Problem 1

a)

Bellman-Ford

> **Function** bellman_ford (G, S)
> $\text{dist}[s] = 0, \text{dist}[x] = \infty$ for other $x \in V$
> while $\exists\, \text{dist}[x]$ is updated
> for each $(u, v) \in E$
> $\text{dist}[v] = \min\{\text{dist}[v], \text{dist}[u] + d(u, v)\}$

Assume there is no update at $|V|$ round, which means:

$$\forall (u, v) \in E, dist(v) \leq dist(u) + dist(u, v)$$

Here (u, v) denotes there is a directed edge $u \to v$.

However, if we apply this to the edge constituting negative cycle C

$$u_1 \to u_2 \to u_3 \to \cdots \to u_n \to u_1$$

then we get:

$$\begin{aligned} dist(u_2) &\leq dist(u_1) + dist(u_1, u_2) \\ dist(u_3) &\leq dist(u_2) + dist(u_2, u_3) \\ &\vdots \\ dist(u_n) &\leq dist(u_{n-1}) + dist(u_{n-1}, u_n) \\ dist(u_1) &\leq dist(u_n) + dist(u_n, u_1) \end{aligned}$$

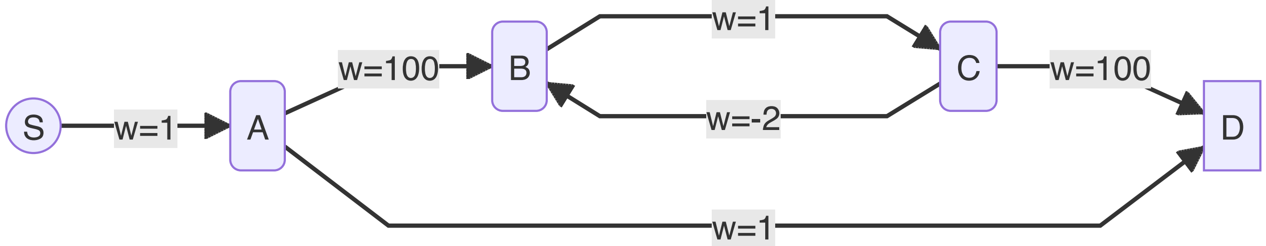
Sum the inequalities above we get:

$$\sum_{i=1}^{n} dist(u_i, u_{i+1}) \geq 0, (u_{n+1} = u_1)$$

which contradicts to the fact that the weight of the cycle is negative.

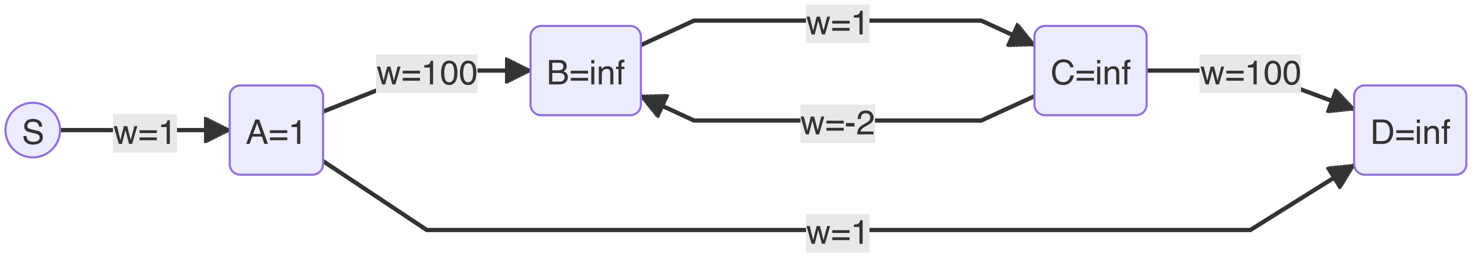
b)

We construct a graph with $|V| = 5$ like below:

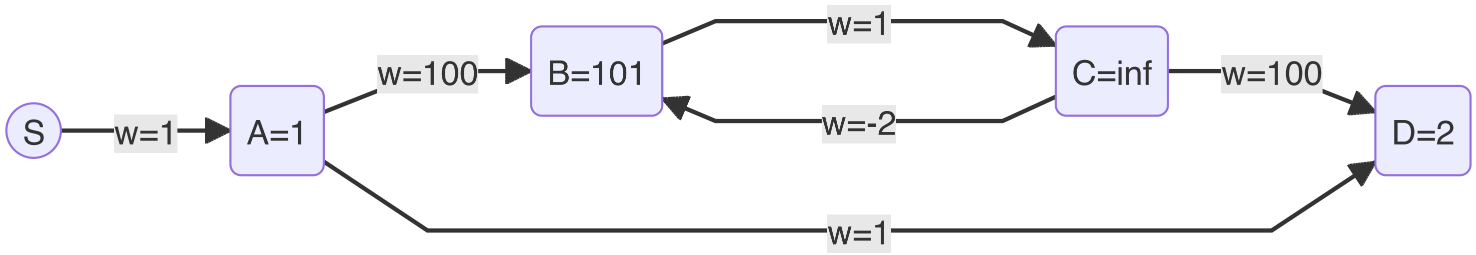


WLOG, we assume the iteration order of the edges is from right to left, and particularly consider the worst case of Bellman-Ford ALG, i.e. every update is based on the distance in last iteration.

