CS 3319 Foundations of Data Science

8. Streaming Algorithms

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Data Streams

- A data stream is a sequence of signals used to transmit or receive information that is in the process of being transmitted.
  - **Infinite**
  - **Non-stationary**

. . . 1, 5, 2, 7, 0, 9, 3

. . . a, r, v, t, y, h, b

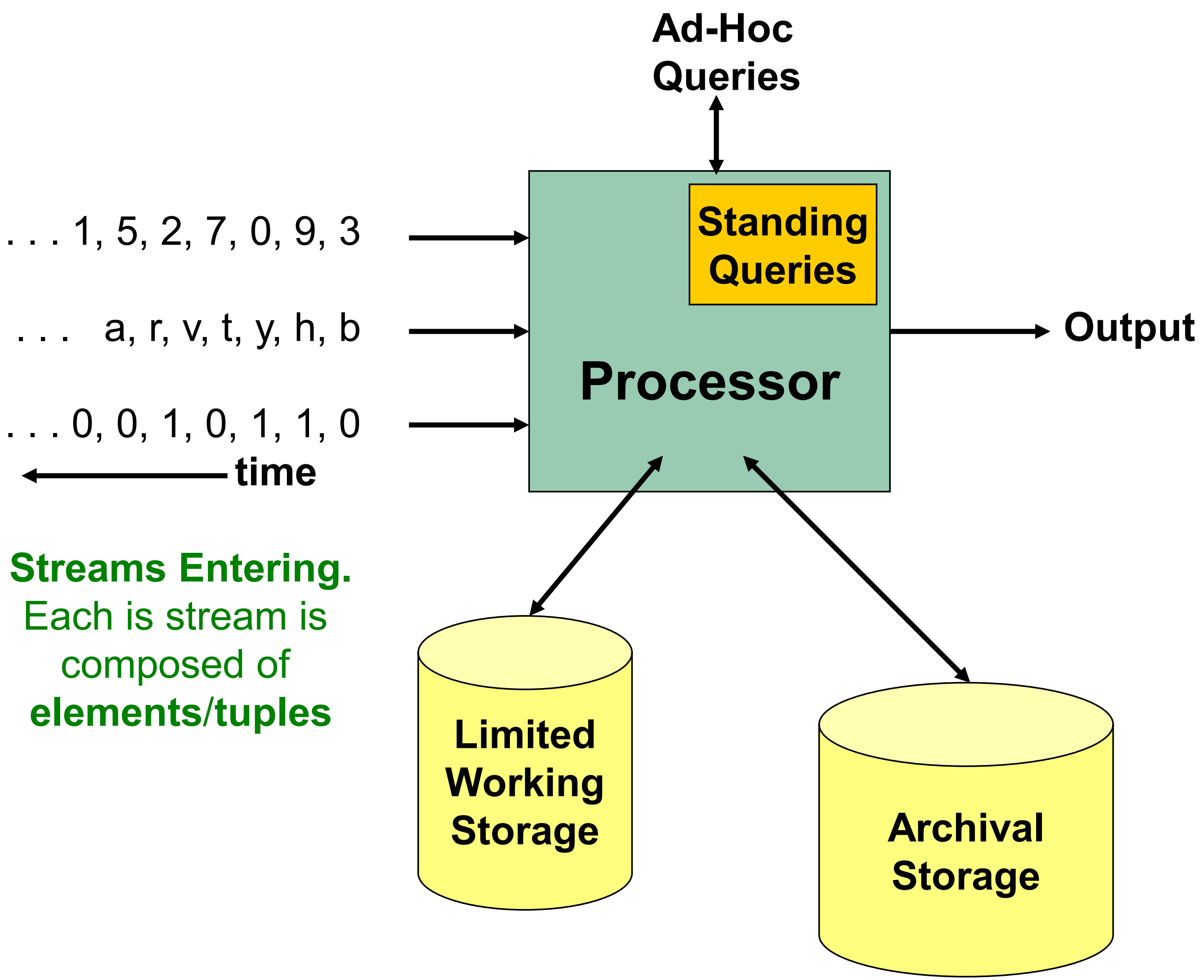
. . . 0, 0, 1, 0, 1, 1, 0

time

- Stream Model
  - The system cannot store the entire stream
  - Input **elements(tuples)** enter at a **rapid** rate, at one or more input ports
  - Make critical calculations about the stream using a limited amount of memory

Adapted from J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

General Stream Processing Model



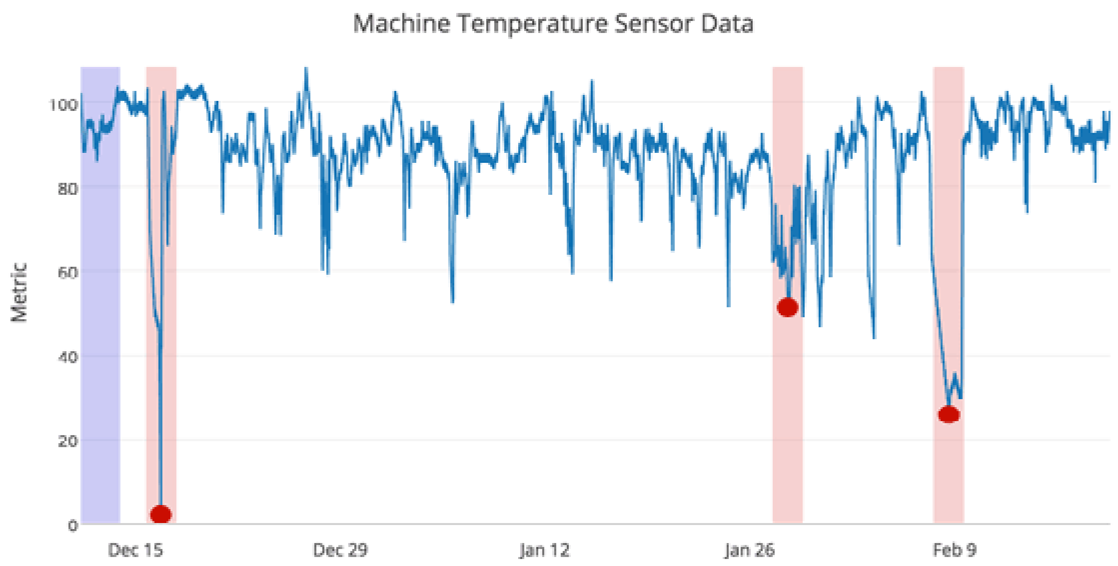
Streams Entering. Each is stream is composed of **elements/tuples**

It is better to use a crude approximation and know the truth, plus or minus 10 percent, than demand an exact solution and know nothing at all.

