Mathematics for Quantum Computing

$2x+(y-x) = 2x+(y+(-x))$ Notation $\rightarrow$ Additive Inverse

2 → Matter, −2 → Anti-matter

$2+(-2)=0 \rightarrow$ Additive identity

$= 2x+((-x)+y)$ Commutativity

$= (2x+(-x))+y$ Associativity

$= (2x+(-1)x)+y$ Notation

$= (2+(-1))x+y$ Distributive property

$= 1x+y$ Closure

$x \in \mathbb{Z}$

$1x \in \mathbb{Z}$

$= x+y$ Multiplicative idutity

$a \,\square\, e = a$

e → identity of that operation

Algebraic Structure

$(R, \underbrace{*_1, *_2, *_3 \ldots}_{\text{Binary operations}})$ → Closure property (by default)

R → Non-empty

$a, b \in R$

$a *_1 b \in R$

A non-empty set, and an operation defined on that set, and that operation satisfies closure property, it is called a groupoid. And if it also satisfies associativity, it is called semi-group.

If the semigroup also has an 'identity' element as well, it's called a Monoid.

If ~~the~~ each element also has an inverse, it's called as Group.

If the group is also commutative, it's called an Abelian Group.

$(R, +, \cdot) \Rightarrow$ A set equipped with two operations; $+, \cdot$

① Closure under addition

$a, b \in R$

$a+b \in R$

② Associativity under addition

$a, b, c \in R$

$(a+b)+c = a+(b+c)$

③ Identity under addition

$\exists\ 0 \in R$ such that

$a+0 = a = 0+a$

④ Additive inverse

$a \in R \quad \exists\ -a \in R$

such that,

$a+(-a) = 0 = (-a)+a$

⑤ Commutativity under Addition

$a, b \in R$ such that

$a+b = b+a$

⑥ Closure under multiplication

$a, b \in R$

$\Rightarrow a \cdot b \in R$

⑦ Associativity under multiplication

$a, b, c \in R$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

⑧ Identity under Multiplication

$a \in R \quad \exists\ 1 \in R$ such that

$a \cdot 1 = 1 \cdot a = a$

⑨ Multiplicative inverse

$a \in R \quad \exists\ a^{-1} \in R$ such that

$a \cdot a^{-1} = a^{-1} \cdot a = 1$

⑩ Commutativity under Multiplication

$a, b \in R$

$a \cdot b = b \cdot a$

⑪ Distributive property

$(a+b)c = ac + bc$

$a(b+c) = ab + ac$

⑫ $0 \neq 1$

(This is just to exclude some trivial cases)

(Identity and inverse element must be unique)

If $(R, +, \cdot)$ ~~satisfies~~ is an Abelian group under addition and has closure and associative multiplication and satisfies the distributive property, this algebraic structure is called a RING.

If a ring also satisfies the identity property under multiplication, it is called Ring with Unity.

If a Ring with unity also satisfies commutativity under Multiplication, it is called Commutative Ring with Unity.

An Abelian group under Addition and Multiplication (satisfying all ⑩ Rules) + satisfies Distributive prop. + ⑫ Rule

⇓

Field

$\boxed{V(F)}$ → Field

↓

Vector Space

$F \longrightarrow \mathbb{R}^*, \mathbb{C}^*, \mathbb{Q}^*$

$\mathbb{R}^* \Downarrow (R - \{0\})$, $\mathbb{C}^* \Downarrow (C - \{0\})$, $\mathbb{Q}^* \Downarrow (\mathbb{Q} - \{0\})$

Note: There are ∞ Extension Fields of $\mathbb{Q}$

★ $\mathbb{Q}(\alpha)$ is an Extension Field of $\mathbb{Q}$ ⇒ "ALGEBRAIC EXTENSION"

Extending $\mathbb{Q}$:

$\sqrt{2}$ is not a rational no.

A Field containing both $\mathbb{Q}$ and $\sqrt{2}$: $\mathbb{Q}(\sqrt{2})$ → + Abelian, • Abelian, Distributive & $0 \neq 1$

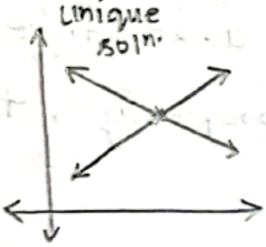
$\mathbb{Q}(\sqrt{2})$ ↓ Extension Field of $\mathbb{Q}$

$\mathbb{Q}(\alpha) \rightarrow$ Extended Field of $\mathbb{Q}$

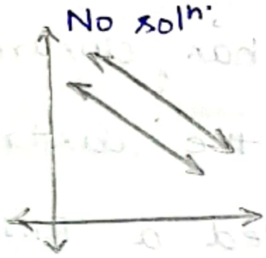
α ↓ Any irrational no.

Linear Transformation and Vector Spaces

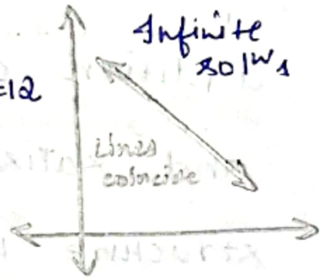
$$x + y = 6$$
$$-2x - y = -3$$
$$\overline{\quad -x = 3 \quad}$$



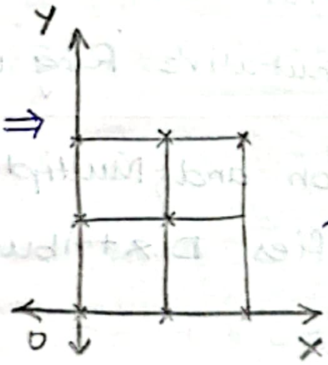
$$x + y = 6$$
$$x + y = 4$$



$$x + y = 6$$
$$2x + 2y = 12$$



(x, y)
$(0,0)$
$(0,1)$
$(1,0)$
$(1,1)$
$(2,0)$
$(0,2)$
$(2,2)$ $\Rightarrow$

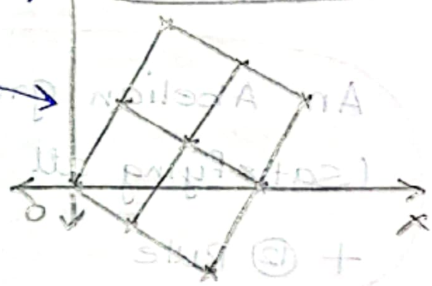


Let's say; $x + y = X$
$2x + y = Y$

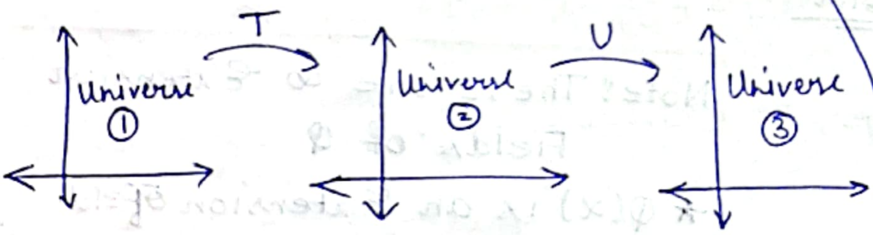
we find the values of X & Y for the corresponding (x,y) points taken before

(X, Y)
$(0,0)$
$(1,1)$
$(1,2)$ $\Rightarrow$

Graphs are drawn just to understand and might not be the actual graphs

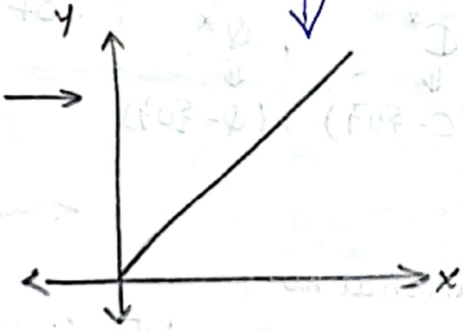


$\Downarrow$ The graph that was a straight square has now become a tilted rectangle
$\Rightarrow$ **TRANSFORMATION**



Let's say $x + 2y = X$
$2x + 4y = Y$ $\rightarrow$

$$\boxed{\begin{matrix} 2(x+2y) = Y \\ 2X = Y \end{matrix}}$$



Notation:

Cayley

$$x + y = 6$$
$$2x - y = 4$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

"Matrix Notation"

