ICE2601 Cheat Sheet

Entropy and AEP

Definition 1. *Let X be a discrete random variable with alphabet $\mathcal{X}$ and probability mass function $p(x), x \in \mathcal{X}$. The entropy of X is*

$$H(X) = -\sum_{x\in\mathcal{X}} p(x)\log p(x)$$

Entropy is a measure of the uncertainty of a random variable, as well as the amount of information required on average to describe the random variable.

- $0\log 0 \to 0$
- $H(X)$ only depends on $p(x)$, so we can write $H(p)$ for $H(X)$
- When X is uniform over $\mathcal{X}$, then $H(X) = \log|\mathcal{X}|$

The entropy can be regard as a form of expectation

$$H(X) = E\left[\frac{1}{p(X)}\right]$$

The joint entropy $H(X,Y)$ of a pair of discrete random variable (X,Y) with joint distribution $p(x,y)$ is defined as

$$H(X,Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y)$$

Similarly we can derive that $H(X,Y) = -E\log p(X,Y)$. Observe that $H(X,X) = H(X)$ and $H(X,Y) = H(Y,X)$.

Definition 2. *If $(X,Y) \sim p(x,y)$, the conditional entropy $H(Y|X)$ is defined as*

$$\begin{aligned}
H(Y|X) &= \sum_{x\in\mathcal{X}} p(x)H(Y|X=x) \text{ (by total probability rule)}\\
&= -\sum_{x\in\mathcal{X}} p(x)\sum_{y\in\mathcal{Y}} p(y|x)\log p(y|x)\\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x)\\
&= -E\log p(Y|X)
\end{aligned}$$

The following are important properties

- $H(X,Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$
- If X,Y are independent, $H(X,Y) = H(X) + H(Y)$
- If X is a function of Y, then $H(X,Y) = H(Y)$
- Bayesian formula: $H(X,Y|Z) = H(X|Z) + H(Y|X,Z)$
- If $H(Y|X) = 0$, then Y is a function of X. The property is referred to as zero entropy. If $H(X|Y) = 0$ also holds, then a bijection f exists $f := X \mapsto Y$. Note that it can be extended to multi-variable cases.

Definition 3. *Relative entropy: A measure of the distance between the two distributes. The relative entropy or KL distance between the two probability mass functions $p(x)$ and $q(x)$ over $\mathcal{X}$ is defined as*

$$D(p\parallel q) = \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} = E_p\left[\log\frac{p(X)}{q(X)}\right]$$

- The relative entropy $D(p\parallel q)$ is a measure of the **inefficiency** of assuming that the distribution is q when the true distribution is p.
- $p\log\frac{p}{0} = \infty$, so $D(p\parallel q)$ can go to infinity
- Important corollary: $D(p\parallel q) \geq 0$ (can use convexity or $\log x \leq x-1$ for proof), equality holds iff $p(x) = q(x)$ for all x
- $D(p\parallel q) = E_p[-\log q(x)] - H(p)$
- $D(p\parallel p) = 0$

Definition 4. *Mutual Information: consider two random variables X,Y with joint probability mass function $p(x,y)$ and marginal probability mass functions $p(x)$ and $p(y)$. The mutual information $I(X;Y)$ is the relative entropy between the joint distribution $p(x,y)$ and the product distribution $p(x)p(y)$.*

$$\begin{aligned}
I(X;Y) &= \sum_X\sum_Y p(x,y)\log\frac{p(x,y)}{p(x)p(y)}\\
&= D[p(x,y)\parallel p(x)p(y)]\\
&= E_{p(x,y)}\left[\log\frac{p(X,Y)}{p(X)p(Y)}\right] \geq 0
\end{aligned}$$

We can interpret mutual interpretation as the reduction in the uncertainty of X due to the knowledge of Y.

- $I(X;Y) = I(Y;X)$
- $I(X;X) = H(X)$
- If X,Y are independent, then $I(X;Y) = 0$

Theorem 1 (Chain rule of entropy)**.** *Suppose $X_1,\ldots,X_n$ are drawn according to $p(x_1,x_2,\ldots,x_n)$, then we have*

$$H(X_1,\ldots,X_n) = H(X_1) + H(X_2|X_1) + \cdots + H(X_n|X_{n-1},\ldots,X_1)$$

Definition 5. *Conditional mutual information: shows the reduction in the uncertainty of X due to the knowledge of Y when Z is given.*

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = E_{p(x,y,z)}\left[\log\frac{p(X,Y|Z)}{p(X|Z)p(Y|Z)}\right]$$

Theorem 2. *Chain rule for information.*

$$I(X_1,\ldots,X_n;Y) = \sum_{i=1}^{n} I(X_i;Y|X_{i-1},\ldots,X_1)$$

Definition 6. *Conditional Relative Entropy*

$$\begin{aligned}
D[p(y|x)\parallel q(y|x)] &= \sum_x p(x)\sum_y p(y|x)\log\frac{p(y|x)}{q(y|x)}\\
&= \sum_x\sum_y p(x)p(y|x)\log\frac{p(y|x)}{q(y|x)}\\
&= E_{p(x,y)}\left[\frac{p(Y|X)}{q(Y|X)}\right]
\end{aligned}$$

Theorem 3. *Chain rule for relative entropy*

$$D[p(x,y)\parallel q(x,y)] = D[p(x)\parallel q(x)] + D[p(y|x)\parallel q(y|x)]$$

Definition 7. *Independence Bound: Let $X_1,\ldots,X_n$ be drawn according to $p(x_1,\ldots,x_n)$. Then*

$$H(X_1,X_2,\ldots,X_n) \leq \sum_{i=1}^{n} H(X_i)$$

with equality iff X_i are independent.

Definition 8. *Markov Chain: Random variables X,Y,Z are said to form a Markov Chain in that order (denoted by $X\to Y\to Z$) if the conditional distribution of Z depends on Y and is conditionally independent of X, i.e.*

$$p(x,y,z) = p(x)p(y|x)p(z|y)$$

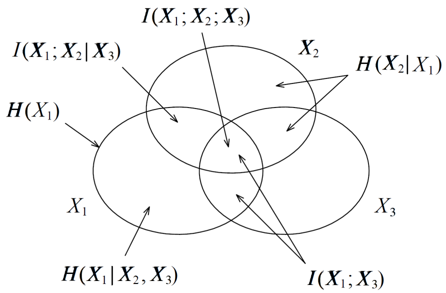
- $X\to Y\to Z$ iff X,Z are conditionally independent given Y, i.e.
$$p(x,z|y) = \frac{p(x,y,z)}{p(y)} = \frac{p(x,y)p(z|y)}{p(y)} = p(x|y)p(z|y)$$
- $X\to Y\to Z$ implies that $Z\to Y\to X$.
- $Z = f(Y)$, then $X\to Y\to Z$
- If $X\to Y\to Z$, then $I(X;Z|Y) = 0$.