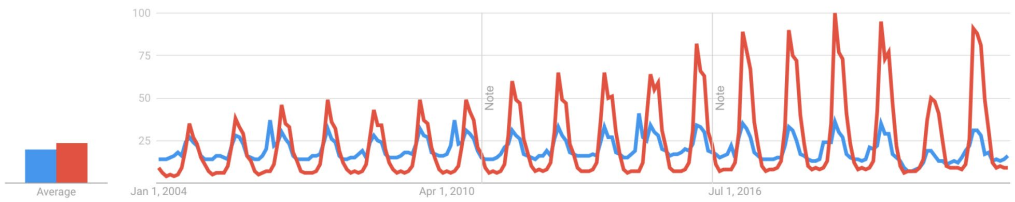
——Arthur Bloch, The Complete Murphy's Law

Applications: Networks

- **Mining network streams**
  - Finding abnormal patterns in sensor reading streams
  - Filtering out spam calls in phone call streams
  - Detect denial-of-service attacks in IP packet streams



Applications: Internet



- **Mining query streams**
  - Google wants to know what **queries** are more **frequent** today than yesterday
- **Mining click streams**
  - Bytedance wants to know which of its pages are getting an **unusual** number of **hits** in the past hour
- **Mining social network news feeds**
  - E.g., look for **trending topics** on Weibo

Problems on Data Streams

- Types of queries one wants on answer on a data stream (element):
  - **Sampling data from a stream**
    - Construct a random sample
  - **Filtering a data stream**
    - Select elements with property x from the stream

Problems on Data Streams

- Types of queries one wants on answer on a data stream (statistics):
  - **Queries over sliding windows**
    - Number of items of type *x* in the last *k* elements of the stream
  - **Counting distinct elements**
    - Number of distinct elements in the last *k* elements of the stream
  - **Estimating moments**
    - Estimate avg./std. dev. of last *k* elements
  - **Finding frequent elements**
    - Estimate the most frequent elements of the last *k* elements

Sampling from a Data Stream: Sampling a fixed-size sample

Maintaining a fixed-size sample

- Suppose we need to maintain a random sample S of size exactly s tuples
  - E.g., main memory size constraint

- Suppose at time n we have seen n items
  - Each item is in the sample S with equal prob. s/n

How to think about the problem: say s = 2
Stream: a x c y z k c d e g…
At **n= 5,** each of the first 5 tuples is included in the sample **S** with equal prob.
At **n= 7,** each of the first 7 tuples is included in the sample **S** with equal prob.

Q: How to achieve?

Solution: Fixed Size Sample

- **Algorithm**

Store all the first s elements of the stream to S

- Suppose we have seen $n-1$ elements, and now the n^{th} element arrives ($n > s$)
  - With probability s/n, keep the n^{th} element, else discard it
  - If we picked the n^{th} element, then it replaces one of the s elements in the sample S, picked uniformly at random

- This algorithm maintains a sample S with the desired property:
  - After n elements, the sample contains each element seen so far with probability s/n

Proof: By Induction

- **We prove this by induction:**
  - Assume that after *n* elements, the sample contains each element seen so far with probability *s/n*
  - We need to show that after seeing element *n+1* the sample maintains the property
    - Sample contains each element seen so far with probability *s/(n+1)*
- **Base case:**
  - After we see **n=s** elements the sample **S** has the desired property
    - Each out of **n=s** elements is in the sample with probability *s/s = 1*

Proof: By Induction

- **Inductive hypothesis:** After *n* elements, the sample *S* contains each element seen so far with prob. *s/n*
- Now element *n+1* arrives
- **Inductive step:** For elements already in *S*, probability that the algorithm keeps it in *S* is:

$$\underbrace{\left(1-\frac{s}{n+1}\right)}_{\text{Element } \mathbf{n+1} \text{ discarded}}+\underbrace{\left(\frac{s}{n+1}\right)}_{\text{Element } \mathbf{n+1} \text{ not discarded}}\underbrace{\left(\frac{s-1}{s}\right)}_{\text{Element in the sample not picked}}=\frac{n}{n+1}$$

- Time *n→n+1*, tuple stayed in *S* with prob. **n/(n+1)**
- At time *n*, tuples in *S* were there with prob. **s/n**
- So prob. tuple is in *S* at time *n+1* = $\frac{s}{n} \cdot \frac{n}{n+1} = \frac{s}{n+1}$

Filtering Data Streams

Applications

- Email spam filtering
  - We know 1 billion “good” email addresses
  - If an email comes from one of these, it is **NOT** spam

- Publish-subscribe systems
  - You are collecting lots of messages
  - People express interest in certain sets of keywords
  - Determine whether each message matches user’s interest

Filtering Data Streams

- Each element of data stream is a tuple
- Given a list of keys $S = [\boldsymbol{key_1}, \boldsymbol{key_2}, \cdots]$
- **Determine which tuples of stream are in *S***

- Obvious solution: store and compare
  - But suppose we **do not have enough memory** to store all of S
  - The **complexity** is $O(S)$, which can be big.