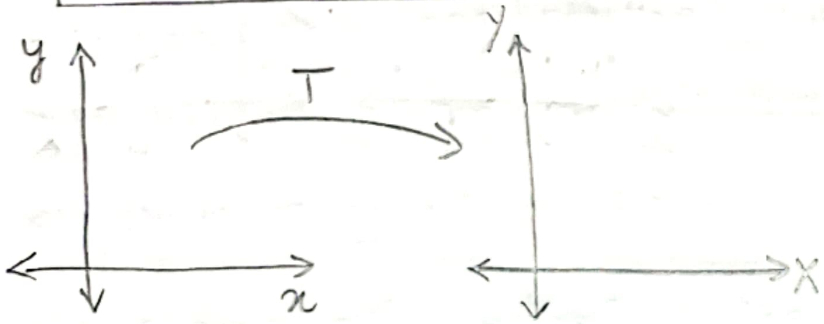
$$x + 2y + z = 0$$
$$x - 3y + z = 3$$
$$2x - y + z = 5$$

$$\boxed{\begin{pmatrix} 1 & 2 & 1 \\ 1 & -3 & 1 \\ 2 & -1 & 1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

$\Downarrow$
Responsible for transformation

$x+y=X$
$2x-y=Y$

$\boxed{T(x,y)=(X,Y)}$ (or) $\boxed{T(x,y)=(x+y,2x-y)}$



Shifting from this plane to . . . . this plane

$G(x,y)=(x,-y)=(X,Y)$
$H(x,y)=(y,-x)=(x',y')$

If $G(x,y)=(x,-y)$

$X=x+0y$
$Y=0x-1y$

$\Rightarrow G=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

If $H(x,y)=(y,-x)$

$x'=0x+1y$
$y'=-1x+0y$

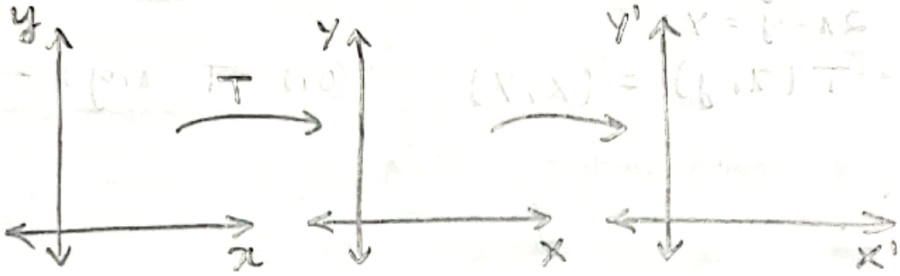
$\Rightarrow H=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$G(x',y')=(x',-y')$
$=(y,-(-x))$
$=(y,x)$

$\Rightarrow GH(x,y)=(y,x)$

Ex:- $GH(1,2)$
$\Rightarrow H(1,2)=(2,-1)$
$\Rightarrow G(2,-1)=(2,1)$

$$X = ax + by$$
$$Y = cx + dy$$



$$x' = AX + BY$$
$$y' = CX + DY$$

$$\Rightarrow x' = A(ax+by) + B(cx+dy)$$
$$= Aax + Aby + Bcx + Bdy$$
$$\boxed{x' = (Aa+Bc)x + (Ab+Bd)y}$$

$$\Rightarrow y' = C(ax+by) + D(cx+dy)$$
$$= Cax + Cby + Dcx + Ddy$$
$$\boxed{y' = (Ca + Dc)x + (Cb + Dd\cancel{o})y}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix} \Rightarrow$$ Matrix <u>Multiplication</u>

From the previous example of G and H transformations;

$$G(x,y) = (x,-y)$$
$$H(x,y) = (y,-x)$$

$$GH = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$GH = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Abstract Formalism :

$(a,b) \in \mathbb{R}^2$ $\quad$ $(c,d) \in \mathbb{R}^2$

$\mathbb{R}^2 \rightarrow$ 2-dimensional plane (SPACE)

$(a,b)+(c,d) \in \mathbb{R}^2$ [Closure]

$r \rightarrow$ Scalar (FIELD) $\downarrow$ no./component

$(a,b)+(c,d) = (a+c, b+d)$

$$\boxed{\begin{aligned} T[(a,b)+(c,d)] &= T(a,b)+T(c,d) \\ T(r(a,b)) &= rT(a,b) \end{aligned}} \Rightarrow \text{Linear Transformation}$$

A space with infinite dimensions $\Rightarrow$ HILBERT SPACE

Extended Fields

* $\underbrace{a_1, a_2, a_3 \ldots L}_{\text{Sequence}}$

$L \rightarrow$ Limit of the sequence

$(\mathbb{Q}+L) = \mathbb{R}$

$\Downarrow$

Field

* Extended $\mathbb{R}$

$\mathbb{R}(i)$ $\quad$ $i \rightarrow$ Imaginary no. $= \sqrt{-1}$

$\hookrightarrow \mathbb{C}$

$\hookrightarrow$ Field

Extension of $\mathbb{C} = \mathbb{C}$

$\mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} \middle| f, g \text{ are polynomials with coeff}^{n} \text{ in } \mathbb{C} \right.$

$\hookrightarrow$ Rational functions

Ex:- $\mathbb{C}(x) = \dfrac{ix+3}{3-ix^3}$

$\mathbb{C}(x) \Rightarrow$ Extended $\mathbb{Q}$

VECTOR SPACE

A vector space is a set of vectors for which addition and scalor multiplication are defined such that if u and v are vectors in the space and k is a scalar, then u+v is in the space and ku is in the space.

What is space in Mathematics?

→ A non-empty set having some $\boxed{\text{mathematical Structure}}$

<u>Ex:-</u> $A = \{1, 2, 3\}$ ⇒ ~~No Simple~~ Ordinary Set (आम सेट)

$(\mathbb{R}, +)$ ⇒ This set follows certain theorems & definitions

∴ This is a space

Definitions : ① Closure
② Associativity
③ Identity
④ Inverse

Group

Theorems : ① If G is a group w.r.t $*$, then identity element is unique.
② If G is a group w.r.t $*$, then inverse element is unique.

<u>Ex:-</u> $X = \mathbb{R}$, $d(x,y) = |x-y|$

(Basically, if any x and y satisfy these 3 conditions, they will come under metric space)

① Let X be a non-empty set and

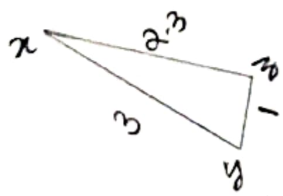
$d : X \times X \to \mathbb{R}$

(i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality)

Then X is metric space (∵ it's equipped with a $\underbrace{\text{machinde 'd'}}_{\text{Mathematical Structure}}$ that calculates distance)



Examples of vector spaces:

① $u = x+1$, $v = 2x-3$ (Polynomials of first degree)

$u+v = 3x-2 \Rightarrow$ (Polynomial of first degree)

$5u = 5(x+1) = 5x+5 \Rightarrow$ (Polynomial of first degree)

② $u = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $v = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$

③ $u = \int f(x)\,dx$, $v = \int g(x)\cdot dx$

④ $u = \frac{d}{dx} f(x)$, $v = \frac{d}{dx} g(x)$

⑤ $u = 1$, $v = 2$

Axioms that a vector space must fulfill:

Let V be a vector space and F a field of scalars if $u, v, w \in V$ and $a, b \in F$, then

1] $u+v = v+u$ → Commutative ppty of addition

2] $(u+v)+w = u+(v+w)$ → Associativity

3] There should be zero vector 0, such that

$0+u = u+0 = u$ → Additive identity

4] For every $u \in V$, $\exists\, u'$ such that $u+u' = 0$ → Additive inverse

5] $\forall\, u \in V$, $1u = u$

6] $a\cdot(bu) = (ab)\cdot u$

7] $(a+b)u = au + bu$

8] $a(u+v) = au + av$

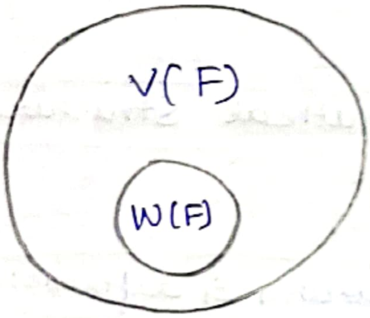
9] There should be a zero vector s.t $u\cdot 0 = 0$