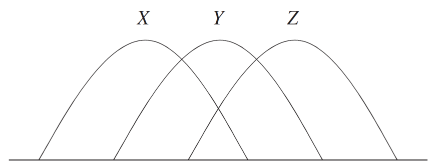
Theorem 4. *Data Processing Inequality: if $X\to Y\to Z$, then $I(X;Y) \geq I(X;Z)$.*

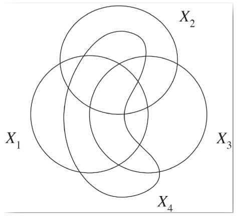
The following are important properties

- $I(X;Y,Z) = I(Y,Z;X) = I(X;Z) + I(X;Y|Z) = I(X;Y) + I(X;Z|Y) = I(X;Y)$
- In particular, if $Z = g(Y)$, we have $I(X;Y) \geq I(X;g(Y))$
- If $X\to Y\to Z$, then $I(X;Y|Z) \leq I(X;Y)$

The most important three figures for this part.







Suppose we want to estimate a random variable X with a distribution $p(x)$. We observe a random variable Y that is related to X by the conditional distribution $p(y|x)$. From Y, we calculate a function $g(Y) = \hat{X}$, where $\hat{X}$ is an estimate of X. To bound the probability $\hat{X} \neq X$, we observe that $X \to Y \to \hat{X}$ forms a Markov chain. Define the probability of error $P_e = Pr(\hat{X} \neq X)$.Fano's Inequality establishes the relationship between P_e and $H(X|Y)$.

Theorem 5. *Fano's Inequality: For any estimator $\hat{X}$ such that $X \to Y \to \hat{X}$, with $P_e = Pr(\hat{X} \neq X)$, we have*

$$H(P_e) + P_e \log|\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y)$$

The second inequality is actually yielded by Data-processing Inequality. This inequality can be weakened to

$$1 + P_e \log|\mathcal{X}| \geq H(X|Y)$$

From Fano's inequality, we can see that $P_e \to 0$ implies $H(X|Y) \to 0$.

Definition 9. *The counterpart of Law of Large Numbers is Asymptotic Equipartition Property (AEP): if $X_1, \ldots, X_n$ are i.i.d $\sim p(x)$, then*

$$-\frac{1}{n} \log p(X_1, \ldots, X_n) \to H(X)$$

We can reinterpret the equation as follows

$$H(X) - \varepsilon \leq -\frac{1}{n} \log p(X_1, \ldots, X_n) \leq H(X) + \varepsilon$$

$$2^{-n(H(X)+\varepsilon)} \leq p(X_1, \ldots, X_n) \leq 2^{-n(H(X)-\varepsilon)}$$

Definition 10. *Typical set $A_\varepsilon^{(n)}$ is a set such that*

$$2^{-n(H(X)+\varepsilon)} \leq p(X_1, \ldots, X_n) \leq 2^{-n(H(X)-\varepsilon)}$$

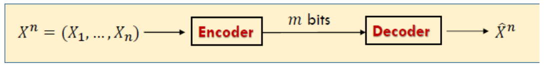
Important properties for typical sets

- All elements of the typical set are nearly equiprobable.
- The typical set has probability nearly 1. $Pr(A_\varepsilon^{(n)}) \geq 1 - \varepsilon$.
- The number of elements in the typical set is nearly 2^{nH} (much smaller than the scale of the alphabet). $|A_\varepsilon^{(n)}| \leq 2^{n(H(X)+\varepsilon)}$ and $|A_\varepsilon^{(n)}| \geq (1-\varepsilon)2^{n(H(X)-\varepsilon)}$ for large n.

Entropy Rates and Data Compression

Intuition

The basic idea for data compression is to **find a shorter sequence to encode the source**. The basic process is shown as follows



- Suppose $X_1, \ldots, X_n$ are i.i.d $\sim p(x)$, $X^n = (X_1, \ldots, X_n)$ denotes the n-tuple that represents a sequence of n source symbols.
- The alphabet $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$ are the possible values that each X_i can take on.
- Encoder and decoder are a pair of functions f, g such that $f : \mathcal{X} \to \{0,1\}^*$ and $g : \{0,1\}^* \to \mathcal{X}$.
- Probability of error $P_e = P(X^n \neq \hat{X}^n)$
- The rate of a scheme: $R = \frac{m}{n}$. Intuitively, $R = \log|\mathcal{X}|$ is a trivial solution, as we can always represent $|\mathcal{X}|$ numbers using $\log|\mathcal{X}|$ bits. If so, then we naturally need $m = n\log|\mathcal{X}|$ bits for encoding n symbols.

Our task is to find an encoder and decoder pair such that $P_e \to 0$ as $n \to \infty$. Note that the typical set accounts for nearly all cases (with probability nearly 1). Hence, we know that we can represent $(A_\varepsilon^{(n)})^c$ with $n\log|\mathcal{X}| + 1 + 1$ bits. For $A_\varepsilon^{(n)}$, it has $2^{n(H+\varepsilon)}$ elements and there are $n(H+\varepsilon) + 1 + 1$ bits. The added two bits are used for reservation and flag (0 for being in the typical set and 1 for the other case) respectively.

