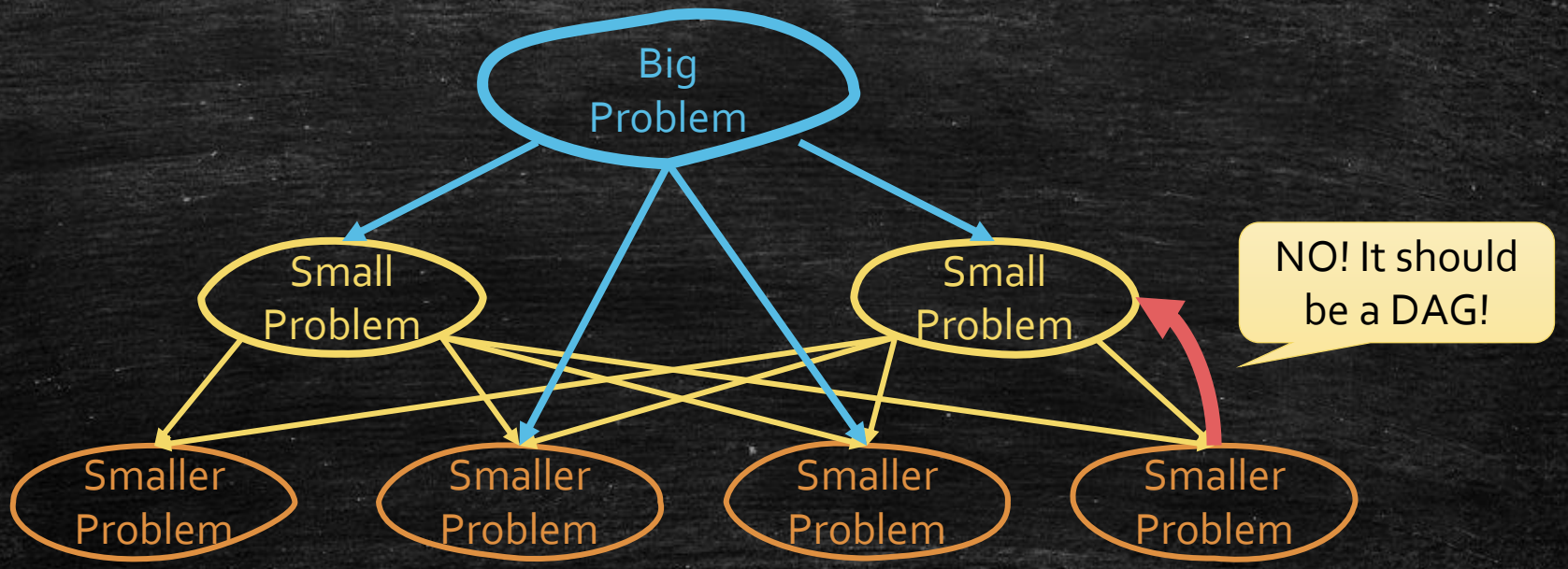
Dynamic Programming

DP on graphs

Dynamic Programming



A simpler guideline

- Find subproblems.
- Check whether we are in a **DAG** and find the **topological order** of this DAG. (Usually, by hand.)
- Solve & store the subproblems by the topological order.

Understand Bellman-Ford as A DP

Bellman-Ford

Function bellman_ford(G, s)
    $dist[s] = 0, dist[x] = \infty$ for other $x \in V$
    while $\exists dist[x]$ is updated
        for each $(u, v) \in E$
            $dist[v] = \min\{dist[v], dist[u] + d(u, v)\}$

Lemma 1

After k rounds, $dist(v)$ is the shortest distance of all **k-edge-path (path with at most k edges)**.

Define subproblems

- $dist[k, v]$: the shortest distance from s to v among all **k-edge-path (path with at most k edges)**.

Observation 2

The shortest distance of all **$|V|$-edge-path** can not be shorter than the shortest distance of all **$(|V| - 1)$ –edge-path** unless there is a **Negative Cycle.**

Bellman-Ford

function bellman_ford(G, s)
    $dist[0, s] = 0, dist[0, x] = \infty$ for other $x \in V$
    for $k = 1$ to $|V|$
        for each $(u, v) \in E$
            $dist[k, v] = \min\{dist[k-1, v], dist[k-1, u] + d(u, v)\}$

Solving Subproblems

- $dist[k, v] = \min\{dist[k-1, v], dist[k-1, u] + d(u, v)\}$

$f[k, v]$	s	v_2	v_3	v_4	v_5	v_6	v_7	…	$v_{\|V\|}$
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1									
2									
3						$f[k, v]$			
…									
$\|V\|$									

All Pair Shortest Path

- **Input:** A directed graph $G(V,E)$, and a weighted function $d(u,v)$ for all $(u,v) \in E$.
- **Output:** Distance $dist(u,v)$, for **all vertex pair** u, v.

What can we do?

- Naïve Plan:
  - Run $|V|$ times Bellman-Ford
  - $O(|V|^2|E|)$
- Improve it by an integrated DP!
  - Floyd-Warshall Algorithm!
  - $O(|V|^3)$
  - History from Wikipedia:

History and naming [edit]

The Floyd–Warshall algorithm is an example of dynamic programming, and was published in its currently recognized form by Robert Floyd in 1962.[3] However, it is essentially the same as algorithms previously published by Bernard Roy in 1959[4] and also by Stephen Warshall in 1962[5] for finding the transitive closure of a graph,[6] and is closely related to Kleene's algorithm (published in 1956) for converting a deterministic finite automaton into a regular expression.[7] The modern formulation of the algorithm as three nested for-loops was first described by Peter Ingerman, also in 1962.[8]

Define subproblems

- Bellman-Ford: $dist[k, v]$: the shortest distance from s to v among all **k-edge-path (path with at most k edges)**.
- A very natural generalization!
- Natural Generalization: $dist[k, u, v]$: the shortest distance from u to v among all **k-edge-path (path with at most k edges)**.

Natural Generalization

- Natural Generalization: $dist[k, u, v]$: the shortest distance from u to v among all **k-edge-path (path with at most k edges)**.
- Transfer:
  - $dist[k, u, v] = \min_{(s,v)\in E}\{dist[k-1, u, s] + d(s, v)\}$
- Time:
  - $|V|$ rounds
  - In one round, an edge can be used to update $|V|$ distance.
  - Totally $O(|V|^2|E|)$!

No improvement!

Solving Subproblems

- $dist[k, u, v] = \min_{(s,v) \in E} \{dist[k-1, u, s] + d(s, v)\}$

$k-1$	v_1	v_2	v_3	…	$v_{\|V\|}$
v_1					
v_2					
v_3					
v_4					
…					
$v_{\|V\|}$					

k	v_1	v_2	v_3	…	$v_{\|V\|}$
v_1					
v_2					
v_3					
v_4			$f[k, u, v]$		
…					
$v_{\|V\|}$					

The point for improvement

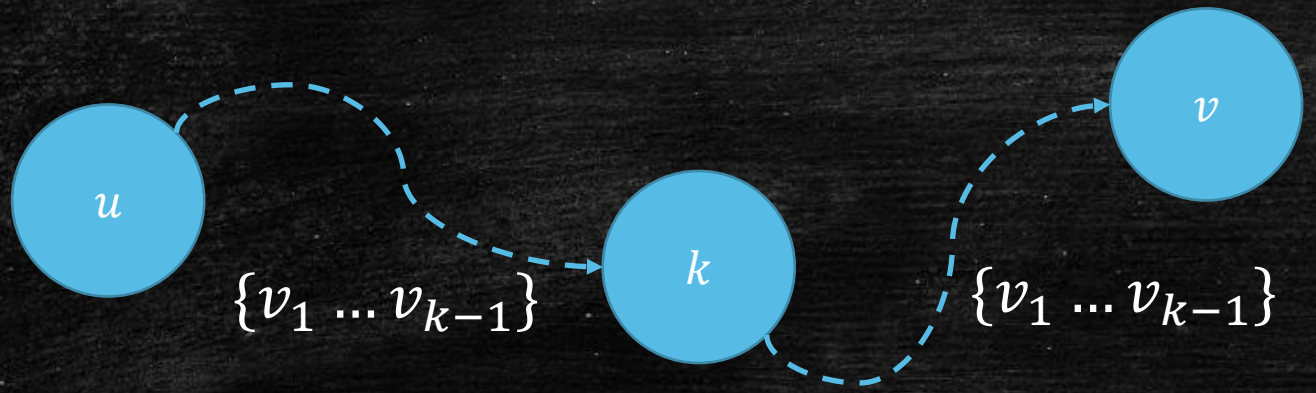
- Transfer:
  - $dist[k, u, v] = \min_{(s,v)\in E}\{dist[k-1, u, s] + d(s, v)\}$
- Problem
  - We know all states of $dist[k-1, u, v]$.
  - We only use $dist[k-1, u, s]$ to update $dist[k-1, u, v]$. (start from u)
  - Different start points are independent.
  - The same as run $|V|$ times Bellman-Ford.

Floyd-Warshall: Subproblems

- Natural Generalization: $dist[k, u, v]$: the shortest distance from u to v among all **k-edge-path (path with at most k edges)**.
- Floyd-Warshall: $dist[k, u, v]$: the shortest distance from u to v that **only across inter-vertices in $\{v_1 \dots v_k\}$.**
- Remark:
  - We can label vertices from 1 to $|V|$.
  - $dist[0, u, v]$ is exactly $d(u, v)$ or ∞. (allow 0 inter-vertex)
  - $dist[|V|, u, v]$ is exactly what we want!

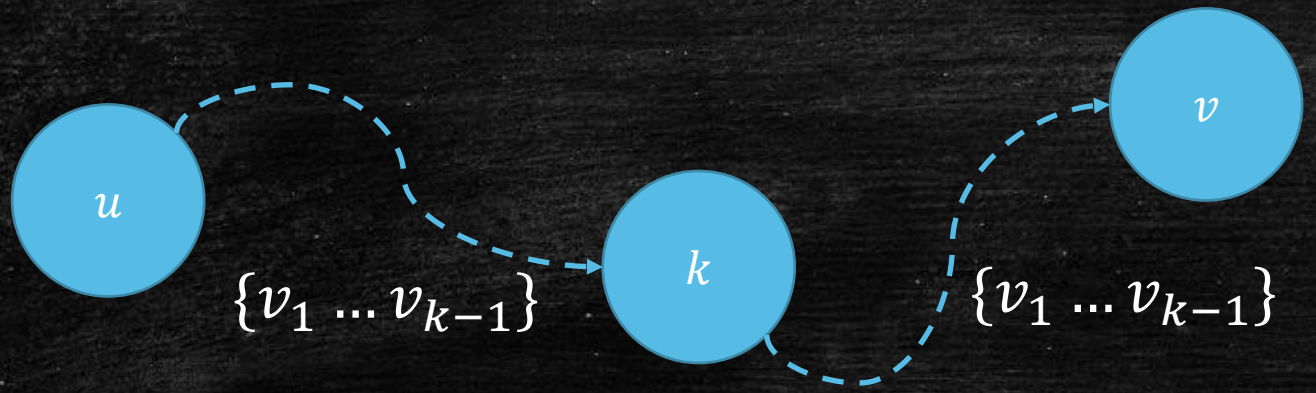
Key Fact

- If there is no negative cycle, v_k only appear once.
- Focusing on positive edge weight.
- v_k on any $dist[k', u, v]$ for $k' \geq k$.



Floyd-Warshall: Solving Subproblems

- $dist[k, u, v]$: the shortest distance from u to v that only **across inter-vertices in $\{v_1 \dots v_k\}$.**
- Solve $dist[k, u, v]$ (give addition power k to all pairs)
  - Case 1: the shortest path do not go across k.
  - Case 2: the shortest path go across k.
  - $dist[k, u, v] = \min\{dist[k-1, u, v], dist[k-1, u, k] + dist[k-1, k, v]\}$



Solving Subproblems

- $dist[k, u, v] = \min\{dist[k-1, u, v], dist[k-1, u, k] + dist[k-1, k, v]\}$

$k-1$	v_1	v_2	v_3	…	$v_{\|V\|}$
v_1					
v_2					
v_3					
v_4					
…					
$v_{\|V\|}$					

k	v_1	v_2	v_3	…	$v_{\|V\|}$
v_1					
v_2					
v_3					
v_4			$f[k, u, v]$		
…					
$v_{\|V\|}$					

DAG and Topological

- $dist[k, u, v]$ only depends
  - $dist[k-1, u, v]$
  - $dist[k-1, u, k]$
  - $dist[k-1, k, v]$
- We initialize $dist[0, u, v] = d(u, v)$ for all (u, v).
- Solve them from $k = 1$ to n is a topological order.
- Running Time: $3 \cdot O(|V| \cdot |V| \cdot |V|)$

Floyd-Warshall

Floyd-Warshall $O(|V|^3)$

function floyd_warshall(G)
    $dist[0, u, v] = d(u, v)$ for all $(u, v) \in E$, $dist[0, u, v] = \infty$ otherwise.
    for $k = 1$ to $|V|$
        for $u = 1$ to $|V|$
            for $v = 1$ to $|V|$
                $dist[k, u, v] = \min\{dist[k-1, u, v], dist[k-1, u, k] + dist[k-1, k, v]\}$

Floyd-Warshall: a simpler implement

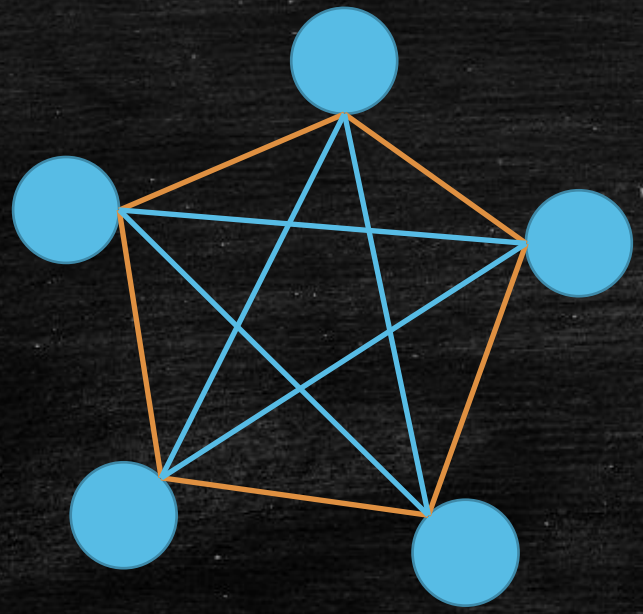
Floyd-Warshall

function floyd_warshall(G)
    $dist[u, v] = d(u, v)$ for all $(u, v) \in E$, $dist[u, v] = \infty$ otherwise.
    for $k = 1$ to $|V|$
        for $u = 1$ to $|V|$
            for $v = 1$ to $|V|$
                $dist[u, v] = \min\{dist[u, v], dist[u, k] + dist[k, v]\}$

$O(|V|^3)$ running time but $O(|V|^2)$ space! Why it is correct?

Traveling Salesman Problem (TSP)

- **Input:** A complete weighted undirected graph G, such that $d(u, v) > 0$ for each pair u, v $(u \neq v)$.
- **Output:** the cycle of n vertices with the minimum weight.

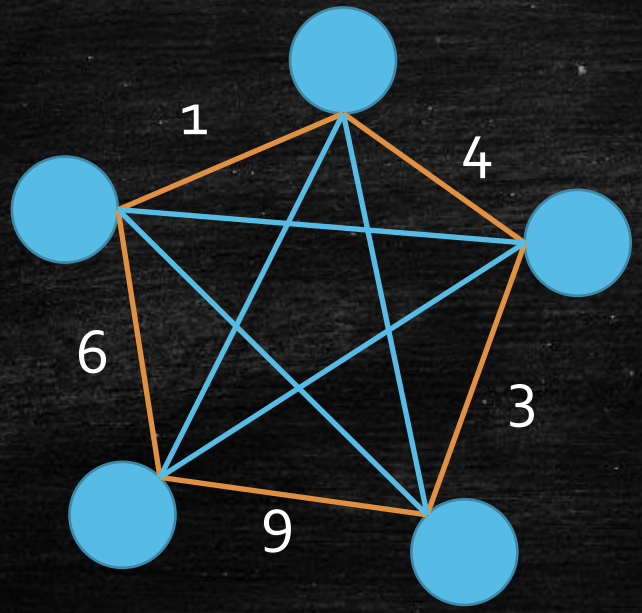


How to brute-force

What is the time complexity?

TSP vs. Shortest Path

- TSP
  - **Output:** the cycle of n vertices with the minimum weight.
- All Pair Shortest Path
  - **Output:** the minimum weight path from u to v.

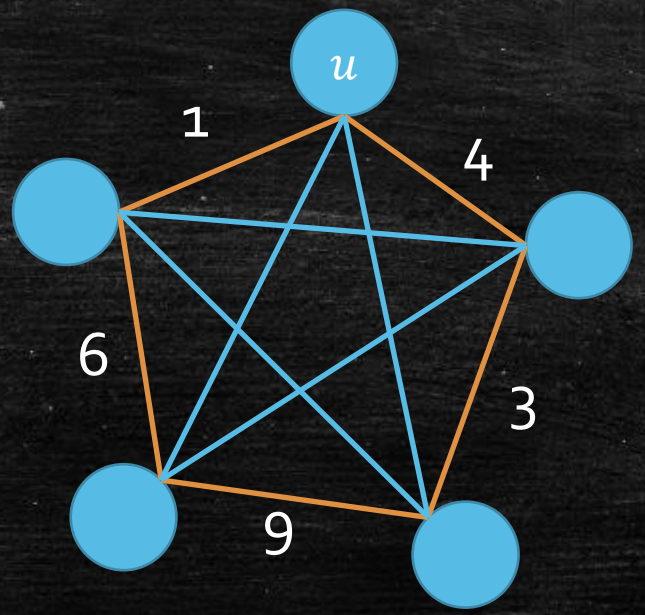


Subproblems in Shortest Path Problem

- $f[k, u, v]$
  - The shortest path from u to v, with inter-vertex chosen in $v_1 \ldots v_k$.
- Is it good for TSP?
- What we should do now?

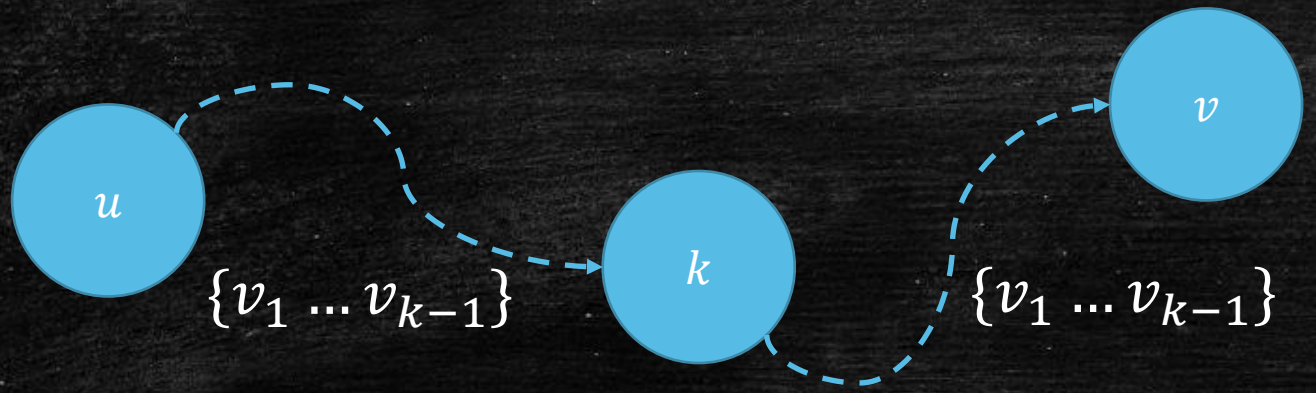
Plan A

- $f[k, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $v_1 \dots v_k$ except u and v.
- How to solve TSP?
  - $\min_u f[|V|, u, u]$ is what we want!
- How to solve $f[k, u, v]$?



