

CS4495/6495

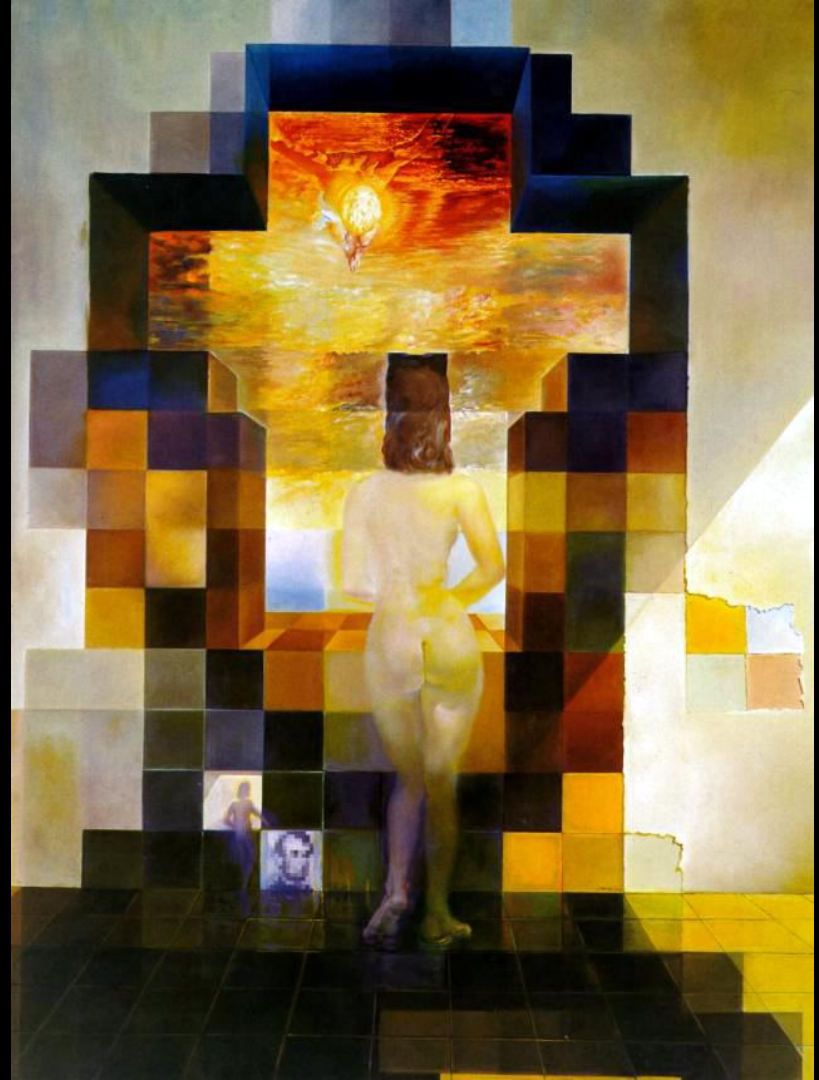
Introduction to Computer Vision

2C-L1 *Fourier transform*

What do you see?

Salvador Dali

*“Gala Contemplating the
Mediterranean Sea,
which at 30 meters
becomes the portrait of
Abraham Lincoln”, 1976*



What do you see?

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Decomposing an image: *basis sets*

A basis set is (edit from Wikipedia):

- A basis B of a vector space V is a linearly independent subset of V that spans V .

Decomposing an image: *basis sets*

Suppose that $B = \{ \mathbf{v}_1, \dots, \mathbf{v}_n \}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers \mathbb{R} or \mathbb{C}). Then B is a basis if it satisfies the following conditions:

- Linear independence:

For all $a_1, \dots, a_n \in F$, if $a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n = \mathbf{0}$, then necessarily

$$a_1 = \dots = a_n = 0$$

Decomposing an image: *basis sets*

Suppose that $B = \{ \mathbf{v}_1, \dots, \mathbf{v}_n \}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers \mathbb{R} or \mathbb{C}). Then B is a basis if it satisfies the following conditions:

- Spanning property,
 - For every \mathbf{x} in V it is possible to choose $a_1, \dots, a_n \in F$ such that
$$\mathbf{x} = a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n = \mathbf{0}.$$

Decomposing an image: *basis sets*

Suppose that $B = \{v_1, \dots, v_n\}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers \mathbb{R} or \mathbb{C}). Then B is a basis if it satisfies the following conditions:

- Not necessarily orthogonal... but helpful if they are.
(Why?)

