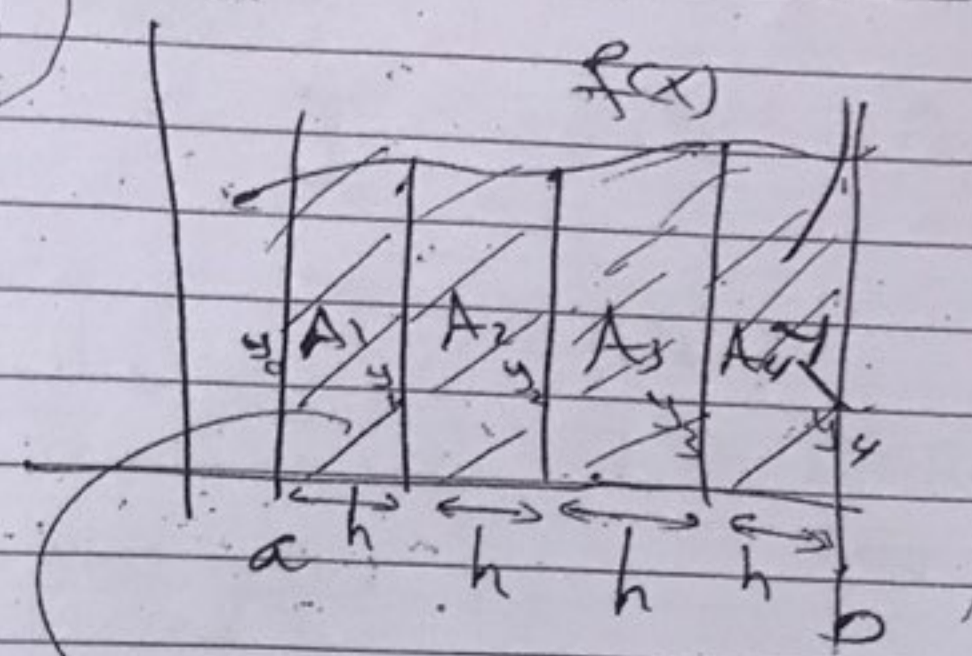
① Kiyam Muhammed

Numerical Integrals

① Trapezoidal method

$$\int_a^b f(x)\,dx = \sum A = T$$

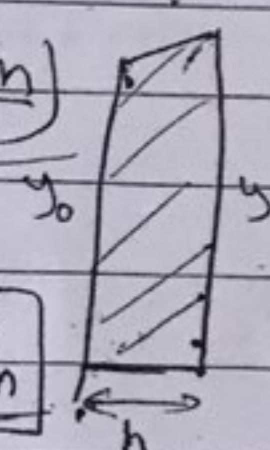


n = no. of divisions

$$h = \frac{b-a}{n}$$

عامة $T = \left(\frac{y_0+y_1}{2}\right)h + \left(\frac{y_1+y_2}{2}\right)h$ أو رسم

$+ \left(\frac{y_2+y_3}{2}\right)h + \cdots h\left(\frac{y_{n-1}+y_n}{2}\right)$



قانون المساحة تحت المنحنى الكلية $T = h/2\left[y_0 + 2y_1 + 2y_2 + 2y_3 \cdots + y_n\right]$

Ex// Use the trapezoidal rule with $n=4$ to estimate $\int_1^2 x^2\,dx$ أنه معطى (a,b) لكن أعطاك n مباشرة

ناطي n = 4

$a = 1 = x_0 \quad , \quad b = x_n = 2$

$y_0 = \frac{16}{16} \quad , \quad y_n = 2$

$h = \Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

$x_1 = a + h = \frac{5}{4} \Longrightarrow y_1 = \frac{25}{16}$

نسبة $\frac{4}{4}$ $1 + \frac{1}{4}$

$x_2 = a + 2h = \frac{6}{4} \Longrightarrow y_2 = \frac{36}{16}$

$x_3 = a + 3h = \frac{7}{2} \Longrightarrow y_3 = \frac{49}{16}$

$x_4 = b = 2 \longrightarrow y_4 = 4$

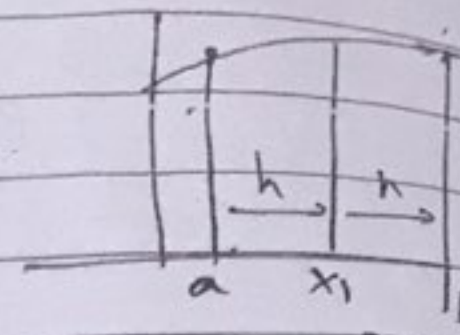
$T = h/2\,(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = \frac{75}{8} \times h$