Theorem 6. *We can represent sequences X^n using $nH(X)$ bits on average. Conversely, for any scheme with rate $r < H(X)$, $P_e \to 1$.*

$$E\left[\frac{1}{n} l(X^n)\right] \leq H(X) + \varepsilon$$

Proof.

$$\begin{aligned}
E(l(X^n)) &= \sum_{x^n} p(x^n) l(x^n) \\
&\leq \sum_{x^n \in A_\varepsilon^{(n)}} p(x^n)[n(H+\varepsilon)+2] + \sum_{x^n \notin A_\varepsilon^{(n)}} p(x^n)(n\log|\mathcal{X}|+2) \\
&= Pr\{A_\varepsilon^{(n)}\}[n(H+\varepsilon)+2] + Pr\{(A_{\varepsilon^{(n)}})^c\}(n\log|\mathcal{X}|+2) \\
&\leq n(H+\varepsilon+2) + \varepsilon n(\log|\mathcal{X}|) + 2 \\
&= n(H+\varepsilon')
\end{aligned}$$

□

Stochastic Process and Entropy Rate

Definition 11 (Stochastic process)**.** *A stochastic process $\{X_i\}$ is an indexed sequence of random variables. It is said to be stationary if*

$$Pr(X_1 = x_1, \ldots, X_n = x_n) = Pr(X_{1+l} = x_1, \ldots, X_{n+l} = x_n)$$

Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process, then we can prove that

$$H(X_0|X_{-1}, \ldots, X_{-n}) = H(X_0|X_1, \ldots, X_n)$$

Recall that $X_1, \ldots, X_n$ is said to be a Markov chain or a Markov process if for any n

$$P(X_{n+1} = x_{n+1}|X_n = x_n, \ldots, X_1 = x_1) = P(X_{n+1} = x_{n+1}|X_n = x_n)$$

The Markov chain is said to be *time invariant* if the conditional probability $p(x_{n+1}|x_n)$ does not depend on n, i.e.

$$Pr(X_{n+1} = b|X_n = a) = Pr(X_2 = b|X_1 = a), \forall a, b \in \mathcal{X}$$

We will assume that the Markov chain is time invariant unless otherwise stated, and it is characterized by its initial state and a probability transition matrix $P = \{P_{ij}\}$, where

$$P_{ij} = Pr\{X_{n+1} = j|X_n = i\}$$

By the definition of stationary, a Markov Chain is stationary iff $p(X_{n+1}) = p(X_n)$. If the probability mass function at a time n is $p(x_n)$, then

$$p(x_{n+1}) = \sum_{x_n} p(x_n) P_{x_n x_{n+1}} \Leftrightarrow x_n^T P = x_{n+1}^T$$

Definition 12. *Entropy rate: the entropy rate of a stochastic process $\{X_i\}$ is defined by*

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n)$$

Theorem 7. *Suppose $H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n|X_{n-1}, \ldots, X_1)$ and $\{X_n\}$ is stationary stochastic process, we have*

$$H(\mathcal{X}) = H'(\mathcal{X}) = H(X_2|X_1) = \sum_i p(x_i) H(X_2|X_1 = x_i)$$

Kraft Inequality and Optimal Codes

Without of loss of generality, we assume that the D-ary alphabet is $\mathcal{D} = \{0, 1 \ldots, D-1\}$. A **source code** C for a random variable X is a mapping from $\mathcal{X}$, the range of X, to D^*, the set of finite-length strings of symbols from a D-ary alphabet. Let $C(x)$ denote the **codeword** corresponding to x and let $l(x)$ denote the length of $C(x)$, then the expected length $L(C)$ of a source code $C(x)$ for a random variable X with probability mass function $p(x)$ is given by

$$L(C) = \sum_{x\in\mathcal{X}} p(x) l(x)$$

To minimize $L(C)$ and construct the code for achieving the minimum values, we have Kraft Inequality and Huffman Encoding. Some concepts are listed as follows.

- A code is *nonsingular* if for all $x \neq x' \Rightarrow C(x) \neq C(x')$
- The *extension* C^* of a code C is the mapping from finite length strings of X to finite length strings of D, defined by

  $$C(x_1 \ldots x_n) = C(x_1) \ldots C(x_n)$$

  where $C(x_1) \ldots C(x_n)$ indicates concatenation of the corresponding codewords
- A code is called *uniquely decodable* if its extension is nonsingular
- A code is called a prefix code or instantaneous code if no codeword is a prefix of any other codeword.

Theorem 8. *Kraft Inequality: for any instantaneous code over an alphabet of size D, the codeword lengths $l_1, \ldots, l_m$ must satisfy the inequality*

$$\sum_{i=1}^{m} D^{-l_1} \leq 1$$

Conversely, given a set of codeword lengths that satisfy this inequality, there exists an instantaneous code with these word lengths.

Note that the inequality can be extended to countable infinite set of codewords. With Kraft inequality, we can formulate the original problem into an optimization problem

$$\min L = \sum_i p_i l_i$$

$$\text{subject to } \sum_i D^{-l_i} \leq 1$$

The result of which is $l_i^* = -\log_D p_i$ (non-integer optimal value). The optimal expected length is thus

$$L^* = \sum_i p_i l_i^* = \sum_i -p_i \log p_i = H_D(X)$$

In general, $H_D(X)$ cannot be attained, and we have $L^* \geq H_D(X)$. We can further derive that $L^* < H_D(X) + 1$, as we might round l_i up to become an integer.

Theorem 9. *Important bound on the optimal code length.*

$$H_D(X) \leq L^* \leq H_D(X) + 1$$

This type of coding is called **Shannon codes**, where

$$l_i = \left\lceil \log_D \frac{1}{p_i} \right\rceil$$

It can be shown that we can actually approach the limit and remove the extra 1 bit of Shannon codes. We have the following theorem: the minimum expected codeword length per symbol satisfies

$$\frac{H(X_1, \ldots, X_n)}{n} \leq L^* < \frac{H(X_1, \ldots, X_n)}{n} + \frac{1}{n}$$

If $X_1, \ldots, X_n$ is a stationary stochastic process, then $L^* \to H(\mathcal{X})$.

Theorem 10. *In some cases, we might design the code based on an estimated (possibly wrong) distribution $q(x)$, the expected length under $p(x)$ of the code assignment $l(x) = \log \frac{1}{q(x)}$ satisfies*

$$H(p) + D(p \parallel q) \leq E[l(x)] < H(p) + D(p \parallel q) + 1$$

Proof. Both sides of the equation can be derived in a similar way, here we consider the right-hand side.

$$\begin{aligned} El(x) &= \sum_x p(x) \left\lceil \log \frac{1}{q(x)} \right\rceil \\ &< \sum_x p(x) \left[\log \frac{1}{q(x)} + 1 \right] \\ &= \sum_x p(x) \log \frac{p(x)}{q(x)} \frac{1}{p(x)} + \sum_x p(x) \\ &= \sum_x p(x) \log \frac{p(x)}{q(x)} + \sum_x p(x) \log \frac{1}{p(x)} + 1 \\ &= D(p \parallel q) + H(p) + 1 \end{aligned}$$

If we slack the conditions from instantaneous codes to uniquely decodable codes, then we have Kraft Inequality is still the tight bound (cannot be shorter).