Subspace

$V(F) \longrightarrow$ Subset of $V(F) \Rightarrow W(F)$

$\downarrow$ also

Vector Space



$W(F)$ is a subspace of $V(F)$

Formal Definitions:

1] Internal Composition:

A composition $*$ is an internal composition in a non-empty set V if $\forall\ \alpha, \beta \in V$

$\Rightarrow \alpha * \beta \in V$

"It is unique"

2] External Composition:

$F \rightarrow$ Field

$\hookrightarrow$ Scalar (number)

$a \in F$ and $\alpha \in V$

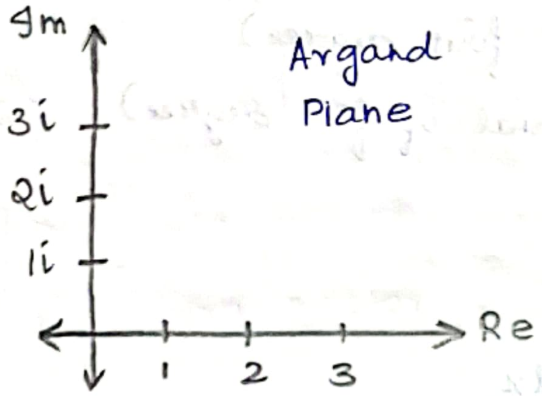
A composition 'o' is an external composition in a non-empty set V over a non-empty field F such that $\forall\ a \in F$ and $\alpha \in V$

$\Rightarrow a \circ \alpha \in V$

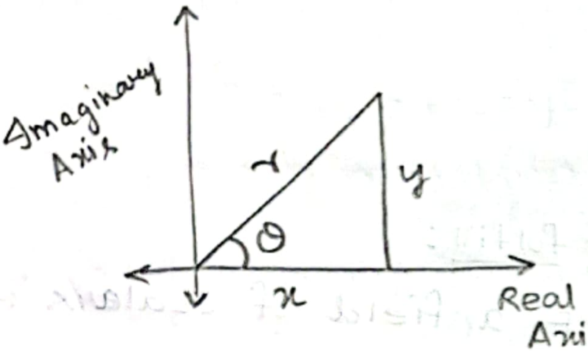
"It is also unique"

Complex Numbers:

$\sqrt{-1} = i \rightarrow$ Imaginary Number



$$\boxed{z = x + iy \rightarrow \text{Complex No.}}$$



$\cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$

$\Rightarrow x = r\cos\theta$ $\quad \Rightarrow y = r\sin\theta$

Complex number $z = x + iy$

$\Rightarrow z = r\cos\theta + ir\sin\theta$

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

$$\boxed{z = re^{i\theta}}$$ Euler notation

* In calculations, we use complex no.s but the final answer we always obtain will be a Real Number

$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

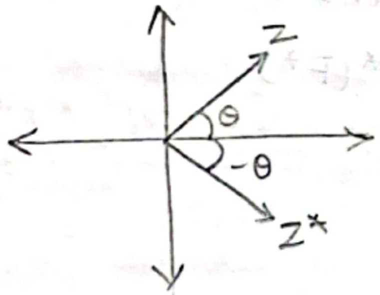
$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta})$

$= (r_1 r_2)(e^{i(\theta_1 + \theta_2)})$

* $V(F) \rightarrow$ Complex Vector Space ($\because$ We will take $\mathbb{C}$ as F)

* For every complex number z, there exists a z^* that is z's complex conjugate

If $z = x + iy$, then $z^* = x - iy$



$$zz^* = (x+iy)(x-iy)$$
$$= (re^{i\theta})(re^{-i\theta})$$
$$= r^2(e^0)$$
$$\underline{\underline{zz^* = r^2}}$$

$z^* \rightarrow$ Complex No.

z & $z^* \rightarrow$ Dual no. System

$\because \forall z \; \exists$ a unique z^*

* Special Complex Number → PHASE FACTOR

Where $r = 1$ $\boxed{z = e^{i\theta}}$

$\boxed{z = \cos\theta + i\sin\theta}$

Notations:

Z Complex No.

Z^* Complex Conjugate

} Dual No. System

$|\psi\rangle$ Ket Vector

$\langle\psi|$ Bra vector

} Vectors

$|\psi\rangle \in V(F)$ → Complex Vector Space

$\langle\psi| \in V^*(F^*)$ → Dual Space/ Complex Conjugate Vector space

$\langle\phi|\psi\rangle$ Inner product → Number

$|\phi\rangle \otimes |\psi\rangle$ Tensor product

A^* Complex conjugate of a matrix

A^T Transpose

$A^\dagger$ Hermitian Conjugate $(A^T)^*$

Topological Space:

↳ Most General Spaces

→ Set + Topology (Collection of open sets) Satisfying

① Union of arbitrary open sets is open

② Intersection of finite no. of open sets is open

→ Ex:- Functional Spaces, Polynomial Spaces

Vector Spaces → Normed Spaces

↓

Set of elements (vectors)

↓

2 Axioms

↓

Abstract Structure w/o any inherent notion of vector, distance or angle b/w vectors

↓

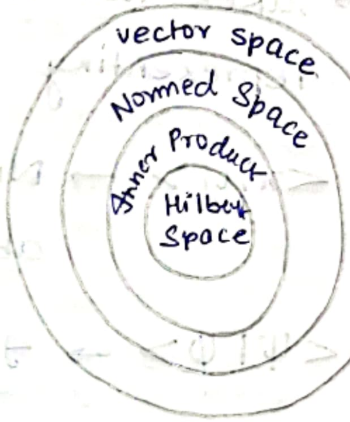
Mainly used to define length of vector

(OR)

$\|\cdot\|$ is also used to calculate distance b/w two vectors

(OR)

$\|\cdot\|$ assigns a non-negative number to each ~~pair~~ vectors (elements of VS) w/o any inherent notion of actual distance



Function $\|\cdot\| : V \to F$ (Only +ve nos) satisfying following properties

① Non-negativity

$\| |\psi\rangle \| \geq 0$

② Homogeneity

$v \in V$, $a \in F$

$\| a v \| = |a| \, \|v\|$

③ Triangular inequality

$\| |\psi\rangle + |\phi\rangle \| \leq \| |\psi\rangle \| + \| |\phi\rangle \|$ (Consider -ve vectors)

$\boxed{d(x,y) = \| x - y \|}$ → Dist. b/w two vectors

Inner Product Space (IPS):

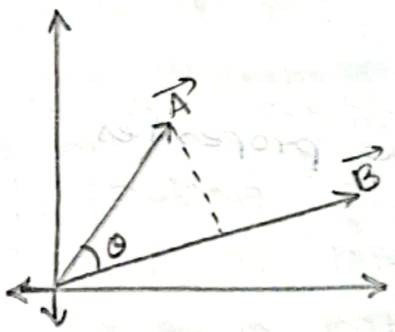
→ Vector Space + Inner Product

→ Takes two vectors and assigns a number ($\mathbb{R}$ or $\mathbb{C}$), representing angles or correlation b/w them

$\langle x, y \rangle$ → No. / Scalar that encodes the information about their alignment or how similar they are

$\langle \psi | \phi \rangle$ → IPS (The answer in generally will be complex)

→ "Geometrically, IP measures how much one vector points in the direction of the other vector and how their magnitudes contribute to their alignment."



