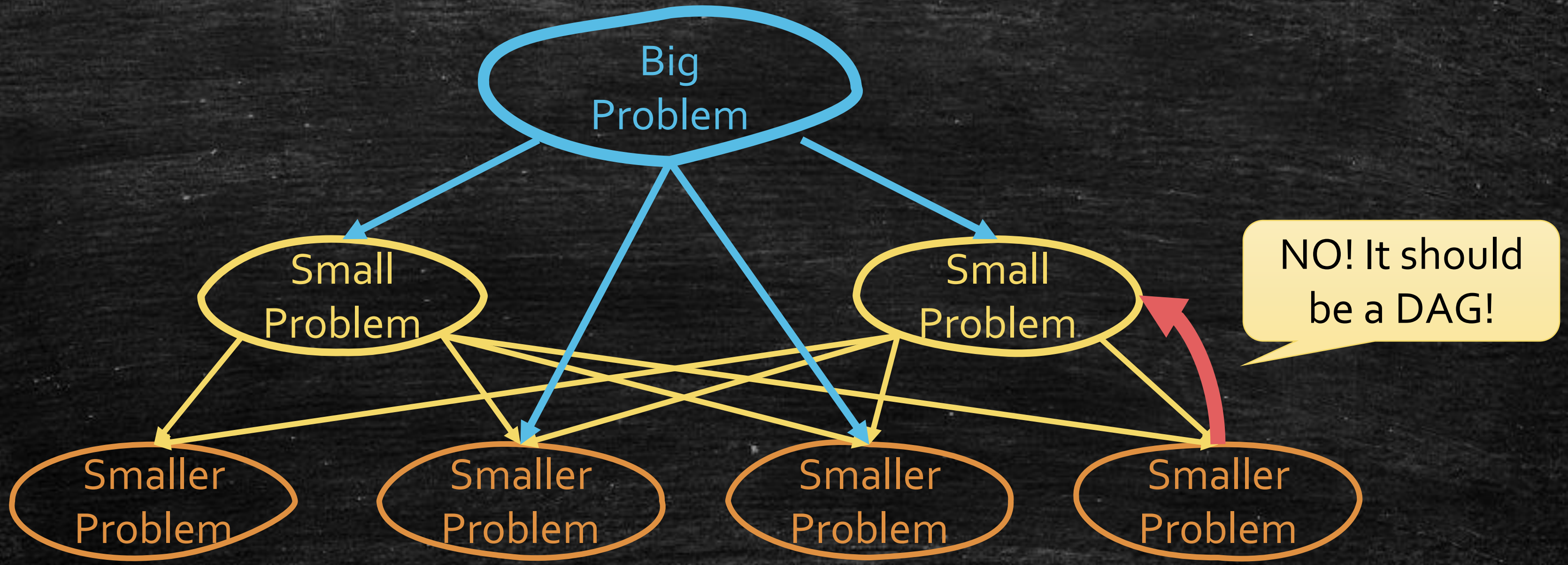
Dynamic Programming

DP improvement

Dynamic Programming



A simpler guideline

- Find subproblems.
- Check whether we are in a **DAG** and find the **topological order** of this DAG. (Usually, by hand.)
- Solve & store the subproblems by the topological order.

Recap the three examples

- Longest Increasing Sequence
  - Subproblem $LIS[i]$: the longest increasing sequence ended by a_i.
- Edit Distance
  - Subproblem $ED[i, j]$: the edit distance for $A[1..i]$ and $B[1..j]$.
- Knapsack
  - Subproblem $f[i, w]$: the maximum value we can get by using first i items and w budget.

How to find these subproblems

- Think from a recursive method
- LIS:
  - We want to find the LIS.
  - It may be ended by every a_i.
  - Solve LIS ended by a_i need to know all LIS ended by $a_{j<i}$.

How to find these subproblems

- Think from a recursive method
- Edit Distance
  - We want to know the Edit Distance.
  - We think how we align the last two character.
  - Different case make us go into different subproblems.
  - We these subproblems can be merged to $ED[i, j]$.

How to find these subproblems

- Think from a recursive method
- Knapsack
  - We want to know the maximum value.
  - We know that for each item, we have two choice: buy it or not.
  - Buy: we have $\mathrm{W} - c_i$ budget for other items.
  - Not Buy: we have W budget for other items.
  - Consider we recursive from a_n.
  - Subproblems can be merged to $f[i, w]$.

A Simple but Useful Data Structure.

Priority Queue

Largest Number in k Consecutive Numbers

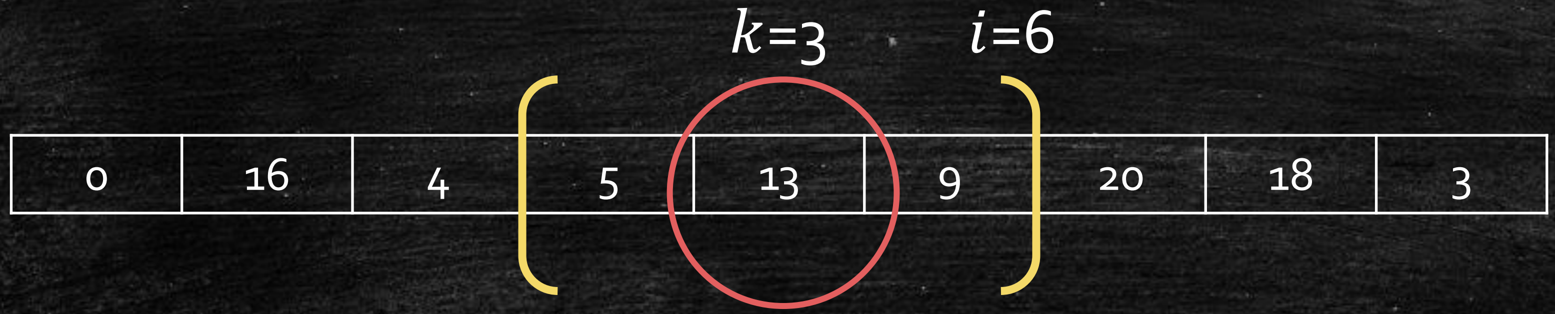
- **Input:** A sequence of numbers $a_1, a_2, \ldots, a_n$, and a number k.
- **Output:** The largest number in every k consecutive numbers.

k=3

0	16	4	5	13	9	20	18	3

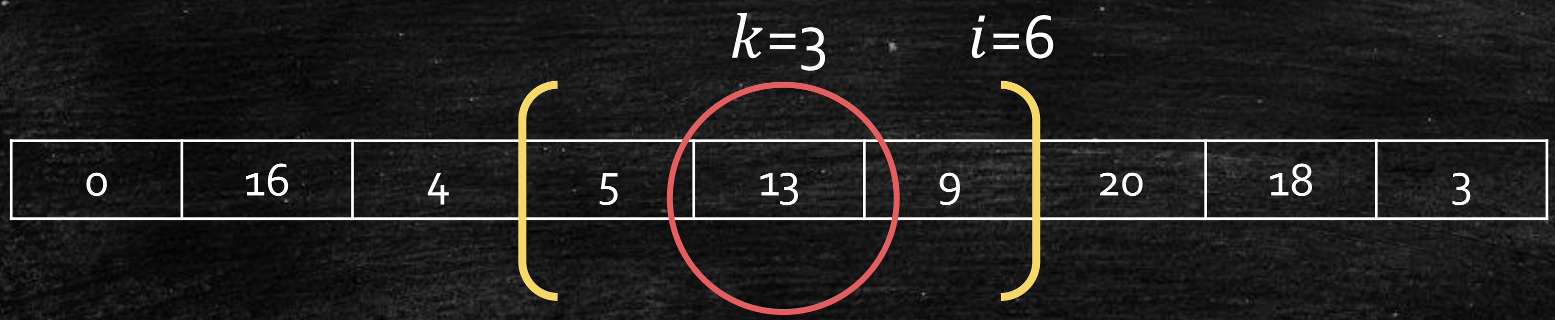
Subproblem Definitions

- $large[i]$: the largest number from a_{i-k+1} to a_i.
- Output: $large[k]$~$large[n]$.



Solving Subproblems

- $large[i]$: the largest number from a_{i-k+1} to a_i.
- Can you find a way to solve $large[i]$ by other subproblems?
  - Brute-force: $large[i] = \max_{j=i-k+1}^{i}\{a_i\}$.



Solving Subproblems

- $large[i]$: the largest number from a_{i-k+1} to a_i.
- Can you find a way to solve $large[i]$ by other subproblems?
  - Brute-force: $large[i] = \max_{j=i-k+1}^{i}\{a_i\}$.
  - Tips: from $large[j], j < i$.

k=3 i=6

0	16	4	5	13	9	20	18	3

Recall Knapsack

- What we always do before:
- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.
- ~~Use $g[i]$ to store how much budge $f[i]$ uses.~~

How to solve $f[i]$ by $f[j < i]$?

$f[i]$	5	10	13	16	21	30	?

We know $f[j]$ but we do not know how much budget it uses!

Key problem: Subproblem definition does not contain enough information!

What kind of information do we need now?

Observation

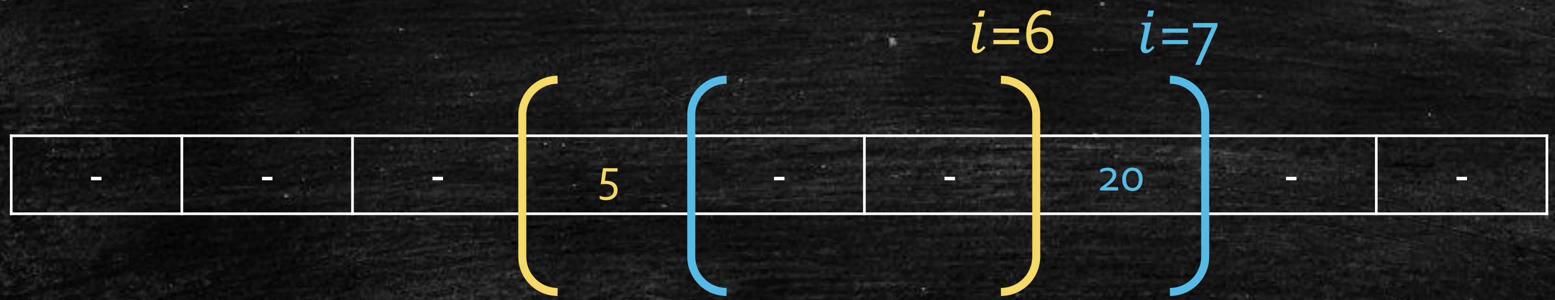
- Compare two $large[i]$ and $large[i-1]$.
- Difference
  - One entering number: 20
  - One outgoing number: 5
  - Question: how they affect the largest number?

i=6 i=7

0	16	4	5	13	9	20	18	3

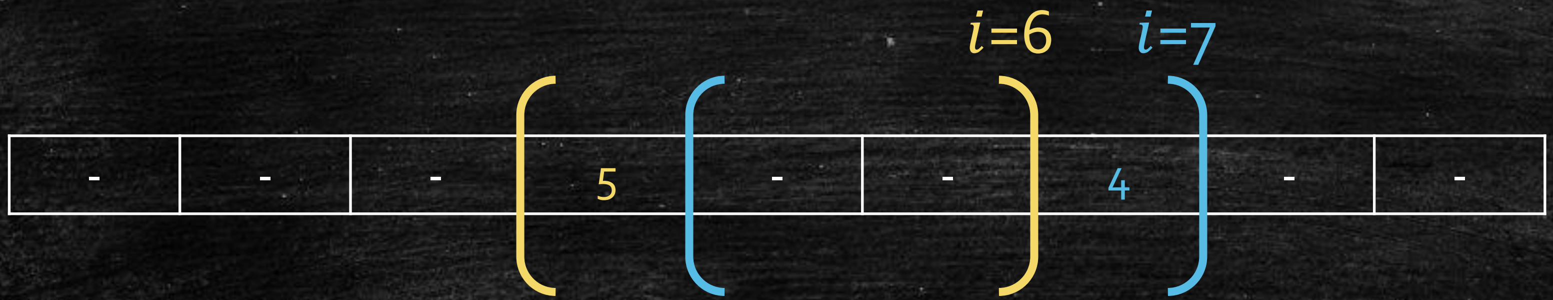
How they affect the largest number

- Difference
  - One entering number: 20
  - One leaving number: 5
  - Question: how they affect the largest number?
  - Case 1: the entering number is the new largest!



How they affect the largest number

- Difference
  - One entering number: 20
  - One leaving number: 5
  - Question: how they affect the largest number?
  - Case 2: the leaving number is the previous largest!

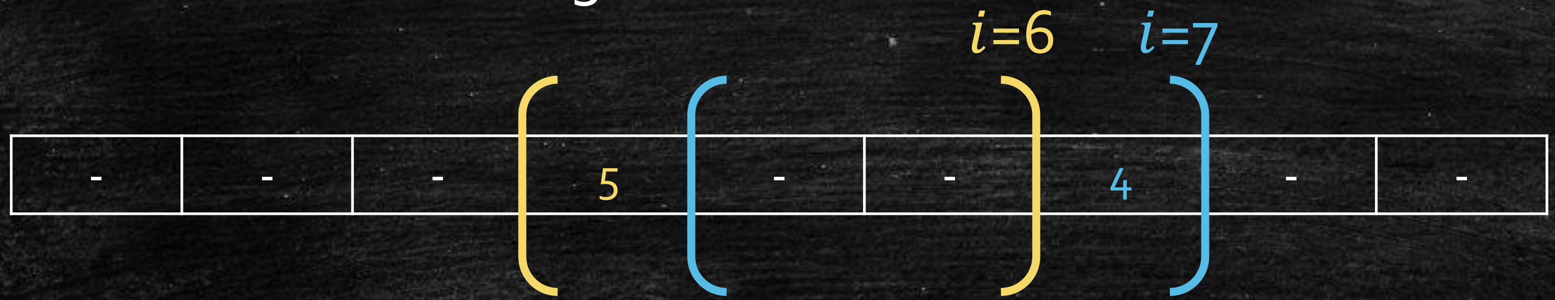


Key problem: We should know what is the previous second largest number.

Ok, let us record it!

How they affect the largest number

- Difference
  - One entering number: 20
  - One leaving number: 5
  - Question: how they affect the largest number?
  - Case 3: the leaving number is the previous second largest!
  - What is the second largest now?

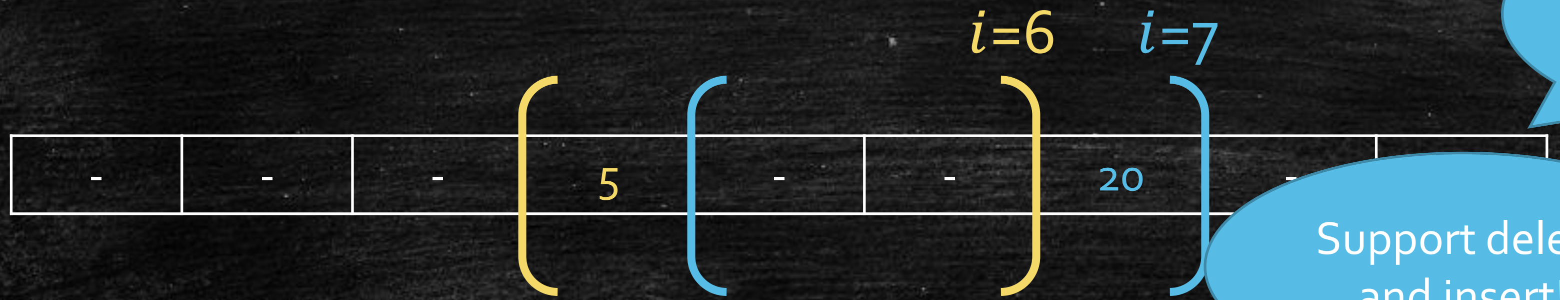


Key problem: We should know what is the previous third largest number.

Ok, let us record it.....

Summarize

- Difference
  - One entering number: 20
  - One leaving number: 5
  - Question: how they affect the largest number?



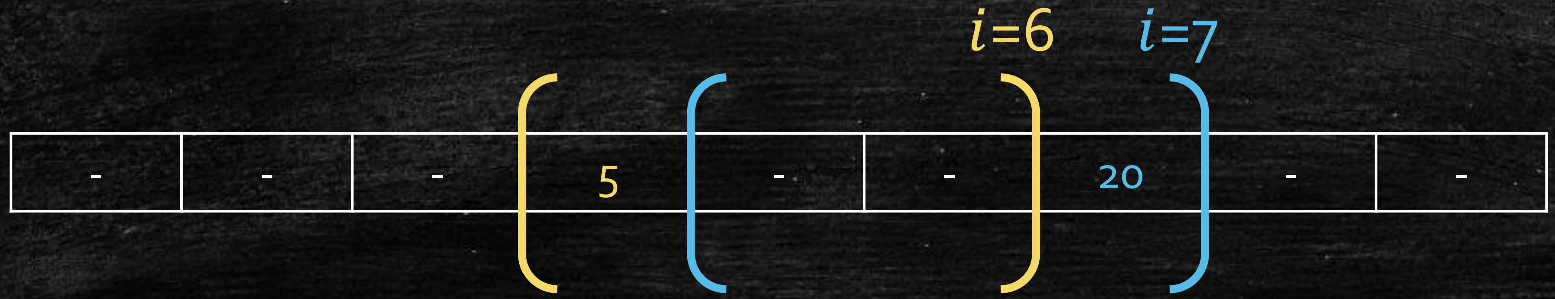
Heap!
$O(n \log k)$!

Support delete
and insert!

Summarize: We should maintain
a data structure!

Let us think more!

- New Subproblem: Solving the Heap of $a_{i-k+1} \sim a_i$.
  - Delete (Update & PopMax)
  - Insert
  - FindMax
  - $O(n \log k)$!
- Is Heap too powerful for this problem?
  - We delete and insert only based on the index!



A new Subproblem!

- Think again: why we need the heap?
  - We need two know who is the largest.
  - We need to know who is the **potential largest**.
  - We need to update the **potential largest list**.
- Do we have a better way to maintain this **potential largest list**?
  - Heap views all k numbers as **potential largest**.

Observation

- Who can be the **potential largest** number?

5

13

9

i=6

0	16	4	5	13	9	20	18	3

Observation

- Who can be the **potential largest** number?

5

13

9

5 is not a potential largest number because 5 is older than 13 and 5<13.

9 is a potential largest number although 13>9 because 9 is younger.

i=6

0	16	4	5	13	9	20	18	3

Key Observation: the potential largest list can be smaller than k.

Potential Largest List

- **Potential Largest List** (PLL)
  - $PLL[i]$: the Potential Largest List for $a_{i-k+1} \sim a_i$.
  - At most k numbers.
  - Sorted by the index.
  - $i - k + 1 \leq \text{Index} \leq i$

$PLL[20]$

Smaller

a_5	a_6	a_8	a_{12}	a_{13}	a_{15}	a_{16}	a_{19}	a_{20}

$\leq k$=15

Key Property:
$a_i \geq a_j$ if $i < j$.