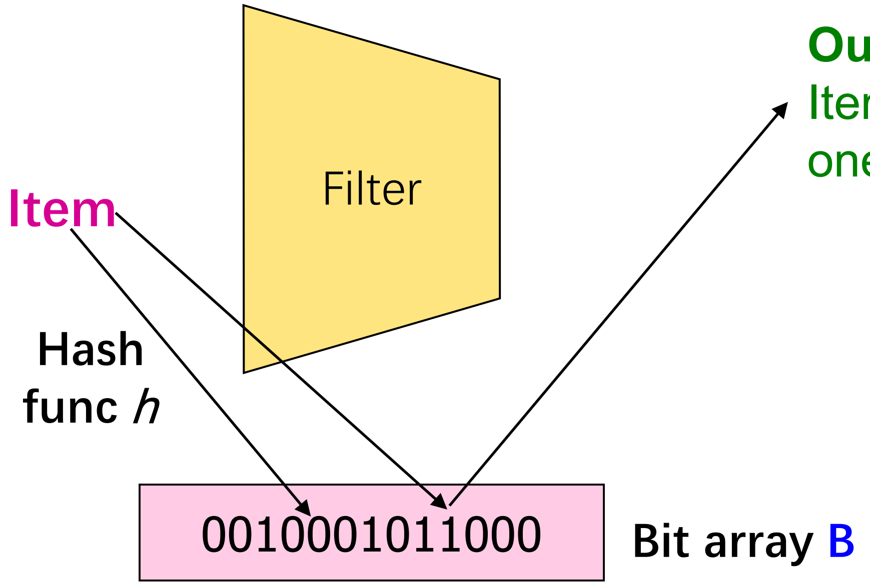
  What can we do?

Adapted from J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

First Cut Solution

- **Given a set of keys S that we want to filter**
- Create a **bit array** B of n bits, initially all 0s
- Choose a **hash function** h with range **[**$0,n$**)**
- Hash each member of **$s \in S$** to one of n buckets, and set that bit to **1**, i.e., $B[h(s)]=1$
- Hash each element a of the stream and output only those that hash to bit that was set to **1**
  - Output a if B[h(a)] = 1

First Cut Solution



Output the item since it may be in *S*.
Item hashes to a bucket that at least one of the items in **S** hashed to.

Drop the item.
It hashes to a bucket set to **0** so it is surely not in ***S***.

- Creates **false positives** but **no false negatives**
  - If the item is in *S* we surely output it, if not we may still output it

First Cut Solution

- |S| = 1 billion email addresses
  |B|= 1GB = 8 billion bits, for the hash values

- If the email address is in ***S***, then it surely hashes to a bucket that has the big set to **1**, so it always gets through (*no false negatives*)

- Approximately **1/8** of the bits are set to **1**, so about **1/8** of the addresses not in ***S*** get through to the output (*false positives*)

Analysis: Throwing Darts

- More accurate analysis for the number of **false positives**
- Consider: If we throw *m* darts into *n* equally likely targets, what is the probability that a target gets at least one dart?
- **In our case:**
  - **Targets** = bits/buckets
  - **Darts** = hash values of items

Analysis: Throwing Darts

- We have *m* darts, *n* targets
- What is the probability that **a target gets at least one dart**?

Equals **1/e** as **n** $\rightarrow\infty$

Equivalent

$$1 - \boxed{(1 - 1/n)^{n}}^{\,(m/n)}$$

$$\mathbf{1 - e^{-m/n}}$$

Probability some target **X** not hit by a dart

Probability at least one dart hits target **X**

Analysis: Throwing Darts

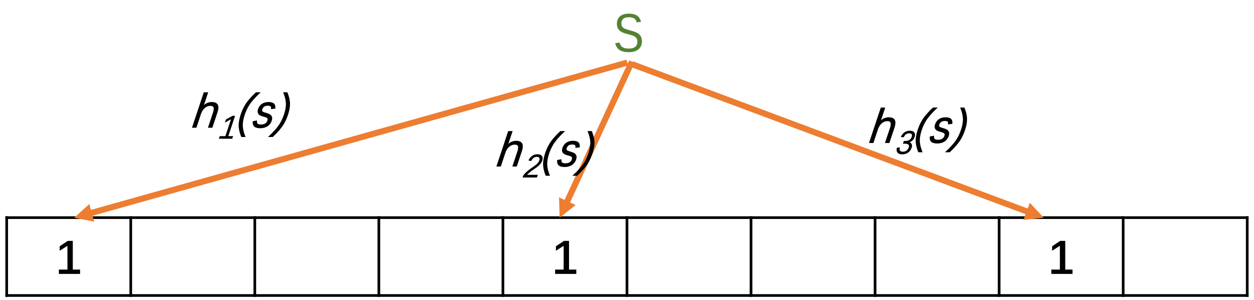
- Fraction of 1s in the array B

= probability of false positive = $1 - e^{-m/n}$

- **Example:** $\mathbf{10^9}$ darts, $\mathbf{8*10^9}$ targets
  - Fraction of **1s** in **B = 1 – $\mathbf{e^{-1/8}}$ = 0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

- How to further **improve** this false positive probability?
- Similar to LSH: Bloom Filter.

Bloom Filter



- Consider: **|S|** = m, **|B|** = n
- Use k independent hash functions $h_1, \cdots, h_k$
- **Initialization:**
  - Set **B** to all **0s**
  - Hash each element $s \in S$ using each hash function h_i, set **B**[$h_i(s)$] = **1** (for each $i = 1,.., k$)
- **Run-time:**
  - When a stream element with key x arrives
    - If **B**[$h_i(x)$] = **1** **for all** i = **1,..., k** then declare that x is in S
      - That is, x hashes to a bucket set to **1** for every hash function $h_i(x)$
    - Otherwise discard the element x

What is the false positive probability?

Bloom Filter — Analysis

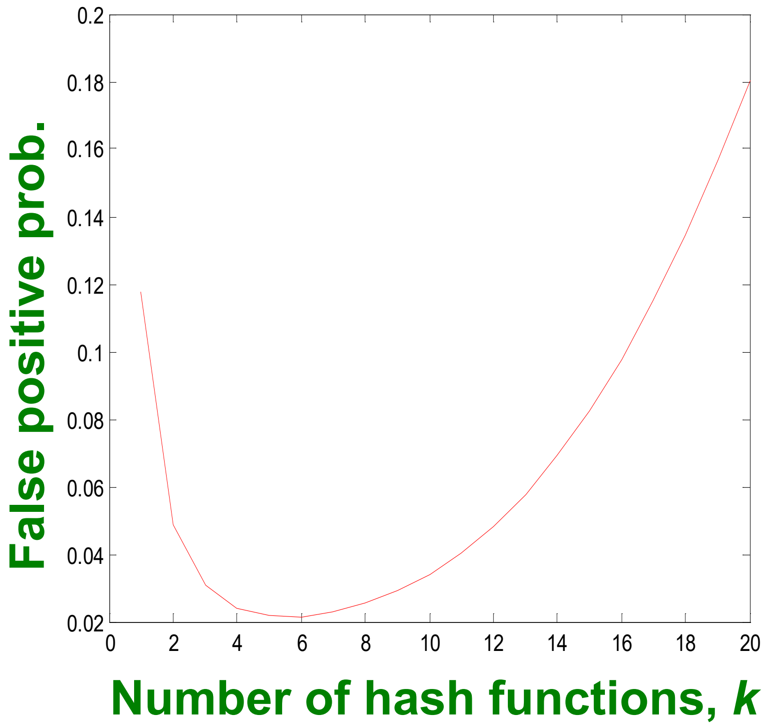
- **What fraction of the bit vector B are 1s?**
  - Throwing $k{\cdot}m$ darts at n targets
  - So fraction of **1**s is $(1 - e^{-km/n})$ **(false positive of 1 hash function)**

