Computing Moments

Generalization: Moments

- Suppose a stream has elements chosen from a set A of N values (say 1 to N)

- Let m_i be the number of times item i occurs in the stream

- The k^{th} *moment* is

$$\sum_{i \in A} (m_i)^k$$

Special Cases

$$\sum_{i \in A} (m_i)^k$$

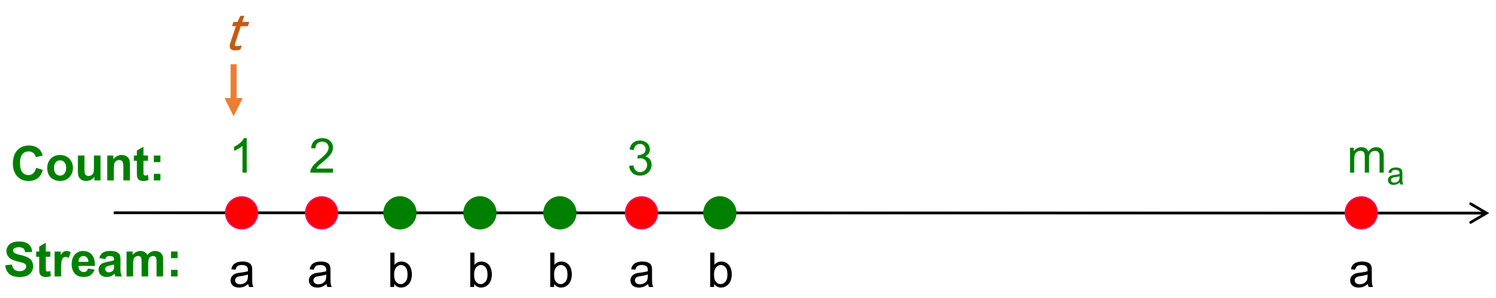
- **0th moment =** number of **distinct** elements (Flajolet-Martin)
- **1st moment =** count of the **numbers** of elements
- **2nd moment =** a measure of how uneven the distribution is (denoted as *S*)
  - E.g. **Stream of length 100, 11 distinct values**
    - Item counts: **10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9** $S = 910$
    - Item counts: **90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1** $S = 8{,}110$

AMS(Alon-Matias-Szegedy) Method

- Gives an **unbiased estimate** for the **2nd moment** $\boldsymbol{S} = \sum_i \boldsymbol{m}_i^2$

by keeping track of just **one variable** *X:*

  - *X.el* corresponds to a item *i*
    - Pick some random time *t* (*t*<*n*) to start, **equally likely** in a stream of length *n*
    - If at time *t* the stream have item *i*, we set *X.el = i*
  - *X.val* corresponds to the **count** of the chosen item *i*
    - Count *c* (*X.val = c*), the number of item *i* starting from the chosen time *t*



AMS(Alon-Matias-Szegedy) Method

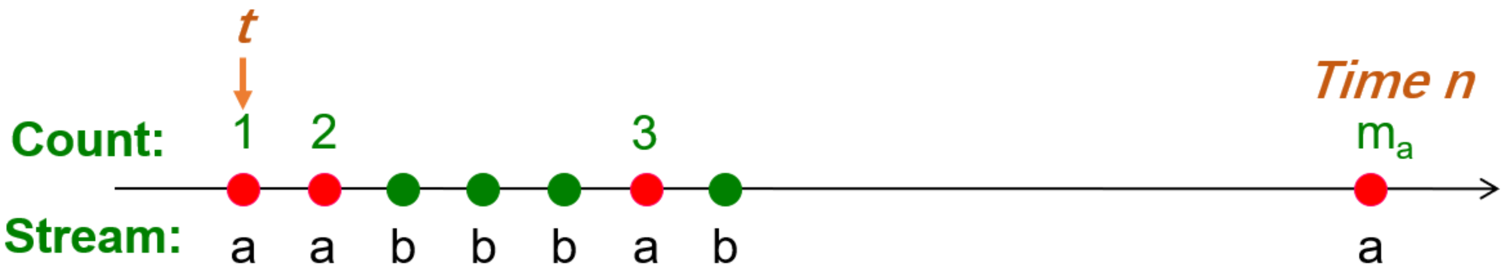
- **The estimate of the 2nd moment ($\sum_i m_i^2$) is:**

$$f(X) = n\,(2 \cdot c - 1)$$

  - Note, we will keep track of multiple Xs, (X_1, X_2,⋯ X_k) and our final estimate will be $S = 1/k \sum_j^k f(X_j)$

