MAE 598 - DESIGN OPTIMIZATION HW-2

① $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

Solving the gradient for the stationary point.

$$\left.\begin{aligned} 4x_1 - 4x_2 &= 0 \\ -4x_1 + 3x_2 + 1 &= 0 \end{aligned}\right\}$$ solving the two equations we get $x_1 = 1$ & $x_2 = 1$

Taylor's series at the saddle point:

$$f(x_1, x_2) \simeq f(x_0) + \nabla f(x_0)^T . (x - x_0) + \frac{1}{2}(x - x_0)^T . H(x_0) . (x - x_0)$$

$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore f(x_0) = 0.5$

$$\nabla f(x_0) = 0$$

$$x - x_0 = \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} = \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x) = 0.5 + \frac{1}{2}\begin{bmatrix} \partial x_1 & \partial x_2 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x_1, x_2) = 0.5 + \frac{1}{2}\left[4\, dx_1^2 - 8\, dx_1 dx_2 + 3 dx_2^2\right]$$

$$f(x_1, x_2) - 0.5 = \frac{1}{2}\left[4\, dx_1^2 - 8 dx_1 dx_2 + 3 dx_2^2\right]$$

This can be factorized as

$$f(x_1, x_2) - 0.5 = \frac{1}{2}(2dx_1 - dx_2)(2dx_1 - 3dx_2)$$

To get the down slopes we can show

$$\boxed{\begin{gathered} 2dx_1 - dx_2 < 0 \ \& \ 2dx_1 - 3dx_2 > 0 \\ \text{or} \\ 2dx_1 - dx_2 > 0 \ \& \ 2dx_1 - 3dx_2 < 0. \end{gathered}}$$

Q2) we can make the objective function as.

$$d^2 = (x+1)^2 + (y)^2 + (z-1)^2$$

we use the constraint $\rightarrow x + 2y + 3z = 1$

$$\therefore z = \frac{1-x-2y}{3}$$

$$d^2 = (x+1)^2 + y^2 + \left(\frac{1-x-2y-3}{3}\right)^2$$

$$\therefore d^2 = x^2 + 2x + 1 + y^2 + \frac{x^2 + 4xy + 4x + 8y + 4y^2 + 4}{9}$$

$$\therefore f(x,y) = d^2 = (10x^2 + 22x + 13y^2 + 8y + 4xy + 13)/9$$

$$\nabla f(xy) = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} 20x + 22 + 4y/9 \\ 26y + 8 + 4x/9 \end{bmatrix} \quad H = \begin{bmatrix} 20/9 & 4/9 \\ 4/9 & 26/9 \end{bmatrix}$$

$\hookrightarrow$ Hence its convex!

To find the x & Y values when gradient is zero.

$$\left.\begin{aligned} 20x + 4y &= -22 \\ 4x + 26y &= -8 \end{aligned}\right\}$$ solving for unknowns x & y we get

$$\boxed{\begin{aligned} x &= -15/14 \\ y &= -1/7 \end{aligned}}$$

when we put x & Y back into equation for Z we get

$$\boxed{z = 11/14.}$$

(3)

$$H = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n \;\middle|\; a_1 x_1 + \cdots\cdots a_n x_n = c \right\}$$

where $a_1, \ldots\ldots, a_n \neq 0$ and $c \in R$

we need to show

$\forall\, X, Y \in H \quad \lambda X + (1-\lambda) Y \in H \qquad 0 \leq \lambda \leq 1$

$$\lambda X + (1-\lambda) Y = \begin{bmatrix} \lambda x_1 + (1-\lambda) y_1 \\ \vdots \\ \lambda x_n + (1-\lambda) y_n \end{bmatrix}$$

we can writ it as

$$a_1 x_1 + \cdots\cdots + a_n x_n = c$$

$$a_1 y_1 + \cdots\cdots a_n y_n = c$$

$\therefore$ for all $\lambda \in [0, 1]$

$$\lambda_1 a_1 x_1 + \cdots\cdots + \lambda a_n x_n = \lambda c$$

$$(1-\lambda) a_1 y_1 + \cdots\cdots + (1-\lambda) a_n y_n = (1-\lambda) c$$

summing up the above equation.

$$a_1 (\lambda x_1 + (1-\lambda) y_1) + \cdots\cdots + a_n (\lambda x_n + (1-\lambda) y_n) = \lambda c + (1-\lambda) c$$

$$a_1 (\lambda x_1 + (1-\lambda) y_1) + \cdots\cdots + a_n (\lambda x_n + (1-\lambda) y_n) = c$$

$$\Rightarrow \boxed{\lambda X + (1-\lambda) Y \in H}$$

4 a) Show that the problem is convex.

$h = I_t / I$

$h(p) = I_t / a^T p$

$\nabla h = I_t \big/ -(a^T p)^2 \cdot a$

$$H = 2\frac{I_t \; a . a^T}{(a^T p)^3}$$ → Since the Hessian $\geq 0$ we can say that the problem is ~~post~~ convex

b) The problem will be convex as the function will become a combination of diffrent half spaces. and hence its convex.

ex: $P_1 + P_2 + \cancel{B} P_3 + \ldots . P_{10} \leq p^*$

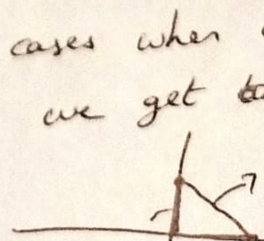
$P_2 + B P_3 + \ldots \ldots + P_{11} < p^*$

Al though the function becomes convex. it is not necessary for it to have a unique solution

c) The problem will not be convex as the function created will not follow the rule of. $\boxed{\lambda x_1 + (1-\lambda) x_2 \in S}$

for a convex set S.

for example. we have two lamps & two cases when one is on & the other is off & vice versa then we get ~~two~~ function that is non convex



line does not pass through the function.

5) $C^*(y) = \max_x \{ xy - C(x) \}$

Show that $C^*(y)$ is convex with respect to $y$.

We can check the second differential of the function

$$\frac{d\,C^*(y)}{dy} = x$$

$$\frac{d^2\,C^*(y)}{dy^2} = 0 \rightarrow$$ Since the second differential is zero we can say that the function is convex.

max of a convex function is also. convex.