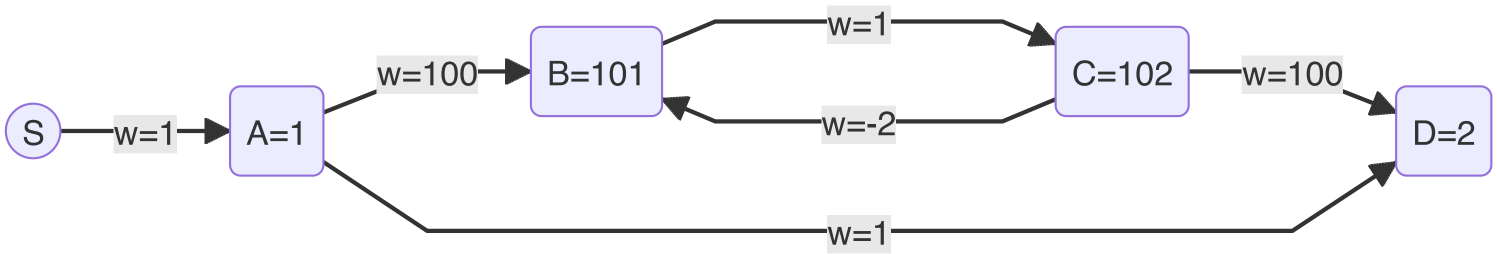
Iteration 1



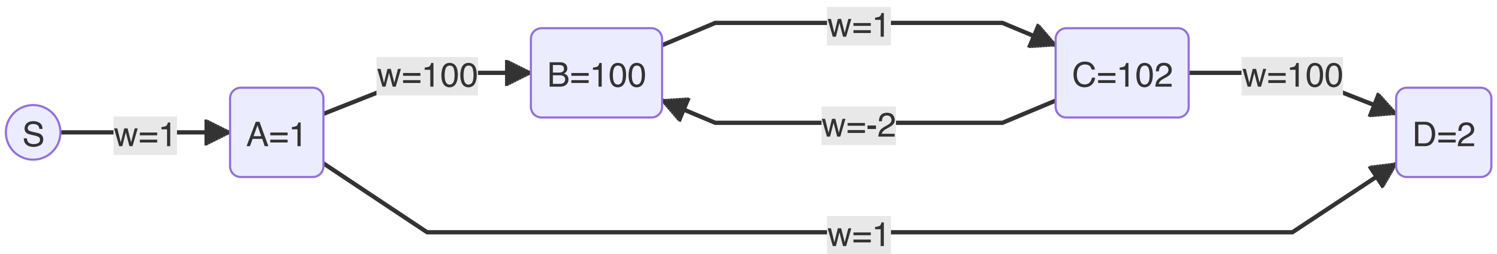
Iteration 2



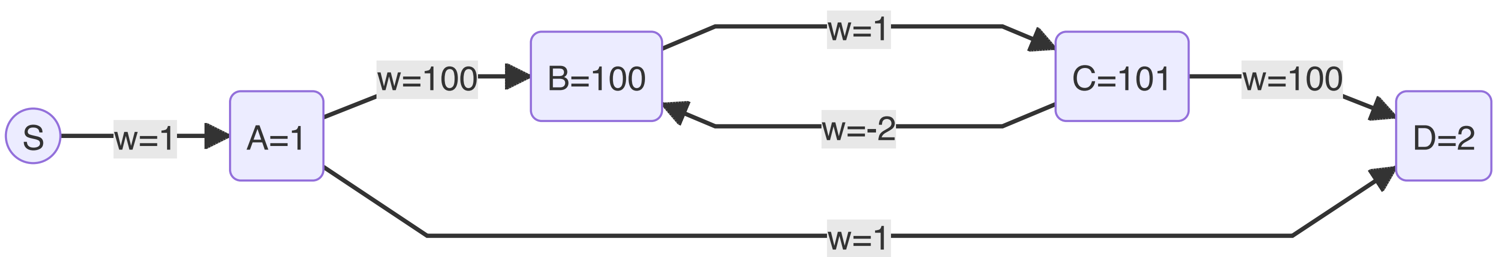
Iteration 3



Iteration 4



Iteration 5



Terminate after $|V| = 5$ rounds.

we find that though there exists a path from S to D containing a negative cycle, for example, $S \to A \to B \to C \to B \to C \to D, dist(D)$ is not updated at $|V| - th$ iteration.

c)

Algorithm

> Modified Bellman-Ford
>
> 1. Apply DFS to the source S on G and store all the vertex visited and all it's reversed edge covering these and only these vertices in a new graph $\tilde{G}$
> 2. Apply DFS to a single vertex t on $\tilde{G}$ and store all the vertex visited and all it's edge covering these and only these vertices in another new graph G'
> 3. Apply Bellman-Ford ALG to G'
>   1. If at the $|V'|$-th round there exists updated vertex then negative cycle exists in the t-to-s path.
>   2. o.w. Such path does not exist.