Images as points in a vector space

- Consider an image as a point in a $N \times N$ size space – can rasterize into a single vector

$$\begin{bmatrix} x_{00} & x_{10} & x_{20} & \dots & x_{(n-1)0} & x_{10} & \dots & x_{(n-1)(n-1)} \end{bmatrix}^T$$

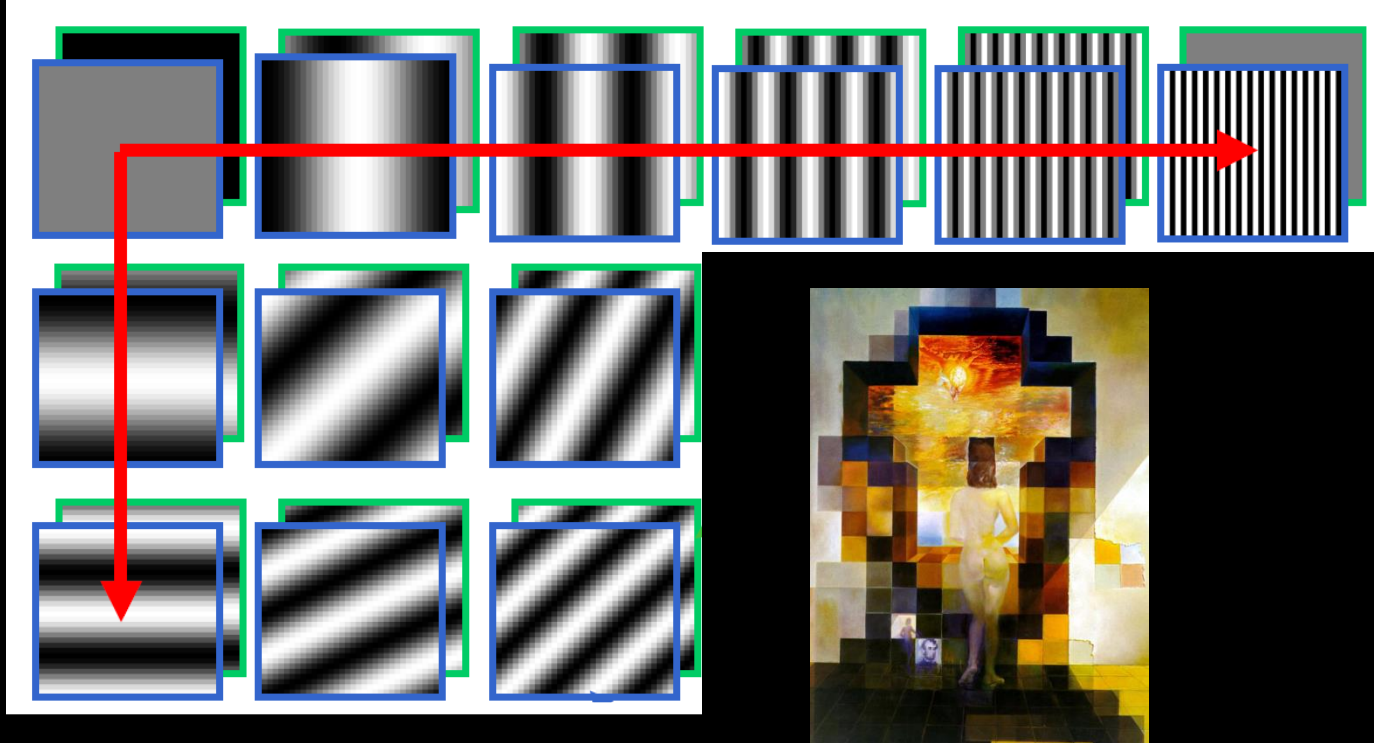
- The “normal” basis is just the vectors:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}^T$$

- Independent
- Can create any image
- Not very helpful...