Two Encoding Methods

D-ary *Huffman codes* for a given contribution: each time combine D symbols with the lowest possibilities into a single source symbol, until there is only symbol. Note that Huffman coding is optimal.
If $D \geq 3$, we may not have a sufficient number of symbols so that we can combine them D at a time. In such a case, we add *dummy variables* to the end of the set of symbols. Since at each stage of the reduction, the number of symbols is reduced by $D - 1$, the total number of symbols should be $1 + k(D - 1)$, where k is the number of merges.
Now we discuss some of the properties of Huffman coding

- Huffman code is not unique.
- If a probability distribution $P(X)$ is called *D-adic* if each of the probabilities $P(X = x_i) = D^{-n}$ for some n. As proved previously, for a *D-adic* distribution, the optimal solution in Lagrange is unique: $l_i = \log \frac{1}{p_i} = n_i$.
- Compare Huffman and Shannon codes, we can find that if the probability distribution is D-adic, Shannon codes are optimal. But Shannon codes may be much worse when $p_i \to 0$.

Finally, we consider Shannon-Fano-Elias coding as a specific way to put Shannon's coding in practice. Without loss of generality, we can take $\mathcal{X} = \{1, 2, \ldots, m\}$. Assume that $p(x) > 0$ for all x. The cumulative distribution function $F(x)$ is defined as
$F(x) = \sum_{a \leq x} p(a)$.
Consider the modified cumulative distribution function [using an idea of taking the average]

$$\overline{F}(x) = \sum_{a<x} p(a) + \frac{1}{2} p(x) = F(x) - \frac{1}{2} p(x)$$

Since $\overline{F}(x)$ is a real number, we truncate $\overline{F}(x)$ to $l(x)$ bits and use the first $l(x)$ bit of $\overline{F}(x)$ as a code for x, denote it by $\lfloor \overline{F}(x) \rfloor_{l(x)}$. Apparently, we have $\overline{F}(x) - \lfloor \overline{F}(x) \rfloor_{l(x)} \leq \frac{1}{2^{l(x)}}$. If $l(x) = \lceil \log \frac{1}{p(x)} \rceil + 1$, we have

$$\frac{1}{2^{l(x)}} \leq \frac{p(x)}{2} = \overline{F}(x) - F(x-1)$$

We can see that $\lfloor \overline{F}(x) \rfloor$ lies within the step corresponding to x. Thus, $l(x)$ bits suffice to describe x, i.e.
$L = \sum p(x) l(x) < H(X) + 2$. The general idea stretches as follows

$$p(x) \Rightarrow F(x) = \sum_{a \leq x} p(a) \Rightarrow \overline{F}(x) = F(x) - \frac{1}{2} p(x) \Rightarrow l(x) + 1 \text{ bits}$$

Optimality: Let $l(x)$ be the codeword lengths associated with the Shannon code, and let $l'(x)$ be the code word length associated with any other uniquely decodable code. Then

$$P[l(X) \geq l'(x) + c] \leq \frac{1}{2^{c-1}}$$

Hence, no other code can do much better than the Shannon code most of the time. □

Channel Capacity

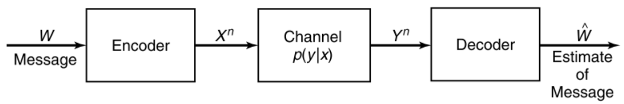
Intuition, Definition and Calculation

For each task of transmitting information, the message M with alphabet $\mathcal{M}$, the input is X with alphabet $\mathcal{X}$, the output is Y with alphabet $\mathcal{Y}$. $\mathcal{X}, \mathcal{Y}$ can be disjoint. The change from $X \to Y$ can be modeled as a transition matrix between X, Y, i.e. $p(Y|X)$.
The channel is just like a phone, each time a user can use it to make a call M. But the message may be too large to send in just one use of channel. Thus, we have

$$M \to X_1 \ldots, X_n$$

That is, the channel is used n times and we use a random process $\{X_i\}$ to denote it. We will now define the discrete memoryless channel (DMC).

Definition 13. *A discrete channel is a system consisting of an input alphabet $\mathcal{X}$ and an output alphabet $\mathcal{Y}$ and a probability transition matrix $p(y|x)$. The channel is said to be memoryless if the probability distribution of the output depends only on the input at that time and is conditionally independent of previous channel inputs or outputs.*



Definition 14. *We define the information channel capacity of a discrete memoryless channel as*

$$C = \max_{p(x)} I(X;Y)$$

where the maximum is taken over all input distributions of $p(x)$.

$I(X;Y) = H(Y) - H(Y|X)$ is the formula that's most frequently used. We can estimate $H(Y|X) = \sum_x H(Y|X=x)p(x)$ by the given transition probability transition matrix. In very few situations, we use $I(X;Y) = H(X) - H(X|Y)$.

- Binary Symmetric Channel. $C = 1 - H(p)$, where p is the crossover probability.
- Binary Erasure Channel. $C = 1 - \alpha$, where α is the erasure probability.
- BSC is a special case of symmetric channel. A channel is said to be symmetric if the rows of the channel transition matrix $p(y|x)$ are permutations of each other and the columns are permutations of each other. It can be shown that

$$C = \log |\mathcal{Y}| - H(\mathbf{r})$$

where $\mathbf{r}$ denotes a row of a transition matrix.