Correctness Analysis

Note there DFS to a single vertex v means that we won't enter the outer loop of DFS.

`step 1` exclude the part of the graph which is actually unreachable from S. (Later we add condtion that S can reach all points, but I think this addition is unnecessary)

After `step 2` we will get a graph G' consists of $t - s$ paths. Because $\forall v \in \tilde{G}$, we could always starts from it and get to S. Given this presiquite, if we apply DFS to t then, we just add t to the head of such path which starts from the neighbor of t and could reach S.

Lemma

> **If there exists a negative cycle in G', then there exists a t-to-s path with a negative cycle.**

> Proof: Since we know that for any vertex v in G', there exists a path p_1 from v to S (S can reach any vertex) and there also exists a path p_2 from t to v, for $v \in$ negative cycle, we add the cycle to make $p_1 \rightarrow cycle \rightarrow p_2$ is also a t to s path.

Then by Bellman-Ford ALG, we could judge whether there exists a negative cycle. Combined with the lemma above and partial symmetry property between G and G' , we could decide if there is an s-t path that contains a negatively weighted cycle in the original graph G.

Time Complexity

1. `DFS to the source and get reversed graph` $O(C(|V|+|E|)) = O(|V|+|E|)$
2. `DFS to a single vertex t and get a new graph` $O(C(|V|+|E|)) = O(|V|+|E|)$
3. `Bellman-Ford` $O(|V||E|)$

Here C is a constant.

Total: $O(|V||E|)$

Problem 2

Denote $SCC(v)$ as the strong connected component including vertex v

Algorithm

> 1. Get SCC graph of $G(V, E)$, denote it as $\tilde{G}(V', E')$
> 2. Assign all the weight of E' to be -1
> 3. Apply Bellman-Ford to $\tilde{G}$ to get the shortest distance from $SCC(s)$ to $SCC(t)$
>   1. if $dis(SCC(s), SCC(t)) = 1 - |V'|$, then such path exists
>   2. o.w. Such path does not exist

Correctness Analysis

Since the vertices in the same SCC are already connected, if we get the SCC graph $\tilde{G}(V', E')$ of $G(V, E)$, then we could transform the original problem to "finding the longest path in from $SCC(s)$ To $SCC(t)$, and judge wether the length of the longest path is $|V'| - 1$". If so, it means there exists a path covering all the SCC vertices in $\tilde{G}$, since there is no cycle inside a SCC graph.

Correspondingly, we could follow such path in $\tilde{G}'$ to find the path in G that cover all the vertices.

By assigning the weight of the edge to be -1, given that there is no cycle inside a SCC graph, we could use Bellman-Ford to find the shortest path from $SCC(s)$ to $SCC(t)$, which is actually the longest length. By checking whether $dis(SCC(s), SCC(t)) = 1 - |V'|$ after $|V'| - 1$ rounds, we could decide wether the length of the longest path is $|V'| - 1$.

Time Complexity

1. `Get SCC graph` $O(|V| + |E|)$
2. `Assign weight` $O(|E|)$
3. `Bellman-Ford` $O(|V||E|)$

Total: $O(|V||E|)$

Problem 3

Assumption

> G is free of multiple edge, as assumed in class

a)

$\Rightarrow$ If G is a tree, then it's DFS and BFS tree must contain all the edges. Since tree has no cycles, which means for $\forall v \in V$, there exists only one path from s to v (otherwise it will form a cycle), which every path is necessary for reaching all the vertices in the search, thus it's DFS and BFS tree must contain all the edges, i.e. G is a good graph.