(How much $\vec{A}$ is contributing in the direc^n^ of $\vec{B}$)

$\langle A | B \rangle = \vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta$

{This is a kind of Inner Product, usually defined for use in Physics}

→ Relation between Norm and Inner Product:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Ex:- $\xrightarrow{\vec{A}}$ $\vec{A} \cdot \vec{A} = |A||A| \cdot \cos 0 = A^2$

$\Rightarrow |A| = \sqrt{\vec{A} \cdot \vec{A}}$

Definition - $\langle \cdot, \cdot \rangle : V \times V \longrightarrow F$ (Only +ve no.s) satisfying

① Linearity

$$\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$$

② Conjugate Symmetry

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

③ Positive definitiveness

$$\langle x, x \rangle \geq 0 \text{ \& } \langle x, x \rangle = 0 \text{ iff } x = 0$$

Hilbert Spaces

ε (Extremely small no. but $!=0$)

Complete IPS

$\downarrow$

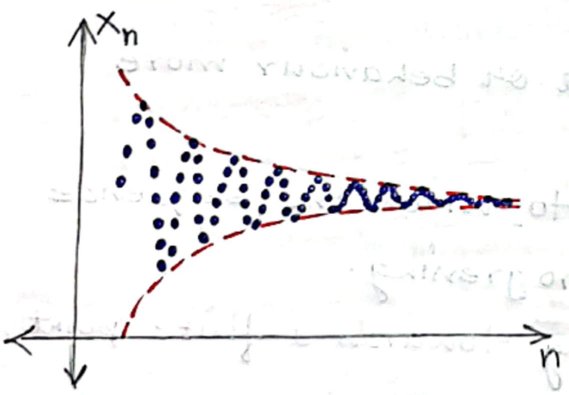
Properties of IPS + Completeness
(Every Cauchy Sequence has a limit in Hilbert Space)

Cauchy Sequence: एक ऐसी सीक्वेन्स जिसमें एक point के बाद, दो consecutive elements के बीच का distance बहुत ही कम हो जाता है ([illegible])

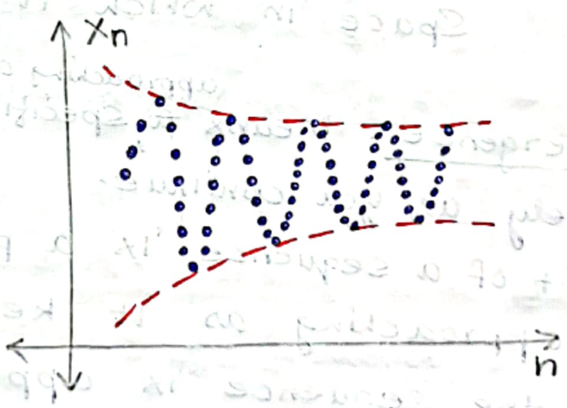
$\{\frac{1}{n}\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\}$

→ After a point, sequence becomes more and more tightly clustered.

Cauchy Sequence: A sequence $\{x_n\}$ in (metric space or Normed Space) $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}\ \forall\ m, n$ such that $|x_m - x_n| < \varepsilon$



A plot of Cauchy Sequence



A sequence that's not Cauchy

Completeness: Normed Space X is complete if every Cauchy Sequence in that space has a limit within the same space.

[If points of sequence become closer, the point of convergence should also be in the same space as the space in which the sequence was defined.]

Ex:- $E_n = \frac{-13.6}{n^2} z^2$

As $n \to \infty$, $\underbrace{E_n \to 0}$

Limit of the E_n sequence is 0

$\Rightarrow$ 0 should also be in the Space in which E_n was defined

$\Downarrow$

Completeness

Ex:- [Not Complete]

Let's take a sequence in Rational Normed Space ~~(R)~~ ($\mathbb{Q}$), $n \in \mathbb{N}$

$\{x_n\} = \{1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} + \ldots\}$

This started from rational no.s but is approaching 'e'. 'e' is an Irrational no.

$\Rightarrow$ Limit of this Cauchy sequence is not in the Same Space in which it was defined

① Convergence: Means approaching a specific value or behaviour more closely as you continue.

Limit of a sequence is a point to which the sequence is approaching as it keeps progressing.

If the sequence is approaching towards a finite point, it is called a convergent sequence.

Ex:- We have a sequence $\{\frac{1}{n}\}$

$\lim_{n \to \infty} \frac{1}{n} = 0$ $\Rightarrow$ This sequence is converging to zero (Although it will never be equal to zero)

Analogy: Dart Board $\to$ Every shot we hit on the dartboard wants to go as near to the centre as possible.

② **Connectedness:** Meaning the space can't be divided into non-overlapping or disjoint non-empty open sets.

③ **Continuity:** Smoothness of function
⇒ Small changes in input results in small changes in O/P.

④ **Compactness:** Space is bounded and closed, making it well behaved for convergence and containment.

$(a, b) \rightarrow$ open ⇒ Not compact → Can't be covered by finite no. of open sets

$[a, b] \rightarrow$ closed ⇒ Compact

⑤ **Subspace:** If W is a subset of V and W follows all properties ~~of~~ as V, then W is called subspace of V.

Quantum Mechanics
↓
Hilbert Space
∞-Dimensional Vector Space
↳ Relates to the range of possible states a single particle can have / occupy, not the no. of particles

$|\psi\rangle \rightarrow$ State of a particle in a Hilbert Space

→ For many Quantum systems, the Hilbert Space is ∞-dimensional.

→ Particle can be described by a potentially ∞ set of basis functions.

→ Spin of $e^{\ominus}$
Polarization of photon } 2 or 3 dimensions

→ Position
Energy
Momentum } ∞ Dimensions

★ Infinite Dimension means the complexity of states available to a particle.

→ It allows a particle's states to be represented by a superposition of ∞ no. of basis states.

Ex:- Different Energy levels in a Quantum Harmonic Oscillator

→ One particle is represented by one vector in Hilbert Space.

→ Space is ∞-dimensional due to the range of possible states available to the particle.

→ For many particles ⇒ Tensor product is used

Tensor product of HS for each particle

Space becomes very large but each particle in HS is still ∞ dimensional, if space is described in terms of continuous spectrum of ∞ states.

Let $V(\mathbb{C})$ be a vector space over field of $\mathbb{C}$

$V(\mathbb{C}) \rightarrow$ Complex Vector Space

$|\psi\rangle \in V(\mathbb{C})$

↓

State of a quantum particle

Axioms followed by a ket vector:

① $|\psi\rangle + |\phi\rangle \in V$ (Closure Under +)

② $|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle$ (Commutativity under +)

③ $|\psi\rangle + \{|\phi\rangle + |\chi\rangle\} = \{|\psi\rangle + |\phi\rangle\} + |\chi\rangle$ (Associativity under +)

④ $|\psi\rangle + 0 = |\psi\rangle$ (Identity under +)

⑤ $|\psi\rangle + (-|\psi\rangle) = 0$ (Inverse under +)

⑥ $|z\psi\rangle = z|\psi\rangle$

⑦ $z\{|A\rangle + |B\rangle\} = z|A\rangle + z|B\rangle$

$(z+w)|A\rangle = z|A\rangle + w|A\rangle$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Linearity

Note: In ordinary space, multiplying by a complex no. doesn't make any sense.
Our final answer must come in $\mathbb{R}$

A ket vector can be defined as a function $A(x)$, that is a continuous, complex valued function such that you can perform addition $(A_1(x) + A_2(x))$ and can perform multiplication by a scalar and that it follows the above 7 axioms, then $A(x)$ is a ket vector.

Ex:- Matrices

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix}$$

$$z\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} z\alpha_1 \\ z\alpha_2 \end{pmatrix}$$

Note:- We normally do not mix vectors of different dimensionality

For every ket vector there exists a bra vector

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \qquad \langle\psi| = (\alpha_1^* \ \alpha_2^* \ \cdots \ \alpha_n^*)$$ (Complex conjugate vector space)

(Bra vector also satisfies 7 axioms)

$z \rightarrow$ complex no. $\qquad z^* \rightarrow$ Dual version

$V(F) \rightarrow$ Complex Vector Space $\qquad V^*(F^*) \rightarrow$ Complex conjugate vector space

$\forall |a\rangle \in V \ \exists \ \langle a| \in V^*$

Note:- ① If $\langle a|$ is a bra vector corresponding to $|a\rangle$

& $\langle b|$ is a bra vector corresponding to $|b\rangle$

then $|a\rangle + |b\rangle \Rightarrow \langle a| + \langle b|$

*② $z|A\rangle \not\Rightarrow \langle A|z$

$\because z \in \mathbb{C}$ and $|A\rangle \in V(\mathbb{C})$

$z^* \in \mathbb{C}^*$ and ~~$\langle A| \in V(\mathbb{C}^*)$~~

$\therefore \langle A| \in V^*(\mathbb{C}^*)$

$z|A\rangle \Rightarrow \langle A|z^*$

Vector Space $\longrightarrow$ Normed Space

*Length of a vector:

$z = x + iy$

$z^* = x - iy$

$zz^* = (x+iy)(x-iy)$

$= x^2 - i^2y^2$

$$\boxed{zz^* = x^2 + y^2}$$

$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$$\boxed{||\psi\rangle| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}$$