- But we have k independent hash functions and we only let the element x through **if all** k hash element x to a bucket of value **1**

- So, **false positive probability =** $(1 - e^{-km/n})^k$

Bloom Filter – Analysis

- m = 1 billion, n = 8 billion
  - **k = 1**: $(1 - e^{-1/8}) =$ **0.1175**
  - **k = 2**: $(1 - e^{-1/4})^2 =$ **0.0493**



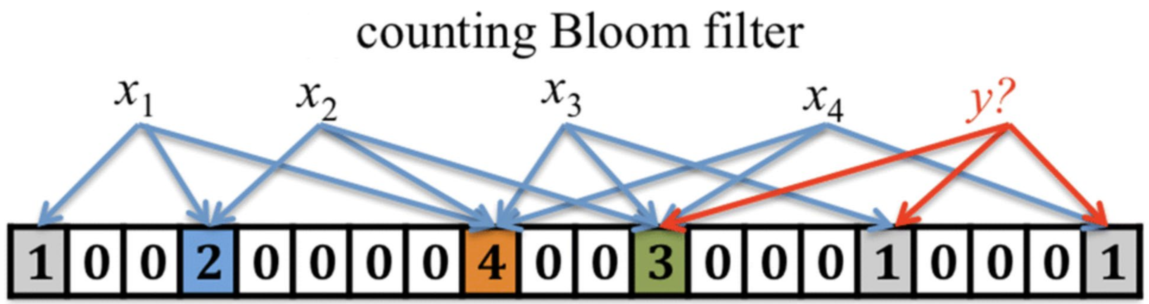
- **What happens as we keep increasing *k*?**

- "Optimal" value of *k*: n/m **ln(2)**
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-1/6})^2 =$ **0.0235**

Bloom Filter: Wrap-up

- Bloom filters guarantee **no false negatives**, and use limited memory
  - Great for pre-processing before more expensive checks
- Suitable for hardware implementation
  - Hash function computations can be parallelized

- Disadvantage: only insertion, no deletion from Bloom Filter.



Count-Min Sketch

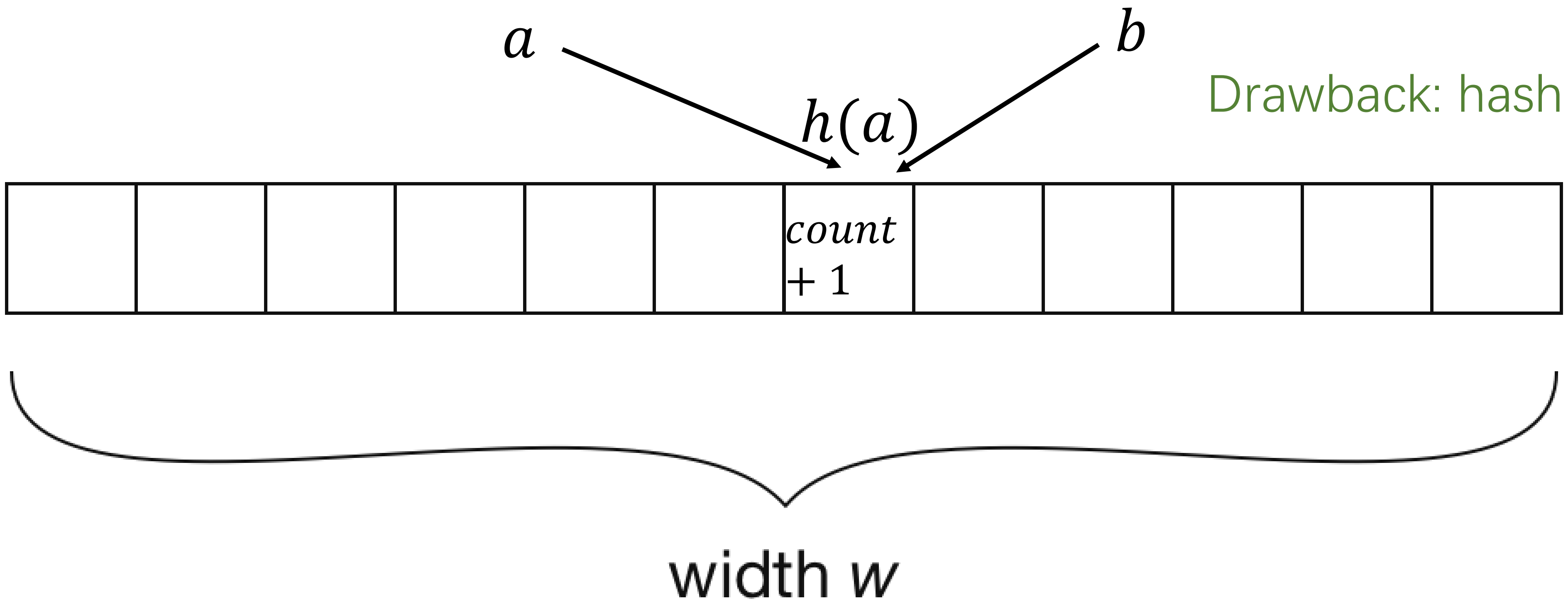
Count Element Frequency

- Faced with big data streams, storing all elements and corresponding frequencies is **impossible**.

- **Approximate** counts are acceptable.
- We can use **hashing** again.

