Algorithm Design and Analysis

Assignment 6

Deadline: Jun 14, 2023

1. (0 points) For practice and for fun, not for credits.

   (a) Given an undirected graph $G = (V, E)$ with $n = |V|$, decide if G contains a clique with size exactly $n/2$. Prove that this problem is NP-complete.

   (b) Consider the decision version of *Knapsack*. Given a set of n items with weights $w_1, \ldots, w_n \in \mathbb{Z}^+$ and values $v_1, \ldots, v_n \in \mathbb{Z}^+$, a capacity constraint $C \in \mathbb{Z}^+$, and a positive integer $V \in \mathbb{Z}^+$, decide if there exists a subset of items with total weight at most C and total value at least V. Prove that this decision version of Knapsack is NP-complete.

   (c) Given two undirected graphs G and H, decide if H is a subgraph of G. Prove that this problem is NP-complete.

   (d) Given an undirected graph $G = (V, E)$ and an integer k, decide if G has a spanning tree with maximum degree at most k. Prove that this problem is NP-complete.

   (e) Given a ground set $U = \{1, \ldots, n\}$, a collection of its subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, and a positive integer k, the *set cover* problem asks if we can find a subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{T}} S = U$ and $|\mathcal{T}| = k$. Prove that set cover is NP-complete.

   (f) Given a collection of integers (can be negative), decide if there is a subcollection with sum exactly 0. Prove that this problem is NP-complete.

   (g) In an undirected graph $G = (V, E)$, each vertex can be colored either black or white. After an initial color configuration, a vertex will become black if all its neighbors are black, and the updates go on and on until no more update is possible. (Notice that once a vertex is black, it will be black forever.) Now, you are given an initial configuration where all vertices are white, and you need to change k vertices from white to black such that all vertices will eventually become black after updates. Prove that it is NP-complete to decide if this is possible.

   (h) Suppose we want to allocate n items $S = \{1, \ldots, n\}$ to two agents. The two agents may have different values for each item. Let $u_1, u_2, \ldots, u_n$ be agent 1's values for those n items, and $v_1, v_2, \ldots, v_n$ be agent 2's values for those n items. An allocation is a partition (A, B) for S, where A is the set of items allocated to agent 1 and B is the set of items allocated to agent 2. An allocation (A, B) is *envy-free* if, based on each agent's valuation, (s)he believes the set (s)he receives is (weakly) more valuable than the set received by the other agent. Formally, (A, B) is envy-free if

$$\sum_{i \in A} u_i \geq \sum_{j \in B} u_j \qquad \text{agent 1 thinks } A \text{ is more valuable}$$

and

$$\sum_{i \in B} v_i \geq \sum_{j \in A} v_j \qquad \text{agent 2 thinks } B \text{ is more valuable.}$$

Prove that deciding if an envy-free allocation exists is NP-complete.

(i) Given an undirected graph $G = (V, E)$, the *3-coloring* problem asks if there is a way to color all the vertices by using three colors, say, red, blue and green, such that every two adjacent vertices have different colors. Prove that 3-coloring is NP-complete.

(j) Given a ground set $U = \{1, \ldots, n\}$ and a collection of its subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, the *exact cover* problem asks if we can find a subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{T}} S = U$ and $S_i \cap S_j = \emptyset$ for any $S_i, S_j \in \mathcal{T}$. Prove that exact cover is NP-complete.

2. (30 points) Solve (a).

3. (30 points) Solve (f).

4. (40 points) Solve (e).

5. (Bonus 20 points) Solve (i) or (j).

6. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.