$= 2.34375$

2 Simpson's 1/3 Rule

n عدد تقسيم
يجب ان يكون زوجي $n = \text{even}$

$$I = \int_a^b f(x)\,dx$$



$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

for general

$$I = 2h\frac{\left(f(x_0) + 4f(x_1) + f(x_2)\right)}{6} + 2h\frac{\left(f(x_2) + 4f(x_3) + \dots\right)}{6}$$

$$+ \dots + 2h\frac{\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)}{6}$$

$$I = h/3\left[f(a) + 4\sum_{i=1,3,5}^{N-1} f(x_i) + 2\sum_{j=2,4,6}^{N-2} f(x_j) + f(b)\right]$$

Ex// Use Simpson 1/3 rule for $\int_1^2 1/x\,dx = \ln 2$

$y = \frac{1}{x}$ $\quad h = \frac{2-1}{4} = 0.25$

	x	$y = 1/x$	weight factor	product
x_0	1	1	1	1
x_1	1.25	0.8	4	3.2
x_2	1.5	0.66667	2	1.33333
x_3	1.75	0.57143	4	2.28572
x_4	2	0.5	1	0.5000

Sum = 8.31905

$$\int_1^2 1/x\,dx = \frac{h}{3} \times 8.31905 = \frac{1}{12} \times 8.31905 = 0.6923$$

Simpson's 3/8 Rule

شرط n تقبل القسمة على (٣) بدون باقي

$$I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$I = \frac{(b-a)}{n} * 3\,\frac{\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]}{8}$$

$$I = h * \frac{3}{8}$$

Ex// find the area under the curve

$$y = \int_1^7 x^2\,dx \qquad \text{must } n = 6$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x_i	1	2	3	4	5	6	7
y_i	1	4	9	16	25	36	49

$$h = \frac{7-1}{n} = \frac{6}{6} = 1$$

$$I = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 3y_3 + 3y_4 + 3y_5 + y_6\right]$$

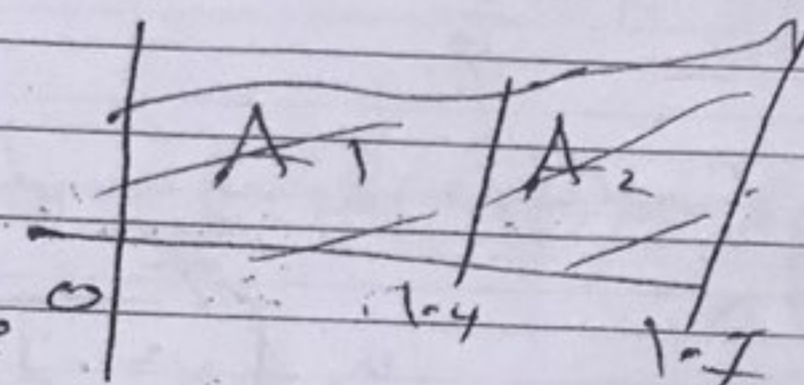
بدون الضرب برقم (3)

$I = 120$ total integral

Combined Simpson's 1/3 & Simpson's 3/8 rule

Ex Integrate the data using Simpson's Rule

i	0	1	2	3	4	5	6	7
x_i	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
f_i	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828



① Simpson's 1/3 rule

$$\int_{1.0}^{1.4} f(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + f_4\right]$$

$$I_1 = \frac{0.1}{3}\left[1.543 + 4(1.669) + 2(1.811) + 4(1.971) + 2.151\right] = 0.729200$$

