P, NP, NP-Completeness

P, NP, NP-Completeness, and Reductions

Introduction

- Some problems can be solved in polynomial time.
  - as most of the problems we have seen in the previous lectures
- You've heard some other problems are "NP-hard" or "NP-complete".
- This lecture:
  - Learn what exactly do we mean by NP-hardness, or NP-completeness.
  - Understand why people believe these problems are hard.

Let's first see some famous NP-hard problems

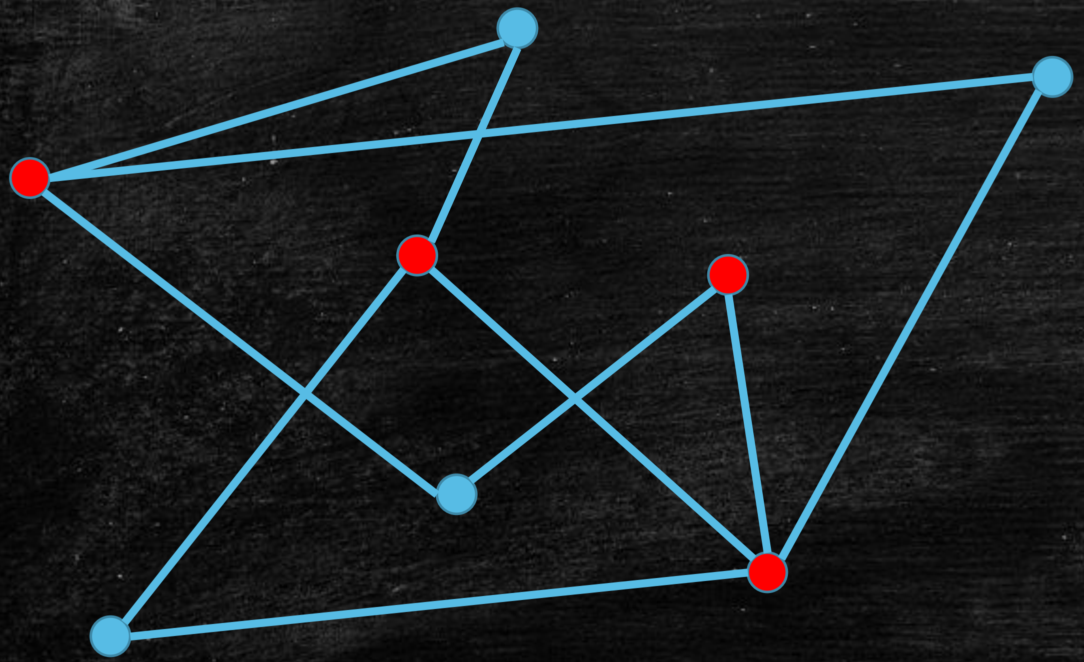
- SAT
- Vertex Cover
- Independent Set
- Subset Sum
- Hamiltonian Path

SAT (Boolean Satisfiability Problem)

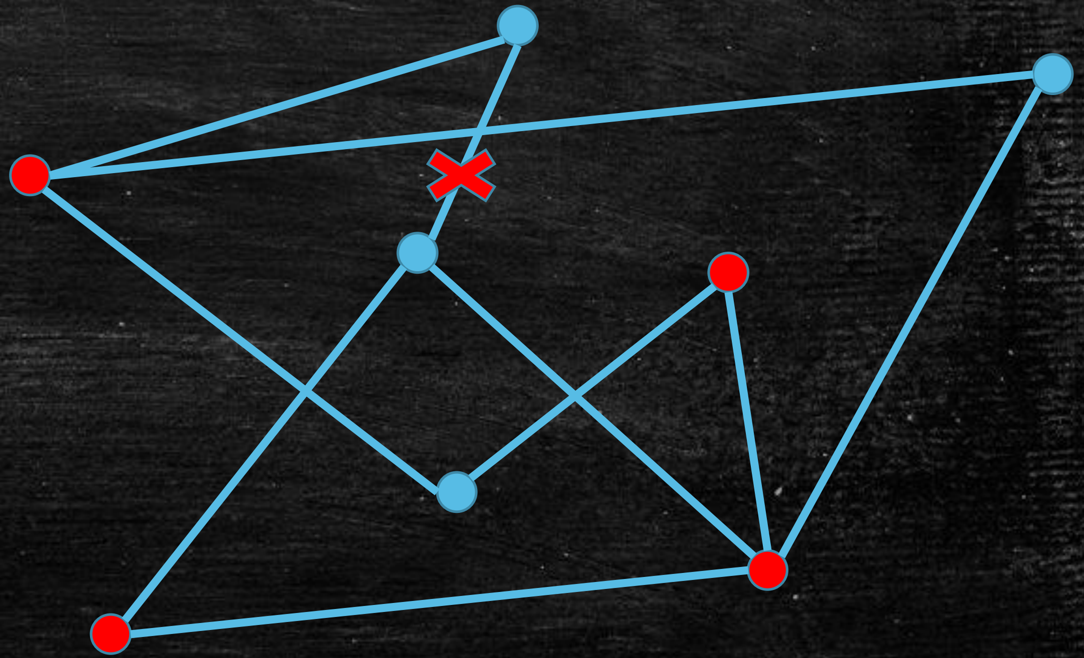
- A Boolean formula is built from variables, operators AND ($\wedge$), OR ($\vee$), NOT ($\neg$), and parentheses.
  - Example: $(x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$
- A Boolean formula is in conjunctive normal form (CNF) if it is an "AND" of many clauses:
  - Each clause contains "OR" of literals:
    - A literal is a variable x_i or its negation $\neg x_i$
  - The example is in CNF; it has three clauses: $(x_1 \vee x_3 \vee \neg x_4)$, $(x_2 \vee \neg x_3)$ and $(\neg x_1 \vee \neg x_2)$
- [SAT Problem] Given a CNF formula ϕ, decide if there is a value assignment to the variables to make ϕ true.
  - This is true for the example above: $x_1 = \text{true}, x_2 = \text{false}, x_3 = \text{false}$.

Vertex Cover

- Given an undirected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is a vertex cover if S contains at least one endpoint of every vertex.



a vertex cover

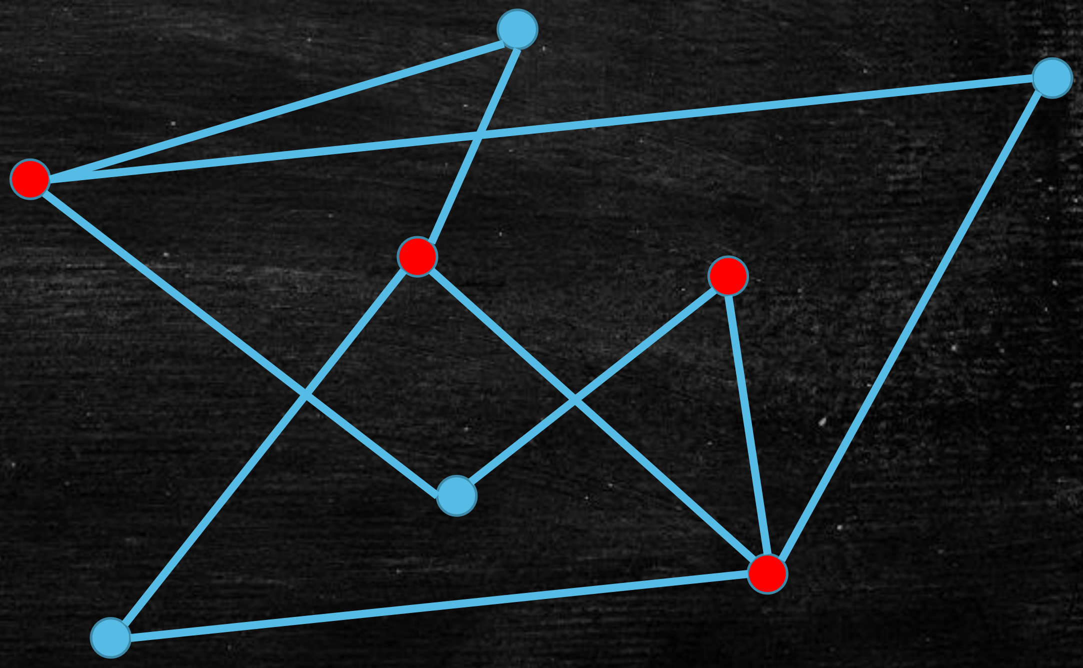


not a vertex cover

Vertex Cover Problem

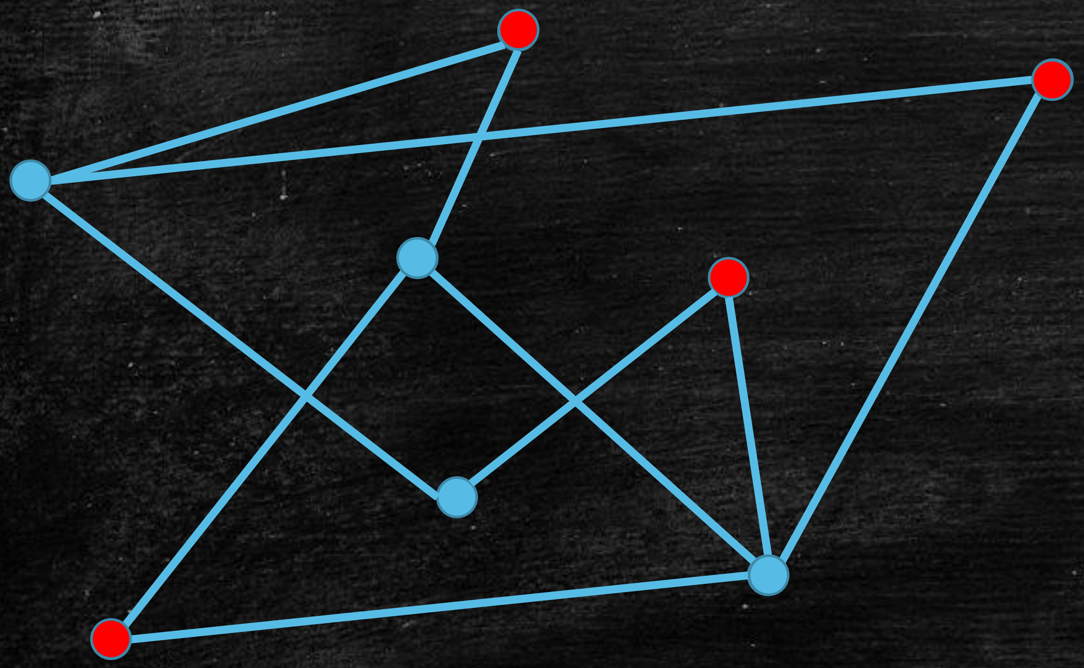
- [Vertex Cover Problem] Given an undirected graph $G = (V, E)$ and $k \in \mathbb{Z}^+$, decide if the graph has a vertex cover of size k.

For this graph and $k = 4$, the output should be yes.

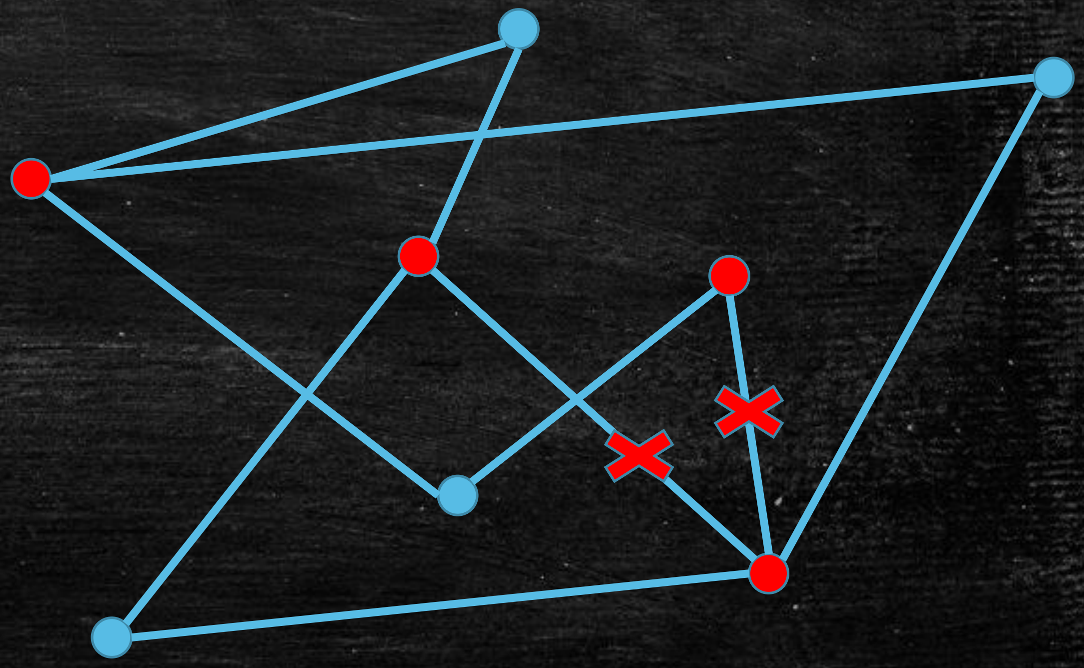


Independent Set

- Given an undirected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is an independent set if there is no edge between any two vertices in S.



an independent set

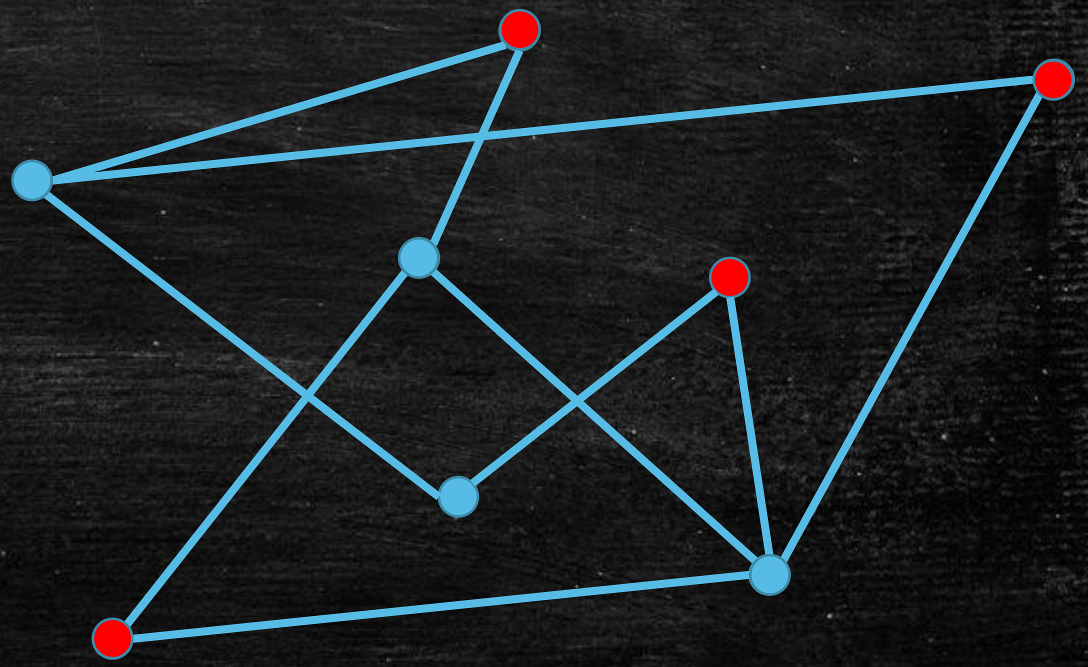


not an independent set

Independent Set Problem

- [Independent Set Problem] Given an undirected graph $G = (V, E)$ and $k \in \mathbb{Z}^+$, decide if the graph has an independent set of size k.

For this graph and $k = 4$, the output should be yes.

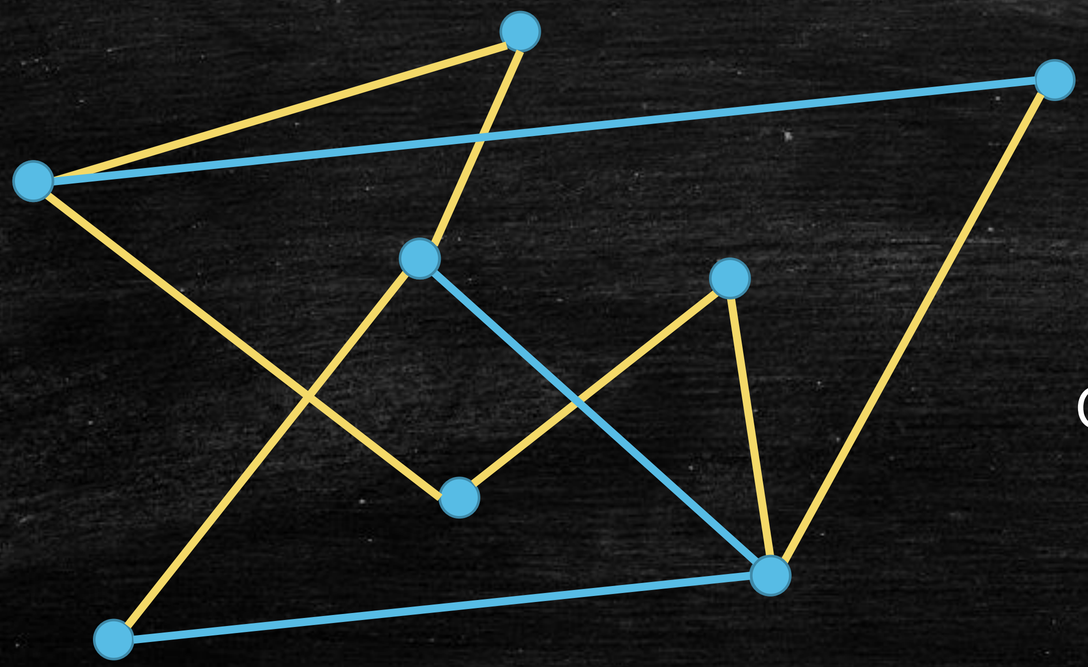


Subset Sum Problem

- [Subset Sum Problem] Given a collection of integers $S = \{a_1, \ldots, a_n\}$ and $k \in \mathbb{Z}^+$, decide if there is a sub-collection $T \subseteq S$ such that $\sum_{a_i \in T} a_i = k$.
- The output should be yes for $S = \{1,1,6,13,27\}$ and $k = 21$, as $1 + 1 + 6 + 13 = 21$.
- The output should be no for $S = \{1,1,6,13,27\}$ and $k = 22$.

Hamiltonian Path Problem

- Given an undirected graph $G = (V, E)$, a Hamiltonian path is a path containing each vertex exactly once.
- [Hamiltonian Path Problem] Given an undirected graph $G = (V, E)$, decide if it contains a Hamiltonian path.



Output should be yes

In this lecture, we will only focus on…

- Decision Problems: those with output yes or no.
- Polynomial Time vs Not Polynomial Time
  - E.g., we will not care about $O(n)$ or $O(n^2)$
  - "Easy" Problems: those can be solved in polynomial time
  - "Hard" problems: those for which people believe cannot be solved in polynomial time

Decision Problem – Formal Definition

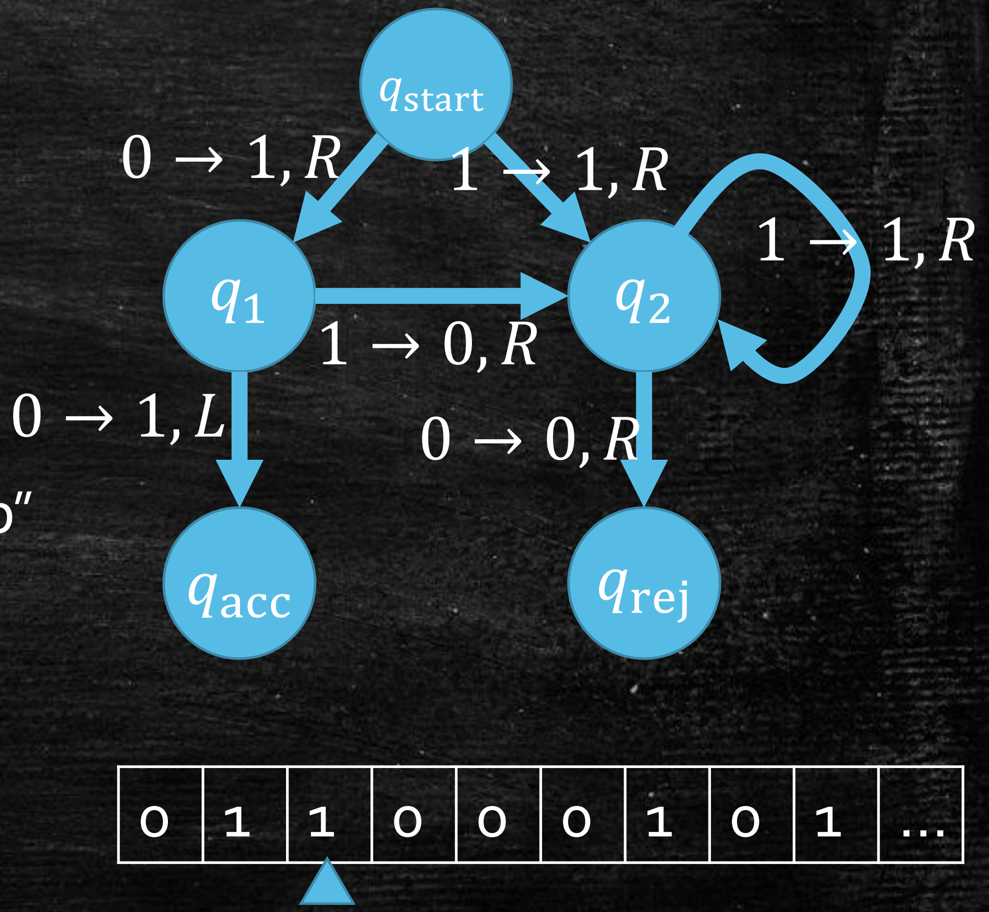
- A decision problem is a function $f: \Sigma^* \to \{0, 1\}$
- Σ - set of alphabets: for example, binary alphabets $\Sigma = \{0, 1\}$
- Σ^n - set of strings using alphabets in Σ with length n
- $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$ - set of all strings with any lengths
- $x \in \Sigma^*$ - an instance
- $f(x) = 1$: x is a yes instance
  - E.g., x encodes G and k where G has a k-vertex cover
- $f(x) = 0$: x is a no instance
  - E.g., x encodes G and k where G does not have a k-vertex cover
  - Or x is not a valid encoding of G and k

Problems That Are "Easy"

- A decision problem $f: \Sigma^* \rightarrow \{0,1\}$ is "easy" if there is a polynomial time algorithm $\mathcal{A}$ that computes it.
- That is, $\mathcal{A}(x) = f(x)$ always holds.
- Polynomial time: $\mathcal{A}(x)$ terminates in $|x|^{O(1)}$ steps.
- But wait! What exactly is an algorithm??