How to maintain PLL?

- How to solve $PLL[i = 21]$ by $PLL[i - 1 = 20]$?
- First, kick the number if $index < i - k + 1 = 6$.

$PLL[20]$

Smaller

a_5	a_6	a_8	a_{12}	a_{13}	a_{15}	a_{16}	a_{19}	a_{20}

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How to maintain PLL?

- How to solve $PLL[i = 21]$ by $PLL[i - 1 = 20]$?
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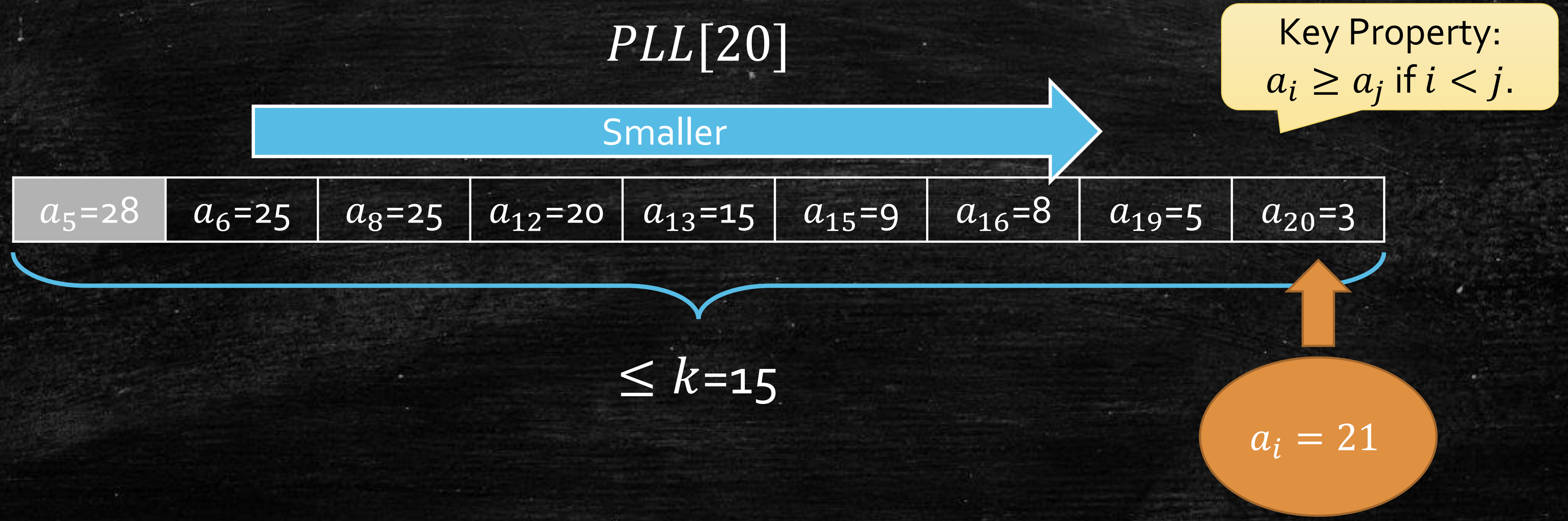
- Second, kick numbers by $a_{i=21}$.



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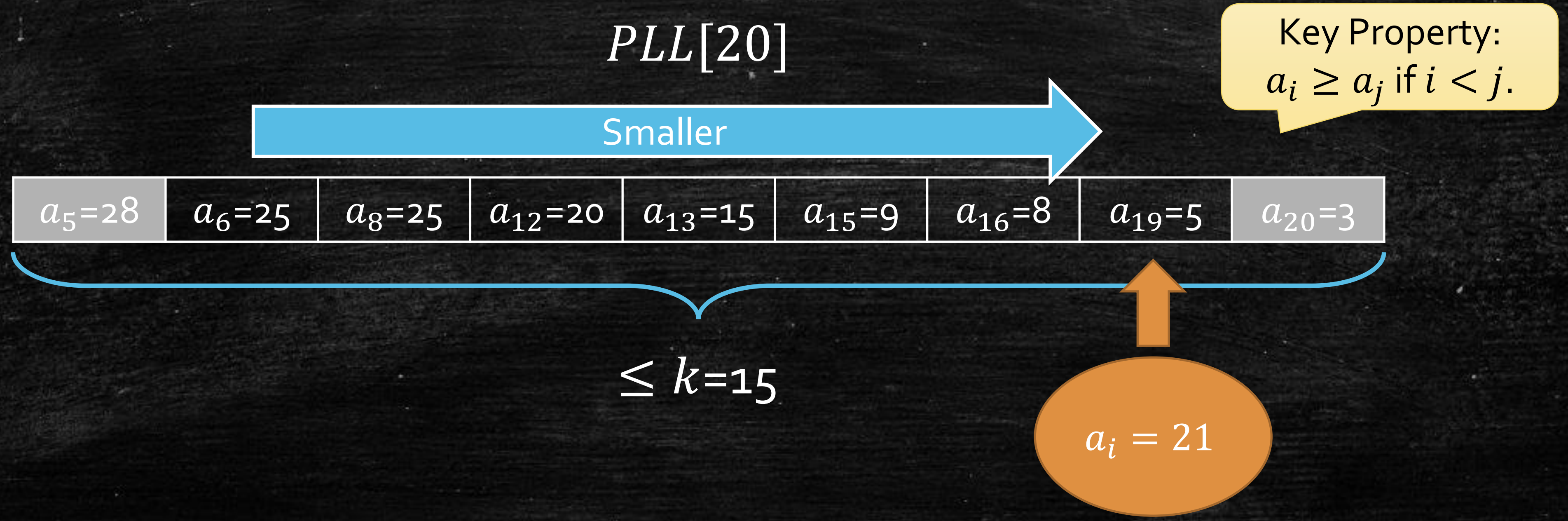

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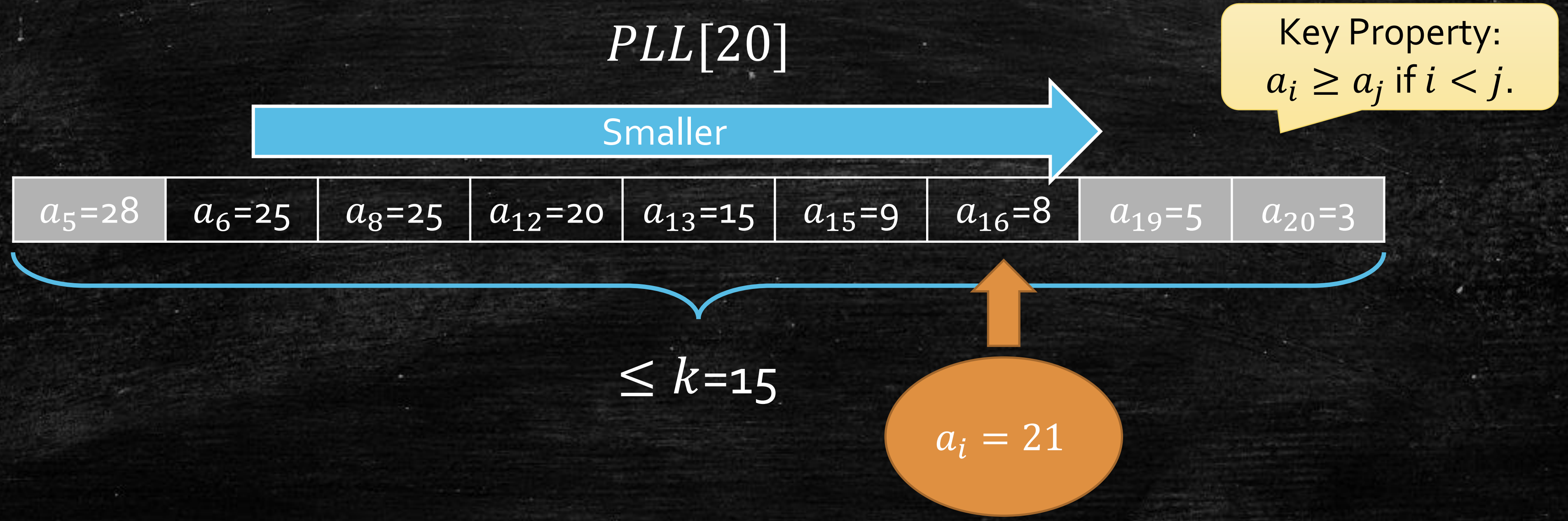

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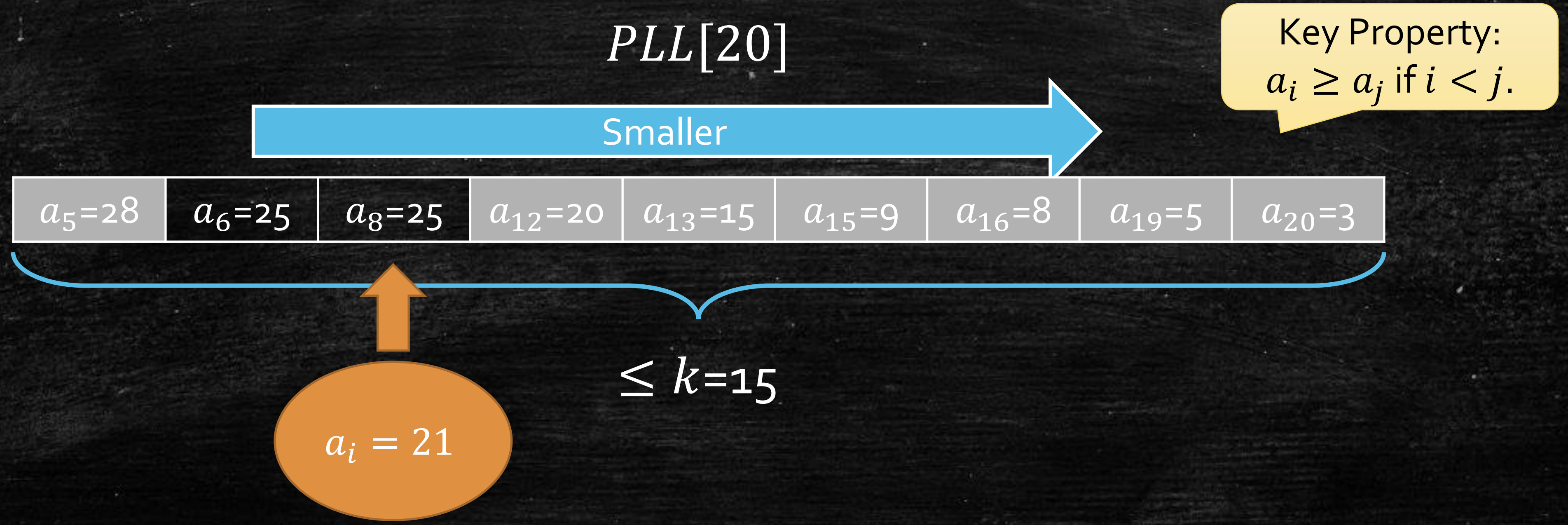

How to maintain PLL?

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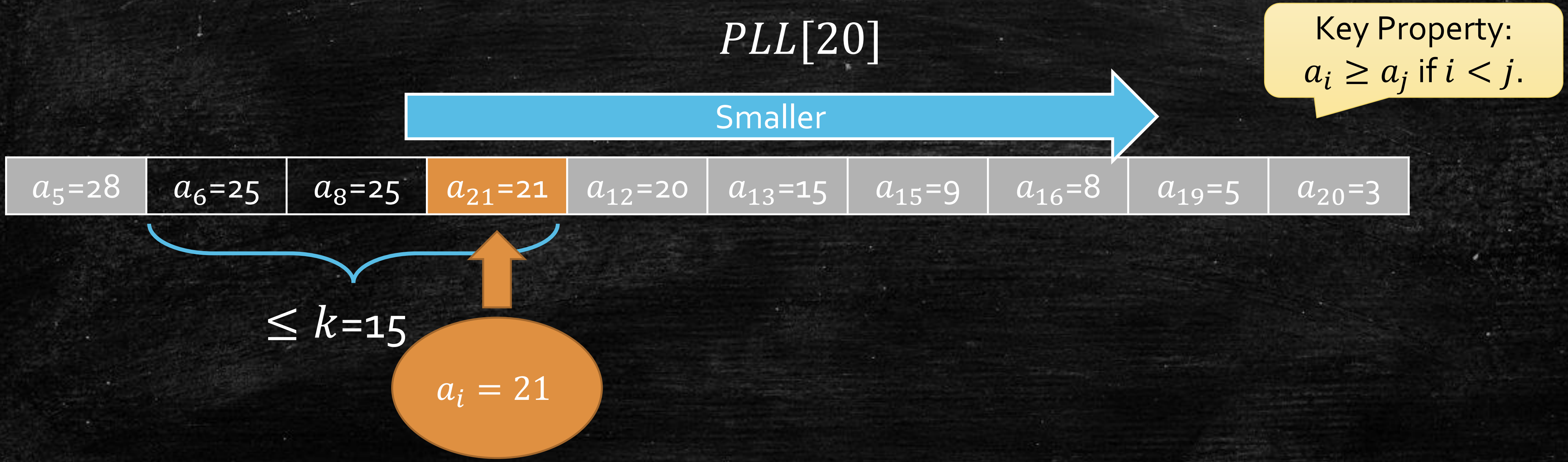

Program: updating priority queue

Updating Priority Queue

function $updating(a[1..n], i, k, PLL)$

  If $PLL.front.index \leq i - k$

    $PopFront(PLL)$

  while $PLL.back.value \leq a[i]$

    $PopBack(PLL)$

  $PushBack(PLL, (index = i, value = a[i]))$

Largest Number in range k

function $largest(a[1..n], k)$

  $PLL = NULL$

  for $i = 1\ to\ n$

    $updating(a, i, k, PLL)$

    output $PPL.front.$

Running Time

Updating Priority Queue

function $updating(a[1..n], i, k, PLL)$

  If $PLL.front.index \leq i - k$

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  while $PLL.back.value \leq a[i]$

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  $PLL = NULL$

  for $i = 1$ to n

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    output $PPL.front.$

$a_i = 21$

Charge to a_5.

a_5=28	a_6=25	a_8=25	a_{12}=20	a_{13}=15	a_{15}=9	a_{16}=8	a_{19}=5	a_{20}=3

$\leq k$=15

Running Time

Updating Priority Queue

function $updating(a[1..n], i, k, PLL)$
  If $PLL.front.index \leq i - k$
    $PopFront(PLL)$
  $\boxed{\textbf{while } PLL.back.value \leq a[i] \quad PopBack(PLL)}$
  $PushBack(PLL, (index = i, value = a[i]))$

Largest Number in range k

function $largest(a[1..n], k)$
  $PLL = NULL$
  for $i = 1$ to n
    $updating(a, i, k, PLL)$
    output $PPL.front.$

$a_i = 21$

a_5=28	a_6=25	a_8=25	a_{12}=20	a_{13}=15	a_{15}=9	a_{16}=8	a_{19}=5	$\boxed{a_{20}=3}$

$\leq k$=15

Charge to a_{20}.

Running Time

Updating Priority Queue

function $updating(a[1..n], i, k, PLL)$

If $PLL.front.index \leq i - k$

$PopFront(PLL)$

$\boxed{\textbf{while}\ PLL.back.value \leq a[i]}$

$PopBack(PLL)$

$\boxed{PushBack(PLL, (index = i, value = a[i]))}$

Largest Number in range k

function $largest(a[1..n], k)$

$PLL = NULL$

for $i = 1\ to\ n$

$updating(a, i, k, PLL)$

output $PPL.front.$

$a_i = 21$

a_5=28	a_6=25	a_8=25	a_{21}=21	a_{12}=20	a_{13}=15	a_{15}=9	a_{16}=8	a_{19}=5	a_{20}=3

$\leq k$=15

Charge to a_{21}.

Running Time

- The cost of n times updating has been charged to numbers!
- Each number
  - Charged **once** when it is **popped out**.
  - Charged **once** when it is **pushed in**.
- Totally: $O(n)$.

Priority Queue Helps DP

Priority Queue

Longest Increasing Sequence Revisit

- **Input:** A sequence $a_1, a_2, \dots, a_n$.
- **Output:** the Longest Increasing Subsequence (LIS)
  - $a_{i_1} < a_{i_2} < a_{i_3} \dots < a_{i_k}$
  - $i_1 < i_2 < i_3 \dots < i_k$

1	5	13	2	6	24	15	23	2	16

Do you feel that we can improve?

Is there any monotone thing?

Previous Transfer

- $lis[i] = \max_{a_j < a_i, j < i} \{lis[j] + 1\}$
- Definition: **Potential Prefix**
  - The set of a_j that is possible to be the best prefix of future numbers.

$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

Who are the Potential Prefix?

Previous Transfer

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$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

It is not because $a[i] > a[j]$ and $lis[i] = lis[j]$

Who are the Potential Prefixes?

New Potential List

- $Sm[len]$: the **smallest ended number** for an increasing subsequence with length len.
- Remark: it is enough to record all Potential Prefixes (length and number).

$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

New Potential List

- $Sm[len]$: the **smallest ended number** for an increasing subsequence with length len.
- Remark: it is enough to record all Potential Prefixes (length and number).

i

$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

Larger

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?
- Difference between $i-1$ and i?
  - a_i comes in.
  - It may **become** a potential prefixes and **kick** some potential prefixes.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

					i					
$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

$a_i = 5$

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

						i				
$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

$a_i = 5$

Case 1: $a_i > sm[i-1, len]$

Case 2: $a_i \le sm[i-1, len]$

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

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$a_i = 5$

Case 1: $a_i > sm[len]$

- it can create a longer LIS.
- it can not update $sm[len]$.

Case 2: $a_i \leq sm[len]$

- It may update $sm[len]$
- it can not create a longer LIS.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

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- It may update $sm[len]$
- it can not create a longer LIS.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

i

$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

$a_i = 5$

Case 1: $a_i > sm[len]$

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	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

i

$a[i]$	1	5	13	2	6	5	15	23	2	16
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$a_i = 5$

Case 1: $a_i > sm[len]$

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	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

i

$a[i]$	1	5	13	2	6	5	15	23	2	16
$lis[i]$	1	2	3	2	3	-	-	-	-	-

$a_i = 5$

Case 1: $a_i > sm[len]$

- it can create a longer LIS.
- it can not update $sm[len]$.

Because we move to here.

Case 2: $a_i \leq sm[len]$

- It **must** update $sm[len]$.
- it can not create a longer LIS.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	6	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

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	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	a_i=5	-	-	-	-	-	-

Updating $sm[len]$!

- How to update $sm[len]$ (Potential Prefixes)?

						i				
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$a_i = 5$

Case 1: $a_i > sm[len]$

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Because we move to here.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[len]$	0	1	2	a_i=5	-	-	-	-	-	-

The DP Algorithm

- Initialize $sm[0] = 0$
- For $i = 1$ to n
  - Update $sm[len]$ by a_i
  - **It requires $O(\max\{len\} = i)$!**
  - **Remark: now we do not kick everything we pass.**
- Output the largest len such that $sm[len] \neq$"-".