② Simpson's 3/8 rule

$$\int_{1.4}^{1.7} f(x)\,dx = \frac{3h}{8}\left[f_4 + 3f_5 + 3f_6 + f_7\right]$$

$$= \frac{3 \times 0.1}{8}\left[2.151 + 3(2.352) + 3(2.577) + 2.828\right]$$

$$I_2 = 0.741225$$

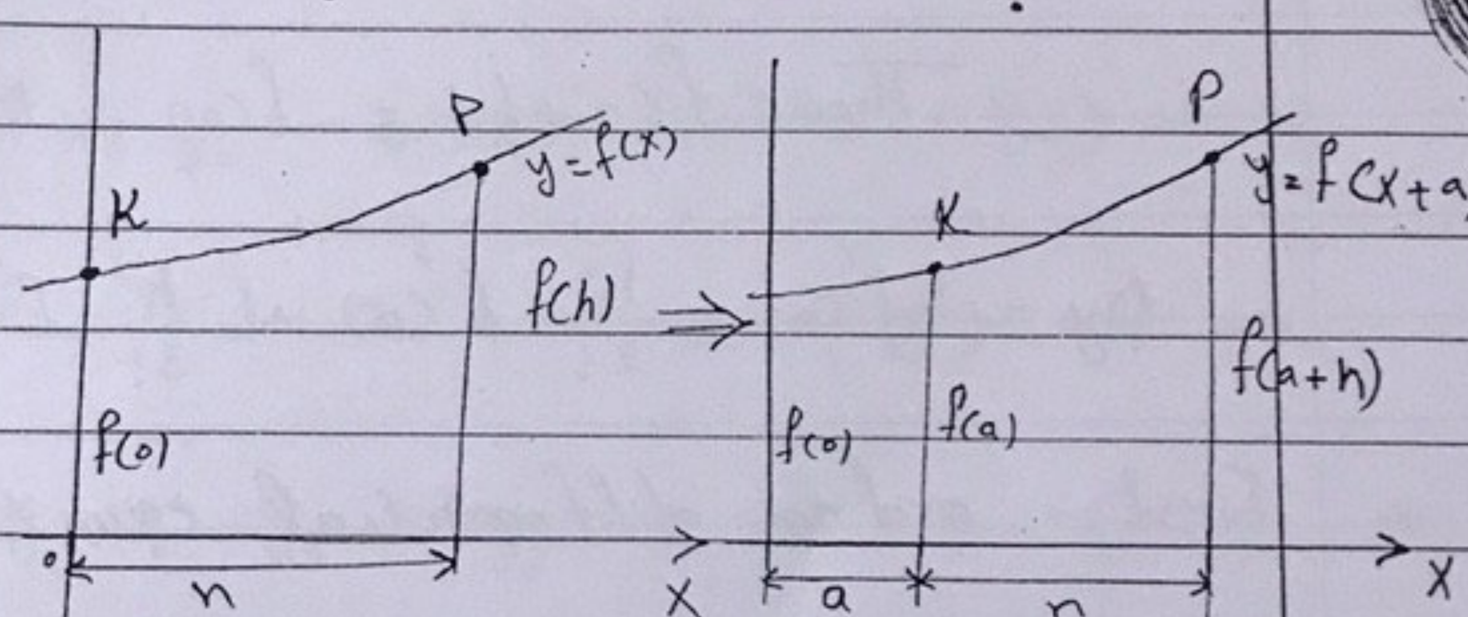
total integral $I = I_1 + I_2 = 1.470425$

"Numerical Solution of ordinary differential equation "ODE"

Taylor's Series :-

Maclaurin's series for f(x) is

$$f(x) = f(0) + f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{n}(0) + \dots$$