Turing Machine (TM)

- An abstract machine that is a prototype of modern computers.
- A Turing Machine is a triple (Q, Σ, δ)
  - one tape: contains infinitely many cells
    - Each cell can store an alphabet
  - A moving head pointing at a cell of the tape
  - Σ: set of alphabets
  - Q: set of states, each state specifying "the current step"
  - Transition function $\delta: Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$
    - instructions on how to move to the next step
    - Input: current state, current alphabet the head is reading
    - Output: next state, new alphabet written on the current position of the head, move to left (L) or right (R) by one cell



Turing Machine: Start and Terminate

- Start:
  - At a special state called starting state: $q_{\text{start}} \in Q$
  - Input is loaded to the tape
  - Moving Head is pointing at the first cell
- Terminate:
  - Two special state called halting states: q_{acc} and q_{rej}
  - TM terminates when reaching a halting state
  - TM accepts a string if q_{acc} is reached
  - TM rejects a string if q_{rej} is reached
  - TM's output is the content on the tape when TM terminates

TM Example: Check if two strings are identical

- Input: a string of format $x\#y$ where $x, y \in \{0,1\}^n$ and $\# \in \Sigma$ is a special separating alphabet
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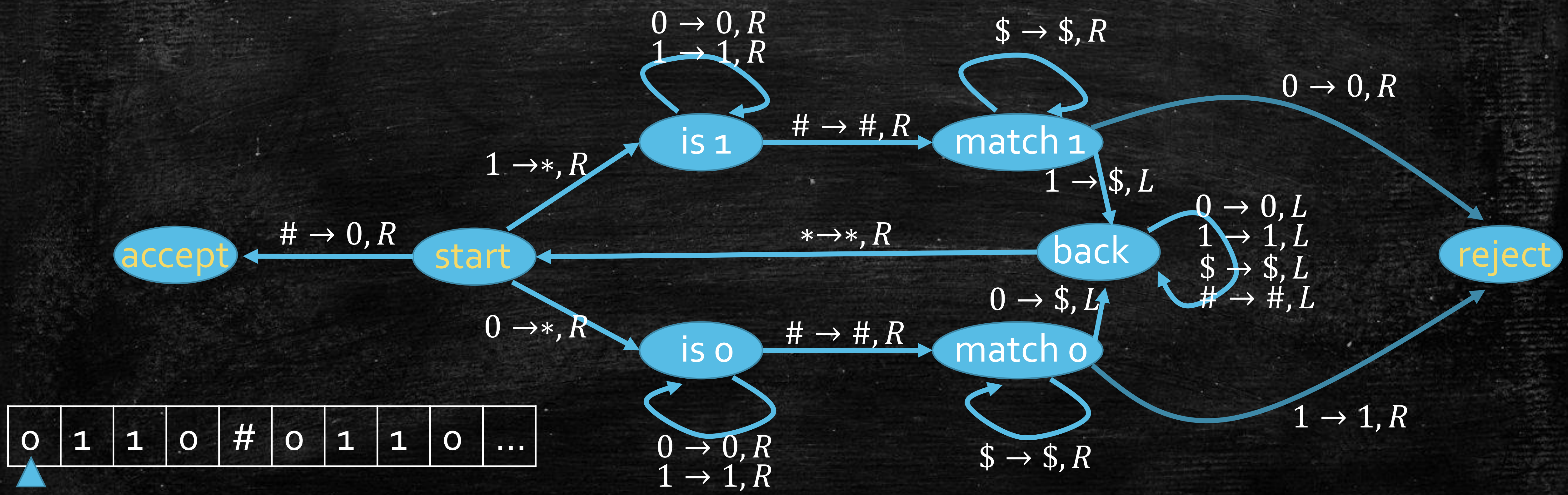

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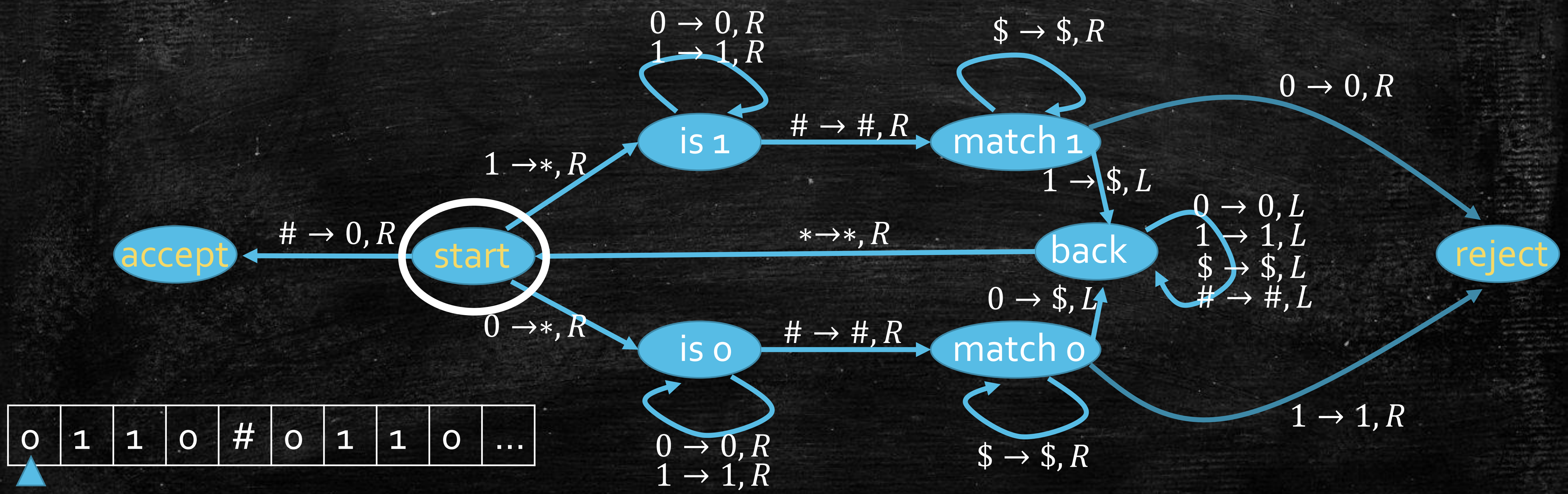

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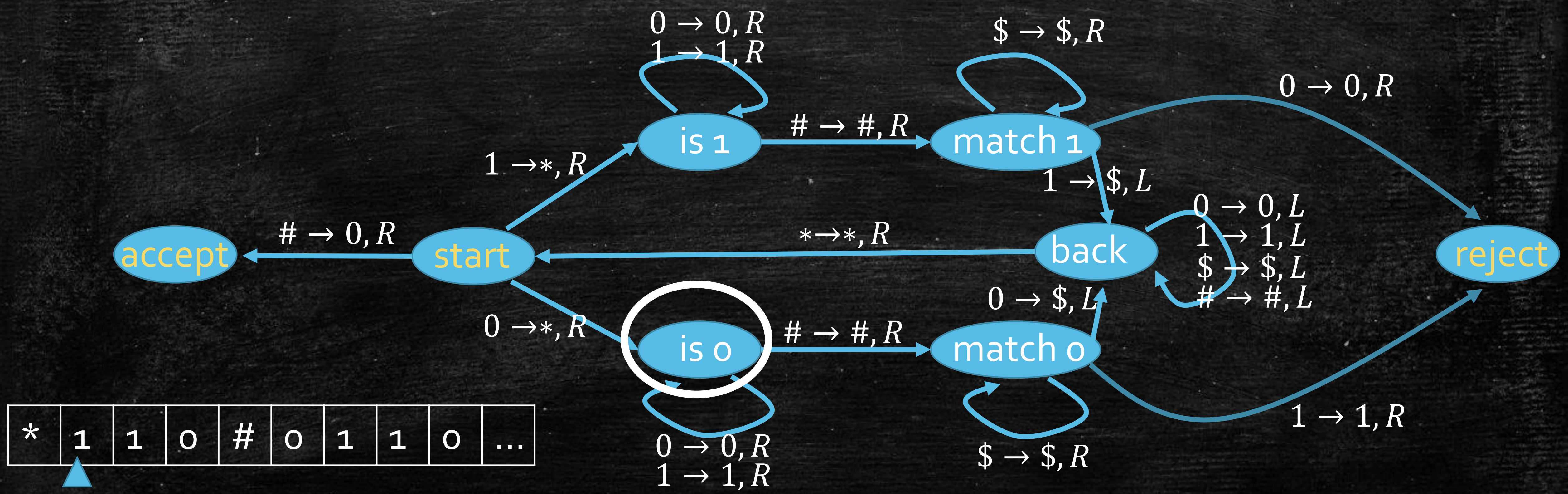

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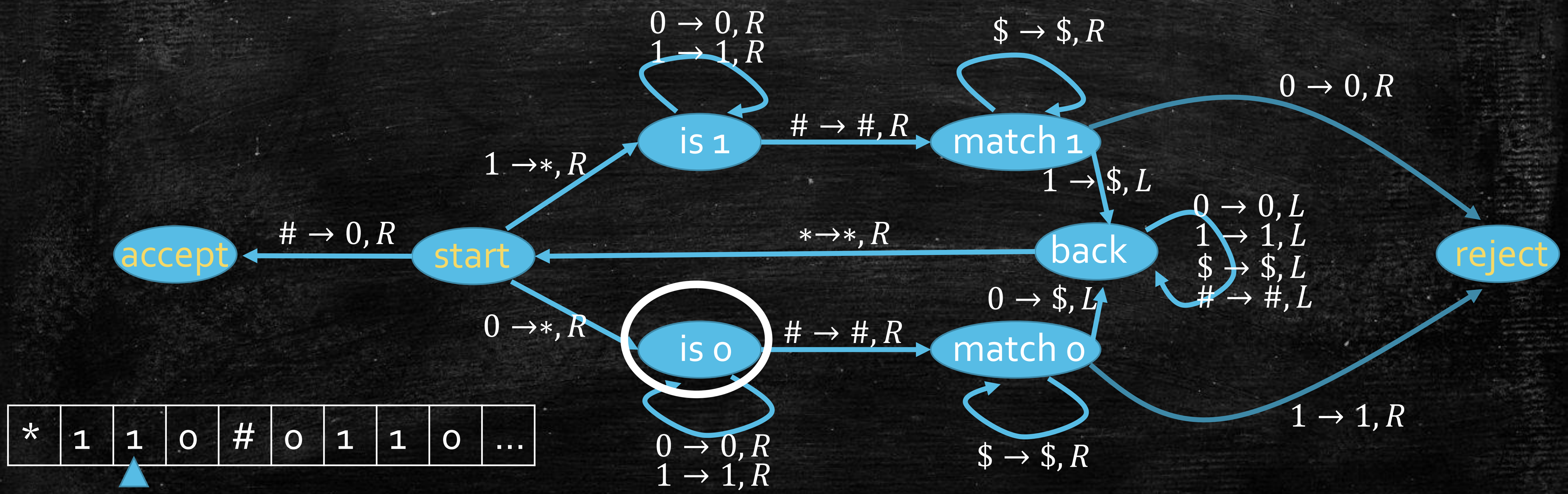

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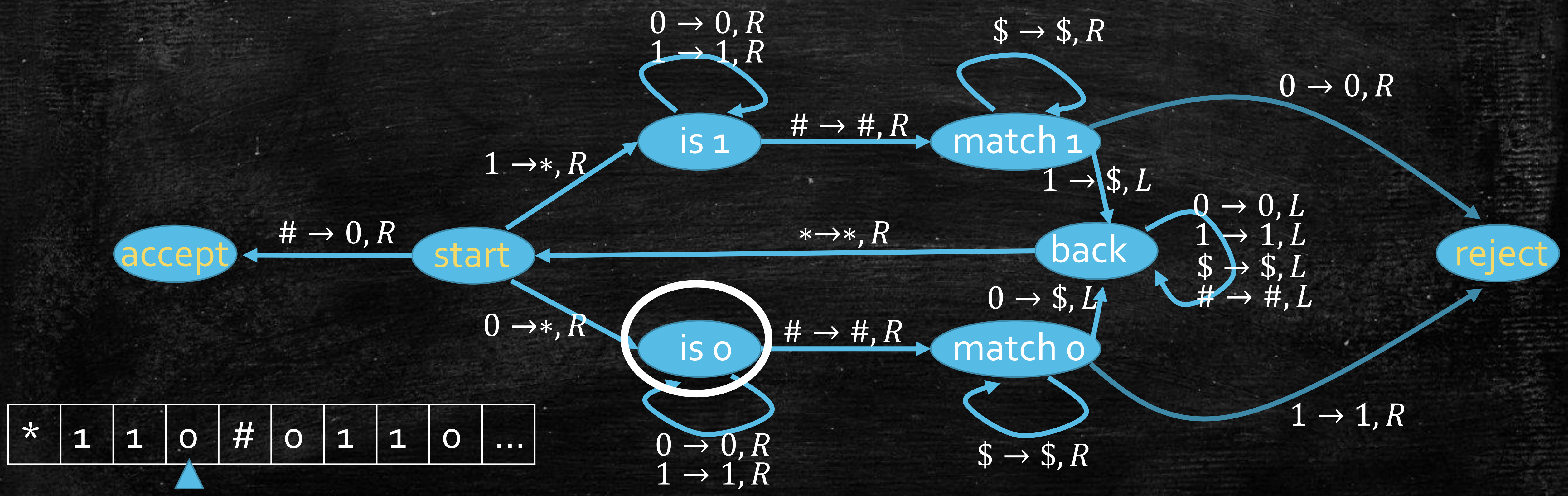

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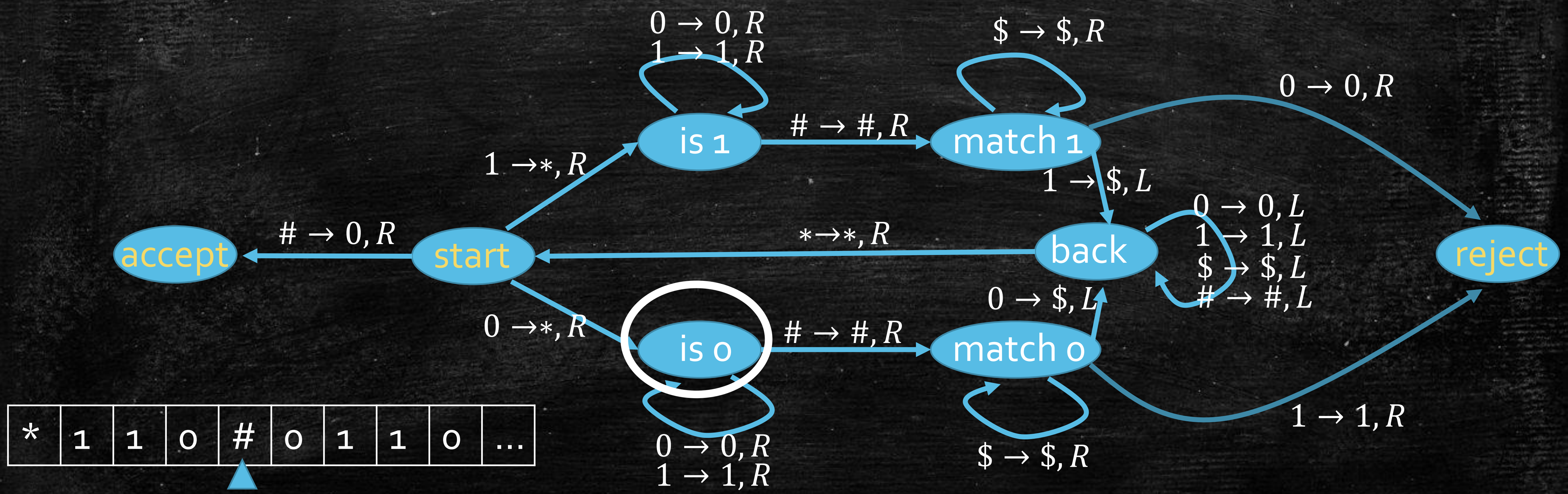

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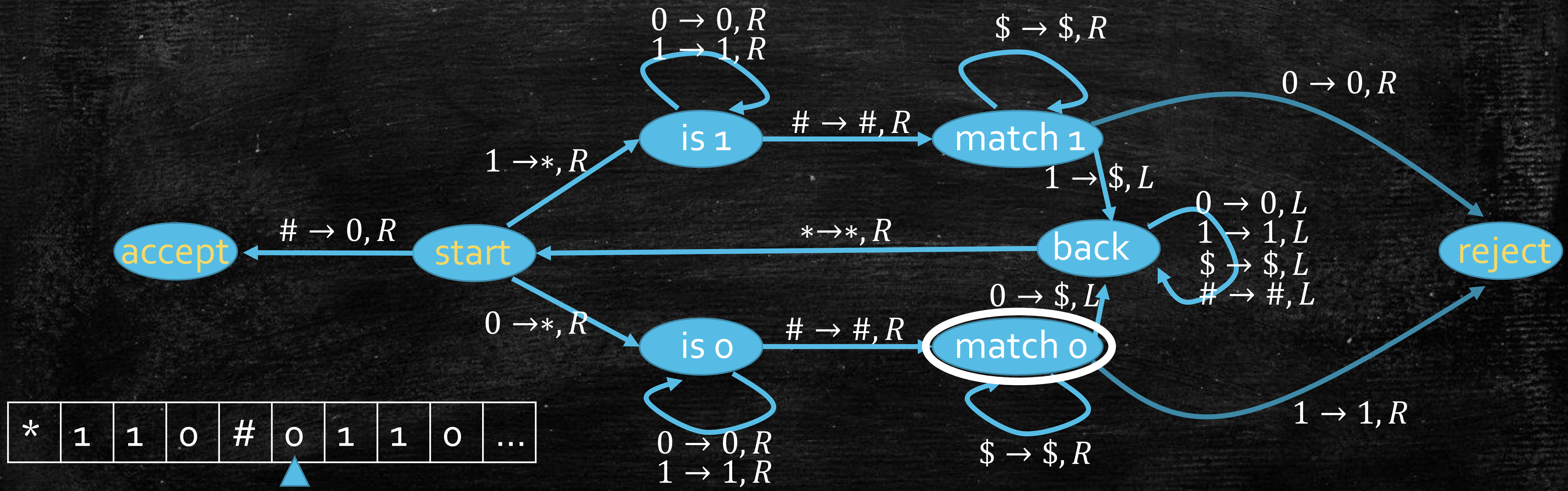

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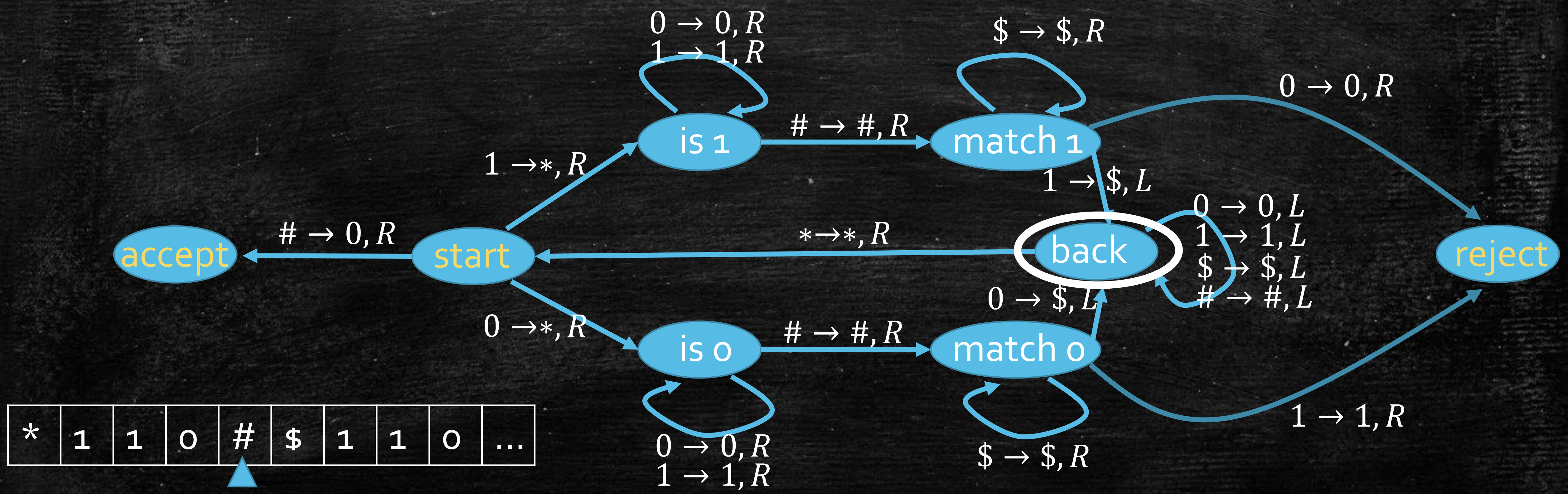

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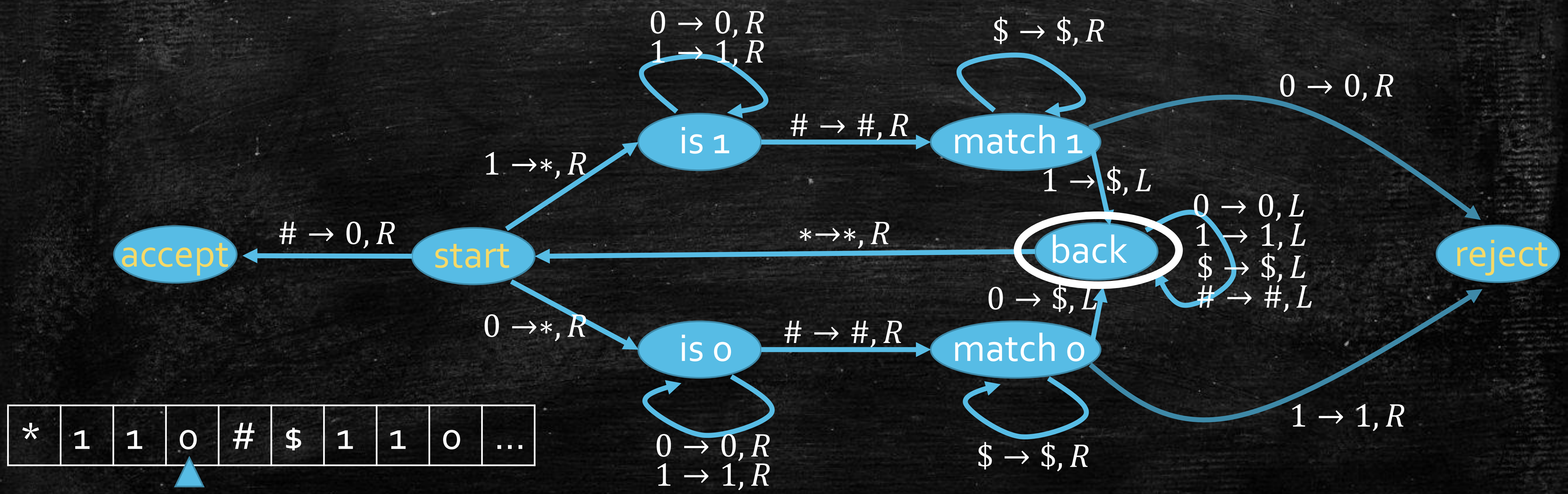

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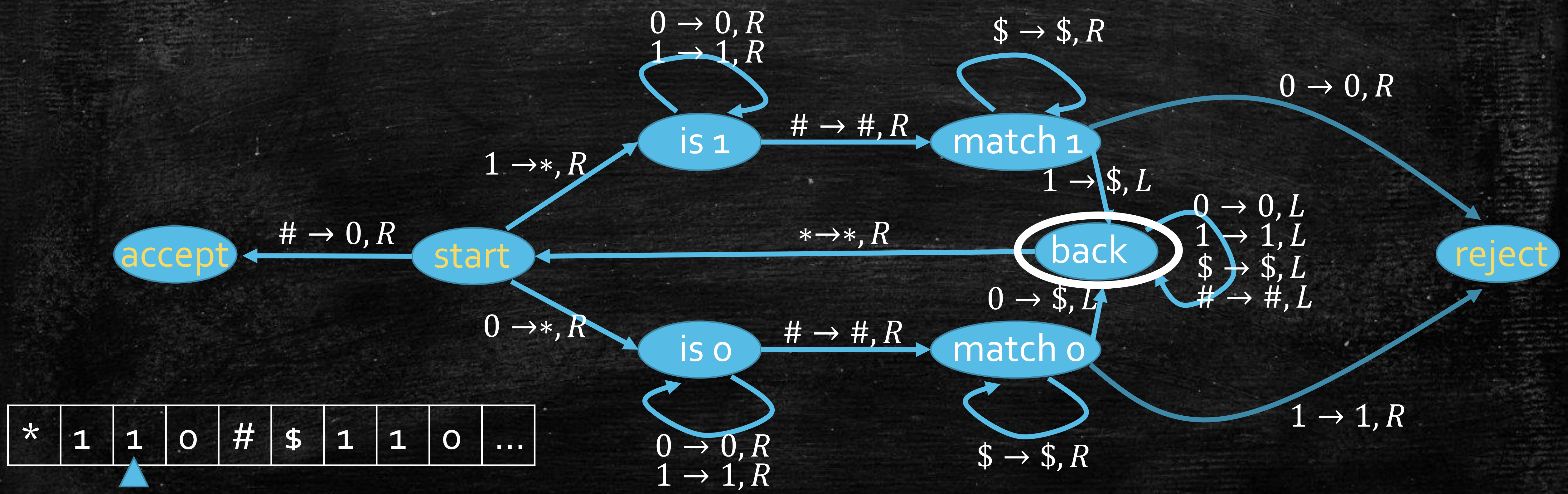

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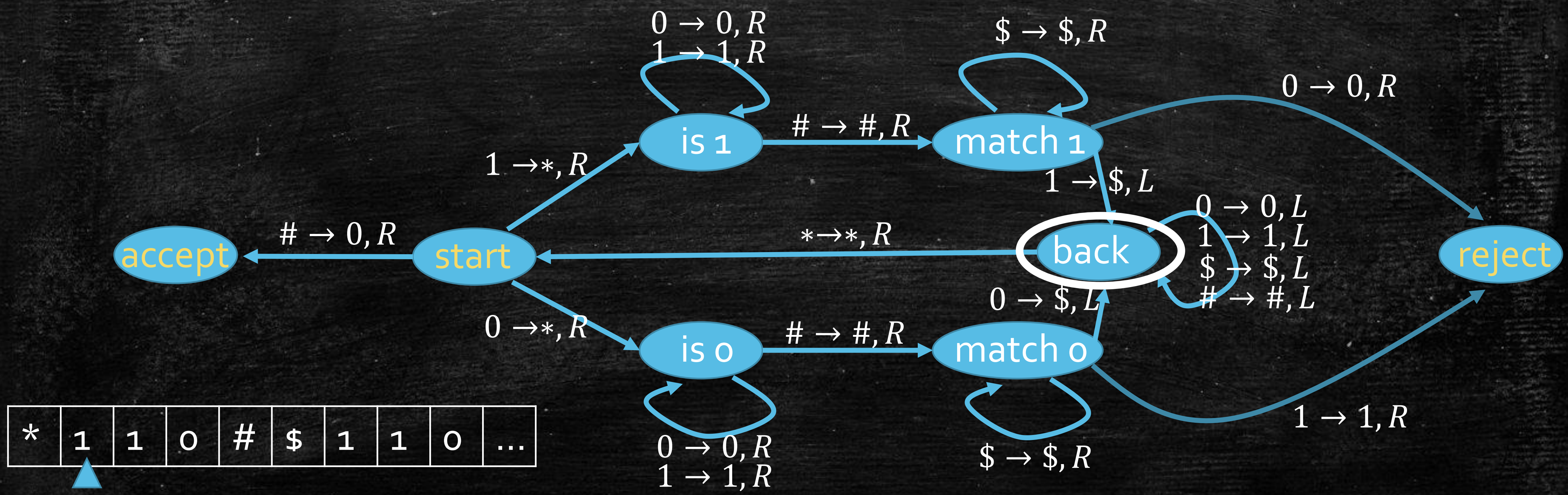

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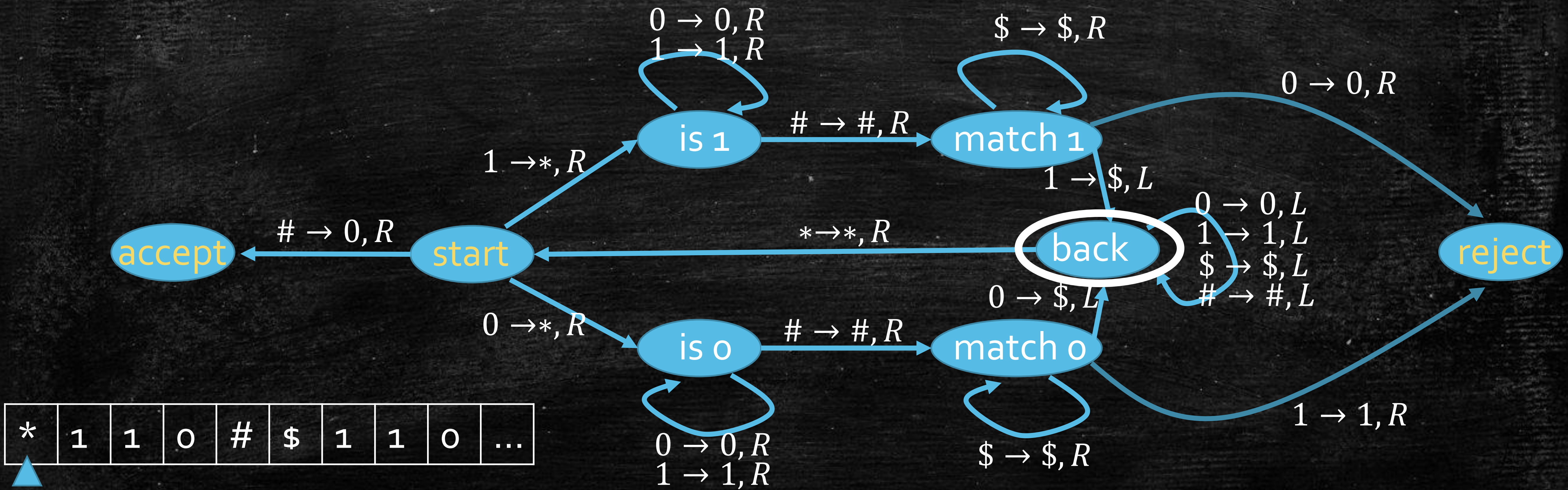