Channel Coding Theorem

The encoding and decoding process (i.e. $W \to X^n$ and $Y^n \to \hat{W}$) could be designed by us. $X^n \to Y^n$ is out of control and depends on $p(y|x)$. A good design should attempt to decrease n, in other word, we try to maximize $\frac{H(W)}{n}$.
For each message w, we can denote it by their index set $\mathcal{M} = \{1, 2, \ldots, M\}$. Apparently, we need to use $\log M$ bits to represent a symbol in $\mathcal{M}$. Due to the length of the message, we need to use the channel n times on average to send an index. That is, for each $w \in \mathcal{M}$, $w = x_1 \ldots x_n \in \mathcal{X}^n \Rightarrow y_1 \ldots y_n \in \mathcal{Y}^n$. By the time of their generation, we can derive

$$w \to x_1 \to y_1 \to x_2 \to y_2 \to \cdots \to x_n \to y_n$$

In general, x_i depends on $w, x_1, \ldots, y_1, \ldots, y_{i-1}$.
The n-th extension of the discrete memoryless channel is the channel $(\mathcal{X}^n, p(y^n|x^n), \mathcal{Y}^n)$, where

$$p(y_k|x^k, y^{k-1}) = p(y_k|x_k)$$

Note that $x^k = x_1, x_2, \ldots, x_k$. When x_k is given, y_k is determined by $p(y|x)$ and is independent of all generated before time k. If the channel is used without feedback, then we have $p(x_k|x^{k-1}, y^{k-1}) = p(x_k|x^{k-1})$. Combining memoryless and no feedback, we have the following property

$$\begin{aligned}
p(y^n|x^n) &= p(y^{n-1}|x^n)p(y_n|y^{n-1}, x^n) \\
&= p(y^{n-1}|x^{n-1}, x_n)p(y_n|y^{n-1}, x^n) \\
&= p(y^{n-1}|x^{n-1})p(y_n|y^{n-1}, x^{n-1}, x_n) \\
&= p(y^{n-1}|x^{n-1})p(y_n|x_n) \\
&= \prod_{i=1}^{n} p(y_i|x_i)
\end{aligned}$$

Thus, we can also derive that

$$H(Y^n|X^n) = \sum_{i=1}^{n} H(Y_i|X_i)$$

So we can regard it as a Markov Chain $W \to X^n \to Y^n \to \hat{W}$.
Now, we focus our attention on the encoding and decoding process. An encoder is a function f such that $f(w) : \mathcal{M} \to \mathcal{X}^n$. f yields a distribution on $\mathcal{X}^n$ ($\mathcal{X}$ if the channel is a DMC). The encoding rule $f(w) = x^n \in \mathcal{X}^n$ generates a codebook. (e.g. $f(\text{hi}) = 01011$)
Decoder received $y^n \sim p(y^n|x^n) = \prod p(y_n|x_n)$. The decoder need to guess the possible x^n by y^n. By the codebook $f^{-1}(x^n) = w$, $\widehat{w}$ could be recovered by decoder. An error would occur if $\widehat{w} = w$.
Formally, an (M, n) code for the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of the following

- An index set $\{1, 2, \ldots, M\}$. (M messages in total).
- An encoding function $X^n : \{1, 2, \ldots, M\} \to \mathcal{X}^n$, yielding codewords $x^n(1), \ldots, x^n(M)$. The set of codewords is called the codebook.
- A decoding function
$$g : \mathcal{Y}^n \to \{1, 2, \ldots, M\}$$
which is a deterministic rule that assigns a guess to each possible received vector.

Definition 15. *We define the conditional probability of error given that the index i was sent*

$$\lambda_i = P(g(Y^n) \neq i | X^n = x^n(i)) := \sum_{y^n} p(y^n|x^n(i)I[g(y^n \neq i)])$$

The maximal probability of error and average probability of error can be easily defined.

Definition 16. *The rate R of (M, n) code is defined by*

$$R = \frac{\log M}{n}$$

A rate R is said to be achievable if there exists a sequence of $(2^{nR}, n)$ codes such that the maximal probability of error $\lambda(n)$ tends to zero as $n \to \infty$. The capacity of a channel is the supremum of all achievable rates.

Theorem 11. *Channel coding theorem: For a discrete memoryless channel, all rates below the capacity C are achievable. Specially, for every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$. Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \to 0$ must have $R \leq C$.*

To minimize the chance of error, our goal is that no two X sequences produce the same Y output sequence. For each *typical* input n-sequence, there are *approximately* $2^{nH(Y|X)}$ possible Y sequences, all of them equally likely. The total *number* of possible *typical* Y sequences is $2^{nH(Y)}$.
This set has to be divided into sets of size $2^{nH(Y|X)}$ corresponding to the different input X sequences. The total number of disjoint sets is less than or equal to $2^{nI(X;Y)}$. Hence, we can send at most $2^{nI(X;Y)}$ distinguishable sequences of length n.
We decode a channel output Y^n as the i-th index if the codeword $X^n(i)$ is jointly typical with the received signal Y^n.

Definition 17. *Jointly typical. The set $A_\epsilon^{(n)}$ of jointly typical sequences $\{(x^n, y^n)\}$ with respect to the distribution $p(x, y)$ is the set of n-sequences with empirical entropies ϵ-close to the true entropies*

$$\begin{aligned}
A_\epsilon = &\{(x^n, y^n) \in X^n \times Y^n | \\
&H(X^n) \to H(X), H(Y^n) \to H(Y), H(X^n, Y^n) \to H(X, Y)\}
\end{aligned}$$

Note that $X^n \in A_\epsilon^{(n)}$ and $Y^n \in A_\epsilon^{(n)}$ cannot imply $(X^n, Y^n) \in A_\epsilon^{(n)}$. The key properties for joint AEP are shown as follows.

- $P(X^n, Y^n) \in A_\epsilon^{(n)} \to 1$ as $n \to \infty$.
- $|A_\epsilon^{(n)}| \leq 2^{n(H(X;Y)+\epsilon)}$
- If $(\tilde{X^n}, \tilde{Y^n}) \sim p(x^n)p(y^n))$, then we have
$$(1-\epsilon)2^{-n(I(X;Y)+3\epsilon)} \leq P((\tilde{X^n}, \tilde{Y^n}) \in A_\epsilon^{(n)}) \leq 2^{-n(I(X;Y)-3\epsilon)}$$
Proof. Consider the RHS, we have
$$\begin{aligned}
P((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}) &= \sum_{(x^n, y^n) \in A_\epsilon^{(n)}} p(x^n)p(y^n) \\
&\leq 2^{n[H(X,Y)+\epsilon]} \cdot 2^{-n(H(X)-\epsilon)} e^{-n(H(Y)-\epsilon)} \\
&= 2^{-n(I(X;Y)-3\epsilon)}
\end{aligned}$$
$\square$

Proof of Channel Coding Theorem

Note that the Channel Coding Theorem has *two directions*, and we will prove the converse first.