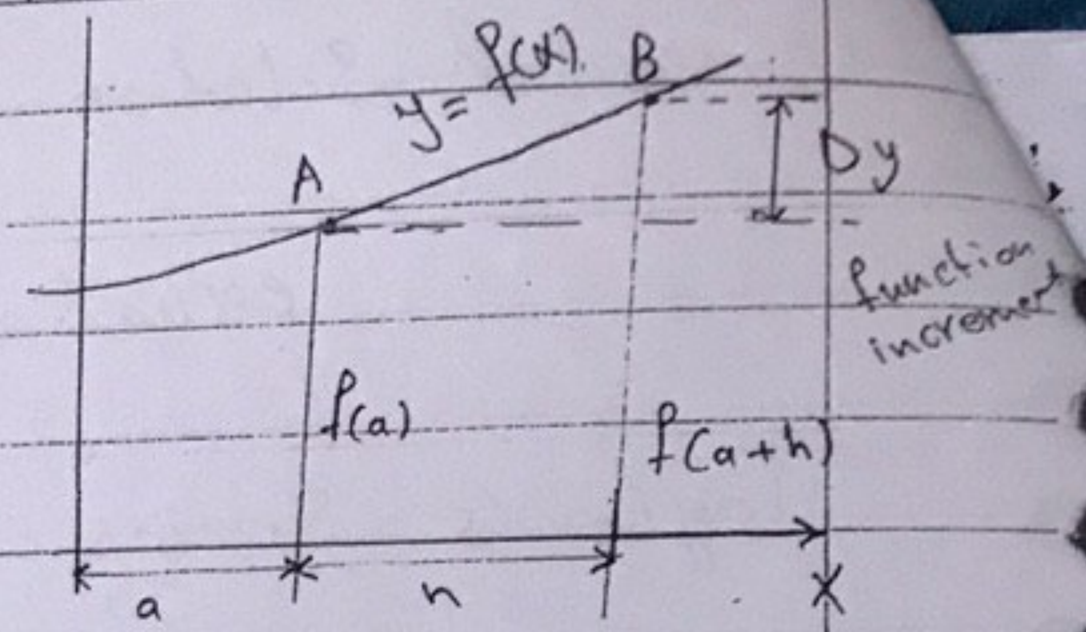
The Series will be :-

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \dots \frac{x^n}{n!} f^{n}(a) + \dots$$

oR

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots \frac{h^n}{n!} f^{n}(a) + \dots$$

(Taylor's Series).



$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\Delta y = f(a+h) - f(a)$$

$$\text{Then } f(a+h) = f(a) + \Delta y$$

$$\Delta y = h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

* First - order differential equation :-

Numerical Solution of $\left\{ \frac{dy}{dx} = f(x,y) \right\}$ with initial condition $x = x_0$; $y = y_0$

- Euler's method :-

The Simplest of the numerical method to Solve first order O.D.E .

$$f(a+h) \simeq f(a) + hf'(a)$$

$$y_1 \simeq y_0 + h y'_0$$

طريقة الحل :-

١- نحدد الدالة $\left(\frac{dy}{dx}\right)$ المعطاة في السؤال.

٢- نحدد القيم الابتدائية (x_0, y_0)

٣- نجد (y'_0) وذلك بتعويض (x_0, y_0) في الدالة المعطاة.

٤- نجد (y_1) من العلاقة التالية

$$y_1 \simeq y_0 + h y'_0$$

حيث ان قيمة (h) هي مقدار الزيادة المقترحة. نقوم

٥- نقوم باضافة (X) جديد ونجد (y'_1) جديدة بتعويض (x_1, y_1)

$$y'_1 = f(x_1, y_1) \qquad x_1 = x_0 + h$$

٦- ثم نجد (y_2) ونستمر حتى نصل لقيمة (X) المطلوبة

$$y_2 = y_1 + h y'_1$$

$$x_n = x_{n-1} + h \quad or \quad x_n = x_0 + n\,h$$

حيث (n) هو مقدار التكرار

Ex:- Given that $\frac{dy}{dx} = 2(1+x) - y$ with the initial condition that at $\underbrace{x=2}_{x_0} \Rightarrow \underbrace{y=5}_{y_0}$ take $h = 0.2$