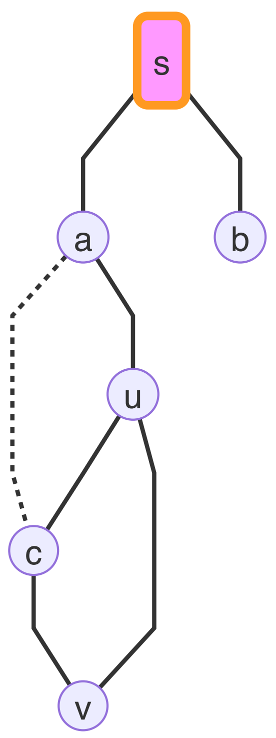
$\Leftarrow$ If G is a good graph, we assume there exists an edge (u, v) outside the tree $T(V, E')$. Since G is free of multiple edge, there can't be another edge from u to v. This implies three facts:

- There exist two path in G connecting u and v, one is (u, v) outside the tree $T(V, E')$, another path $p(V')$ is inside the $T(V, E')$, here V' is the subset of V.
- Path(a single edge) (u, v) is shorter than $p(V')$, since the latter contains more than one vertex.
- There exist a cycle containing u and v

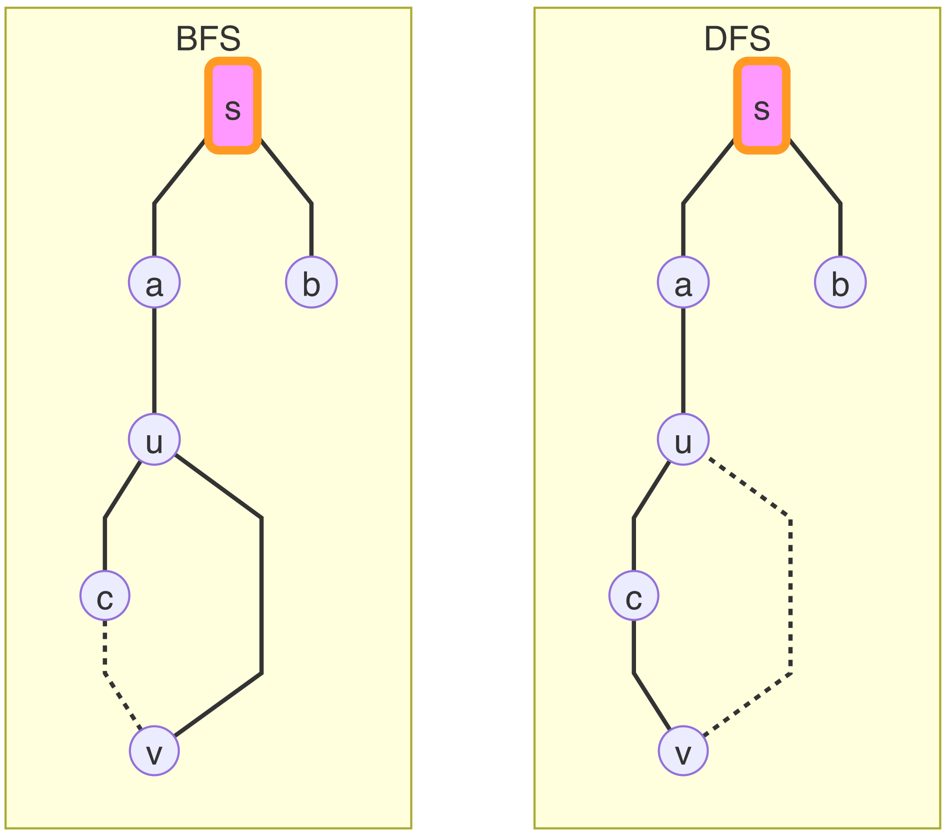
To prove the proposition, we actually need to consider the case when there are more than one path like (u, v). Let's focus on the cycle it forms, and we could always find the closest cycle to s, here distance between a cycle to s is measured by $\min\{dist(u, s), u \in cycle\}$.

We denote this cycle as C. If we want to reach C starting from s, the "entering point" (denoted as u in the illustration below and will be continually used) is the same no matter in DFS or BFS, since the path to the cycle is exclusive, otherwise C is not the closest cycle to s, as shown below with **dotted line** representing the existence of another cycle.

> cycle C: $u - v - c - u$ is the closest one to s in this graph. The dotted line shows otherwise case.



Then in DFS we notice that the adjacent edges of u are not all visited, while in BFS all the adjacent edges of u will be visited.

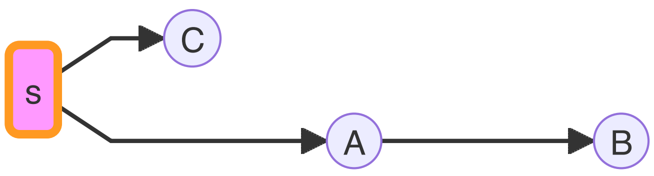


Then we could claim that G is not a good graph, contradicted!

b)

The statement is not true.

In this case, the topological order is $[s, A, B, C]$, but an ascending distance array obviouly won't end with C. However,



is not only a BFS tree, but also a DFS tree, which means G is a good graph.