A nice set of basis

Teases away fast vs. slow changes in the image.



Jean Baptiste Joseph Fourier (1768-1830)

Had crazy idea (1807):

- Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.



Jean Baptiste Joseph Fourier (1768-1830)

Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!



Jean Baptiste Joseph Fourier (1768-1830)

But it's true!

- Called Fourier Series



A sum of sines

Our building block:

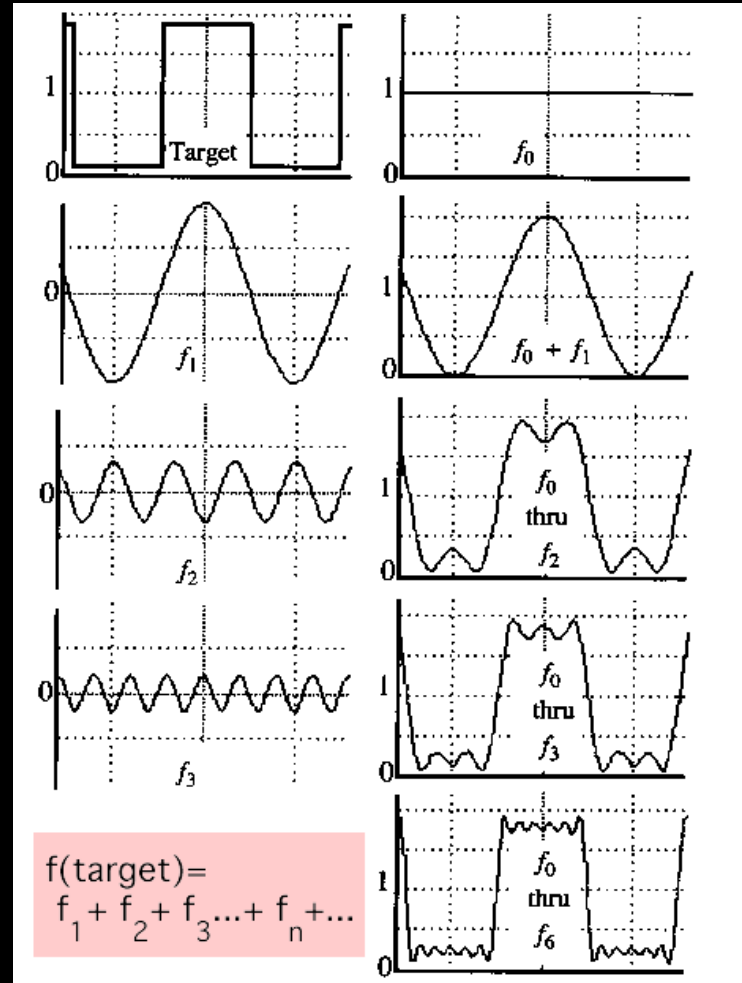
$$A \sin(\omega x + \varphi)$$

Add enough of them to get any signal $f(x)$ you want!

How many degrees of freedom?

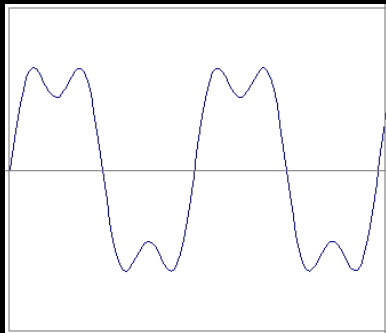
What does each control?

Which one encodes the coarse vs. fine structure of the signal?