Until $x = 3$ find y

Sol :-

$$\frac{dy}{dx} = y' = 2(1+x) - y \quad ; \quad x_0 = 2 \; ; \; y_0 = 5$$

$$y'_0 \Big|_{\substack{x_0=2 \\ y_0=5}} = 2(1+x_0) - y_0 = 2(1+2) - 5 = 1$$

$$y_1 = y_0 + h y'_0 = 5 + (0.2 * 1) = 5.2$$

n	x_n	y_n	y'_n
0	2	5	1
1	2.2	5.2	1.2
2	2.4	5.44	1.36
3	2.6	5.712	1.488
4	2.8	6.0096	1.5904
5	3	6.32768	1.67232

$$\frac{dy}{dx}\Big|_{\substack{x=x_1 \\ y=y_1}} = y'_1 = 2(1+2.2) - 5.2$$

$$y'_1 = 1.2$$

$$y_2 = y_1 + h y'_1$$

$$= 5.2 + 0.2 \times 1.2$$

$$= 5.44$$

Ex:- obtain a numerical Solution of the equation

$$\frac{dy}{dx} = 1 + x - y$$ with initial Condition

for the range $y_0 = 2$ at $x_0 = 1$.

$$X = \underbrace{1.0}_{x_0} \; \underbrace{(0.2)}_{h} \; \underbrace{3.0}_{x_n}$$

n	x_n	y_n	y'_0
0	1.0	2	0
1	1.2	2	0.2
2	1.4	2.04	0.36
3	1.6	2.112	0.488
4	1.8	2.2096	0.5904
5	2	2.32768	0.67232
6	2.2	2.462144	⋮
7	2.4	2.6097152	⋮
8	2.6	2.76777216	
9	2.8	2.934217728	
10	3	3.1073774824	

$y_1 = y_0 + h y'_0$

$y_2 = y_1 + h y'_1$

$y_n = y_{n-1} + h y'_{n-1}$

H.W obtain $\frac{dy}{dx} = x + y$ with initial Condition

$x_0 = 0$; $y_0 = 1$ For the range $x = \underbrace{0}_{x_0}\ \underbrace{(0.1)}_{h}\ \underbrace{0.5}_{x_n}$

* The Euler – Cauchy method :-

OR

The Improved Euler method.

- خطوات الحل :

١- نبدأ من (y') المعطاة في السؤال مع القيم الابتدائية (x_0, y_0)

$$y_1' = f(x, y)$$

٢- نضع الجدول التالي

n	x_n	y_n	y_0'

٣- نقوم بزيادة (x) حسب القانون التالي.

$$x_n = x_{n-1} + h$$

$$\text{OR} \quad x_n = x_0 + nh$$

٤- نجد (y_0')

$$y_0' = f(x_0, y_0)$$

٥- نجد (y_1'') من القانون التالي :

$$y_1'' = y_0 + h y_0'$$

6- نجد (y_1) من القانون التالي :-

$$y_1 = y_0 + \frac{1}{2} h \{ (y_0') + \underbrace{f(x_1, y_1^{\prime\prime})}_{\frac{dy}{dx}\big|_{x=x_1,\ y=y_1''}} \}$$

حيث أن (f) هي $\frac{dy}{dx}$ المعطاة في السؤال عندما $x = x_1$; $y = y_1''$

7- نجد (y_1')

$$y_1' = \underbrace{f(x_1, y_1)}_{\frac{dy}{dx}\big|_{x=x_1,\ y=y_1}}$$

8- نعيد الخطوات $(4-7)$ لـ x الجديد حتى نصل للقيمة

النهائية

Ex:- Apply the Euler - Cauchy method to Solve

$y' = x + y$ with initial Condition

at $x = 0$; $y = 1$ For range $x = \underbrace{0}_{x_0} \, (\underbrace{0.1}_{h}) \, \underbrace{1}_{x_n}$