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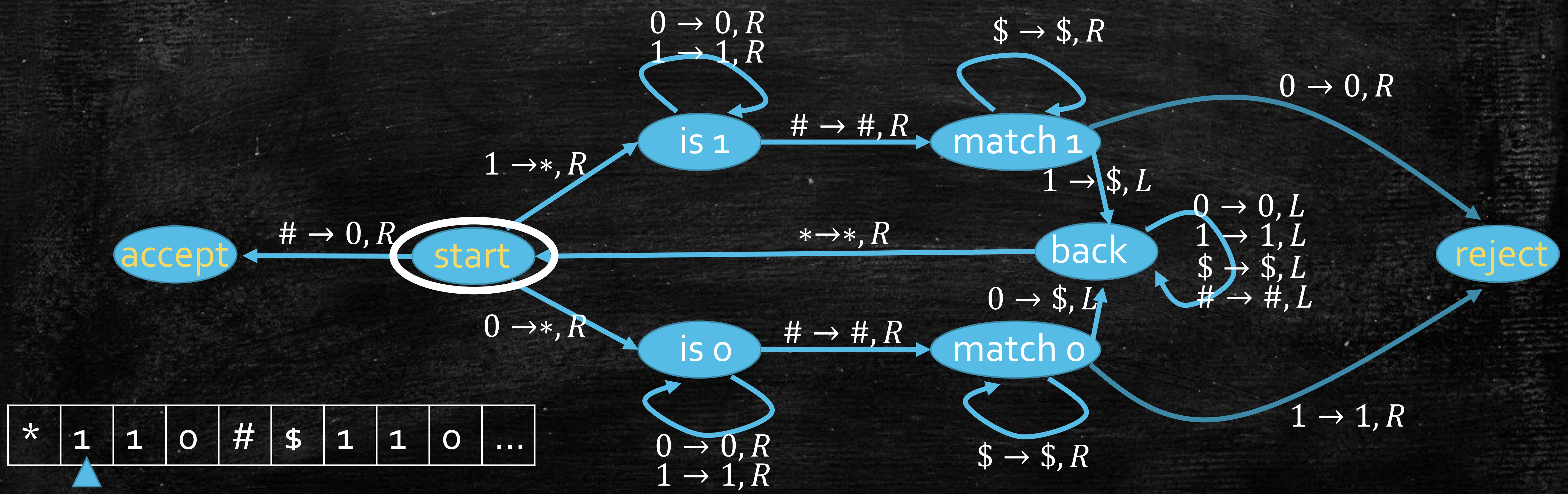

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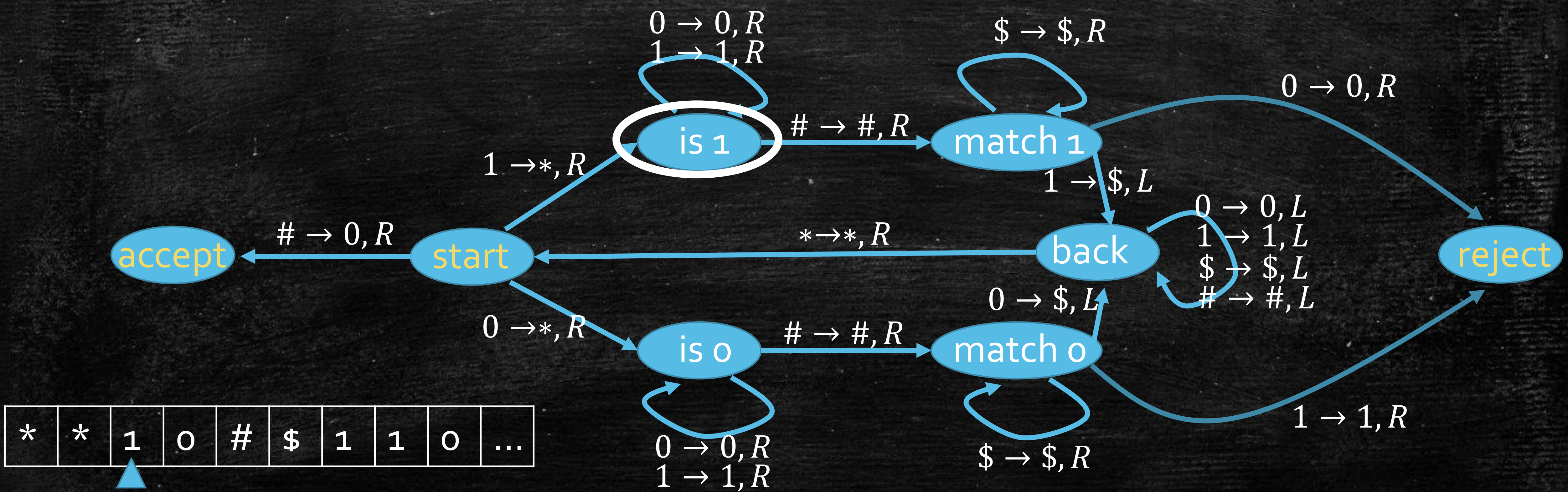

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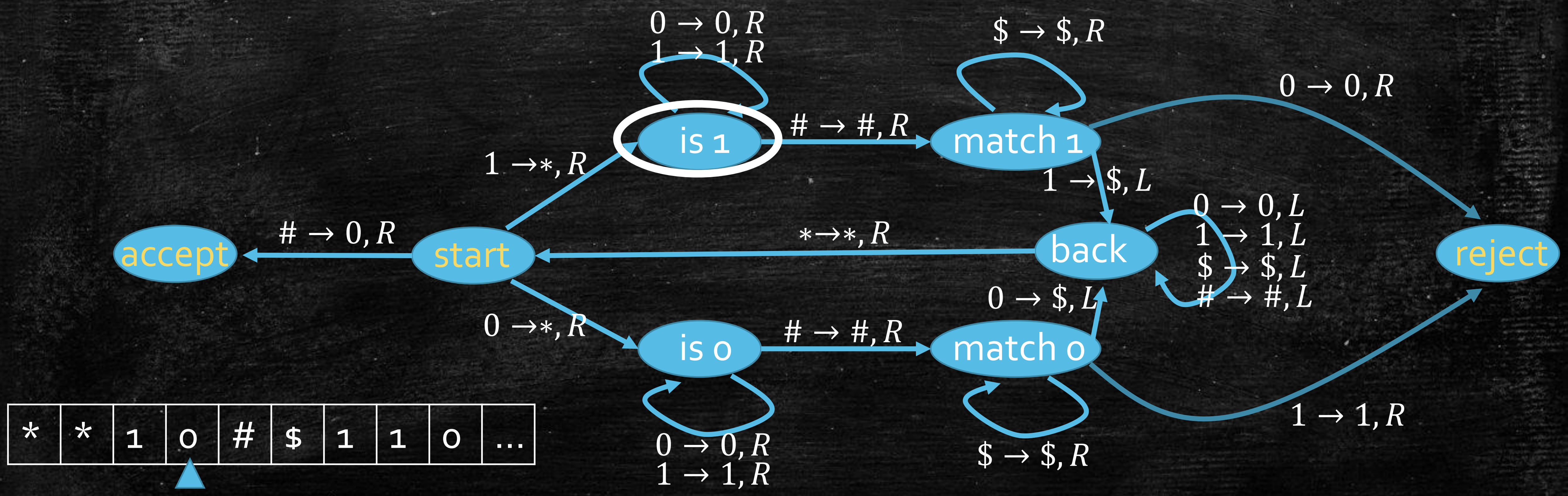

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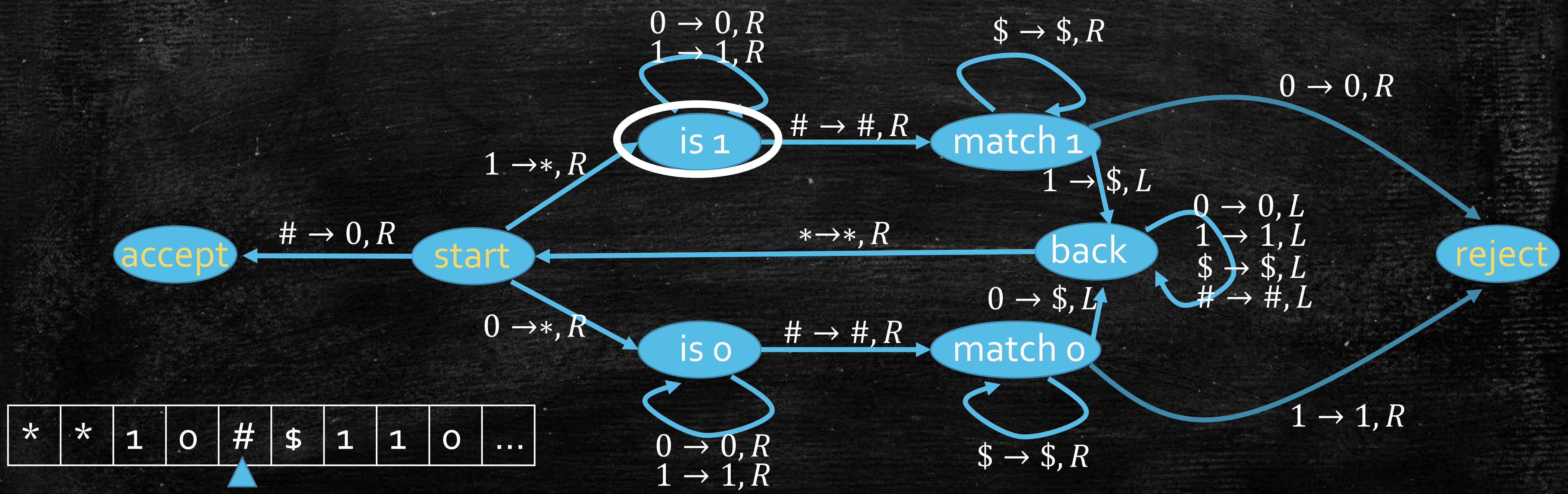

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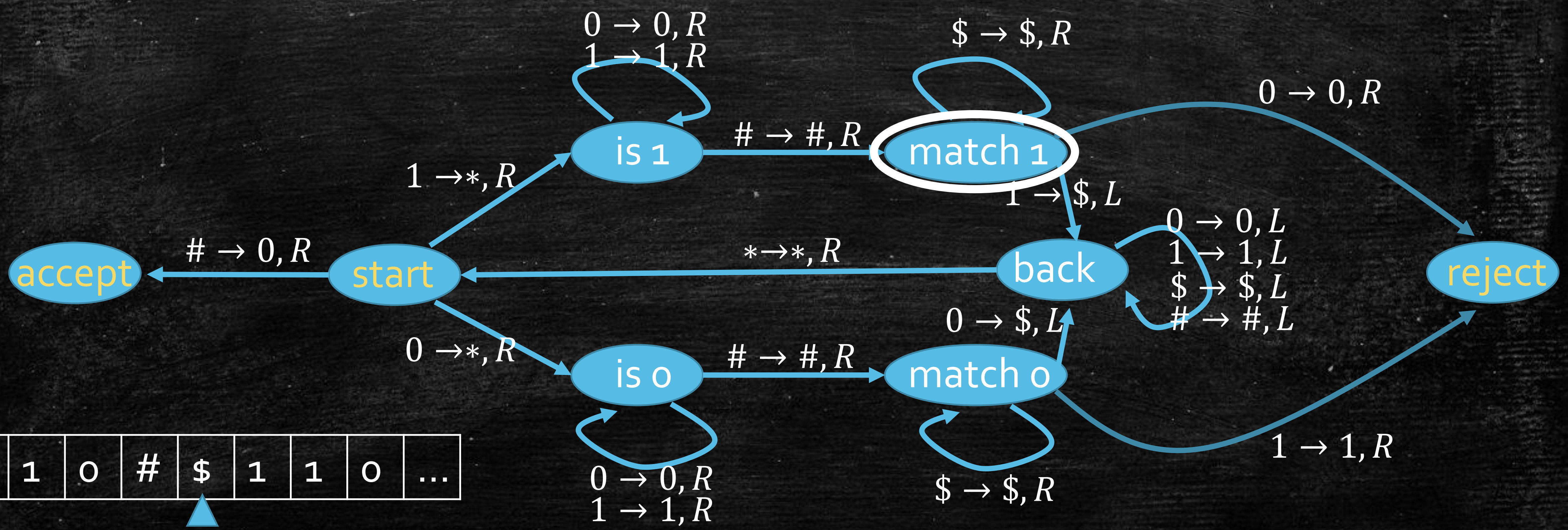

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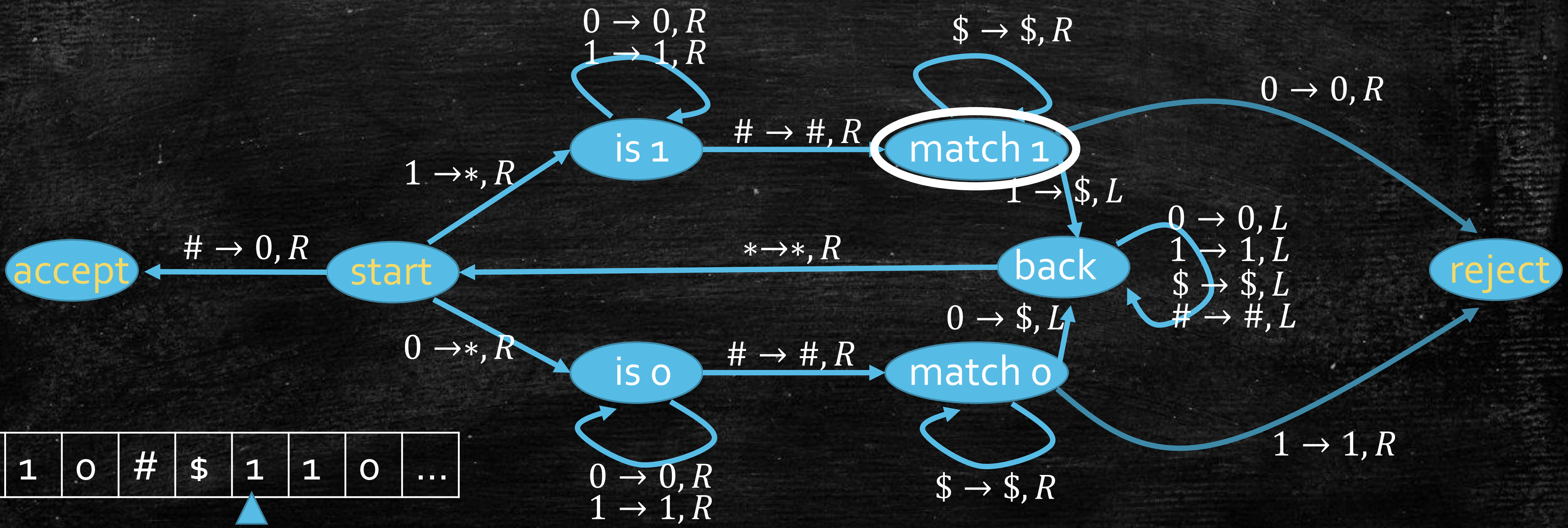

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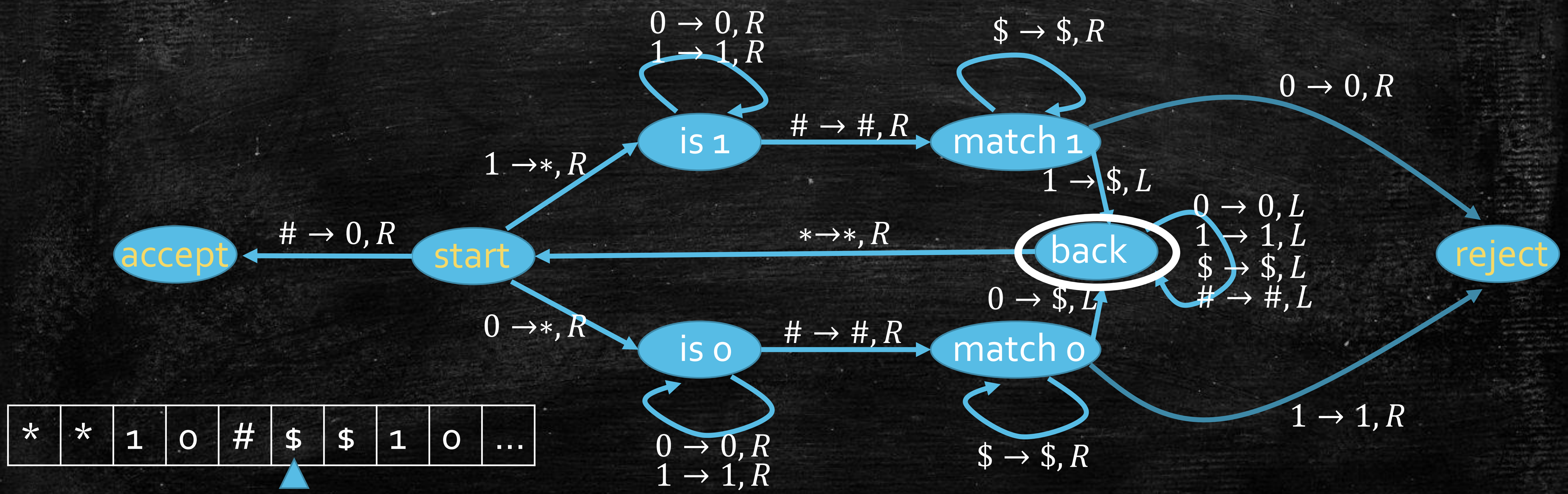

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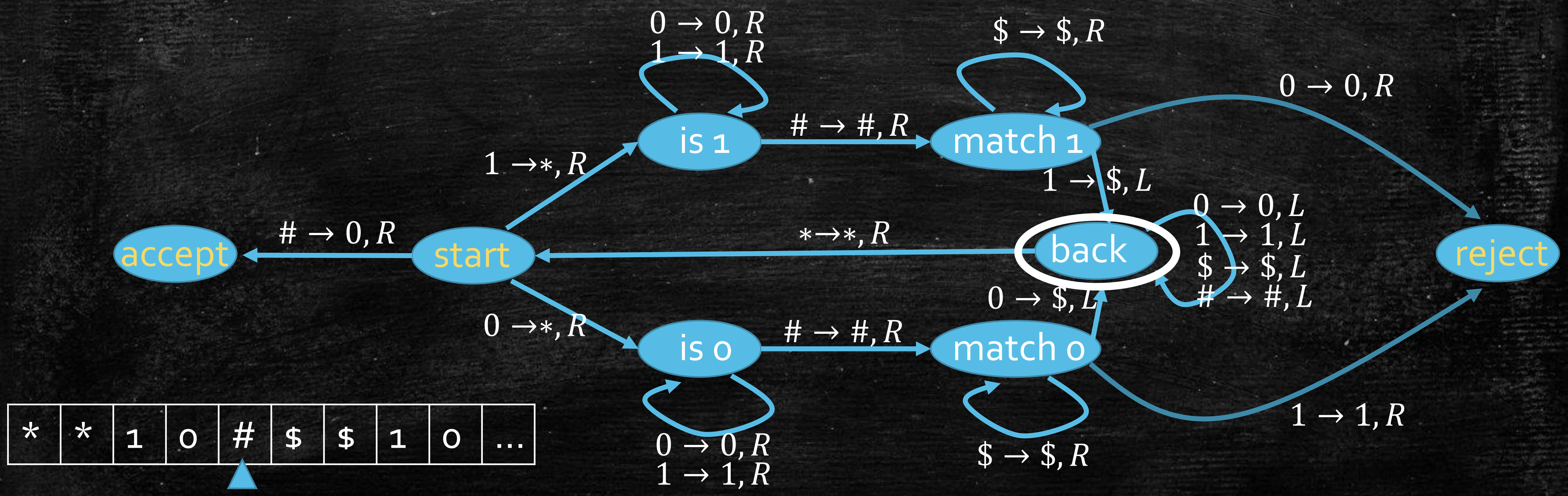

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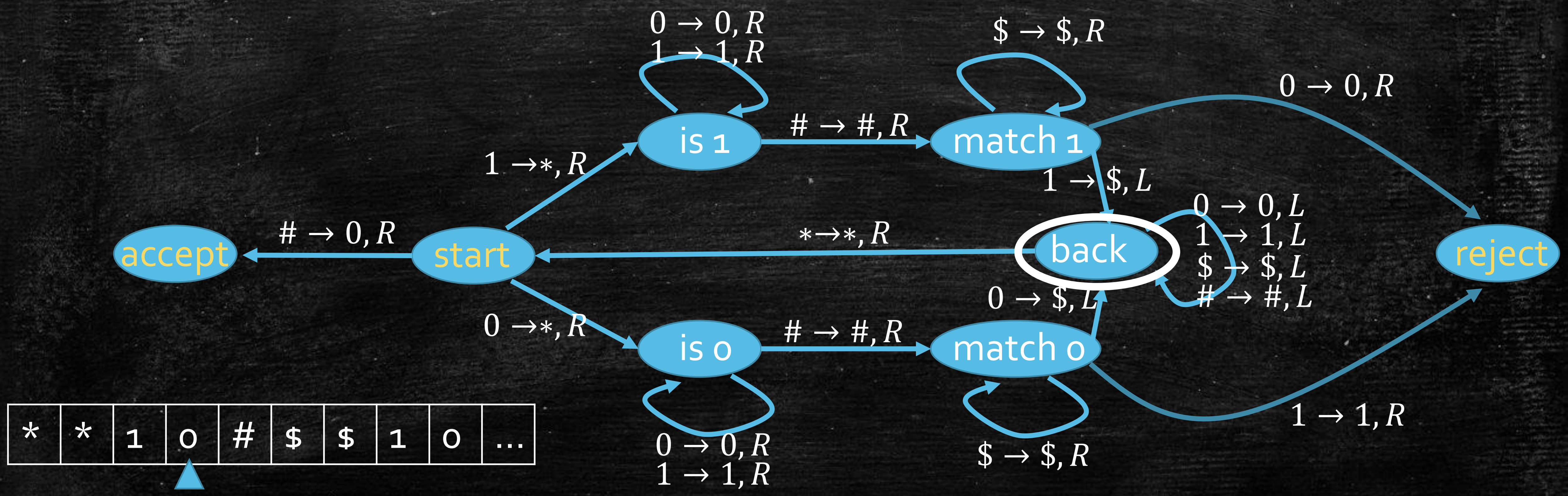

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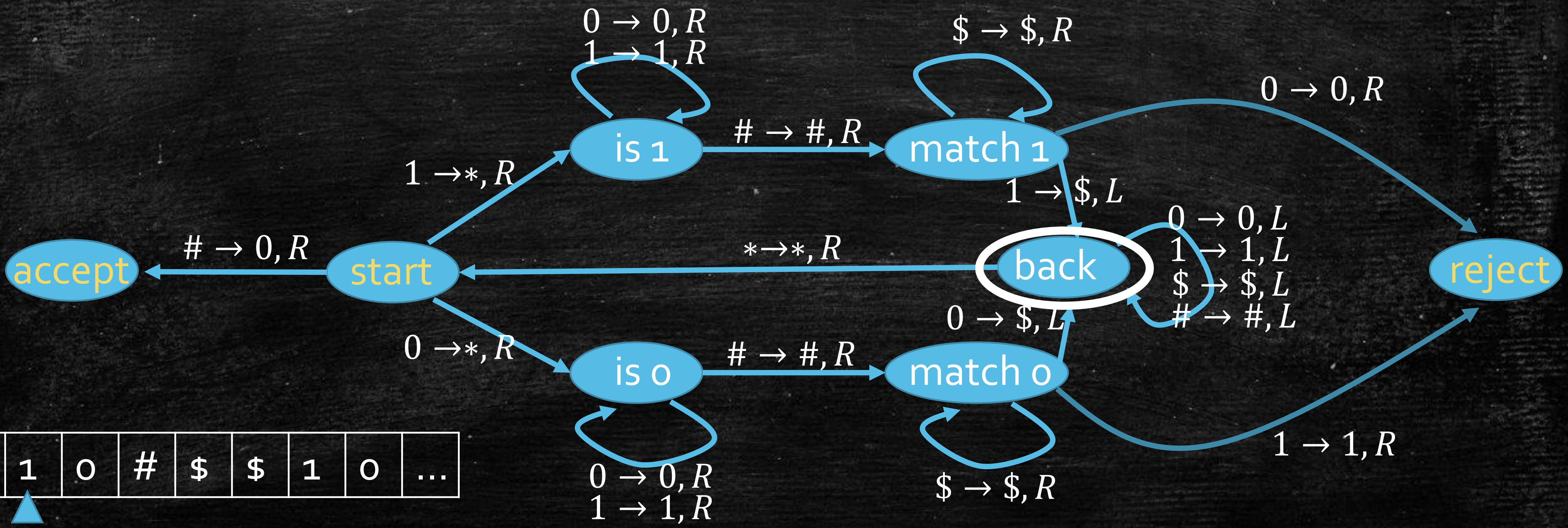

