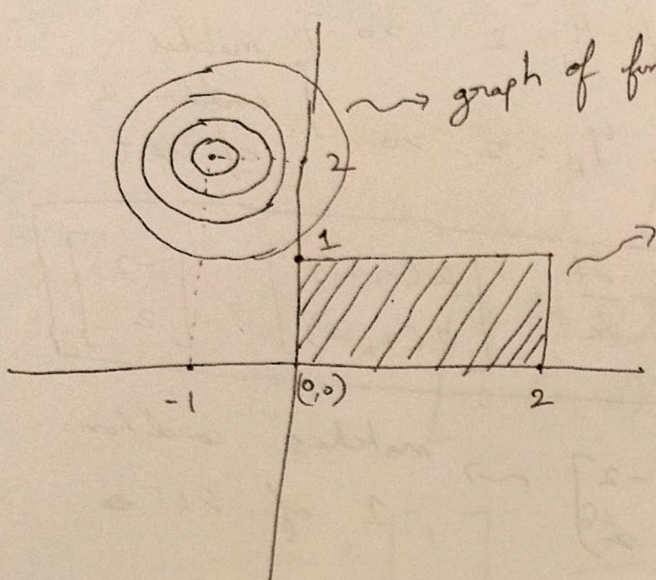
MAE-598 — HW 3

① $\min f(x) = (x_1+1)^2 + (x_2-2)^2$

subject to. $g_1 = x_1 - 2 \le 0$, $g_2 = x_2 - 1 \le 0$, $g_3 = -x_1 \le 0$, $g_4 = -x_2 \le 0$.



graph of function where minimum value is $(-1, 2)$

due to constraints our solution must be on or within the rectangular region.

$$L(x,\mu) = (x_1+1)^2 + (x_2-2)^2 + \mu_1(-x_1) + \mu_2(-x_2) + \mu_3(x_1-2) + \mu_4(x_2-1)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) - \mu_1 + \mu_3 \\ 2(x_2-2) - \mu_2 + \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4 cases of active vs inactive

$\mu_1 > 0 \iff -x_1 = 0$		$\mu_1 = 0$	$-x_1 < 0$
$\mu_2 > 0 \iff -x_2 = 0$		$\mu_2 = 0$	$-x_2 < 0$
$\mu_3 > 0 \iff x_1 - 2 = 0$		$\mu_3 = 0$	$x_1 - 2 < 0$
$\mu_4 > 0 \iff x_2 - 1 = 0$.		$\mu_4 = 0$	$x_2 - 1 < 0$.

Case 1: $x_1 = 0 \ (\mu_1 > 0)$, $\mu_2 = 0$

$x_2 = 1 \ (\mu_4 > 0)$, $\mu_3 = 0$ $\Rightarrow$ Point $[0, 1]$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2 - \mu_1 \\ -2 + \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left\{ \begin{array}{ll} \mu_1 = 2 & > 0 \\ \mu_4 = 2 & > 0 \end{array} \right\}$$ Matches condition 2 of KKT

$$\boxed{\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}}$$

$$-\frac{\partial f}{\partial x} = \frac{\partial g_4}{\partial x}\mu_4 + \frac{\partial g_1}{\partial x}\mu_1$$

$$\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} 2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$ $\rightsquigarrow$ matches condition 1 of KKT

Case 2: $x_1 = 0 \ (\mu_1 > 0)$, $\mu_3 = 0$, $\mu_4 = 0$

$x_2 = 0 \ (\mu_2 > 0)$

$$\therefore \frac{\partial L}{\partial x} = \begin{bmatrix} 2 - \mu_1 + \cancel{} \\ -4 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \cancel{\text{solution}}$$

$$\begin{matrix} \mu_1 \\ \mu_2 \end{matrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

since $\mu_2 = -4$ it contradicts KKT condition $\boxed{\mu_2 > 0}$

$$-\frac{\partial f}{\partial x} = \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$ $\rightsquigarrow$ gradient direction

$$\frac{\partial g_1}{\partial x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$-\frac{\partial f}{\partial x} = \mu_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ +4 \end{bmatrix}$$