* Vector with length $= 1 \Rightarrow$ Unit Vector

$aa^* = a^2$

Ex:- $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$||\psi\rangle| = \sqrt{1^2 + 0^2} = 1$

Few more examples of unit vectors:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\leftarrow\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\nearrow\rangle = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$

$$|\rightarrow\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \qquad |\swarrow\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

A vector can be made a unit vector by —

$$|\psi\rangle = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$||\psi\rangle| = \sqrt{10}$$

$$\text{Unit vector} = \underbrace{\{\,\}}_{\text{Normalizing factor}} |\psi\rangle$$

$$\text{Unit vector} = \frac{1}{||\psi\rangle|}\,|\psi\rangle$$

$$\text{unit vector} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$ → Scalar multiplication allowed because $|\psi\rangle$ is an element of V.S

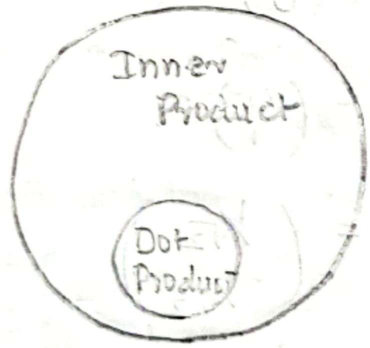
$$|\text{unit vector}| = \sqrt{\frac{9}{10} + \frac{1}{10}} = \underline{\underline{1}}$$

Vector Space $\rightarrow$ Normed Space $\rightarrow$ Inner product Space

★ $\langle \phi | \psi \rangle = C$

$\downarrow$

This encodes information about how similar the two vectors (bra & ket) are.



★ Follows two properties:

① Linearity:

$$\langle \psi | \{ |\phi\rangle + |\chi\rangle \} = \langle \psi | \phi \rangle + \langle \psi | \chi \rangle$$

② Conjugate property:

$$\langle A | B \rangle = \langle B | A \rangle^{*}$$ (Complex conjugation)

★ $\langle A | A \rangle$ is always a real number (Some exceptions)

Important points:

① Orthogonality:

★ $\langle A | B \rangle = 0 \Rightarrow \langle A |$ & $| B \rangle$ are ~~a~~ independent of each other / not similar

② Normalization:

If $\langle A | A \rangle = 1$, then A is a normalized vector

③ Orthonormality:

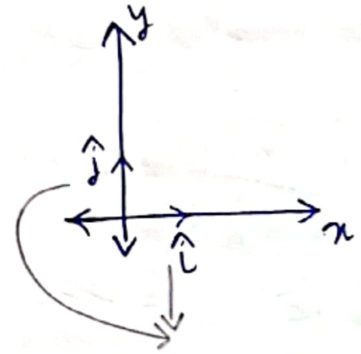
$\langle A | B \rangle = \delta_i^j$ if $0 \Rightarrow$ A & B are different

$1 \Rightarrow$ A & B are same

★ Basis = Spanning Set

$<\hat{i}|\hat{i}> = 1$

$<\hat{i}|\hat{j}> = 0$



Basis Vectors

Let V(F) is a vector space over the field F

$\{|i>\} \longrightarrow$ Orthonormal Basis (Sequence of orthonormal vectors) (In Quantum Mechanics, basis vector must be orthonormal)

(Now, any vector can be written in terms of the basis)

$$|A> = \sum_i \alpha_i |i> \longrightarrow ①$$

$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{A}\cdot\hat{i} = 2\hat{i}\cdot\hat{i} + \cancel{3\hat{j}\cdot\hat{i}} + \cancel{4\hat{k}\cdot\hat{i}}$

$\vec{A}\cdot\hat{i} = 2$

$\Rightarrow \vec{A} = (\vec{A}\cdot\hat{i})\hat{i} + (\vec{A}\cdot\hat{j})\hat{j} + (\vec{A}\cdot\hat{k})\hat{k}$

$\times^{ly}$ both sides with $<j|$

$$<j|A> = <j| \sum_i \alpha_i |i>$$

$$<j|A> = \sum_i \alpha_i <j|i>$$

$<j|i> = 1$ if $j = i$

$\Downarrow$

$\boxed{\text{Collapse}}$

$$<j|A> = \alpha_j$$

Substituting in eq^n ①

$$\Rightarrow \boxed{|A> = \sum_i <i|A> |i>}$$

Some example questions:

① $|\uparrow> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|\downarrow> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

} Standard Basis $\rightarrow$ used for spin of $e^{\ominus}$

$$<\uparrow|\uparrow> = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{1}$$

$\langle\downarrow|\downarrow\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\underline{1}}$

$\langle\uparrow|\downarrow\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{0}$

② $|\rightarrow\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$|\leftarrow\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

③ $|\nearrow\rangle = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$

$|\swarrow\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$

<u>H.W</u> $\langle\rightarrow|\rightarrow\rangle = \langle\leftarrow|\leftarrow\rangle = \langle\nearrow|\nearrow\rangle = \langle\swarrow|\swarrow\rangle = 1$

$\langle\leftarrow|\rightarrow\rangle = \langle\swarrow|\nearrow\rangle = 0$

★ Orthonormal Basis → Span the complete Vector Space.

⇒ Using orthonormal basis, any vector in the vector space can be written.

Linear Combination:

Any vector from a vector space can be written as the linear combination of the basis vectors of the vector Space.

$|v\rangle \in V(F)$

$\{|\uparrow\rangle, |\downarrow\rangle\} \rightarrow$ Orthonormal Basis

<u>Ex:-</u> ① $|v\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

$|v\rangle = x_1|\uparrow\rangle + x_2|\downarrow\rangle$

$\begin{pmatrix} c \\ d \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow x_1 = c$$
$$x_2 = d$$

Another example of basis:

$$|\rightarrow\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \qquad |\leftarrow\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Let $|v\rangle = x_1|\rightarrow\rangle + x_2|\leftarrow\rangle$

$\times^{ly}$ both sides by $\langle\leftarrow|$

$$\Rightarrow \langle\leftarrow|v\rangle = \cancel{x_1\langle\leftarrow|\rightarrow\rangle}^{0} + x_2\langle\leftarrow|\leftarrow\rangle$$

$$\Rightarrow x_2 = \langle\leftarrow|v\rangle$$
$$= (1/\sqrt{2} \;\; 1/\sqrt{2})\begin{pmatrix} c \\ d \end{pmatrix}$$
$$= \frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}}$$
$$= \frac{c+d}{\sqrt{2}}$$

$|||^{ly}$ $x_1 = \langle\rightarrow|v\rangle$
$$= (1/\sqrt{2} \;\; -1/\sqrt{2})\begin{pmatrix} c \\ d \end{pmatrix}$$
$$= \frac{c-d}{\sqrt{2}}$$

$$\Rightarrow |v\rangle = \left(\frac{c-d}{\sqrt{2}}\right)|\rightarrow\rangle + \left(\frac{c+d}{\sqrt{2}}\right)|\leftarrow\rangle$$

★ Considering ∞-dimensional vector space :

$$|\psi\rangle = a_1|b_1\rangle + a_2|b_2\rangle + \dots + a_n|b_n\rangle \longrightarrow ①$$

X^{ly} $\langle b_1|$ on b.s

$$\langle b_1|\psi\rangle = a_1\langle b_1|b_1\rangle + \underbrace{a_2\langle b_1|b_2\rangle + \dots + a_n\langle b_1|b_n\rangle}_{\text{(collapse)}} \rightarrow 0$$

$$\langle b_1|\psi\rangle = a_1$$

⇒ Eqn ① becomes

$$\boxed{|\psi\rangle = \langle b_1|\psi\rangle|b_1\rangle + \langle b_2|\psi\rangle|b_2\rangle + \dots + \langle b_n|\psi\rangle|b_n\rangle}$$

Probability Amplitude

$$\underbrace{\langle b_2|\psi\rangle\langle\psi|b_2\rangle^*}_{\text{Square of Probability Amplitude}} \rightarrow \text{Probability of } |\psi\rangle \text{ jumping to } |b_2\rangle.$$

★ Linear Dependence :

A set of vectors $\{|v_1\rangle, |v_2\rangle, \dots |v_n\rangle\}$ is Linearly Dependent if ∃ scalars $c_1, c_2, \dots c_n$ such that at least one of them is non-zero and $c_1|v_1\rangle + c_2|v_2\rangle + \dots + c_n|v_n\rangle = 0$, then the set of vectors are linearly dependent.