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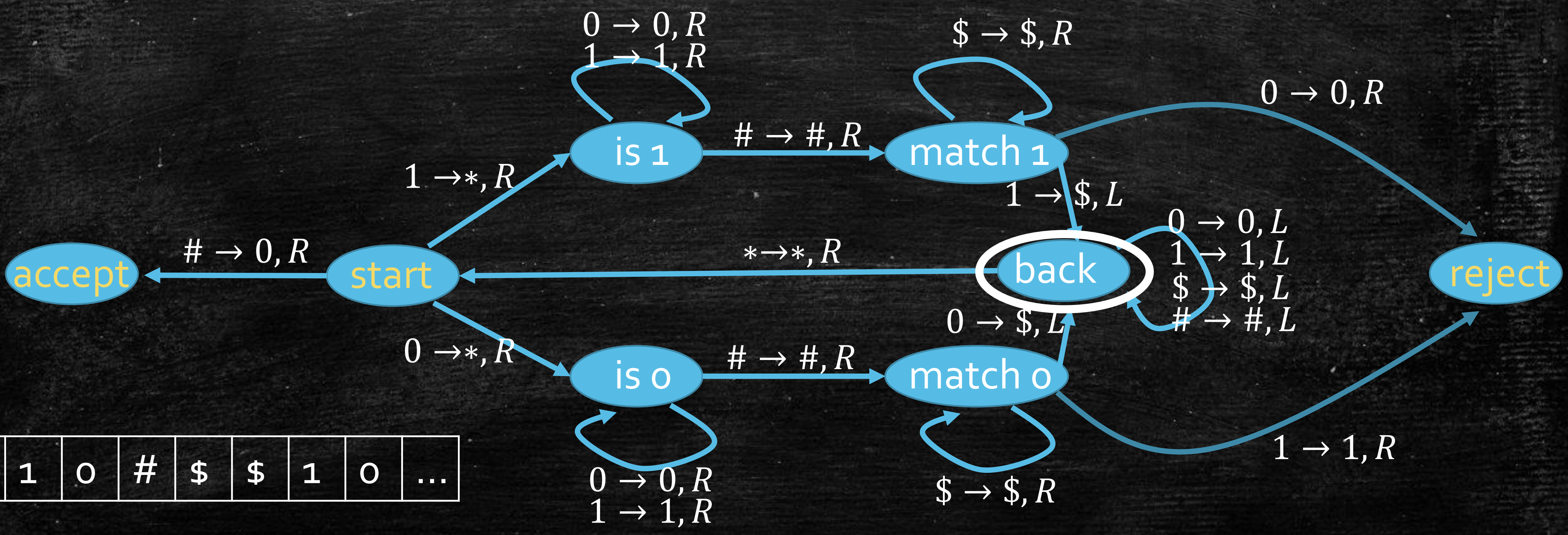

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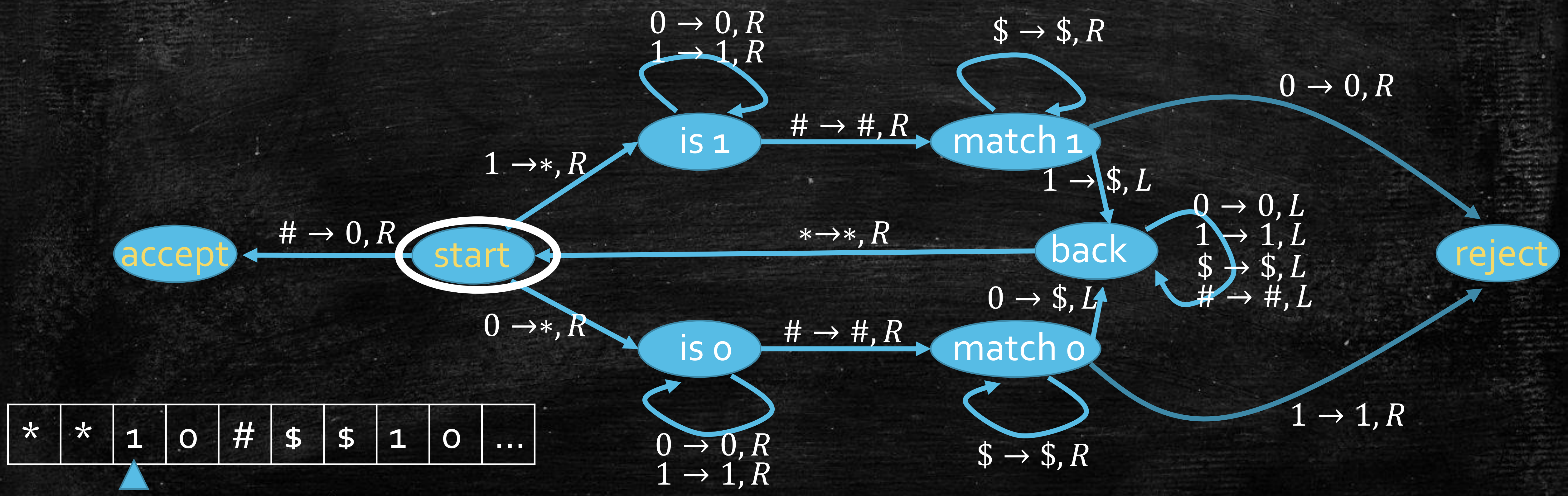

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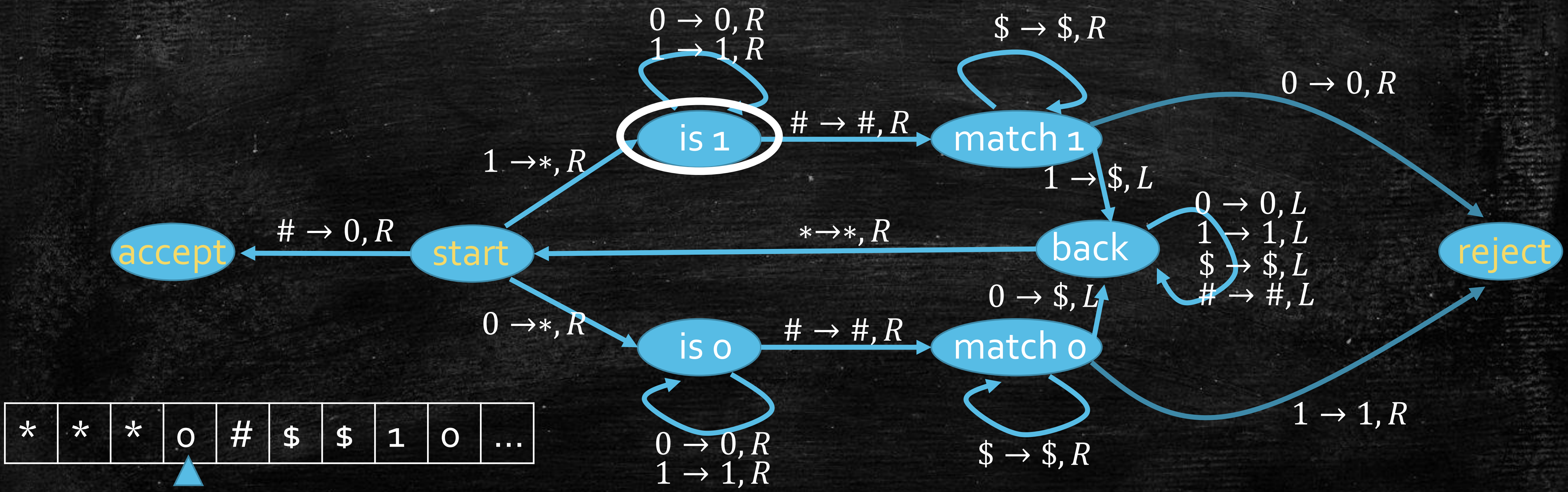

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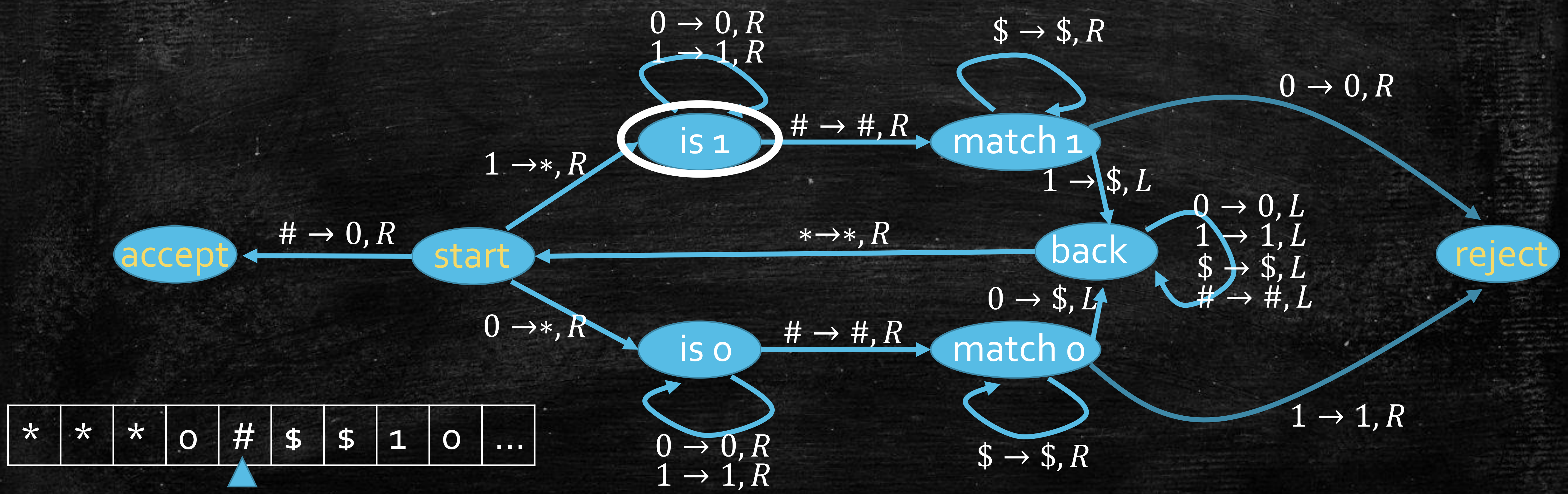

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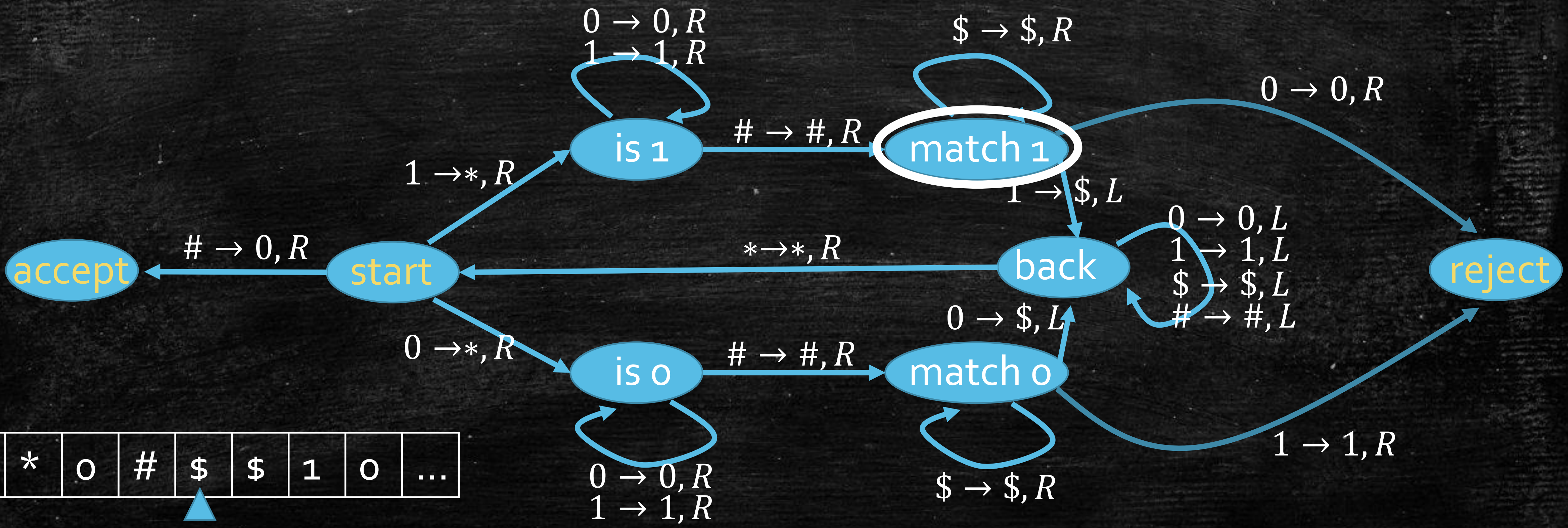

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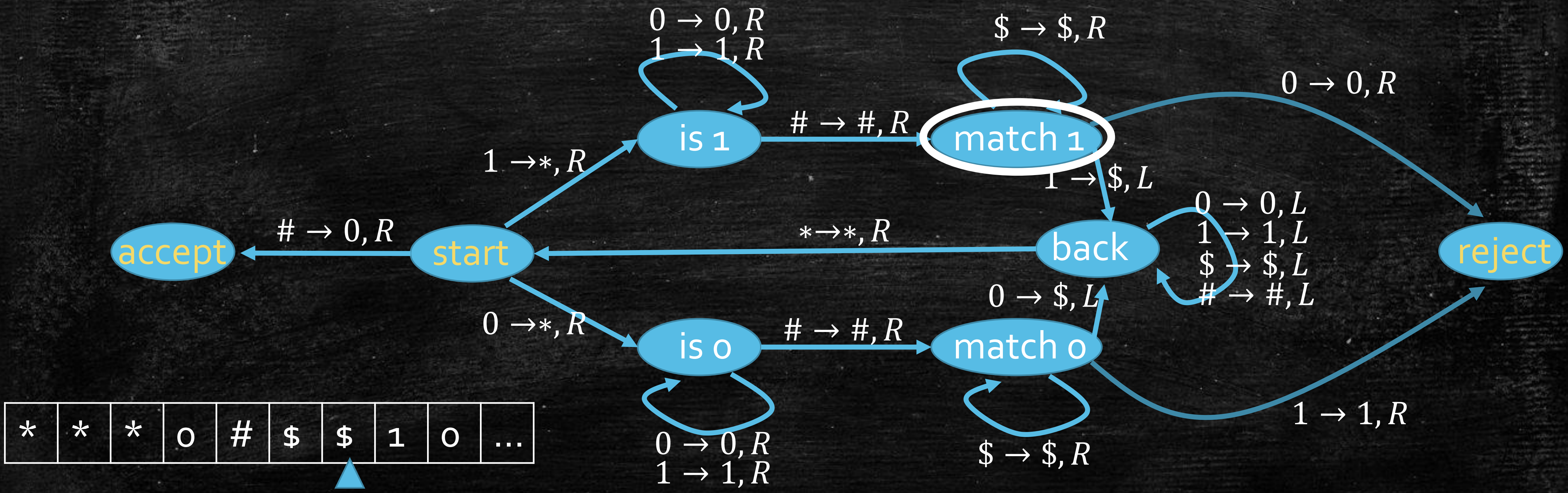

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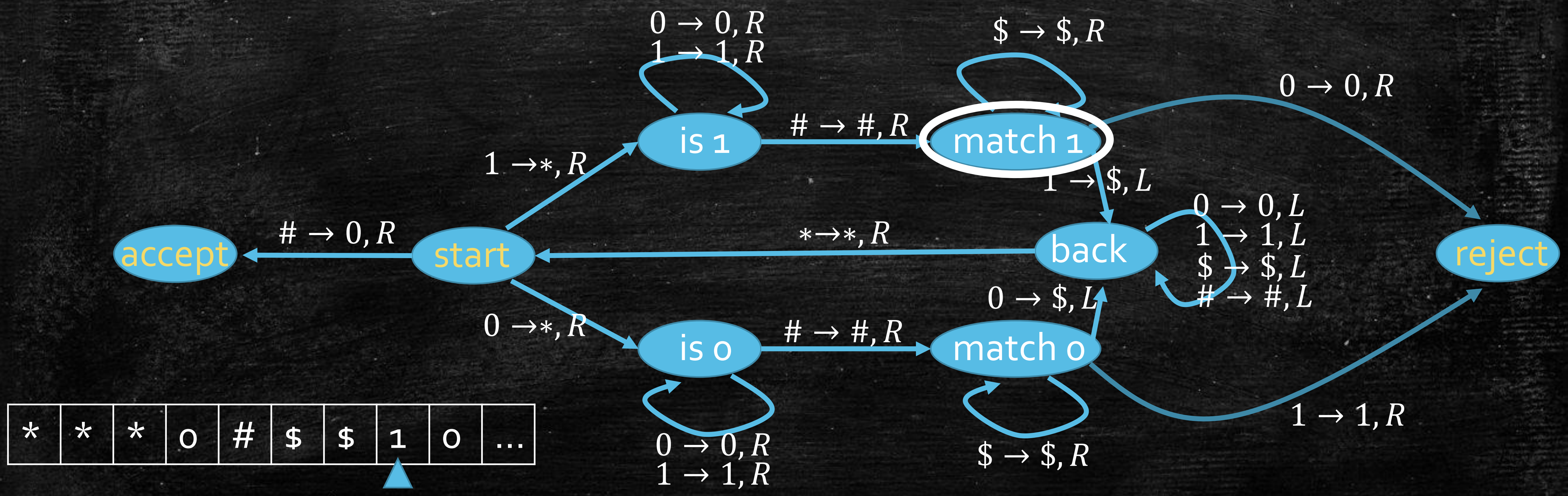

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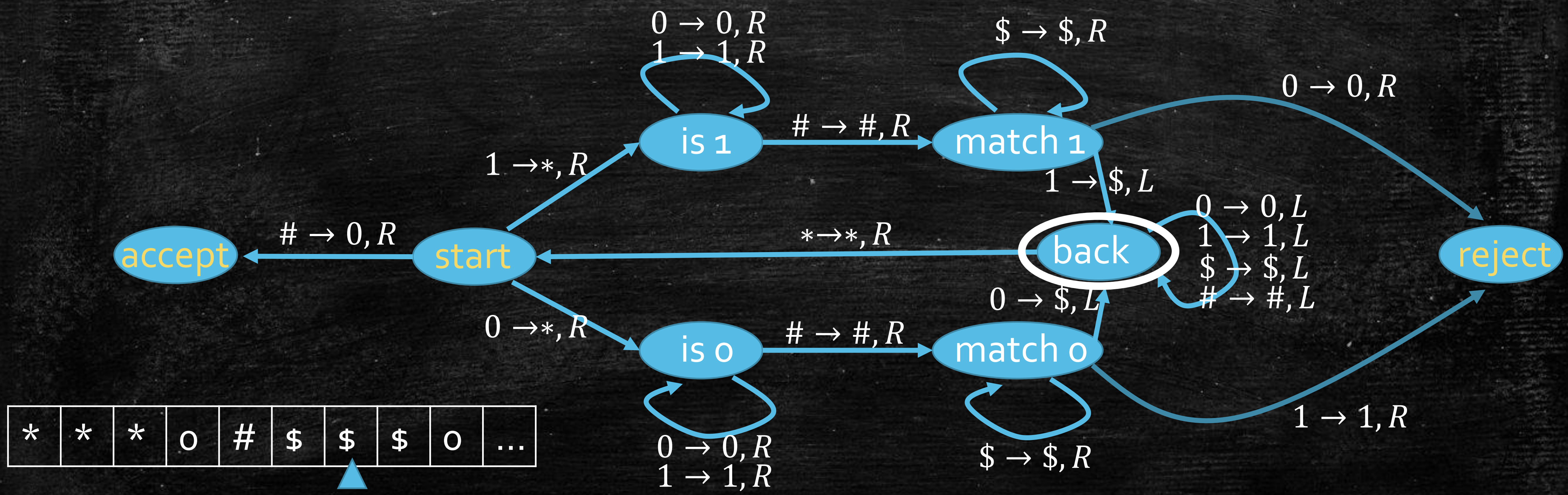

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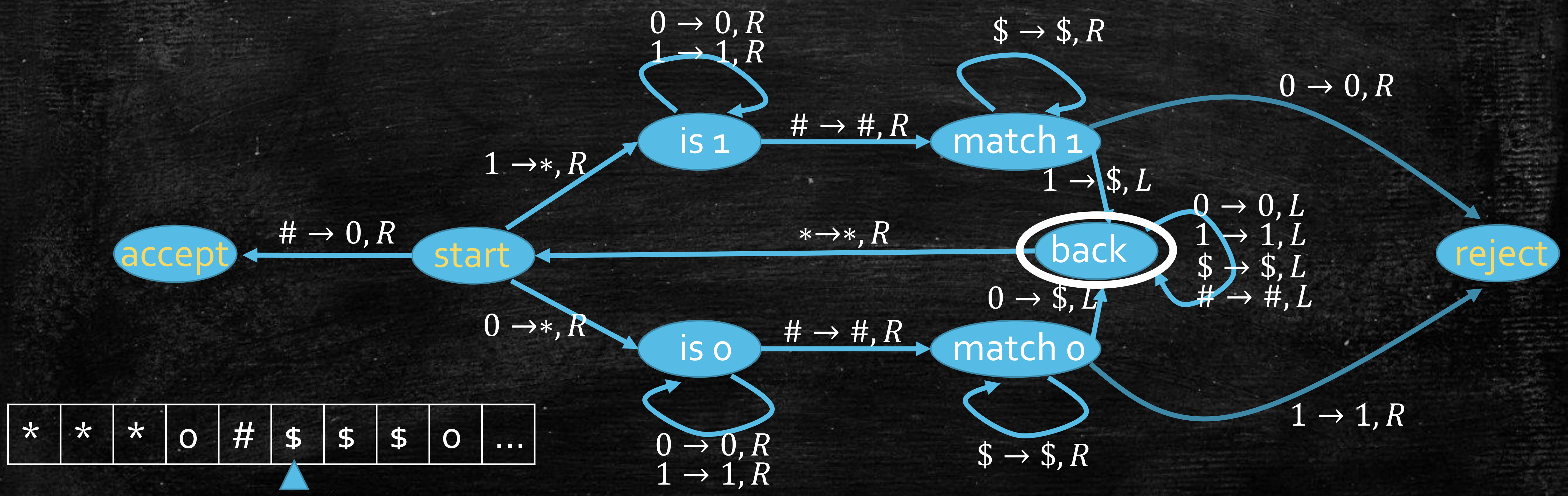

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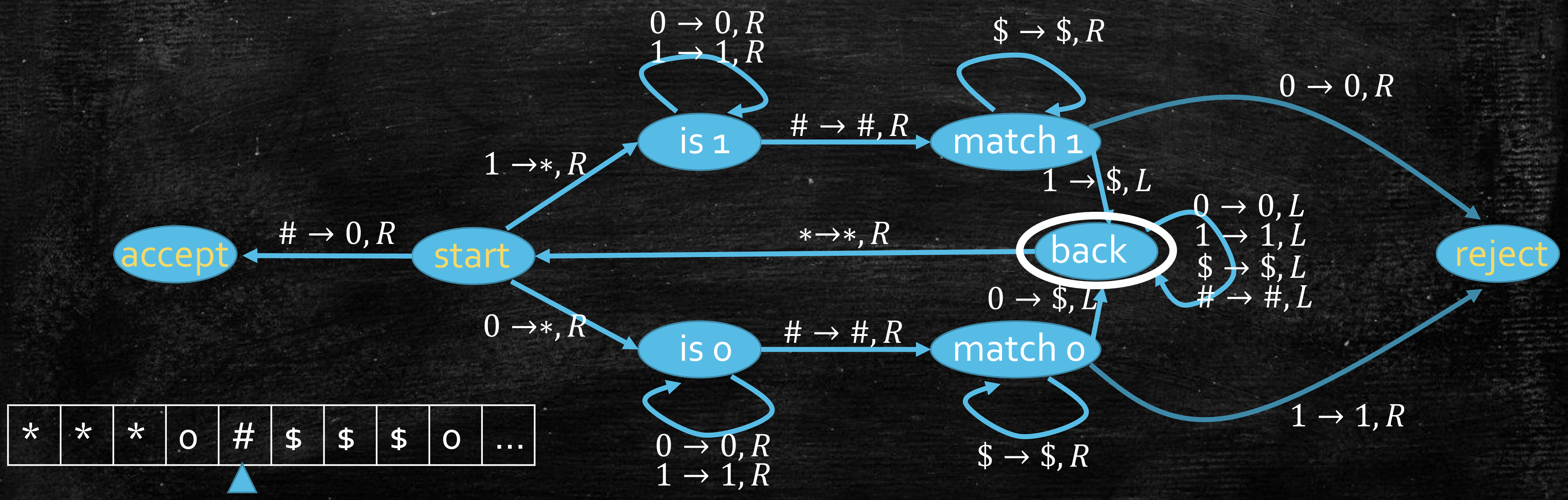

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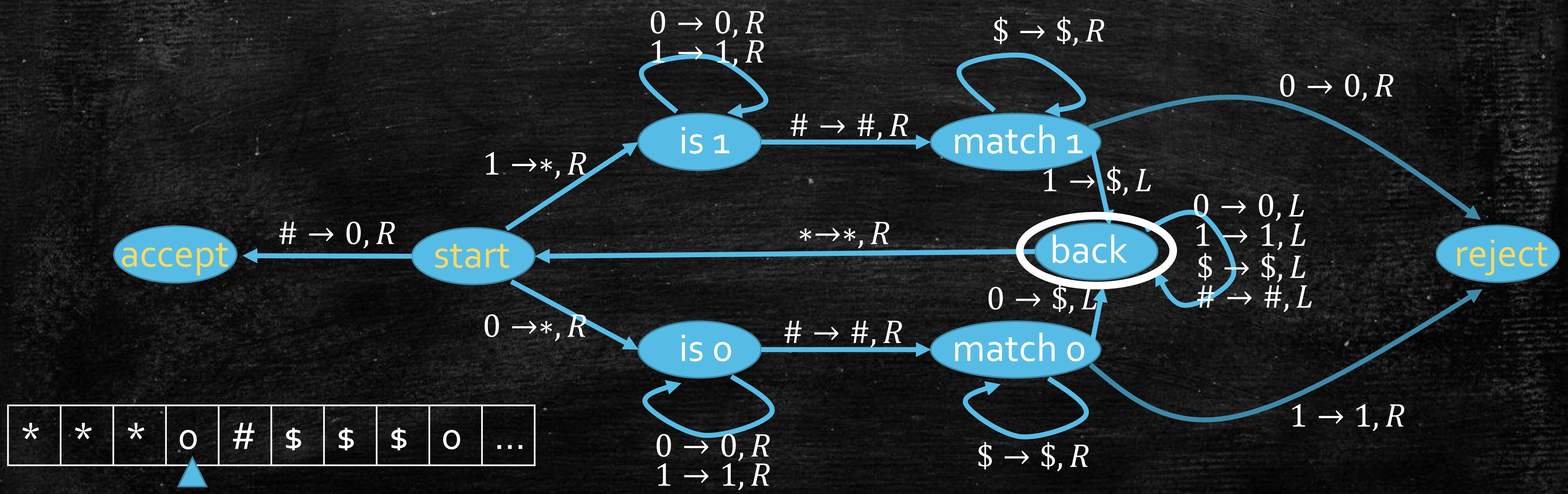

