HOMEWORK FOUR

Notice

In the `lecture 4_convfun_part2` , we have introduced that the nonnagative combinations of a series convex function is also convex function, which is in the form like:

$$f(\boldsymbol{x})=\sum_{i=1}^{n} c_i f_i(\boldsymbol{x}) \tag{1}$$

From this thereom, we can actually derive that

$$f(\boldsymbol{x})=\sum_{i=1}^{n} c_i f_i(x_{k_i}) \tag{2}$$

is also convex function, where $\boldsymbol{x}=(x_1,x_2,\cdots,x_m), k_i \in \{t|t\in[1,m]\cap\mathbb{N}\}$. Considering conciseness, I only go through it briefly during the proof of Question2 (d), and will simply use it in other parts. (Click here)

QUESTION 1

(a).

First, convert the function into quadratic form.

$$\begin{aligned} f(\boldsymbol{x}) = f(x_1,x_2,x_3) &= x_1^2+x_1x_2+x_2^2+x_1x_3+2x_3^2 \\ &= \boldsymbol{x}^T\cdot\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 2 \end{pmatrix}\cdot\boldsymbol{x} \\ &= \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \end{aligned} \tag{4}$$

since $\mathrm{dom}\, f=\mathbb{R}^3$ is open covex set and $\nabla^2 f(\boldsymbol{x})=2\cdot\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 2 \end{pmatrix}$ is positive definite at every $\boldsymbol{x}\in\mathrm{dom}\, f$,

$f(\boldsymbol{x})$ **is <u>convex</u>.**

(b).

$f(\boldsymbol{x})=f(x_1,x_2)=(x_1x_2)^{-2}=e^{-2ln(x_1x_2)}=e^{g(\boldsymbol{x})}=h(g(\boldsymbol{x}))$

First, $h(x)=e^x$ is convex function on $\mathbb{R}$, and is increasing. $-ln(x)$ is convex function on $(0,+\infty)$.

As the nonnegative compostion of $-ln(x)$, $g(\boldsymbol{x})$ is also convex.

Therefore, $f(\boldsymbol{x})=h(g(\boldsymbol{x}))$ is **<u>convex.</u>**

(c).

$f(x_1, x_2) = x_1^2 x_2^3$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

$\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 2x_2^3 & 6x_1x_2^2 \\ 6x_1x_2^2 & 6x_1^2x_2 \end{pmatrix}$ is indefinite since $|\nabla^2 f(\boldsymbol{x})| < 0$ while single elements in $\nabla^2 f(\boldsymbol{x})$ is positive.

Therefore $f(\boldsymbol{x})$ is **neither convex nor concave.**

(d).

$f(x_1, x_2) = x_1^{1/2} x_2^{-1/2}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

First we set $x_2 = 1$, then we get $F(x) = x^{1/2}$. Since $F''(x) < 0$, $F(x)$ is concave.

Second, we set $x_1 = 1$, then we get $G(x) = x^{1/2}$. Since $G''(x) > 0$, $G(x)$ is convex.

Therefore $f(\boldsymbol{x})$ is **neither convex nor concave.**

(e).

$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} \alpha \cdot (\alpha - 1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha \cdot (1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha \cdot (1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & \alpha \cdot (\alpha - 1) x_1^{\alpha} x_2^{-1-\alpha} \end{pmatrix} \tag{5}$$

Since $\alpha \in [0, 1]$, $\nabla^2 f(x_1, x_2)$ is negative semidefinite

$\therefore f(x_1, x_2)$ **is concave** on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$.

(f).

$f(\boldsymbol{x}) = ||\boldsymbol{Ax} + \boldsymbol{b}||^5$ on $\mathbb{R}^n$

Since $|| \cdot ||$ is convex function, and $\boldsymbol{Ax} + \boldsymbol{b}$ is affine function of $\boldsymbol{x}$, so $\boldsymbol{x} = ||\boldsymbol{Ax} + \boldsymbol{b}||$ is convex function.

Since $g(x) = x^5$ is convex and increasing on $(0, +\infty)$, we have $f(\boldsymbol{x}) = ||\boldsymbol{Ax} + \boldsymbol{b}||^5$ **is convex** on $\mathbb{R}^n$.

QUESTION 2

(a).

$\forall \boldsymbol{x}, \boldsymbol{y} \in S, \forall \theta \in (0, 1)$, we have:

$$\theta x_2 + \bar{\theta} y_2 \ge \theta |x_1| + \bar{\theta} |y_1| \ge |\theta x_1 + \bar{\theta} y_2| \tag{6}$$

$\therefore \boldsymbol{z} = \theta \boldsymbol{x} + \bar{\theta} \boldsymbol{y} = (\theta x_2 + \bar{\theta} y_2, \theta x_1 + \bar{\theta} y_2) \in S$

$\therefore S$ **is a convex set.**

(b).

We choose $\boldsymbol{x} = (0,0) \in S, \boldsymbol{y} = (-1,-1) \in S$. Then we find that $\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y} \notin S$ since $-\frac{1}{2} \leq (-\frac{1}{2})^3$.