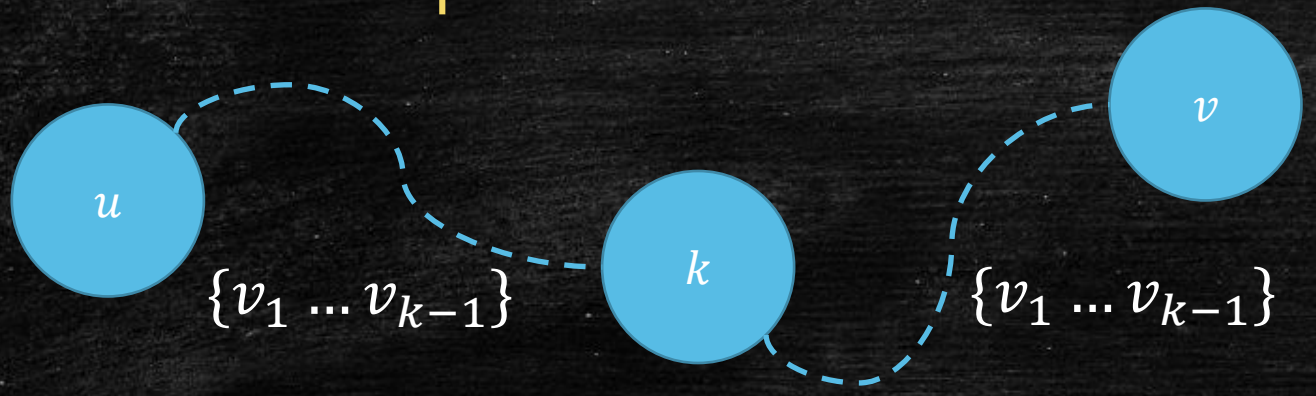
Floyd-Warshall: Solving Subproblems

- $dist[k, u, v]$: the shortest distance from u to v that only **across inter-vertices in $\{v_1 \dots v_k\}$.**
- Solve $dist[k, u, v]$ (give addition power k to all pairs)
  - Case 1: the shortest path do not go across k.
  - Case 2: the shortest path go across k.
  - $dist[k, u, v] = \min\{dist[k-1, u, v], dist[k-1, u, k] + dist[k-1, k, v]\}$



Plan A: Subproblem Definition

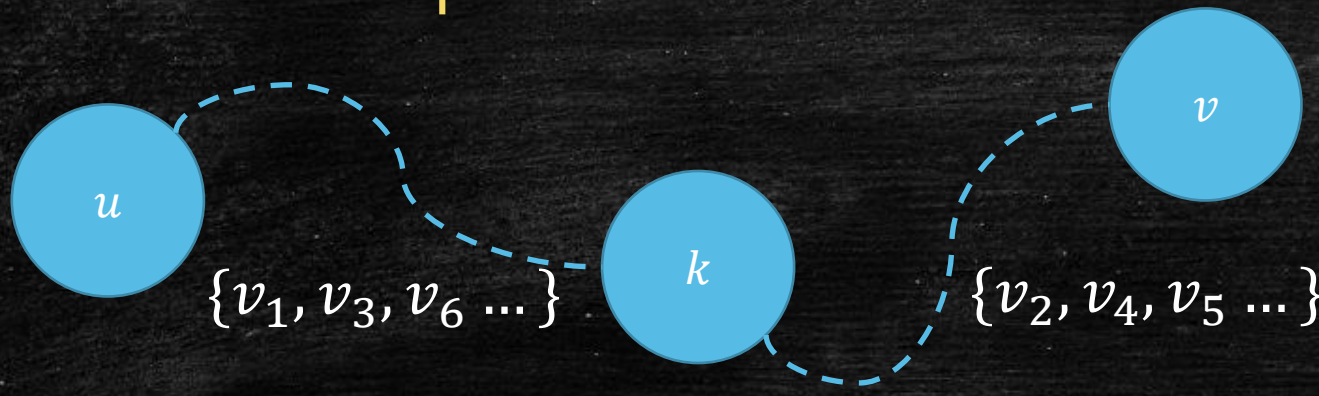
- $f[k, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $v_1 \dots v_k$ except u and v.
  - $\min_u f[|V|, u, u]$ is what we want!
- How to solve $f[k, u, v]$?
- What is the problem now?



Two sub paths can not contain same vertices.

Plan A: Why it is not enough?

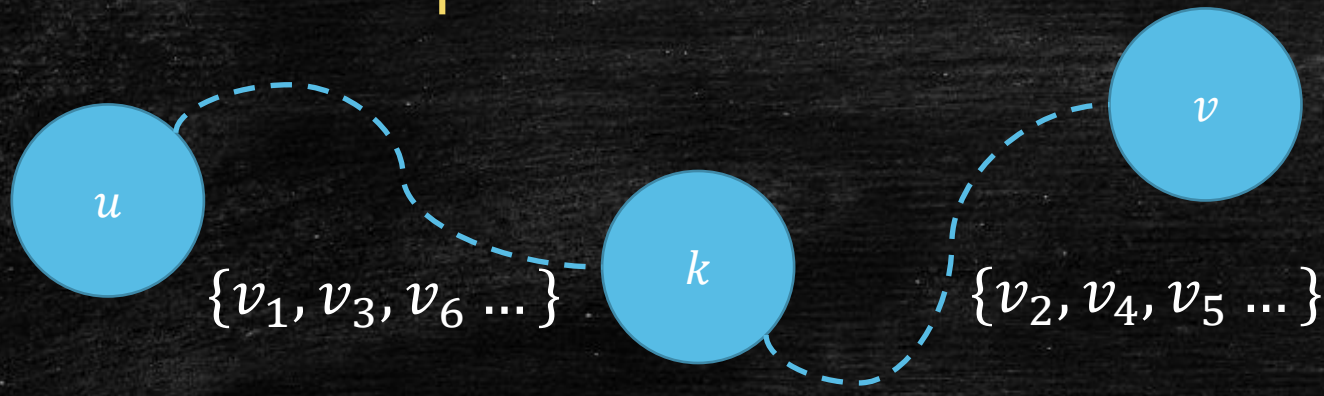
- $f[k, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $v_1 \dots v_k$ except u and v.
  - $\min_u f[|V|, u, u]$ is what we want!
- How to solve $f[k, u, v]$?
- What is the problem now?



We need to know
- what vertices $u \to k$ use?
- what vertices $k \to v$ use?

Plan A: Why it is not enough?

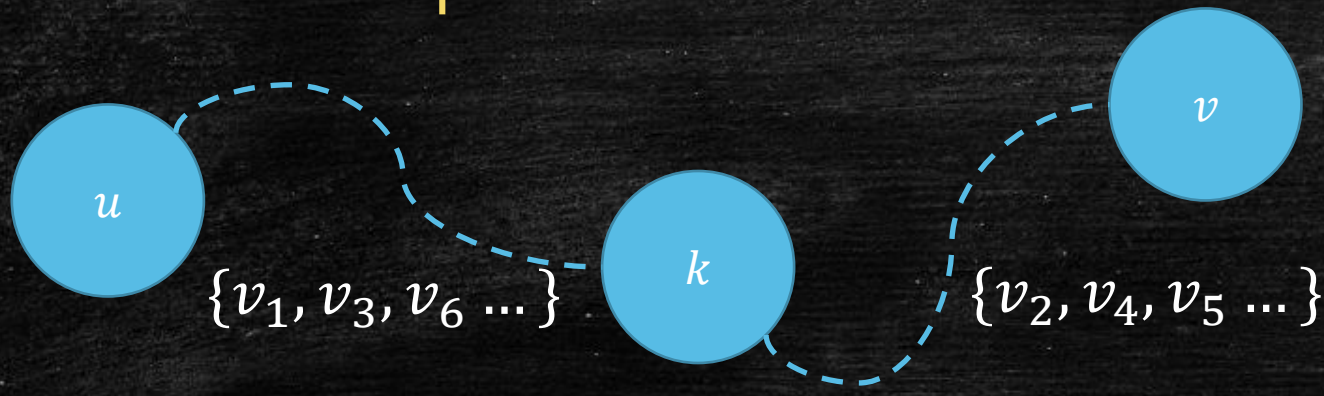
- $f[k, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $v_1 \dots v_k$ except u and v.
  - $\min_u f[|V|, u, u]$ is what we want!
- How to solve $f[k, u, v]$?
- What is the problem now?



We need to know
- what vertices $u \to k$ use?
- what vertices $k \to v$ use?

Plan A: Why it is not enough?

- $f[k, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $v_1 \dots v_k$ except u and v.
  - $\min_u f[|V|, u, u]$ is what we want!
- How to solve $f[k, u, v]$?
- What is the problem now?



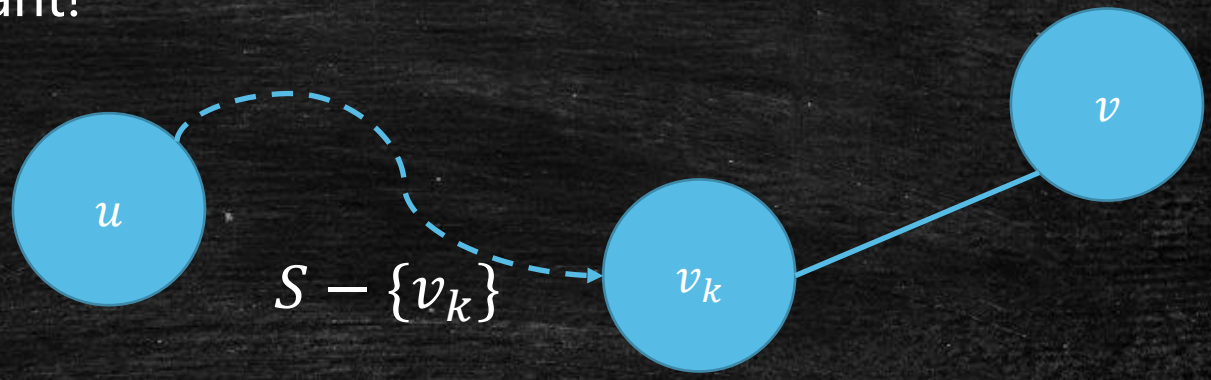
We need to know
- what vertices $u \to k$ use?
- what vertices $k \to v$ use?

We do not solve the subproblem $u \to k$ with $\{v_1, v_3, v_6 \dots\}$

Do you know how to fix?

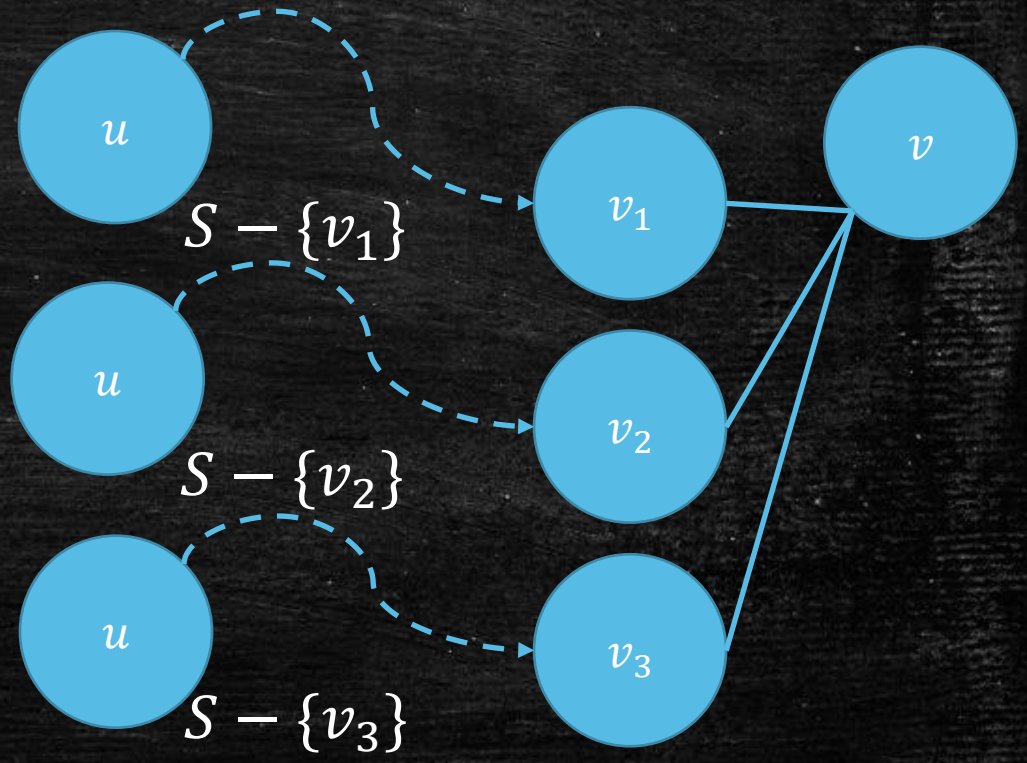
Plan B: Subproblem Definition

- $f[S, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $S \subset V$ except u and v.
  - $\min_u f[V, u, u]$ is what we want!
- How to solve $f[S, u, v]$?



Plan B: Solving Subproblems

- $f[\boldsymbol{S}, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $\boldsymbol{S} \subset \boldsymbol{V}$ except u and v.
  - $\min_u f[\boldsymbol{V}, u, u]$ is what we want!
- How to solve $f[\boldsymbol{S}, u, v]$?
- $f[S, u, v] = \min_{k \in S} f[S - \{k\}, u, k] + d(k, v)$

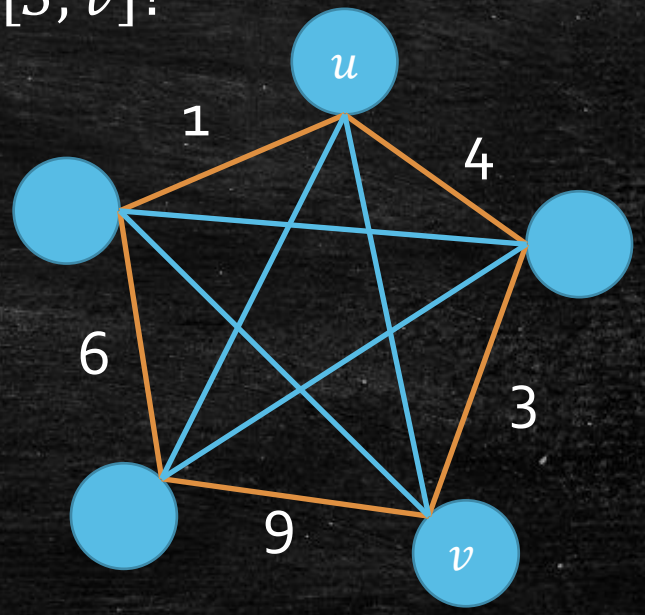


Summarize

- $f[S, u, v]$
  - The shortest path from u to v with inter-vertex **exactly** $S \subset V$ except u and v.
  - $\min_u f[V, u, u]$ is what we want!
- $f[S, u, v] = \min_{k \in S} f[S - \{k\}, u, k] + d(k, v)$
- Do you know the topological order of the DP?

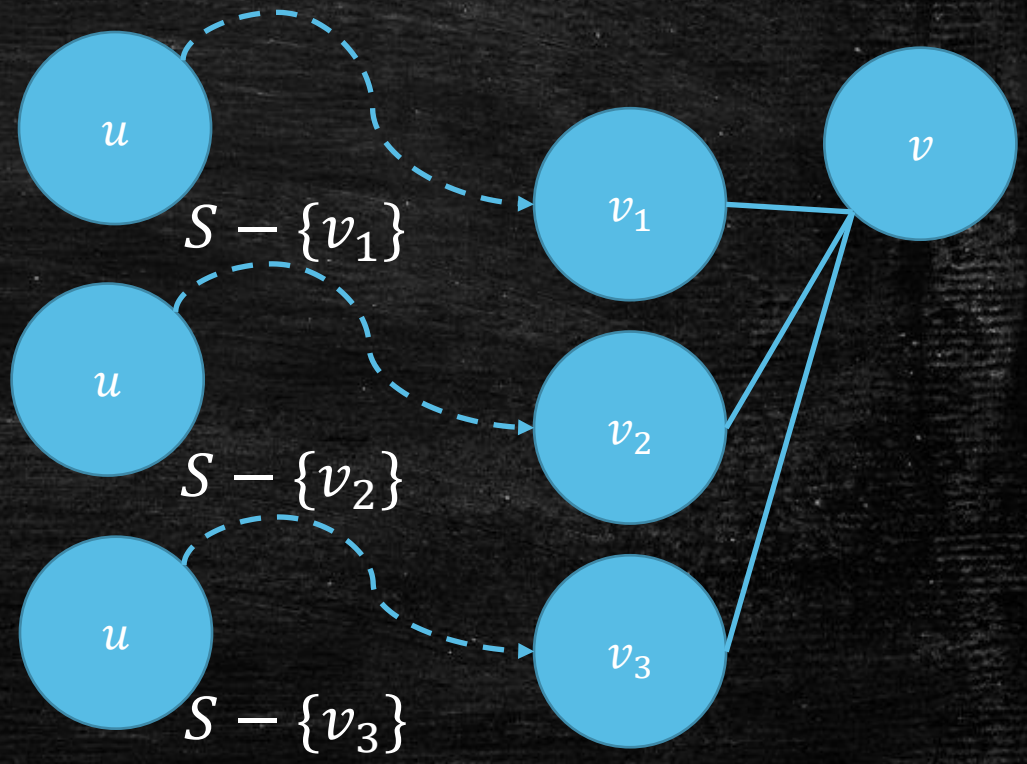
A little improvement:

- $\forall u, v, f[V, u, u] = f[V, v, v]$!
- We only need to know one fixed $f[V, u, u]$.
- Can we fix an arbitrary u and only solve $f[S, v]$?



Solve $f[S, v]$!

- $f[S, u, v] = \min_{k \in V} f[S - \{k\}, u, k] + d(k, v)$
- $f[S, u, v]$ only comes from $f[S - \{k\}, u, k]$.
- It is enough for us to only record $f[S, v]$.
- $f[S, v] = \min_{k \in V} f[S - \{k\}, k] + d(k, v)$.



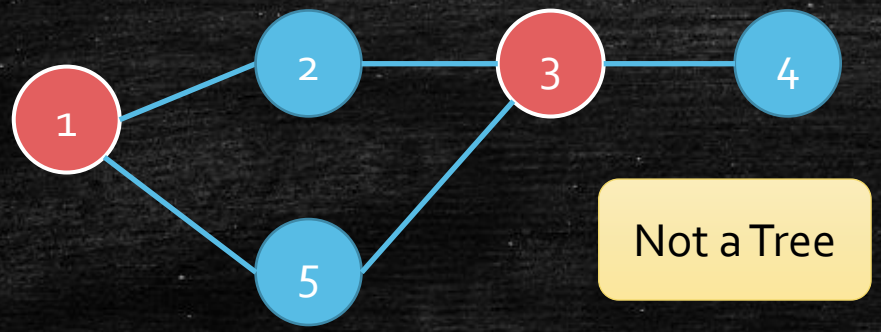
Time Complexity

- Time complexity $(n = |V|)$?
  - $O(n2^n)$ subproblems.
  - $O(n)$ solving.
  - $\boldsymbol{O(n^2 2^n)}$ totally!
- Comparing to Brute-force
  - Brute-force: $\boldsymbol{O(n!)}$
  - Do you know why $O(n^2 2^n)$ is better than $O(n!)$?
  - Do you know why DP is better than brute-force?
- Do you know how to implement $f[S, v]$?
  - S is a set.
  - Use $v(s) \in (0 \dots 2^n)$ to represent!

