file should include all necessary figures, the outputs of your Python code, and your answers to the questions. Do NOT submit your figures in separate files. Your answers in any of the .py or .ipynb files will NOT be graded.

1. Consider the LP in standard form,

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

We are going to solve it using the barrier method.

(a). Write down the approximating equality constrained problem.

(b). Write down the gradient and Hessian matrix of the objective function you find in (a).

(c). Implement the barrier method for solving a generic LP in standard form with a given feasible initial point. Complete the functions `centering_step` and `barrier` in `LP.py`. For the centering step, i.e. line 3 on slide 14 of §13, you can use your implementation of constrained Newton's method in HW 9 Problem 4(c) by defining the penalized objective function and its derivatives inside `centering_step`.

(d). Consider the following LP

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Convert it to standard form and then use your implementation in (c) to solve it. You can find a feasible initial point by setting $x_1 = 2, x_2 = 1$ and the slack variables appropriately. Show the output. Note `p2.py` also plots the projection of the iterates onto the x_1, x_2 coordinates.

2. Consider the LP in Problem 1(d).

(a). Find the dual LP in the standard form, i.e. with all four dual variables.

(b). Find the symmetric dual LP, i.e. without the dual variables for the primal nonnegativity constraints.

(c). Solve the dual LP in (b) graphically. Compare the dual optimal value and the primal optimal value computed in Problem 1(d).

(d). Solve the dual LP in (a) using your implementation in Problem 1(c). Note you need to convert the maximization problem into a minimization problem. You can use the feasible initial point $\boldsymbol{\mu}_0 = (4, 1, 2, 6)^T$. Show the dual optimal solution and dual optimal value.

3. Consider the following optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & f(x) = \log(2 + e^x) \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

(a). Find the optimal solution and the optimal value.

(b). Find the dual function and the dual problem.

(c). Find the dual optimal solution and the dual optimal value. Does strong duality hold?

4. Consider the optimization problem ,

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 2)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 2)^2 + (x_2 + 1)^2 \leq 1 \end{aligned}$$

(a). Find the Lagrange dual function and the dual problem.

(b). Find the dual optimal value ϕ^*. Does strong duality hold?

(c). Does Slater's condition hold? What can you conclude about the necessity of Slater's condition for strong duality?

(d). Is the dual optimal value ϕ^* attained by any dual feasible point? What does this say about whether KKT conditions hold at the primal optimal solution? Explain your answer.

5. Consider the following minimization problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad & f(\boldsymbol{x}) = \begin{cases} x_1^5 + x_2^5, & \text{if } \boldsymbol{x} \geq \mathbf{0} \\ +\infty, & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \end{aligned} \tag{P1}$$

Note the domain of f is $\operatorname{dom} f = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} \geq \mathbf{0}\}$ and the domain of the problem is $D = \operatorname{dom} f$.

(a). Since D is not the entire space, the dual function of this problem is defined by

$$\phi(\mu) = \inf_{\boldsymbol{x} \in D} \{f(\boldsymbol{x}) + \mu(1 - x_1 - x_2)\}$$

Find the explicit expression of $\phi(\mu)$.

(b). Find the dual optimal solution.

(c). What is the primal optimal value? Hint: Note f is convex on its domain.

(d). Note the primal problem (P1) is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f_1(\boldsymbol{x}) = x_1^5 + x_2^5 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned} \tag{P2}$$

What's the dual function of this equivalent problem (P2)? Does strong duality hold for (P2)?

Remark. Note $\operatorname{dom} f_1 = \mathbb{R}^2$ and f_1 is not convex. This problem shows that equivalent primal problems can have very different dual problems. Not all dual problems are equally useful.