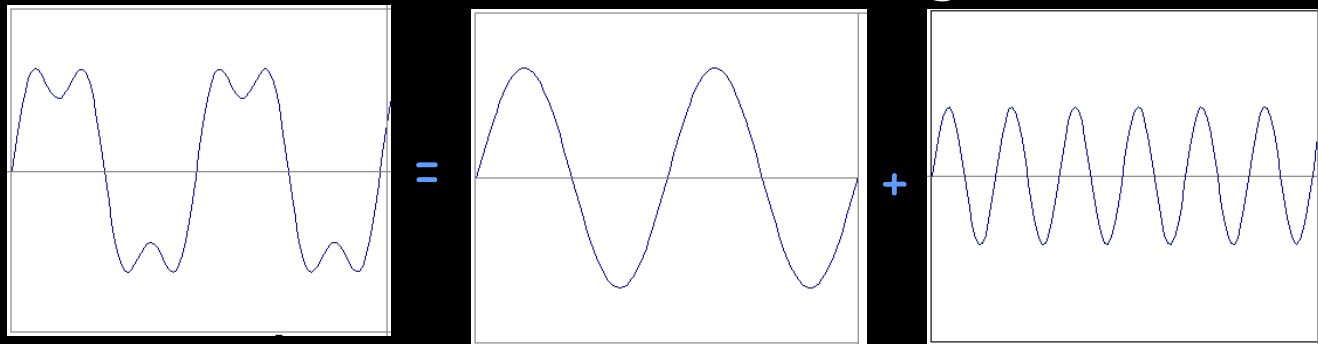
Time and Frequency

- Example: $g(t) = \sin(2\pi ft) + \frac{1}{3}\sin(2\pi(3f)t)$



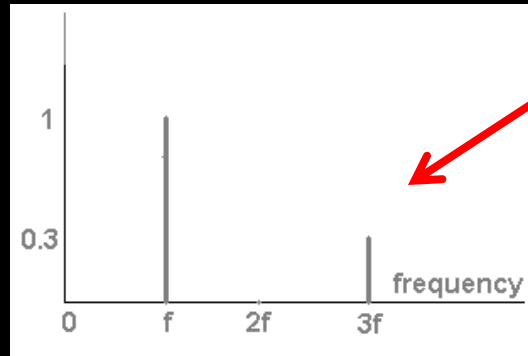
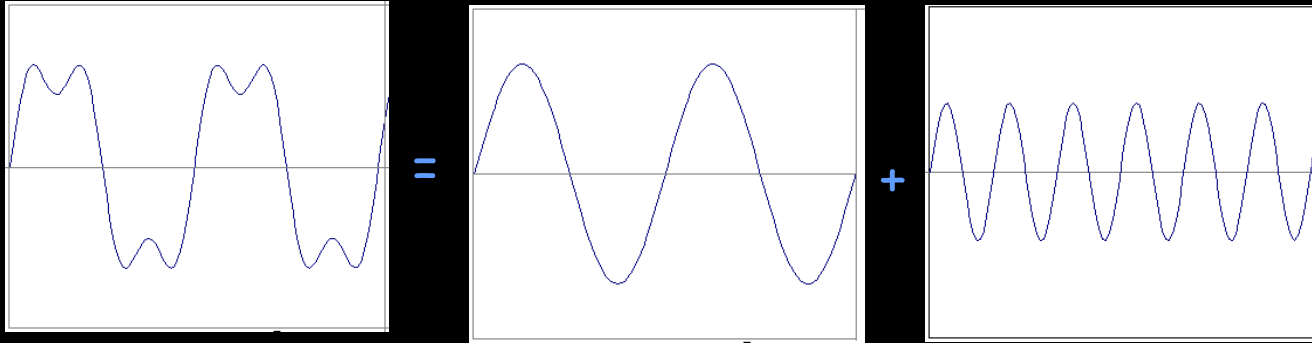
Time and Frequency

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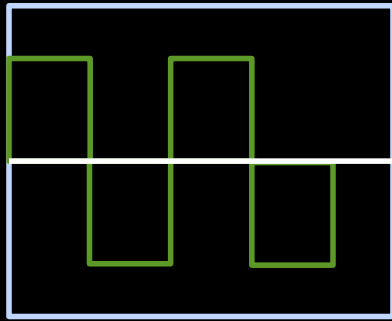
Frequency Spectra - Series

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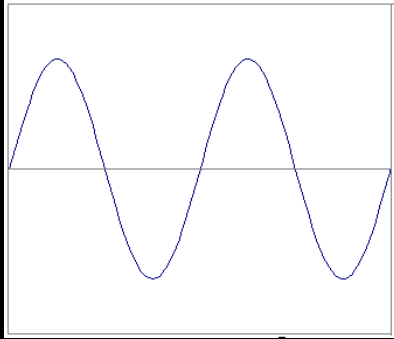