Solve Problems on Trees

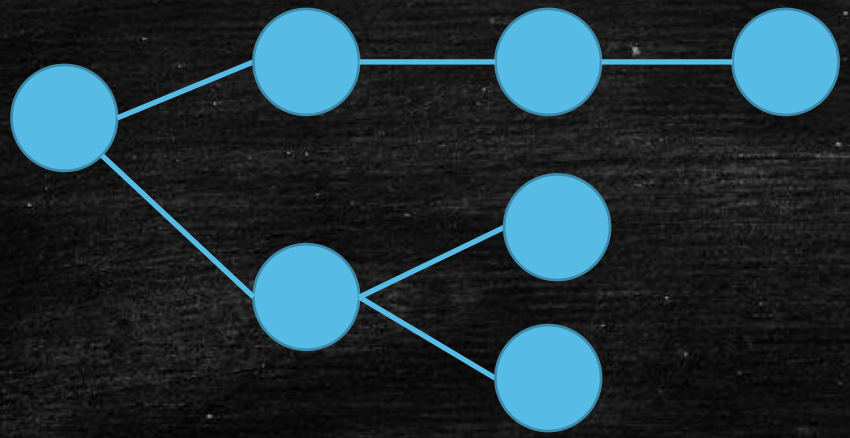
Maximize Independent Set on Trees

- **Input:** an undirected tree $G = (V, E)$.
- **Output:** an independent set with maximum cardinality (number of vertices)
- **Independent Set:** a set S of vertices:
  - $\forall u, v \in S$, we have $(u, v) \notin E$



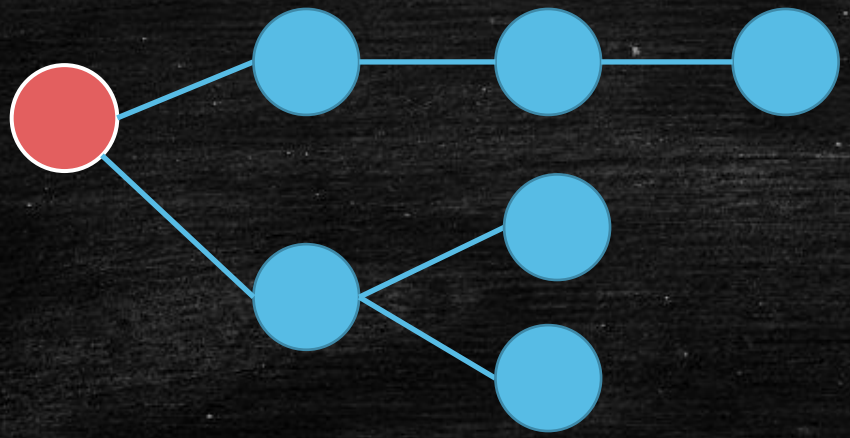
Maximize Independent Set ~~on Trees~~

- Maximize Independent Set ~~on Trees~~ is NP-hard.
- Is the tree special case easier?



Solve it Recursively

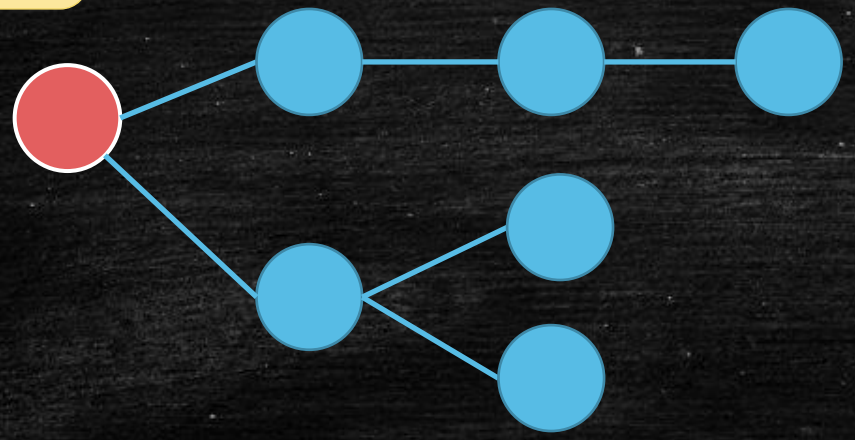
- Start from the root
  - Case 1: We choose the root, what happens?
  - Case 2: We do not choose the root, what happens?



Start from recursive

- Start from the root
  - Case 1: We choose the root, what happens?
  - Case 2: We do not choose the root, what happens?

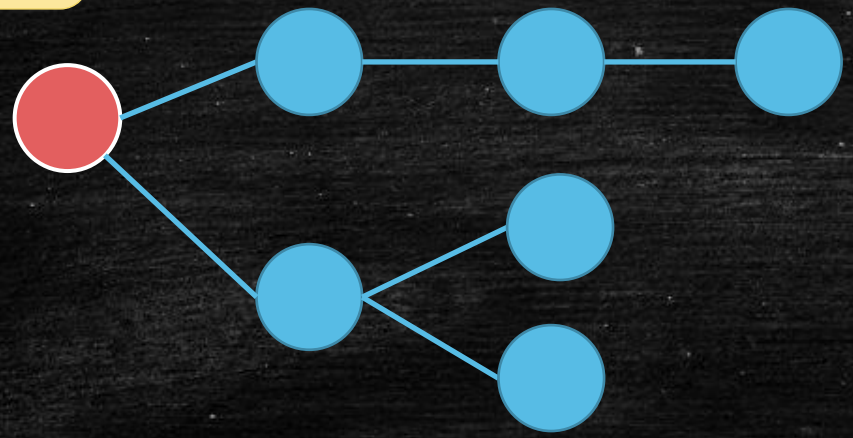
Case 1: We can not choose its children.



Start from recursive

- Start from the root
  - Case 1: We choose the root, what happens?
  - Case 2: We do not choose the root, what happens?

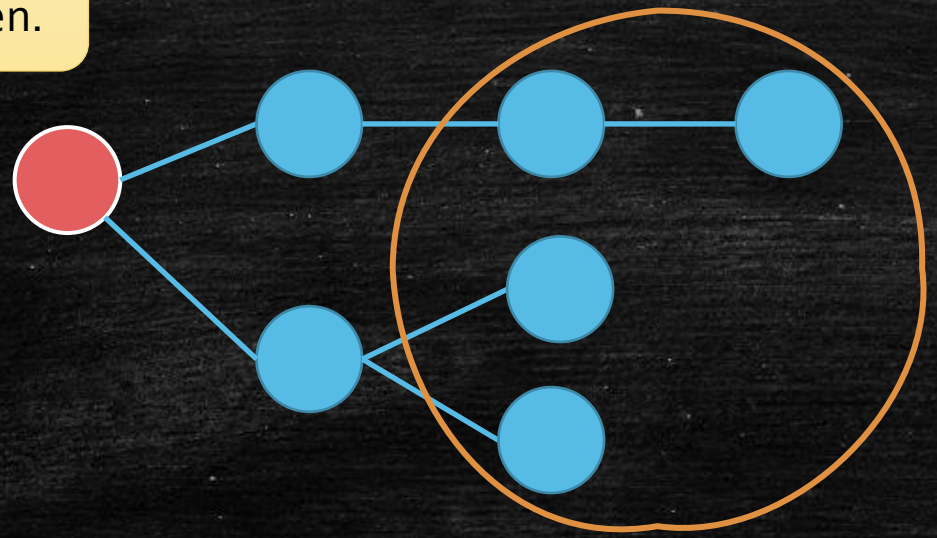
Case 2: We can choose its children.



What subproblems do we need to solve?

Case 1: We can not choose its children.

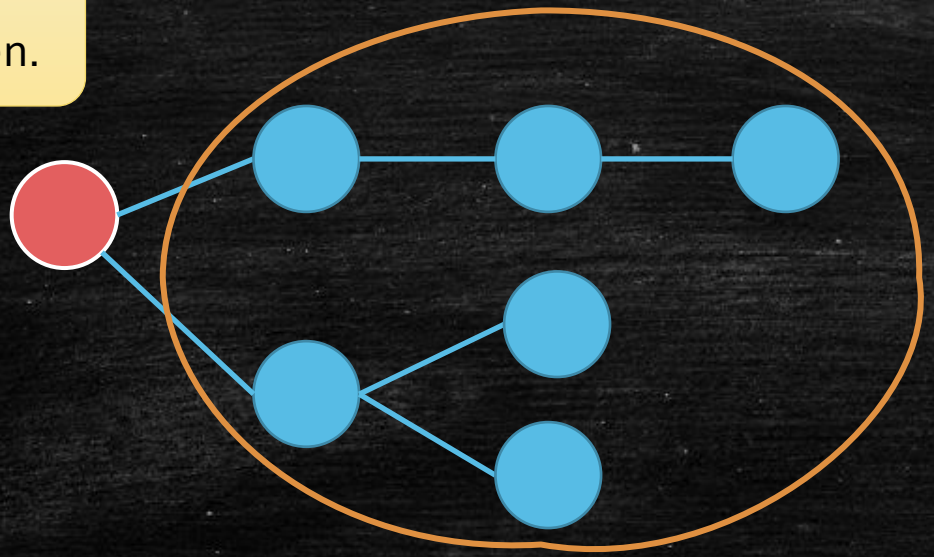
We need to know the max independent set here.



What subproblems do we need to solve?

Case 2: We can choose its children.

We need to know the max independent set here.

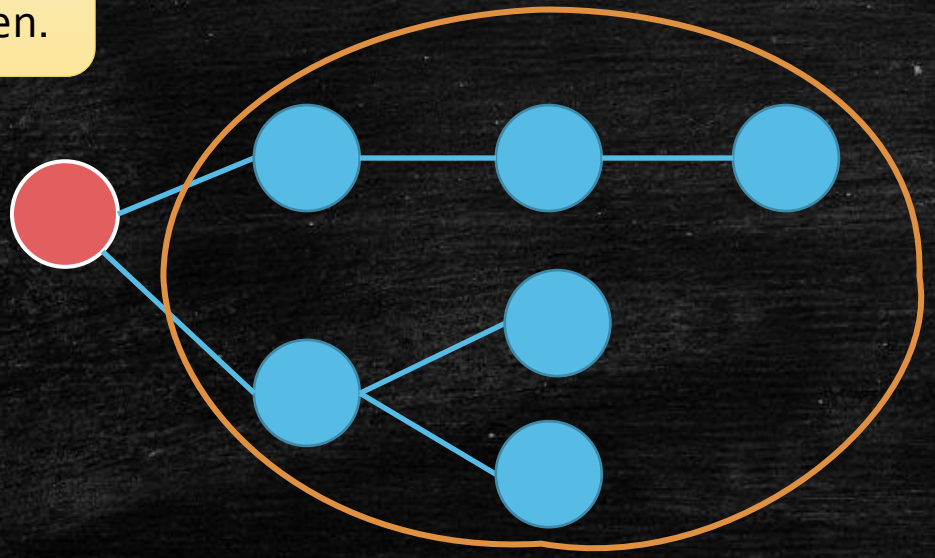


How to define subprobelms?

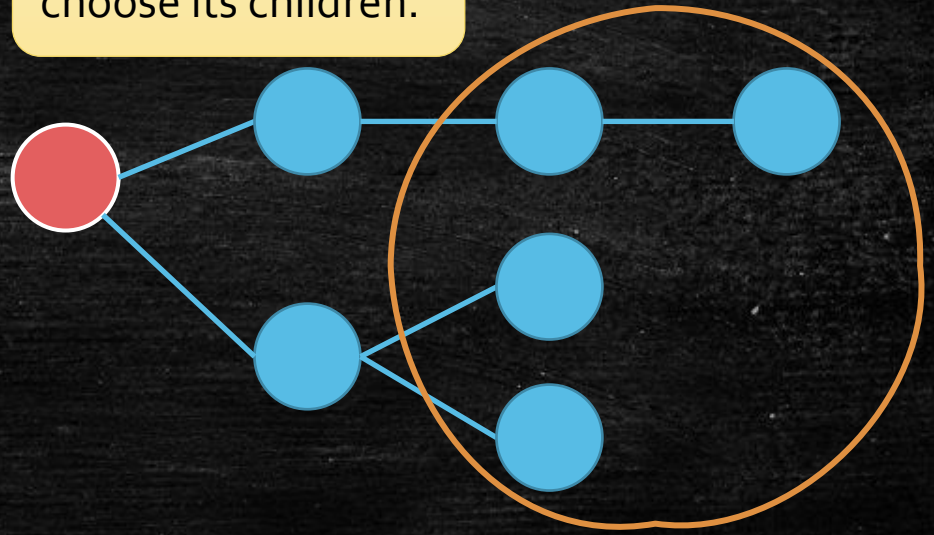
Subproblem Definition

- Subproblem $f[v]$: the maximized size of independent set of the subtree rooted at v.

Case 2: We can choose its children.



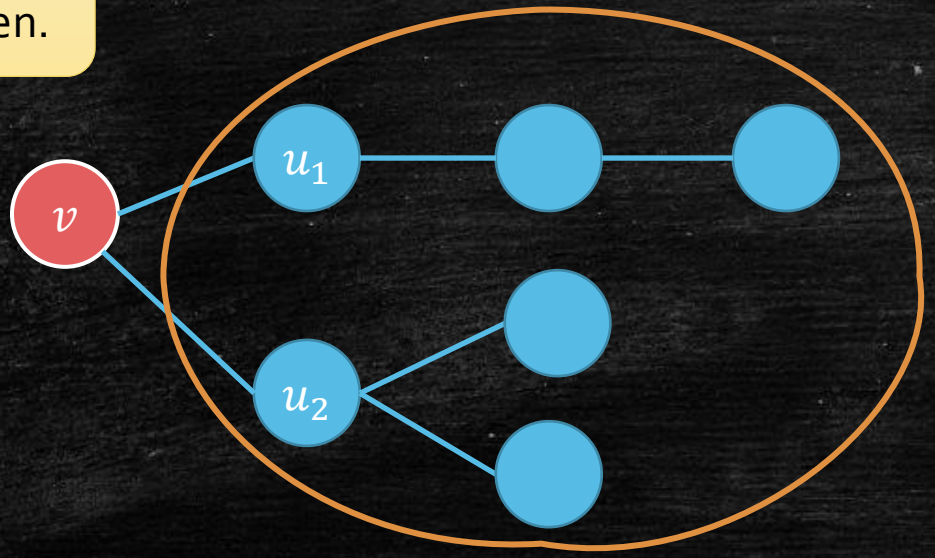
Case 1: We can not choose its children.



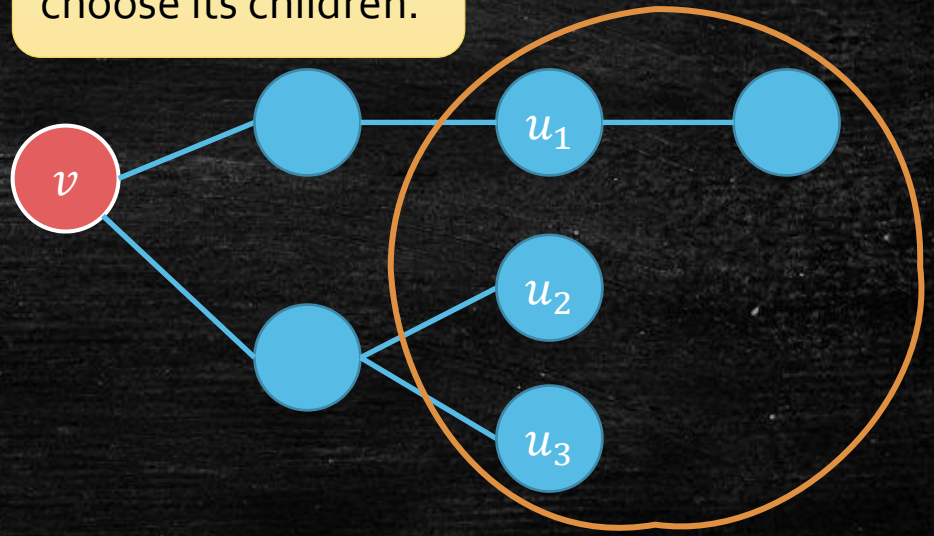
Subproblem Solving

- Subproblem $f[v]$: the maximized size of independent set of the subtree rooted at v.
- $f[v] = \max\{\sum_{u \in children(v)} f[u], \sum_{u \in grandchildren(v)} f[u] + 1\}$

Case 2: We can choose its children.



Case 1: We can not choose its children.



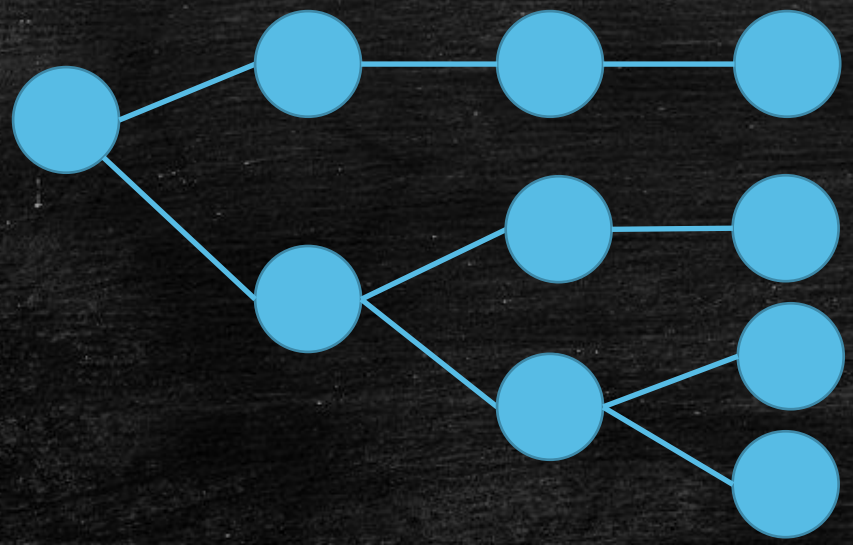
Running Time

- $f[v] = \max\{\sum_{u \in children(v)} f[u], \sum_{u \in grandchildren(v)} f[u] + 1\}$
- Looks $O(n^2)$.
  - We have n subproblems.
  - Each take $O(n)$ times.
- But it is $O(n)$.
  - Each of its children and its grandchildren cost one.
  - On other words, each vertex only need to pay one for its **parent** and one for its **grandparent**.
  - Totally $O(n)$.
  - Question: how to find a bottom-up order?

What is the topological order of the DP?

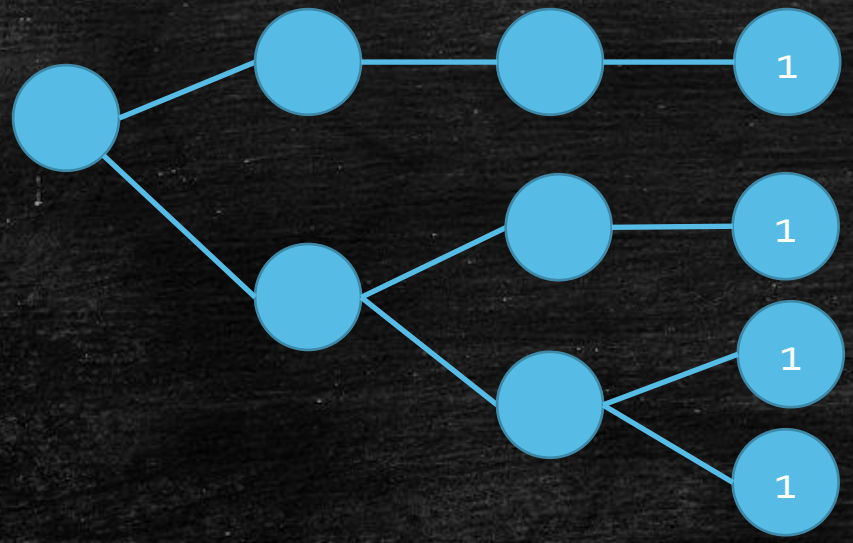
It is also a greedy algorithm!

- Try to solve it bottom-up!



It is also a greedy algorithm!

- Try to solve it bottom-up!