Approximate Counts with Hashing

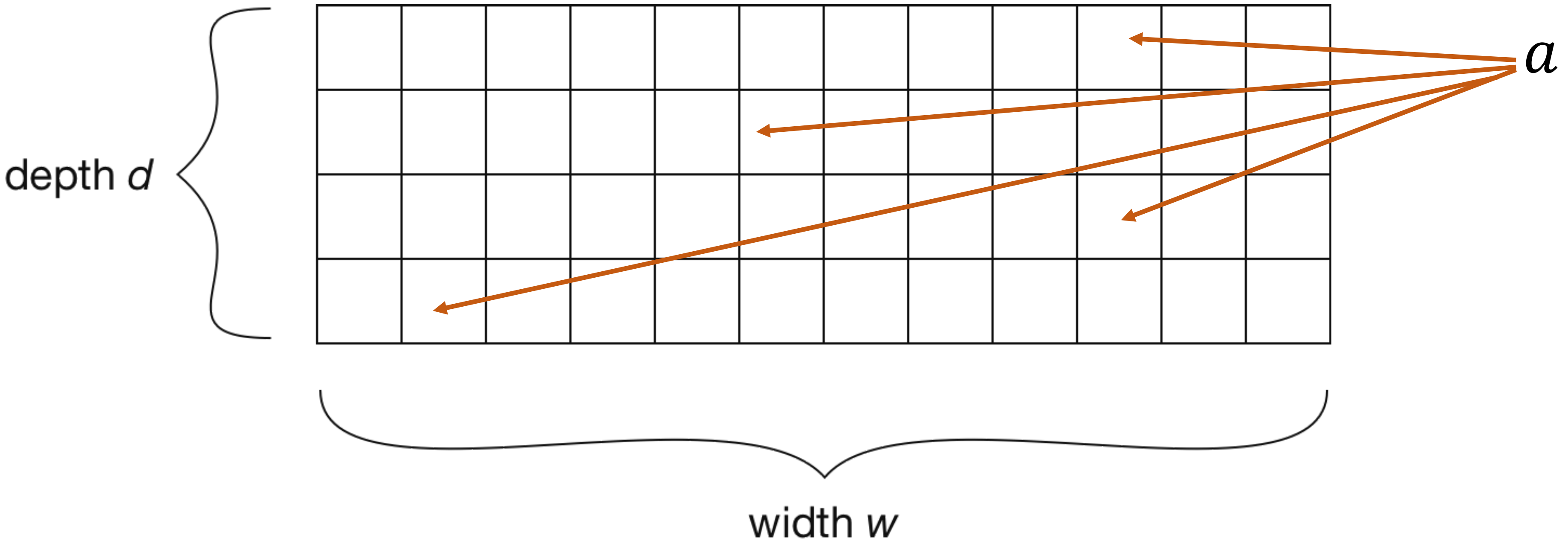
- **Initialization**: $count[i] = 0$, for $i \in [1, w]$
- **Increment** count of element a: $count[h(a)] \mathrel{+}= 1$
- **Retrieve** count of element a: $count[h(a)]$



Drawback: hash conflicts

Improvement: More Hash Functions

- We use *d* pairwise independent hash functions
- **Increment** count of element a: $count[i, h_i(a)] \mathrel{+}= 1$ for $i \in [1, d]$
- **Retrieve** count of element a: $\min_{i \in [1,d]} count[i, h_i(a)]$



Guarantees

- Theorem[1]: with probability $1-\delta$, the error is at most $\varepsilon * count$. Concrete values for these error bounds can be chosen by setting $w = \left\lceil \frac{e}{\varepsilon} \right\rceil$ and $\mathrm{d} = \left\lceil \ln(\frac{1}{\delta}) \right\rceil$, e ≈2.718.
  - Adding another **hash** function **exponentially** decreases the chance of hash conflicts
  - Increasing the **width** helps spread up the counts with a **linear** effect

[1]Graham Cormode and S. Muthukrishnan. 2005. An improved data stream summary: the count-min sketch and its applications. Journal of Algorithms 55, 1 (2005), 58–75

Queries over a Sliding Window

Streaming Binary Counting

Sliding Windows

- A useful model of stream processing is that queries are within a *window* of length N – the N most recent elements received
  - **Amazon example:** For every product **X** we keep 0/1 stream of whether that product was sold in the **n**-th transaction. We want answer queries, how many times we sold **X** in the last **k** sales.

Item sold
in transactions: 0 1 0 0 1 1 0 1 1 1 0 1 0 1 0 1 1 0 $\boxed{1\ 1\ 0\ 1\ 1\ 0}$

← Past Future →

Suppose we keep a window with length N=6,
we can query on the last k transactions, for $k \leq N$.

Counting Bits

- **Problem:**
  - Given a stream of **0**s and **1**s
  - Be prepared to answer queries of the form
    How many 1s are in the last *k* bits? where $k \leq N$

- **Obvious solution:**
  Store the most recent N bits
  - When new bit comes in, discard the N**+1**st bit
  - **Not feasible** when N is so **large** that the data cannot be stored in memory, or even on disk

0 1 0 0 1 1 0 1 1 1 0 1 0 1 0 1 1 0 1 1 0 1 1 0

← Past Future →

Counting Bits

- **Real Problem:**
  What if we cannot afford to store or compute N bits?
  - **E.g.**, we're processing 1 billion streams and N **= 1 billion**

0 1 0 0 1 1 0 1 1 1 0 1 0 1 0 1 1 0 ~~1 1 0 1 1 0~~

← Past Future →

- But we are happy with an **approximate** answer

An attempt: Simple solution

- **Q: How many 1s are in the last *N* bits?**
- A simple solution that does not really solve our problem: **Uniformity assumption**

$\longleftarrow$ ***N*** $\longrightarrow$

0 1 0 0 1 1 1 0 0 0 1 0 1 0 0 1 0 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0

$\longleftarrow$ Past Future $\longrightarrow$

- **Maintain 2 counters:**
  - *S*: number of 1s from the beginning of the stream
  - *Z*: number of 0s from the beginning of the stream
- How many 1s are in the last N bits? $\boldsymbol{N \cdot \frac{S}{S+Z}}$
- But, what if stream is **non-uniform?**
  - What if distribution changes over time? This is always true in reality.