Recap The Updating

- We need to find the largest len such that $a_i > sm[i-1, len]$.
- Then we update: $sm[i, len+1] = a_i$.

$a_i = 5$

Case 1: $a_i > sm[i-1, len]$

- it can create a longer LIS.
- it can not update $sm[i, len]$.

Case 1: $a_i \le sm[i-1, len]$

- It **must** update $sm[i, len]$
- it can not create a longer LIS.

	len=0	len=1	len=2	len=3	len=4	len=5	len=6	len=7	len=8	len=9
$sm[i-1, len]$	0	1	2	a_i=5	-	-	-	-	-	-

Larger

How to do it efficiently?

Yes! Binary Search!

Now it is better!

- Plan
  - Initialize $sm[0,0] = 0$
- Solve $sm[i, len]$ from $sm[i-1, len]$ by a_i.
  - It requires $O(\log(len)) = O(\log n)$.
- Output the largest len such that $sm[n, len] \neq "\text{-}"$.
- Totally $O(n\log n)$.

The DP Algorithm

- Initialize $sm[0] = 0$
- For $i = 1$ to n
  - Update $sm[len]$ with a_i by binary search.
  - ~~**It requires $O(\max\{maxlen\} = i)$**~~**!**
  - **Remark: now we do not kick everything we pass.**
  - It requires $O(\log(maxlen)) = O(\log n)$.
- Output the largest len such that $sm[len] \neq$"-".

Priority Queue Can Be Stronger

Minimizing Manufacturing Cost

- **Input:** A sequence of items with cost $a_1, a_2, \dots, a_n$.
- Need to Do:
  - Manufacture these items.
  - Operation $\mathrm{man}(l, r)$: manufacture the items from l to r.
  - $cost(l, r) = C + \left(\sum_{i=l}^{r} a_i\right)^2$.
- **Output:** The **minimum** cost to manufacture all items.

Discussion

- Cost function: $cost(l,r) = C + (\sum_{i=l}^{r} a_i)^2$.
- Cost function: $cost(l,r) = C + \sum_{i=l}^{r} a_i$.
- Cost function: $cost(l,r) = C + (\sum_{i=l}^{r} a_i)^2$, with $C = 0$.
- Only the first one need to optimize!

Define subproblems

- $f[i]$: the minimum cost for manufacturing item 1 to i.
- How to solve $f[i]$?

0	16	4	5	13	9	20	18	3

i

Solving $f[i]$

- $f[i]$: the minimum cost for manufacturing item 1 to i.
- How to solve $f[i]$?
- We can manufacture item i alone.

0	16	4	5	13	9	20	18	3

i

Solving $f[i]$

- $f[i]$: the minimum cost for manufacturing item 1 to i.
- How to solve $f[i]$?
- We can also manufacture i along with an interval.
- $f[i] = \min\limits_{j<i} f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2$

$f[j]$

0	16	4	5	13	9	20	18	3
	j		i					

$cost(j+1, i)$

DP algorithm

- Define $f[0] = 0$.
- Solve $f[i]$ from 1 to n, and output $f[n]$.
- $f[i] = \min_{j<i} f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2$.

$O(n^2)$

$f[j]$

0	16	4	5	13	9	20	18	3

j i

$cost(j+1, i)$

The Potential Idea Again!

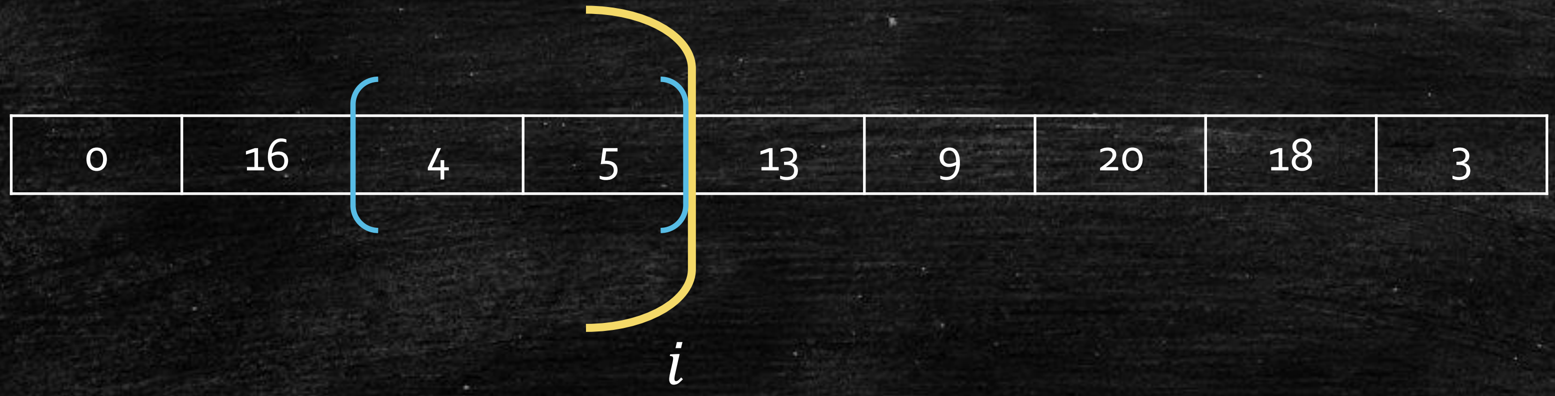
- Question: Can every j be a potential prefix for the future?

0	16	4	5	13	9	20	18	3

i

The Potential Idea Again!

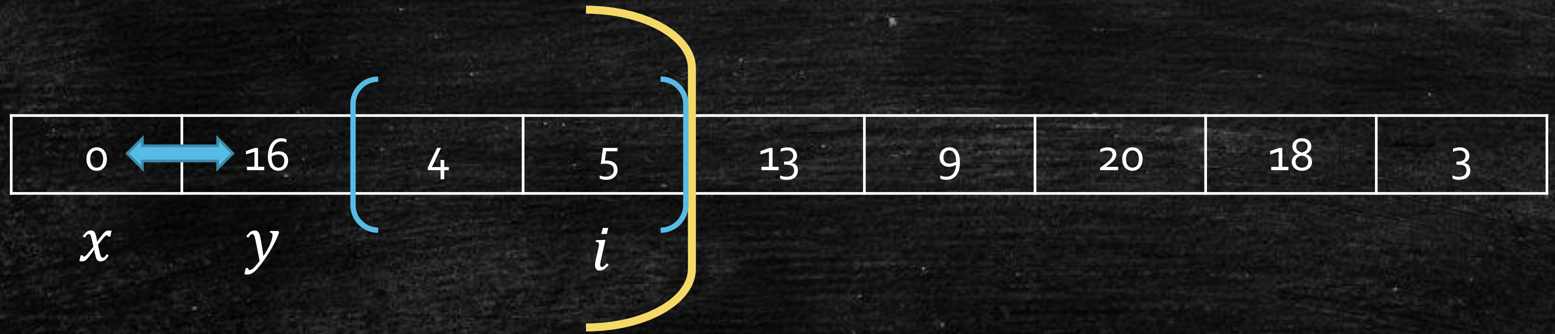
- Question: Can every j be a potential prefix for the future?
- Trade-off
  - Smaller j is better for paid cost.
  - Larger j is better for future cost.



Let us do some math!

General Question

- $f[i] = \min\limits_{j<i} f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2$.
- When $j = y$ is better than $j = x$ when calculate $f[i]$?



Math Time!

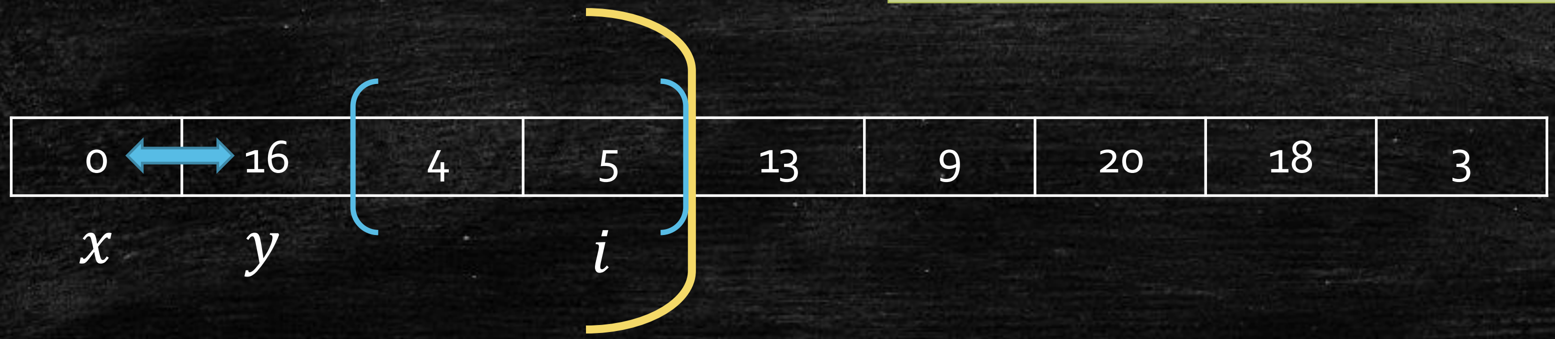
- $f[i] = \min_{j<i} f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2.$
- When $j = y$ is better than $j = x$ when calculate $f[i]$?
- $f[x] + C + \left(\sum_{k=x+1}^{i} a_k\right)^2 > f[y] + C + \left(\sum_{k=y+1}^{i} a_k\right)^2$

0	16	4	5	13	9	20	18	3
x	y		i					

Math Time!

- $f[i] = \min\limits_{j<i} f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2.$
- When $j = y$ is better than $j = x$ when calculate $f[i]$?
- Let $s(i) = \sum_{k=1}^{i} a_k$

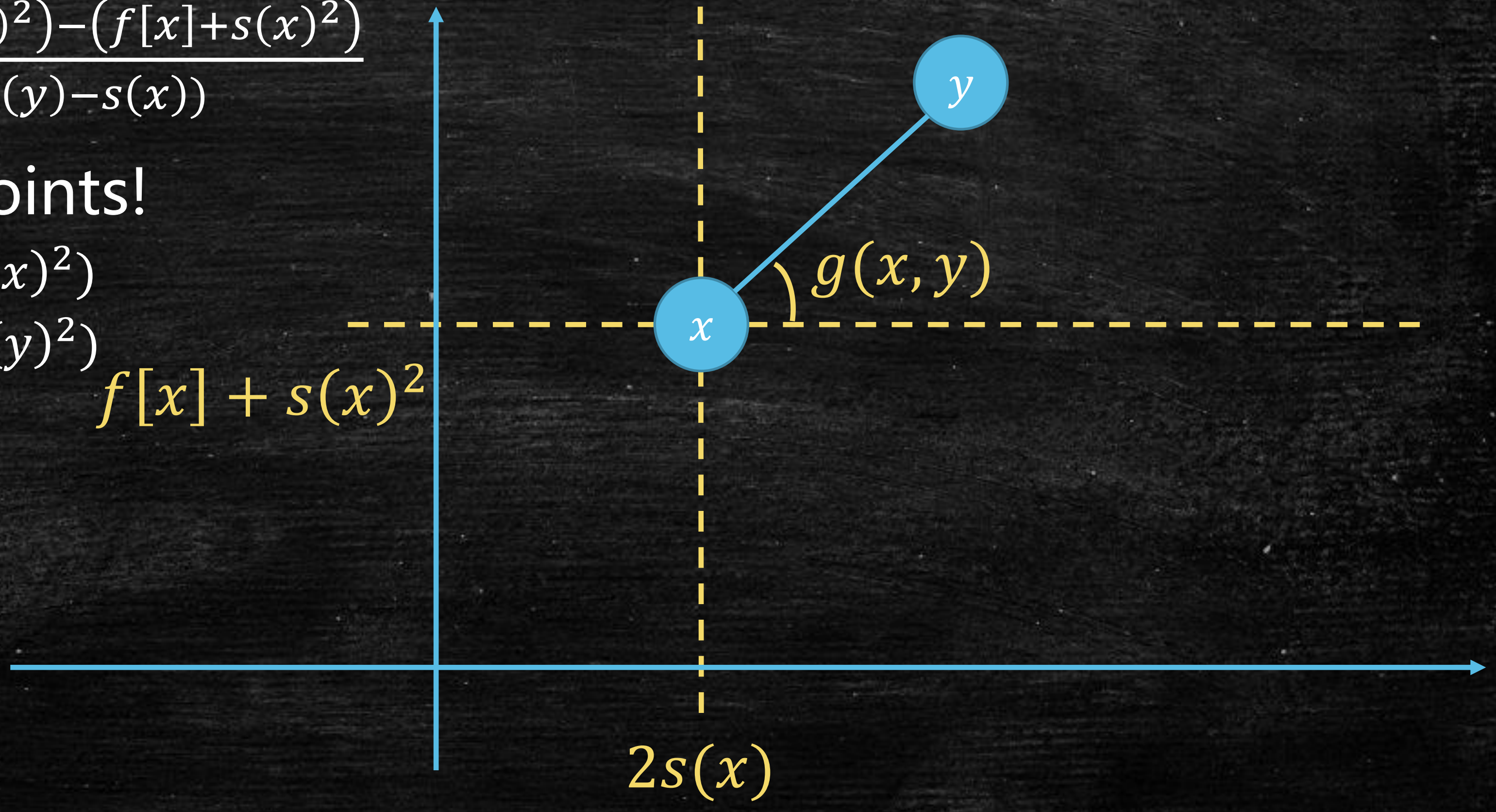
$$f[x] + C + (s(i) - s(x))^2 > f[y] + C + (s(i) - s(y))^2$$
$$f[x] - f[y] > \left(s(i) - s(y)\right)^2 - \left(s(i) - s(x)\right)^2$$
$$= s(y)^2 - s(x)^2 - 2s(i)(s(y) - s(x))$$
$$\frac{(f[y]+s(y)^2)-(f[x]+s(x)^2)}{2(s(y)-s(x))} < s(i)$$



Math Time!

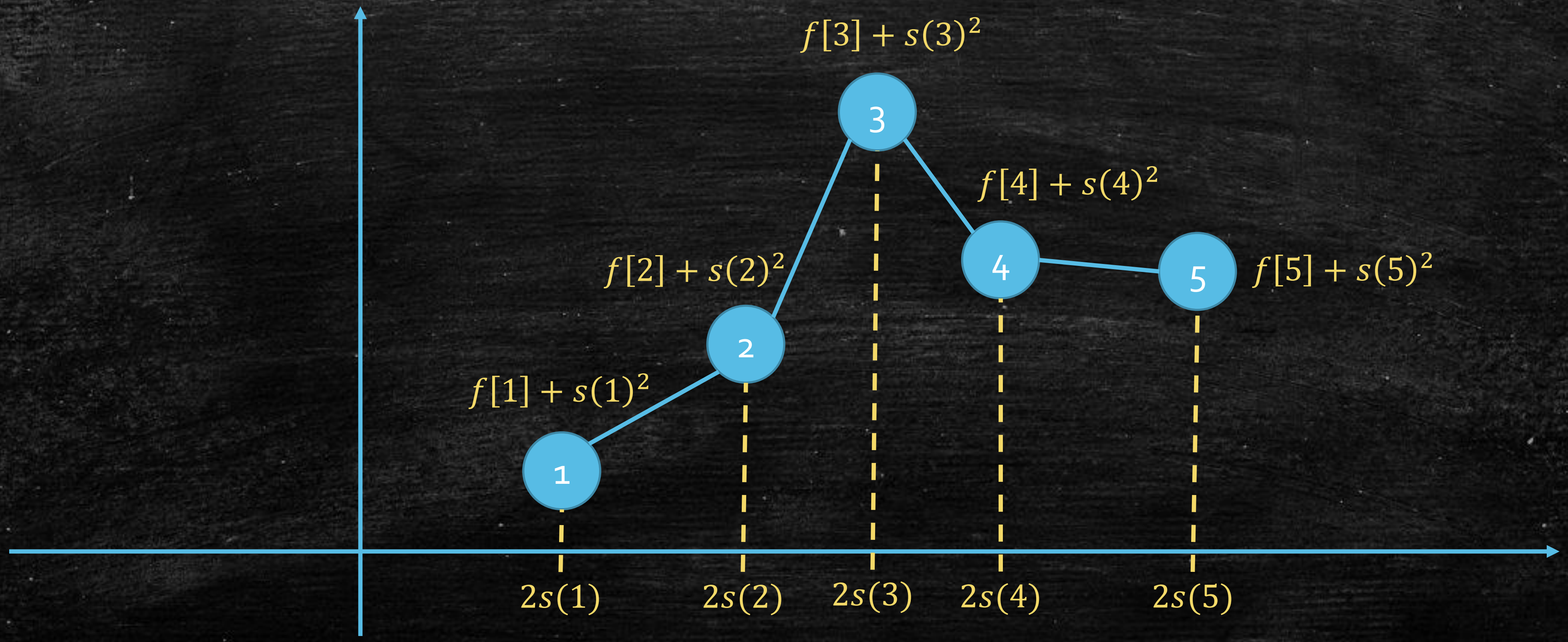
- $\frac{(f[y]+s(y)^2)-(f[x]+s(x)^2)}{2(s(y)-s(x))} < s(i)$
- $g(x,y) = \frac{(f[y]+s(y)^2)-(f[x]+s(x)^2)}{2(s(y)-s(x))}$
- View it as two points!
  - x: $(2s(x), f[x]+s(x)^2)$
  - y: $(2s(y), f[y]+s(y)^2)$

y is better than x for i means the gradient of x->y: smaller than $s(i)$.