Case 3:

Q5

$x_1 = 2 \quad \mu_3 > 0 \quad \mu_1 = 0 \quad \mu_4 = 0.$

$x_2 = 0. \quad \mu_2 > 0$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) + \mu_3 \\ 2(x_2-2) - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + \mu_3 \\ -4 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore \quad \left.\begin{matrix} \mu_3 = -6 \\ \mu_2 = -4 \end{matrix}\right\}$$

Both ~~and~~ are < 0 $\therefore$ does not satisfy KKT condition

$$-\frac{\partial f}{\partial x} = -\begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

$$\frac{\partial g_3}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$-\frac{\partial f}{\partial x} = -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + -4\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} \checkmark$$

Case 4:

$x_1 = 2 \quad \mu_3 > 0 \quad \mu_1 = 0 \ \& \ \mu_2 = 0$

$x_2 = 1 \quad \mu_4 > 0$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(3) + \mu_3 \\ 2(-1) + \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left.\begin{matrix} \mu_3 = -6 \\ \mu_4 = 2 \end{matrix}\right\}$$

$\mu_3 < 0$ $\therefore$ does not satisfy KKT

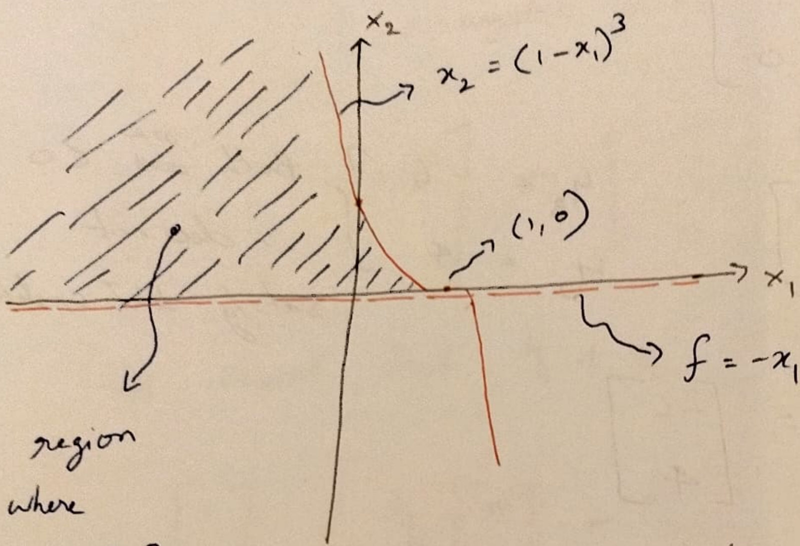
$$-\frac{\partial f}{\partial x} = -\begin{bmatrix} 2(3) \\ 2(-1) \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} \quad \frac{\partial g_3}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{\partial g_4}{\partial x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$-\frac{\partial f}{\partial x} = -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} \checkmark$$

2.

min $f = -x_1$, subject to

$g_1 = x_2 - (1-x_1)^3 \leq 0$ & $g_2 = x_2 \geq 0$. or $\underline{\underline{-x_2 \leq 0}}$.



$f = -x_1 \rightarrow$ This is a contour line that decreases as x_1 increases.

region where $x_2 \geq 0$ & $x_2 \leq (1-x_1)^3$

From the graph we can visually see that the minimum point is at $(1,0)$

$$L(x, \mu) = -x_1 + \mu_1 \left(x_2 - (1-x_1)^3\right) + \mu_2(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -1 + \mu_1 \cdot 3(1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>active</u>

$\mu_1 > 0 \quad x_2 = (1-x_1)^3$

when $x_2 = 0 \quad x_1 = 1$

$\mu_2 > 0 \quad x_2 = 0.$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} -1 + \mu_1 \cdot 3(0)^2 \\ \mu_1 - \mu_2 \end{bmatrix} \quad \therefore \quad \mu_1 = \mu_2.$$

From the optimality conditions we can see that $\mu_1 = \mu_2$

but we cannot tell whether μ_1 & μ_2 are positive or negative.