If $c_1|v_1\rangle + c_2|v_2\rangle + \dots + c_n|v_n\rangle = 0$ if $c_1 = c_2 = \dots = c_n = 0$, then the set of vectors is called Linearly Independent.

Ex:- $(2\ -1)$, $(3\ 2)$, $(4\ 3) \longrightarrow$ L.I ?

$$c_1(2\ -1) + c_2(3\ 2) + c_3(4\ 3) = (0, 0)$$

$$(2c_1 + 3c_2 + 4c_3 \quad -c_1 + 2c_2 + 3c_3) =$$

* Basis
  - Spans the entire V.S $\Rightarrow$ Orthonormality
  - Linear Independence

<u>Note</u>:- * Any two sets of linearly independent elements which span the Vector Spaces, they have same number of elements.

* A vector space can have ∞ no. of distinct basis vectors, but they all have same no. of elements

<u>Ex</u>:- $|\nearrow\rangle, |\swarrow\rangle$ $\quad$ $|\uparrow\rangle, |\downarrow\rangle$ $\quad$ $|\leftarrow\rangle, |\rightarrow\rangle$

* No. of elements in Basis set = Dimension of Vector Space

* In Quantum Computing $\rightarrow$ Finite Dim. Space
  Quantum Mechanics $\rightarrow$ ∞ Dim. Space

* Ordered Basis:
  $\{|\uparrow\rangle, |\downarrow\rangle\} \neq \{|\downarrow\rangle, |\uparrow\rangle\}$

MATRICES

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{3\times 3}$$

① Square Matrix

② Row Matrix

$\langle a| = [a \quad b \quad c]$

③ Column Matrix

$$|a\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

④ Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AI = A = IA$ $\quad$ ($AB \neq BA$ in general)

⑤ Transpose

$$A^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

⑥ Trace of Matrix

$Tr(A) =$ Sum of diagonal elmts
$= a + e + i$

⑦ Symmetric Matrix:

$A^T = A$

<u>Ex:-</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

⑧ Hermitian Matrix

$A^{\dagger} = A$

$A^{\dagger} = (\bar{A})^T$

<u>Ex:-</u> $A = \begin{pmatrix} 3 & 1-i \\ 1+i & -2 \end{pmatrix}$

$\bar{A} = \begin{pmatrix} 3 & 1+i \\ 1-i & -2 \end{pmatrix}$

$(\bar{A})^T = \begin{pmatrix} 3 & 1-i \\ 1+i & -2 \end{pmatrix}$

$= \underline{A}$

<u>Note:-</u> Diagonal elements of HM is always real

⑨ Orthogonal Matrices:

$AA^T = I = A^T A$

Rows & columns form an orthonormal set of vectors:

<u>Ex:-</u> ① $A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$

$\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} (-\sin x \quad \cos x)$

$= -\cos x \sin x + \sin x \cos x = \underline{0}$

$\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} (\cos x \quad \sin x) = \cos^2 x + \sin^2 x = \underline{1}$

$A^T = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$

$AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ex:- ② $A = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$ $A^T = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$

$$A A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⑩ Unitary Matrices

$$A A^{+} = I = A^{+} A$$

Ex:- ① $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$

$$A^{+} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

$$A A^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex:- ② $B = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \Rightarrow \bar{B} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \Rightarrow (\bar{B})^T = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$

Q. $A = \{|b_1\rangle, |b_2\rangle \dots |b_n\rangle\}$

Check if A forms basis set

① Check normalization

$\langle b_1 | b_1 \rangle = 1$

② Orthogonality

$A^{+} A = I$

$$\underset{n\times 1}{\begin{bmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_n| \end{bmatrix}} \begin{bmatrix} |b_1\rangle & |b_2\rangle & \dots & |b_n\rangle \end{bmatrix} = \underset{n\times n}{\begin{bmatrix} \overset{1}{\cancel{\langle b_1|b_1\rangle}} & \langle b_1|b_2\rangle & \dots & \langle b_1|b_n\rangle \\ \langle b_2|b_1\rangle & \overset{1}{\cancel{\langle b_2|b_2\rangle}} & \dots & \\ \vdots & & & \\ \langle b_n|b_1\rangle & \dots & \dots & \overset{1}{\cancel{\langle b_n|b_n\rangle}} \end{bmatrix}}$$

$= I$

LINEAR OPERATORS :

→ States in Quantum Mechanics are represented by vectors in V.S

→ Physical observables are represented by Linear Operators in Vector Space.

↓

For now we will take it as an axiom

Energy, position, angular momentum, momentum

→ Linear operators are also associated with a vector Space but they are not vectors.

(Think of Linear operators as a Machine that transforms)

→ Not every operator is linear

→ If anything goes into the linear operator (machine), then something must come out.

If it doesn't give an output, it isn't a linear operator.

Axioms for linearity:

① $M\{z|A\rangle\} = zM|A\rangle$

② $M\{|A\rangle + |B\rangle\} = M|A\rangle + M|B\rangle$

★ Row and column representation of vectors and linear transformations is dependent on the chosen basis

$$|A\rangle = \sum_i a_i |i\rangle$$

$$M|A\rangle = |B\rangle$$

~~X'lying both sides with k~~

$$M\sum_i a_i|i\rangle = \sum_i b_i|i\rangle$$

X'ly both sides with $\langle k|$

$$\sum_i a_i\langle k|M|i\rangle = \sum_i b_i\langle k|i\rangle$$

vector

$\langle k|M|i\rangle$

Complex Number

$$\sum_i a_i m_{ki} = b_k$$

Now m_{ki} is an element of the matrix M, where M is the Linear Operator.

Dimensions of M depend on the basis vectors we choose.

$$\underbrace{\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{|A\rangle} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{|B\rangle}$$

$$M \; |A\rangle = |B\rangle$$

Note:- Relationship b/w vectors and operators is independent of basis we choose when dealing with abstract notations.

★ Identity Operator

$I_v |v\rangle = |v\rangle \quad , \quad |v\rangle \in V(F)$

★ zero Operator

$O|v\rangle = O \quad , \quad |v\rangle \in V(F)$

★ $V: A \rightarrow B$, $W: B \rightarrow C$ where V and W both are linear operators

$AB(|v\rangle) = A(B|v\rangle)$

Linear operators $\equiv$ Matrices (But for this we have to have specific input and output basis of vector Space).

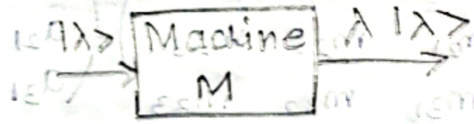
EIGEN VALUES & EIGEN VECTORS:

→ In general, a linear operator changes the direction of vector (input)

→ But there are certain Linear Operators that preserve the direction of Input Vector

$$\boxed{M|\lambda\rangle = \lambda|\lambda\rangle}$$

λ → complex no.



Ex: ① $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad |v\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$M|v\rangle = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M|v\rangle = 3|v\rangle$$

3 → Eigen value

② $|u\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$M|u\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$M|u\rangle = -1|u\rangle$$

$\frac{-1\times i}{i\times i} = \frac{+i}{+1}$

③ $|u'\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$N|u'\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ \frac{-1}{i} \end{pmatrix}$$

$$= -i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\Rightarrow N|u'\rangle = -i|u'\rangle$$

* $M|v\rangle \not\Rightarrow \langle v|M$

We must find the complex conjugate of M

Dual space में जाते ही conjugate बन जाते हैं

$z|A\rangle = |B\rangle \Rightarrow \langle A|z^* = \langle B|$

Finding complex conjugation of M

$M|A\rangle = |B\rangle$

$M \sum_i a_i |i\rangle = \sum_i b_i |i\rangle$

$\sum a_i M|i\rangle = \sum b_i |i\rangle$

$\sum a_i \langle j|M|i\rangle = \sum b_i \langle j|i\rangle$

$\sum a_i m_{ji} = b_j$

In dual space;

$\sum a_i^* m_{ji}^* = b_j^*$

$\Rightarrow$ $M|A\rangle = |B\rangle$

$\Downarrow$

$\langle A|M^\dagger = \langle B|$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

M dagger

Hermitian conjugate

Note:- ① Outcome of an experiment is always a real no.

② Observables in Quantum Mechanics are Hermitian Operators.