Put everything on the graph

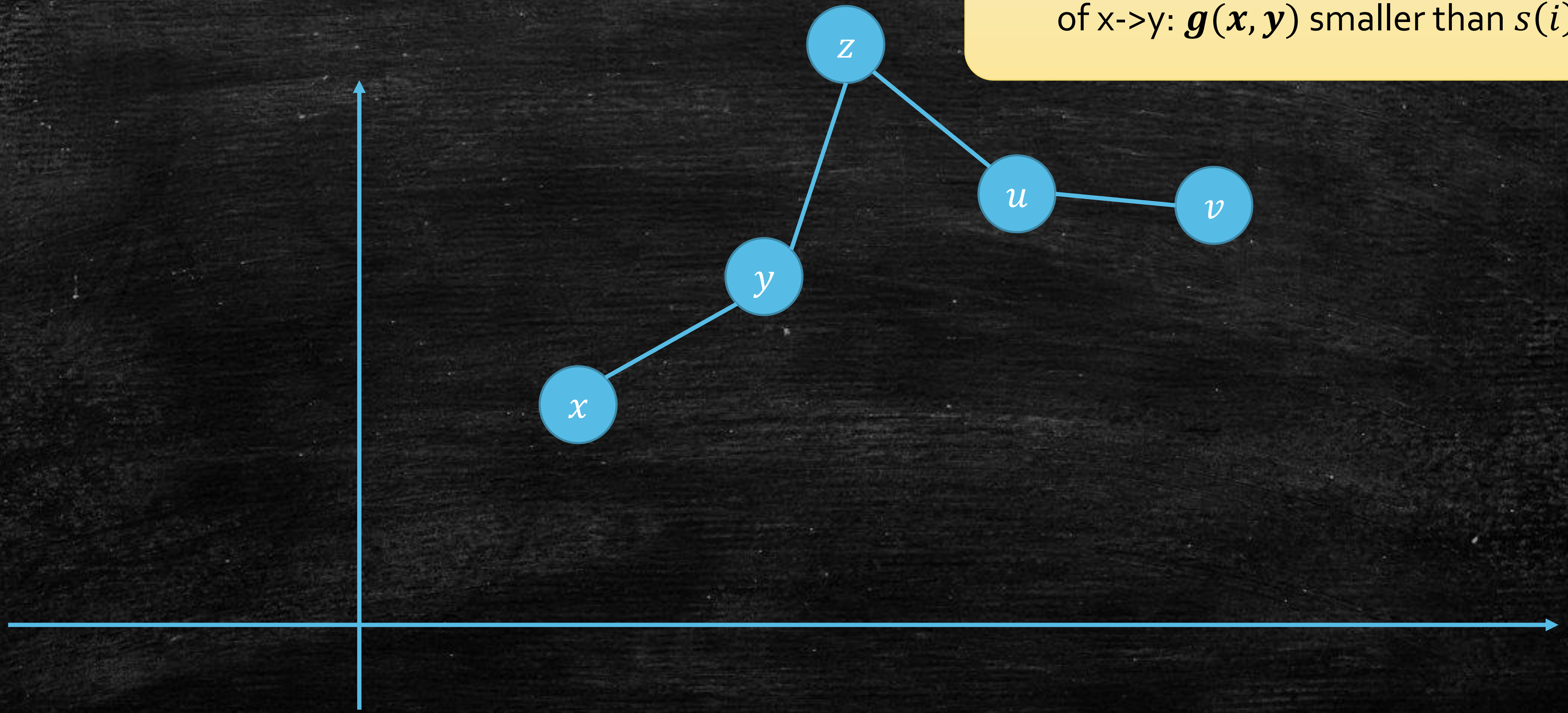
$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$				
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



Who can be kicked out?

Who can be kicked out?

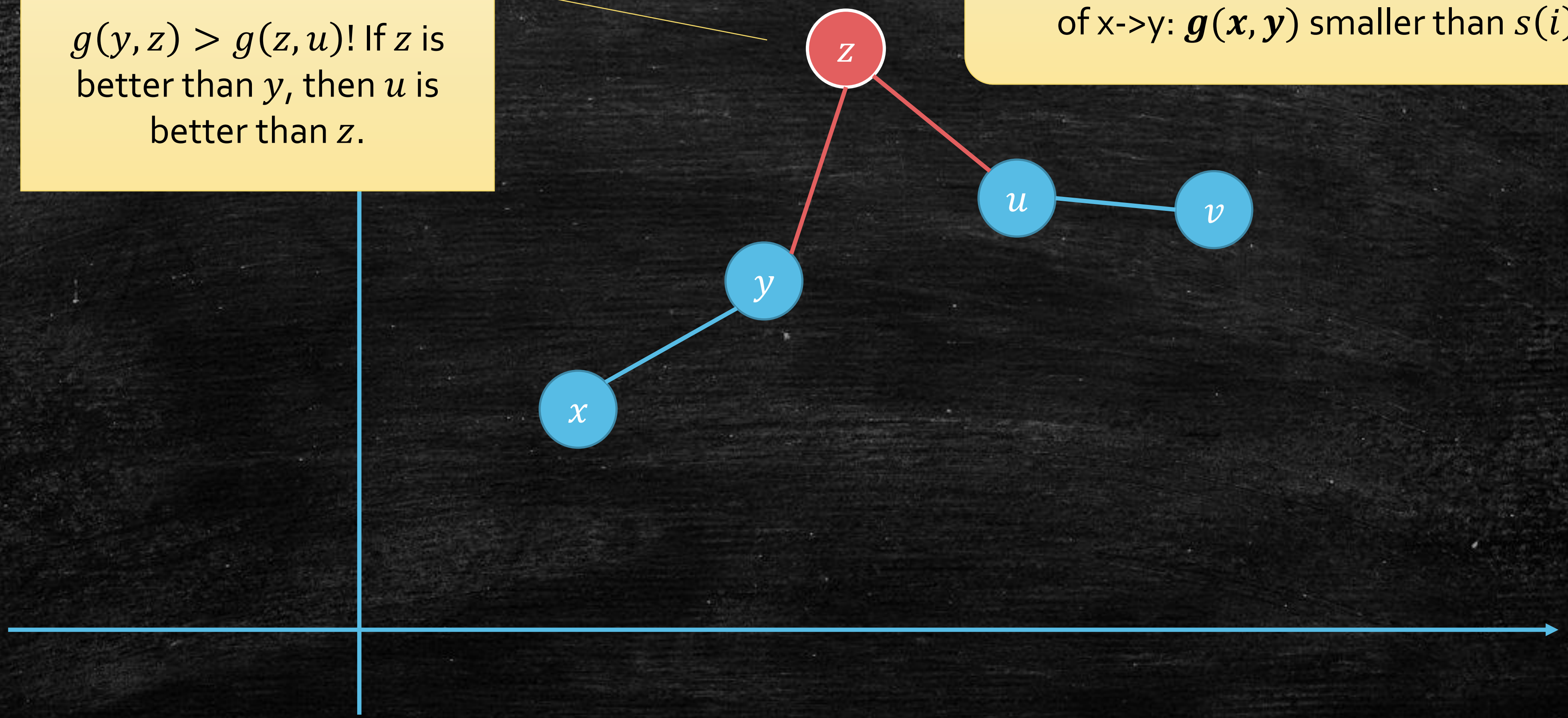
y is better than x for i means the gradient of x->y: $\boldsymbol{g(x,y)}$ smaller than $s(i)$.



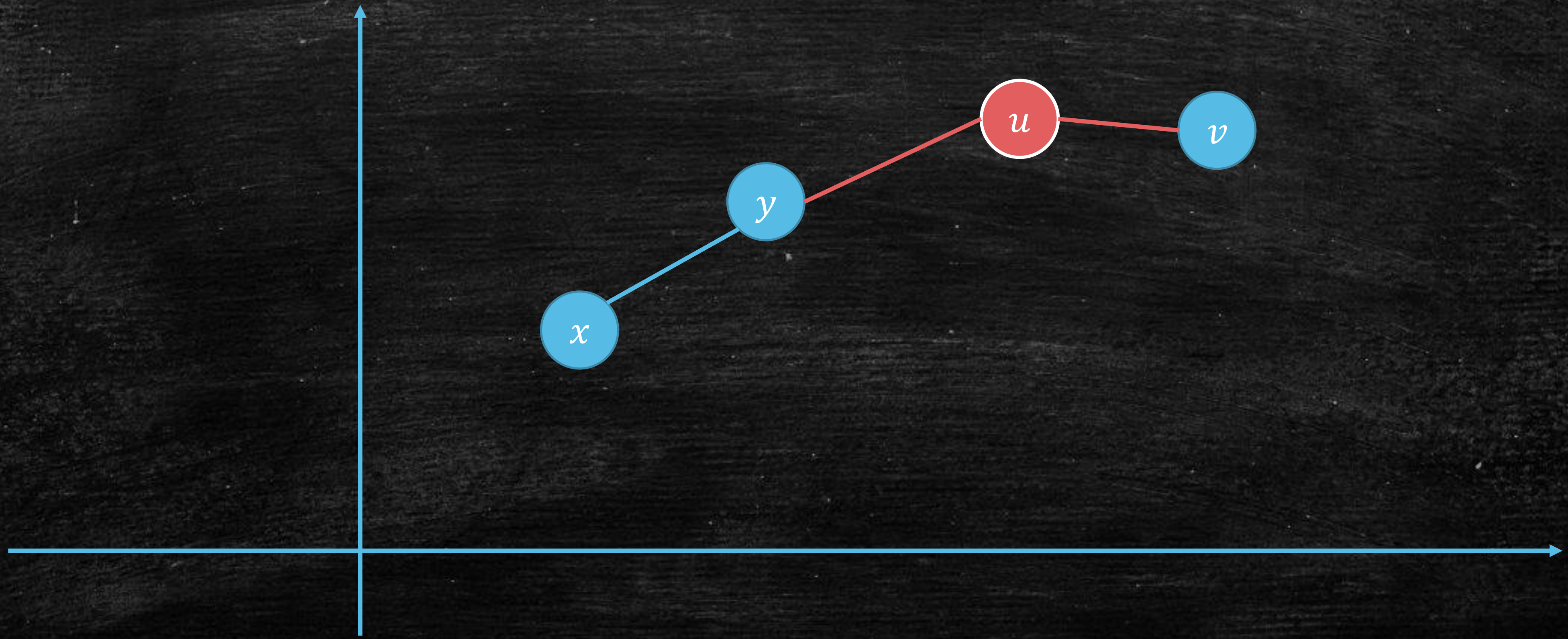
Who can be kicked out?

$g(y,z) > g(z,u)$! If z is better than y, then u is better than z.

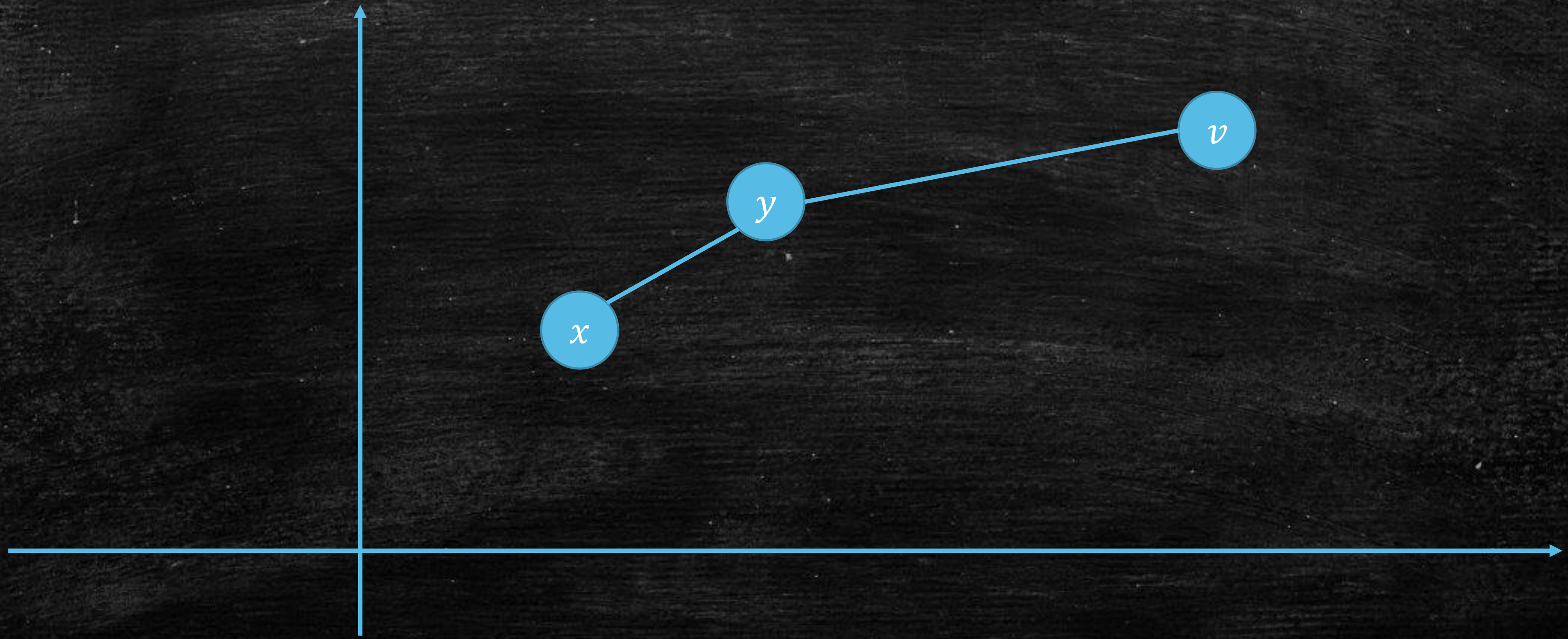
y is better than x for i means the gradient of x->y: $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$ smaller than $s(i)$.



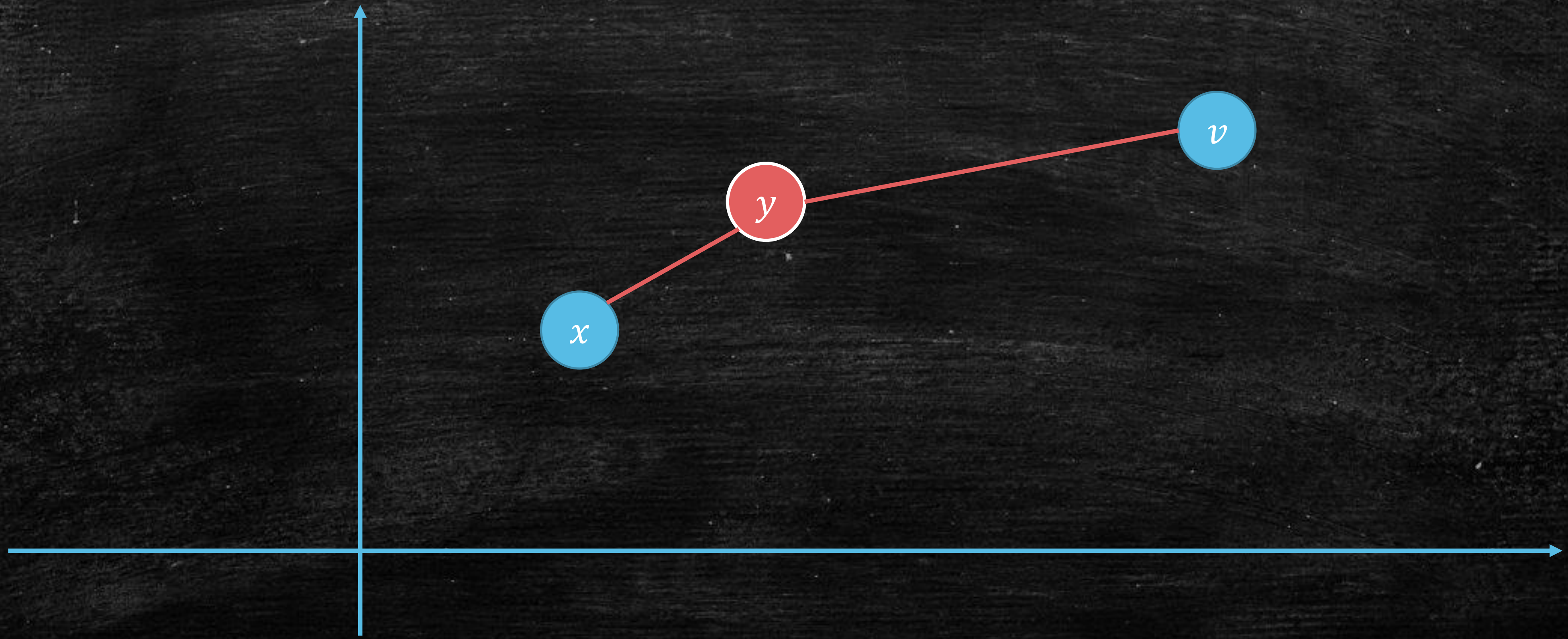
Who can be kicked out?



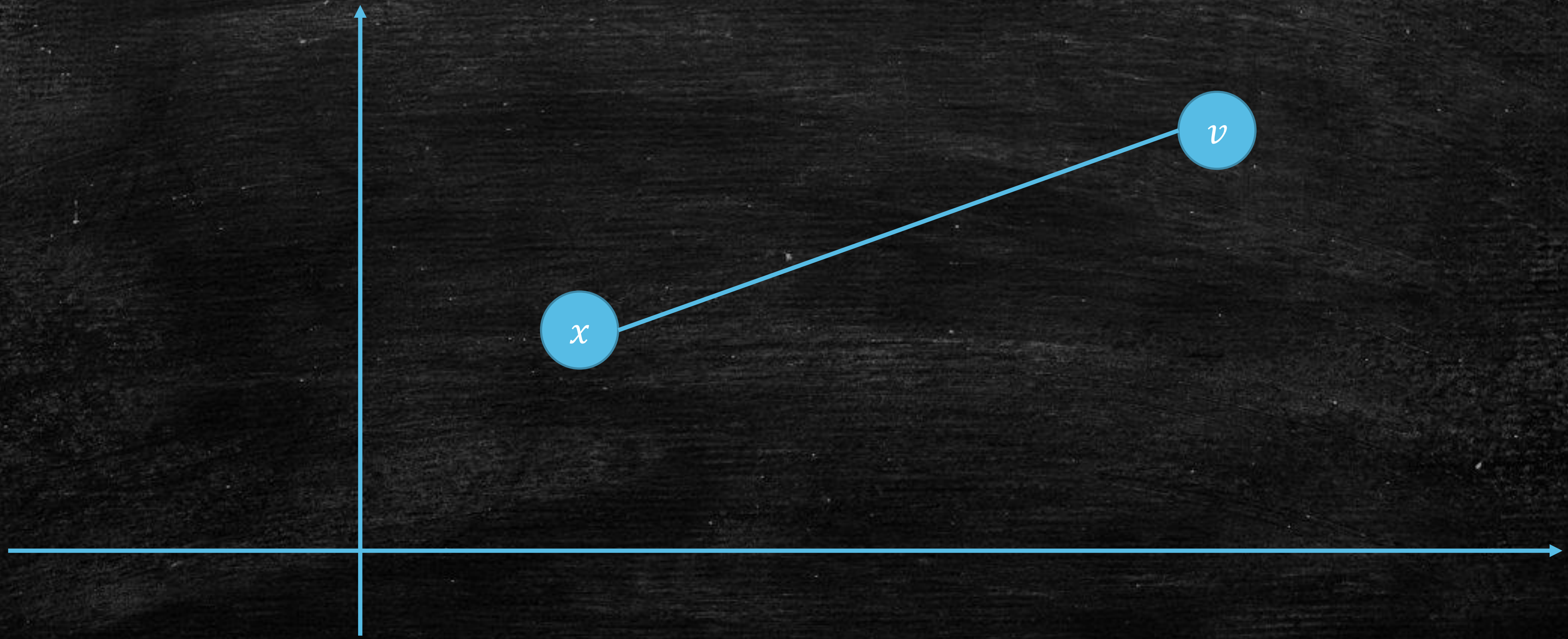
Who can be kicked out?



Who can be kicked out?