- **Let's prove** $\mathbf{E}[f(\mathbf{X})] = \sum_i (m_i)^2 = S$

Expectation Analysis



- c_t … number of times item at time t appears from time **t** to n ($c_1 = m_a$, $c_2 = m_a - 1$, $c_3 = m_b$)

m_i … total count of item i in the stream

- $$E[f(X)] = \frac{1}{n}\sum_{t=1}^{n} n(2c_t - 1)$$

$$= \frac{1}{n}\sum_{i \in A} n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$$

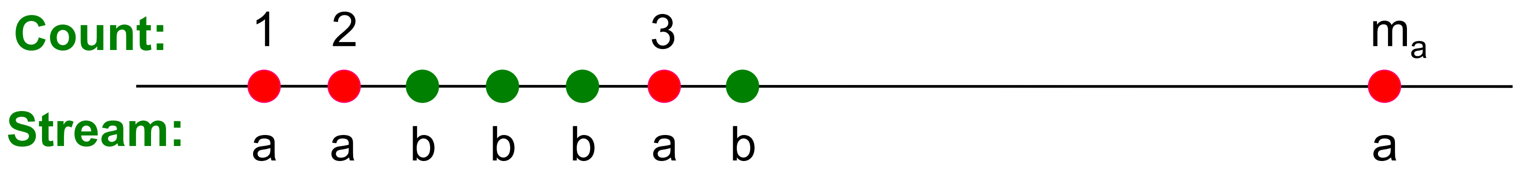
Group times by the value seen

Time t when the last i is seen ($c_t = 1$)

Time t when the penultimate i is seen ($c_t = 2$)

Time t when the first i is seen ($c_t = m_i$)

Expectation Analysis



- $E[f(X)] = \frac{1}{n} \sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$
  - calculation: $(1 + 3 + 5 + \cdots + 2m_i - 1) = \sum_{i=1}^{m_i} (2i - 1) = 2\frac{m_i(m_i+1)}{2} - m_i = (m_i)^2$
- **Then** $\mathbf{E}[\boldsymbol{f}(\mathbf{X})] = \frac{\mathbf{1}}{\boldsymbol{n}} \sum_{\boldsymbol{i}} \boldsymbol{n}\,(\boldsymbol{m_i})^{\mathbf{2}} = \boldsymbol{S}$

- **We have the second moment (in expectation)!**

Higher-Order Moments

- **For estimating k^{th} moment we essentially use the same algorithm but change the estimate:**
  - For **k=2** we used ***n*** **(2·c – 1)** (where **c=X.val**)
  - For **k=3** , can you try to find out what we use?
    - ***n*** **(3·c^2 – 3c + 1)**
- **Why?**
  - **For k=2:** Remember we had $(1+3+5+\cdots+2m_i-1)$ and we showed terms *2c-1* (for **c=1,⋯,m**) sum to *m^2*
    - $\sum_{c=1}^{m} 2c-1 = \sum_{c=1}^{m} c^2 - \sum_{c=1}^{m} (c-1)^2 = m^2$
    - So: $\mathbf{2c-1 = c^2-(c-1)^2}$
  - **For k=3:** **c^3 - $(c-1)^3$ = $3c^2$ - 3c + 1**
- **Generally:** Estimate $= n\,(c^k-(c-1)^k)$

Combining Samples

- **In practice:**
  - Compute $f(X) = n(2c - 1)$ for as many variables X as you can fit in memory
  - Average them in groups
  - Take median of averages

- **Problem: Streams never end**
  - We assumed there was a number n, the number of positions in the stream
  - But real streams go on forever, so n is a variable – the number of inputs seen so far

Streams Never End: Fixups

(1) The variables X have n as a factor –
keep n separately; just hold the count in X

(2) Suppose we can only store k counts.
We must throw some Xs out as time goes on:

- **Objective:** Each starting time t is selected with probability k/n
- **Solution: (fixed-size sampling)**
    - Choose the first k times for k variables
    - When the n^{th} element arrives ($n > k$), choose it with probability k/n
    - If you choose it, throw one of the previously stored variables **X** out, with equal probability

Summary of Streaming Algorithms

- Queries
  - Filtering a data stream
  - Queries over a sliding window
  - Estimating statistics

- Key techniques
  - Hashing functions
  - Approximation with sketch/summarization
  - Theoretical analysis