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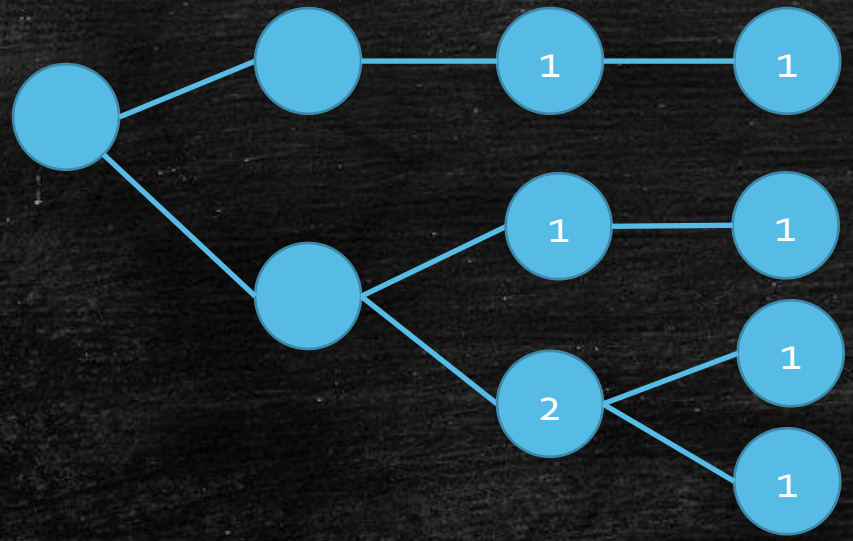

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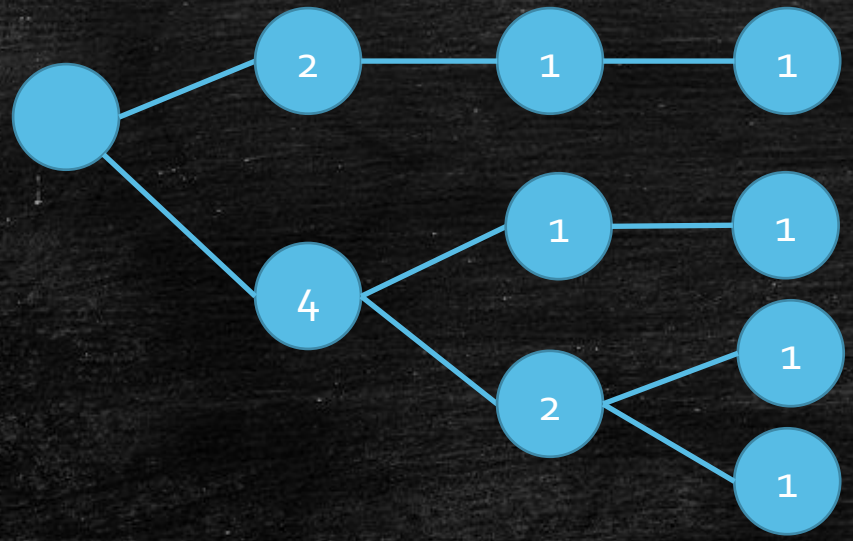

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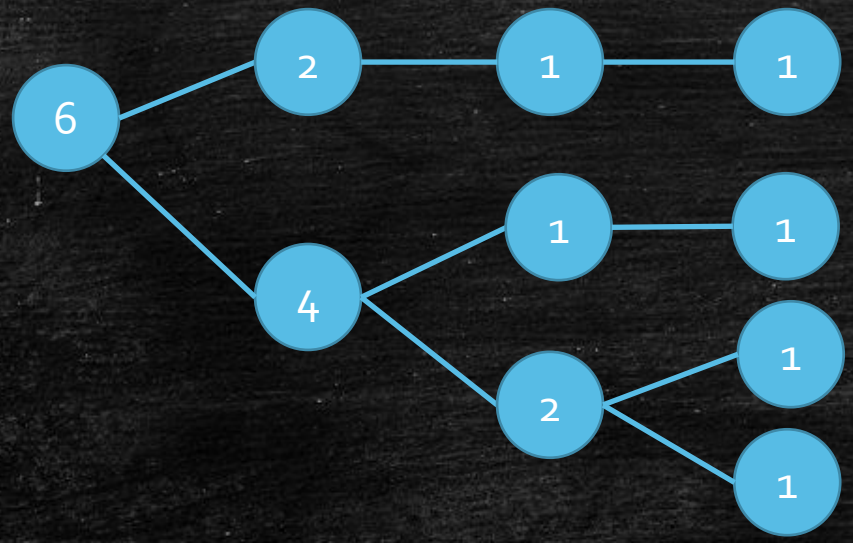

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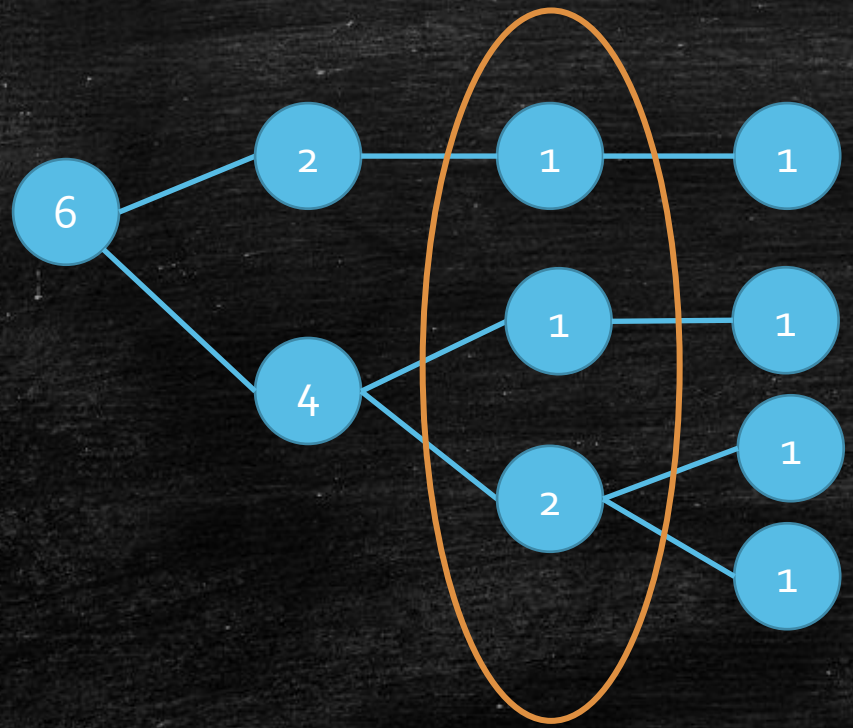
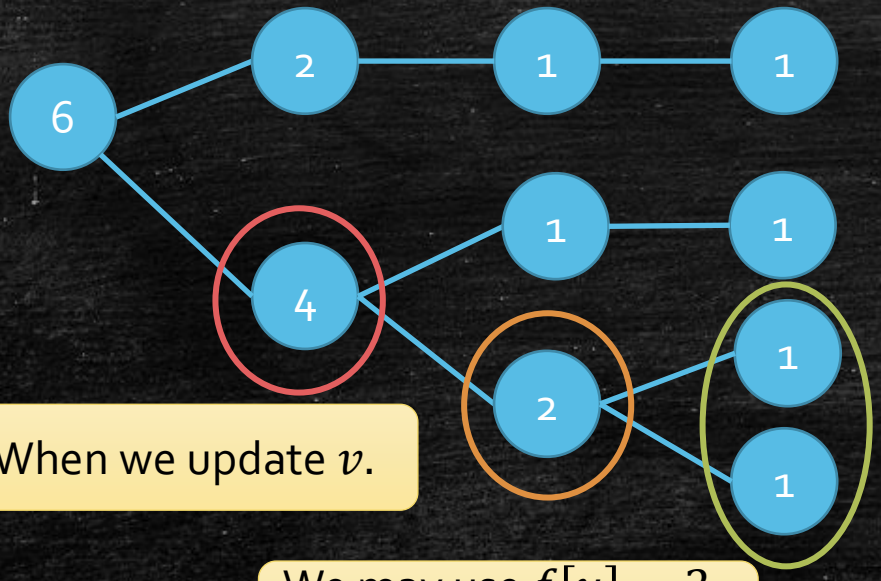
- Try to solve it bottom-up!



They are useless!

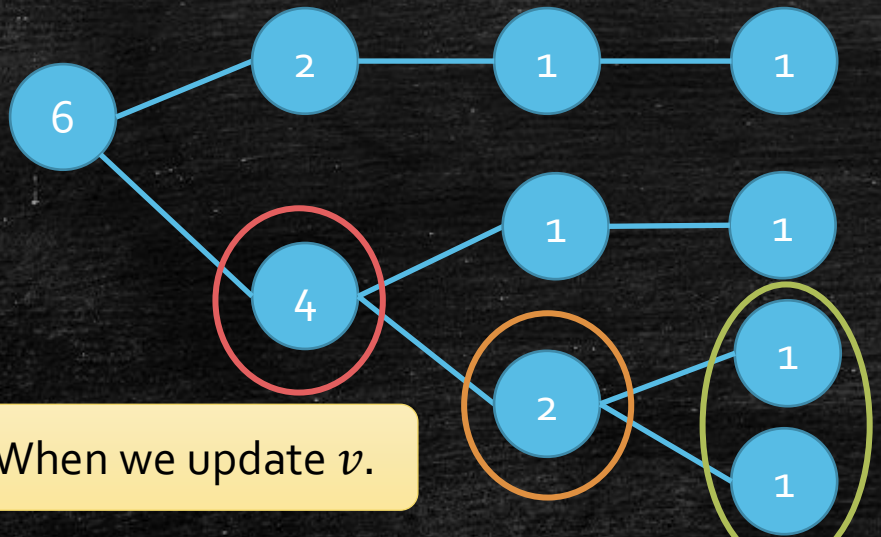
It is also a greedy algorithm!

- Try to solve it bottom-up!



It is also a greedy algorithm!

- Try to solve it bottom-up!
- $\sum_{u \in children(v)} f[u]$



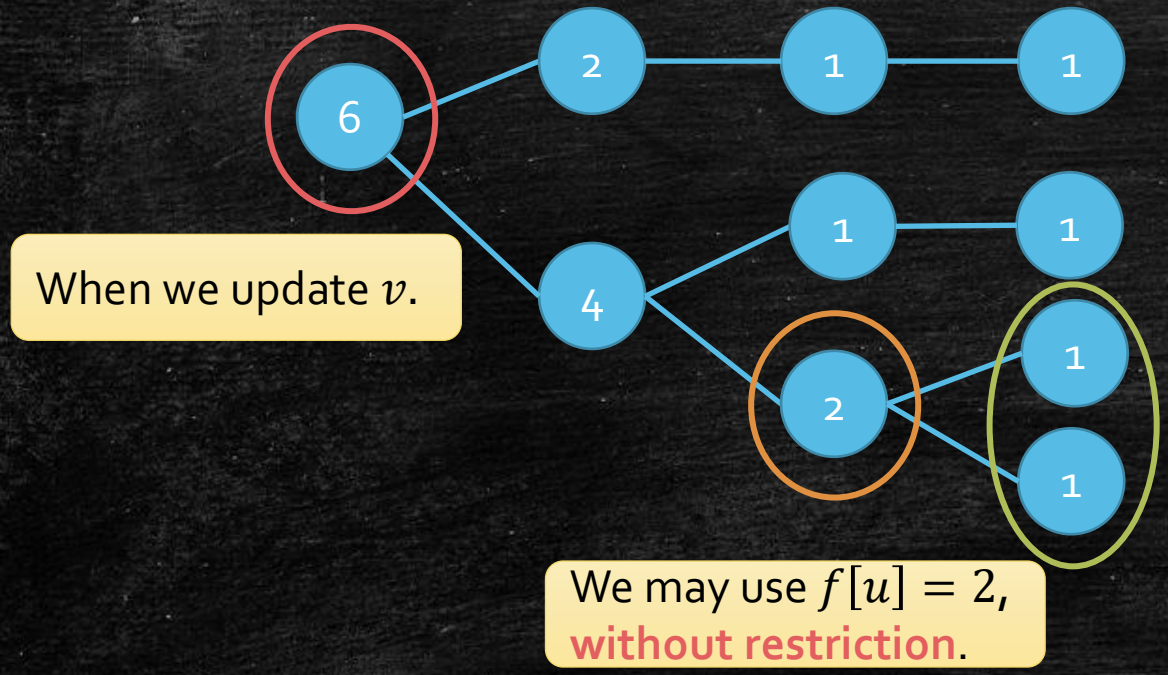
When we update v.

We may use $f[u] = 2$, with restriction.

Why not use $f[u_1] + f[u_2]$ without restriction?

It is also a greedy algorithm!

- Try to solve it bottom-up!
- $\sum_{u \in grandchildren(v)} f[u] + 1$



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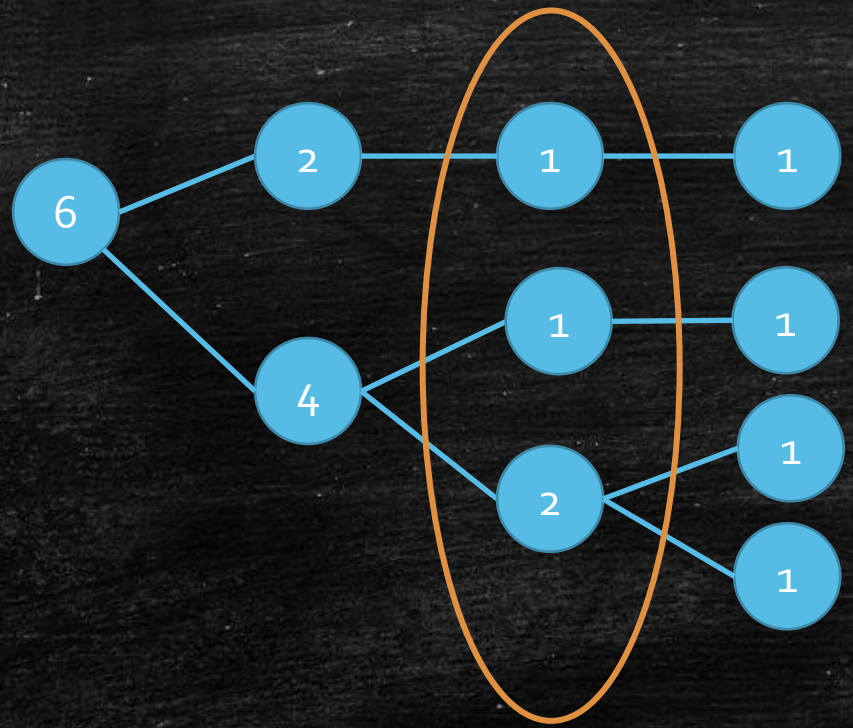
- Try to solve it bottom-up!



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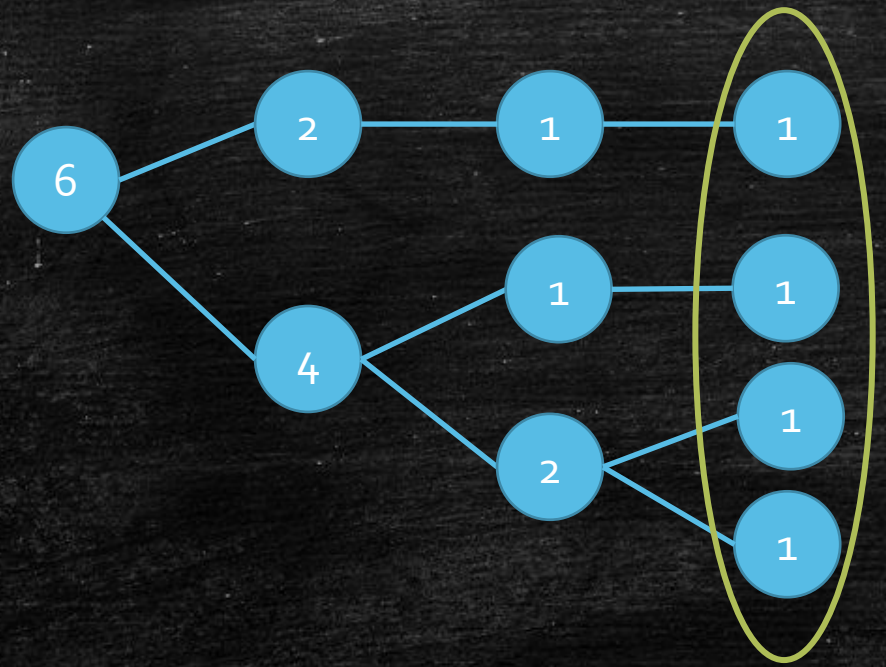
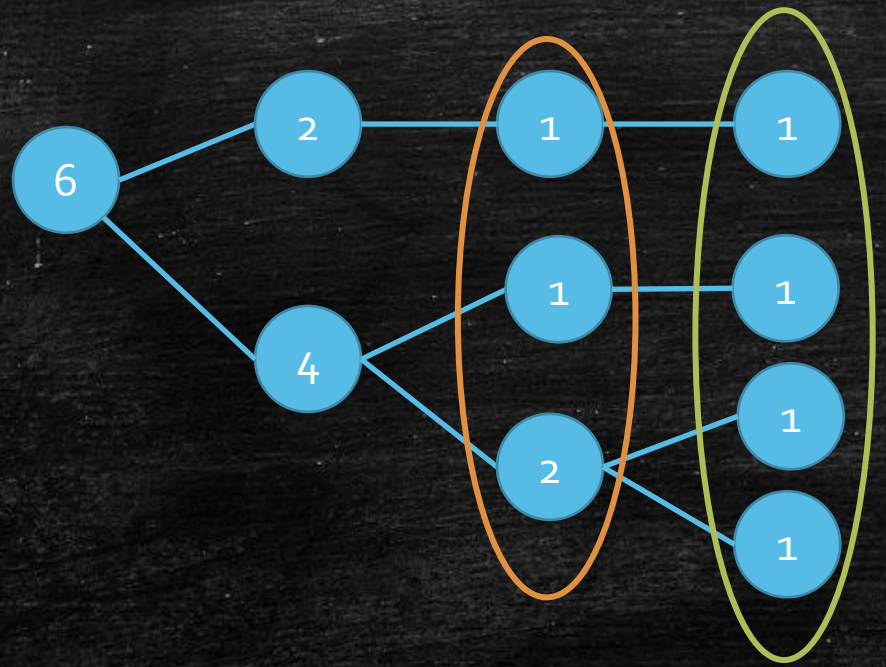


They are super useful!
They can update their ancestor **without restriction**!

It is also a greedy algorithm!

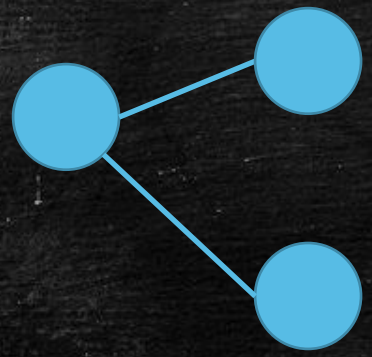
- Try to solve it bottom-up!

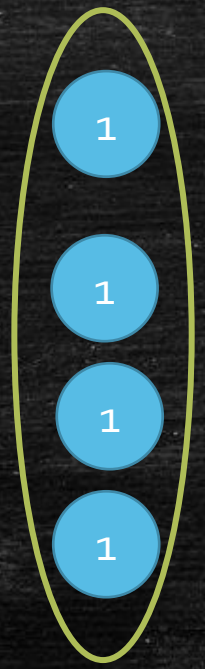


It is same to say, we choose Green and remove Orange.

It is also a greedy algorithm!

- Try to solve it bottom-up!