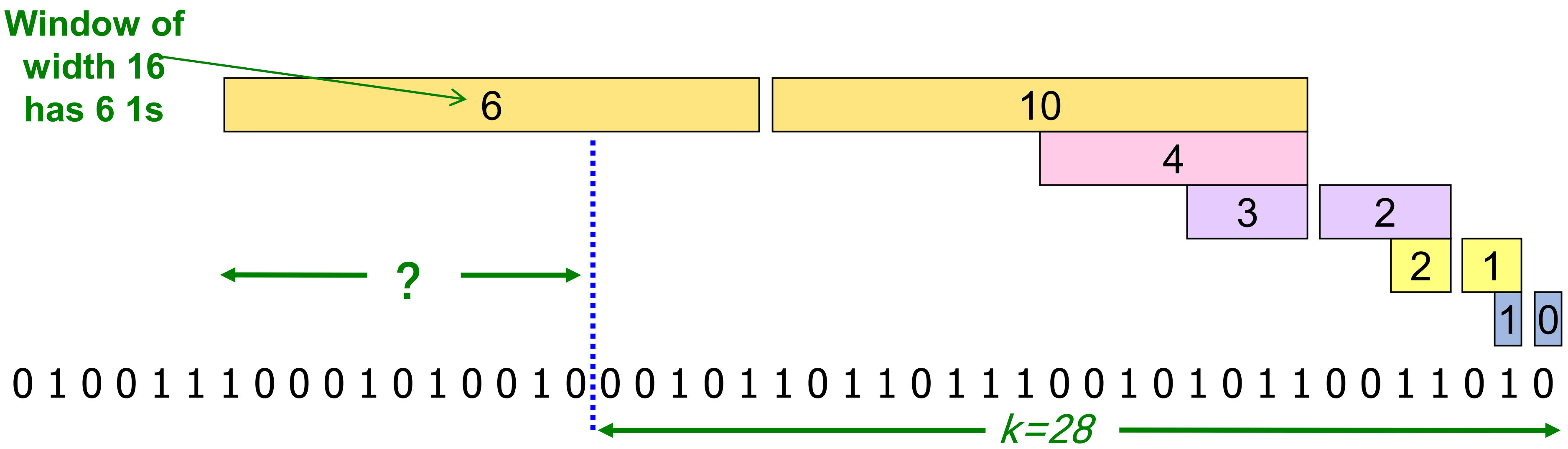
DGIM Method

- **DGIM(*Datar-Gionis-Indyk-Motwani Algorithm)*** solution that does <u>not</u> assume uniformity
- We store $\boldsymbol{O}(\log^2 \boldsymbol{N})$ bits per stream
- Solution gives **approximate** answer, **never off** by more than **50%**
  - Error factor can be reduced to any fraction > 0, with more complicated algorithm and proportionally more stored bits

Idea: Exponential Windows

1. when a bit comes in, create a bucket of length 1 with the proper count (0 or 1).
2. If any level has 3 buckets:
   a) add the rightmost two and create a bucket at the next higher level (twice the length) with that sum.
   b) delete the leftmost two buckets, keeping only the rightmost of the three.
3. Repeat (2) recursively for progressively higher levels.

- **First trial:**
  - Summarize **exponentially increasing** regions of the stream, looking backward, to answer queries over last k items $(k \leq N)$.
  - Drop small regions if there are more than two on the same level(keep the rightmost)



We can reconstruct the count of the last ***k*** bits, except we are not sure how many of the last **6 1s** are included in the ***k=28*** window

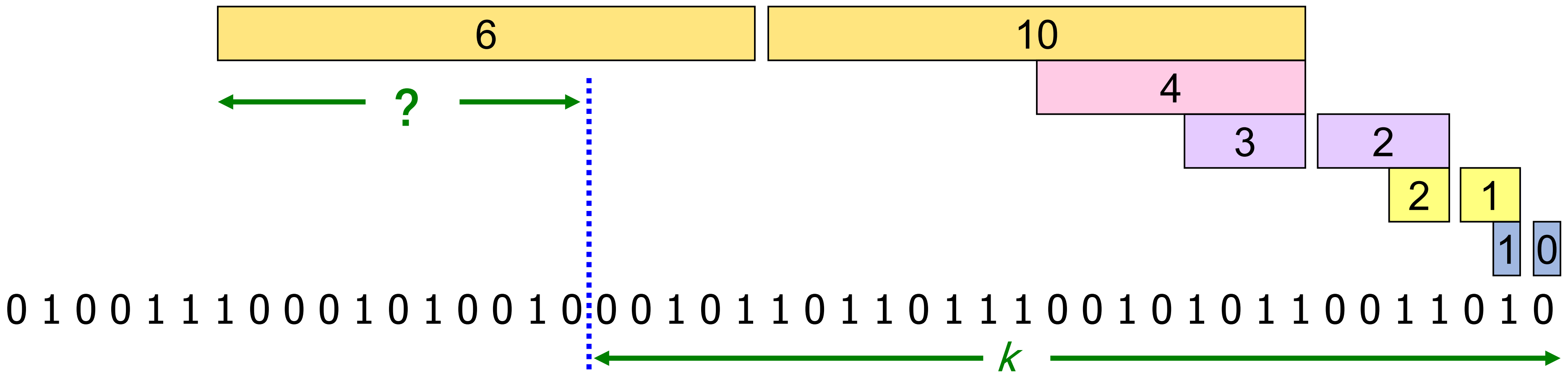
What's Good?

- **Stores only O($\log^2 N$) bits**
  - $\boldsymbol{O}(\log \boldsymbol{N})$ counts of $\log_2 \boldsymbol{N}$ bits each
- **Easy update** as more bits enter
- **Error** in count no greater than the number of **1s** in the "**unknown**" area

What's Not So Good?

Relative error=error/true count

- As long as the **1s** are fairly evenly distributed, the error due to the unknown region is small
- **What is the bound of the relative error?**
  - Consider the case that all the **1s** are in the unknown area(**?** part) and the rest are all 0s. Here the relative error is infinite.