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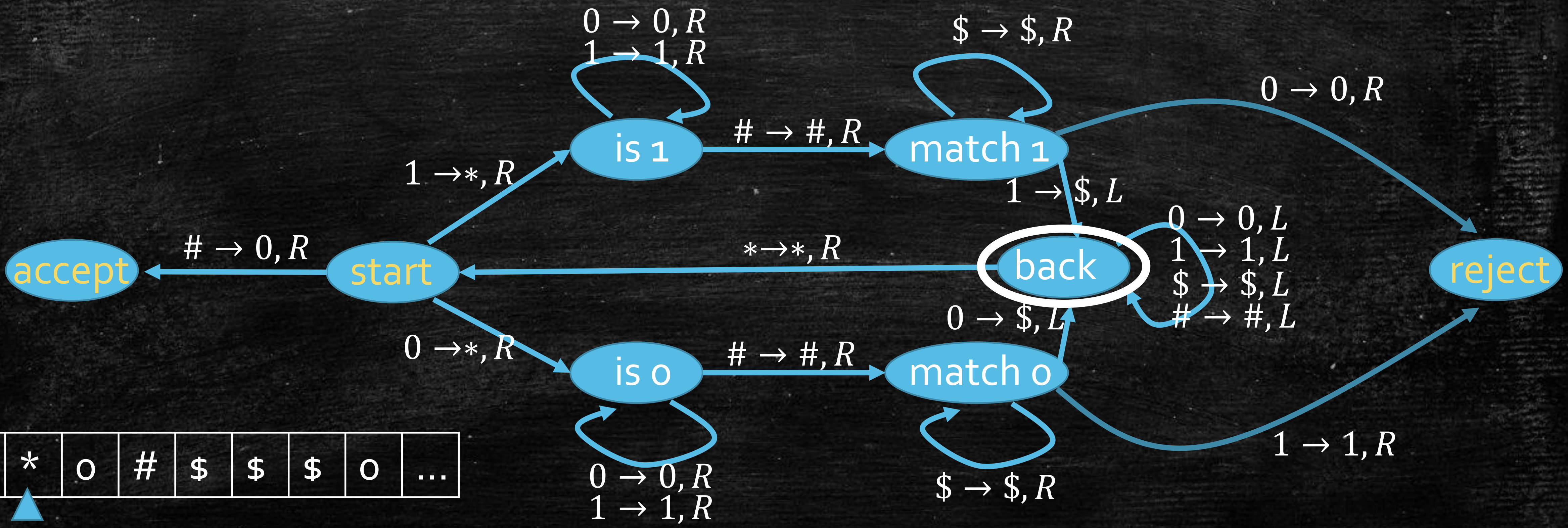

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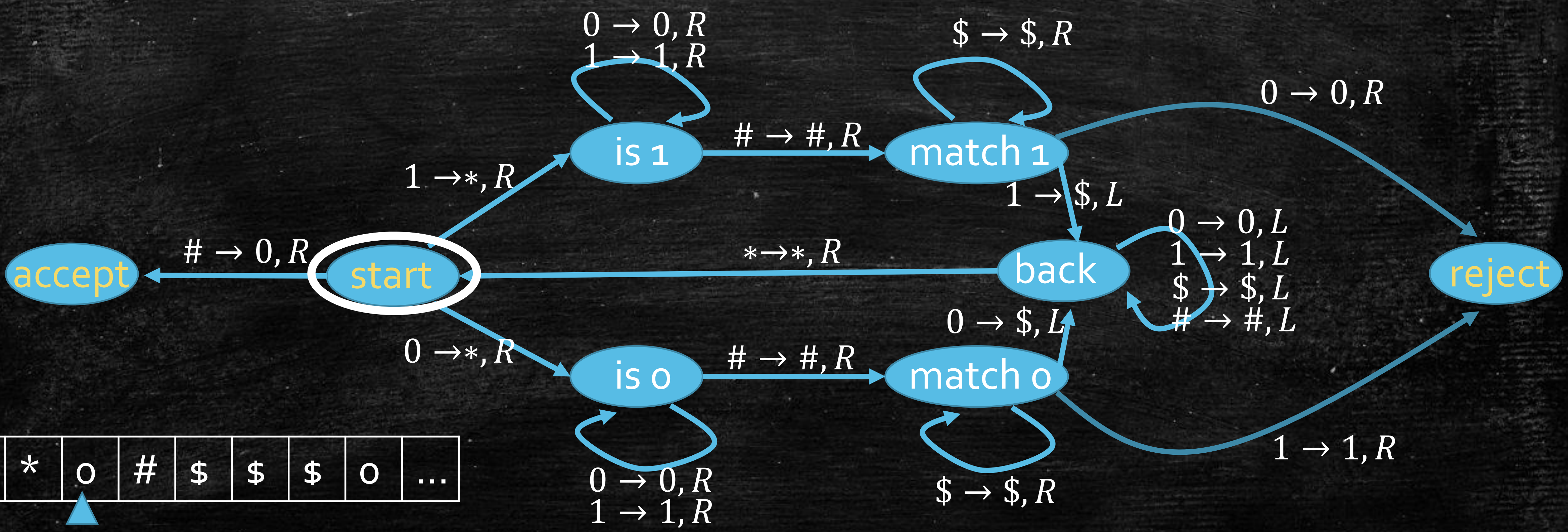

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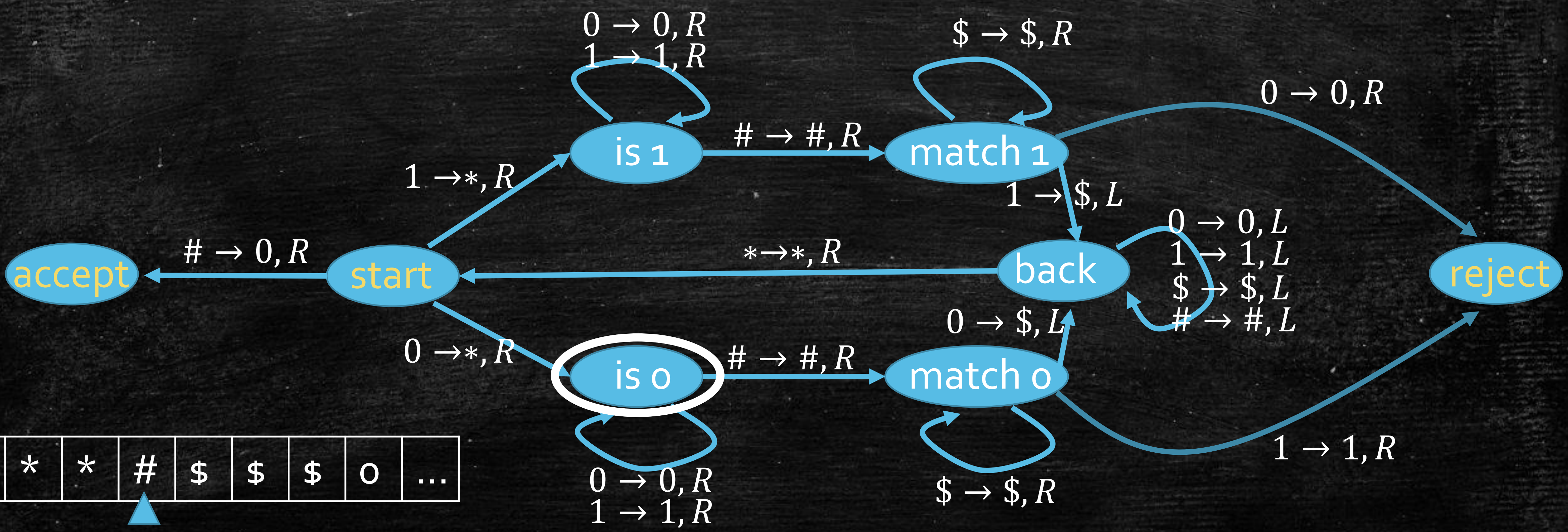

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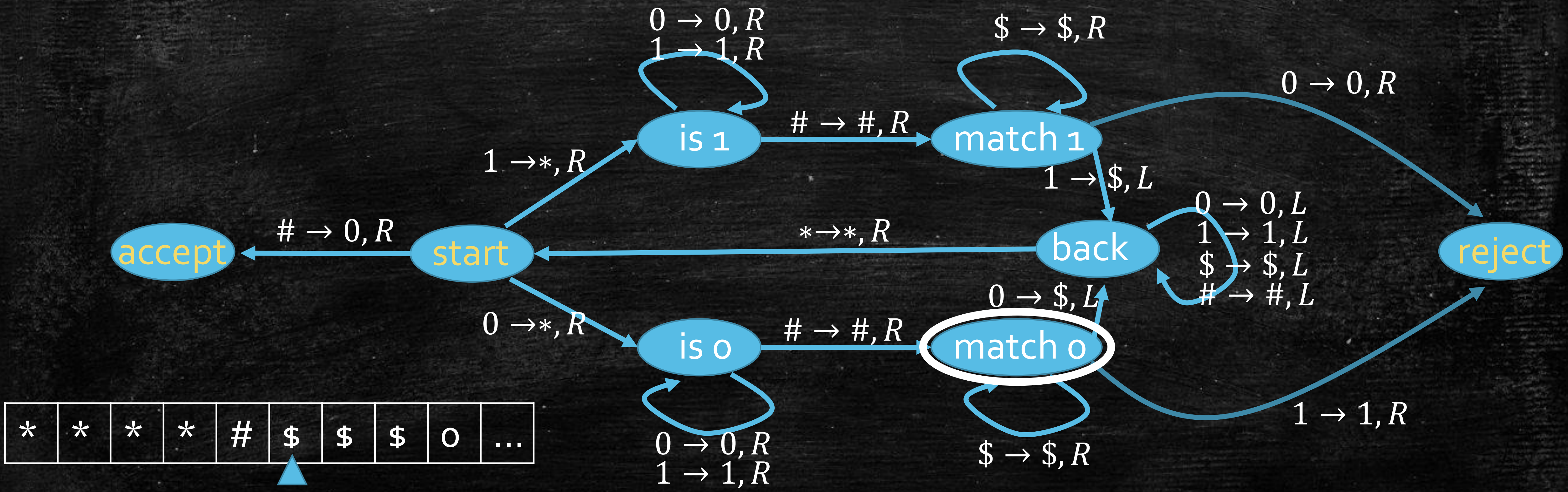

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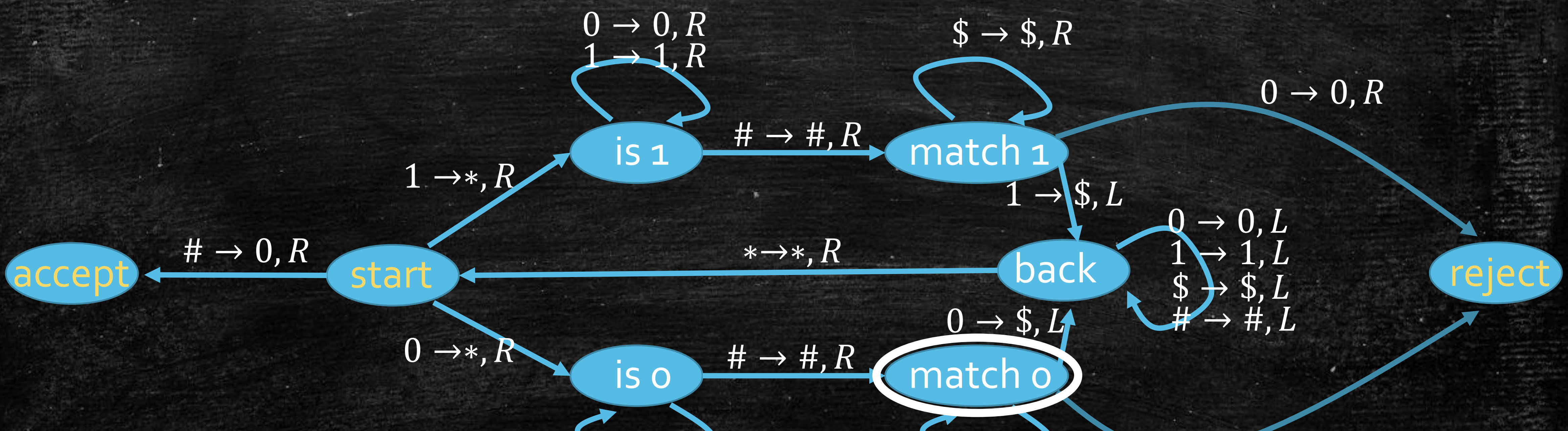

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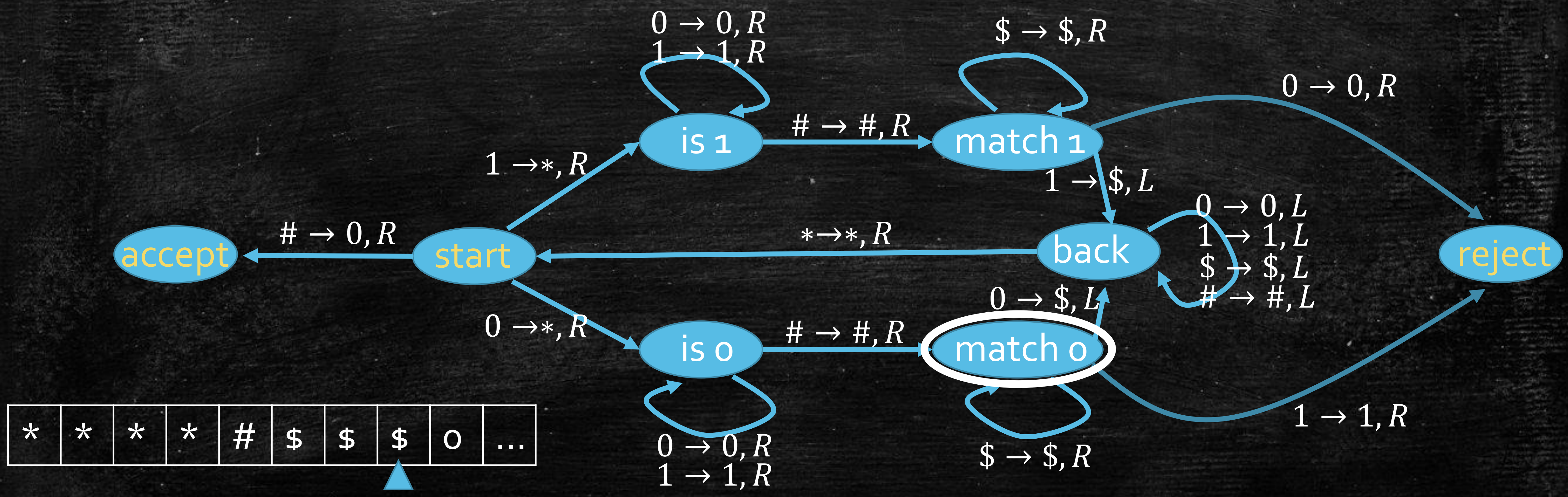

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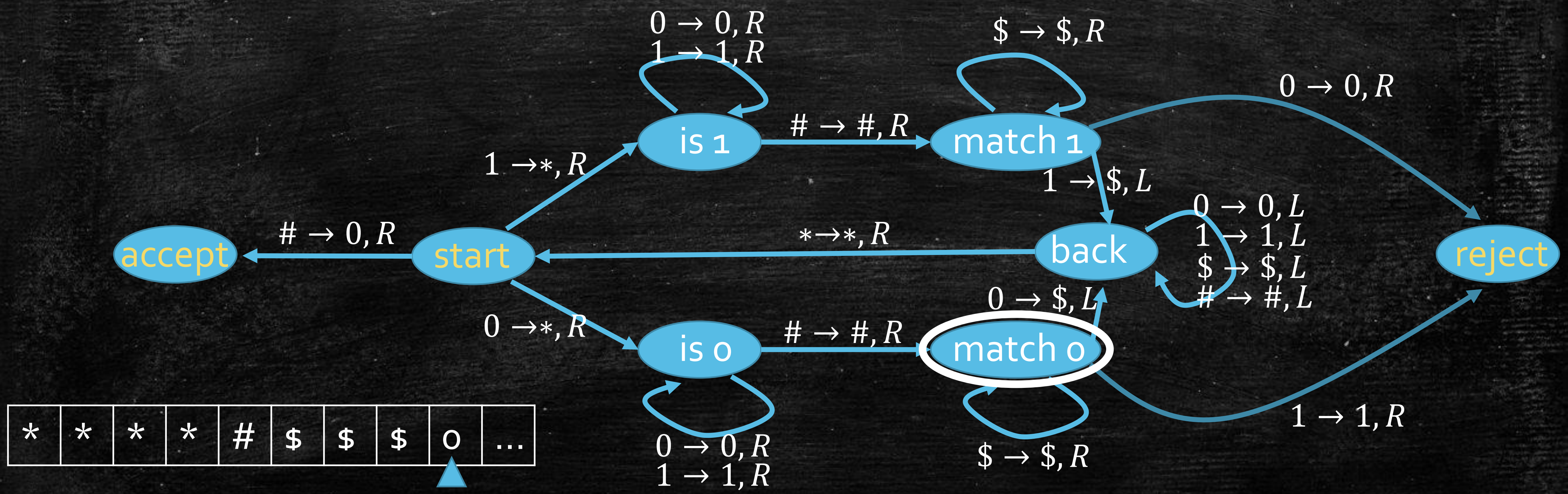

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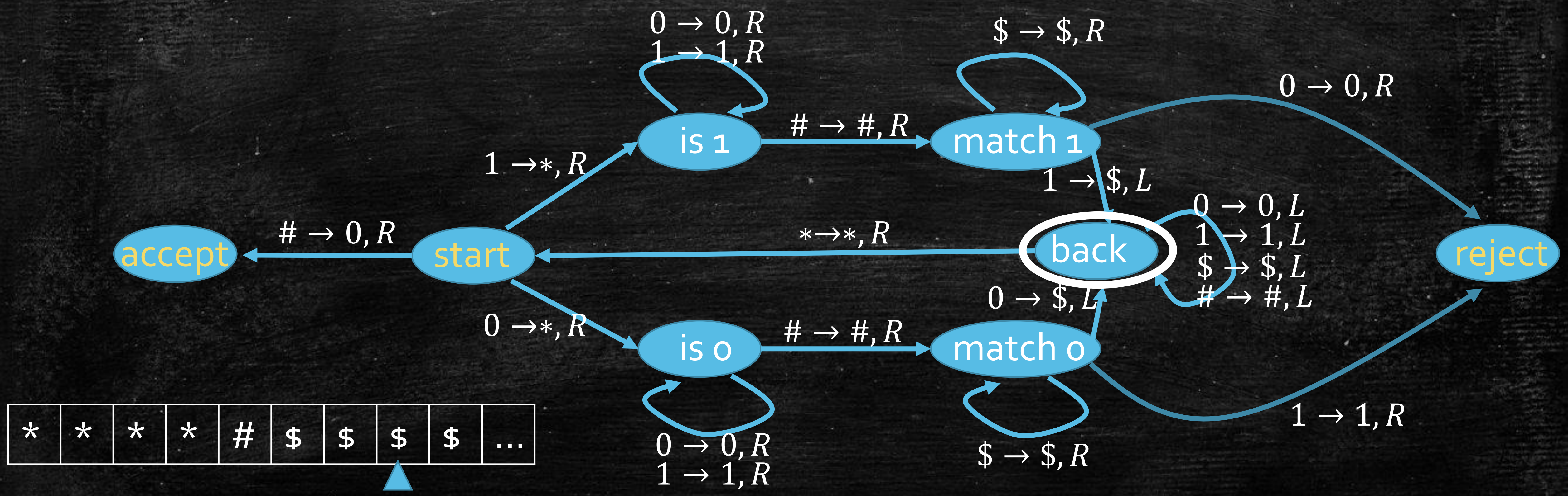

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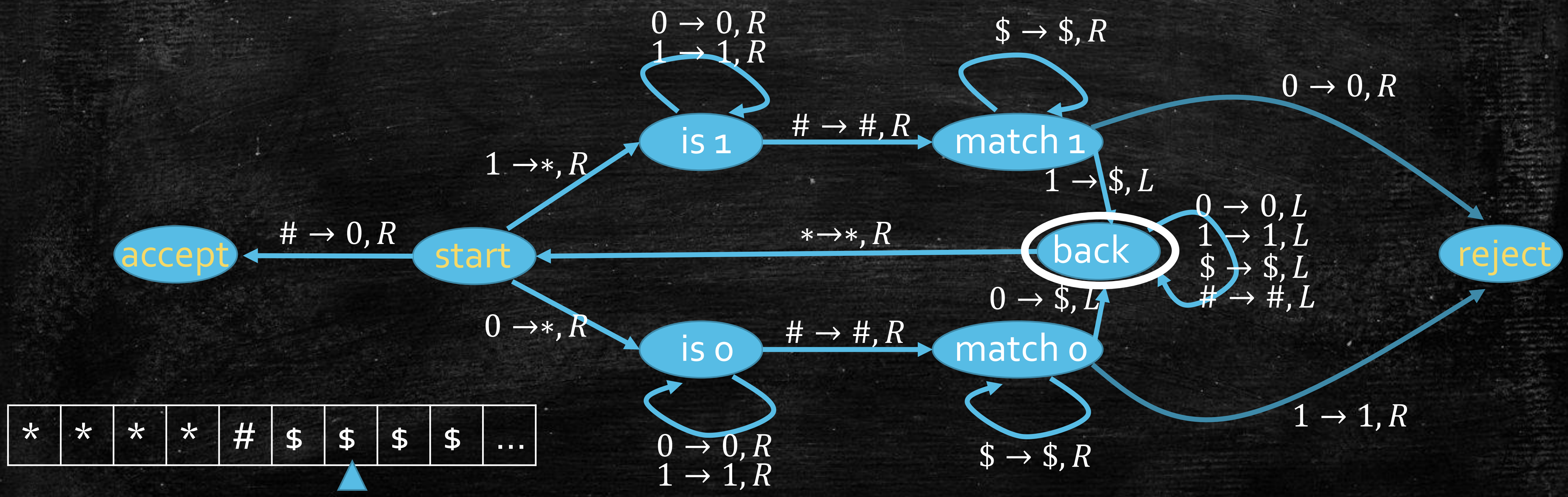

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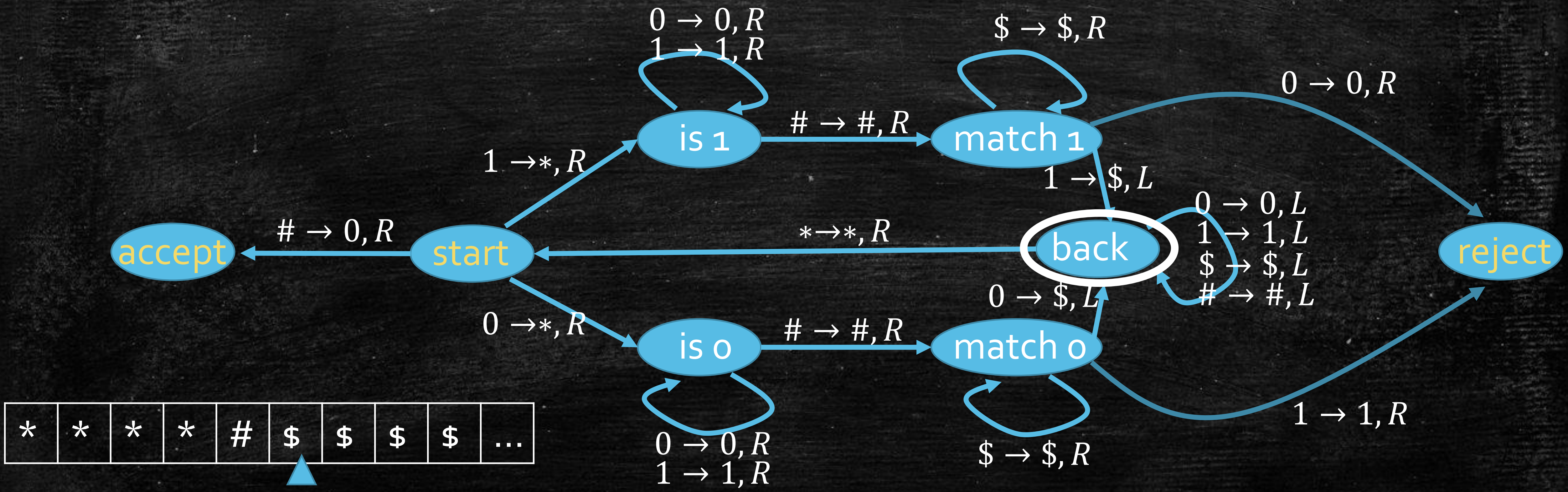

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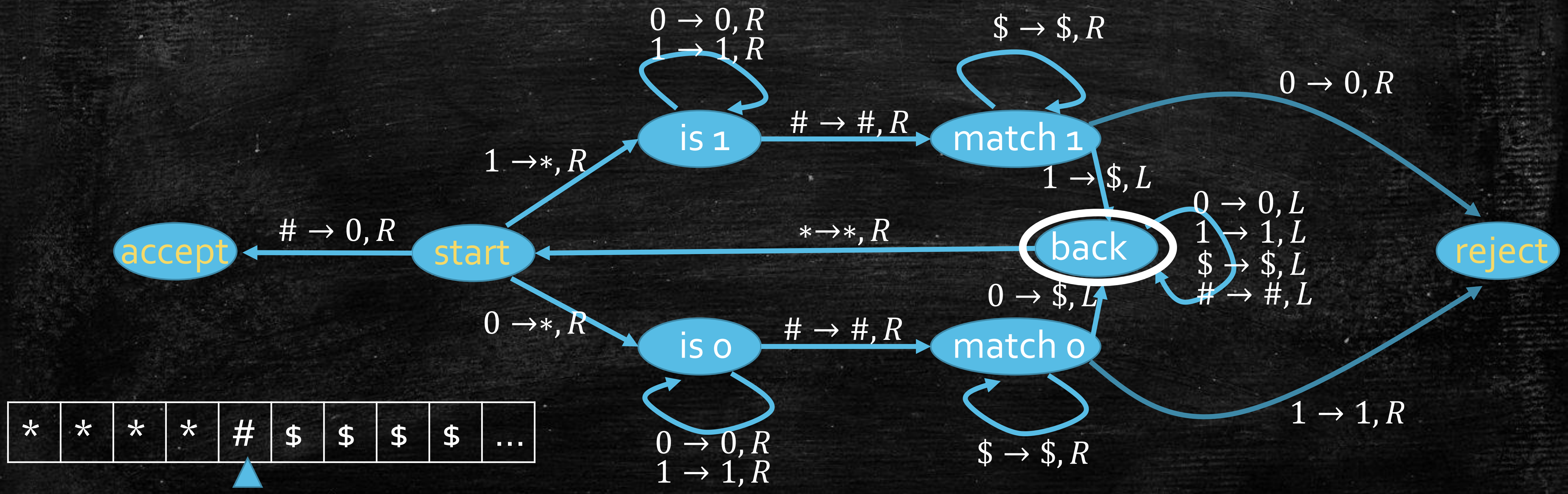