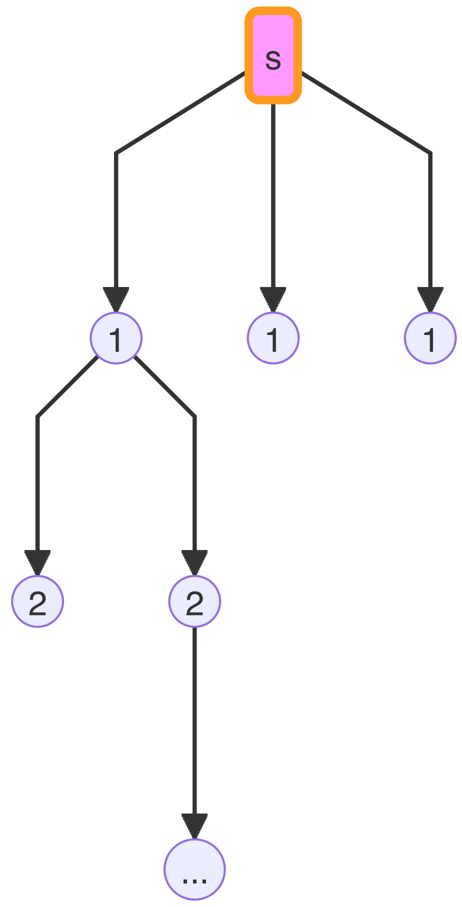
c)

The statement is true. An ascending order of distance is corresponding to the visited order of BFS, which is also corresponding to a BFS tree.

observation

> For a vertex v, if $dist(s, v) = t + 1$, then there should exist a vertex u, such that $dist(u, v) = t$.
>
> Meanwhile, there exists no other path from s to v with smaller $dist$ than $t + 1$.

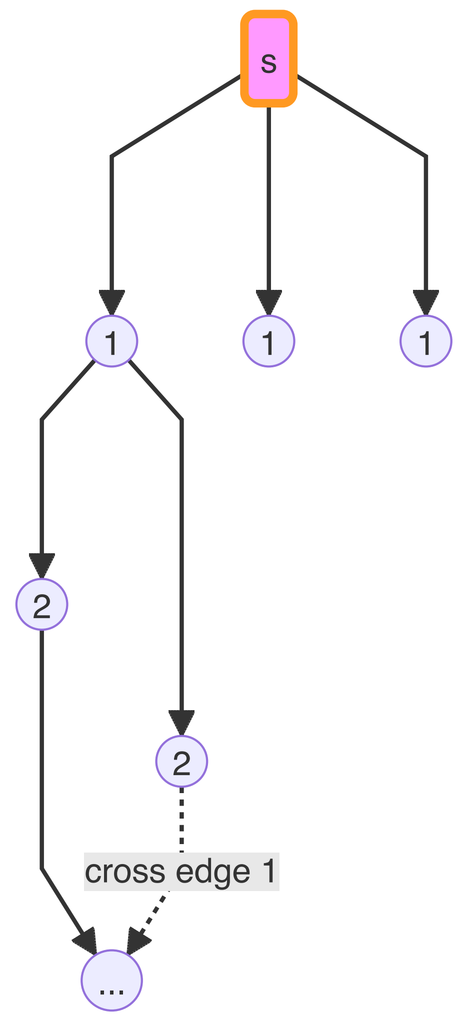
From the observation above, we then make u the farther of v. If the satifying u is more than one, just randomly choose one. Repeat the procedure from the vertex with longest distance from s and to the smallest, we can get a BFS tree.



> Here vertex with "..." means **could be more vertices or simply no more**

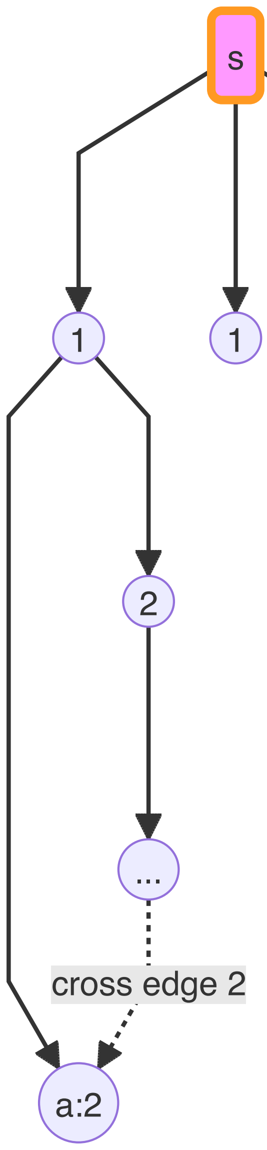
If G is simply such BFS tree, then G is obvioul y good. Now, let's add other edges in G to this BFS tree and discuss different cases. We try to prove in each possible case, G is a good graph.