Sol :-

$x_n = x_{n-1} + h$

$x_1 = x_0 + h$ oR $x_n = x_0 + nh$

$x_2 = x_1 + h$ $\qquad x_1 = x_0 + h$

$\vdots \qquad x_2 = x + 2h$

$x_3 = x + 3h$

$y'_0 = f(x_0, y_0)$

$y''_1 = y_0 + h y'_0$

$y_1 = y_0 + \frac{1}{2} h \{ y'_0 + f(x_1, y''_1) \}$

$y'_0 = f(x_0, y_0) \Big|_{\substack{x = x_0 = 0 \\ y = y_0 = 1}} = x + y = 0 + 1 = 1$

$y''_1 = y_0 + h * y'_0 = 1 + 0.1 * 1 = 1.1$

$y_1 = 1 + \frac{1}{2} * 0.1 \{ 1 + \underbrace{x_1}_{0.1} + \underbrace{y''_1}_{1.1} \} = 1.11$

$$y'_1 \Big|_{\substack{x=x_1 \\ y=y_1}} = x + y = 0.1 + 1.11 = 1.21$$

n	X_n	y_0	y'_0
0	0	1	1
1	0.1	1.11	1.21
2	0.2	1.24205	1.44205
3	0.3	1.39846025	1.69846525
4	0.4	|	
5	0.5	|	
6	0.6	|	
7	0.7	|	
8	0.8	|	
9	0.9	|	
10	1	3.4281516932	4.4281616932

H.W Solve by modified Euler method:-

a- $y' = 2(1+x) - y$

$x_0 = 2 \quad ; \quad y_0 = 5$

range $\quad 2 \; (0.2) \; 3$

b- $y' = y^2 + xy$

$x_0 = 1 \quad ; \quad y_0 = 1$

الموضوع

range

$$x = 1\ (0.1)\ 1.5$$

الموضوع

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- Runge - Kutta method :-

1- $X_n = X_0 + nh$

oR

$X_n = X_{n-1} + h$

2- $y' = X_0 , y_0$

$y_1 = y_0 + \Delta y_0$

3- $\Delta y_0 = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4$

$K_1 = h * f (x_0 , y_0)$

$K_2 = h * f (X_0 + \frac{1}{2} h , y_0 + \frac{1}{2} K_1)$

$K_3 = h * f (X_0 + \frac{1}{2} h , y_0 + \frac{1}{2} K_2)$

$K_4 = h * f (X_0 + h , y_0 + K_3)$

4- تعاد الخطوات السابقة حتى تطلع (y_2).

$x_1 , y_1 \Rightarrow y'_1 = f (x_1 , y_1)$

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الموضوع

$K_1 = h * f(x_1, y_1)$

$K_2 = h * f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1)$

$K_3 = h * f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_2)$

$K_4 = h * f(x_1 + h, y_1 + K_3)$

$$y_2 = y_1 + \Delta y_1$$

$$\Delta y_1 = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Ex :- Find the numerical Solution of

$y' = x + y$ using the Runge - Kutta method with $y = 1$ and $x = 0$ for the Values in the range $x = 0\ (0.1)\ 1.0$

Sol :-

$$K_1 = 0.1 * (\underbrace{0}_{x} + \underbrace{1}_{y}) = 0.1$$

$$K_2 = 0.1 * \left((\underbrace{0 + \tfrac{1}{2} * 0.1}_{x}) + (\underbrace{1 + \tfrac{1}{2} * 0.1}_{y})\right) = 0.11$$

