MAE 598 HW 4

Q2) $\dot{h}(t) = v(t)$ $h \to$ altitude

$\dot{v}(t) = -g + \dfrac{a(t)}{m(t)}$ $v \to$ velocity

$m \to$ mass of moon lander.

$\dot{m}(t) = -k\,a(t)$

with initial conditions

$h(0) = h_0 > 0$

$v(0) = v_0$

$m(0) = m_0 > 0.$

The goal is to land on the moon safely, maximizing the remaining fuel $m(\tau)$, where $\tau = \tau[a(\cdot)]$ is the first time $h(\tau) = v(\tau) = 0$

Since $a = -\dfrac{\dot{m}}{k}$, our intention is equivalently to mimize the total applied thrust before landing, so that

$$P[a(\cdot)] = -\int_0^t a(t)\,dt$$

This is so since

$$\int_0^\tau a(t)\,dt = \frac{m_0 - m(\tau)}{k}$$

Introducing the maximum principle. In terms of the general notation we have

$$x(t) = \begin{pmatrix} h(t) \\ v(t) \\ m(t) \end{pmatrix}, \quad f = \begin{pmatrix} v \\ -g + a/m \\ -ka \end{pmatrix}$$

Here the Hamiltonian is

$$H(x, p, a) = f \cdot p + r$$

$$= (v, -g + a/m, -ka) \cdot (p_1, p_2, p_3) - a$$

$$= -a + p_1 v + p_2 (-g + a/m) + p_3 (-ka)$$

We next figure out the adjoint dynamics (ADJ). for our particular hamiltonian,

$$H_{x_1} = H_h = 0; \quad H_{x_2} = H_v = p_1, \quad H_{x_3} = H_m = \frac{-p_2 a}{m^2}$$

Therefore,

$$\begin{cases} \dot{p}^1(t) = 0 \\ \dot{p}^2(t) = -p^1(t) \\ \dot{p}^3(t) = \dfrac{p^2(t)\, a(t)}{m(t)^2} \end{cases}$$

The maximization conditions (M) reads.

$$H(x(t), p(t), a(t)) = \max_{0 \le a \le 1} H(x(t), p(t), a)$$

$$= \max_{0 \le a \le 1} \left\{ -a + p^1(t)\, v(t) + p^2(t)\left[-g + \frac{a}{m(t)}\right] + p^3(t)(-ka) \right\}$$

$$= p^1(t)\, v(t) - p^2(t) g + \max_{0 \le a \le 1} \left\{ a\left(-1 + \frac{p^2(t)}{m(t)} - k p^3(t)\right) \right\}$$

Thus the optimal control law is given by the rule.

$$a(t) = \begin{cases} 1 & \text{if } 1 - \frac{p^2(t)}{m(t)} + k p^3(t) < 0. \\ 0 & \text{if } 1 - \frac{p^2(t)}{m(t)} + k p^3(t) > 0. \end{cases}$$

Using the maximum principle. Now we will attempt to figure out the form of the solution and check it accords with the max principle.

$$a(t) = \begin{cases} 0 & \text{for } 0 \le t \le t^* \\ 1 & \text{for } t^* \le t \le \tau \end{cases}$$

Therefore, for times $t^* \le t \le \tau$ our ODE becomes

$$\begin{aligned} \dot{h}(t) &= v(t) \\ \dot{v}(t) &= -g + 1/m(t) \qquad (t^* \le t \le \tau) \\ \dot{m}(t) &= -k \end{aligned}$$

Now put $t = t^*$:

$$m(t^*) = m_0$$

$$v(t^*) = g(\tau - t^*) + \frac{1}{k} \log\left[\frac{m_0 + k(t^* - \tau)}{m_0}\right]$$

$$h(t^*) = -g\frac{(t^* - \tau)^2}{2} - \frac{m_0}{k^2} \log\left[\frac{m_0 + k(t^* - \tau)}{m_0}\right] + \frac{t^* - \tau}{k}$$

Before time t^* $\alpha = 0$ Then ODE reads

$$\begin{cases} \dot{h} = v \\ \dot{v} = -g \\ \dot{m} = 0 \end{cases} \quad \text{\& thus} \quad \begin{cases} m(t) = m_0 \\ v(t) = -gt + v_0 \\ h(t) = -\frac{1}{2}gt^2 + t v_0 + h_0 \end{cases}$$

We combine the formulas for $v(t)$ & $h(t)$, to discover

$$h(t) = h_0 - \frac{1}{2g}(v^2(t) - v_0^2) \quad (0 \leq t \leq t^*)$$

$\therefore$ we can say that during free fall we get a parabola.

now we find the costate $p(\cdot)$

$$p^1(0) = \lambda_1, \quad p^2(0) = \lambda_2, \quad p^3(0) = \lambda_3$$

We solve (ADJ) for $a(\cdot)$ as above and find

$$p^1(t) = \lambda_1$$

$$p^2(t) = \lambda_2 - \lambda t$$

$$p^3(t) = \begin{cases} \lambda_3 & 0 \le t \le t^* \\ \lambda_3 + \int_\tau^t \frac{\lambda_2 - \lambda_1 s}{(m_0 + k(\tau - s))^2}\, ds & (t^* \le t \le \tau) \end{cases}$$

Define

$$r(t) := 1 - \frac{p^2(t)}{m(t)} + p^3(t)k;$$

$$\dot r = -\frac{\dot p^2}{m} + \frac{p^2 \dot m}{m^2} + \dot p^3 k = \frac{\lambda_1}{m} + \frac{p^2}{m^2}(-ka) + \left(\frac{p^2 a}{m^2}\right)k = \frac{\lambda_1}{m(t)}$$

Choose $\lambda_1 < 0$, so that r is decreasing. We calculate

$$r(t^*) = 1 - \frac{(\lambda_2 - \lambda_1 t^*)}{m_0} + \lambda_3 k$$

and then adjust λ_2, λ_3 so that $r(t^*) = 0$.

Then r is nonincreasing, $r(t^*) = 0$, and consequently $r > 0$ on $[0, t^*)$, $r < 0$ on $(t^*, \tau]$. But (M) says

$$a(t) = \begin{cases} 1 & \text{if } r(t) < 0 \\ 0 & \text{if } r(t) > 0. \end{cases}$$

Thus an optimal control changes just once from 0 to 1; and so our earlier guess of $a(\cdot)$ does indeed satisfy the Pontryagin maximum principle.