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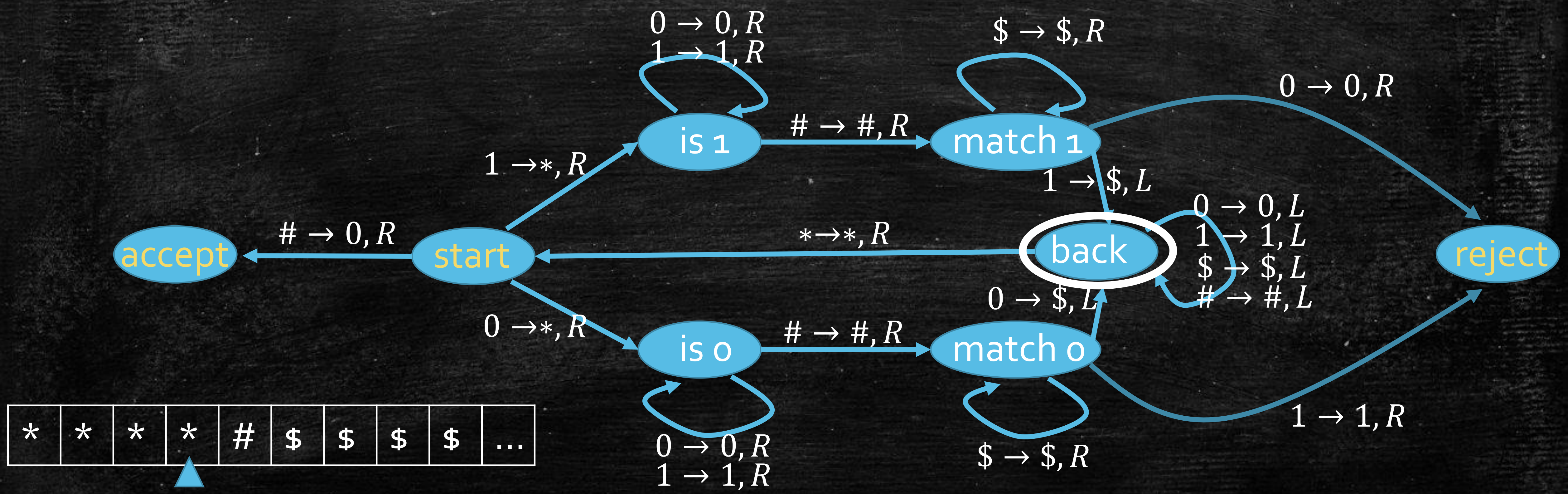

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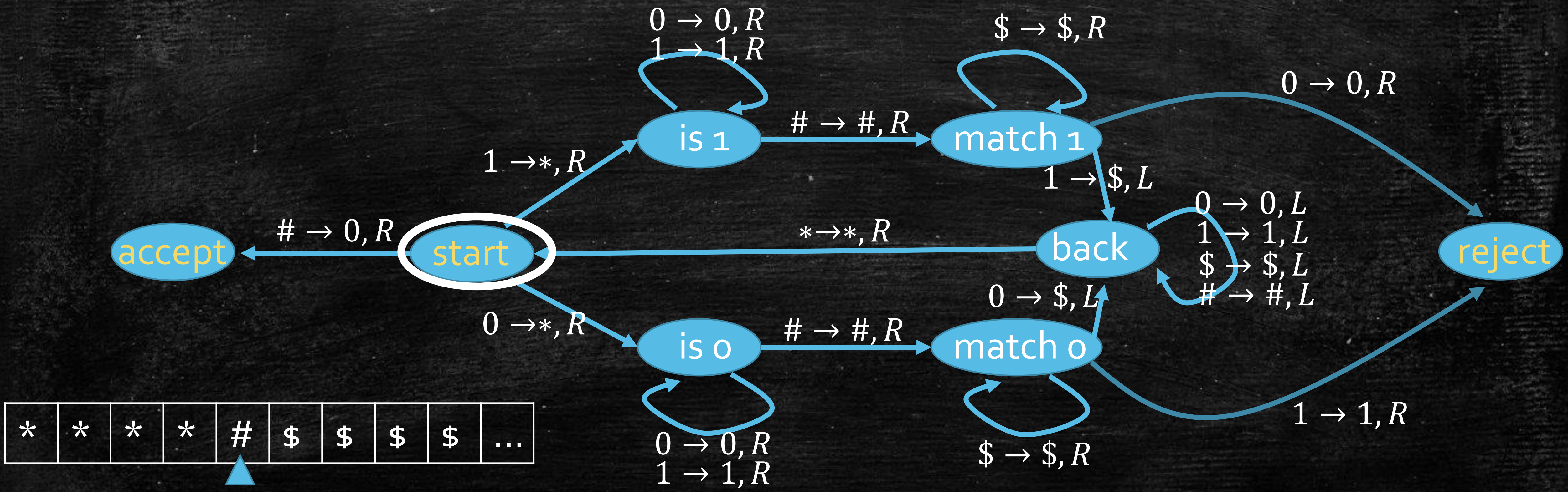

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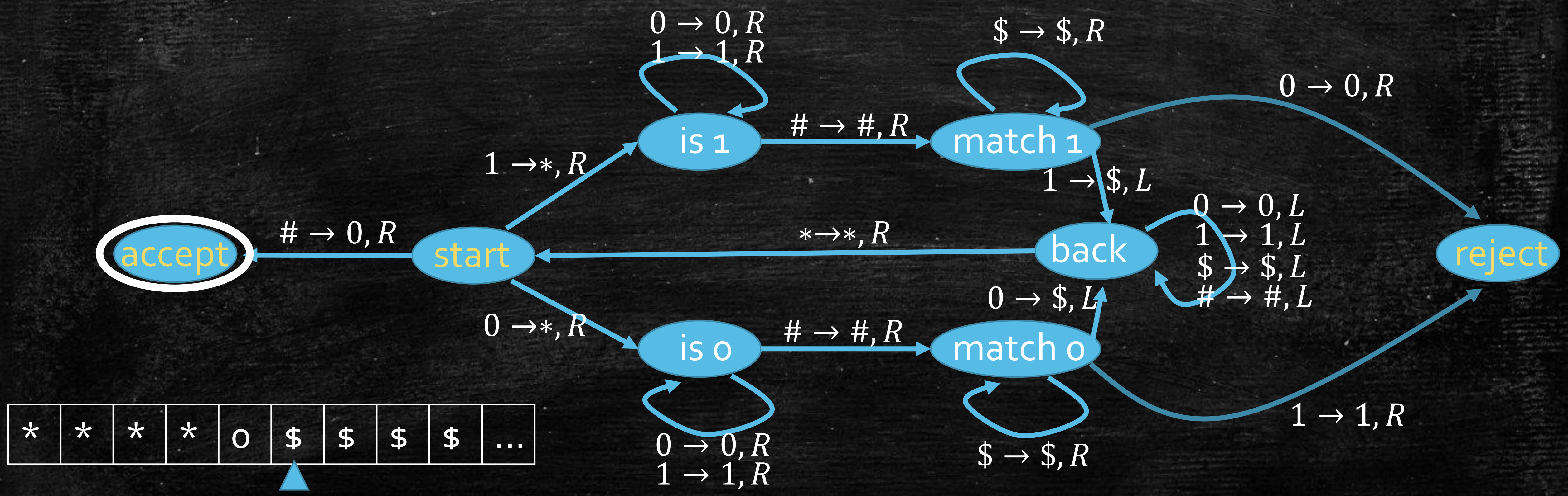


Turing Machine

- If you do not appreciate a Turing machine, in this course, just treat it as a computer program or an algorithm (that outputs "accept" or "reject" as well as an output string)...
- Turing machine has the same power as a computer program or an algorithm, in the following sense:
- Whatever can be computed in polynomial time by a computer program or an algorithm can also be computed in polynomial time by a Turing machine.

Polynomial Time TM

- **Definition.** A Turing Machine $\mathcal{A}$ is a polynomial time TM if there exists a polynomial p such that $\mathcal{A}$ always terminates within $p(|x|)$ steps on input x.

The Complexity Class **P**

- A decision problem $f: \Sigma^* \to \{0, 1\}$ is in **P**, if there exists a polynomial time TM $\mathcal{A}$ such that
  - $\mathcal{A}$ accepts x if $f(x) = 1$
  - $\mathcal{A}$ rejects x if $f(x) = 0$
- Problems in **P** are those "easy" problems that can be solved in polynomial time.

Examples for Problems in **P**

- [PATH] Given a graph $G = (V, E)$ and $s, t \in V$, decide if there is a path from s to t.
  - Build a TM that runs BFS or DFS at s; accept if t is reached; reject if the search terminates without reaching t.
  - PATH ∈ **P**
- [k-FLOW] Given a directed graph $G = (V, E)$, $s, t \in V$, a capacity function $c: E \to \mathbb{R}^+$, and $k \in \mathbb{R}^+$, decide if there is a flow with value at least k.
  - Build a TM that implements Edmonds-Karp, Dinic's, or other algorithms.
  - k-FLOW ∈ **P**
- [PRIME] Given $k \in \mathbb{Z}^+$ encoded in binary string, decide if k is a prime number.
  - [Agrawal, Kayal & Saxena, 2004] PRIME ∈ **P**

The Complexity Class **NP**

- A commonality with SAT, VertexCover, IndependentSet, SubsetSum, HamiltonianPath:
  - For a yes instance, it can be easily verified if a hint is given.
- SAT: a hint can be a valid assignment to the variables
- VertexCover/IndependentSet: a hint can be a valid set of k vertices
- SubsetSum: a hint can be a sub-collection with sum k
- HamiltonianPath: a hint can be an encoding of a valid path

The Complexity Class **NP**

- **NP**: Problems whose yes instances can be efficiently verified if hints are given.
- Formal Definition. A decision problem $f: \Sigma^* \to \{0,1\}$ is in **NP** if there exist a polynomial q and a polynomial time TM $\mathcal{A}$ such that
  - If x is a yes instance ($f(x) = 1$), there exists $y \in \Sigma^*$ with $|y| \le q(|x|)$ such that $\mathcal{A}$ accepts the input (x, y)
  - If x is a no instance ($f(x) = 0$), for all $y \in \Sigma^*$ with $|y| \le q(|x|)$ such that $\mathcal{A}$ rejects the input (x, y)
- The string y is called a certificate.
- SAT, VertexCover, IndependentSet, SubsetSum, HamiltonianPath are all in **NP**.

SAT is in **NP**

- **Proof**. Define a Turing machine $\mathcal{A}$ that takes two strings x and y as inputs and does the following job:
  - Reject if x does not encode a CNF formula ϕ or y does not encode a valid Boolean assignment
  - Check if y makes ϕ evaluated to true. Accept if it does, and reject if it does not.
- $\mathcal{A}$ clearly runs in polynomial time.
- If x is a yes instance (i.e., ϕ is satisfable), let y be the encoding of a satisfying assigment, and $\mathcal{A}$ will accept (x, y).
- If x is a no instance (i.e., x is not a valid encoding of a CNF formula, or x encodes a CNF formula ϕ that is not satisfable), then no y can make $\mathcal{A}$ accept (x, y).

Can you prove the following problems are all in NP?

- SAT
- Vertex Cover
- Independent Set
- Subset Sum
- Hamiltonian Path

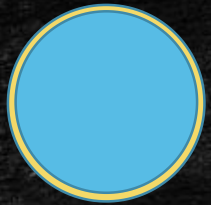
Theorem. P ⊆ NP

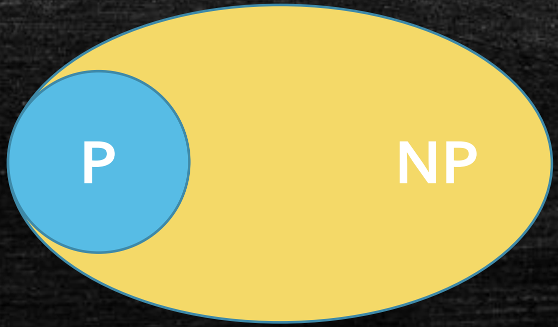
- Proof. If a decision problem $f\colon \Sigma^* \to \{0,1\}$ is in **P**, we will show it is in **NP**.
- By definition of **P**, there exists a polynomial time TM $\mathcal{A}$ such that $\mathcal{A}$ accepts x if and only if $f(x) = 1$.
- Let $\mathcal{A}'$ be a TM such that it outputs $\mathcal{A}(x)$ on input (x, y). That is, $\mathcal{A}'$ implements $\mathcal{A}$ and ignore y.
- If $f(x) = 1$, there exists y, say, $y = \emptyset$, such that $\mathcal{A}'$ accepts (x, y).
- If $f(x) = 0$, for all y, $\mathcal{A}'$ rejects (x, y).
- Thus, $f \in$ **NP**.

Central Open Problem: **P** vs. **NP**

- Central Open Problem: Does **P** equals **NP**?
- Most research believes no…
  - If **P** = **NP**, we do not need the certificate: we can just “guess” it correctly and efficiently… This doesn’t seem possible.
  - Given an exam question, do you believe solving the question is much harder than checking if someone’s solution to the question is correct? **P** = **NP** would suggest they are equally easy…

P = NP





P $\subsetneq$ NP

Class Activity

Which one or more of the following problems are in **NP**?

A. Decide if the polytope $P = \{\boldsymbol{x}: A\boldsymbol{x} \leq \boldsymbol{b}\}$ contains an integral point.

B. Decide if $G = (V, E)$ does not contain an independent set of size k.

C. Find the size of the maximum independent set in $G = (V, E)$.

D. Decide if $G = (V, E)$ contains a cycle of length at most k.

NP Problems

- We have seen many **NP** problems not known in **P**
  - SAT
  - VertexCover
  - IndependentSet
  - SubsetSum
  - HamiltonianPath
- Are some of these problems "more difficult" than the others?

3SAT

- A 3-CNF formula is a CNF formula where each clause contains at most three literals:
  - a 3-CNF formula: $(x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$
  - Not a 3-CNF formula: $(x_1 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4)$
- [3SAT] Given a 3-CNF formula, decide if there is a value assignment to the variables to make the formula true.
- Clearly, 3SAT is at most as hard as SAT, as it is a special case.
- We will prove 3SAT is also at least as hard as SAT.
  - so that SAT and 3SAT are "equally hard"

Proving 3SAT is “weakly harder than” SAT

- Idea: given a CNF formula ϕ, construct a 3-CNF formula ϕ' such that ϕ is a yes SAT instance if and only if ϕ' is a yes 3SAT instance.
- If converting ϕ to ϕ' can be done in polynomial time, being able to solve 3SAT in polynomial time implies being able to solve SAT in polynomial time.
  - That is, 3SAT is weakly harder than SAT.

Proving 3SAT is "weakly harder than" SAT

- We can "break" a long clause in ϕ to shorter clauses by introducing new variables:
- $(x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4) = (x_1 \vee x_2 \vee y_1) \wedge (\neg y_1 \vee \neg x_3 \vee \neg x_4)$
  - For example, if $x_2 = \text{true}$ is the one making LHS true, we can set $x_2 = \text{true}$, $y_1 = \text{false}$ to make RHS true.
  - If $x_1 = x_2 = \text{false}$ and $x_3 = x_4 = \text{true}$ so that LHS is false, at least one of the two clauses on RHS is false.
- We can "break" a even longer clause to clauses with at most three literals:
- $(x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4 \vee x_5 \vee x_6) = (x_1 \vee x_2 \vee y_1) \wedge (\neg y_1 \vee \neg x_3 \vee y_2) \wedge (\neg y_2 \vee \neg x_4 \vee y_3) \wedge (\neg y_3 \vee x_5 \vee x_6)$
  - For example, if $x_4 = \text{false}$ is the one making LHS true, we can set $y_3 = \text{false}$, $y_2 = \text{true}$, $y_1 = \text{true}$ to guarantee RHS is true.

Proving 3SAT is “weakly harder than” SAT

In general:

- $(\ell_1 \vee \cdots \vee \ell_k) = (\ell_1 \vee \ell_2 \vee y_1) \wedge (\neg y_1 \vee \ell_3 \vee y_2) \wedge \cdots \wedge (\neg y_{k-2} \vee \ell_{k-1} \vee \ell_k)$
- If a literal ℓ_i is true, we can make all RHS clauses true by properly setting y_i’s

$$(\ell_1 \vee \ell_2 \vee \underset{\substack{\uparrow\\ \text{true}}}{y_1}) \wedge \cdots \wedge (\underset{\substack{\uparrow\\ \text{false}}}{\neg y_{i-3}} \vee \ell_{i-1} \vee \underset{\substack{\uparrow\\ \text{true}}}{y_{i-2}}) \wedge (\underset{\substack{\uparrow\\ \text{false}}}{\neg y_{i-2}} \vee \overset{\substack{\text{true}\\ \downarrow}}{\ell_i} \vee \underset{\substack{\uparrow\\ \text{false}}}{y_{i-1}}) \wedge (\underset{\substack{\uparrow\\ \text{true}}}{\neg y_{i-1}} \vee \ell_{i+1} \vee \underset{\substack{\uparrow\\ \text{false}}}{y_i}) \wedge \cdots \wedge (\underset{\substack{\uparrow\\ \text{true}}}{\neg y_{k-2}} \vee \ell_{k-1} \vee \ell_k)$$

- If all of ℓ_i’s are false, we cannot make all RHS clauses true:
  - We have to set $y_1 = \text{true}$ to make the first clause true
  - After that, we have to make $y_2 = \text{true}$ to make the second clause true
  - ……
  - We have to make $y_{k-2} = \text{true}$; however, this will make the last clause false

Proving 3SAT is "weakly harder than" SAT

- We have described how to convert a CNF formula ϕ to a 3-CNF formula ϕ'.
- The conversion can clearly done in polynomial time.
- We have shown that ϕ is a yes SAT instance if and only if ϕ' is a yes 3SAT instance.
- If we have a polynomial time algorithm for 3SAT, we have a polynomial time algorithm for SAT:
  - Given input ϕ, compute ϕ'
  - Solve 3SAT instance ϕ' and obtain answer yes or no
  - Output the same answer for ϕ

Last Lecture Recaps

- We have defined decision problems, Turing Machines, the complexity class **P** and the complexity class **NP**.
- **P**: decision problems solvable in polynomial time
- **NP**: decision problems whose yes instances are verifiable in polynomial time if a hint/certificate is given
- **P** $\subseteq$ **NP**, and it is a central open problem if **P** = **NP**.
- It is obvious that 3SAT is no harder than SAT, but we have also proved SAT is also no harder than 3SAT.
- This lecture: explore the hardest problems in **NP**.

IndependentSet is "weakly harder" than 3SAT

- Same Idea before: Given a 3SAT instance ϕ, construct a IndependentSet instance $(G = (V, E), k)$ such that ϕ is a yes instance if and only if $(G = (V, E), k)$ is a yes instance.
- If construction can be done in polynomial time, this implies IndependentSet is weakly harder than 3SAT.

IndependentSet is “weakly harder” than 3SAT

Here is how we do it:

$$\phi = (x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

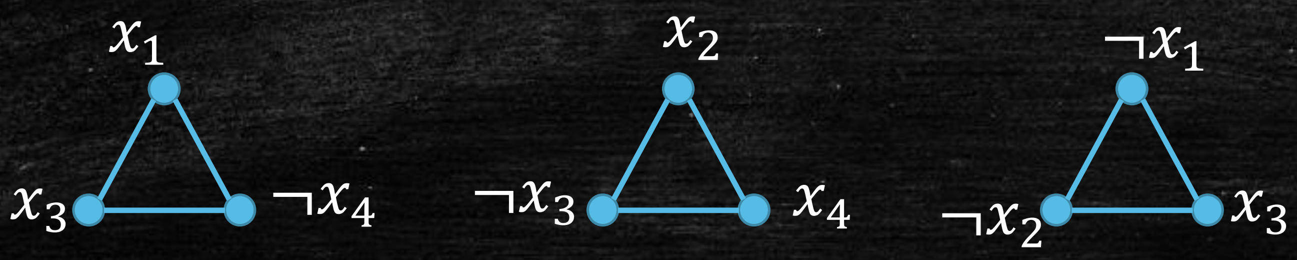
IndependentSet is “weakly harder” than 3SAT

Here is how we do it:

- For each clause, construct a triangle where three vertices represent three literals.

$$\phi = (x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

G:

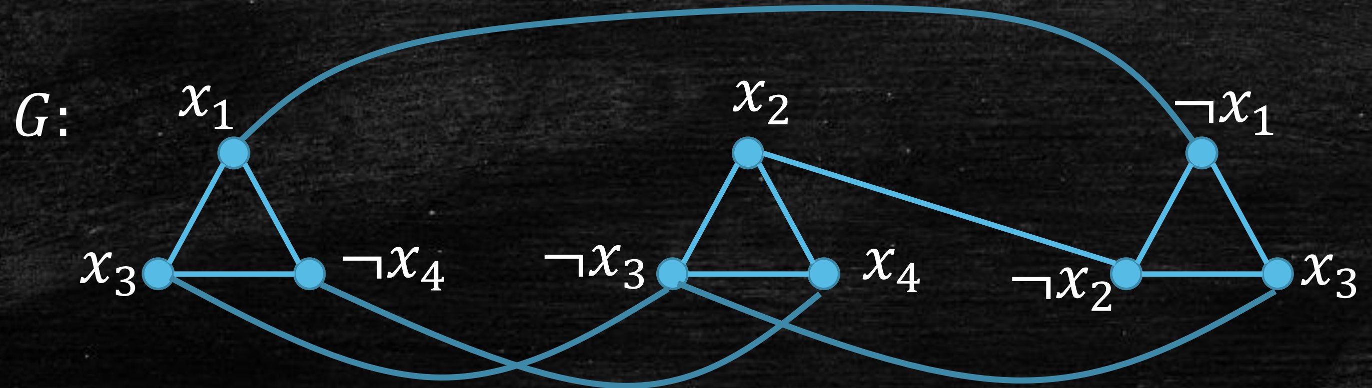


IndependentSet is “weakly harder” than 3SAT

Here is how we do it:

- For each clause, construct a triangle where three vertices represent three literals.
- Connect two vertices if one represents the negation of the other.

$$\phi = (x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$



IndependentSet is “weakly harder” than 3SAT

Here is how we do it:

- For each clause, construct a triangle where three vertices represent three literals.
- Connect two vertices if one represents the negation of the other.
- Set k in IndependentSet instance to the number of clauses

$$\phi = (x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

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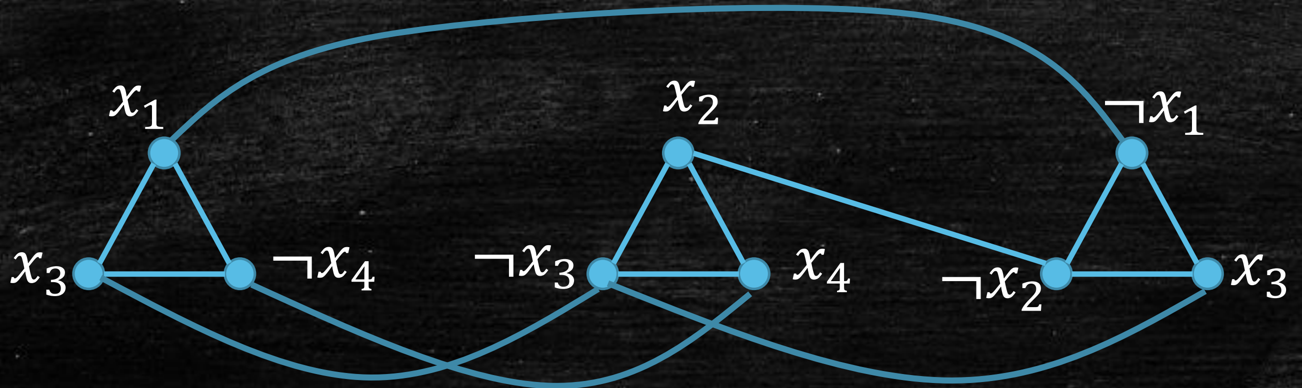

$k = 3$

IndependentSet is "weakly harder" than 3SAT

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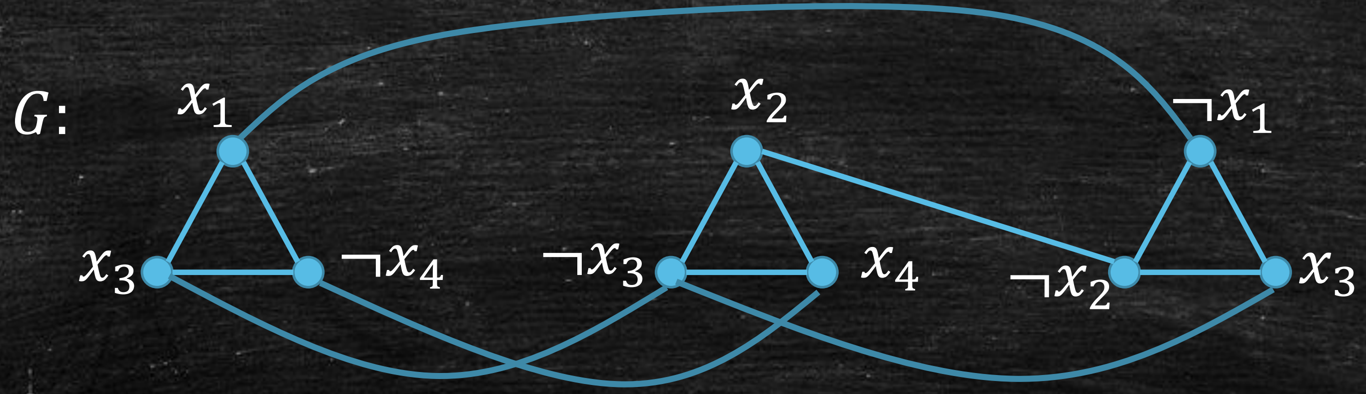
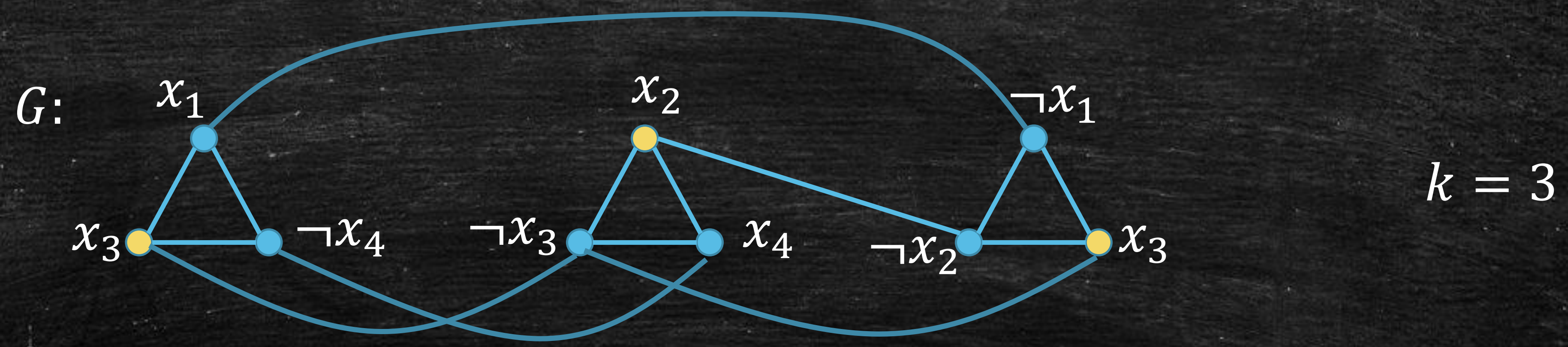


$k = 3$

- If ϕ is a yes instance, each clause must have a literal with value true.
- For each triangle in G, pick exactly one vertex representing a true literal in S.
- S is an independent set and $|S| = k$. So (G, k) is a yes instance.