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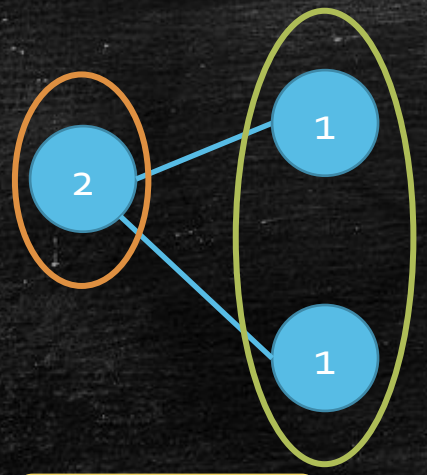
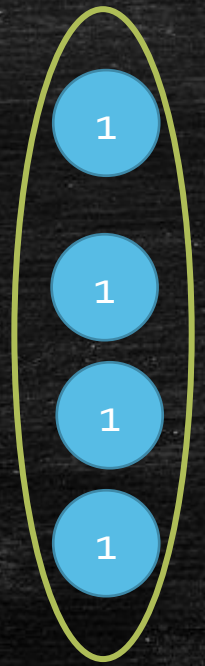


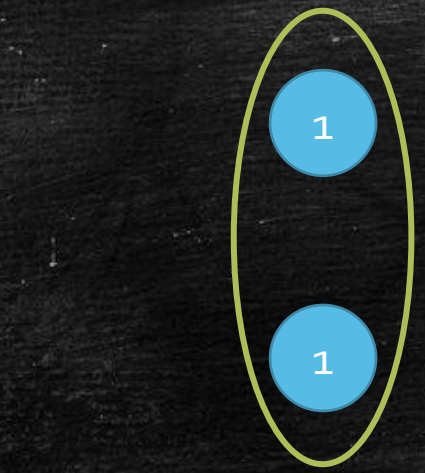
Do it again



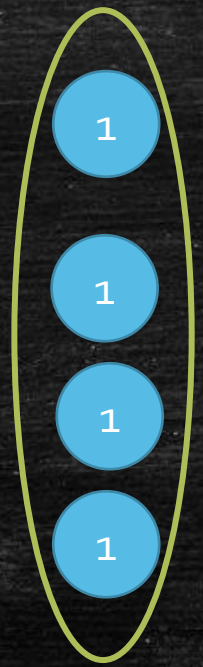
It is same to say, we choose Green and remove Orange.

It is also a greedy algorithm!

- Try to solve it bottom-up!



Do it again



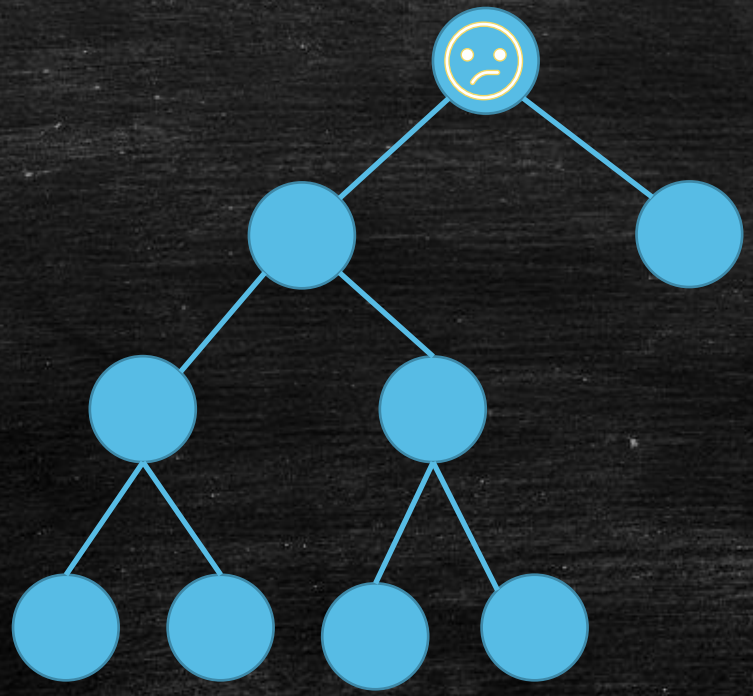
It is same to say, we choose Green and remove Orange.

The Greedy Algorithm

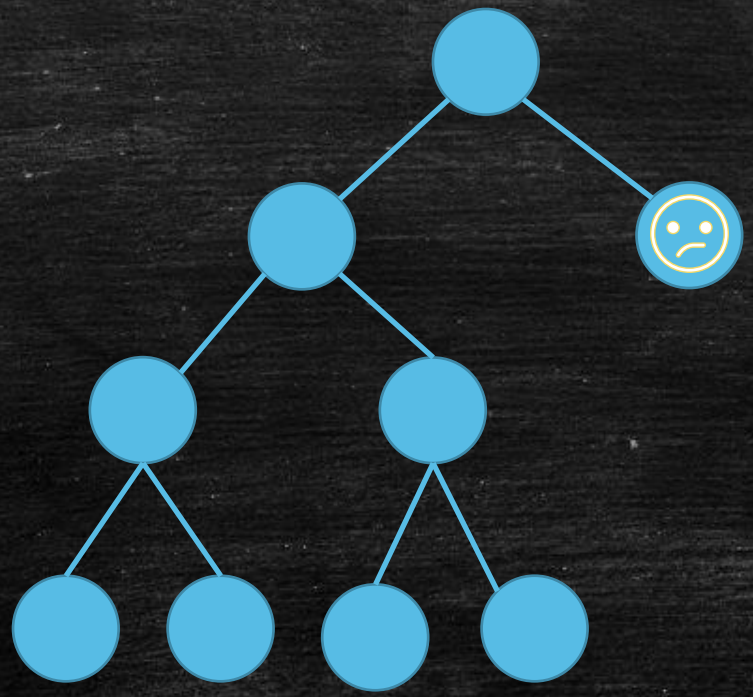
1. Choose all Leaves.
2. Remove all leaves' parents.
3. Repeat 1 again.

- How to implement it in $O(|V|)$?

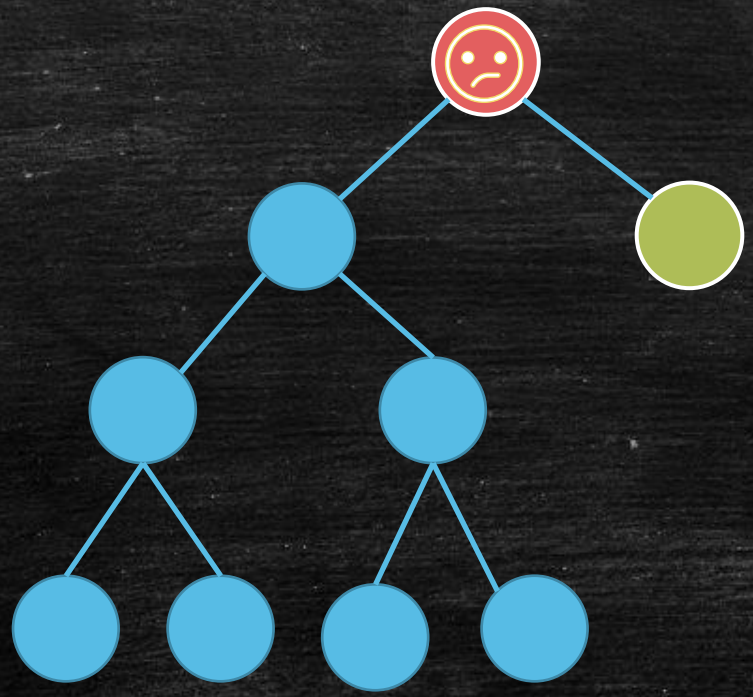
Greedy Algorithm Implementation



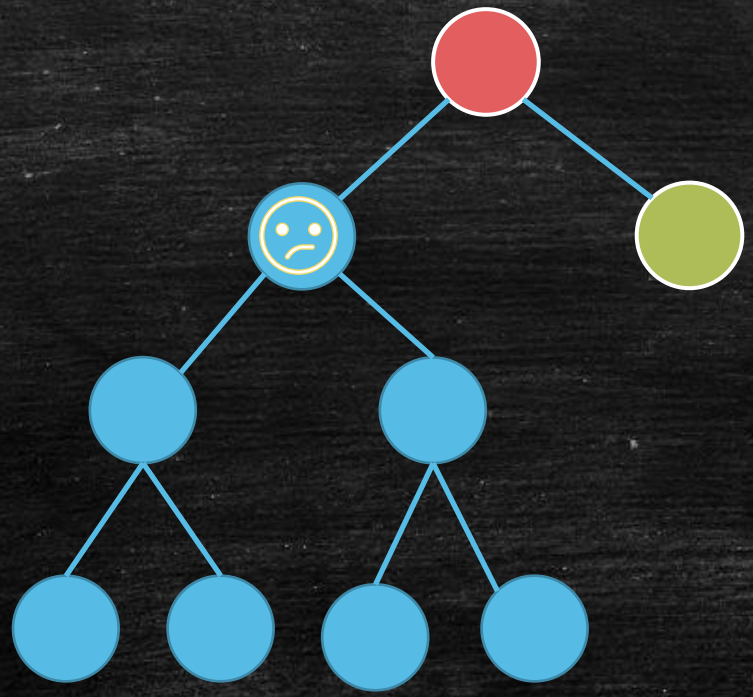
Greedy Algorithm Implementation



Greedy Algorithm Implementation



Greedy Algorithm Implementation



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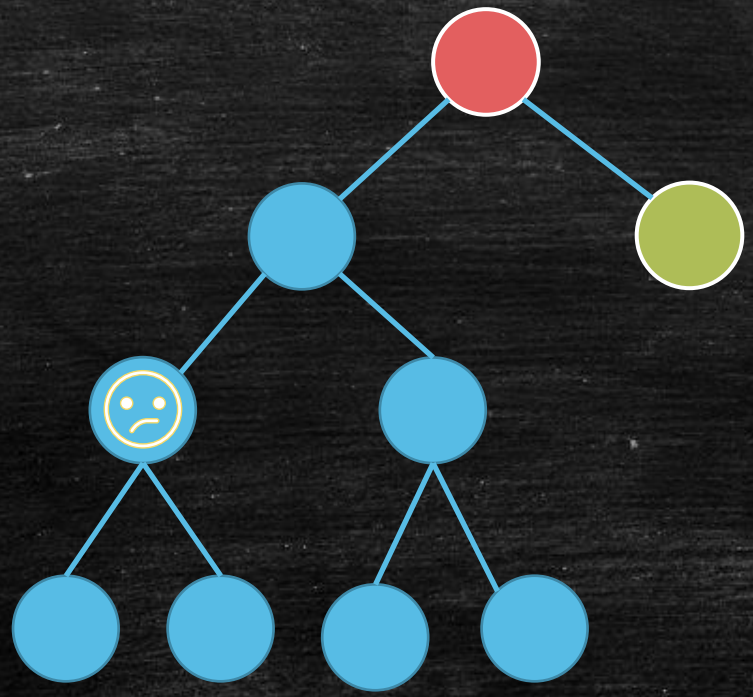
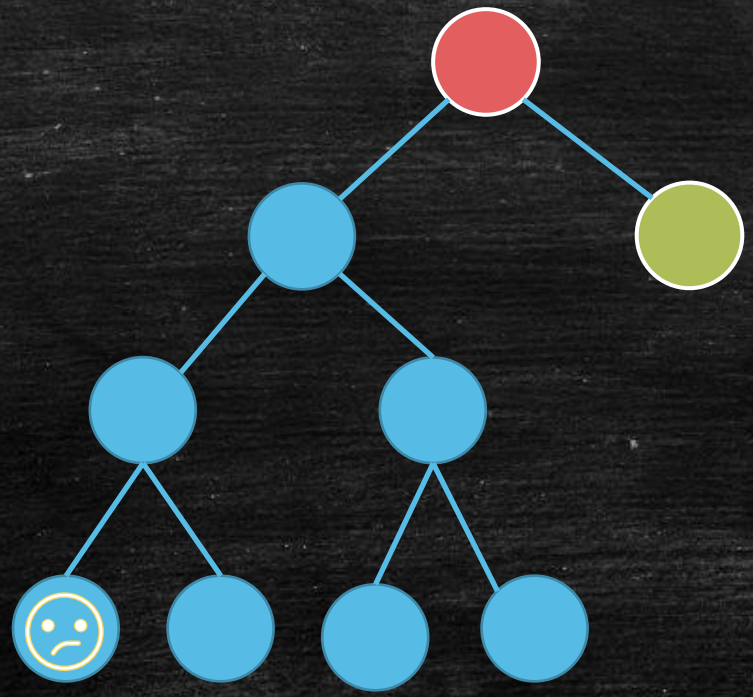


Greedy Algorithm Implementation



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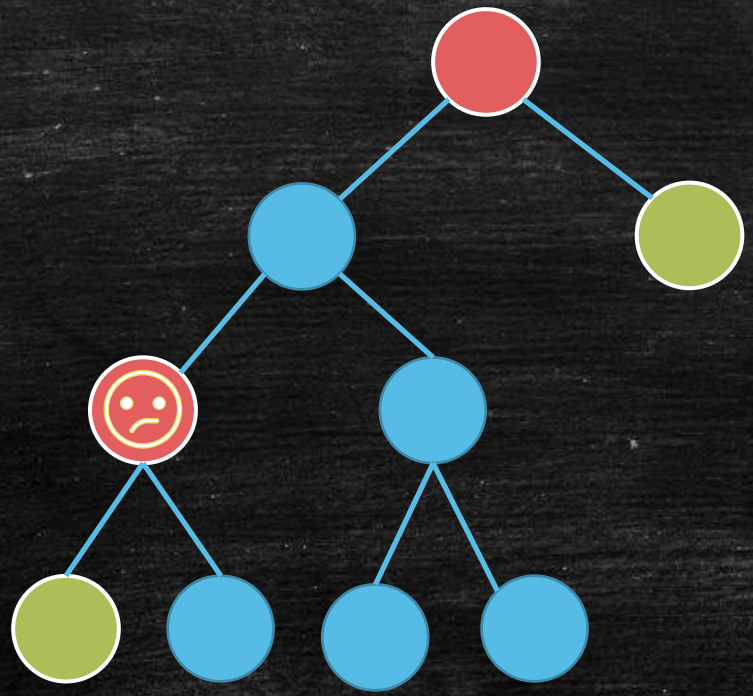

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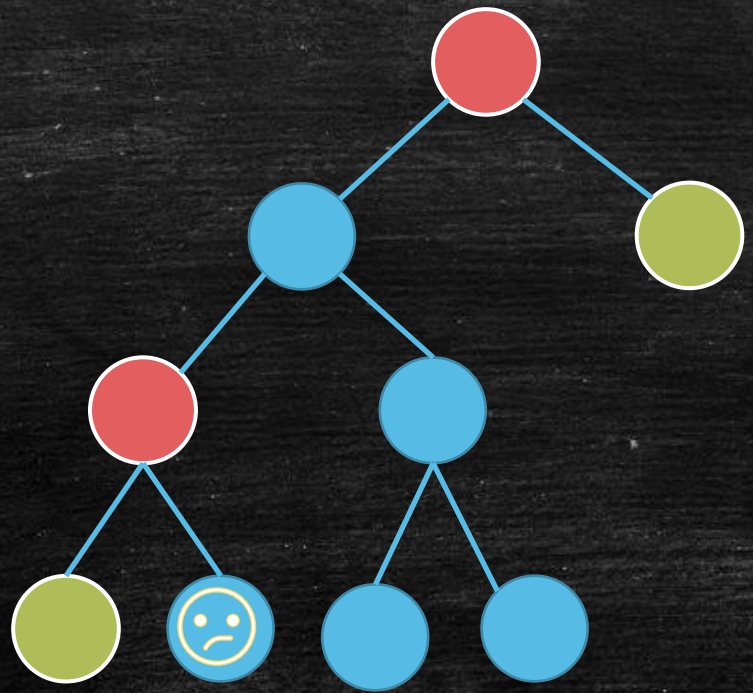
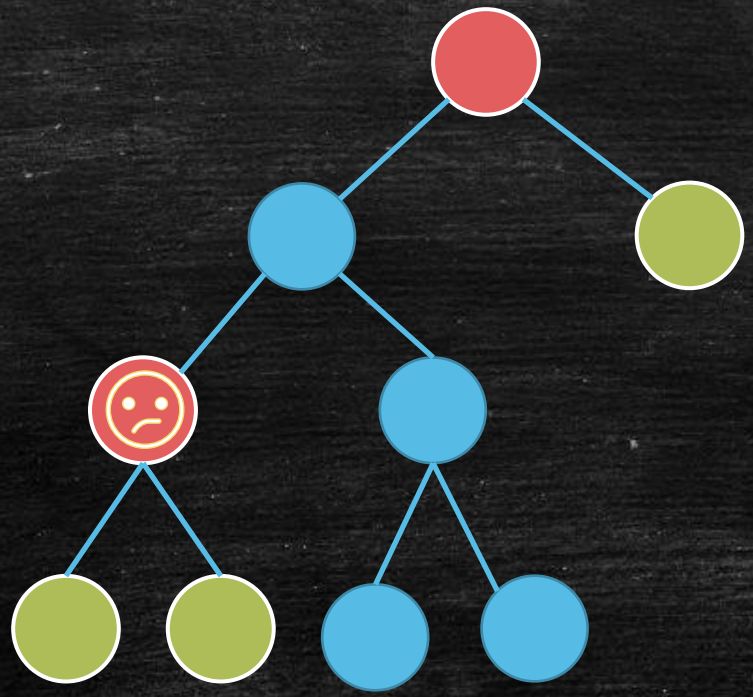
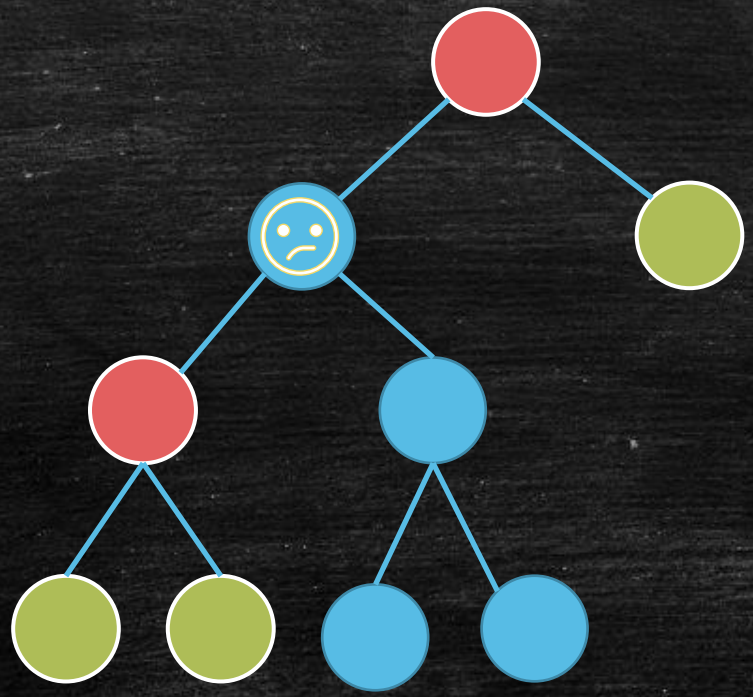


Greedy Algorithm Implementation



Greedy Algorithm Implementation



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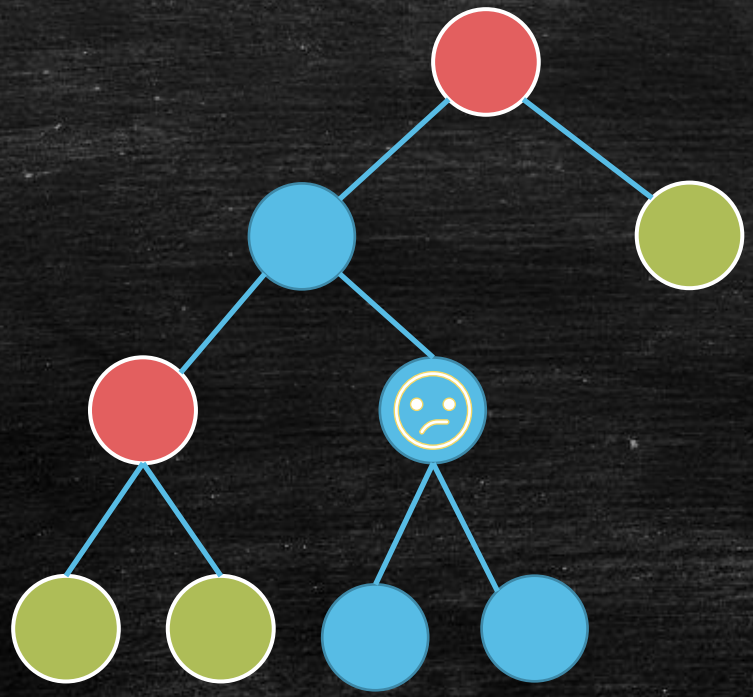