Cross Edge 1



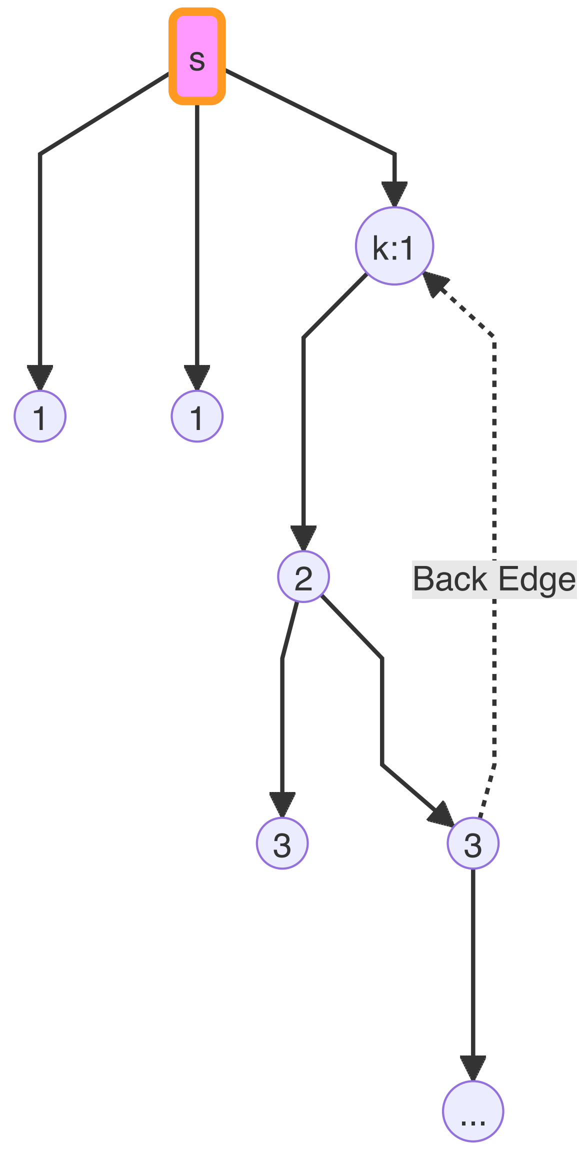
In this case, the topological order could be still the same with BFS order. And by choosing traversing order, we could make BFS tree as same as DFS tree.

Cross Edge 2

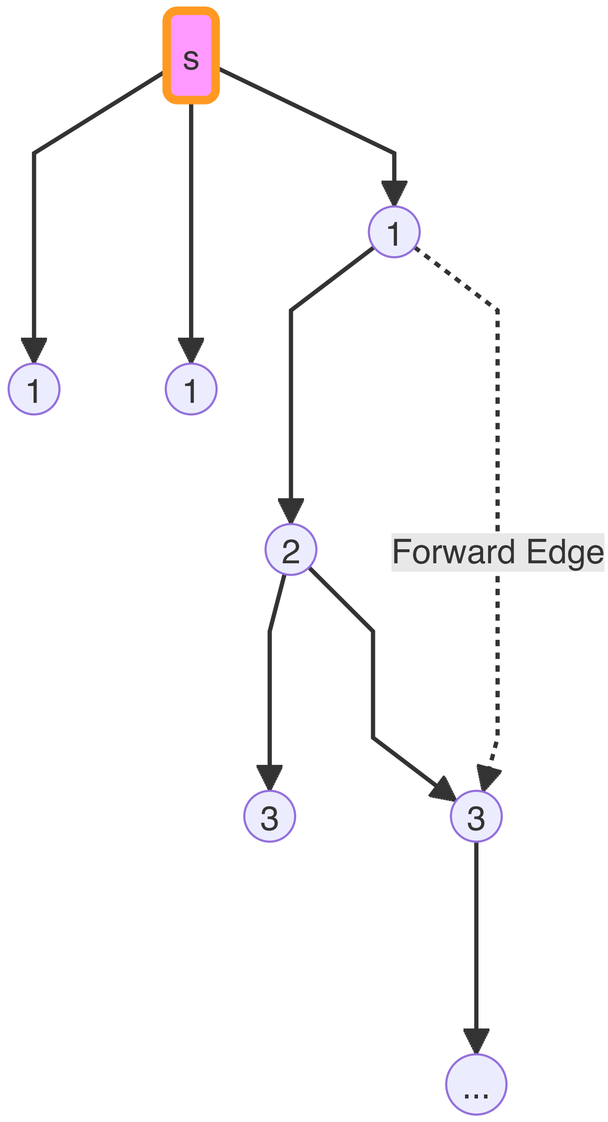


In this case, the topological order cannot be the same with BFS order because of vertex a. It's in the last of topological order while apparently not the last one in BFS order. Such edge doesn't exist in G.

Back Edge Since we have add the conditions G is acyclic, we could exlude this case.



Forward Edge



Still, In this case, In this case, the topological order could be still the same with BFS order. And by choosing traversing order, we could make BFS tree as same as DFS tree.