$M = M^\dagger$

$\Rightarrow$ All Eigen values are real

Proof: $M|\lambda\rangle = \lambda|\lambda\rangle$

$\langle\lambda|M^\dagger = \lambda^* \langle\lambda|$

x'lying $|\lambda\rangle$ on b.s

$\langle\lambda|M^\dagger|\lambda\rangle = \lambda^* \langle\lambda|\lambda\rangle$

$\ominus \langle\lambda|M|\lambda\rangle \overset{\ominus}{=} \lambda\langle\lambda|\lambda\rangle$ ($\because$ M is Hermitian $\Rightarrow M^\dagger = M$)

$0 = \langle\lambda|\lambda\rangle(\lambda^* - \lambda)$

$\Rightarrow 0 = (\lambda^* - \lambda)(\langle \lambda | \lambda \rangle)$

$\Rightarrow \lambda^* = \lambda$

$\Rightarrow$ All Eigen values are real

FOURIER TRANSFORM

↳ A way to see a funcⁿ from different perspective

Fourier Series :

→ Fourier series is used to express a complex periodic function into the sum of sines and cosines, each with its own amplitude & frequencies.

$$\underbrace{f(x) = f(x + T)}_{\text{Periodic function}} \quad T \rightarrow \text{Time period}$$

Ex: $\sin(0) = 0$

$\sin(2\pi + 0) = 0$

(Here sines and cosines work as basis)

A

Ex:- $f(x) = 1 \sin \omega t + \frac{1}{2} \sin 2\omega t + \frac{3}{2} \cos \omega t$

$\omega = 2\pi \nu$

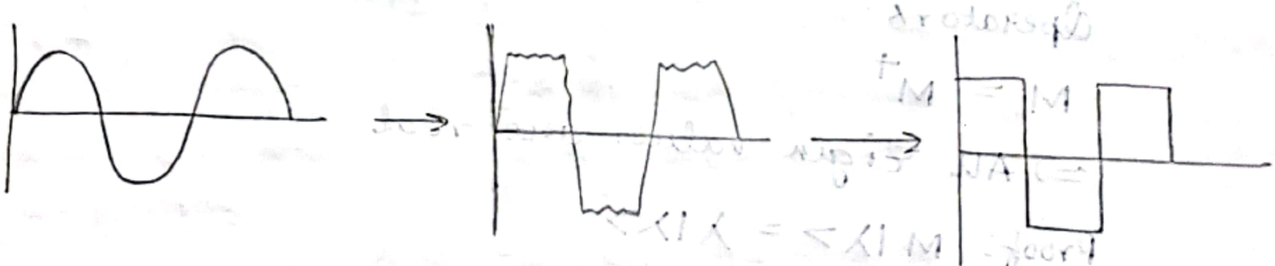
ω → Angular velocity

ν → Frequency (no. of waves per second)

$c = \lambda \nu$

$\omega = \frac{\theta}{t}$ (θ → Angular displacement)

$\Rightarrow \theta = \omega t$



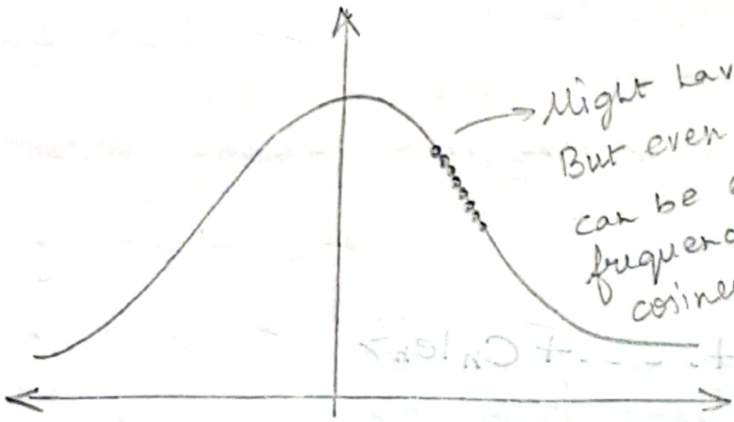
Heisenberg's uncertainty principle ⟶ Heart of Q.M

Fourier Transform ⟶ Heart of Heisenberg principle

~ Rajan Chopra

Fourier Transform:

→ Expressing a non-periodic function in terms of its frequency content.



→ Might have to consider ∞ points
But even this non-periodic func. can be expressed in terms of frequency content (using sines, cosines or e)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$z = x + iy \Rightarrow |z| = \sqrt{x^2+y^2} = 1$

x	y	Power of i
1	0	1
0	1	i
-1	0	-1
0	-1	$-i$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$(x-h)^2 + (y-k)^2 = r^2$

$x^2 + y^2 = 1$

↳ When using Fourier Transform we express funcⁿ in terms of exponential (e).

→ A non-periodic function can be thought of as having a continuous spectrum of frequencies.

↓

can be represented as a continuous sum (integral) of sines, cosines or exponential ($e^{i\theta}$)
(even maybe by using almost all possible frequencies)

L^2 Space

→ Square Integrable functions

$$\int_{-\infty}^{\infty} |f|^2 \, dx < \infty$$

(Finite)

In Linear Algebra,

$$|v\rangle = c_1|e_1\rangle + c_2|e_2\rangle + \dots + c_n|e_n\rangle$$

$|e_i\rangle$ → Basis element

c_i → Coefficients

↓

These coefficients tell us how much each basis element contributes to regenerate the original vector

In Fourier Series,

$f(t)$ → Periodic function

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \sin(n\omega_0 t) + b_n \cos(n\omega_0 t)]$$

* Why use just sines and cosines?

Orthogonality

↓

Sine and cosine are independent of each other (Like Orthogonal vectors in v.s)

Completeness

↓

Set of sine & cosine is a complete set

↓

Can generate a wide variety of functions, specially from L^2 space

$\star\ e^{i\omega t} = \cos \omega t + i \sin \omega t$

$$\star\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}$$

$$= \cos\theta$$

$$\star\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \sin(n\omega_0 t) + b_n \cos(n\omega_0 t)]$$

$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) + b_n \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}\right) + b_n \left(\frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n}{2i} e^{in\omega_0 t} - \frac{a_n}{2i} e^{-in\omega_0 t} + \frac{b_n}{2} e^{in\omega_0 t} + \frac{b_n}{2} e^{-in\omega_0 t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n + ib_n}{2i} e^{in\omega_0 t} + \frac{ib_n - a_n}{2i} e^{-in\omega_0 t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{ia_n - b_n}{-2} e^{in\omega_0 t} + \frac{-b_n - ia_n}{-2} e^{-in\omega_0 t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{b_n - ia_n}{2} e^{in\omega_0 t} + \frac{b_n + ia_n}{2} e^{-in\omega_0 t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[C_n e^{in\omega_0 t} + C_{-n} e^{-in\omega_0 t}\right]$$

$$\boxed{f(t) = \sum_{-\infty}^{\infty} C_n e^{in\omega_0 t}}$$

where, $C_n \rightarrow$ Complex Exponential Fourier coefficient

★ Derive $C_n = \frac{1}{T}\int_0^T f(t) e^{-in\omega_0 t}\, dt$.

W.K.T $f(t) = \sum_{-\infty}^{\infty} C_n e^{in\omega_0 t}$

Soln. Let $f(t)$ be a periodic function with period T

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega_0 t}$$

① x'ly by $e^{-in\omega_0 t}$ (for a fixed integer n) and integrate over a period 0 to T

$$\Rightarrow e^{-in\omega_0 t} \cdot f(t) = \sum_{k=-\infty}^{\infty} C_k e^{i(k-n)\omega_0 t}$$

$$\int_0^T e^{-in\omega_0 t} \cdot f(t) \cdot dt = \int_0^T \left(\sum_{k=-\infty}^{\infty} C_k e^{i(k-n)\omega_0 t}\right) \cdot dt$$

② Exchange sum and integral (Fubini's theorem):

$$\int_0^T e^{-in\omega_0 t} \cdot f(t) \cdot dt = \sum_{k=-\infty}^{\infty} C_k \int_0^T e^{i(k-n)\omega_0 t} \cdot dt$$

③ Use $\underbrace{\text{orthogonality of complex exponentials}}$

★ Any two complex exponentials $e^{ik\omega_0 t}$ and $e^{in\omega_0 t}$ are orthogonal over a period T.