One form of
spectrum – more
in a bit

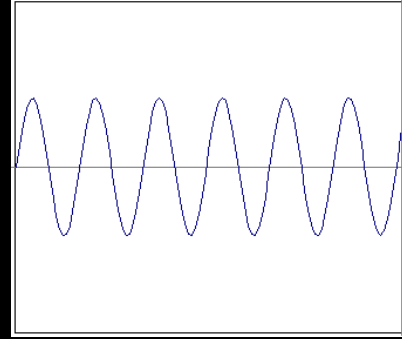
Frequency Spectra - Series



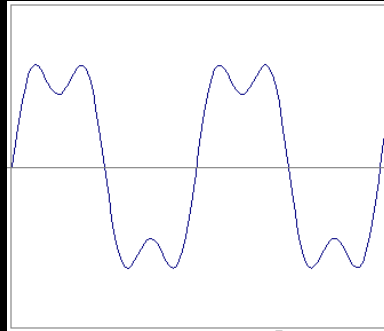
\approx



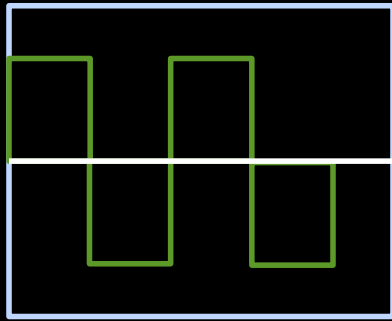
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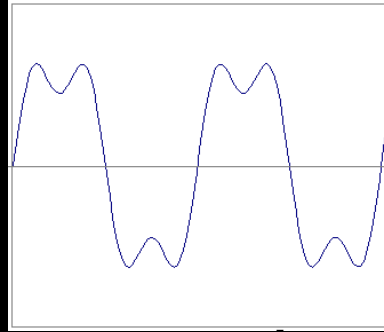
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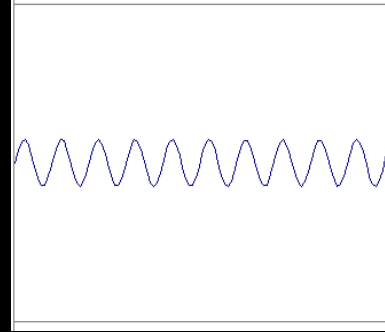
Frequency Spectra - Series



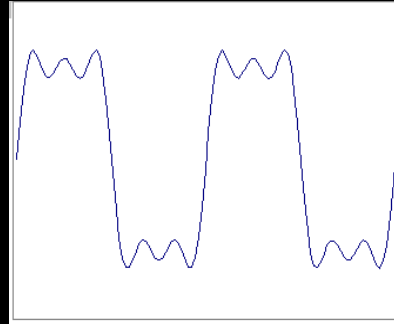
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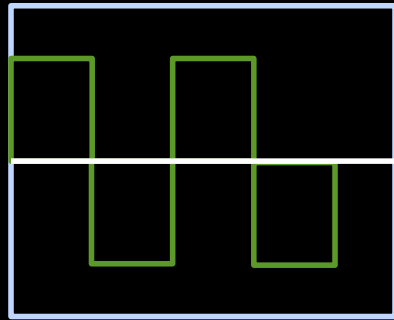
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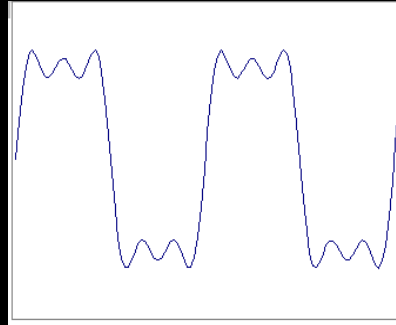
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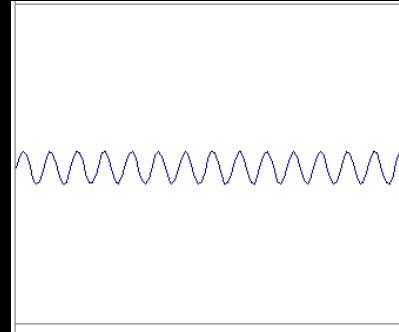
Frequency Spectra - Series