Fixup: DGIM method

- **Idea:** Instead of summarizing fixed-length blocks, summarize blocks with specific number of **1s**:
  - Let the block **sizes** (number of **1s**) increase exponentially
  - Data dependent
- When there are few 1s in the window, block sizes stay small, so errors are small

1001010110001011010101010101011010101010101011101010101110101000101100010

$\longleftarrow$ *N* $\longrightarrow$

DGIM: Timestamps

- Each bit in the stream has a *timestamp*, starting **1**, **2,** ⋯
- Record timestamps modulo *N* (**the window size**), so we can represent any **relevant** timestamp in $\boldsymbol{O(log_2 N)}$ bits

E.g. Real Timestamp: 105,
N=40, we only store 105 mod 40=25.

1001010110001011 0 101010101010101 1 0 101010101010111 0 1010101 110101 000 101 1 00 1 0

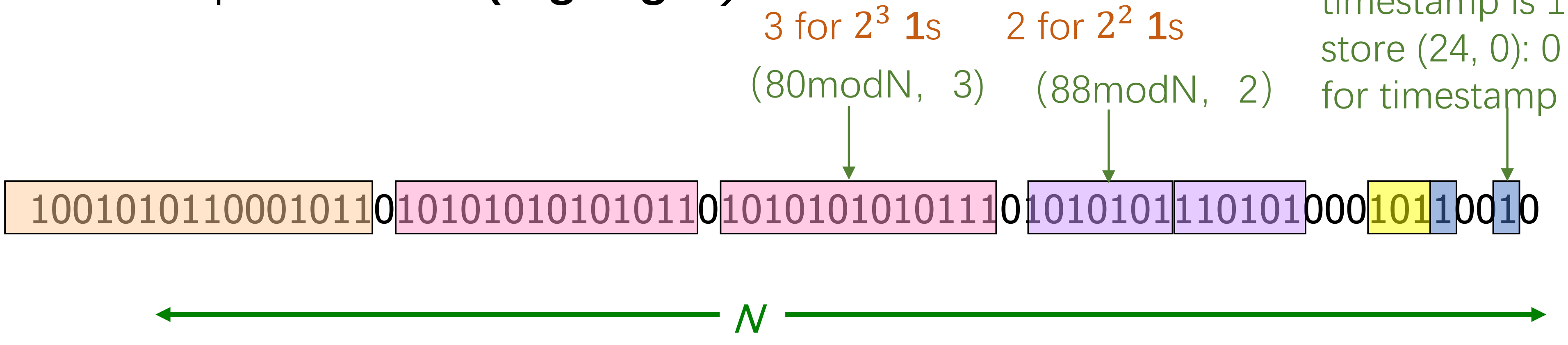
N

DGIM: Buckets

- A *bucket* in the DGIM method is a record consisting of:
  - **The timestamp of its end [O(log *N*) bits]**
  - **The number of 1s between its beginning and end [O(log log *N*) bits]**

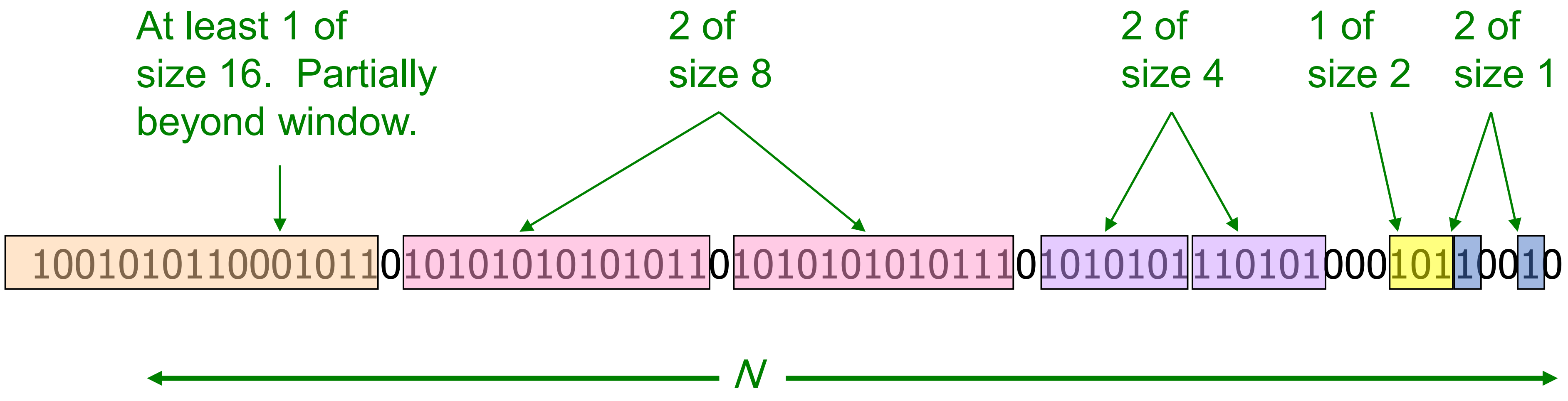
- **Constraint on buckets:**
  Number of **1s** must be a power of **2**
  - That explains the **O(log log *N*)**

E.g. In this window, if the timestamp of the last timestamp is 105, we actually store (24, 0): 0 for $2^0 = 1$, 24 for timestamp 104 mod 40.



Representing a Stream by Buckets

- Either **one** or **two** buckets with the same **power-of-2 number** of **1s**
- Buckets **do not overlap** in timestamps
- Buckets are **sorted** by **size**
  - Earlier buckets are not smaller than later buckets
- Buckets **disappear** when their end-time is **> *N*** time units in the past



Updating Buckets

- When a new bit comes in, **drop** the last (oldest) bucket if its end-time is **prior to** N time units before the current time
- **2 cases:** Current bit is **0** or **1**
- **If the current bit is 0:**
  no other changes are needed

Updating Buckets

- **If the current bit is 1:**
  - **(1)** Create a new bucket of size **1**, for just this bit
    - **End timestamp = current time**
  - **(2)** If there are now three buckets of size 1, combine the oldest two into a bucket of size 2
  - **(3)** If there are now three buckets of size 2, combine the oldest two into a bucket of size 4
  - **(4)** And so on …

Example: Updating Buckets

Current state of the stream:

1001010110001011010101010101011010101010101110101010111010100010110010

Bit of value 1 arrives

00101011000101101010101010101101010101010111010101011101010001011001 01

Two smallest buckets get merged into a size-2 bucket

0010101100010110101010101010110101010101011101010101110101000101100101

Next bit 1 arrives, new size-1 bucket is created, then 0 comes, then 1:

