HOMEWORK 3

QUESTION 1

According to the result in Problem 5(a) of Hw2, int C_1 is convex since C_1 is a convex set.

- First, we know there exists a $w^T \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $b \in \mathbb{R}$ which satisfies:

$$\begin{aligned} w^T x \leq b, \forall x \in intC_1 \\ w^T x \geq b, \forall x \in C_2 \end{aligned} \tag{1}$$

- We also know the fact that a point of C_1 is the limit of points in int C_1 (by the lemma on slide 34 of § 3).
- $f(x) = w^T x$ **is continuous on the closure of** C_1 .
- Then $\forall x' \in C_1$, we could find a series $\{x_k\}$ which converges to x'. And $\forall \epsilon > 0$, we could find a $\delta > 0$ s.t. if $|x' - x_k| < \delta$, then $|w^T x' - w^T x_k| < \epsilon$. Since ϵ can be arbitrarily small and $\{x_k\}$ converges to x', we could say

$$w^T x' \leq b \tag{2}$$

- Therefore the conclusion holds.

QUESTION 2

(a.)

if $\theta = 1$ or $\theta = 0$, it's a trivial case. Else:

$\forall x, y \in S_\alpha, x \neq y$ and $\forall \theta \in (0, 1)$ we have:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) < \alpha \tag{3}$$

$\therefore \theta x + \bar{\theta} y \in S_\alpha$

by definition, S_α is convex.

Similarly, if $\theta = 1$ or $\theta = 0$, it's a trivial case. Else:

$\forall x, y \in C_\alpha, x \neq y$ and $\forall \theta \in (0, 1)$ we have:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) \leq \alpha \tag{4}$$

$\therefore \theta x + \bar{\theta} y \in C_\alpha$

by definition, C_α is convex.

(b.)

The effective domain of f is $S = \{x : f(x) < +\infty\}$

if $\theta = 1$ or $\theta = 0$, it's a trivial case. Else:

$\forall x, y \in S, x \neq y$ and $\forall \theta \in (0, 1)$ we have:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) < +\infty \tag{5}$$

$\therefore \theta x + \bar{\theta} y \in S$

by definition, the effective domain of f is convex.

(c.)

For a certain X we set $\alpha = f(x^*)$, then we have $M = C_\alpha \cap X$. Let's prove this:

In one side,$\forall x^* \in M$, we have $x^* \in X$ and $f(x^*) \leq \alpha$

$\therefore x^* \in C_\alpha \cap X$

On the other side, $\forall x' \in C_\alpha \cap X$, we have $x' \in X$ and $f(x') \leq \alpha \leq f(x)$ $\forall x \in X$.

$\therefore x' \in M$

$\therefore M = C_\alpha \cap X$

since we have proven that C_α is convex in 2. (a), and we know the fact that the intersect of two convex sets is convex, we can derive that M is also convex.

QUESTION 3

$\because f$ is convex

$\therefore$ its domain dom$f = S$ is convex and $\forall x, y \in S$ and $\theta \in [0, 1]$, Jensen's inequality holds:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta} f(y) \tag{6}$$

First, we choose a line segment l_{xy}

Suppose $f(\theta_0 x + \bar{\theta}_0 y) < \theta_0 f(x) + \bar{\theta}_0 f(y)$ for some $\theta_0 \in (0, 1)$, we consider the case where $\theta \in (0, \theta_0)$.

Assume that there exists a series $\{\theta_k\}$ $s.t. f(\theta_k x + \bar{\theta}_k y) = \theta_k f(x) + \bar{\theta}_k f(y)$

In $\{\theta_k\}$ we choose θ as the nearest one to θ_0, then we could find a small enough ϵ $s.t.$

$$
\begin{aligned}
f(\theta_1 x + \bar{\theta}_1 y) < \theta_1 f(x) + \bar{\theta}_1 f(y) \\
f(\theta_2 x + \bar{\theta}_2 y) < \theta_2 f(x) + \bar{\theta}_2 f(y)
\end{aligned}
\tag{7}
$$

Where $\theta_1 = \theta - \epsilon, \theta_2 = \theta + \epsilon$.

$\because f$ is convex, $u_1 = \theta_1 x + \bar{\theta}_1 y$ and $u_2 = \theta_2 x + \bar{\theta}_2 y$ are all in S.

However,

$$
f(\frac{1}{2}u_1 + \frac{1}{2}u_2) = f(\theta x + \bar{\theta} y) = \theta f(x) + \bar{\theta} f(y) > \frac{1}{2}[f(u_1) + f(u_2)] \tag{8}
$$

Which contradicts condition (5)

$\therefore \theta \notin \{\theta_k\}$, and we can keep repeating the same operation through $\{\theta_k\}$ and find that $\forall \theta \in (0, \theta_0)$ s.t.

$$
f(\theta x + \bar{\theta} y) < \theta f(x) + \bar{\theta} f(y) \tag{9}
$$

And when $\theta \in (\theta_0, 1)$, the proof method is the same.

Therefore the conclusion holds.

QUESTION 4

According to the First-order condition for convexity, $\forall x, y \in \mathbf{dom}\, f$, f should satisfy:

$$
f(y) \geq f(x) + \nabla f(x)^T (y - x) \tag{10}
$$

$$
f(y) - \nabla f(y)^T (y - x) \leq f(x) \tag{11}
$$

From equations (9) and (10), we can immediately derive that:

$$
0 \geq (\nabla f(x)^T - \nabla f(y)^T)(y - x) \tag{12}
$$

Which means:

$$
\langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f \tag{13}
$$