$-\frac{\partial f}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ negative gradient direction @ $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{\partial g_1}{\partial x} = \begin{bmatrix} 3 \cdot (1-x_1)^2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

we cannot find μ_1 & μ_2 that satisfy this equation

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mu_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad \boxed{\frac{\partial g_1}{\partial x} \text{ \& } \frac{\partial g_2}{\partial x} \text{ are linearly dependant}}$

(3)

$\underbrace{\max f = x_1 x_2 + x_2 x_3 + x_1 x_3}$

$\downarrow$

$\min f = -x_1 x_2 - x_2 x_3 - x_1 x_3$

$h = x_1 + x_2 + x_3 - 3 = 0.$

$L(x, \lambda) = -x_1 x_2 - x_2 x_3 - x_1 x_3 + \lambda (x_1 + x_2 + x_3 - 3)$

$$\frac{\partial L}{\partial \lambda} = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0.$$

solving the above 4 equations for 4 unknowns

$x_1 = 1, \ x_2 = 1, \ x_3 = 1 \quad \lambda = 2.$

Checking $dx^T L_{xx} dx > 0 \quad \forall\, dx \neq 0$ $\qquad \underbrace{\frac{\partial h}{\partial x}} \cdot \partial x = 0.$

$$L_{xx} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ \partial x_3 + \partial x_2 + \partial x_1 = 0$

(or) $\partial x_3 = -\partial x_2 - \partial x_1$

$\partial x^T L_{xx} \partial x > 0.$

$$\begin{bmatrix} \partial x_1 & \partial x_2 & \partial x_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = -2\,\partial x_2 \partial x_1 - 2\,\partial x_1 \partial x_3 - 2\,\partial x_2 \partial x_3$$

$$= -2\,\partial x_1 \partial x_2 + 2\,\partial x_1(\partial x_1 + \partial x_2) + 2\,\partial x_2(\partial x_1 + \partial x_2)$$

$$= 2\,\partial x_1^2 + 2\,\partial x_2^2 + 2\,\partial x_1 \partial x_2$$

$$= 2\left(\left(\partial x_1 + \frac{1}{2}\partial x_2\right)^2 + \frac{3}{4}\partial x_2^2\right) > 0.$$

$\therefore$ it is a local minimum solution

Q5)

(*) N is the total number of sites (including truck station)

(*) C_{ij} is the cost of moving from site i to site j. If there is no edge, ~~for there is~~ between the nodes, then $C_{ij} = \infty$

(*) x_{ij} be a binary value variable variable which is equal to 1 if the truck moves from site i to j and 0 otherwise

(*) u_i be a variable that represents the order in which sites are visited

Objective function:

$$\text{Minimize} \sum_{i=0}^{N} \sum_{j=0}^{N} C_{ij} \cdot x_{ij} \quad , \text{where } i \neq j$$

Subject to constraints

Each site should be visited once + left once $\sum_{i=0, i \neq j}^{N} x_{ij} = 1$, for $j = 1, 2, \ldots N$

$$+ \sum_{j=0\, j \neq i}^{N} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots N.$$

$$\sum_{j \in N/i} x_{ij} = \sum_{j \in N/i} x_{ji} \quad \forall i \in N \setminus 0$$ → Numbers of times going into a node should be equal to times going out of a node.

$$\sum_{j \in N/i} x_{ij} \geq 1 \quad \forall i \in N \setminus 0$$ → All nodes must be visited

$x_{ij} \in \{0, 1\} \quad \forall i,j \in N, i \neq j.$