0101100010110101010101010110101010101011101010101110101000101100101101

Buckets get merged…

0101100010110101010101010110101010101011101010101110101000101100101101

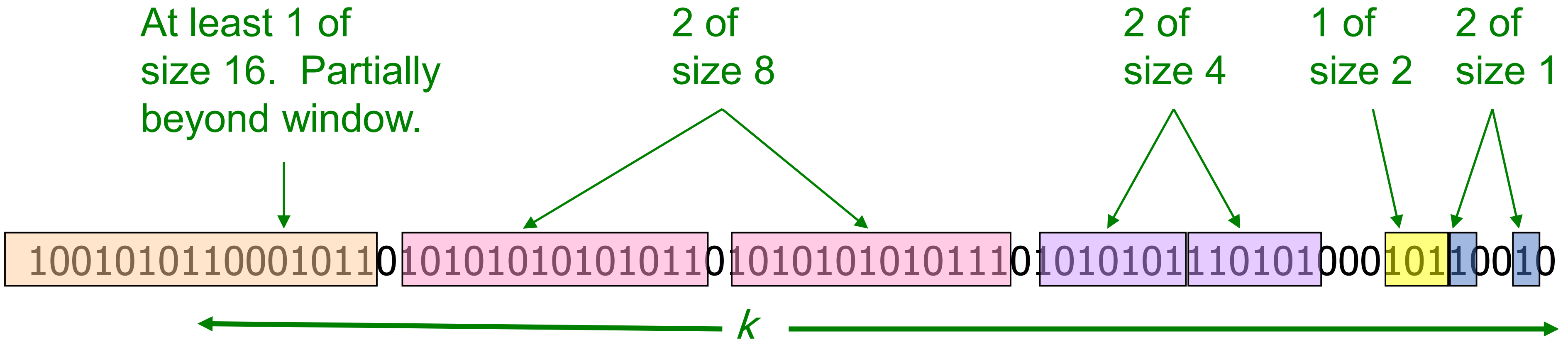
State of the buckets after merging

0101100010110101010101010110101010101011101010101110101000101100101101

How to Query?

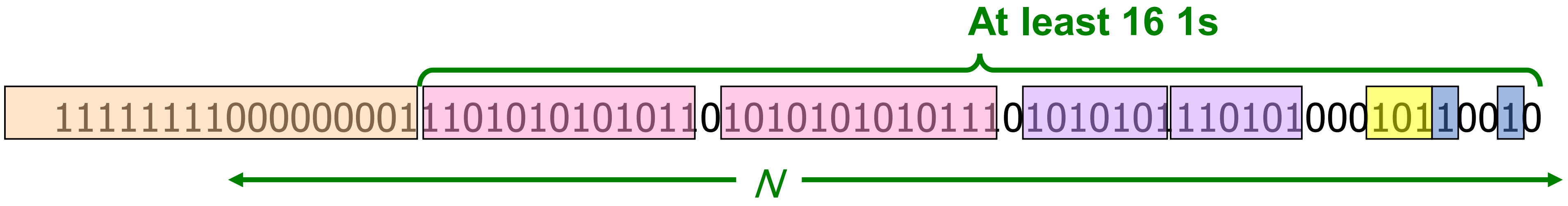
- To estimate the number of 1s in the most recent *k* bits:
  1. Sum the sizes of **all** buckets **but the last**

  2. Add **half** the size of the last bucket
- **Remember:** We do not know how many **1s** of the last bucket are still within the wanted window



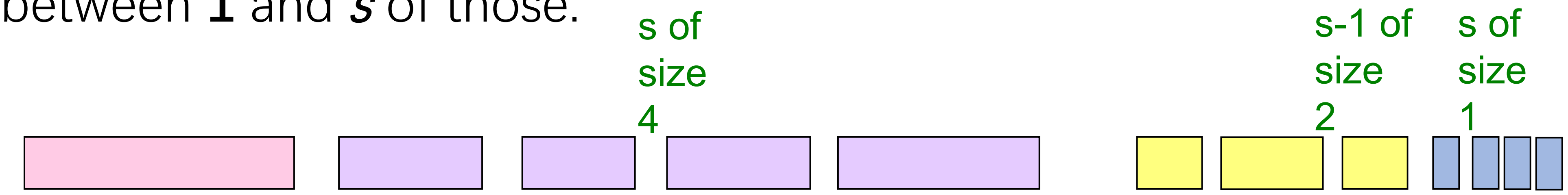
Error Bound: Proof

- **Why is error 50%? Let's prove it!**
- Suppose the last bucket has size $\mathbf{2^{r}}$
- Then by assuming $\mathbf{2^{r-1}}$ (i.e., half) of its **1s** are still within the window, we make an error of at most $\mathbf{2^{r-1}}$
- Since there is at least one bucket of each of the sizes less than $\mathbf{2^{r}}$, the true sum is at least
  $$\mathbf{1 + 2 + 4 + .. + 2^{r-1} = 2^{r} - 1}$$
- Thus, relative error at most **50%**



Further Reducing the Error

- Instead of maintaining **1** or **2** of each size bucket, we allow either ***s*-1** or ***s*** buckets (***s* > 2**)
  - Except for the largest size buckets, where we can have any number between **1** and ***s*** of those.



- **Error is at most $\frac{2^{r-1}}{(s-1)(2^r-1)}$ =O(1/s)**
- By picking ***s*** appropriately, we can tradeoff between number of bits we store and the error

Extensions

- **Can we handle the case where the stream is not bits, but integers, and we want the sum of the last *k* elements?**
- **We want the sum of the last *k* elements**
  - **Amazon:** Avg. price of last **k** sales
- **Solution:**
  - **If you know all have at most *m* bits**
    - Treat *m* bits of each integer as a separate stream
    - Use DGIM to count **1s** in each integer
    - The sum is $= \sum_{i=0}^{m-1} c_i 2^i$

Two streams represent 1

1111111100000000111010101010110101010101011101010101110101000101100$\boxed{0}$

1111111100000000111010101010110101010001011101010101110101000111100$\boxed{1}$