$\therefore S$ **<u>is not a convex set.</u>**

(c).

$\because xlogx$ is strict convex function on $(0, +\infty)$

$\therefore$ its nonnegative combination $f(\boldsymbol{x}) = x_1 log(x_1) + x_2 log(x_2)$ is also convex.

$\because$ sublevel set of a convex function is a convex set.

$\therefore S$ **<u>is a convex set.</u>**

(d).

We go through two parts to prove $S = \left\{\boldsymbol{x} \in \mathbb{R}^2 : \log\left(1 + \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3\right) \leq 3, x_1 \geq \log\left(1 + e^{x_1 + 5x_2}\right)\right\}$ is convex set.

Part1:

$\log\left(1 + \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3\right) \leq 3 \iff \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3 \leq e^3 - 1$

$\because \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3$ is convex function

$\therefore$ the sublevel set of $\|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3$ is **<u>convex set.</u>**

Part2:

$x_1 \geq \log\left(1 + e^{x_1 + 5x_2}\right) \iff 1 \geq e^{-x_1} + e^{5x_2} = F(\boldsymbol{x})$

PROOF HERE! $\downarrow$ return

We first prove $g(\boldsymbol{x}) = e^{x_1} + 0 \cdot x_2$ (We can pretend there exists a x_2) is a convex function on $\mathbb{R}^2$. ($\boldsymbol{x} = (x_1, x_2)$).

$\because f(x) = e^x$ is convex on $\mathbb{R}$

$\therefore \forall \theta \in (0,1), \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, $\theta g(\boldsymbol{x}) + \bar{\theta} g(\boldsymbol{y}) = \theta e^{x_1} + \theta \bar{e}^{y_1} \geq e^{\theta x_1 + \bar{\theta} y_1} = g(\theta \boldsymbol{x} + \bar{\theta} \boldsymbol{y})$

$\therefore g(\boldsymbol{x})$ is convex

Similarly, we can prove $k(\boldsymbol{x}) = e^{-x_2}$ is convex function on $\mathbb{R}^2$. ($\boldsymbol{x} = (x_1, x_2)$)

$\because h(\boldsymbol{x}) = e^{x_1} + e^{-x_2}$ is the nonnagative combination of $g(\boldsymbol{x}), k(\boldsymbol{x})$

$\therefore h(\boldsymbol{x}) = e^{x_1} + e^{-x_2}$ is convex function on $\mathbb{R}^2$. ($\boldsymbol{x} = (x_1, x_2)$)

$\because F(\boldsymbol{x})$ can be rewritten as the form of $h(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$

$\therefore F(\boldsymbol{x})$ is also convex function, its sublevel set is convex set.

<u>According to the fact that the intersection of two convex sets is convex set, we derive that S is convex set.</u>

QUESTION 3

$$KL(\boldsymbol{x}\|\boldsymbol{y}) = \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i} = \sum_{i=1}^{n} y_i \cdot \frac{x_i}{y_i} \log \frac{x_i}{y_i} \tag{7}$$

Here $\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} y_i = 1, x_i > 0, y_i > 0, \forall i \in [1, n]$

$\because x \log x$ is strictly convex (according the page 20 in `lecture 4_convfun_part1`)

$\therefore \sum_{i=1}^{n} y_i \cdot \frac{x_i}{y_i} \log \frac{x_i}{y_i} \geq (\sum_{i=1}^{n} x_i) \log(\sum_{i=1}^{n} x_i) = 0$

$\therefore KL(\boldsymbol{x}||\boldsymbol{y}) \geq 0$

QUESTION 4

(a).

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \\ \text{s.t.} \quad & x_1 e^{-(x_1+x_2)} \leq 0 \\ & x_1^2 - 3x_2 = 0 \end{aligned} \tag{8}$$

Problem (a). is not a convex optimation problem since $x_1^2 - 3x_2$ is not a affine function.

(b).

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + x_2^4 \\ \text{s.t.} \quad & (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2} \leq 0 \\ & x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \leq 0 \\ & 6x_1 - 7x_2 = 0 \end{aligned} \tag{9}$$

$f(\boldsymbol{x}) = x_1^2 + x_2^4$ can be deemed as the nonnagative combination of two convex fucntion, so it self is convex.

$g_1(\boldsymbol{x}) = (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2}$

$\nabla^2 g_1(\boldsymbol{x}) = \begin{pmatrix} e^{x_1+x_2} + 2 & e^{x_1+x_2} + 2 \\ e^{x_1+x_2} + 2 & e^{x_1+x_2} + 2 \end{pmatrix}$ is positive semidefinite, $\therefore g_2(\boldsymbol{x})$ is convex.

$g_2(\boldsymbol{x}) = x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2$

$\nabla^2 g_2(\boldsymbol{x}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is positive semidefinite, $\therefore g_2(\boldsymbol{x})$ is convex.

$h(\boldsymbol{x}) = 6x_1 - 7x_2$ is an affine function.

$\therefore$ problem (b). Is a convex optimation problem.