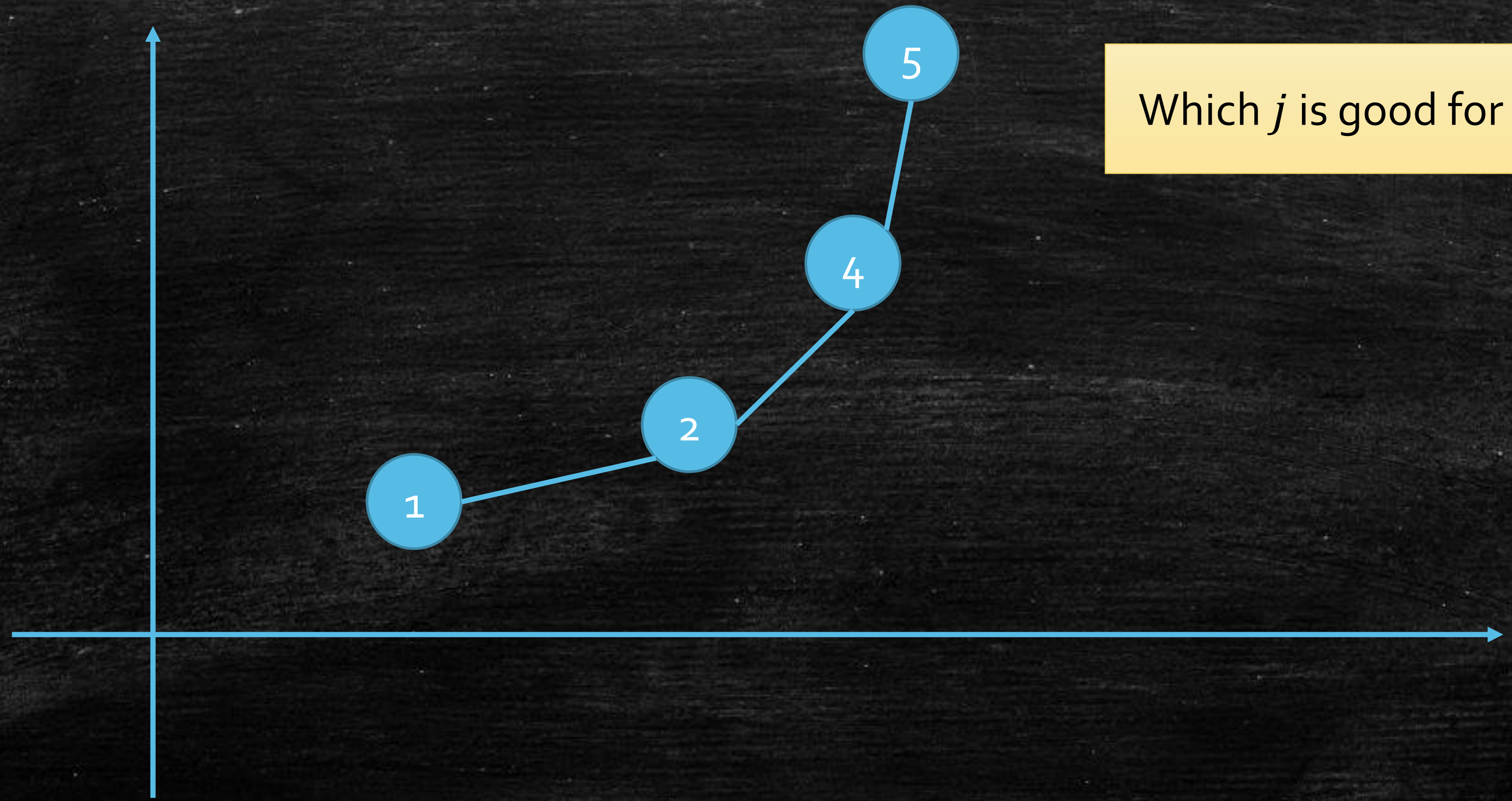


Who can be kicked out?



After Kicking: A Convex Hull.

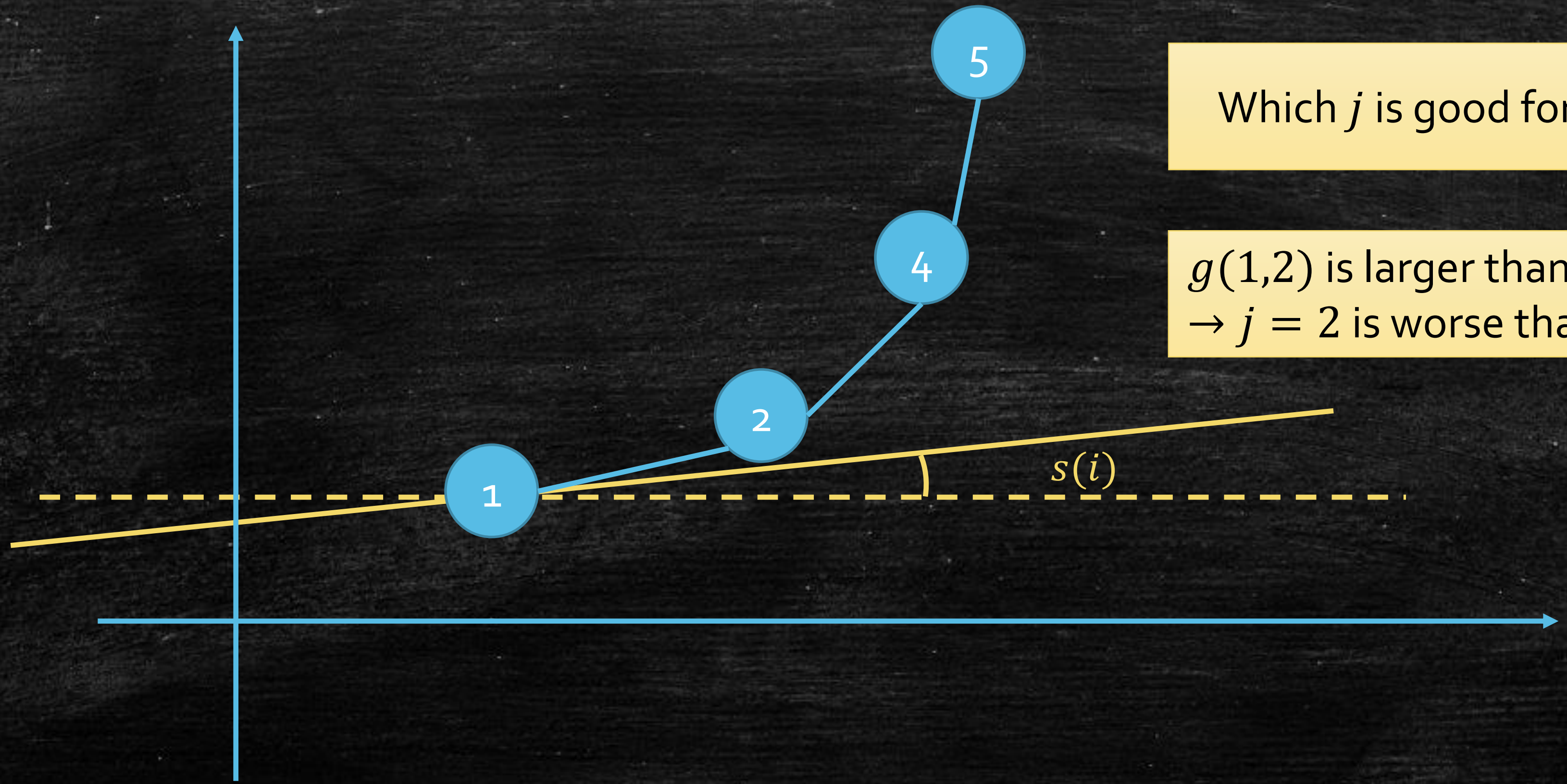
$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



Which j is good for $i = 6$?

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



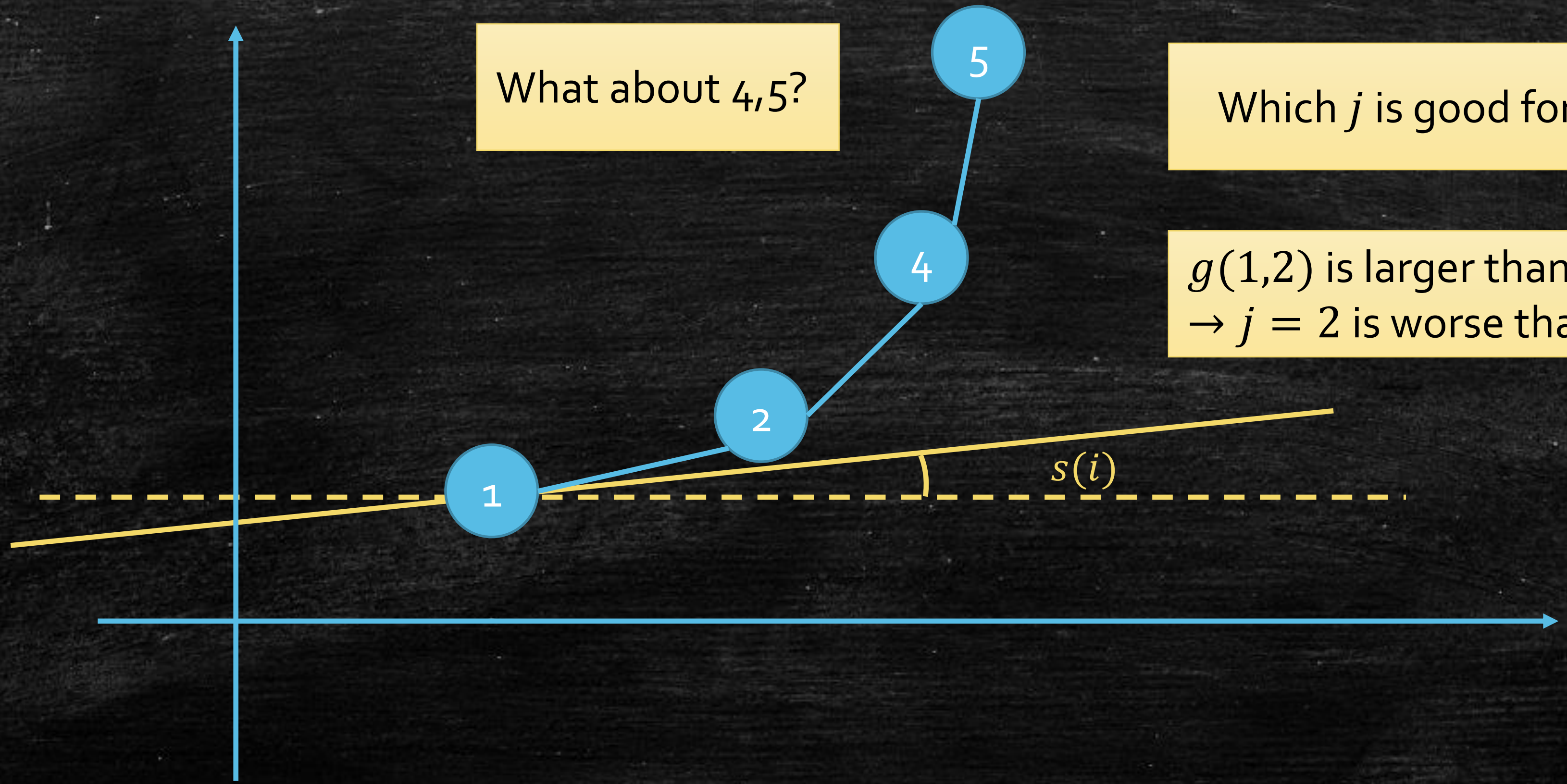
Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is worse than $j = 1$.

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9

What about 4,5?

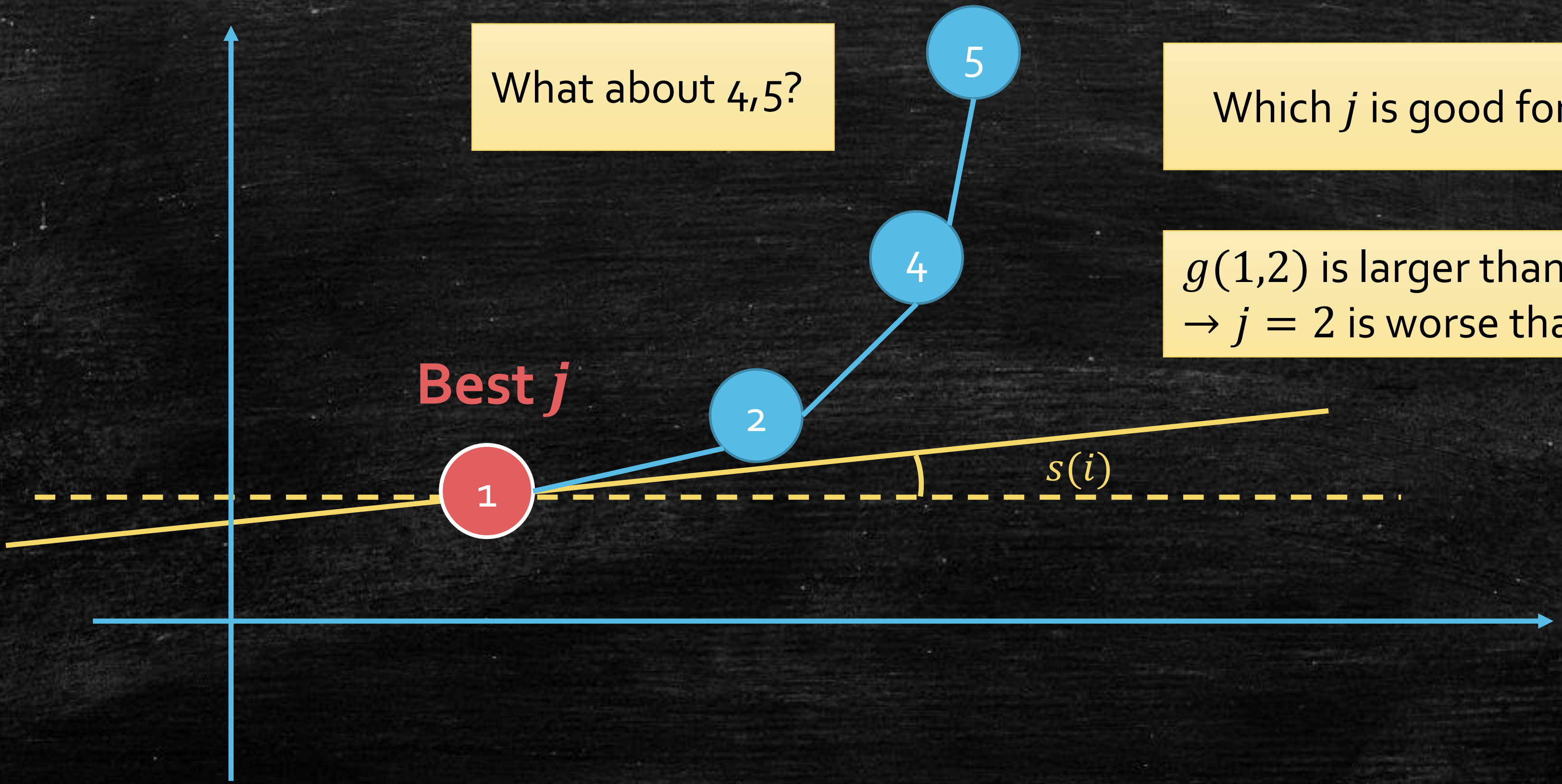


Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is worse than $j = 1$.

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9

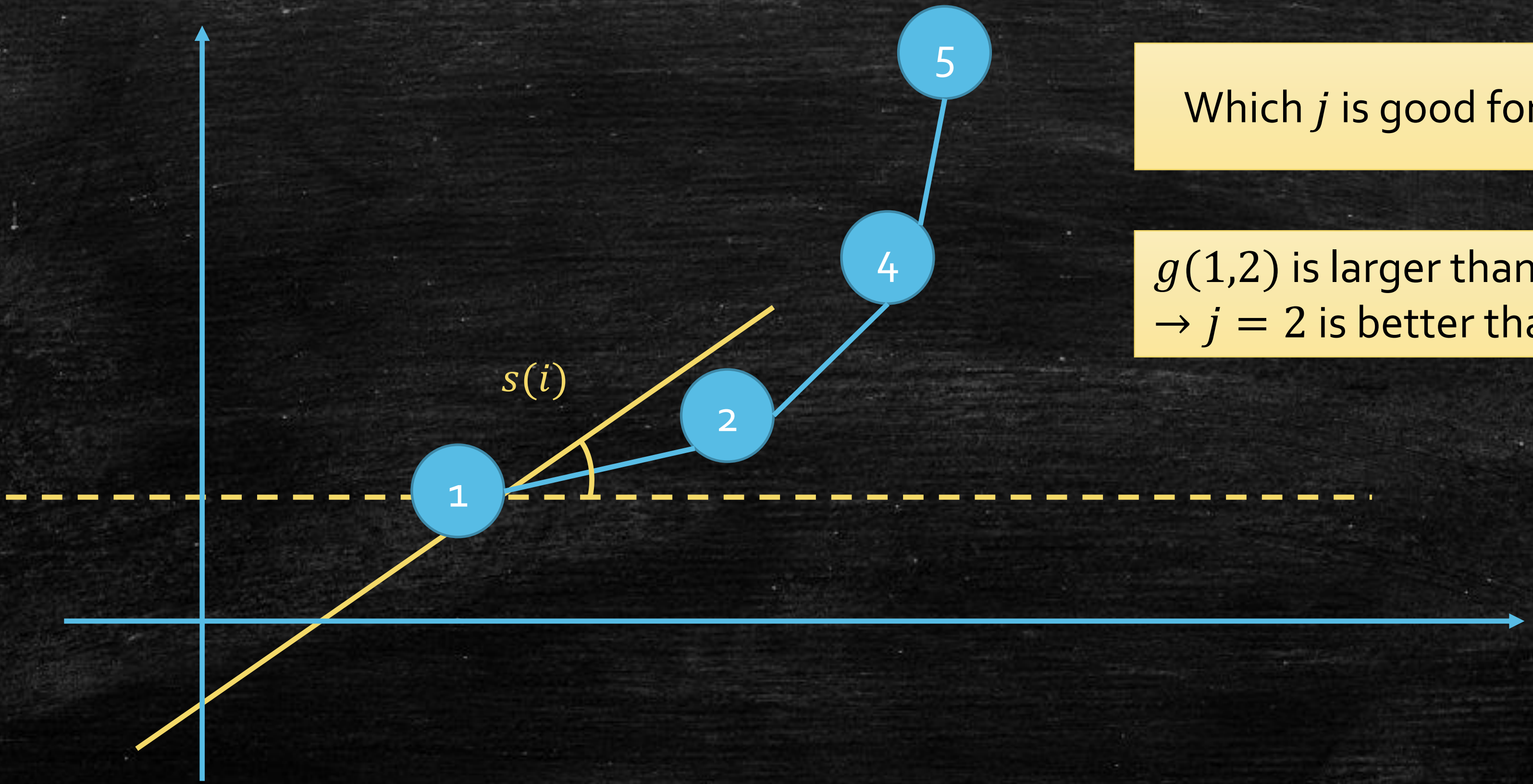


Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is worse than $j = 1$.

