CS4495/6495
Introduction to Computer Vision

2C-L2 *Convolution in frequency domain*

Fourier Transform and Convolution

Let $g = f * h$

Fourier Transform and Convolution

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$= F(u) H(u)$$

Convolution in spatial domain

$\Leftrightarrow$ *Multiplication in frequency domain*

Fourier Transform and Convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = F \cdot H$
$g = f \cdot h$	$\longleftrightarrow$	$G = F * H$

Fourier Transform and Convolution

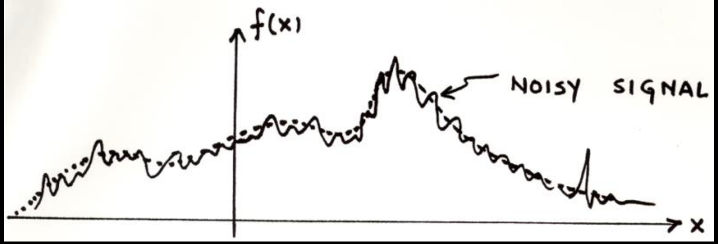
So, we can find *g(x)* by Fourier transform

$$\begin{tikzcd} g & = & f \arrow[d, "\mathrm{FT}"] & * & h \arrow[d, "\mathrm{FT}"] \\ G \arrow[u, "\mathrm{IFT}"] & = & F & * & H \end{tikzcd}$$

Example use: Smoothing/Blurring

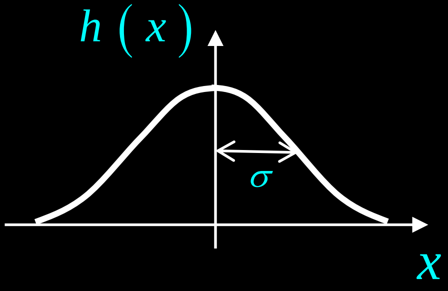
- We want a smoothed function of *f(x)*

$$g(x) = f(x) * h(x)$$



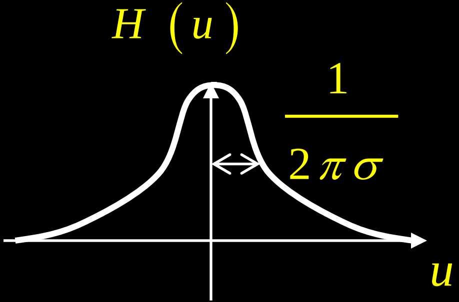
- Let us use a Gaussian kernel

$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2}\right]$$



- The Fourier transform of a Gaussian is a Gaussian

$$H(u) = \exp\left[-\frac{1}{2}(2\pi u)^2 \sigma^2\right]$$



Example use: Smoothing/Blurring

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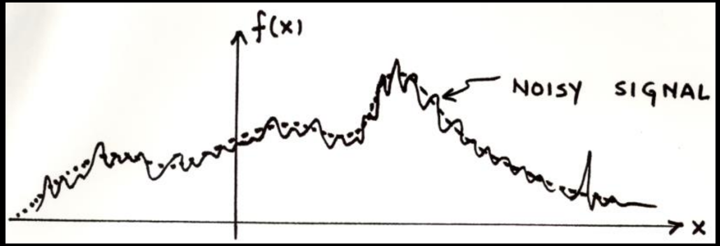
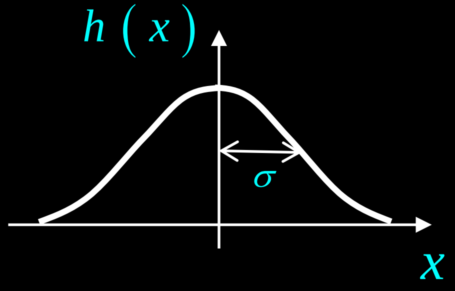