Theorem 12. *(Converse Theorem) If $\lambda^{(n)} \to 0$ (maximum probability of error) for a $(2^{nR}, n)$ code, then $R \leq C$.*

If **no errors** are allowed, then we have

$$\begin{aligned}
nR = \log M = H(W) &= H(W|Y^n) + I(X; Y^n) \\
&= I(W; Y^n) \\
&\leq I(X^n; Y^n) \quad \text{(Markov Chain)} \\
&\leq \sum_i I(X_i; Y_i) \quad \text{(To be proved soon)} \\
&\leq nC
\end{aligned}$$

Proof. $I(X^n; Y^n) \leq \sum_i I(X_i; Y_i)$

$$\begin{aligned}
I(X^n; Y^n) &\leq H(Y^n) - H(Y^n|X^n) = H(Y^n) - \sum_i H(Y_i|X_i) \\
&\leq \sum_i H(Y_i) - \sum_i H(Y_i|X_i) = \sum_i I(X_i; Y_i)
\end{aligned}$$

$\square$

For the no-error case, we have proved that $R \leq C$. Generally, $H(W|Y^n) \neq 0$ and $H(W|\hat{W}) \neq 0$. By Fano's Inequality we have $1 + P_\epsilon^n H(W) \geq H(W|\hat{W})$. Hence the case becomes

$$\begin{aligned}
nR = H(W) &= H(W|\hat{W}) + I(W; \hat{W}) \\
&\leq 1 + P_\epsilon^{(n)} nR + I(W; \hat{W}) \\
&\leq 1 + P_\epsilon^{(n)} nR + I(X^n; Y^n) \\
&\leq 1 + P_\epsilon^{(n)} nR + nC \text{ (proved previously)}
\end{aligned}$$

Thus we have $R \leq P_\epsilon^{(n)} R + \frac{1}{n} + C \to C$.

Theorem 13. *(Forward Theorem). For a DMC, all rates below capacity C are achievable. For every rate $R < C$, there exists a $(2^{nR}, n)$ code such that $\lambda^{(n)} \to 0$.*

We can write the codebook in the form of matrix $\mathcal{C}$. 2^{nR} rows represent M messages in total, n columns denotes the D-nary code (e.g. $\{0, 1\}$ in binary case) for each use of the channel.

$$\mathcal{C} = \begin{bmatrix} x_1(1) & x_2(1) & \ldots & x_n(1) \\ \vdots & \vdots & \vdots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \ldots & x_n(2^{nR}) \end{bmatrix}$$

The general process of the transmission is summarized as follows.

- Fix $p(x)$, generate a $(2^{nR}, n)$ code at random according to $p(x)$

$$p(x^n) = \prod_{i=1}^{n} p(x_i)$$

- The probability we generate a particular code C is

$$P(C) = \prod_{w=1}^{2^{nR}} \prod_{i=1}^{n} p(x_i(w))$$

- The code C is revealed to both the sender and the receiver. Both of them know $p(y|x)$. A message W is chosen according to a **uniform distribution**.

$$P(W = w) = 2^{-nR}, \quad w = 1, 2, \ldots, 2^{nR}$$

- The w-th codeword $X^n(w)$ is sent over the channel.
- The receiver receives a sequence Y^n according to the distribution

$$p(y^n|x^n(w)) = \prod_{i=1}^{N} p(y_i|x_i(w))$$

The receiver guess which message was sent. In jointly typical decoding, the receiver declares that the index $\widehat{W}$ was sent if the following conditions are satisfied:

- $(X^n(\hat{W}), Y^n)$ is jointly typical
- There is no other index $W' \neq W$, such that $(X^n(W'), Y^n) \in A_\epsilon^{(n)}$.

If no such $\widehat{W}$ exists or if there is more than one such, an error is declared. Let ε be the event $\{\hat{W} \neq W\}$, we need to show that

$$P(\varepsilon) \to 0$$

Main idea: If we could prove that for all codebooks (all the possible C), the average $P(\varepsilon) \leq \epsilon$, then the error probability of the best code $\leq \epsilon$. We let W be drawn according to a uniform distribution over $\{1, 2, \ldots 2^{nR}\}$ and use jointly typical decoding $\hat{W}(y^n)$. The following the average of *all codewords in all codebooks.*

$$\begin{aligned} P(\varepsilon) &= \sum_C P(C) P_e^{(n)}(C) = \sum_C P(C) \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \lambda_w(C) \\ &= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_C P(C) \lambda_w(C) \end{aligned}$$

Consider a specific codeword

$$P(\varepsilon) = \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} P(\varepsilon|W = w)$$

Take $P(\varepsilon|W = 1)$ for example, we have

$$\sum_C P(C) \lambda_1(C) = P(\varepsilon|W = 1)$$

$$E_i = \{(X^n(i), Y^n) \text{ is in } A_\epsilon^{(n)}, \quad i \in \{1, 2, \ldots, 2^{nR}\}\}$$

Note that an error occurs in decoding scheme if either E_1^c occurs or $E_2 \cup E_3 \cup \cdots \cup E_{2^{nR}}$. occurs. Then we have

$$\begin{aligned} P(\varepsilon|W = 1) &= P(E_1^c \cup E_2 \cdots \cup E_{2^{nR}}|W = 1) \\ &\leq P(E_1^c|W = 1) + \sum_{i=2}^{2^{nR}} P(E_i|W = 1) \end{aligned}$$

By joint AEP, $P(\varepsilon|W = 1) \to 0$, and hence $P(E_1^c|W = 1) \leq \epsilon$ for n sufficiently large. For $i \geq 2$, $(E_i|W = 1)$, since by the code generation process, $X^n(1)$ and $X^n(i)$ are independent for $i \neq 1$, so are Y^n and $X^n(i)$. Hence the probability that $X^n(i)$ and Y^n are jointly typical is smaller or equal to $2^{-n(I(X;Y)-3\epsilon)}$ by joint AEP. Here we have used the property that if $(\tilde{X}, \tilde{Y}) \sim p(x^n)p(y^n)$, then $P((\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}) \leq 2^{-n(I(X;Y)-3\epsilon)}$

$$\begin{aligned} P(\varepsilon|W = 1) &\leq \epsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + (2^{nR} - 1)2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + 2^{-n(I(X;Y)-R-3\epsilon)} \end{aligned}$$

If n is sufficiently large and $R < I(X;Y) - 3\epsilon$, $P(\varepsilon|W = 1) \leq 2\epsilon$ and $P(\varepsilon) \leq 2\epsilon$. Hence, there exists a best codebook C^* such that

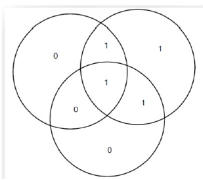
$$P(\varepsilon|C^*) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(C^*) \leq 2\epsilon$$

Without loss of generality, assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2^{nR}}$, we can prove by contradiction that **at least half of them must all be smaller than** 4ϵ.