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9

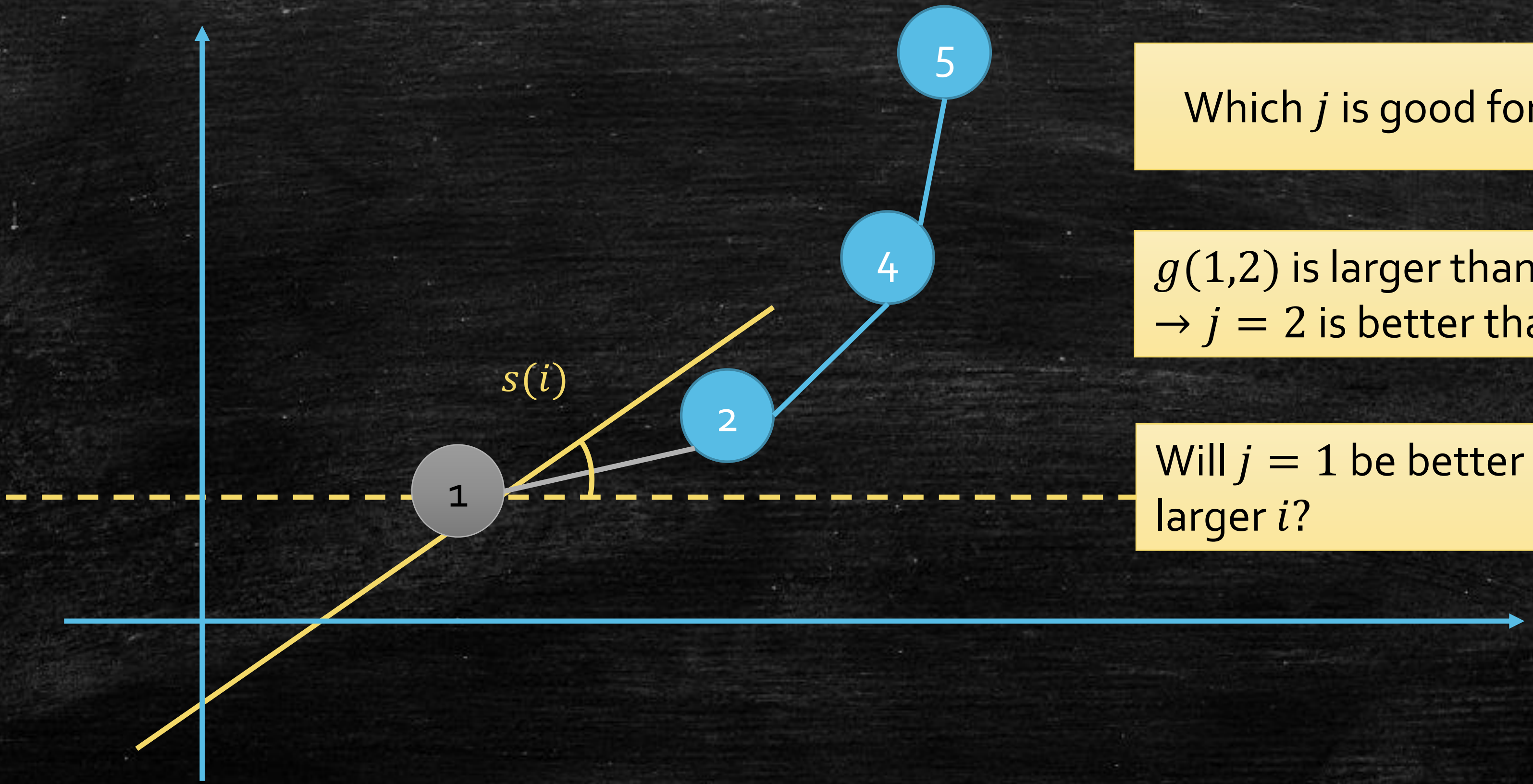


Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is better than $j = 1$.

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



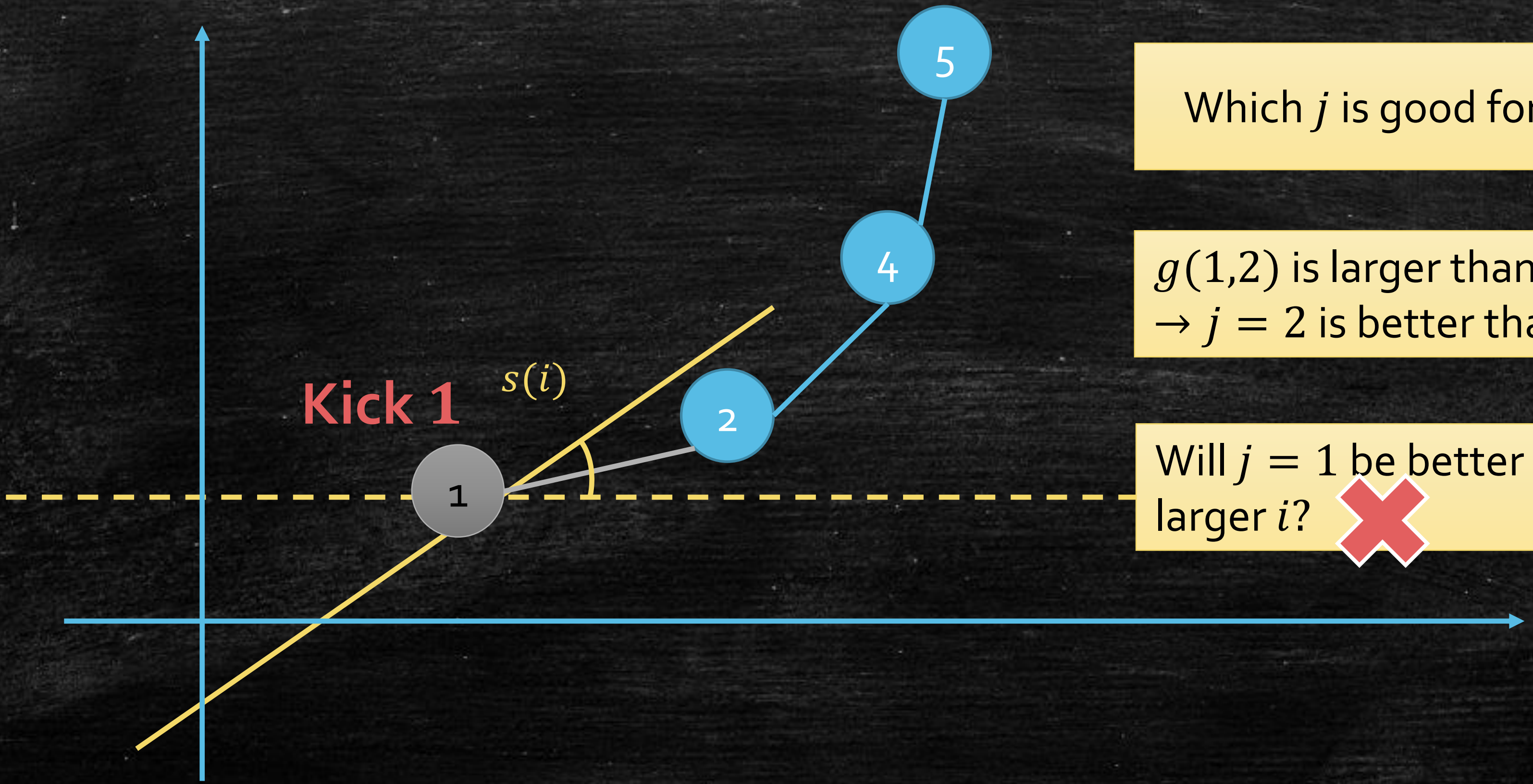
Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is better than $j = 1$.

Will $j = 1$ be better again for larger i?

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



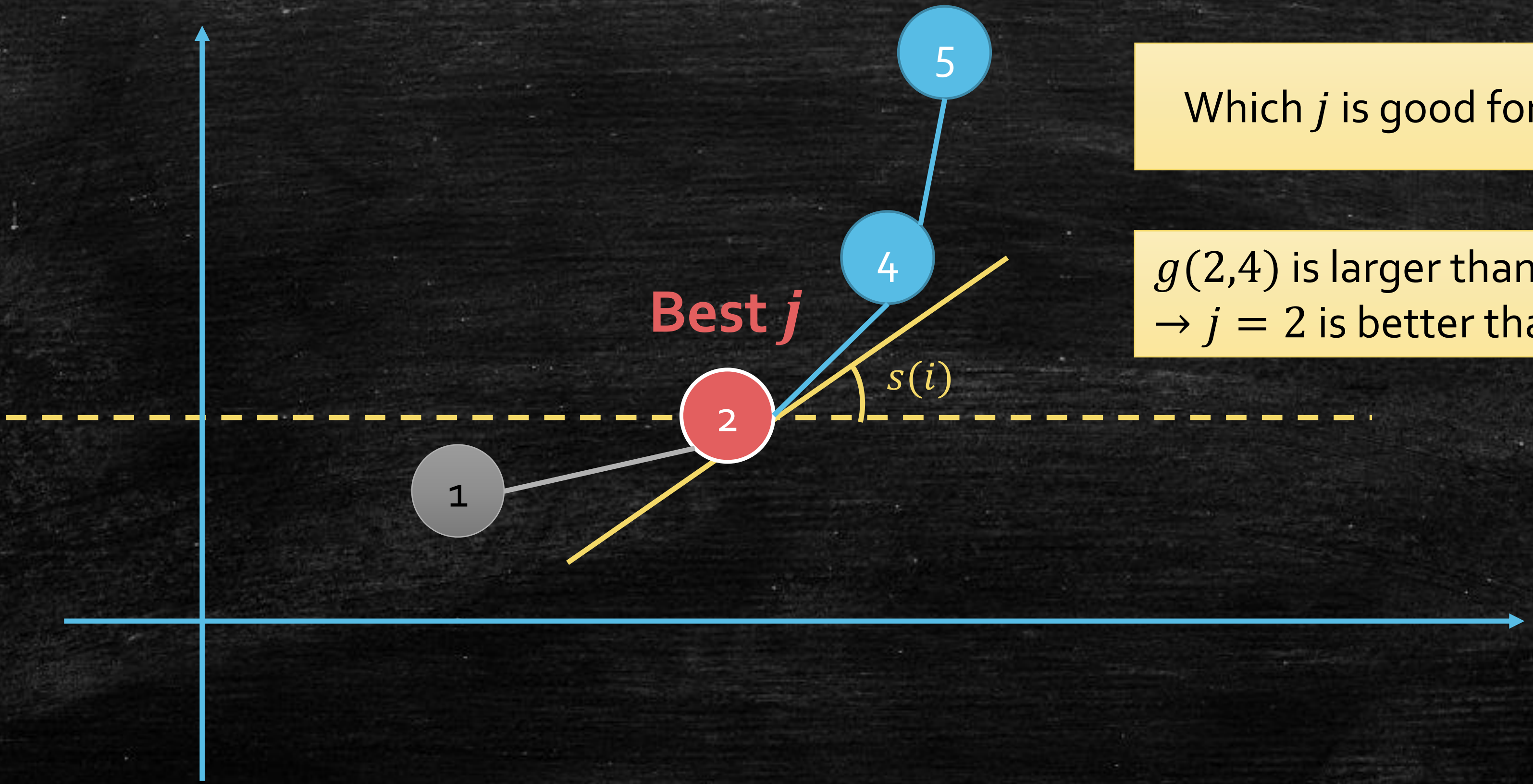
Which j is good for $i = 6$?

$g(1,2)$ is larger than $s(i)$
$\rightarrow j = 2$ is better than $j = 1$.

Will $j = 1$ be better again for larger i?

After Kicking: A Convex Hull.

$f[1]$	$f[2]$	$f[3]$	$f[4]$	$f[5]$	?			
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9



Which j is good for $i = 6$?

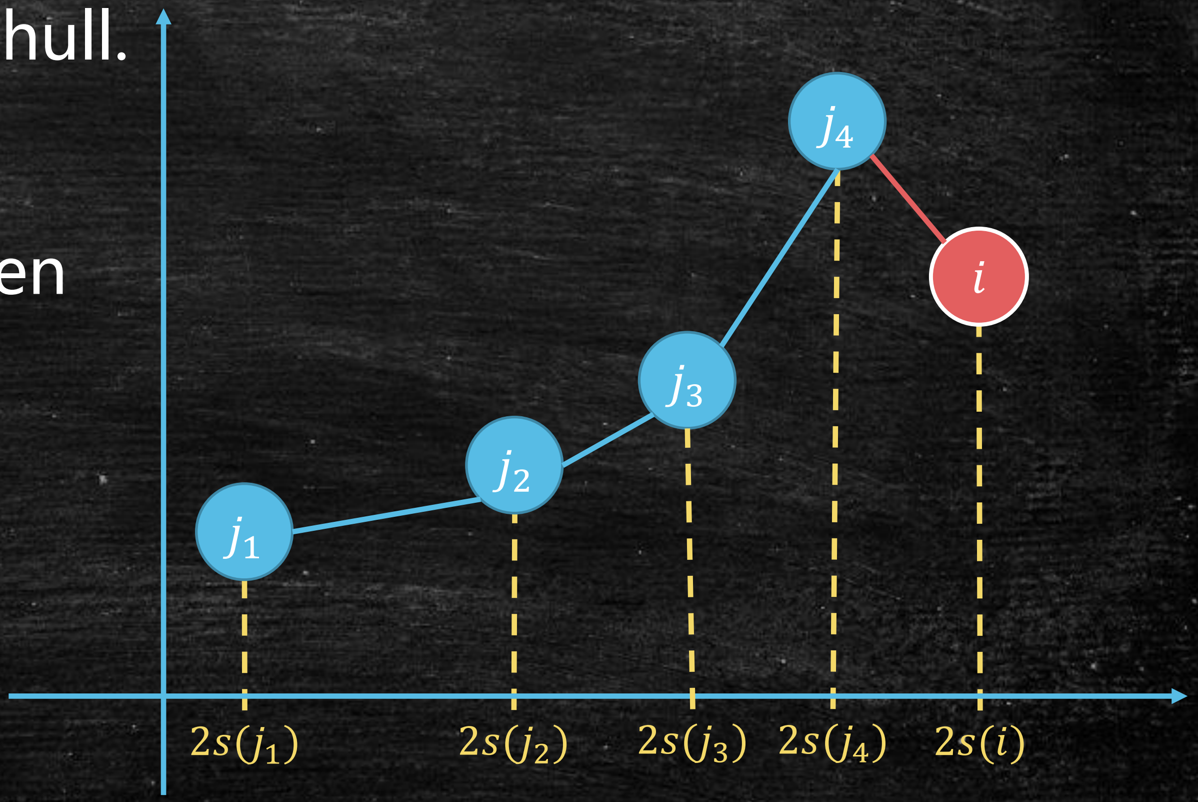
$g(2{,}4)$ is larger than $s(i)$
$\rightarrow j = 2$ is better than $j = 4$.

Algorithm for updating $f[i]$.

- $Let\ j_1, j_2, \ldots j_m$ be the convex hull.
- Loop form $k = 1$
- While $g(j_k, j_{k+1}) \leq s(i)$ then
  - kick j_k
  - $k++$
- Until $g(j_k, j_{k+1}) > s(i)$
- j_k is the **best**!
- $f[i] = f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2 = f[j] + C + (s(i) - s(j))^2$

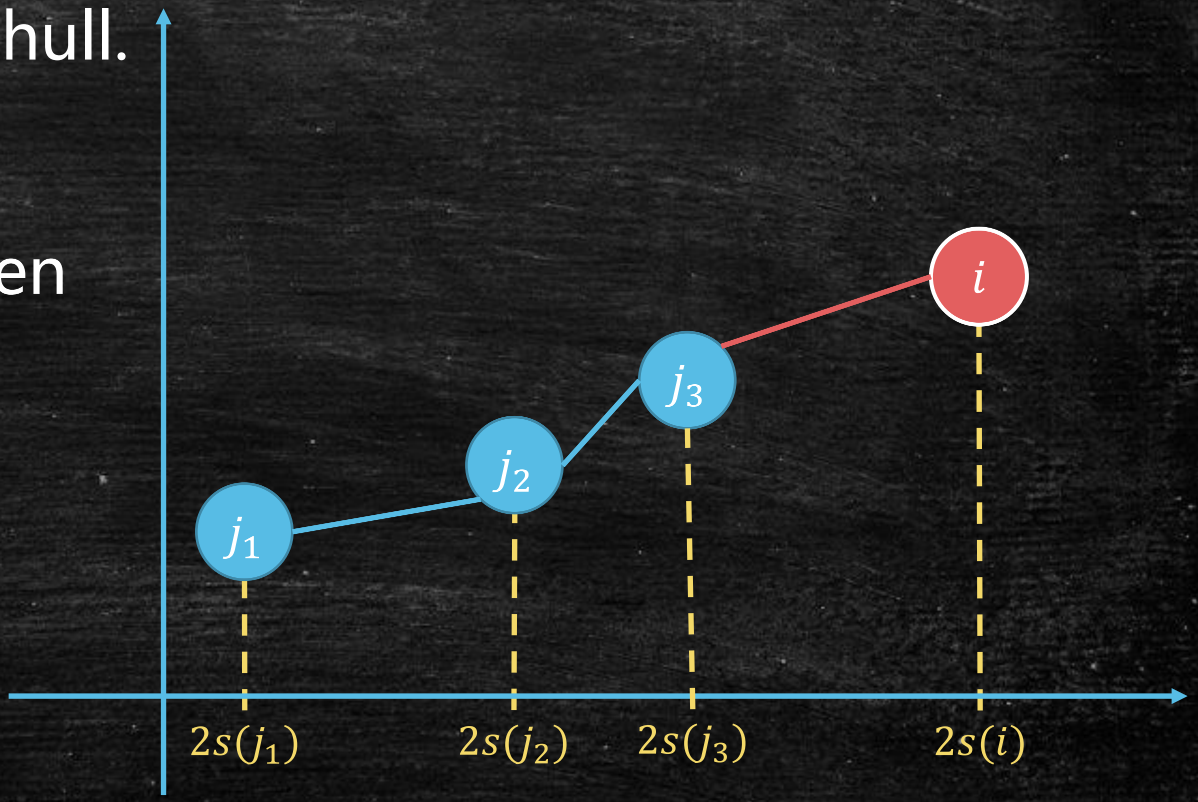
Algorithm for updating the convex hull

- $Let\ j_1, j_2, \dots j_m$ be the convex hull.
- Loop form $k = m$
- While $g(j_{k-1}, j_k) \geq g(j_k, i)$ then
  - kick j_k
  - $k--$
- Until $g(j_k, j_{k+1}) < g(j_k, i)$
- $j_{k+1} \leftarrow i$