By checking all these cases above, we know that in possible cases, we could always find a tree $T(V,E')$ which is a BFS tree meanwhile DFS tree.

Problem 4

Algorithm

1. **Initialize**

   $i \leftarrow 0$

   $distance \leftarrow dist[|V|+1]$

   $path \leftarrow pre[|V|+1]$

   $marks \leftarrow marked[|V|+1]$

   $recordedEdge \leftarrow (0,0)$

2. **Function** $findCycle(s)$:

   $V_0 \leftarrow \{\}$

   for $(v,s) \in E$: *// outer loop*

   $V_0.add(v)$

   $v.root = v$

   $dist[v] = 1$

   while V_i is not empty:

   for each $u \in V_i$:

   for each $(u,v) \in E$

   if $(marked[v] = false)$:

   $marked[v] = true$

   $pre[v] = u$

   $dist[v] = dist[u] + 1$

   add v to V_{i+1}

   $v.root = u.root$

   else if $(marked[v] = true)$:

   s if$(v.root! = u.root)$:

   if$(dist[v] + dist[u] = 2i - 2)$: *// We have found a cycle with length $2i - 1$*

   $recordedEdge = (u,v)$

   jump to `Combine`

   $recordedEdge = (u,v)$ *// a cycle with length $2i$*

   $i \leftarrow i + 1$

   if $(u,v) = (0,0)$: no such cycle, Return

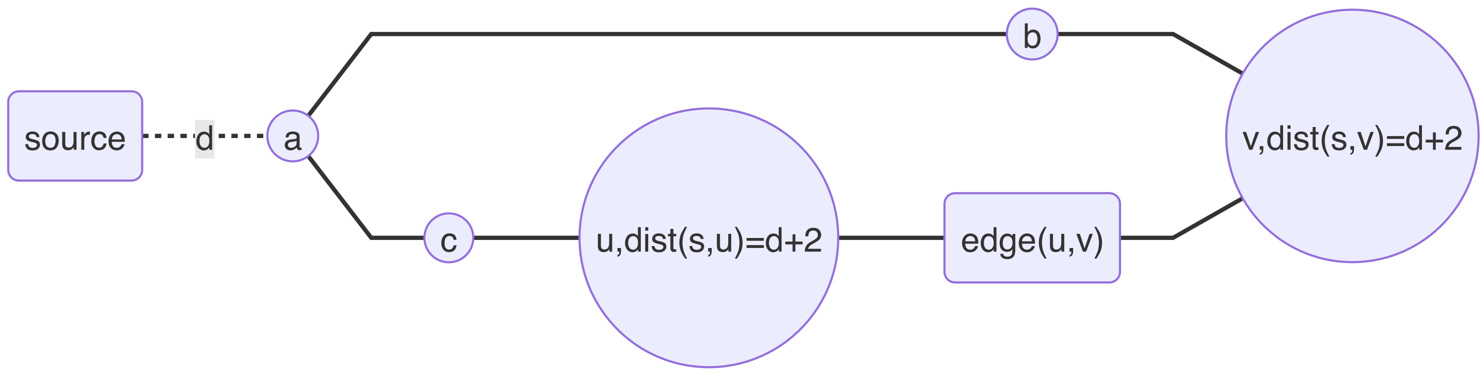
   `Combine` : combine the path from s to v and s to u to print the circle, Return

Correctness Analysis

> **Lemma 1** There exists a cycle in G, if and only if during the procedure of BFS we could encounter a vertex u which has been visited.

$\Rightarrow$ If during the procedure of BFS we encounter a vertext u which has been visited, this means there exist two path from s to u, combine them and we get a cycle.

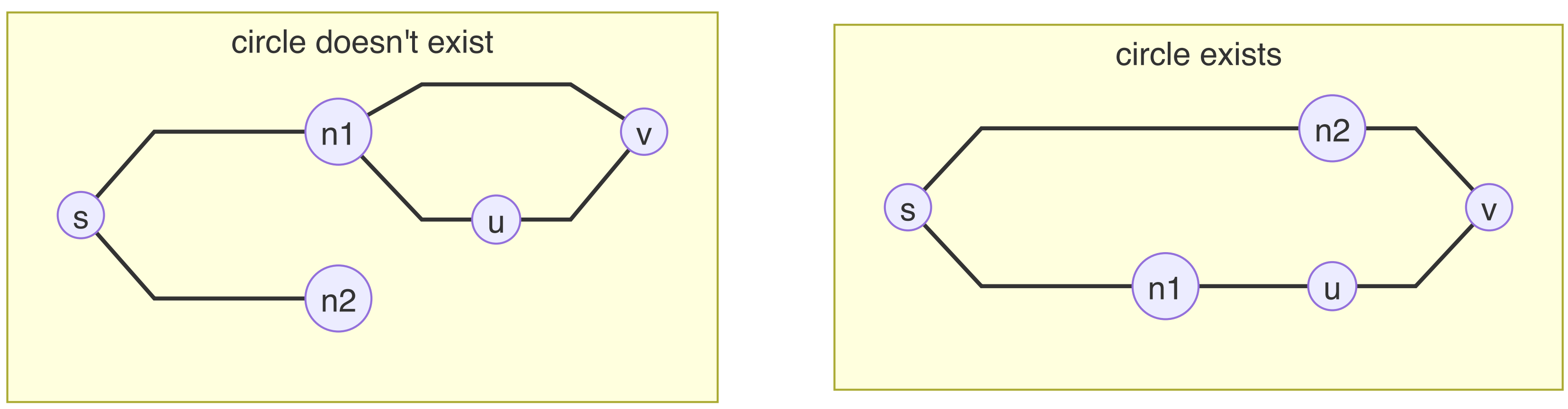
$\Leftarrow$ If there exists a cycle C in G, we denote the distance from s to C as d, then by the symmetry of the circle, we could always find an edge (u, v), with $|dist(s,v) - dist(s,u)| \leq 1$, which means the next iteration of the BFS we will encounter a vertex which has been visited.