We can then further refine the codebook C^* by throwing away the worst half of the codewords in the best codebook C^*. The best half have a maximal probability of error less than 4ϵ. If we reindex these codewords, we have 2^{nR-1} codewords. Throwing out half the codewords has changed the rate from R to $R - \frac{1}{n}$, which is negligible for large n.

Hamming Code

Hamming Codes are described by (n, k, d), where the first k bits in each codeword represent the message and the last $n - k$ bits are parity check bits. A common example is $(7, 4, 3)$ Hamming Code. We can use information graph for the purpose, as we plug in the 4 numbers in the middle. Values for parity check bits are derived using the fact that every *big circle* must have sum $0 \mod 2$.



Differential Entropy

Concepts and Basic Properties

Definition 18. *The differential entropy $h(X)$ of a continuous random variable X with density $f(x)$ is defined as*

$$h(X) = -\int_S f(x) \log f(x) dx$$

where $S = \{x|f(x) > 0\}$ is the support set of the random variable.

- Translation does not change the differential entropy, i.e. $h(X + c) = h(X)$
- $H(X)$ is always non-negative, but $h(X)$ may be negative.
- If $X \sim N(\mu, \sigma^2)$, then we have

$$h(f) = \frac{1}{2} \log 2\pi\sigma^2 e$$

- $h(aX) = h(X) + \log |a|$

Suppose we divide the range of X into bins of length Δ, by the mean value theorem, there exits a value x_i within each bin such that

$$f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x) dx$$

Consider the quantized random variable X^Δ, which is defined by

$$X^\Delta = x_i \quad i\Delta \leq x < (i+1)\Delta$$

Then the probability that $X^\Delta = x_i$ is

$$p_i = \int_{i\Delta}^{(i+1)\Delta} f(x) dx = f(x_i)\Delta$$

Yielding $H(X^\Delta) = -\sum \Delta f(x_i) \log f(x_i) - \log \Delta$, which can be simplified to

$$H(X^\Delta) + \log \Delta = -\sum \Delta f(x_i) \log f(x_i) \to h(f)$$

Theorem 14. *AEP for Differential Entropy.Let $X_1, \ldots, X_n$ be a sequence of random variables drawn i.i.d according to the density $f(x)$, then we have*

$$-\frac{1}{n} \log f(X_1, \ldots, X_n) \to E(-\log f(X)) = h(f)$$

For $\epsilon > 0$ and any n, we define the typical set $A_\epsilon^{(n)}$ with respect to $f(x)$ as follows

$$A_\epsilon^{(n)} = \left\{(x_1, \ldots, x_n) \in S^n : \left|-\frac{1}{n} \log f(x_1, \ldots, x_n) - h(X)\right| \leq \epsilon\right\}$$

We can see that $\mathbb{P}(A_\epsilon^{(n)}) > 1 - \epsilon$ for n sufficiently large. Note that for continuous cases, we cannot use the number of elements to illustrate how big a set is. Instead, we use *volume.* The results are that $\mathrm{Vol}(A_\epsilon^{(n)}) \leq 2^{n(h(X)+\epsilon)}$ and $\mathrm{Vol}(A_\epsilon^{(n)}) \geq (1 - \epsilon)2^{n(h(X)-\epsilon)}$.

Definition 19. *The differential entropy of a set $X_1, \ldots, X_n$ with density $f(x_1, \ldots, x_n)$ is defined as*

$$h(X_1, \ldots, X_n) = -\int f(x^n) \log f(x^n) dx^n$$

Definition 20. *If X, Y have a joint density function $f(x,y)$, we can define the conditional differential entropy $h(X|Y)$ as*

$$h(X|Y) = -\int f(x,y)\log f(x|y)dxdy$$

Note that $h(X|Y) \leq h(X)$ with equality iff X, Y are independent.

Theorem 15. *The Chain rule for differential entropy*

$$h(X_1,\dots,X_n) = \sum_{i=1}^{n} h(X_i|X_1,\dots,X_{i-1})$$

Theorem 16. *Let $N(\mu, K)$ denote the multivariate Gaussian distribution with mean μ and covariance matrix K, then we have*

$$f(x) = \frac{1}{(\sqrt{2\pi})^n |K|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T K^{-1}(x-\mu)}$$

Its entropy and property are similar to the previous case

$$h(X_1,\dots,X_n) = \frac{1}{2}\log(2\pi e)^n |K|$$
$$h(AX) = h(X) + \log|\det(A)|$$

Definition 21. *The relative entropy between two densities f and g is defined by*

$$D(f||g) = \int f \log\frac{f}{g}$$

Definition 22. *The mutual information $I(X;Y)$ between two random variables with joint density $f(x,y)$ is defined as*

$$I(X;Y) = \int f(x,y)\log\frac{f(x,y)}{f(x)f(y)}dxdy$$

We still have the following properties

- $I(X;Y) = h(X) + h(Y) - h(X,Y) = D(f(x,y)||f(x)f(y))$
- $D(f||g) \geq 0$
- $I(X;Y) \geq 0$ (mutual information is still non-negative, but the entropy can be negative)

Now we introduce the *master definition* between two random variables. First, we can prove that the mutual information between two random variables is the limit of mutual information between their quantized versions [two mutual information are essentially equal]

$$\begin{aligned} I(X^\Delta; Y^\Delta) &= H(X^\Delta) - H(X^\Delta|Y^\Delta) \\ &\approx h(X) - \log\Delta - h(X|Y) + \log\Delta \text{ (shown previously)} \\ &= I(X;Y) \end{aligned}$$

Now we generalize and formalize the statement. The mutual information between two random variables X and Y (whether it is discrete or continuous) is given by

$$I(X;Y) = \sup_{\mathcal{P},\mathcal{Q}} I([X]_\mathcal{P}; [Y]_\mathcal{Q})$$

where the supremum is over all finite partitions $\mathcal{P}, \mathcal{Q}$. $[X]_\mathcal{P}$ denotes the quantization of X by $\mathcal{P}$, where

$$P([X]_P = i) = P(X \in P_i) = \int_{P_i} dF(x)$$

Maximum Entropy Principle

We consider the case with constraints. Let the random variable $X \in \mathbb{R}$ have mean μ and variance σ^2, then

$$h(X) \leq \frac{1}{2}\log 2\pi e\sigma^2$$

with equality iff $X \sim N(\mu, \sigma^2)$
Let the random variable $X \in \mathbb{R}$ satisfy $E(X^2) \leq \sigma^2$, then

$$h(X) \leq \frac{1}{2}\log 2\pi e\sigma^2$$

with equality iff $X \sim N(0, \sigma^2)$. Other common properties are listed as follows for reference.