IndependentSet is "weakly harder" than 3SAT

$$\phi = (x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$



- Example: $x_1 = x_2 = x_3 = x_4 =$ true makes $\phi =$ true
- We choose exactly one true literal in each clause, for example,
  - $(x_1 \vee x_3 \vee \neg x_4)$
  - $(x_2 \vee \neg x_3 \vee x_4)$
  - $(\neg x_1 \vee \neg x_2 \vee x_3)$

IndependentSet is “weakly harder” than 3SAT

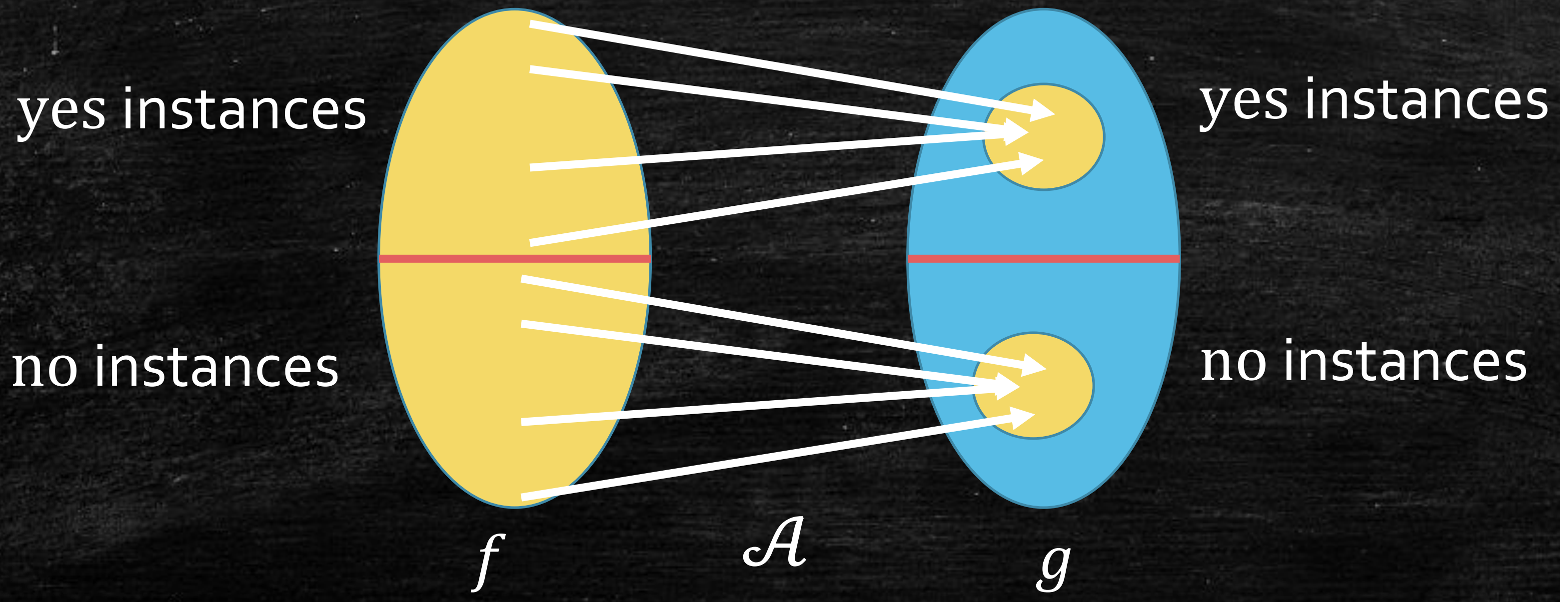
- If ϕ is a no instance, for contradiction, assume (G, k) is a yes instance. Let S with $|S| = k$ be the independent set.
- S must contain exactly one vertex in each triangle.
  - because any two vertices in a triangle is connected
- Assign true to the literals representing the chosen vertices.
  - We will not assign both true and false to a same literal, as x_i and $\neg x_i$ is connected.
- For variables not yet assigned a value, assign values to them arbitrarily.
- The resultant assignment makes ϕ true (as each clause has a true literal), contradicting to that ϕ is a no instance!

Reduction

- A decision problem f Karp reduce to (or simply, reduce to) a decision problem g if there is a polynomial time TM $\mathcal{A}$ such that
  - $\mathcal{A}$ outputs a yes instance of g if a yes instance of f is input
  - $\mathcal{A}$ outputs a no instance of g if a no instance of f is input
- Denoted as $f \leq_k g$
  - Very intuitive: the difficulty level of f is weakly less than that of g
- We have just proved:
  - $\text{SAT} \leq_k \text{3SAT}$
  - $\text{3SAT} \leq_k \text{IndependentSet}$

Reduction

- In the reduction, $f \leq_k g$, the TM $\mathcal{A}$ defines a mapping.
- The mapping needs not to be one-to-one.
- The mapping needs not to be onto.



Reduction $f \leq_k g$

- Transform f to g
- Show f is essentially a special case of g

Transitivity of Reduction

- **Theorem.** If $f \leq_k g$ and $g \leq_k h$, then $f \leq_k h$.
- If g is (weakly) harder than f and h is (weakly) harder than g, then h is (weakly) harder than f.
- Proof. Let $\mathcal{A}_1$ be the polynomial time TM doing $f \leq_k g$ and $\mathcal{A}_2$ be the polynomial time TM doing $g \leq_k h$.
- Let $\mathcal{A} = \mathcal{A}_1 \circ \mathcal{A}_1$ be the TM that first executes $\mathcal{A}_1$ and then executes $\mathcal{A}_2$ (using the output of $\mathcal{A}_1$ as input of $\mathcal{A}_2$).
- Then $\mathcal{A}$ does the job of $f \leq_k h$.
- $\mathcal{A}$ runs in polynomial time: the time complexity of $\mathcal{A}$ is the sum of the time complexities of $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\mathcal{A}_1$ and $\mathcal{A}_2$ are polynomial time TMs.

More Results in Reduction

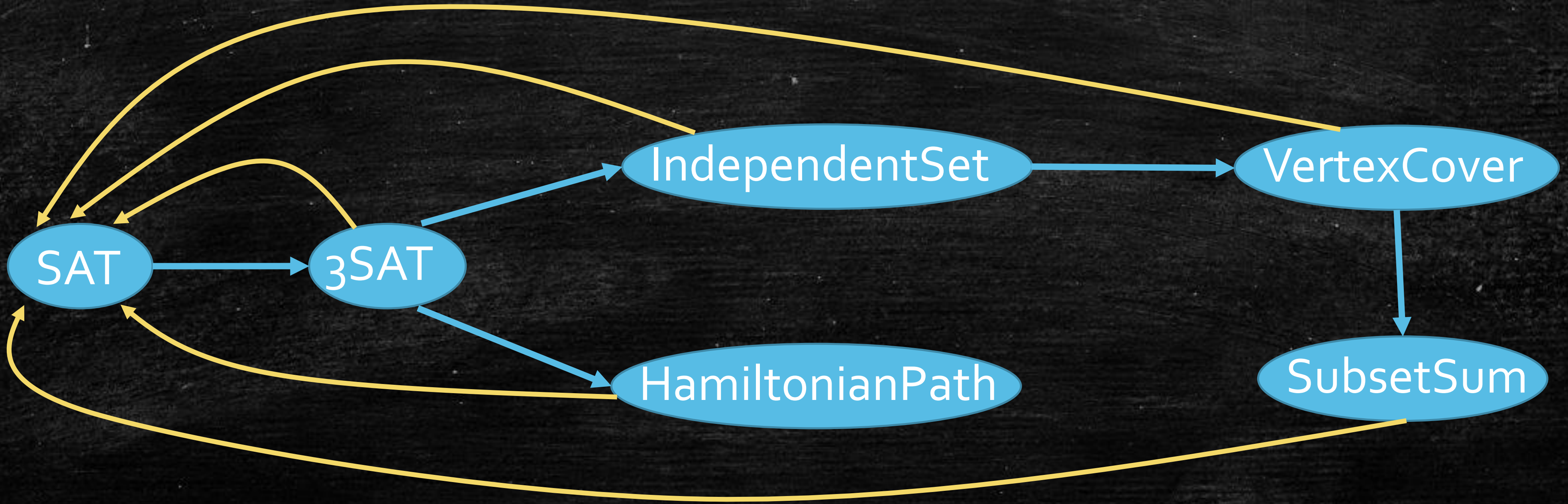
- We have proved S is an independent set of $G = (V, E)$ if and only if $V \setminus S$ is a vertex cover.
- Thus, $\text{IndependentSet} \leq_k \text{VertexCover}$
  - The reduction $\mathcal{A}$ simply maps $(G = (V, E), k)$ to $(G = (V, E), |V| - k)$
- It is also true that:
  - $\text{VertexCover} \leq_k \text{SubsetSum}$
  - $\text{3SAT} \leq_k \text{HamitonianPath}$

The Hardest Problem in **NP**

- We have built difficulty relations between many problems in **NP**.
- Does there exist a problem in **NP** that is the hardest?
- **Definition.** A decision problem f is NP-hard if $g \leq_k f$ for any problem $g \in$ **NP**.
- **Definition.** A decision problem f is NP-complete if $f \in$ **NP** and $g \leq_k f$ for any problem $g \in$ **NP**.
- **[Cook-Levin Theorem] SAT is NP-complete.**

More NP-Complete Problems

- Cook-Levin Theorem implies the yellow arrows, since all the problems below are in **NP**.
- Each problem is NP-complete
  - By transitivity: any **NP** problem reduce to SAT, and SAT reduce to each of these problems.
- These problems are "equally hard", and are the hardest problems in **NP**.



Intuition behind Cook-Levin Theorem

- We have seen SAT is in **NP**.
- Consider an arbitrary **NP** problem f. We will show $f \leq_k \mathrm{SAT}$.
- For a yes instance x, there exist a polynomial time TM $\mathcal{A}$ and a polynomial length certificate y such that $\mathcal{A}$ accepts (x, y).
- Consider a computation tableau that records the tape at every step of $\mathcal{A}$'s execution.

	x_0	x_1	x_2	…	x_n	y_0	y_1	y_2	…	y_m
Step 0	x_0	x_1	x_2	…	x_n	y_0	y_1	y_2	…	y_m
Step 1	1	1	0	0	0	1	1	1	1	0
Step 2	1	1	1	0	0	1	1	1	1	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Final Step	0	1	1	0	0	0	1	1	0	0

Intuition behind Cook-Levin Theorem

	x					y				
Step 0	x_0	x_1	x_2	…	x_n	y_0	y_1	y_2	…	y_m
Step 1	1	1	0	0	0	1	1	1	1	0
Step 2	1	1	1	0	0	1	1	1	1	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Final Step	0	1	1	0	0	0	1	1	0	0

- For each y_i and each cell in the tape from Step 1 to the final step, construct a Boolean variable for the SAT instance.
- We can use clauses to ensure the tableau gives a valid TM computation.
- E.g., we can use two clauses $(x \vee \neg y) \wedge (\neg x \vee y)$ to enforce $x = y$.

Intuition behind Cook-Levin Theorem

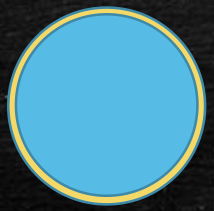
- High-level Intuition: a CNF formula is sufficient to simulate the execution of a Turing Machine!
- If x for the **NP** problem f is a yes instance, the CNF formula constructed can be satisfied:
  - Assign $y_i = \mathrm{true}$ if and only if the i-th bit of y is 1.
  - Assign each other variable the value corresponding to the value of the cell in the computation tableau.
- If x for the **NP** problem f is a no instance, the CNF formula constructed <u>cannot</u> be satisfied:
  - Otherwise, we can find a certificate $y = y_1 y_2 \cdots y_m$ that fools the TM to accept (x, y).

Solving a NP-complete problem implies **P** = **NP**

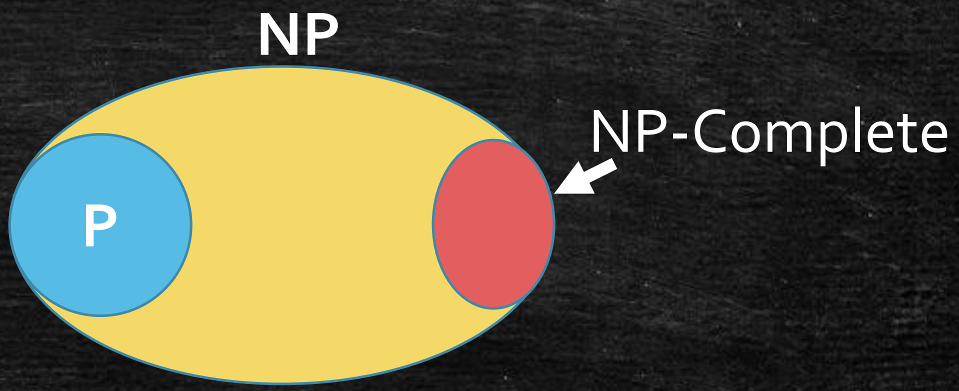
- **Theorem.** If f is NP-complete and $f \in$ **P**, then **P** = **NP**.
- Proof. Suppose there is a polynomial time TM $\mathcal{A}$ that decides f. We will show $g \in$ **P** for any $g \in$ **NP**.
- Since f is NP-hard, $g \leq_k f$, and let $\mathcal{A}'$ be the polynomial time TM that does the reduction.
- Then $\mathcal{A} \circ \mathcal{A}'$ is the polynomial time TM that decides g.
- Thus, $g \in$ **P**.

- If you solve any of SAT, 3SAT, IndependentSet, VertexCover, SubsetSum, HamiltionianPath, you will be the greatest person in the 21st century!

P vs NP



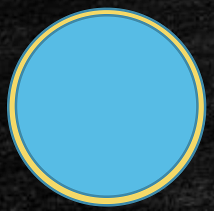
P = NP



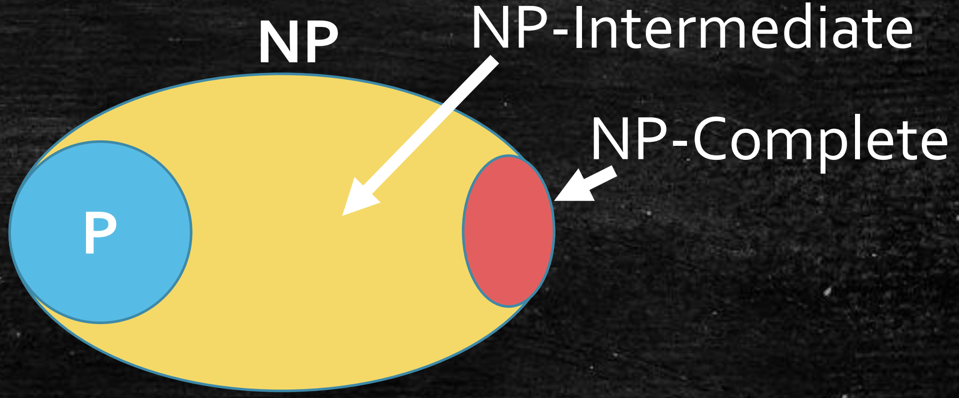
$P \subsetneq NP$

NP-Intermediate

- **[Ladner's Theorem]** If $P \neq NP$, then there exist decision problems that are neither in P nor NP-complete.
- Such problems are called NP-intermediate.
- However, we do not know any "natural" NP-intermediate problems.



$P = NP$



$P \subsetneq NP$

NP-Hard vs NP-Complete

Difference between NP-hardness and NP-completeness:

- For decision problems: NP-complete = NP-hard + (in **NP**)
  - There are NP-hard problems that are not in **NP**; these problems are even harder than NP-complete problems.
- NP-hardness can describe optimization problems:
  - Maximum Independent Set is NP-hard
  - Minimum Vertex Cover is NP-hard
  - Max-3SAT is NP-hard
  - Finding a longest simple path is NP-hard
  - Etc.

VertexCover $\leq_k$ SubsetSum

- We first consider the following "vector version" of SubsetSum.
- [VectorSubsetSum] Given a collection of integer vectors $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n : \mathbf{a}_i \in \mathbb{Z}^m\}$ and a vector $\mathbf{k} \in \mathbb{Z}^m$, decide if there exists $T \subseteq S$ with $\sum_{\mathbf{a}_i \in T} \mathbf{a}_i = \mathbf{k}$.
- We will show that
  1. VertexCover $\leq_k$ VectorSubsetSum
  2. VectorSubsetSum $\leq_k$ SubsetSum

VertexCover $\leq_k$ VectorSubsetSum

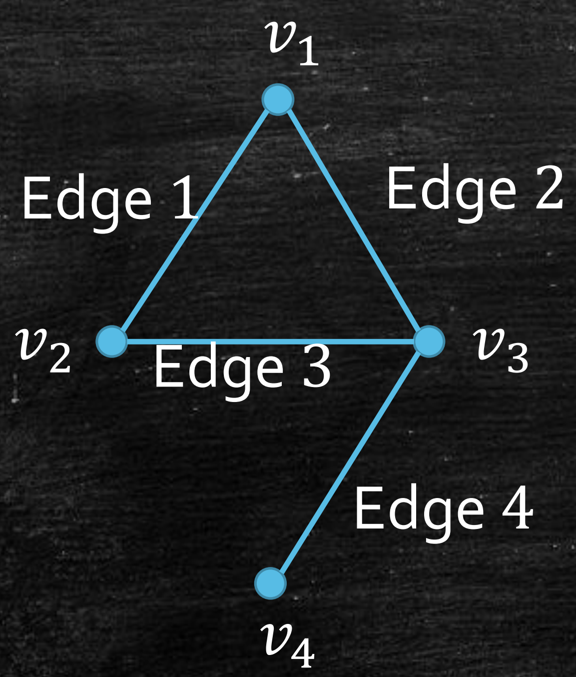
- Given a VertexCover instance $(G = (V, E), k)$, we will construct a VectorSubsetSum instance $(S, \mathbf{k})$.
- First, we label the edges with $1, 2, \dots, |E|$ (in arbitrary order).
- For each $v_i \in V$, construct a $(|E| + 1)$-dimensional vector $\mathbf{a}_i \in S$ such that $\mathbf{a}_i[0] = 1$ and for each $j = 1, \dots, |E|$:

$$\mathbf{a}_i[j] = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of edge } j \\ 0 & \text{otherwise} \end{cases}$$

- For each edge j, construct $\mathbf{b}_j \in S$ where $\mathbf{b}_j[j] = 1$ is the only non-zero entry.
- Let $\mathbf{k} = (k, 2,2, \dots, 2)$.

Example

$G =$



$k = 3$

a VertexCover instance

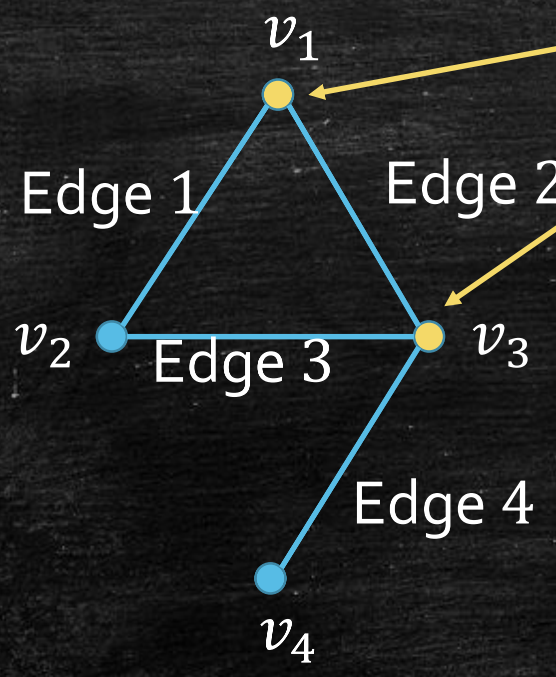
$$
\begin{aligned}
\mathbf{a}_1 &= (1, & 1, & 1, & 0, & 0) \\
\mathbf{a}_2 &= (1, & 1, & 0, & 1, & 0) \\
\mathbf{a}_3 &= (1, & 0, & 1, & 1, & 1) \\
\mathbf{a}_4 &= (1, & 0, & 0, & 0, & 1) \\
\mathbf{b}_1 &= (0, & 1, & 0, & 0, & 0) \\
\mathbf{b}_2 &= (0, & 0, & 1, & 0, & 0) \\
\mathbf{b}_3 &= (0, & 0, & 0, & 1, & 0) \\
\mathbf{b}_4 &= (0, & 0, & 0, & 0, & 1) \\
\mathbf{k} &= (3, & 2, & 2, & 2, & 2)
\end{aligned}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

Picking $\mathbf{a}_i \in T$ represents picking v_i in the vertex cover.

$G =$



$$
\begin{aligned}
\mathbf{a}_1 &= (1, \quad 1, \quad 1, \quad 0, \quad 0) \\
\mathbf{a}_2 &= (1, \quad 1, \quad 0, \quad 1, \quad 0) \\
\mathbf{a}_3 &= (1, \quad 0, \quad 1, \quad 1, \quad 1) \\
\mathbf{a}_4 &= (1, \quad 0, \quad 0, \quad 0, \quad 1) \\
\mathbf{b}_1 &= (0, \quad 1, \quad 0, \quad 0, \quad 0) \\
\mathbf{b}_2 &= (0, \quad 0, \quad 1, \quad 0, \quad 0) \\
\mathbf{b}_3 &= (0, \quad 0, \quad 0, \quad 1, \quad 0) \\
\mathbf{b}_4 &= (0, \quad 0, \quad 0, \quad 0, \quad 1) \\
\mathbf{k} &= (3, \quad 2, \quad 2, \quad 2, \quad 2)
\end{aligned}
$$

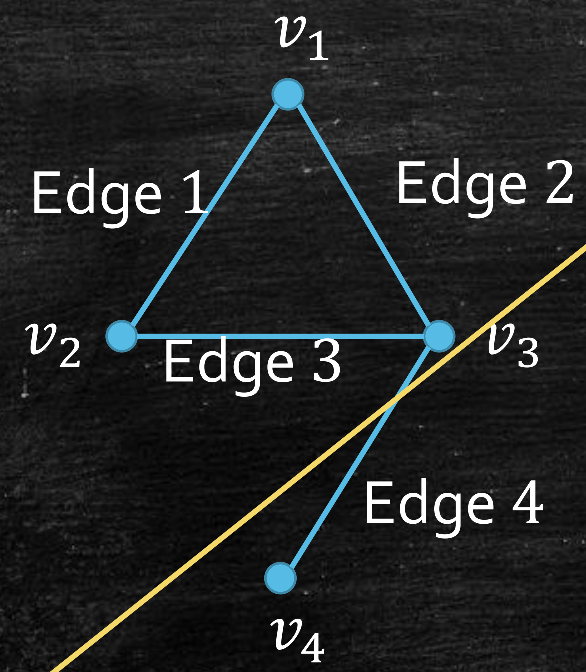
$k = 3$

a VertexCover instance

a VectorSubsetSum instance

Ideas Behind the Reduction

The 0-th entry of $\mathbf{k}$ is set to k, enforcing exactly k vertices must be picked.