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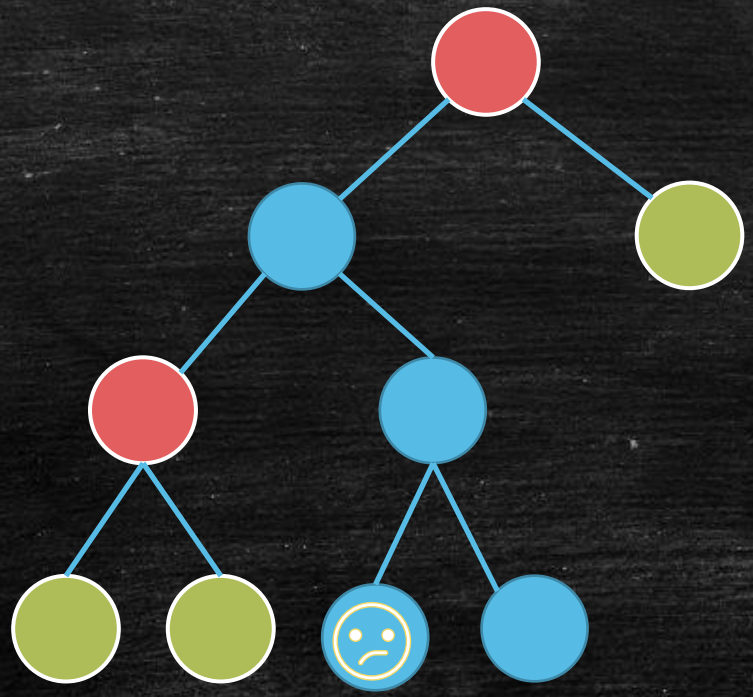

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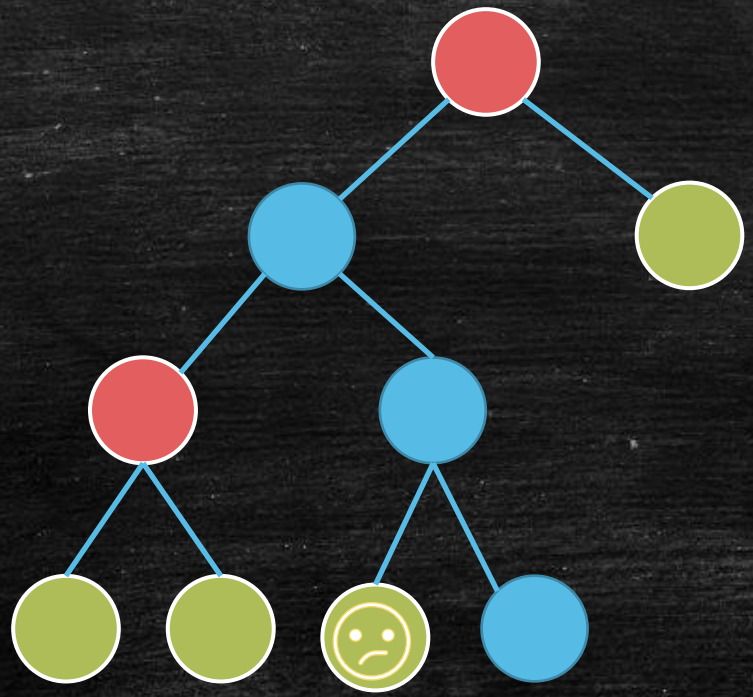

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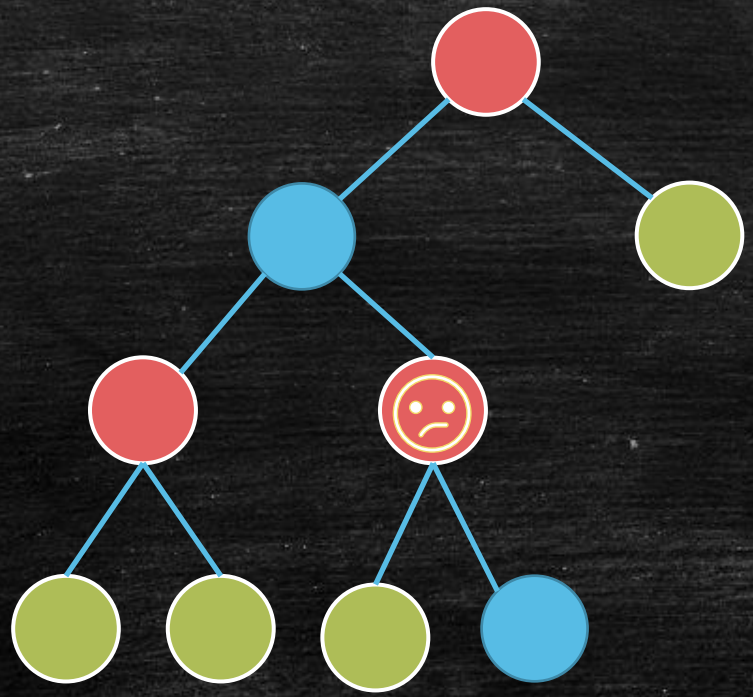

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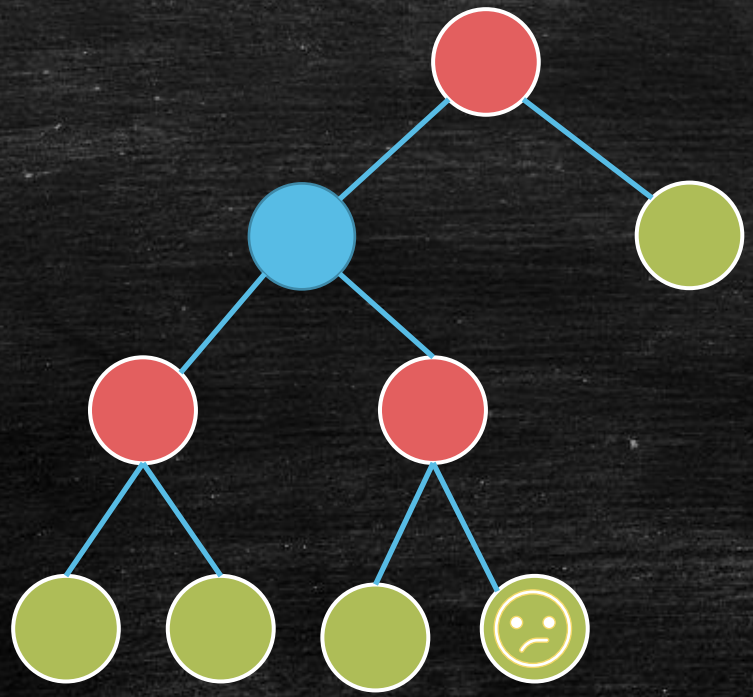

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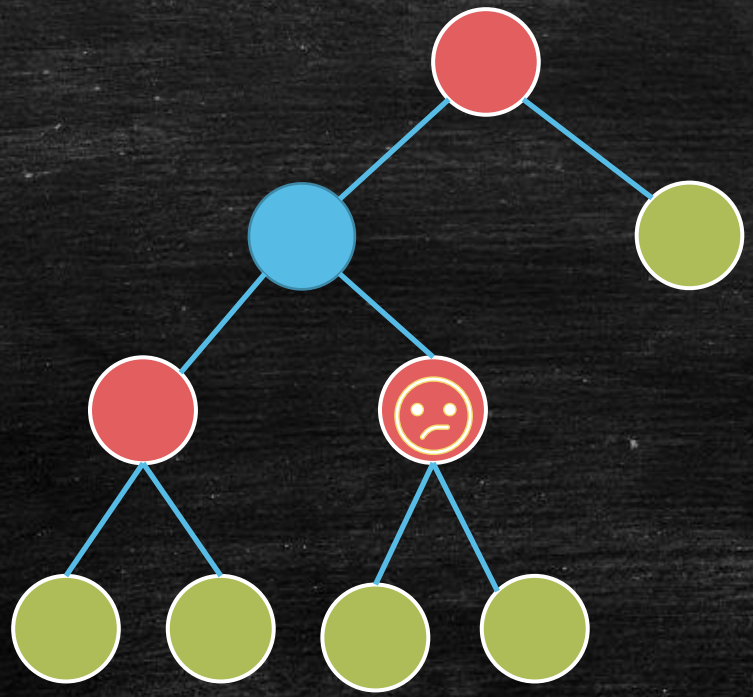

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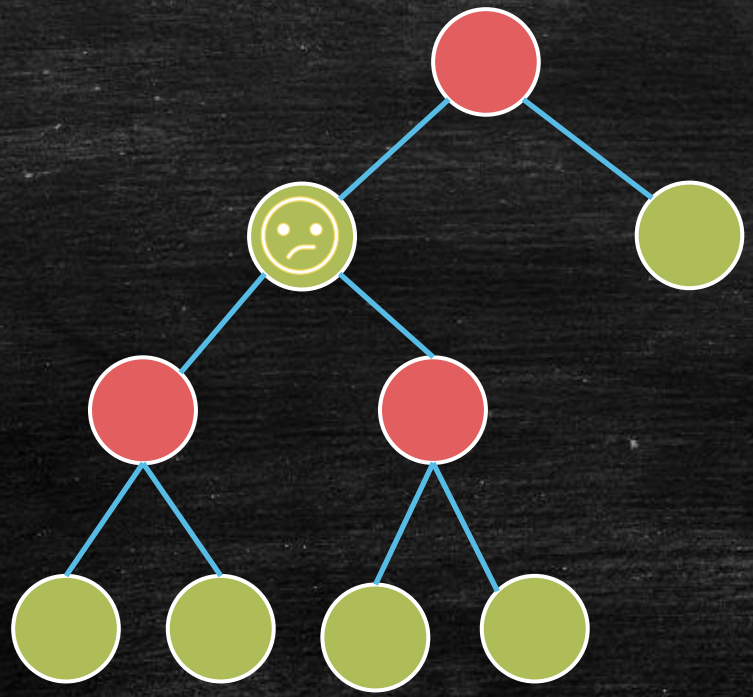
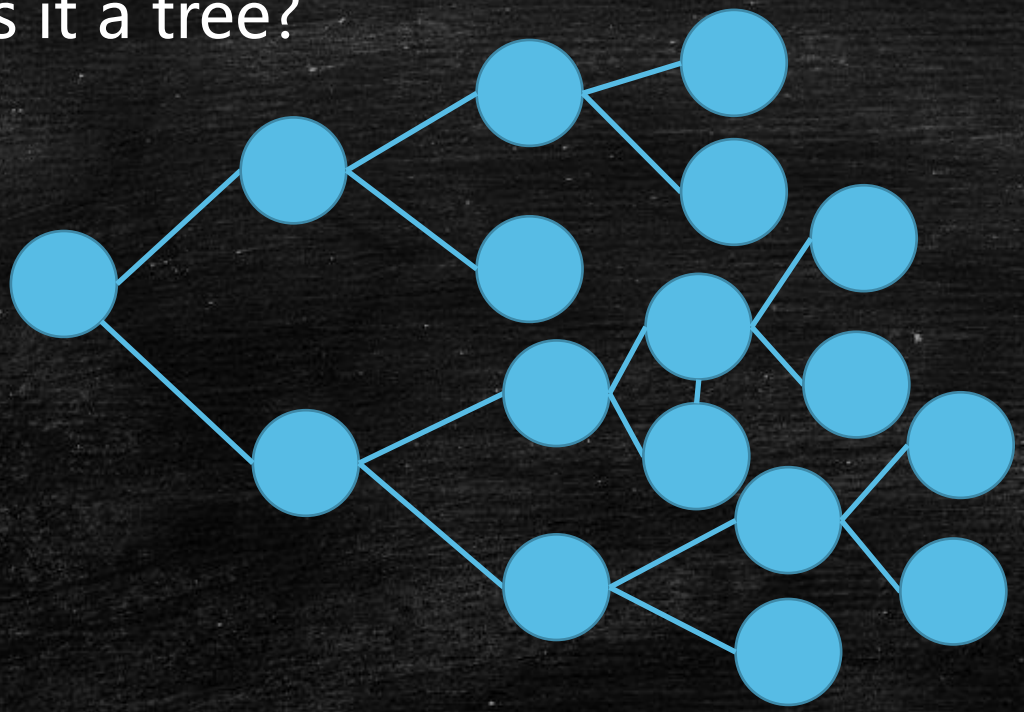


Independent Set on General Graphs

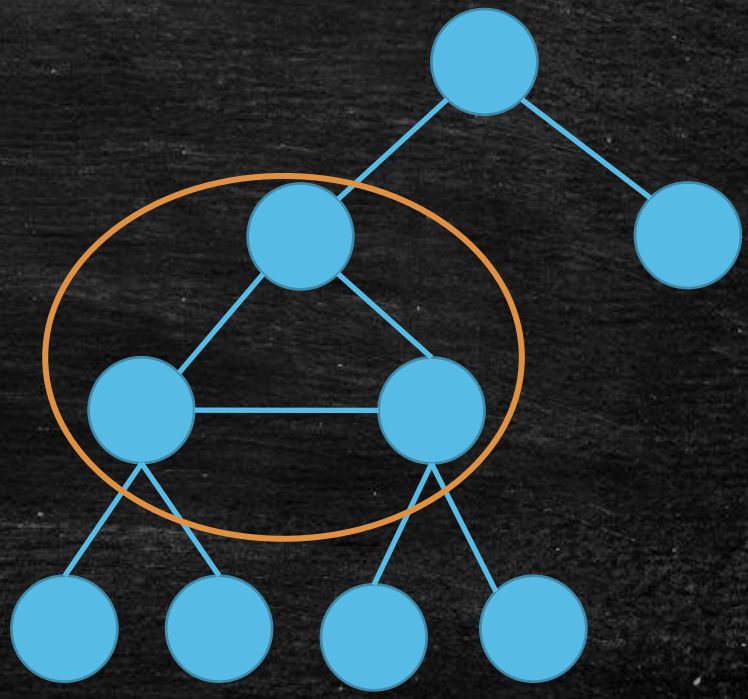
- Look at the following graph.
- Is it a tree?



It looks like a tree!

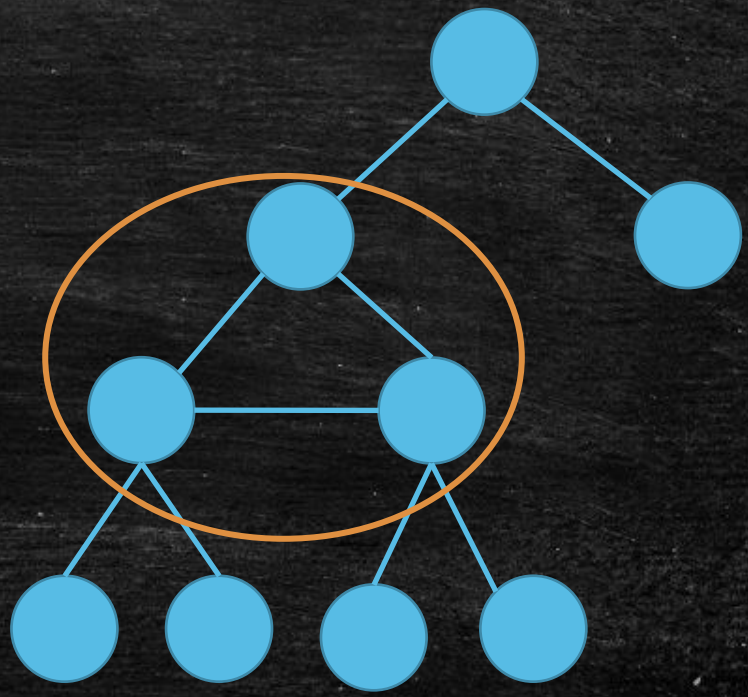
A Tree-like Graph

- It is a tree if we view the triangle as a **super node**.
- Question: Can we still use DP?
- Consider the subtree rooted at the **orange super node**.
  - Case 1: We choose the **super node**, what happens?
  - Case 2: We do not choose the **super node**, what happens?



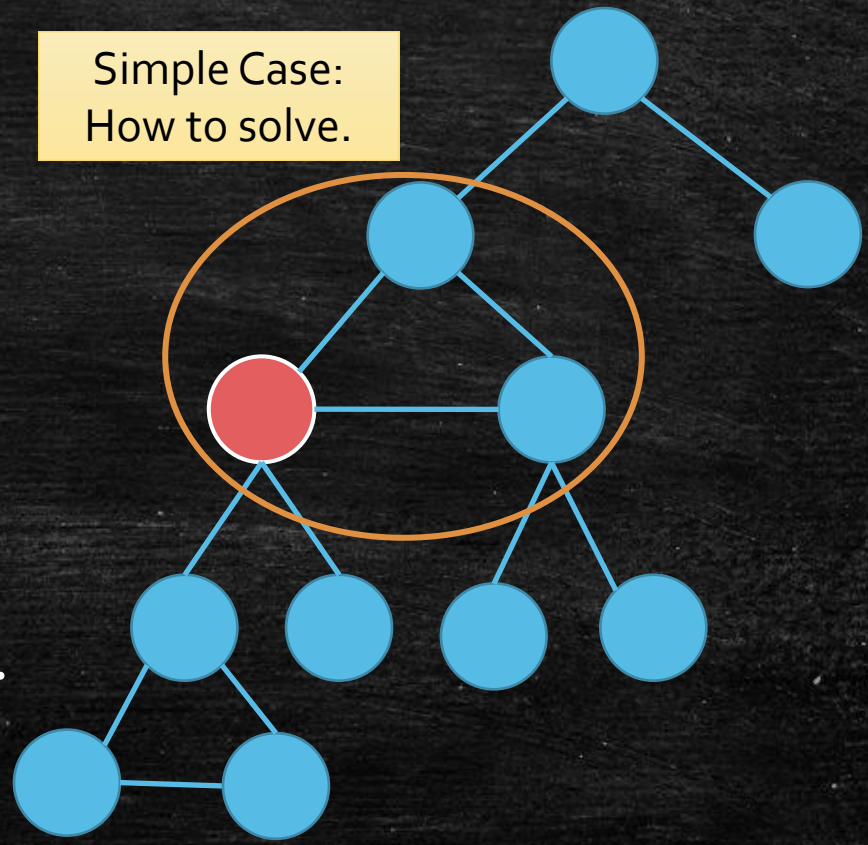
“Choose” a Super Node.

- “Choose” a Super Node.
- How many cases?
- We have at most 2^3 possible way!
- Different way means different restriction for next level selection.