- Let $S = [a,b]$ with no other constraints, then the maximum entropy distribution is the uniform distribution over this range.
- $S = [0, +\infty)$ and $E(X) = \mu$, the the entropy-maximizing distribution is
$$f(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}, \quad x \geq 0$$
- $S = \mathbb{R}$, $E(X) = \alpha_1$ and $E(X^2) = \alpha_2$, then the maximum entropy distribution is $N(\alpha_1, \alpha_2 - \alpha_1^2)$.

Some Inequalities

Definition 23. *K is a nonnegative definite symmetric $n \times n$ matrix. Let $|K|$ denote the determinant of K. Hadamard's Inequality states that $|K| \leq \prod K_{ii}$ with equality iff $K_{ij} = 0, i \neq j$.*

Proof. Suppose $X \sim N(0, K)$, then

$$\frac{1}{2}\log(2\pi e)^n|K| = h(X_1,\dots,X_n) \leq \sum h(X_i) = \sum_{i=1}^{n}\frac{1}{2}\log 2\pi e|K_{ii}|$$

with equality iff $X_1,\dots,X_n$ are independent, i.e. $K_{ij} = 0, i \neq j$. □

To determine whether we can generalize a discrete entropy inequality to differential cases, we introduce balanced information entropy.
Let $[n] := \{1, 2, \dots, n\}$. For any $\alpha \subseteq [n]$, denote $\{X_i, i \in \alpha\}$ by X_α. For example, $\alpha = \{1,3,4\}$, we denote X_1, X_3, X_4 by $X_{\{1,3,4\}}$ for simplicity. We could write any information inequality in the form $\sum_\alpha w_\alpha H(X_\alpha) \geq 0$ or $\sum_\alpha w_\alpha h(X_\alpha) \geq 0$.
Any information inequality is called balanced if for any $i \in [n]$, the net weight of X_i is zero. The linear continuous inequality $\sum_\alpha w_\alpha h(X_\alpha) \geq 0$ is valid iff its corresponding discrete counterpart $\sum_\alpha w_\alpha H(X_\alpha) \geq 0$ is valid and balanced.

Definition 24 (Han's Inequality)**.** *Let $(X_1,\dots,X_n)$ have a density, and for every $S \subseteq \{1,2,\dots,n\}$. We denote the subset $\{X_i := i \in S\}$ by $X(S)$. Han's Inequality states that*

$$h_1^{(n)} \geq h_2^{(n)} \geq \cdots \geq h_n^{(n)} = \frac{H(X_1,\dots,X_n)}{n} = g_n^{(n)} \geq \cdots \geq g_2^{(n)} \geq g_1^{(n)}$$

where

$$h_k^{(n)} = \frac{1}{\binom{n}{k}}\sum_{S:|S|=k}\frac{h(X(S))}{k}, \quad g_k^{(n)} = \frac{1}{\binom{n}{k}}\sum_{S:|S|=k}\frac{h(X(S)|X(S^c))}{k}$$

Definition 25. *Energy Power inequality. If X, Y are independent random n-vectors with densities, then we have*

$$e^{\frac{2}{n}h(X+Y)} \geq e^{\frac{2}{n}h(X)} + e^{\frac{2}{n}h(Y)}$$

Energy Constraints

Gaussian Channel

The most important continuous channel is the Gaussian channel. The noise Z_i is drawn i.i.d. from a Gaussian distribution with variance N, and it is assumed to be independent of the signal X_i. It is a time-discrete channel with output Y_i at time i such that

$$Y_i = X_i + Z_i, \quad Z_i \sim N(0, N)$$

Without further conditions, the capacity of the channel may be ∞. Assume the variance of N is neglected compared to the distances of the values of X, then $Y = X + Z \approx X$, yielding $H(X;Y) \approx H(X)$, which might be ∞.
Usually, the most common limitation on the input is an energy or power constraint. For any codeword $(x_1,\dots,x_n)$ transmitted over the channel, we require that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \leq P$$

The information capacity of the Gaussian channel with power constraint P is

$$\begin{aligned} C &= \max_{f(x):E(X^2)\leq P} I(X;Y) \\ I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z|X) \\ &= h(Y) - h(Z) \end{aligned}$$

Note that $h(Z) = \frac{1}{2}\log 2\pi eN$, and we also have

$$E(Y^2) = E[(X+Z)^2] = E(X^2) + 2E(X)E(Z) + E(Z^2) \leq P + N$$

Therefore, we have the following inequality (by Maximum Entropy Principle), equality attained when $X \sim N(0, P)$.

$$h(Y) \leq \frac{1}{2}\log 2\pi e(P+N)$$
$$I(X;Y) = h(Y) - h(Z) = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Hence, we can conclude that

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Worst Additive Noise

Under the energy constraint P, the channel capacity of additive channel $Y = X + Z$ is

$$C(Z) = \max_{X:E(X)^2\leq P} I(X;Y) = \max_{X:E(X)^2\leq P} = h(X+Z) - h(Z)$$

We now consider what is the minimum of $C(Z)$, if we could choose $Z := E(Z^2) \leq N$. This is intended to give a lower bound of channel capacity with respect to all Z. The problem can be formalized as follows

$$\max_{Z:E(Z^2)\leq N} C(Z) := \min_{E(Z^2)\leq N}\max_{E(X^2)\leq P} I(X;X+Z)$$

We need to find a Z^*. When $C(Z^*)$ is attained by X^*, we have

$$I(X^*;X^*+Z^*) \leq \max_{X:E(X^2)\leq P} I(X;X+Z)$$

Shannon proposed that $\min_{Z:E(Z^2)\leq N} C(Z)$ is attained iff $Z = Z_G \sim N(0, \sigma^2)$, which proved to be true (Gaussian noise is the worst additive noise).