$G =$ 

$k = 3$

$$
\begin{aligned}
\mathbf{a}_1 &= (1,\ 1,\ 1,\ 0,\ 0)\\
\mathbf{a}_2 &= (1,\ 1,\ 0,\ 1,\ 0)\\
\mathbf{a}_3 &= (1,\ 0,\ 1,\ 1,\ 1)\\
\mathbf{a}_4 &= (1,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{b}_1 &= (0,\ 1,\ 0,\ 0,\ 0)\\
\mathbf{b}_2 &= (0,\ 0,\ 1,\ 0,\ 0)\\
\mathbf{b}_3 &= (0,\ 0,\ 0,\ 1,\ 0)\\
\mathbf{b}_4 &= (0,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{k} &= (3,\ 2,\ 2,\ 2,\ 2)
\end{aligned}
$$

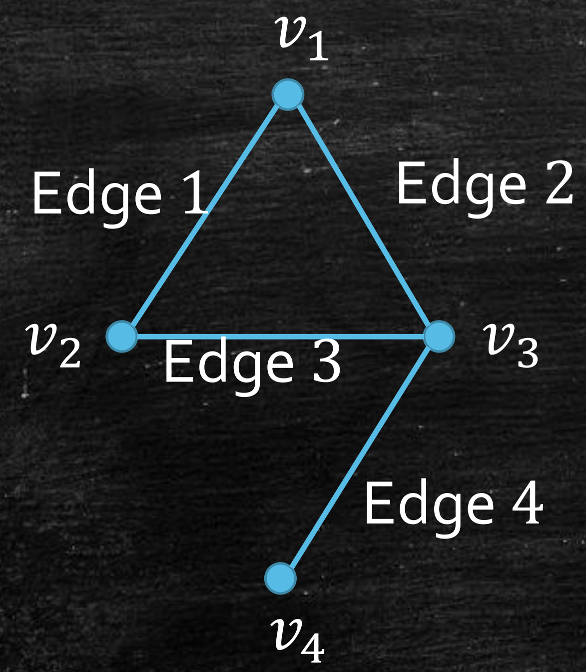
a VertexCover instance

a VectorSubsetSum instance

Ideas Behind the Reduction

The j-th entry of $\mathbf{k}$ is set to 2 enforcing edge j must be covered

$G =$



$k = 3$

a VertexCover instance

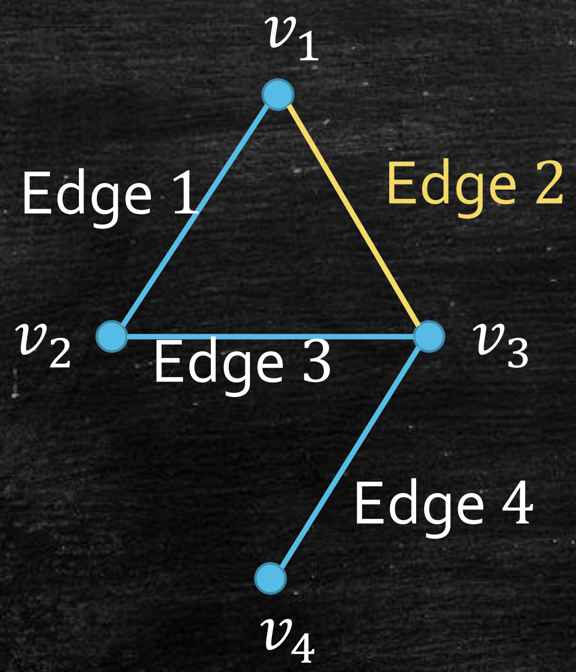
$$
\begin{aligned}
\mathbf{a}_1 &= (1,\ 1,\ 1,\ 0,\ 0)\\
\mathbf{a}_2 &= (1,\ 1,\ 0,\ 1,\ 0)\\
\mathbf{a}_3 &= (1,\ 0,\ 1,\ 1,\ 1)\\
\mathbf{a}_4 &= (1,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{b}_1 &= (0,\ 1,\ 0,\ 0,\ 0)\\
\mathbf{b}_2 &= (0,\ 0,\ 1,\ 0,\ 0)\\
\mathbf{b}_3 &= (0,\ 0,\ 0,\ 1,\ 0)\\
\mathbf{b}_4 &= (0,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{k} &= (3,\ 2,\ 2,\ 2,\ 2)
\end{aligned}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

Consider "Edge 2" (v_1, v_3) for example…

$G =$



$k = 3$

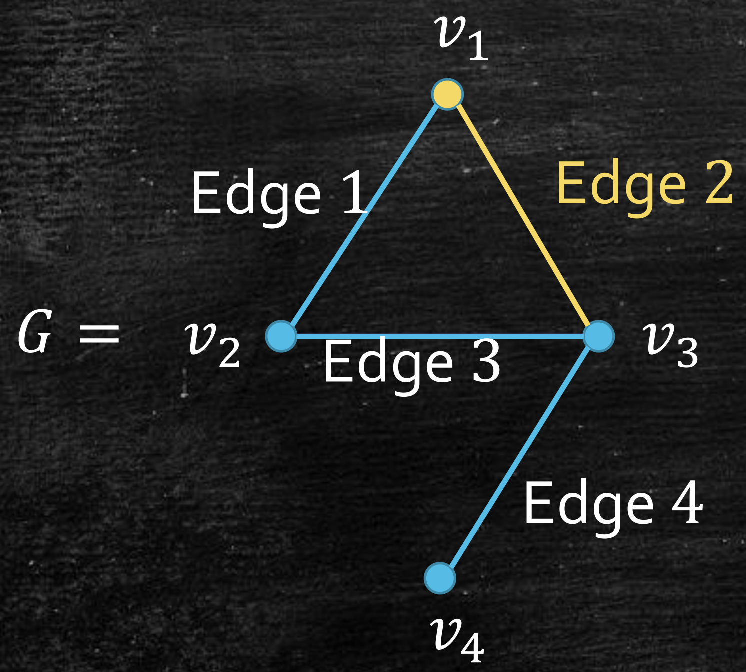
a VertexCover instance

$$
\begin{aligned}
\mathbf{a}_1 &= (1,\ 1,\ 1,\ 0,\ 0)\\
\mathbf{a}_2 &= (1,\ 1,\ 0,\ 1,\ 0)\\
\mathbf{a}_3 &= (1,\ 0,\ 1,\ 1,\ 1)\\
\mathbf{a}_4 &= (1,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{b}_1 &= (0,\ 1,\ 0,\ 0,\ 0)\\
\mathbf{b}_2 &= (0,\ 0,\ 1,\ 0,\ 0)\\
\mathbf{b}_3 &= (0,\ 0,\ 0,\ 1,\ 0)\\
\mathbf{b}_4 &= (0,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{k} &= (3,\ 2,\ 2,\ 2,\ 2)
\end{aligned}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

If $\mathbf{a}_1$ is chosen, we can choose $\mathbf{b}_2$; we are fine!



$k = 3$

a VertexCover instance

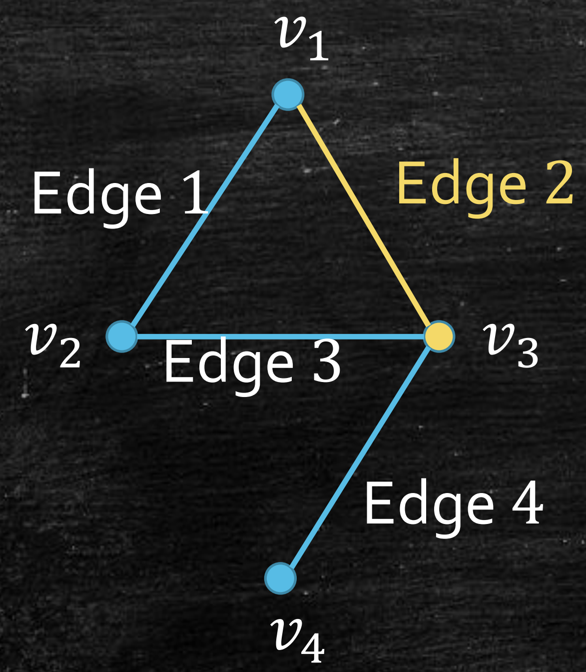
$$
\begin{array}{lllllll}
\mathbf{a}_1 & = (1, & 1, & 1, & 0, & 0) \\
\mathbf{a}_2 & = (1, & 1, & 0, & 1, & 0) \\
\mathbf{a}_3 & = (1, & 0, & 1, & 1, & 1) \\
\mathbf{a}_4 & = (1, & 0, & 0, & 0, & 1) \\
\mathbf{b}_1 & = (0, & 1, & 0, & 0, & 0) \\
\mathbf{b}_2 & = (0, & 0, & 1, & 0, & 0) \\
\mathbf{b}_3 & = (0, & 0, & 0, & 1, & 0) \\
\mathbf{b}_4 & = (0, & 0, & 0, & 0, & 1) \\
\mathbf{k} & = (3, & 2, & 2, & 2, & 2)
\end{array}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

If $\mathbf{a}_3$ is chosen, we can choose $\mathbf{b}_2$; we are fine!

$G =$



$k = 3$

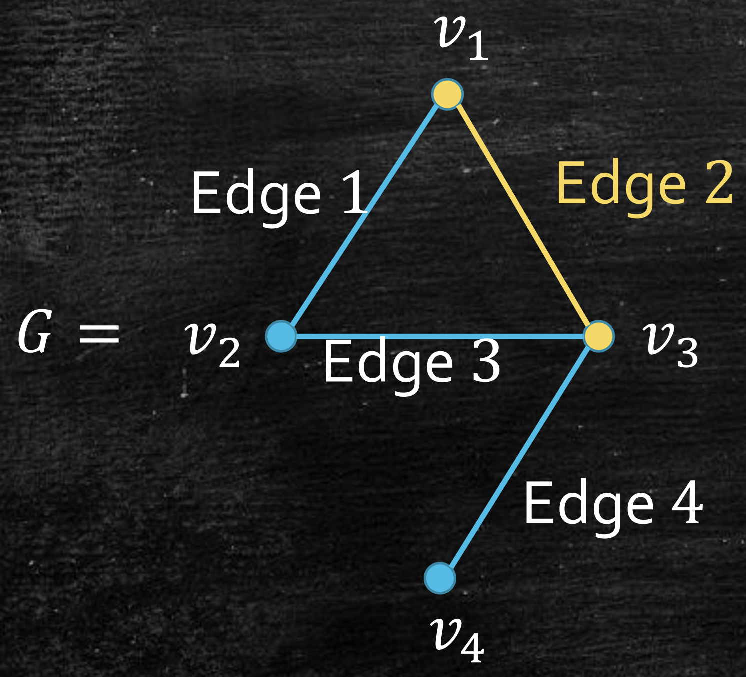
a VertexCover instance

$$
\begin{aligned}
\mathbf{a}_1 &= (1,\ 1,\ 1,\ 0,\ 0)\\
\mathbf{a}_2 &= (1,\ 1,\ 0,\ 1,\ 0)\\
\mathbf{a}_3 &= (1,\ 0,\ 1,\ 1,\ 1)\\
\mathbf{a}_4 &= (1,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{b}_1 &= (0,\ 1,\ 0,\ 0,\ 0)\\
\mathbf{b}_2 &= (0,\ 0,\ 1,\ 0,\ 0)\\
\mathbf{b}_3 &= (0,\ 0,\ 0,\ 1,\ 0)\\
\mathbf{b}_4 &= (0,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{k} &= (3,\ 2,\ 2,\ 2,\ 2)
\end{aligned}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

If $\mathbf{a}_1$ and $\mathbf{a}_2$ are both chosen, we do not choose $\mathbf{b}_2$; we are fine!



$$\begin{aligned}
\mathbf{a}_1 &= (1,\ 1,\ 1,\ 0,\ 0)\\
\mathbf{a}_2 &= (1,\ 1,\ 0,\ 1,\ 0)\\
\mathbf{a}_3 &= (1,\ 0,\ 1,\ 1,\ 1)\\
\mathbf{a}_4 &= (1,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{b}_1 &= (0,\ 1,\ 0,\ 0,\ 0)\\
\mathbf{b}_2 &= (0,\ 0,\ 1,\ 0,\ 0)\\
\mathbf{b}_3 &= (0,\ 0,\ 0,\ 1,\ 0)\\
\mathbf{b}_4 &= (0,\ 0,\ 0,\ 0,\ 1)\\
\mathbf{k} &= (3,\ 2,\ 2,\ 2,\ 2)
\end{aligned}$$

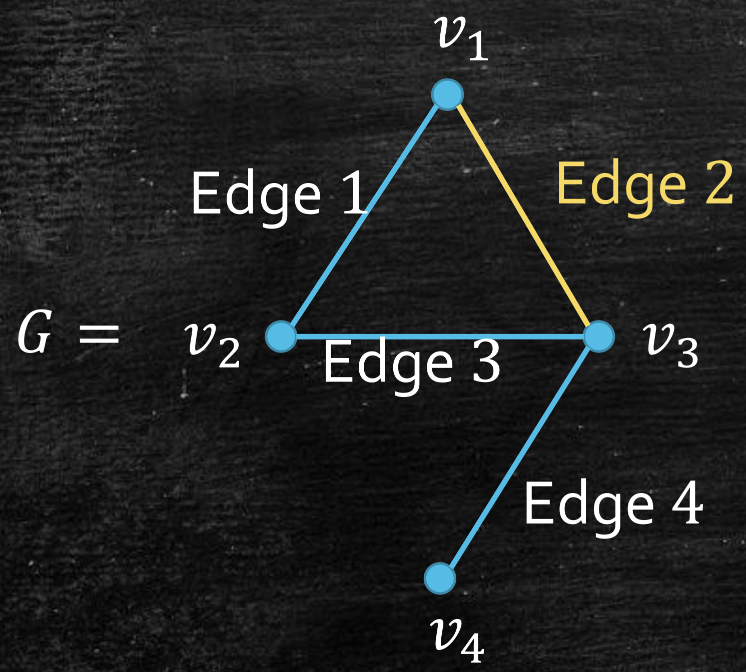
$k = 3$

a VertexCover instance

a VectorSubsetSum instance

Ideas Behind the Reduction

If neither of $\mathbf{a}_1$ and $\mathbf{a}_3$ is chosen, we are <u>not</u> fine: choosing $\mathbf{b}_2$ will not make it.



$k = 3$

a VertexCover instance

$$
\begin{array}{lcl}
\mathbf{a}_1 & = & (1,\ 1,\ \cancel{1},\ 0,\ 0) \\
\mathbf{a}_2 & = & (1,\ 1,\ 0,\ 1,\ 0) \\
\mathbf{a}_3 & = & (1,\ 0,\ \cancel{1},\ 1,\ 1) \\
\mathbf{a}_4 & = & (1,\ 0,\ 0,\ 0,\ 1) \\
\mathbf{b}_1 & = & (0,\ 1,\ 0,\ 0,\ 0) \\
\mathbf{b}_2 & = & (0,\ 0,\ 1,\ 0,\ 0) \\
\mathbf{b}_3 & = & (0,\ 0,\ 0,\ 1,\ 0) \\
\mathbf{b}_4 & = & (0,\ 0,\ 0,\ 0,\ 1) \\
\mathbf{k} & = & (3,\ 2,\ 2,\ 2,\ 2)
\end{array}
$$

a VectorSubsetSum instance

Ideas Behind the Reduction

- Picking $\mathbf{a}_i \in T$ represents picking v_i in the vertex cover.
- The 0-th entry of $\mathbf{k}$ is set to k, enforcing exactly k vertices must be picked.
- The j-th entry of $\mathbf{k}$ is set to 2 enforcing edge j must be covered:
  - Say, edge j is (v_{i_1}, v_{i_2})
  - If $\mathbf{a}_{i_1}, \mathbf{a}_{i_2} \in T$, we are fine, as the j-th entries already add up to 2.
  - If one of $\mathbf{a}_{i_1}, \mathbf{a}_{i_2}$ is chosen in T, we are also fine, as we can include $\mathbf{b}_j \in T$.
  - If $\mathbf{a}_{i_1}, \mathbf{a}_{i_2} \notin T$, we are not fine: the j-th entries add up to at most 1 even if we include $\mathbf{b}_j \in T$.
- We are done! $\mathrm{VertexCover} \leq_k \mathrm{VectorSubsetSum}$

$\mathrm{VectorSubsetSum} \leq_k \mathrm{SubsetSum}$

- We can convert a vector $\mathbf{a} = (\mathbf{a}[0], \dots, \mathbf{a}[m])$ to a large number.
- For example, convert $\mathbf{a} = (1, 4, 5, 3)$ to number 1453
  - $1453 = \mathbf{a}[0] \times 1000 + \mathbf{a}[1] \times 100 + \mathbf{a}[2] \times 10 + \mathbf{a}[3] \times 1$
- We are using decimal representation in the above example…
- To avoid carry, use N-ary representation instead (for sufficiently large N)?
- Additions with vectors are now equivalent to additions with numbers, since we do not have carry issue.
- $\mathrm{VectorSubsetSum} \leq_k \mathrm{SubsetSum}$

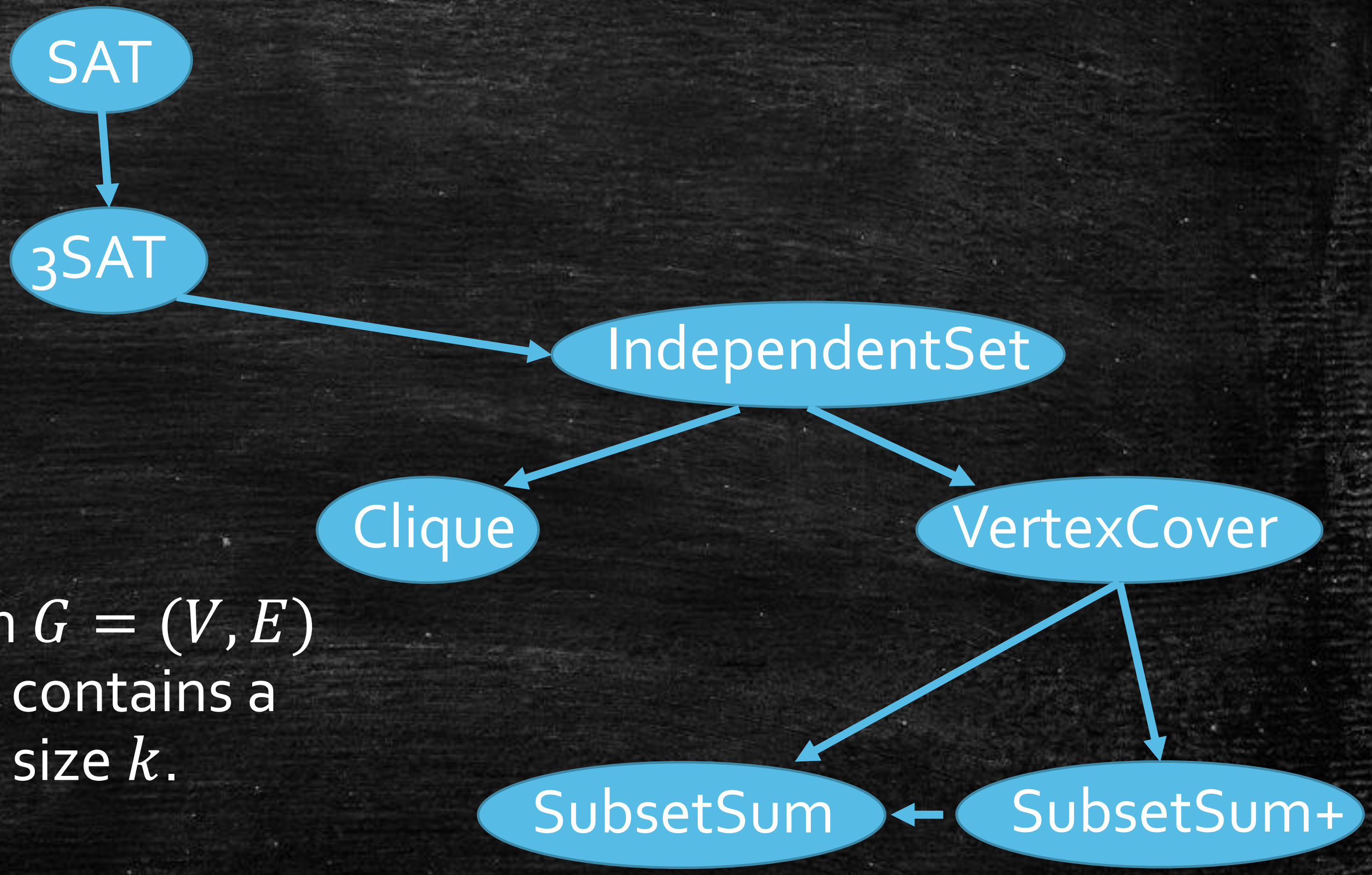
SubsetSum is NP-complete

- We have seen SubsetSum is in NP.
- We have proved
  1. $\text{VertexCover} \leq_k \text{VectorSubsetSum}$
  2. $\text{VectorSubsetSum} \leq_k \text{SubsetSum}$

SubsetSum+

- [SubsetSum+] Given a collection of positive integers $S = \{a_1, \ldots, a_n\}$ and $k \in \mathbb{Z}^+$, decide if there is a sub-collection $T \subseteq S$ such that $\sum_{a_i \in T} a_i = k$.
- SubsetSum+ is NP-complete
  - The same proof for SubsetSum can prove this!
- Test your "sense of direction": Which one holds trivially?
  A. SubsetSum $\leq_k$ SubsetSum+
  B. SubsetSum+ $\leq_k$ SubsetSum

Web of NP-complete Problems



[Clique] Given an undirected graph $G = (V, E)$ and an integer $k \in \mathbb{Z}^+$, decide if G contains a clique (a complete subgraph) with size k.

Partition Problem

- [Partition] Given a collection of integers S, decide if there is a partition of S to A and B such that $\sum_{a\in A} a = \sum_{b\in B} b$.
- [Partition+] Given a collection of positive integers S, decide if there is a partition of S to A and B such that $\sum_{a\in A} a = \sum_{b\in B} b$.
- Exercise: Prove that both Partition and Partition+ are NP-complete.

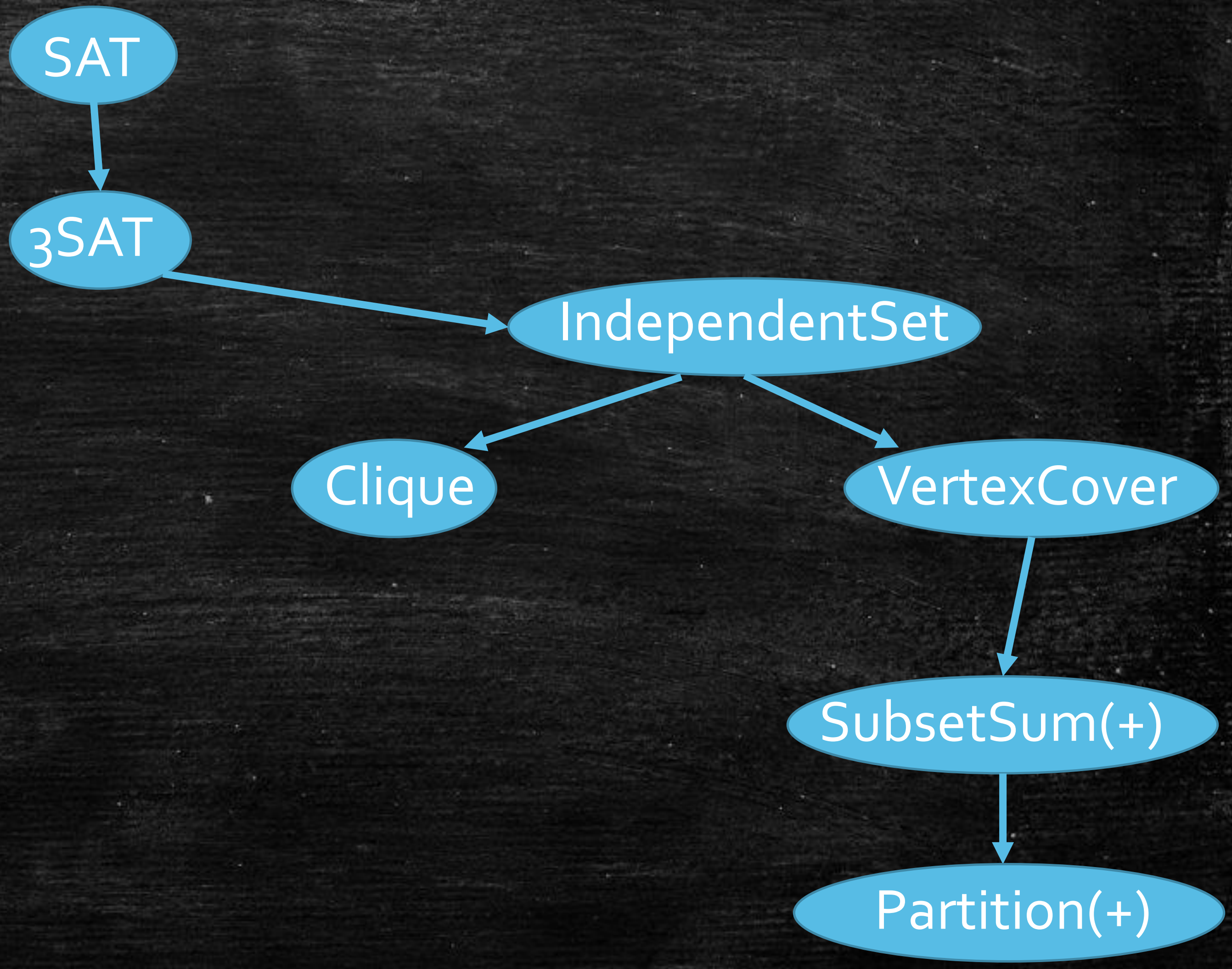
SubsetSum+ $\leq_k$ Partition+

- Given a SubsetSum+ instance $(\{a_1, \dots, a_n\}, k)$, construct a Partition+ instance as follows.
- If $k = \frac{1}{2}\sum_{i=1}^{n} a_i$, then the Partition+ instance is just $\{a_1, \dots, a_n\}$.
- If $k > \frac{1}{2}\sum_{i=1}^{n} a_i$, then the Partition+ instance is $\{a_1, \dots, a_n, b\}$, where $b = 2k - \sum_{i=1}^{n} a_i$.
- If $k < \frac{1}{2}\sum_{i=1}^{n} a_i$, then the Partition+ instance is $\{a_1, \dots, a_n, b\}$, where $b = -2k + \sum_{i=1}^{n} a_i$.
- Can you complete the remaining details?

Partition+ $\leq_k$ Partition

- Do you see the reduction?

Web of NP-complete Problems



This Lecture

- Learn what are P and NP
- Cook-Levin Theorem and NP-complete problems
- Reduction

Take Home Messages

- SAT (3SAT), VertexCover, IndependentSet, SubsetSum, HamiltonianPath are the hardest problems in **NP**, and they are NP-complete.
- Reduction is a effective tool to show one problem is “weakly harder” than another.