Phys512: Assign-4

Hussain Rasiwala

October 2020

Problem 5 Part-a

Formula for sum of a geometric series follows as follows:

$$\sum_{n=0}^{N-1} c\alpha^n = c\frac{\alpha^N - 1}{\alpha - 1} \tag{1}$$

Similarly we are asked to calculate:

$$\sum_{x=0}^{N-1} exp(-2\pi\iota kx/N) = ? \tag{2}$$

So for,

$$\alpha = exp(-2\pi\iota k/N) \tag{3}$$

$$\sum_{x=0}^{N-1} exp(-2\pi\iota kx/N) = \sum_{x=0}^{N-1} exp(-2\pi\iota k/N)^x = \sum_{x=0}^{N-1} c\alpha^x = c\frac{\alpha^N - 1}{\alpha - 1} \tag{4}$$

Therefore, (with c = 1)

$$\sum_{x=0}^{N-1} exp(-2\pi\iota k/N)^x = \frac{exp(-2\pi\iota k/N)^N - 1}{exp(-2\pi\iota k/N) - 1} \tag{5}$$

or,

$$\sum_{x=0}^{N-1} exp(-2\pi\iota k/N)^x = \frac{1 - exp(-2\pi\iota k/N)^N}{1 - exp(-2\pi\iota k/N)} \tag{6}$$

Problem 5 Part-b

To show the value of the series sum in the limit k-¿0, we can simply apply L'Hospital rule as follow:

$$
\begin{aligned}
\lim_{k\to 0}\sum_{x=0}^{N-1} exp(-2\pi\iota xk/N) &= \frac{\frac{d}{dk}(1-exp(-2\pi\iota k/N)^N)}{\frac{d}{dk}(1-exp(-2\pi\iota k/N))}\\
&= \frac{-(-2\pi\iota)exp(-2\pi\iota k)}{-(-2\pi\iota)exp(-2\pi\iota k)/N}\\
&= \frac{-(-2\pi\iota)}{-(-2\pi\iota/N)}\\
&= N
\end{aligned}
$$

Here in the third step we took $exp(-2\pi\iota k/N) = exp(-2\pi\iota k) = 1$ which happens for k = 0,N due to the periodicity of N. For any other value of k we need not take any limit. For integer values of k (other than 0,N): $exp(-2\pi\iota k) = 1$ which makes the numerator go to zero, while the denominator term doesnt due to 1/N in the exponent. Thus, for integral values of k (other than 0,N,2N,...)

$$
\begin{aligned}
\sum_{x=0}^{N-1} exp(-2\pi\iota xk/N) &= \frac{\frac{d}{dk}(1-exp(-2\pi\iota k/N)^N)}{\frac{d}{dk}(1-exp(-2\pi\iota k/N))}\\
&= \frac{1-1}{1-exp(-2\pi\iota k)/N}\\
&= 0
\end{aligned}
$$