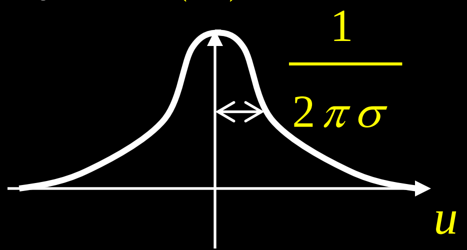
$$g(x) = f(x) * h(x)$$

- Let us use a Gaussian kernel



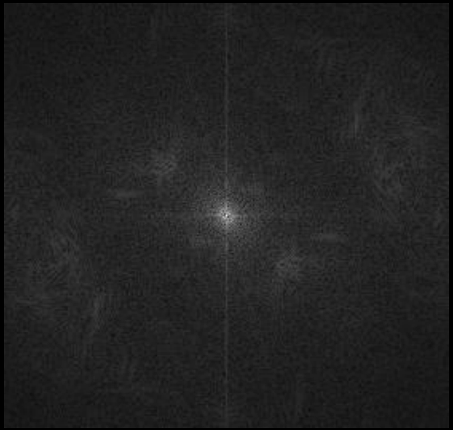
$$h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2}\right]$$

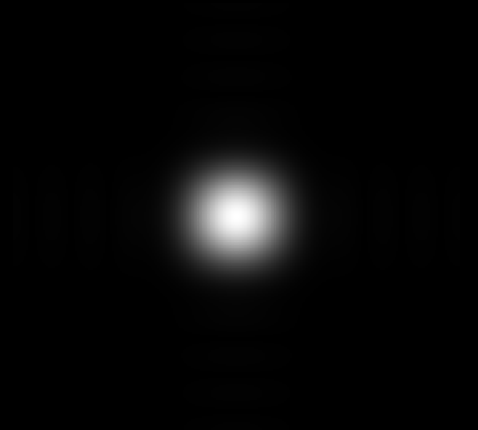
- Convolution in space is multiplication in freq: $H(u)$