$\Rightarrow$ Inner product (integral of their product over one period) is zero if $k \neq n$ (they have different frequencies)

$\Rightarrow$ Only when $k = n$ (same frequency), their inner product is non-zero ($= T$).

$$\Rightarrow \int_0^T e^{i(k-n)\omega_0 t} \cdot dt = \begin{cases} T, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

★ If $k \neq n$:

$$\Rightarrow \int_0^T e^{i(k-n)w_0 t}\cdot dt = \frac{\left[e^{i(k-n)w_0 t}\right]_0^T}{i(k-n)w_0}$$

$$= \frac{e^{i(k-n)w_0 T} - e^0}{i(k-n)w_0}$$

$$= \frac{e^{i(k-n)\frac{2\pi}{\cancel{T}}\cdot \cancel{T}} - 1}{i(k-n)w_0}$$

$$= \frac{e^{i(k-n)2\pi} - 1}{i(k-n)w_0}$$

$\cos(n\pi) = (-1)^n$

$\sin(n\pi) = 0$

$$= \frac{\cos 2\pi(k-n) + i\sin 2\pi(k-n) - 1}{i(k-n)w_0}$$

$$= \frac{1+0-1}{i(k-n)w_0}$$

$$= \underline{\underline{0}}$$

$\Rightarrow$ When $k = n$

④ $$\int_0^T e^{-inw_0 t}\cdot f(t)\cdot dt = \sum_{k=-\infty}^{\infty} C_k \int_0^T 1\cdot dt$$

$$\int_0^T e^{-inw_0 t}\cdot f(t)\cdot dt = C_n \cdot T$$

(All terms vanish except when $k=n$).

⑤ $$\boxed{C_n = \frac{1}{T}\int_0^T e^{-inw_0 t}\cdot f(t)\cdot dt}$$

★ Why $e^{i\omega t}$ is chosen?

Orthogonality
→ $e^{i\omega_0 t}, e^{-2i\omega_0 t}$ ---
all are independent of each other

Completeness
→ Any well behaved func^n can be expressed as the Linear Combination of $e^{in\omega_0 t}$

Periodic nature
→ It traces a circle in the complex plane as t varies

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

$$|v\rangle = C_1 |e_1\rangle + C_2 |e_2\rangle + \cdots$$

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$e^{i\omega t}$ → basis func^n's (combination of sines and cosines)

$\tilde{f}(\omega)$ → Coefficient (tells us how much a particular $e^{i\omega t}$ contributes to regenerate the original function)
f tilde

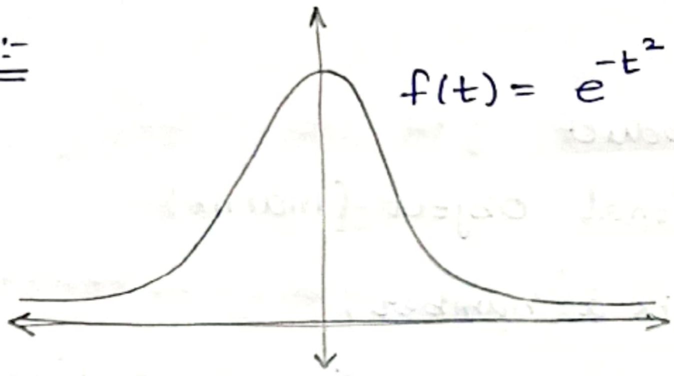
$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Original function:

$$\rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$\frac{1}{2\pi}$ → Normalizing factor

Ex:-



$f(t) = e^{-t^2}$

Amplitude of freq component in original function

$$\tilde{f}(w) = \int_{-\infty}^{\infty} f(t) \cdot e^{-iwt} \cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-iwt} \cdot dt$$

$$\tilde{f}(w) = \frac{1}{\sqrt{\pi}} e^{\frac{-w^2}{4}}$$

$$\Rightarrow e^{-t^2} = \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{\pi}} e^{\frac{-w^2}{4}}}_{\text{coeff}^{n}} \underbrace{e^{iwt}}_{\text{basis function}} \cdot dw$$

Like $\hat{i}, \hat{j}$ and $\hat{k}$ span all 3D space

e^{iwt} can span all functions of L^2 space (especially)

$$\boxed{\tilde{f}(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt} \cdot dt}$$

→ This transforms time domain func^n to frequency domain function

Inverse fourier transform;

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(w)\, e^{iwt} \cdot dw}$$

→ This transforms frequency domain func^n to time domain func^n

Outer Product

→ Specific type of tensor product

→ Results a higher dimensional object (Matrix)

→ Unlike inner product results a number,
Outer Product results a matrix (higher dimensional object)

→ Ex:- $|\psi\rangle\langle\phi|$
↳ Higher dimensional space

→ $|u\rangle \in \mathbb{R}^n$

$|v\rangle \in \mathbb{R}^m$

$|u\rangle\langle v| \in \mathbb{R}^{n+m}$

$$(\ )_{n\times 1}\ (\ )_{1\times m} = (\ )_{n\times m}$$

→ Outer product preserves the full structure of both the vectors (Vectors do not lose their properties)

→ It's a way of "spreading out" the interaction between two vectors in higher dimensional space.

→ Unlike dot product which collapses into a single number.

→ In QM, it is used to represent Projection Operator.

$|\psi\rangle$ → Represents state of a particle in Hilbert Space.

$|\psi\rangle\langle\phi|$ → Outer product

Ex:- $(|\psi\rangle\langle\phi|)|\chi\rangle = |\psi\rangle \langle\phi|\chi\rangle$ → number

→ First $\langle\phi|$ is projected on $|\chi\rangle$ and then the vector $|\psi\rangle$ is scaled using the number obtained after projection ($\langle\phi|\chi\rangle$)

→ Projection operator performs two tasks; projection + scaling

Representation of Linear Operator as Outer Product

$|u\rangle \in$ IPS $\quad V(F)$

$|v\rangle \in$ IPS $\quad V(F)$

$$(|u\rangle\langle v|)|w\rangle = \langle v|w\rangle |u\rangle$$

$|u\rangle\langle v|$ acting on $|w\rangle$

Scaling $|u\rangle$ by $\langle v|w\rangle$

→ We will define OP such that both definitions are equivalent

→ Considering $\boxed{\sum_i a_i |u_i\rangle\langle w_i|}$ as a linear operator acting on $|v\rangle$ then $\left(\sum_i a_i |u_i\rangle\langle w_i|\right)|v\rangle$ gives $\sum a_i \langle w_i|v\rangle |u_i\rangle$

→ Let $|i\rangle$ → Orthonormal Basis for VS $V(F)$

⇒ Any vector $|v\rangle \in V(F)$ can be written as

$|v\rangle = \sum v_i |i\rangle$, $v_i \in \mathbb{C}$ (Assume $|v\rangle$ is an orthogonal vector)

↳ Dot product = 0, Perpendicular, Linear independence

$\langle i|v\rangle = v_i$

$$\left(\sum |i\rangle\langle i|\right)|v\rangle = \sum |i\rangle\langle i|v\rangle = \sum v_i |i\rangle = |v\rangle$$

$$\Rightarrow \boxed{\left(\sum |i\rangle\langle i|\right) = I} \rightarrow \text{Completeness Relation}$$

1. Completeness Relation is used to give the means of representing a Linear Operator as Outer Product.

Let A be a Linear operator mapping vectors from Vector Space V to Vector Space W

$A : V \rightarrow W$

$|v_i\rangle \rightarrow$ Orthonormal basis of V

$|w_j\rangle \rightarrow$ Orthonormal basis of W

$$A = I_V \, A \, I_W$$

$$= \left(\sum_i |v_i\rangle\langle v_i|\right) A \left(\sum_j |w_j\rangle\langle w_j|\right)$$

Matrix w.r.t input & output basis

$$A = \sum_{ij} |v_i\rangle\langle w_j| \, \langle v_i|A|w_j\rangle$$

(Can combine because indices are different (i and j))

↳ Outer product representation of A

So far, Linear Operators can be represented as;

① Abstract Notation ② Matrix Notation ③ Outer product

2. Cauchy Schwarz Inequality:

→ Tells an important point about Hilbert Space

→ Any two vectors that belong to the Hilbert

$|u\rangle$ & $|v\rangle \in$ H.S

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$$