HOMEWORK 8

QUESTION 1

(a.)

$$g(x_2) = f(1-2x_2, x_2) = 3x_2^2 - 4x_2 \tag{1}$$

Let $g'(x_2) = 6x_2 - 4 = 0 \to x_2^* = \frac{2}{3}, x_1^* = -\frac{1}{3}$

the global minimum is $g(\frac{2}{3}) = -\frac{4}{3}$

(b.)

Lagrangian multipliers method. The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 + \lambda(x_1 + 2x_2 - 1) \tag{2}$$

By the Lagrange condition,

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 + x_2 + \lambda - 1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = x_1 + 2x_2 + 2\lambda - 3 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x_1 + 2x_2 - 1 = 0 \end{cases} \implies \begin{cases} x_1^* = -\frac{1}{3} \\ x_2^* = \frac{2}{3} \\ \lambda^* = 1 \end{cases} \tag{3}$$

The minimum is $-\frac{4}{3}$.

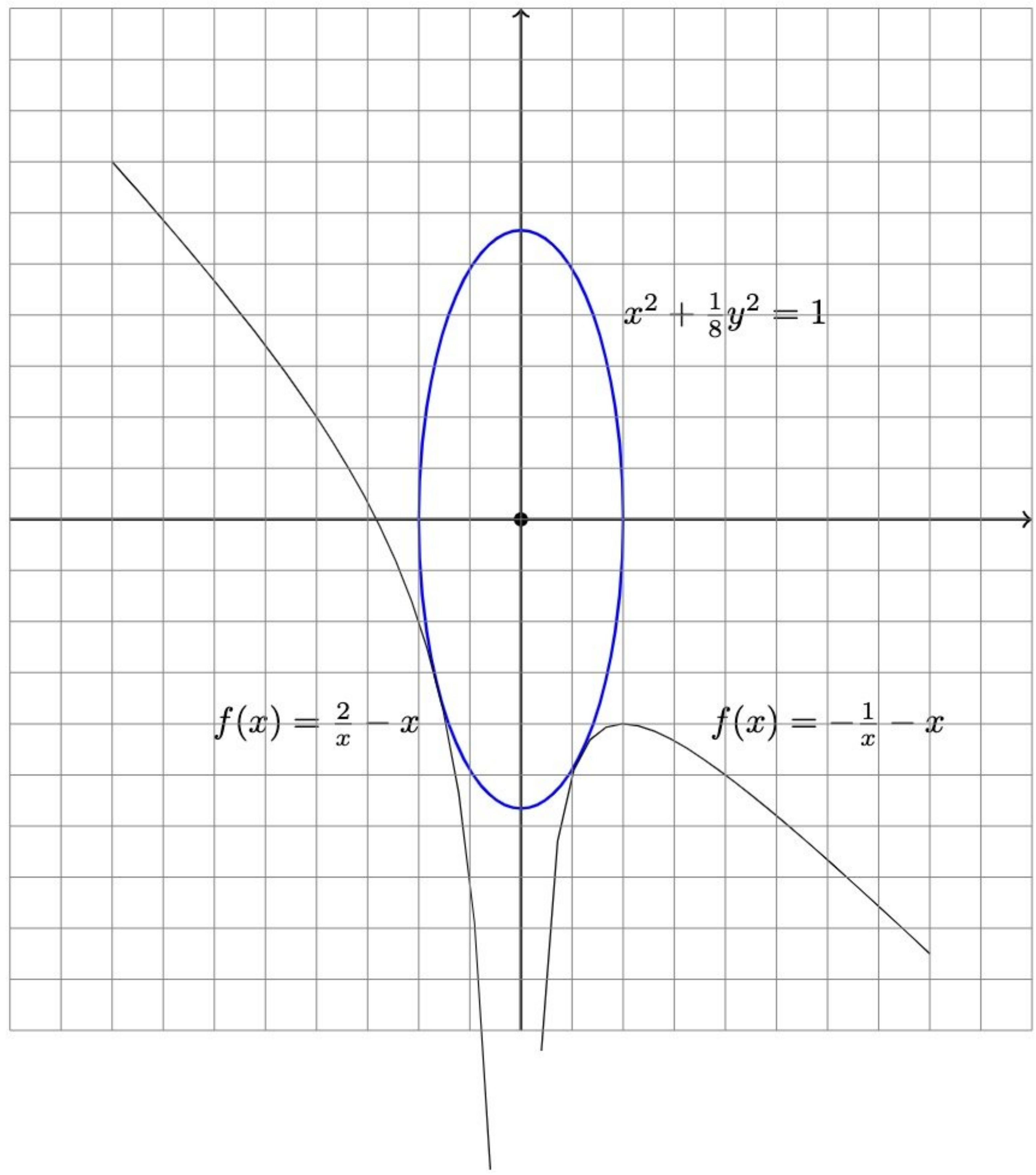
QUESTION 2

Lagrange condition

$$\begin{cases} \frac{\partial f(x_1,x_2)}{\partial x_1} + \lambda \frac{\partial h f(x_1,x_2)}{\partial x_1} = x_2 + 2x_1 + 2\lambda x_1 = 0 \\ \frac{\partial f(x_1,x_2)}{\partial x_2} + \lambda \frac{\partial h(x_1,x_2)}{\partial x_2} = x_1 + \frac{1}{4}\lambda x_2 = 0 \\ x_1^2 + \frac{1}{8}x_2^2 - 1 = 0 \end{cases} \tag{4}$$