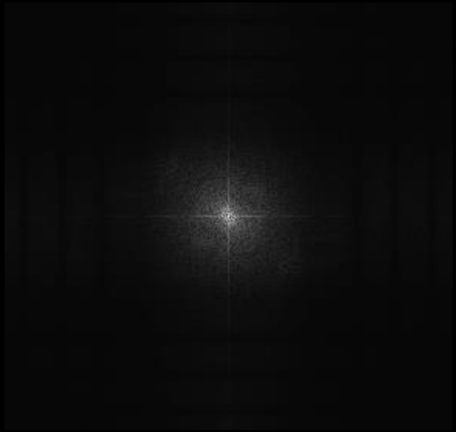


$$G(u) = F(u)\,H(u)$$

$f(x,y) \quad * \quad h(x,y) \quad = \quad g(x,y)$

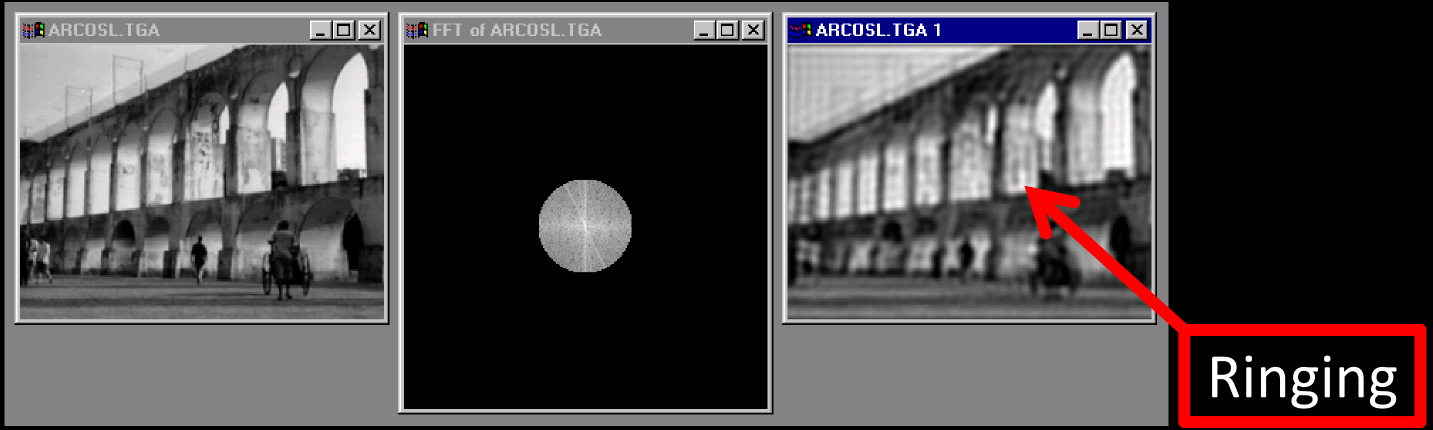


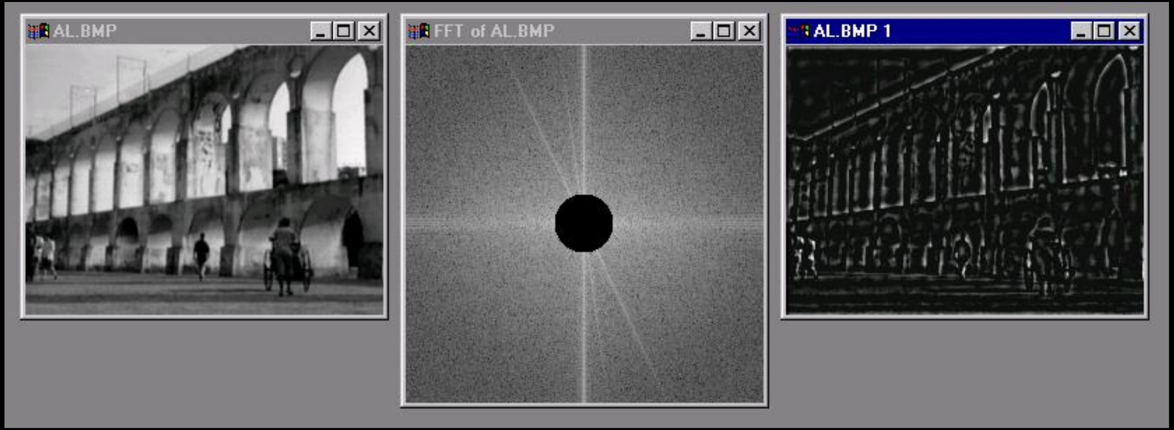




$|F(u,v)| \quad \times \quad |H(s_x,s_y)| \quad \Rrightarrow \quad |G(s_x,s_y)|$

Low and High Pass filtering





Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

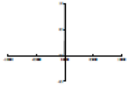
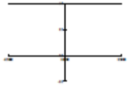
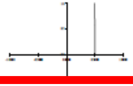
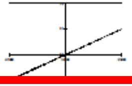
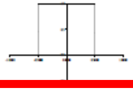
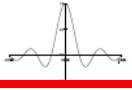
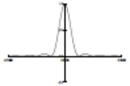
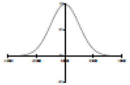
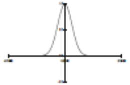
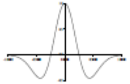
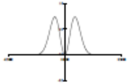
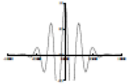
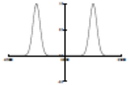
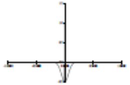
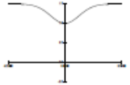
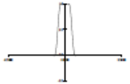
Shrink