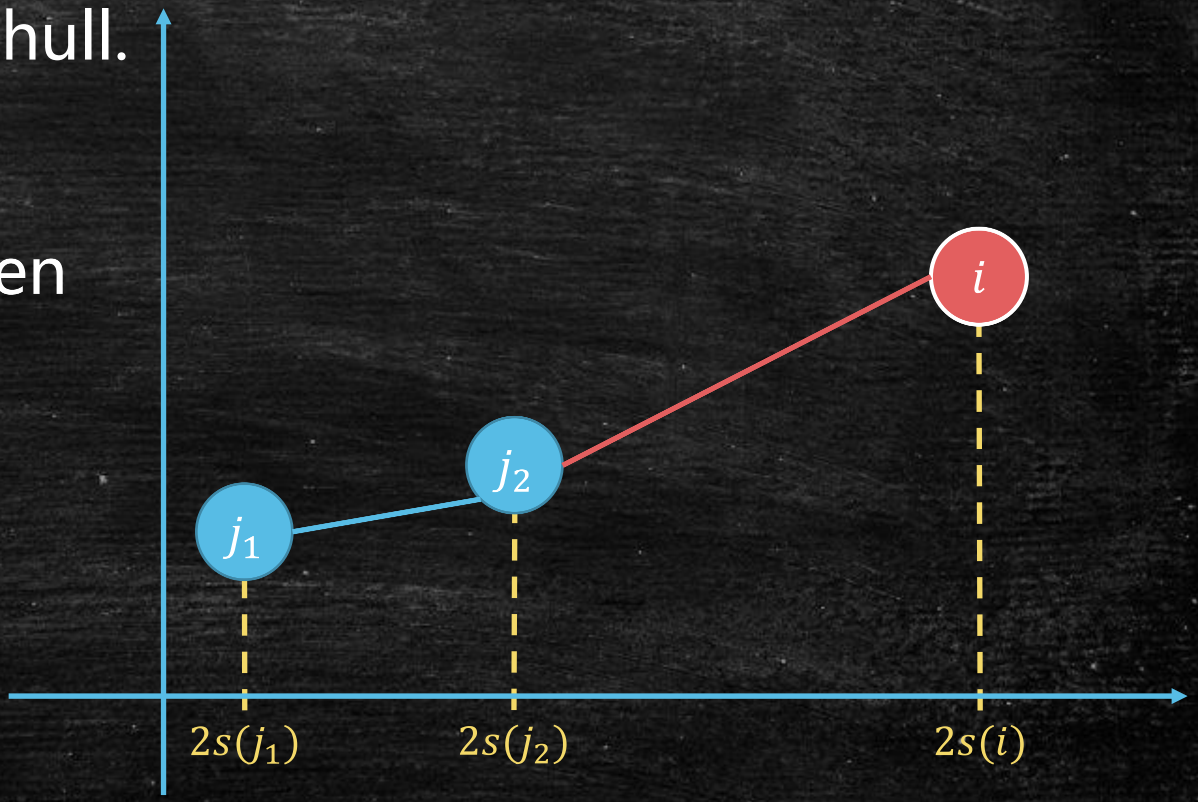
Algorithm for updating the convex hull

- $Let\ j_1, j_2, \ldots j_m$ be the convex hull.
- Loop form $k = m$
- While $g(j_{k-1}, j_k) \geq g(j_k, i)$ then
  - kick j_k
  - $k--$
- Until $g(j_k, j_{k+1}) < g(j_k, i)$
- $j_{k+1} \leftarrow i$



Algorithm for updating the convex hull

- $Let\ j_1, j_2, \ldots j_m$ be the convex hull.
- Loop form $k = m$
- While $g(j_{k-1}, j_k) \geq g(j_k, i)$ then
  - kick j_k
  - $k--$
- Until $g(j_k, j_{k+1}) < g(j_k, i)$
- $j_{k+1} \leftarrow i$



The DP Algorithm

- Complete the DP
  - $f[0] = 0$
  - For $i = 1$ to n
    - Calculate $f[i]$ from the potential convex hull.
    - Insert i into the convex hull.

Running time?

- Complete the DP
  - $f[0] = 0$
  - For $i = 1$ to n
    - Calculate $f[i]$ from the potential convex hull.
    - Insert i into the convex hull.

Algorithm for updating $f[i]$.

- $Let\ j_1, j_2, \ldots j_m$ be the con

Charge to i

1. Loop form $k = 1$

Charge to j_k

2. While $g(j_k, j_{k+1}) \leq s(i)$ then
   - kick j_k
   - $k++$

Charge to i

3. Until $g(j_k, j_{k+1}) > s(i)$
4. j_k is the **best**!
5. $f[i] = f[j] + C + \left(\sum_{k=j+1}^{i} a_k\right)^2 = f[j] + C + (s(i) - s(j))^2$

Algorithm for updating the convex hull

- $Let\ j_1, j_2, \dots j_m$ be the co

Charge to j_k

Charge to i

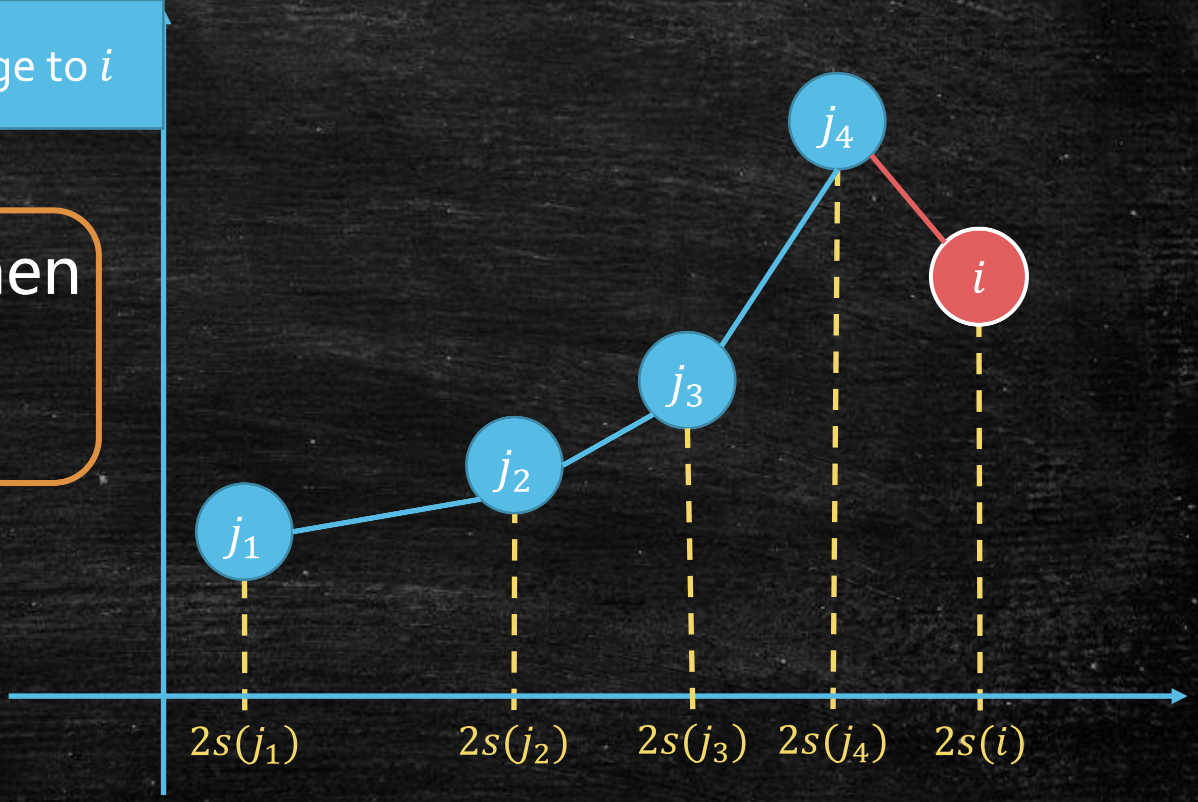
Loop form $k = m$

2. While $g(j_{k-1}, j_k) \geq g(j_k, i)$ then
   - kick j_k
   - $k--$

3. Until $g(j_k, j_{k+1}) < g(j_k, i)$

4. $j_{k+1} \leftarrow i$

Charge to i



Running time?

- Complete the DP
  - $f[0] = 0$
  - For $i = 1$ to n
    - Calculate $f[i]$ from the potential convex hull.
    - Insert i into the convex hull.
- Total Charged Cost for Each i
  - When it is kicked → once
  - When it is calculated → once
  - When it is inserted → once

Product of Sets

- **Input:** n sets $L_1, L_2, \dots, L_n$.
- **Output:** The minimum number of operations to calculate $L_1 \times L_2 \dots \times L_n$.
- What is $L_1 \times L_2$?
- $L_1 = \{a, b, c\},\ L_2 = \{x, y\}$
- $L_1 \times L_2 = \{(a,x), (a,y), (b,x), (b,y), (c,x), (c,y)\}$
- General Form: $L_1 \times L_2 = \{(a,b) \mid a \in L_1, b \in L_2\}$

- We want $L_1 \times L_2 \times L_3$
- Two different calculations
  - $(L_1 \times L_2) \times L_3$
  - $L_1 \times (L_2 \times L_3)$
- Two different costs
  - $m_1 m_2 + m_1 m_2 m_3$
  - $m_1 m_2 m_3 + m_2 m_3$
- $m_1, m_2, m_3, \dots, m_n$ are crucial!

What is the cost of different calculation?

- We want $L_1 \times L_2 \times L_3$
- Two different calculations
  - $(L_1 \times L_2) \times L_3$
  - $L_1 \times (L_2 \times L_3)$
- Two different costs
  - $m_1 m_2 + m_1 m_2 m_3$
  - $m_1 m_2 m_3 + m_2 m_3$
- $m_1, m_2, m_3, \dots, m_n$ are crucial!

Can you design a DP for it?

A simple DP

- **Subproblem:** $c(i,j)$ is the cost for calculating $L_i \times L_2 \dots \times L_j$.
- How to update $c(i,j)$?
  - Case 1: $c(i,j) = c(i+1,j) + m_i m_{i+1} \dots m_j$
  - Case 2: $c(i,j) = c(i,i+1) + c(i+2,j) + m_i m_{i+1} \dots m_j$
  - …
  - Case ?: $c(i,j) = c(i,j-1) + m_i m_{i+1} \dots m_j$
  - Case k: $c(i,j) = c(i,k) + c(k+1,j) + m_i m_{i+1} \dots m_j$
- **Transfer:** $c(i,j) = m_i m_{i+1} \dots m_j + \min_{i \le k < j} \{c(i,k) + c(k+1,j)\}$
- Remark: we use $w(i,j)$ to denote $m_i m_{i+1} \dots m_j$.

Running Time

- What about the running time?
- n^2 subproblems, we use n iterations to calculate each.
- $\boldsymbol{O(n^3)}$
- The topological order.

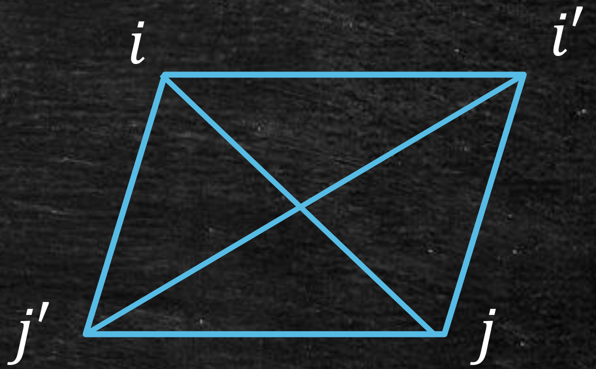
$c(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$					
$i=2$					
$i=3$					
$i=4$					
$i=5$					

Improvement!

F. Frances Yao STOC 1980

Key Property We Observe

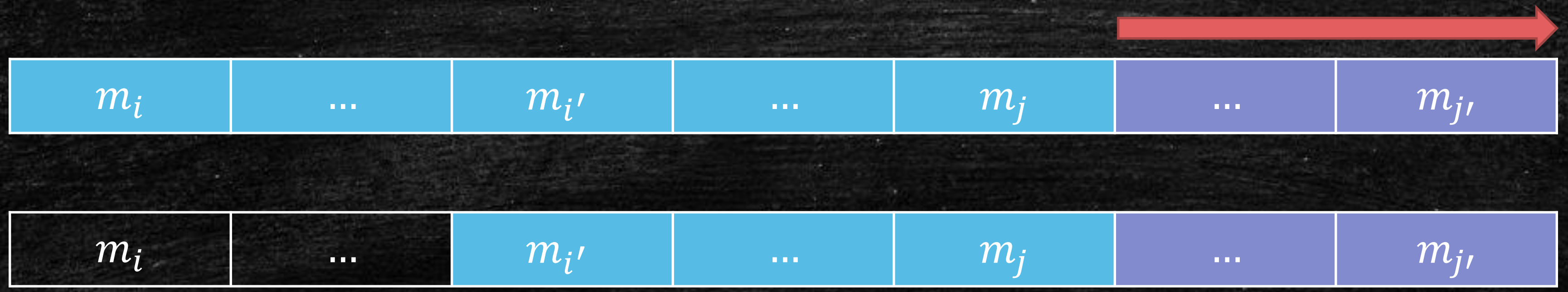
- DP formula
  - $c(i,j) = w(i,j) + \min_{i \le k < j}\{c(i,k) + c(k+1,j)\}$
- Weight function
  - $w(i,j) = m_i m_{i+1} \dots m_j$
  - Quadrangle Inequality (QI)
    - $\forall i \le i' \le j \le j',\ w(i,j) + w(i',j') \le w(i',j) + w(i,j')$
  - Monotonicity
    - $\forall i \le i' \le j \le j',\ w(i',j) \le w(i,j')$



m_i	…	$m_{i'}$	…	m_j	…	$m_{j'}$

Understand QI and Monotonicity

- Monotonicity
  - $\forall i \le i' \le j \le j',\ \ w(i',j) \le w(i,j')$
  - The weight function is increasing.
- Quadrangle Inequality (QI)
  - $\forall i \le i' \le j \le j',\ w(i,j) + w(i',j') \le w(i',j) + w(i,j')$
  - $w(i,j') - w(i,j) \ge w(i',j') - w(i',j)$
  - Larger size w increase larger.



Check $w(i,j)$

- Monotonicity
  - $\forall i \le i' \le j \le j',\ \ w(i',j) \le w(i,j')$
- Quadrangle Inequality (QI)
  - $\forall i \le i' \le j \le j',\ w(i,j) + w(i',j') \le w(i',j) + w(i,j')$
- Prove QI
  - $AB + BC \le B + ABC$
  - $AB(C-1) \ge B(C-1)$

	A		B		C	
m_i	…	$m_{i'}$	…	m_j	…	$m_{j'}$

QI for w implies QI for c

- It implies……
- Quadrangle Inequality (QI) for $c(i,j)$

$$\forall i \le i' \le j \le j', \qquad c(i,j) + c(i',j') \le c(i',j) + c(i,j')$$

$c(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$			$c(i,j)$		$c(i,j')$
$i=2$					
$i=3$			$c(i',j)$		$c(i',j')$
$i=4$					
$i=5$					

Hold on

Quadrangle Inequality (QI): $w(i,j) + w(i',j') \leq w(i',j) + w(i,j')$.
Monotonicity: $w(i',j) \leq w(i,j')$

$$\Big\Downarrow ?$$

Quadrangle Inequality (QI) for $c(i,j)$

$$\forall i \leq i' \leq j \leq j', \qquad c(i,j) + c(i',j') \leq c(i',j) + c(i,j')$$

Assume we have it!

- Quadrangle Inequality (QI) for $c(i,j)$
  - $\forall i \le i' \le j \le j',\ c(i,j) + c(i',j') \le c(i',j) + c(i,j')$
- How to design a better DP?

$c(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$			$c(i,j)$		$c(i,j')$
$i=2$					
$i=3$			$c(i',j)$		$c(i',j')$
$i=4$					
$i=5$					

Go back to the potential idea.

What is the best k for $c(i, j)$?

- Consider again when k_2 is better than k_1.
- $c(i, k_1) + c(k_1 + 1, j) > c(i, k_2) + c(k_2 + 1, j)$
- $c(k_1 + 1, j) - c(k_2 + 1, j) > c(i, k_2) - c(i, k_1)$
- Fix i, which one is better depends on
  - $c(k_1 + 1, j) - c(k_2 + 1, j) > c(i, k_2) - c(i, k_1)$
- Fix j, which one is better depends on
  - $c(i, k_2) - c(i, k_1) < c(k_1 + 1, j) - c(k_2 + 1, j)$

i	…	k_1	…	k_2	…	j

The Monotonicity when Comparing k_1, k_2

- The condition of k_2 is better than k_1
  - $c(k_1+1,j) - c(k_2+1,j) > c(i,k_2) - c(i,k_1)$
- Fix i, consider what happens for different j.
  - $c(k_1+1,j) - c(k_2+1,j)$ is non-decreasing on j!

k_1	…	k_2	…	j

k_1	…	k_2	…	j	…	j'

The Monotonicity when Comparing k_1, k_2

$c(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$			k_1, k_2	k_1, k_2	
$i=2$					
$i=3$					
$i=4$					
$i=5$					

- $c(k_1+1,j) - c(k_2+1,j)$ is non−decreasing on j!
- If k_2 is better than k_1 at $j=3$, it is also better at $j=4$.
- $c(i,k_2) - c(i,k_1) < c(k_1+1,j) - c(k_2+1,j) \le c(k_1+1,j') - c(k_2+1,j')$

What does it mean

- The best k have some monotonicity w.r.t. j.
- Let $k(i, j)$ be the best k $c(i, j)$ selected.

Monotonicity for $k(i,j)$.

$k(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$			Non-decreasing		
$i=2$					
$i=3$					
$i=4$					
$i=5$					

- $c(i,j) = w(i,j) + \min\limits_{k(i,j-1) \le k < j} \{c(i,k) + c(k+1,j)\}$
- Is that enough?

Is that enough?

- $c(i,j) = w(i,j) + \min_{k(i,j-1) \le k < j} \{c(i,k) + c(k+1,j)\}$
- No!
- The cost maybe $1 + 2 + 3 + 4 + \cdots + n - 1 + n$ for each row.
- Still $O(n^3)$!

Monotonicity for $k(i,j)$.

$k(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$					
$i=2$					
$i=3$			k'	k	
$i=4$				k''	
$i=5$					

Non-decreasing

- $c(i,k_2) - c(i,k_1)$ is non−increasing on i.
- k_2 is better? $c(i,k_2) - c(i,k_1) < c(k_1+1,j) - c(k_2+1,j)$

Monotonicity for $k(i,j)$.

Non-decreasing

$k(i,j)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$					
$i=2$					
$i=3$			k'	k	
$i=4$				k''	
$i=5$					

Non-decreasing

- $c(i,k_2) - c(i,k_1)$ is non−increasing on i.
- k_2 is better? $c(i,k_2) - c(i,k_1) < c(k_1+1,j) - c(k_2+1,j)$
- If k_2 is better at i, then it is still better at $i' > i$.