$$(1) \begin{cases} x_1^* = \frac{\sqrt{3}}{3} \\ x_2^* = -\frac{4\sqrt{3}}{3} \\ \lambda^* = 1 \end{cases} \quad (2) \begin{cases} x_1^* = -\frac{\sqrt{3}}{3} \\ x_2^* = \frac{4\sqrt{3}}{3} \\ \lambda^* = 1 \end{cases} \quad (3) \begin{cases} x_1^* = \frac{\sqrt{6}}{3} \\ x_2^* = \frac{2\sqrt{6}}{3} \\ \lambda^* = -2 \end{cases} \quad (3) \begin{cases} x_1^* = -\frac{\sqrt{6}}{3} \\ x_2^* = -\frac{2\sqrt{6}}{3} \\ \lambda^* = -2 \end{cases} \tag{5}$$

The optimal solution is (1) and (2), and the minimum is -1.



QUESTION 3

(a.)

The Lagrangian is

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}^T\boldsymbol{x} + c + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \tag{6}$$

<u>Lagrange condition</u>

$$\begin{cases} \nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g} + \boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{0} \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} = \boldsymbol{0} \end{cases} \tag{7}$$

(b.)

First, let's show the fact that $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$.

Since rank $\boldsymbol{A} = k$,

$\therefore$ the column vector of $\boldsymbol{A}^T$ is linear independent.

$\therefore \forall \boldsymbol{x} \in \mathbb{R}^{k\times 1}, \boldsymbol{x} \neq \boldsymbol{0}$ we have $\boldsymbol{A}^T\boldsymbol{x} \neq \boldsymbol{0}$.

As we know the fact that the postive-definite matrix is invertable, and **the inverse of postive-definite matrix is also positive-definite**, then $\forall \boldsymbol{x} \in \mathbb{R}^{k\times 1}$

we have

$$(\boldsymbol{x}^T\boldsymbol{A})\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\boldsymbol{x}) > 0 \tag{8}$$

$\therefore$ by definition $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$

Then we can solve the Lagrange condition

$$\begin{aligned} \boldsymbol{x}^* &= -\boldsymbol{Q}^{-1}\boldsymbol{g} - \boldsymbol{Q}^{-1}\boldsymbol{A}^T\lambda^* \\ \lambda^* &= (\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T)^{-1}(-\boldsymbol{b} - \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{g}) \end{aligned} \tag{9}$$

(c.)

transform the problem into the form like

$$\begin{array}{ll} \min_{\boldsymbol{x}} & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}^T\boldsymbol{x} + c \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array} \tag{10}$$

Where $\boldsymbol{Q} = \boldsymbol{I}, \boldsymbol{g}^T = -\boldsymbol{x_0}^T, c = \frac{1}{2}\boldsymbol{x_0}^T\boldsymbol{x_0}$

By applying the results from (b.), we know the optimal solution is

$$\begin{aligned} \lambda^* &= (\boldsymbol{A}\boldsymbol{A}^T)^{-1}(-\boldsymbol{b} + \boldsymbol{A}\boldsymbol{x_0}) \\ \boldsymbol{x}^* &= \boldsymbol{x_0} - \boldsymbol{A}^T\lambda^* = \boldsymbol{x_0} - \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}(-\boldsymbol{b} + \boldsymbol{A}\boldsymbol{x_0}) \end{aligned} \tag{11}$$

(d.)

The problem is equal to the problem

$$\begin{array}{ll} \min_{\boldsymbol{x}} & \frac{1}{2}\left\|\boldsymbol{x} - \boldsymbol{x}_0\right\|_2^2 \\ \text{s.t.} & \boldsymbol{w}^T\boldsymbol{x} = b \end{array} \tag{12}$$

By applying the results from (c.), the optimal solution is

$$\begin{aligned} \boldsymbol{x}^* &= \boldsymbol{x_0} - \boldsymbol{w}\lambda^* \\ \lambda^* &= (\boldsymbol{w}^T\boldsymbol{w})^{-1}(-b + \boldsymbol{w}^T\boldsymbol{x_0}) \end{aligned} \tag{13}$$

Then we have

$$\begin{aligned} d(x_0, P) &= \left\|\boldsymbol{x}^* - \boldsymbol{x}_0\right\|_2 \\ &= \left\|\boldsymbol{w}\lambda^*\right\|_2 \\ &= \frac{\left\|-b + \boldsymbol{w}^T\boldsymbol{x_0}\right\|_2}{||\,\boldsymbol{w}\,||} \end{aligned} \tag{14}$$