Stretch

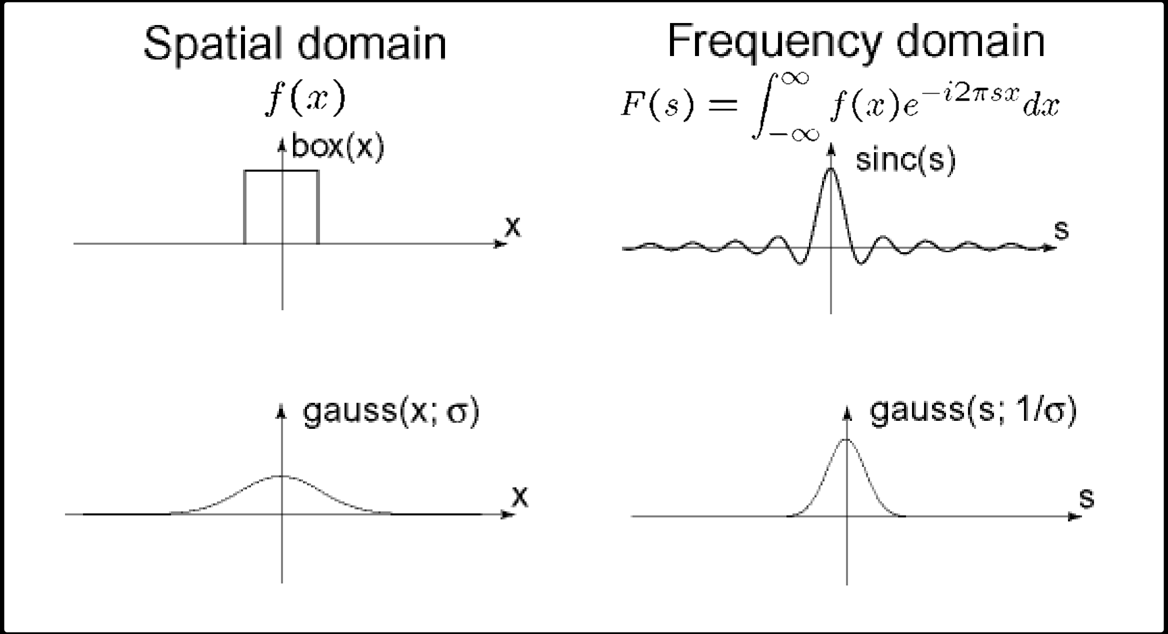
Differentiate

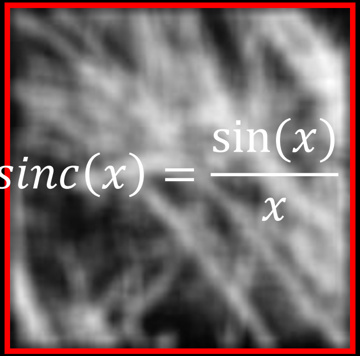
Multiply by u

Fourier Pairs (from Szeliski)

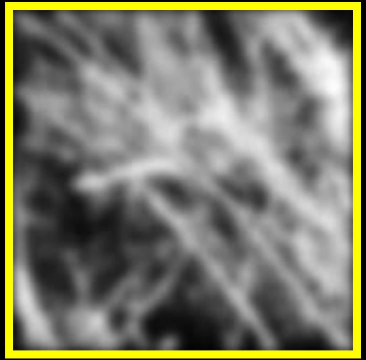
Name	Signal			Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\text{box}(x/a)$	$\Leftrightarrow$	$a\text{sinc}(a\omega)$	
tent		$\text{tent}(x/a)$	$\Leftrightarrow$	$a\text{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2 G(\omega;\sigma^{-1})$	
Gabor		$\cos(\omega_0 x)G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega \pm \omega_0;\sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $-\gamma G(x;\sigma)$	$\Leftrightarrow$	$(1+\gamma)-$ $\frac{\sqrt{2\pi}\gamma}{\sigma}G(\omega;\sigma^{-1})$	
windowed sinc		$\text{rcos}(x/(aW))$ $\text{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

Fourier Transform smoothing pairs





$$sinc(x) = \frac{\sin(x)}{x}$$



Fourier Transform smoothing pairs