A new DP approach

$k(i,j)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$					
$i = 2$					
$i = 3$			k_1	$k_1 \le k \le k_2$	
$i = 4$				k_2	
$i = 5$					

- $c(i,j) = w(i,j) + \min_{k(i,j-1) \le k < k(i+1,j)} \{c(i,k) + c(k+1,j)\}$
- What is a good order for this approach?

A new DP approach

$k(i,j)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$					
$i = 2$					
$i = 3$			k_1	$k_1 \leq k \leq k_2$	
$i = 4$				k_2	
$i = 5$					

- $c(i,j) = w(i,j) + \min_{k(i,j-1) \leq k < k(i+1,j)} \{c(i,k) + c(k+1,j)\}$
- We know k_1 and k_2 in the topological order!

DP algorithm

- Intialize $c[i,i] = 0$ for all i
- For $\Delta = 1$ to $n - 1$
  - For $i = 1$ to $n - \Delta$
    - $j = i + \Delta$
    - $c[i,j] = w(i,j) + \min\{c(i,k) + c(k+1,j)\}$.
    - Searching range k from $k(i,j-1)$ to $k(i+1,j)$
    - $k(i,j) \leftarrow$ the best k

Running Time

- Intialize $c[i,i] = 0$ for all i
- For $\Delta = 1$ to $n - 1$
  - For $i = 1$ to $n - \Delta$
    - $j = i + \Delta$
    - $c[i,j] = w(i,j) + \min\{c(i,k) + c(k+1,j)\}$.
    - Searching range k from $k(i, j-1)$ to $k(i+1, j)$
    - $k(i,j) \leftarrow$ the best k
- $Time = \sum_{i=1}^{n-\Delta} k(i+1, i+\Delta) - k(i, i+\Delta-1)$

$$= k(n-\Delta+1, n) - k(1,\Delta) \le n$$

Don’t forget to check why QI is correct for c.

Prove QI for c

- Given
  - $w(i,j) = m_i m_{i+1} \dots m_j$
  - Quadrangle Inequality (QI)
    - $\forall i \le i' \le j \le j',\ w(i,j) + w(i',j') \le w(i',j) + w(i,j')$
  - Monotonicity
    - $\forall i \le i' \le j \le j',\ \ w(i',j) \le w(i,j')$
- To prove
  - $c(i,j) = w(i,j) + \min_{i \le k < j} \{c(i,k) + c(k+1,j)\}$
  - Quadrangle Inequality (QI)
    - $\forall i \le i' \le j \le j',\ c(i,j) + c(i',j') \le c(i',j) + c(i,j')$

Prove by Induction

- Prove by Induction with $(j' - i)$
- **Base case:** $(i = j')$: easy to check.
- **Inductive:** $j' - i = \Delta$
  - Hypothesis
    - $\forall j' - i < \Delta,\ i \leq i' \leq j \leq j',\ c(i,j) + c(i',j') \leq c(i',j) + c(i,j')$
  - To prove
    - $\forall j' - i = \Delta,\ i \leq i' \leq j \leq j', c(i,j) + c(i',j') \leq c(i',j) + c(i,j')$

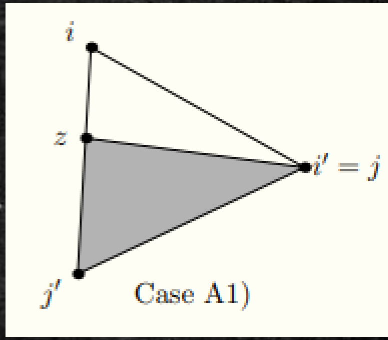
A special case: $i' = j$

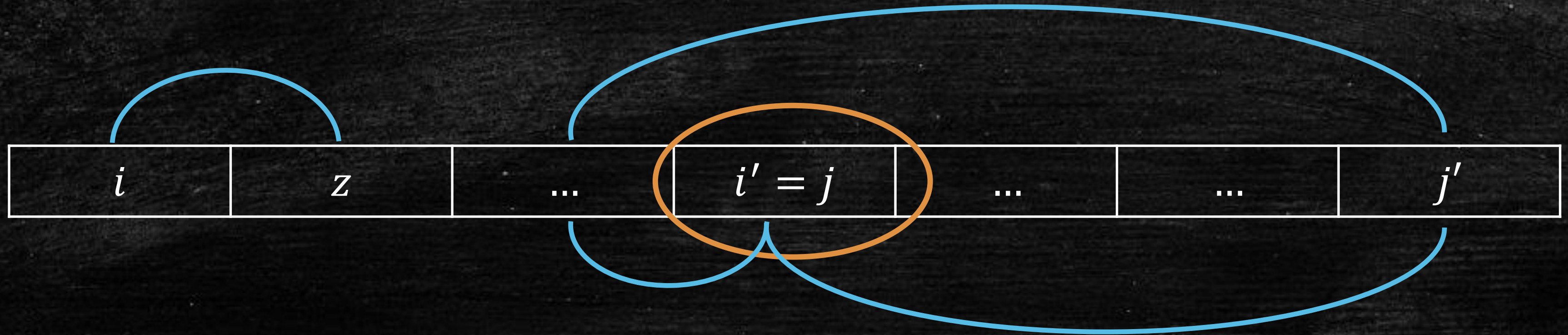
- Quadrangle Inequality (QI) for $c(i,j)$

$$\forall i \le i' \le j \le j', \qquad c(i,j) + c(i',j') \le c(i',j) + c(i,j')$$

- To prove $c(i,j) + c(j,j') \le c(i,j')$
- $c(i,j')$ is minimized at a point in $[i,j']$.
- $c(i,j') = c(i,z) + c(z+1,j') + w(i,j')$

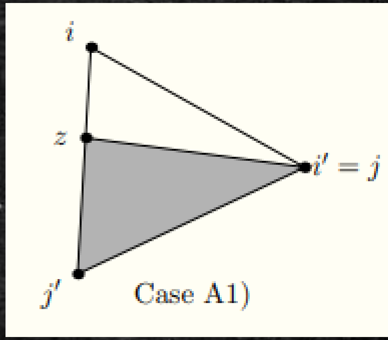
$$\ge c(i,z) + \boldsymbol{c(z+1,j) + c(j,j')} + w(i,j')$$

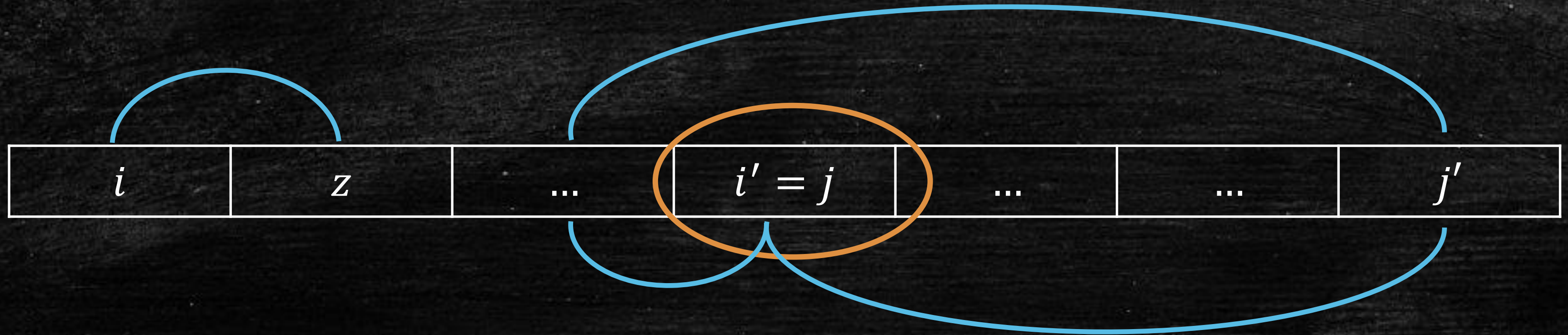




A special case: $i' = j$

- Quadrangle Inequality (QI) for $c(i,j)$
  $$\forall i \le i' \le j \le j', \qquad c(i,j) + c(i',j') \le c(i',j) + c(i,j')$$
- To prove $c(i,j) + c(j,j') \le c(i,j')$
- $c(i,j')$ is minimized at a point in $[i,j']$.
- $c(i,j') = c(i,z) + c(z+1,j') + w(i,j')$
  $$\ge \mathbf{c(i,z)} + \mathbf{c(z+1,j)} + c(j,j') + \mathbf{w(i,j')} \ge c(i,j) + c(j,j')$$





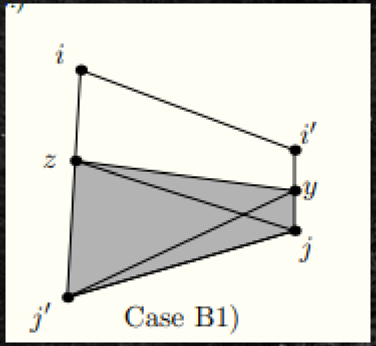
General case: $i' \neq j$

- Quadrangle Inequality (QI) for $c(i,j)$

$$\forall i \le i' \le j \le j', \qquad c(i,j) + c(i',j') \le c(i',j) + c(i,j')$$

- $c(i,j') = c(i,z) + c(z+1,j') + w(i,j')$
- $c(i',j) = c(i',y) + c(y+1,j) + w(i',j)$
- $c(i,j') + c(i',j) = \quad c(i,z) + c(z+1,j') + w(i,j')$

$$+ c(i',y) + c(y+1,j) + w(i',j)$$



General case: $i' \neq j$

- $c(i,j') + c(i',j) = \quad c(i,z) + c(z+1,j') + w(i,j')$

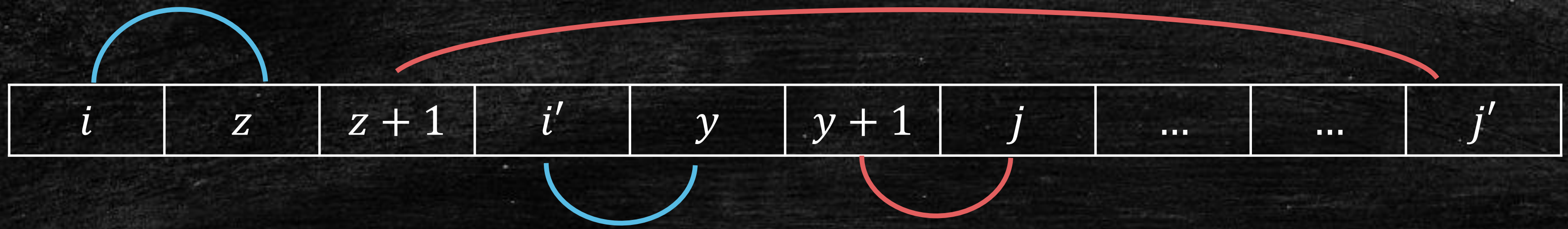
$$+\, c(i',y) + c(y+1,j) + w(i',j)$$

i	z	$z+1$	i'	y	$y+1$	j	…	…	j'

General case: $i' \neq j$

- $c(i,j') + c(i',j) = \quad c(i,z) + c(z+1,j') + w(i,j')$

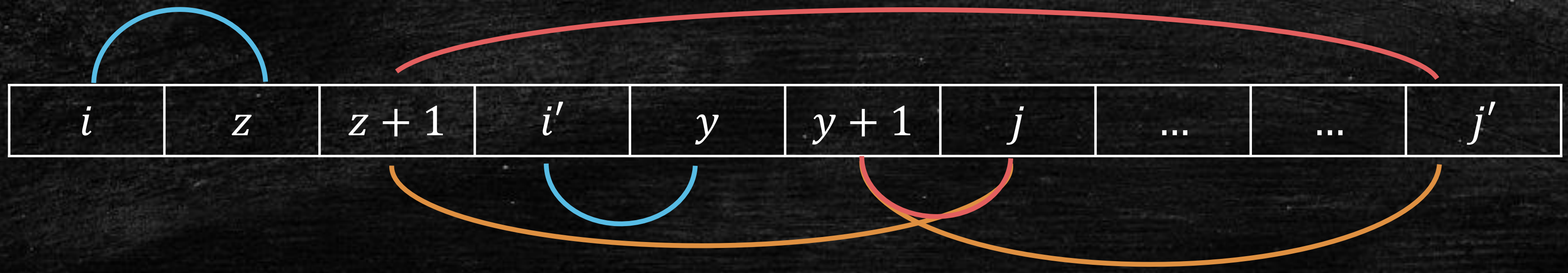
$$+\, c(i',y) + c(y+1,j) + w(i',j)$$



General case: $i' \neq j$

- $c(i,j') + c(i',j) = \quad c(i,z) + c(z+1,j') + w(i,j')$

  $+ c(i',y) + c(y+1,j) + w(i',j)$
- Inductive Hypothesis
  - $c(z+1,j') + c(y+1,j) \geq c(z+1,j) + c(y+1,j')$

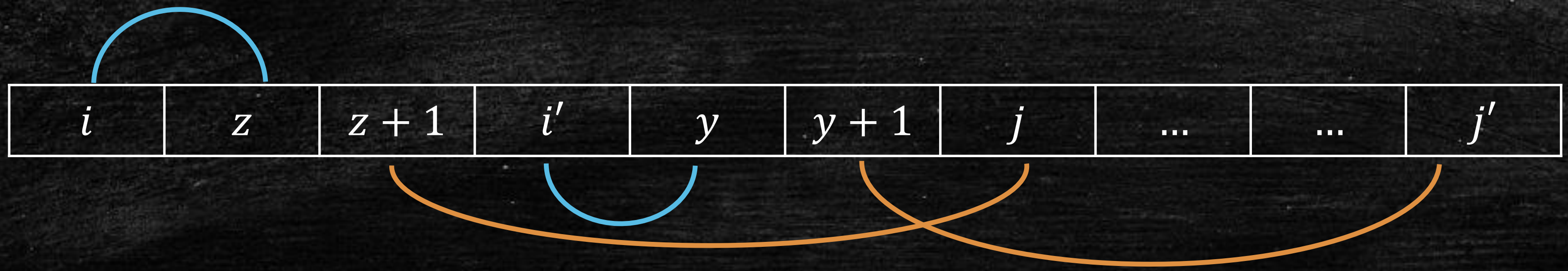


General case: $i' \neq j$

- $c(i,j') + c(i',j) \geq \quad c(i,z) + c(z+1,j) + w(i,j')$

  $$+ c(i',y) + c(y+1,j') + w(i',j)$$

- Inductive Hypothesis
  - $c(z+1,j') + c(y+1,j) \geq c(z+1,j) + c(y+1,j')$

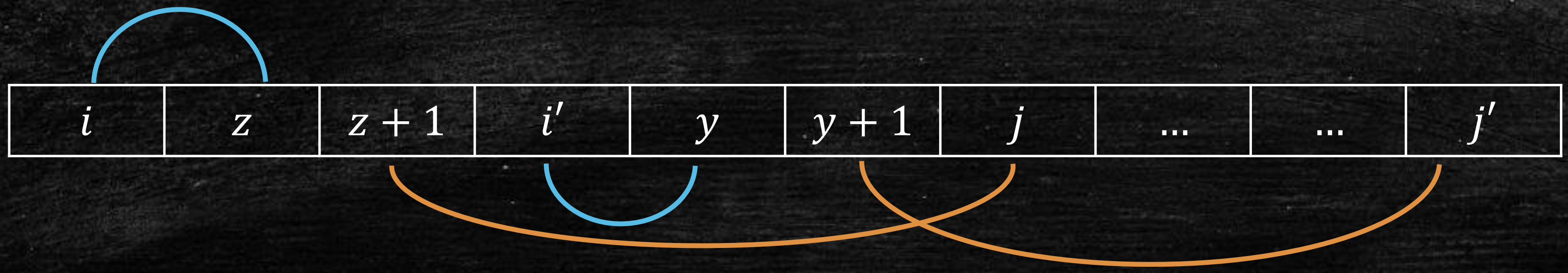


General case: $i' \neq j$

- $c(i,j') + c(i',j) \geq \quad c(i,z) + c(z+1,j) + w(i,j')$

  $$+ c(i',y) + c(y+1,j') + w(i',j)$$

- Inductive Hypothesis
  - $c(z+1,j') + c(y+1,j) \geq c(z+1,j) + c(y+1,j')$
- QI of w
  - $w(i,j') + w(i',j) \geq w(i,j) + w(i',j')$



General case: $i' \neq j$

- $c(i,j') + c(i',j) \geq \; c(i,z) + c(z+1,j) + w(i,j)$

  $$+ c(i',y) + c(y+1,j') + w(i',j')$$

- Inductive Hypothesis
  - $c(z+1,j') + c(y+1,j) \geq c(z+1,j) + c(y+1,j')$
- QI of w
  - $w(i,j') + w(i',j) \geq w(i,j) + w(i',j')$

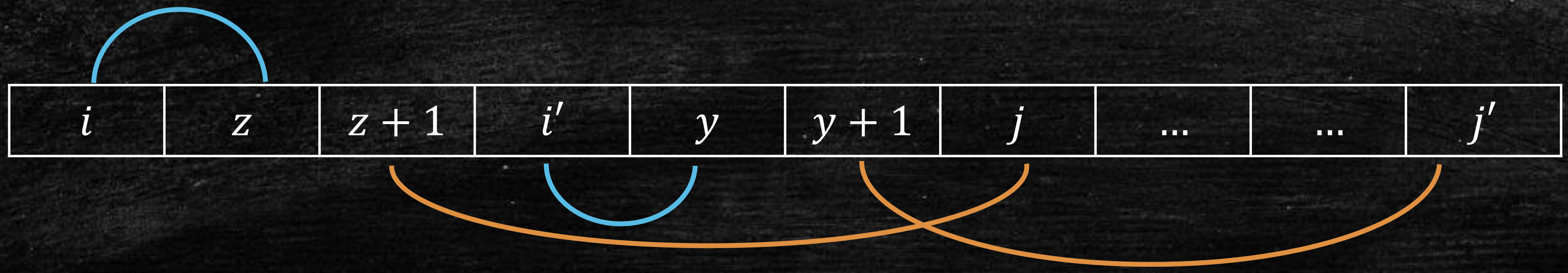
i	z	$z+1$	i'	y	$y+1$	j	...	...	j'

General case: $i' \neq j$

- $c(i,j') + c(i',j) \geq \quad c(i,z) + c(z+1,j) + w(i,j) \geq c(i,j)$

  $$+ c(i',y) + c(y+1,j') + w(i',j') \geq c(i',j')$$

- Inductive Hypothesis
  - $c(z+1,j') + c(y+1,j) \geq c(z+1,j) + c(y+1,j')$
- QI of w
  - $w(i,j') + w(i',j) \geq w(i,j) + w(i',j')$



Today's goal

- Recap the **Guideline** of DP! (Most Important)
- Learn how to improve DP by **Priority Queue**!
- Learn the tool: **Priority Queue**.
- Example
  - Largest Number in k Consecutive Numbers
  - Longest Increasing Sequence
  - Minimizing Printing Cost