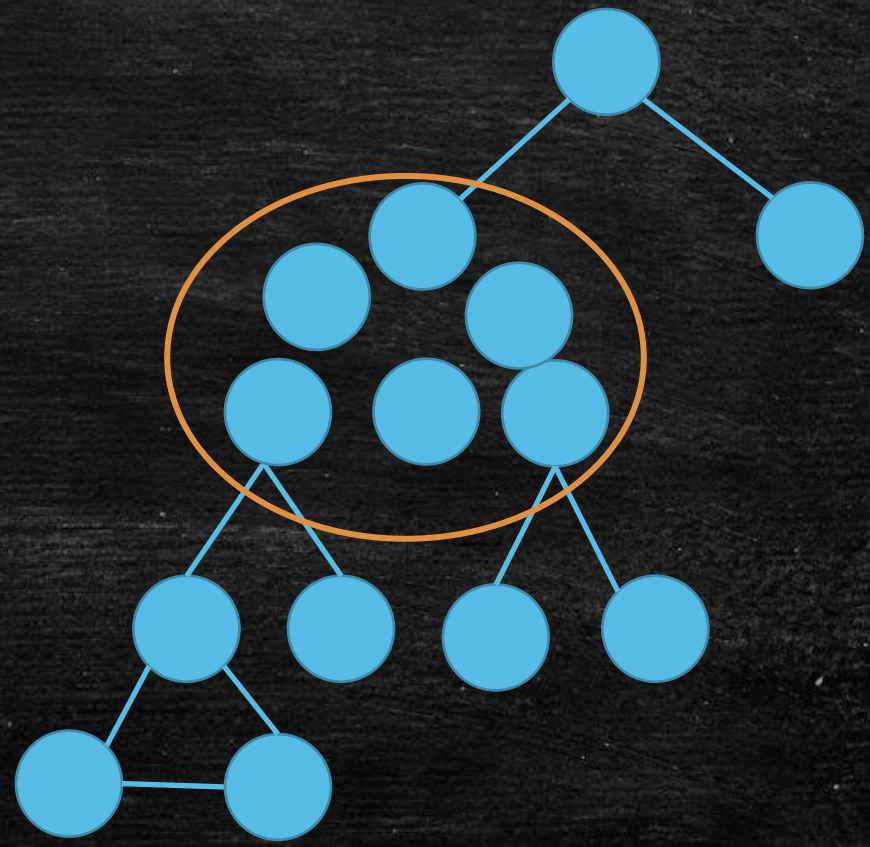
“Choose” a Super Node.

- “Choose” a Super Node.
- How many cases?
- We have at most 2^3 possible way!
- Different way means different restriction for next level selection.
- We can design a DP for $f[i, way]$.
- Time: $O(2^{3+3} \cdot n)$



“Choose” a Super Node.

- What if super node has k vertices?
- We have at most 2^k possible way!
- Time: $O\left(2^{O(k)} \cdot n\right)$
- It hold when the largest super node has k vertices!



Treewidth (Idea)

- The best way to make a graph to tree-like!
- Best: minimize the number of vertices (k) in the largest super node.
- Tree-width: $k - 1$
  - Cycle: 2
  - Clique: n-1
  - Tree: 1
  - series-parallel graphs: ≤ 2
- Many Optimization Problem in these graphs can use tree DP to get $O\left(2^{O(k)} \cdot poly(n)\right)$.

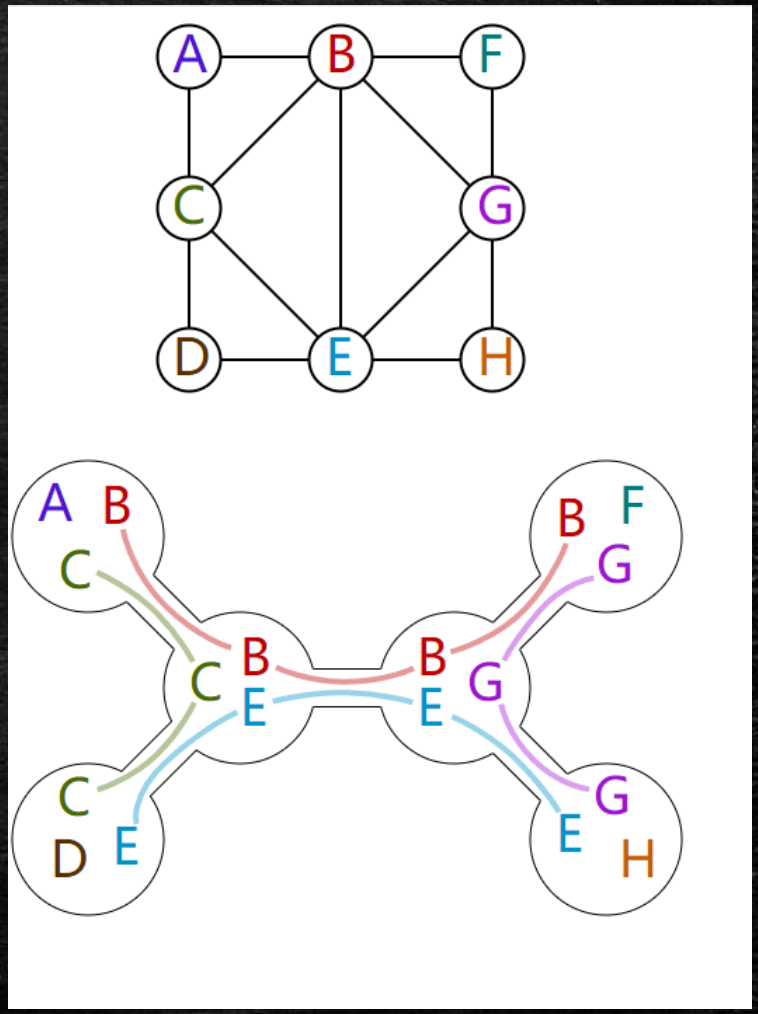
Fixed-Parameter Tractable

- $O(f(k) \cdot n^c)$
- $f(k)$ do not need to be polynomial.
- Many Optimization Problem in these graphs can use tree DP to get $O\left(2^{O(k)} \cdot poly(n)\right)$!
- They are FPT in terms of the treewidth!
- Compare to Approximation Algorithms!

Tree Decomposition and Treewidth

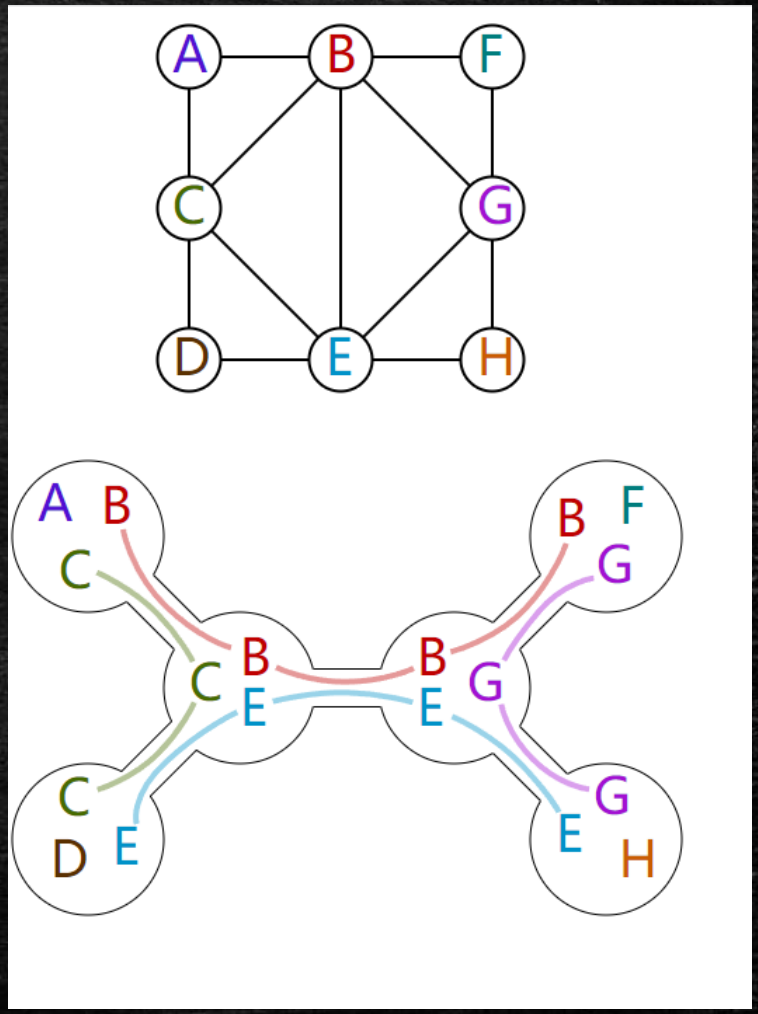
- A tree decomposition
- A Tree node: a **Bag** of vertices.
- Requirement
  - Any vertex must be included in one **Bag**.
  - The **Bags** contains u are connected.
  - An edge (u, v): A bag contains both u and v.

- Treewidth: Size of largest bag – 1.



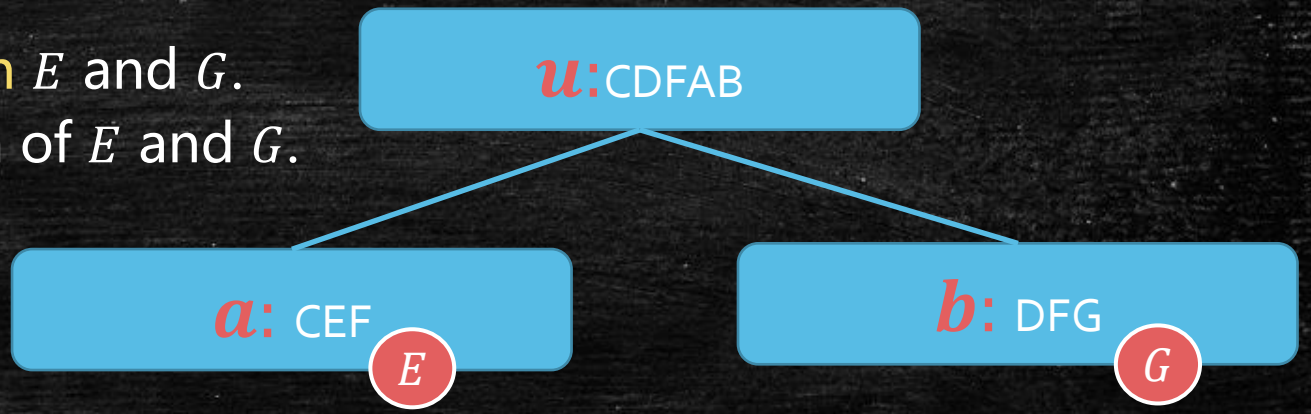
Nice Tree Decomposition

- Number of **Bags** at most $4n$.
- Binary Tree.
- **Theorem:** If G has a tree decomposition with treewidth k, then it admits a nice tree decomposition with treewidth k.
- It is enough to remember we are talking about a tree with size $O(n)$.

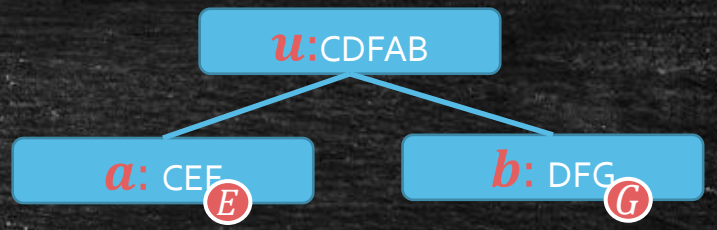


Separation Property

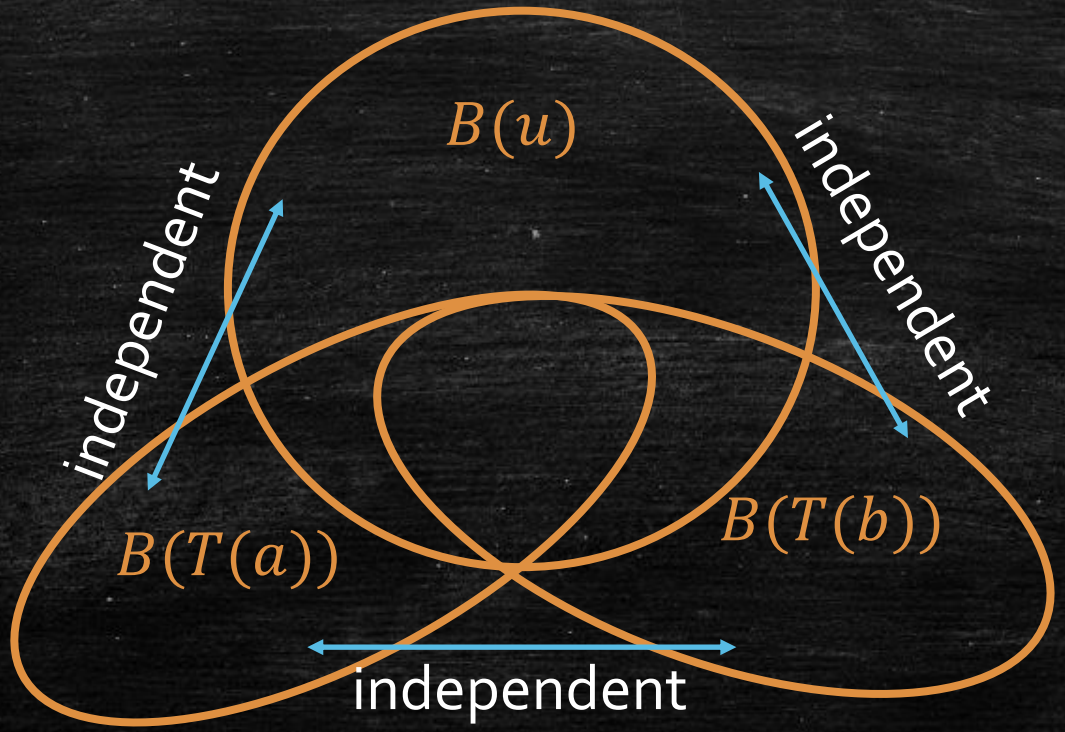
- $B(u)$ separates the two subtree $T(a)$ and $T(b)$.
- $B\big(T(a)\big) - B(u)$ and $B\big(T(b)\big) - B(u)$ are independent!
- Notation: $B(u), T(u), B(T(u))$.
- Example:
  - If (E, G) is an edge.
  - There is a Bag contains both E and G.
  - But u blocks the connection of E and G.



Separation Property

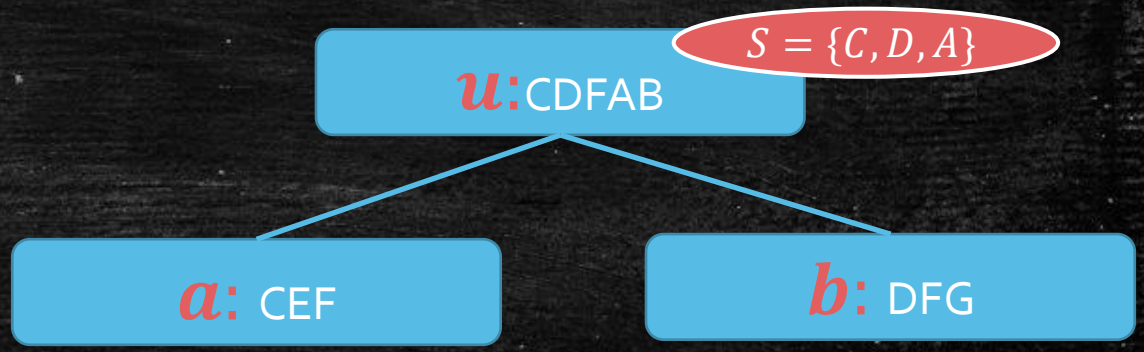


- $B(u)$ separates the two subtree $T(a)$ and $T(b)$.
- $B\big(T(a)\big) - B(u)$ and $B\big(T(b)\big) - B(u)$ and are independent!



DP on tree decomposition

- Solve max independent set on a graph with treewidth k.
- Subproblem $f(S, u)$: The size of max independent set $I \subset B(T(u))$, where $I \cap B(u) = S$.
- $x \in S \rightarrow choosed; x \in B(u) - S \rightarrow not\ choosed$.
  - $C \in I$
  - $D \in I$
  - $A \in I$
  - $F \notin I$
  - $B \notin I$

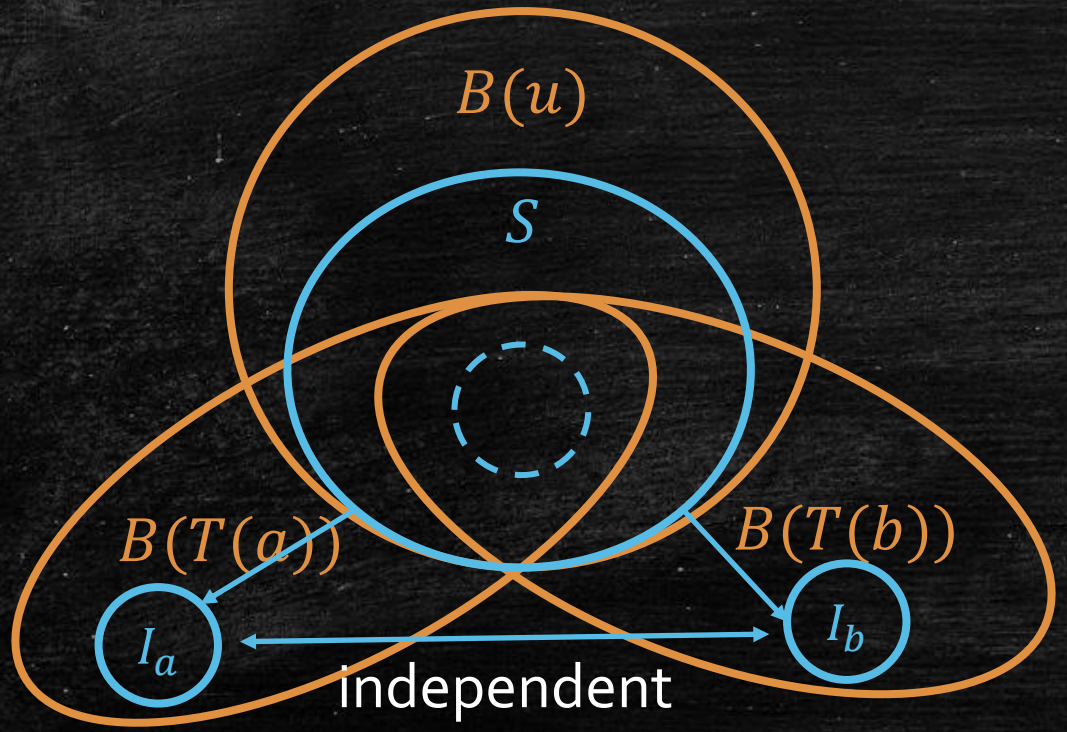


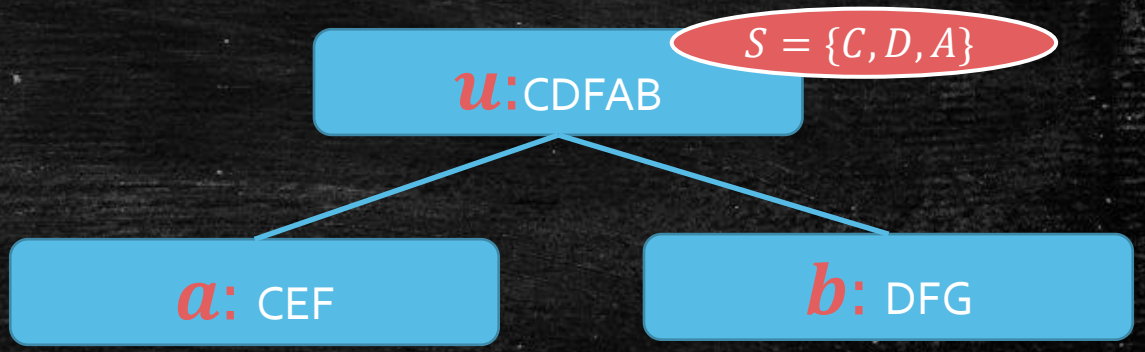
DP on tree decomposition

- Solve max independent set on a graph with treewidth k.
- $f(S, u)$: The size of max independent set $I \subset B(T(u))$, where $I \cup B(u) = S$.
- How to solve $f(S, u)$?
- Bottom-up!
- Base Case (Leaves)
  - $f(S, u) = |S|$ if $|S|$ is independent.
  - $f(S, u) = 0$ if $|S|$ is not independent.
- Output: the largest $f(S, r)$ for root r!