KORANI

الموضوع

$K_3 = 0.1 * ((0 + \frac{1}{2} * 0.1) + (1 + \frac{1}{2} * 0.1)) = 0.1105$

$K_4 = 0.1 * ((0 + 0.1) + (1 + 0.1105)) = 0.12105$

$\Delta y_0 = \frac{1}{6}(0.1 + 2 * 0.11 + 2 * 0.1105 + 0.12105)$

$\Delta y_0 = 0.1103417$

$y_1 = y_0 + \Delta y_0 = 1 + 0.1103417 = 1.1103417$

$x_1 = x_0 + h$ 0.1324635

$= 0 + 0.1 = 0.1$

$K_1 = h * f(x_1, y_1) = 0.1 * (0.1 + 1.1103417) = 0.1210342$

$K_2 = 0.1 * f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1)$

$= 0.1 * (0.1 + \frac{1}{2} * 0.1, 1.1103417 + \frac{1}{2} * 0.1210342)$

$K_2 = 0.1320859$

$K_3 = 0.1326385$

$K_4 = 0.144298$

$\Delta y_1 = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1324635$

الموضوع

$y_2 = y_1 + \Delta y_1 = 1.24280 52$

n	x_n	y_n
0	0	1
1	0.1	1.1103417
2	0.2	1.2428051
3	0.3	1.3997170
4	0.4	1.5836485
5	0.5	1.7974413
6	0.6	2.0442359
7	0.7	2.3275033
8	0.8	2.6510791
9	0.9	3.0192028
10	1	3.43655951

Ex:- Solve $y' = \sqrt{x^2 + y}$ for $x = 0\ (0.2)\ 2$ given that $x = 0$ at $y = 0.8$

Sol :-

$x_0 = 0 \quad ; y_0 = 0.8$

$y_1 = y_0 + \Delta y_0$

$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

KORANI

$\underline{\underline{K_1}} = h * f(x_0, y_0)$

$K_1 = 0.2 * \sqrt{(0)^2 + 0.8} = 0.1788854$

$\underline{\underline{K_2}} = 0.2 * \sqrt{(x_0 + \frac{1}{2}h)^2 + (y_0 + \frac{1}{2}K_1)}$

$K_2 = 0.1896779$

$\underline{\underline{K_3}} = h * f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2)$

$K_3 = 0.1 * \sqrt{(0.1 * \frac{1}{2} * 0.2)^2 + (0.8 + \frac{1}{2} * 0.1896779)}$

$K_3 = 0.190246$

$\underline{\underline{K_4}} = 0.2 * \sqrt{(0 + 0.2)^2 + (0.8 + 0.190246)}$

$K_4 = 0.2030021$

$y_1 = y_0 + \Delta y_0 = 0.9902892$

الموضوع

n	x	Y_n
0	0	0.8
1	0.2	0.9902892
2	0.4	1.2082838
3	0.6	1.459816
4	0.8	1.7490394
5	1	2.0788983
6	1.2	2.4515074
7	1.4	2.868417
8	1.6	3.3307894
9	1.8	3.8395148
10	2	4.3952888

* Second order differential equations:-

In general we have

$$y'' = f(x, y, y') \quad \text{with}$$

$x = x_0$

$y = y_0$ and the range $x = x_0 (h) x_n$

$y' = y'_0$

1- Euler's Second - order method :-

a- $y_1 = y_0 + h y'_0 + \frac{h^2}{2} y''_0$

$$y''_0 = f(x_0, y_0, y'_0)$$

b- $y'_1 = y'_0 + h y''_0$

Ex :- Solve the equation $y'' = x y' + y$ for

$x = 0(0.2)2$ at $x_0 = 0,\ y_0 = 1,\ y'_0 = 0$

using the Euler second order method to solve this equation :-

Sol :-

$$y_1 = y_0 + h y'_0 + \frac{h^2}{2} y''_0$$

$$y''_0 \Big|_{\substack{x=0 \\ y=1 \\ y'=0}} = x y' + y = 0 * 0 + 1 = 1$$

$$y_1 = 1 + 0.2 * 0 + \frac{(0.2)^2}{2} * 1 = 1.02$$

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$$y_2 = y_1 + h y_1' + \frac{h^2}{2} y_1''$$

$$y_1' = y_0' + h y_0'' = 0 + 0.2 * 1 = 0.2$$

$$y_1'' \Big|_{\substack{x_1 = 0.2 \\ y_1 = 1.02 \\ y_1' = 0.2}} = x y' + y = 0.2 * 0.2 + 1.02 = 1.06$$

$$y_2 = 1.02 + 0.2 * 0.2 + \frac{(0.2)^2}{2} * 1.06 = 1.0812$$

n	x_n	y_n
0	0	1.02
1	0.2	1.0812
2	0.4	1.18852
3	0.6	1.3524648
4	0.8	1.5908157
5	1	
6	1.2	
7	1.4	
8	1.6	
9	1.8	
10	2	