> **Lemma 2** There exists a cycle containing s in G, if and only if there exists an $edge(u, v)$, which satisfies: $\exists u' \in u.root, v_i \in v.root$ satisfy $u'! = v'$. Here a vertex v's $root$ is the neighbour of source s who is in the shortest path from s to v. **Note here root could be more than one vertex!** However, later we could see just consider the case when root is a single vertex is enough.

$\Leftarrow$ If there doesn't exist such edge, we are unable to find the cycle containing s, obviously.

$\Rightarrow$ From the condition we know that there exists two path: a u to s path and a v to s path, whose second vertices (counting starts from s) are different. Thus combine them together with (u, v) we could get a cycle containing s.

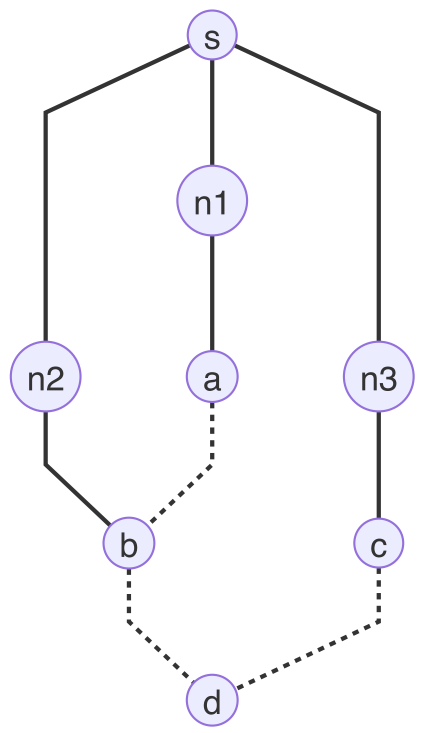


> **Lemma 3** If before $i-th$ iteration we haven't found the cycle we want, and we find a cycle containing s at $i-th$ iteration, then it's at least the second shortest cycle containing s with shortest length.

If before $i-th$ iteration we haven't found the cycle we want, by lemma 1 and lemma 2, we know that either we haven't encounter a vertex that has been visited, or we haven't found an $edge$ satisfying the condition of lemma 2.

More, we also know that there doesn't exist a circle containing s with length smaller than or equal to $2i$ (we perceive outer loop as iteration 1), by the step-by-step growing length of path in BFS ALG.

Then in the next iteration, if we found a circle, it's length is either $2i-1$ or $2i$, as we can see from the graph:



Here dotted line denote the edge visited at $i-th$ iteration.

Given the fact that the length of cycle (if any) found in next iteration is at least $2i-1$, we immediately get the conclusion below:

corollary

if we find a cycle containing s in $i-th$ iteration, then we will find the cycle we want at $i-th$ iteration.

> Lemma 4 In the procedure of the algorithm, the case that "the root of a vertex u has more than one vertex" won't consistently exist

During the procedure of ALG, or at $i-th$ iteration, if we encounter the case a vertex u seems to have two root, which means:

1. there are two shortest path from s to u with different second vertex (counting starts from s), which corresponding to a cycle with length $2i$.
2. We haven't found the cycle we want at $i-th$ iteration
3. Given the corollary of lemma 3, we know after this iteration, we will find the cycle we want, and we will quit the ALG at this iteration.

From all these lemmas above, we have already proved the correctness of the ALG.

Time Complexity

The same as the analysis in BFS, and after considering the additional comparison and value assignment operation, we have a total complexity: $O(C_1|V|+C_2|E|)=O(|V|+|E|)$. Here C_1 and C_2 are constant.

Problem 5

Time : 11 hr

Difficulty Score: 4

Collaborator： 李佳鑫，陈奕锦