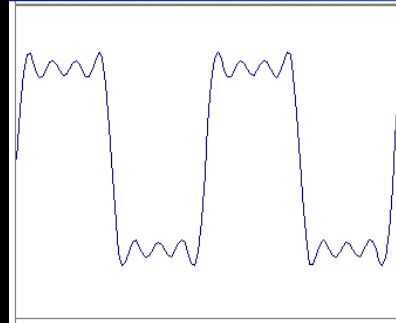
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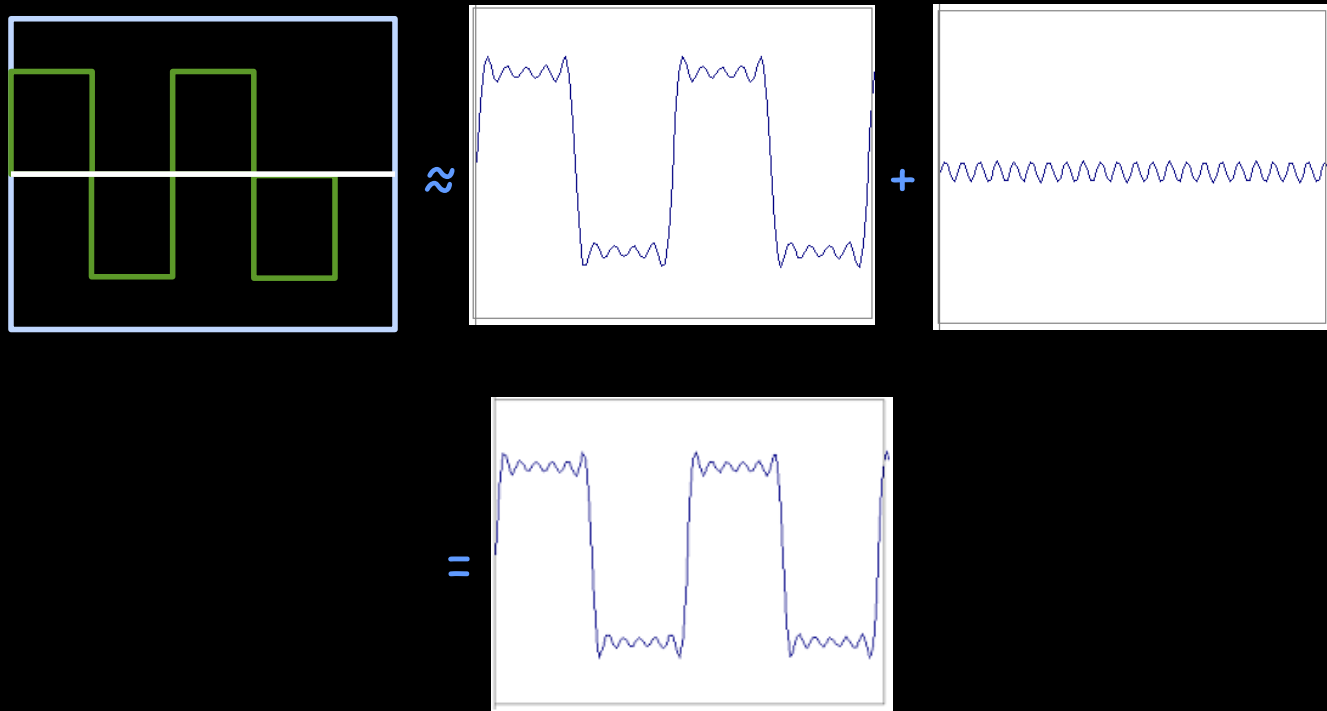
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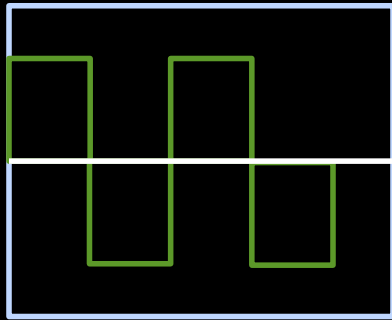
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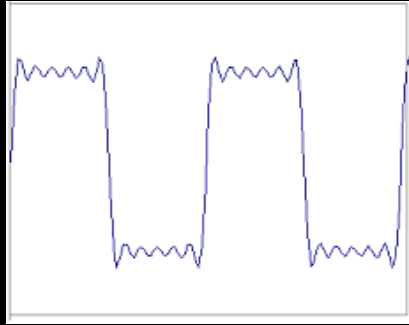
Frequency Spectra - Series