DP on tree decomposition

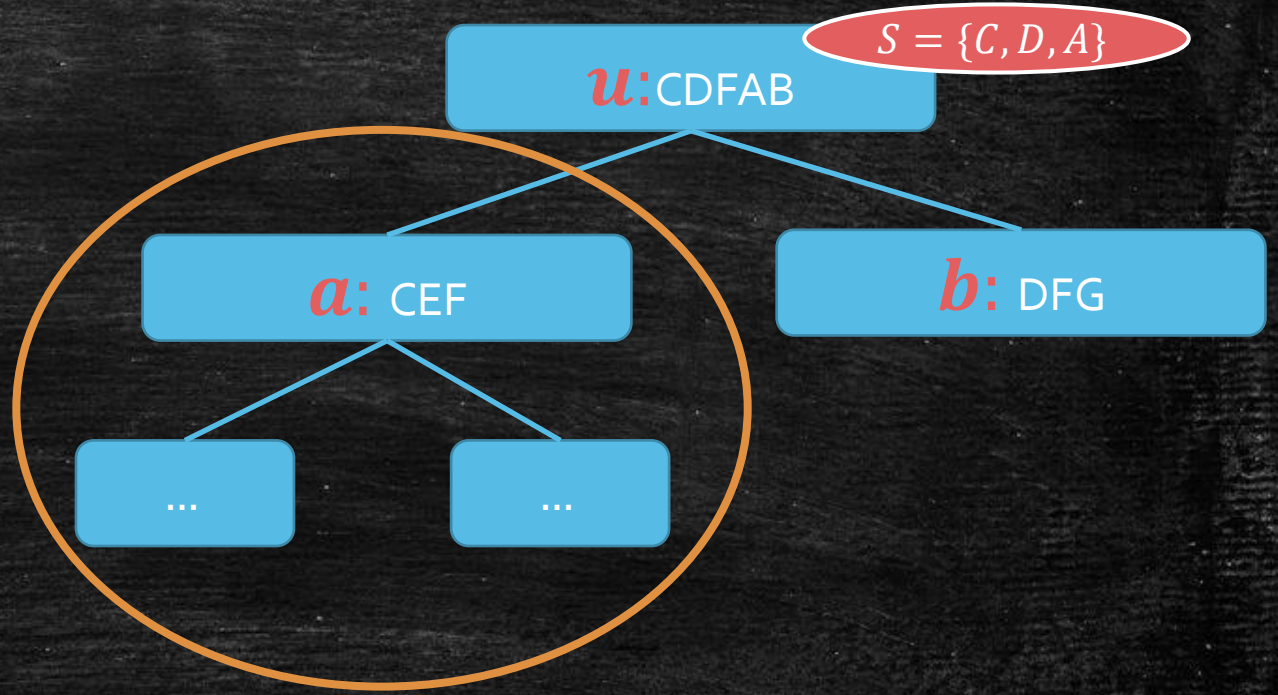
- How to solve $f(S, u)$?
  - If $|S|$ is not independent → $f(S, u) = 0$.
  - How to grow S as large as possible?

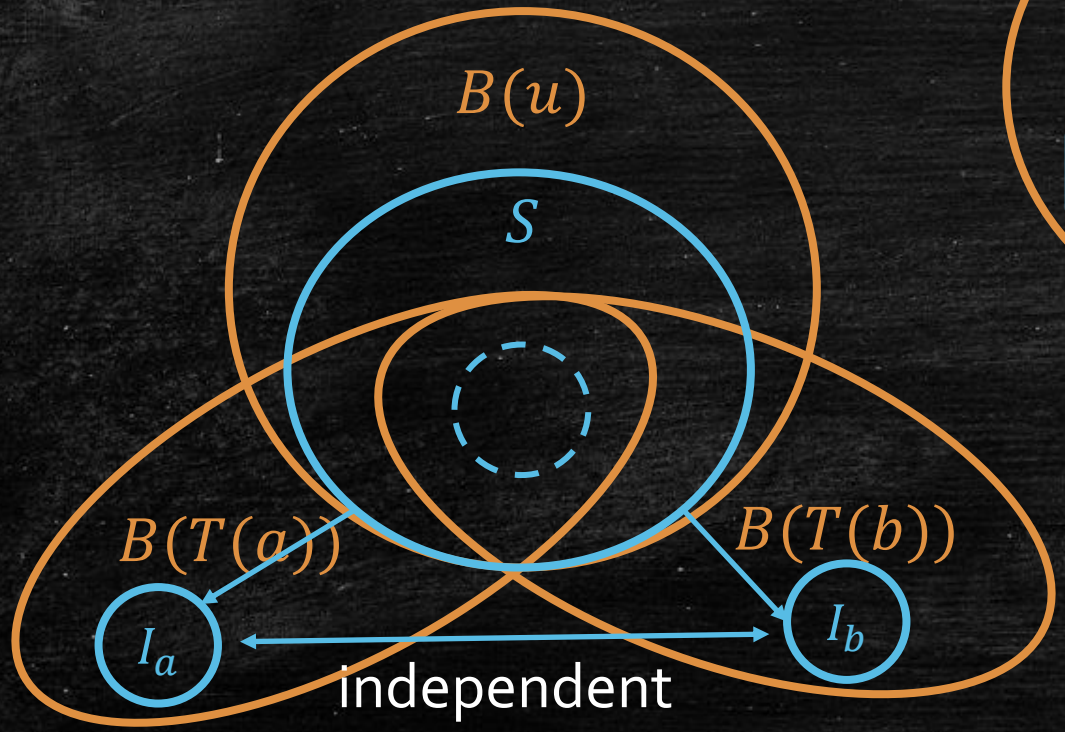




Grow in $T(a)$ first.

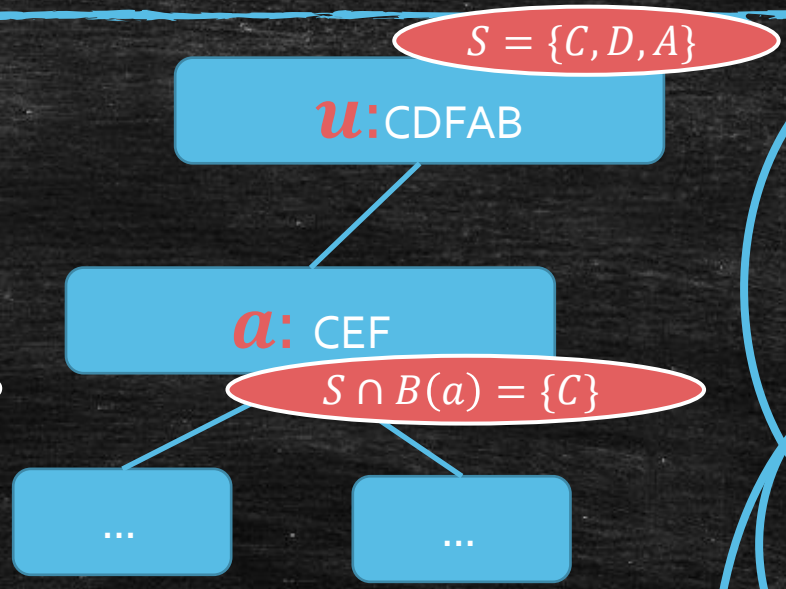
- How large I_a we can get?

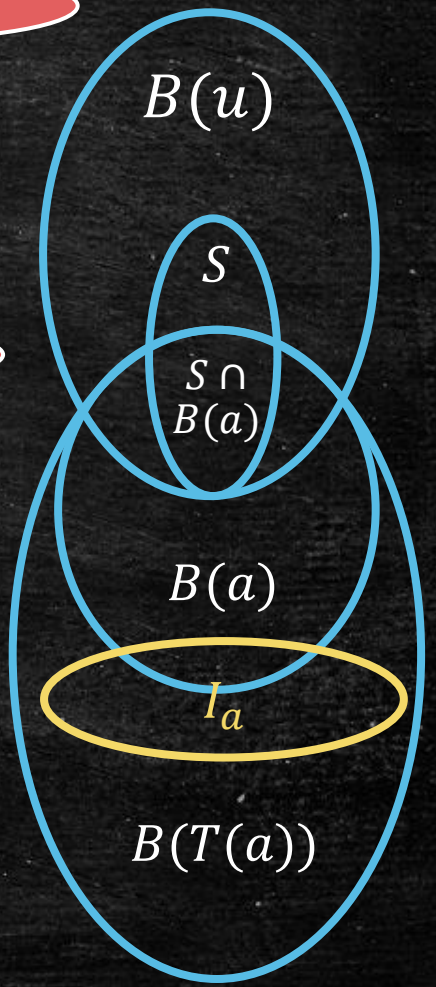




Question: How to calculate?

- How large I_a we can get?
- Requirement
  - I_a is independent with S.
  - ? I_a is independent with $S - B(a)$?
  - ✓ I_a is independent with $S \cap B(a)$?

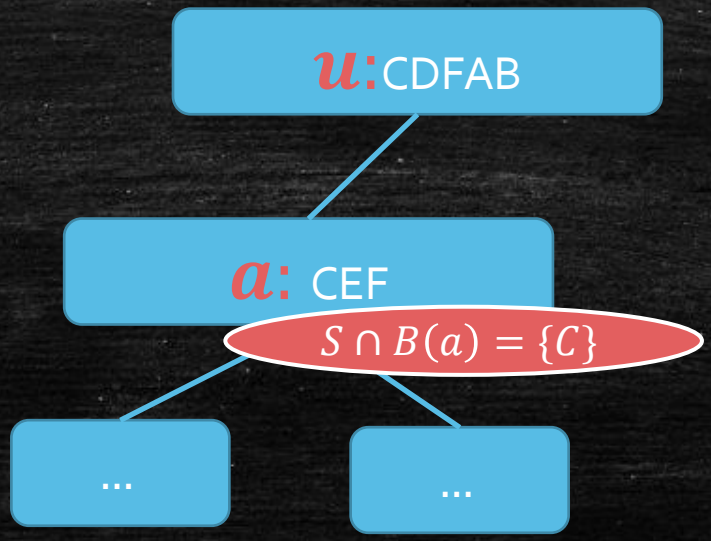


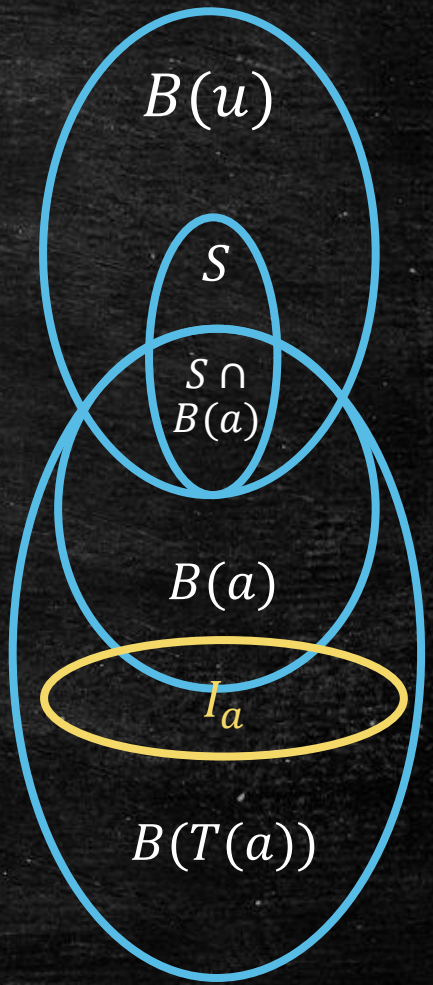


- Find a maximized independent set growing from $S \cap B(a) \rightarrow$ maximized $I_a \cup (S \cap B(a))$.
- Question: is it solved in previous subproblems?
  - Candidate: $f(S \cap B(a), a)$

Question: How to calculate?

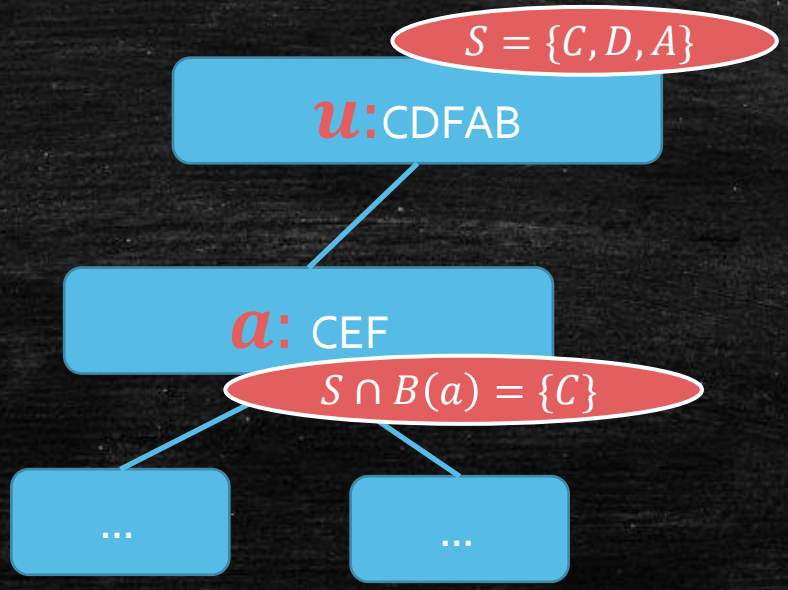
- Find a maximized independent set growing from $S \cap B(a) \rightarrow$ maximized $I_a \cup S \cap B(a)$.
- Question: is it solved in previous subproblems?
- Cases of a
  - Choose $\{C\}$ in $B(a)$.
  - Choose $\{C, E\}$ in $B(a)$.
  - Choose $\{C, F\}$ in $B(a)$.
  - Choose $\{E\}$ in $B(a)$.
  - Choose $\{C, E, F\}$ in $B(a)$.

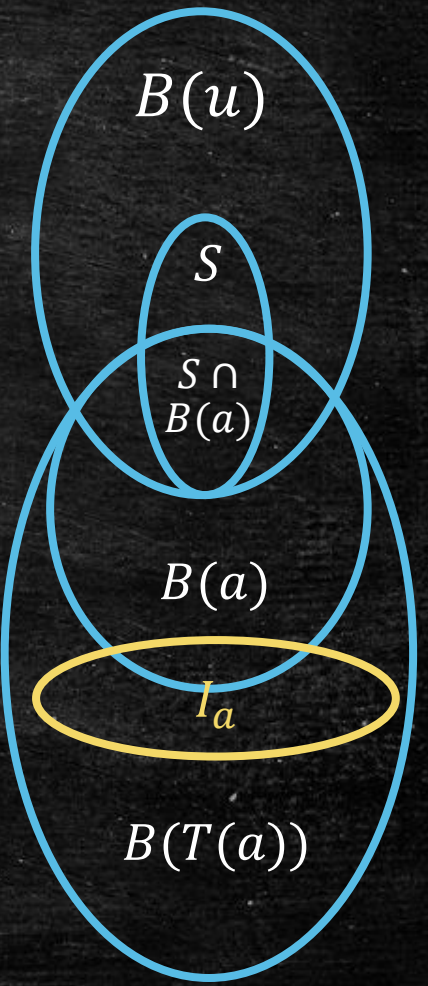




Question: How to calculate?

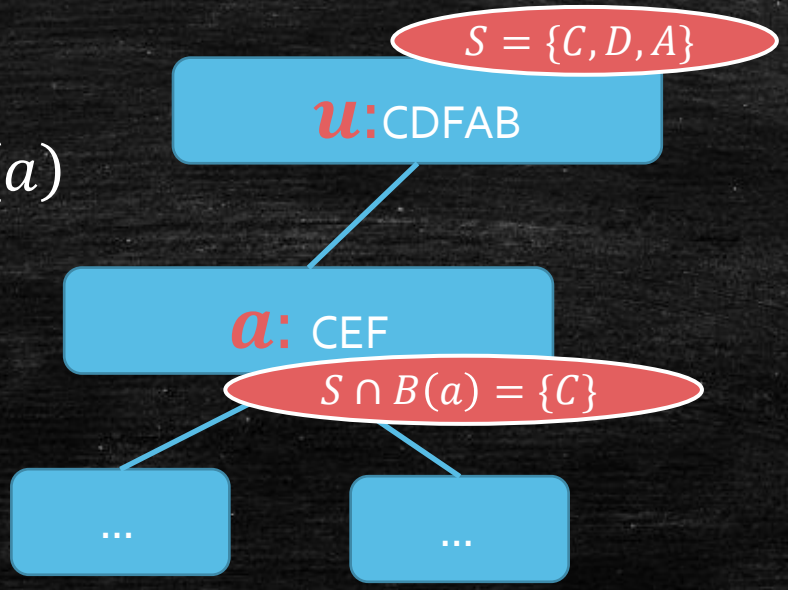
- Find a maximized independent set growing from $S \cap B(a) \rightarrow$ maximized $I_a \cup S \cap B(a)$.
- Question: is it solved in previous subproblems?
- Cases of a
  - Choose $\{C\}$ in $B(a)$.
  - Choose $\{C, E\}$ in $B(a)$.
  - Choose $\{C, F\}$ in $B(a)$.
  - Choose $\{E\}$ in $B(a)$.
  - Choose $\{C, E, F\}$ in $B(a)$.
- Choices of a
  - $S' \cap B(u) = S \cap B(a)$

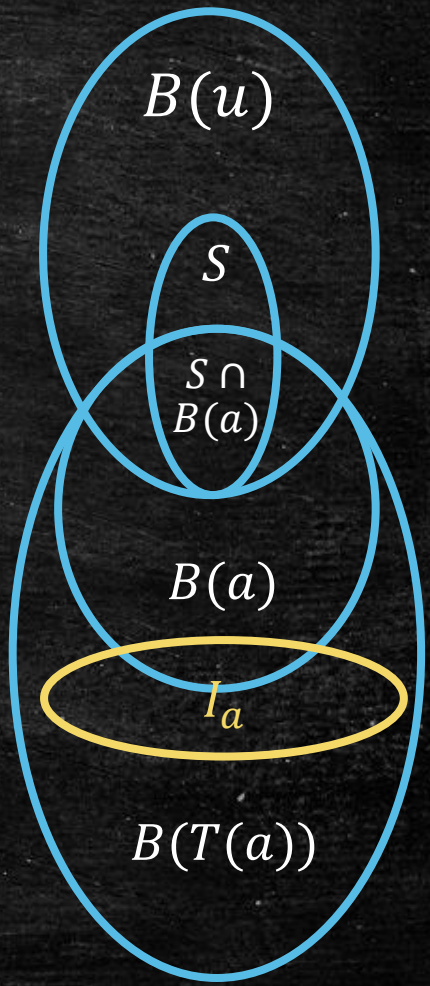




Question: How to calculate?

- Find a maximized independent set growing from $S \cap B(a) \rightarrow$ maximized $I_a \cup S \cap B(a)$.
- Question: is it solved in previous subproblems?
- Choices of a
  - $S' \cap B(u) = S \cap B(a)$
- MIS growing from $S \cap B(a)$
  - $\max_{s' \subset B(a), S' \cap B(u) = S \cap B(a)} f(S', a)$
  - $I_a = \max f(S', a) - |S \cap B(a)|$





Do the same thing for b!

- Choices of b
  - $\forall S' \subset B(b)$
  - $S' \cap B(u) = S \cap B(b)$
- MIS growing from $S \cap B(b)$
  - $\max_{s' \subset B(b), S' \cap B(u) = S \cap B(b)} f(S, b)$
  - $I_b = \max f(S, b) - |S \cap B(b)|$

Combining

- Grow s as large as possible in every direction!
- $f(S,u) =$

$$|S| + \sum_{a \in child(u)} \left(\max_{S' \in B(a), S' \cap B(u) = S \cap B(a)} f(S', a) - |S \cap B(a)| \right)$$

- Subproblem number: $O(2^k n)$
- Transfer Cost: $O(2^k)$
- Totally $O(4^k n)$

What we get!

- Tree-width: $k - 1$
  - Cycle: 2
  - Clique: n-1
  - Tree: 1
  - series-parallel graphs: ≤ 2
- $O(n)$ algorithm for Cycle, Tree, and series-parallel graphs!