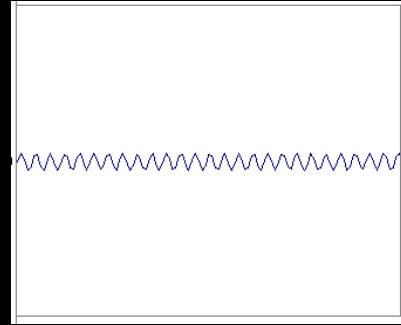
Frequency Spectra - Series



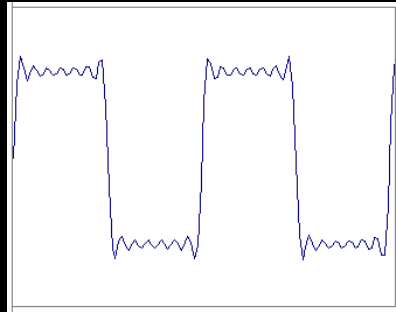
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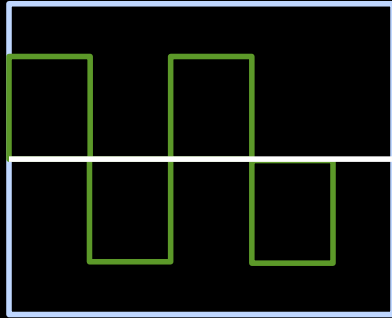
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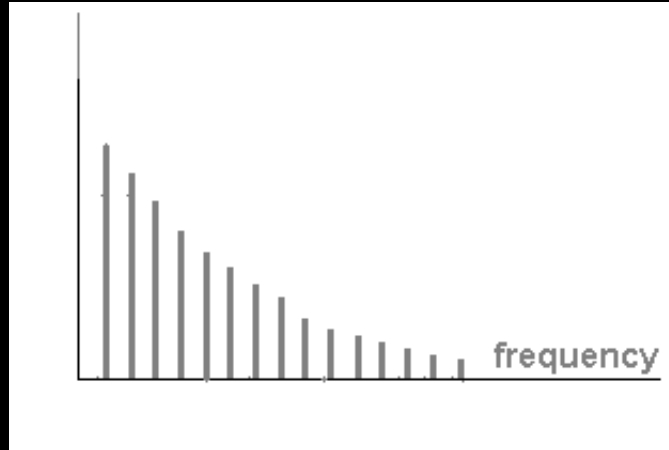


Frequency Spectra - Series



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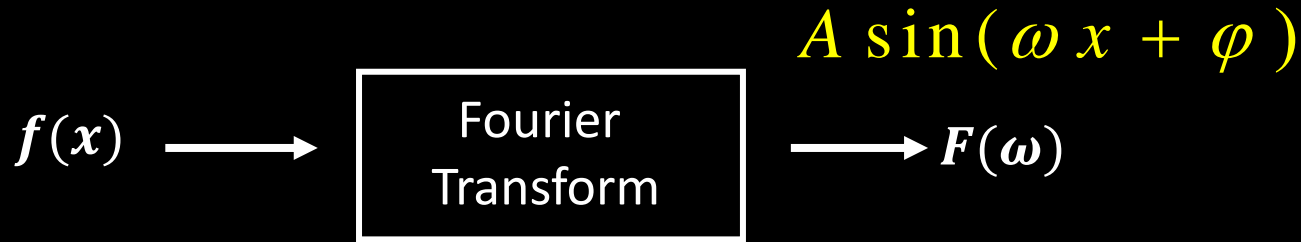
$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k t)$$



Usually, frequency magnitude is more interesting than the phase for CV because we're not reconstructing the image

Fourier Transform

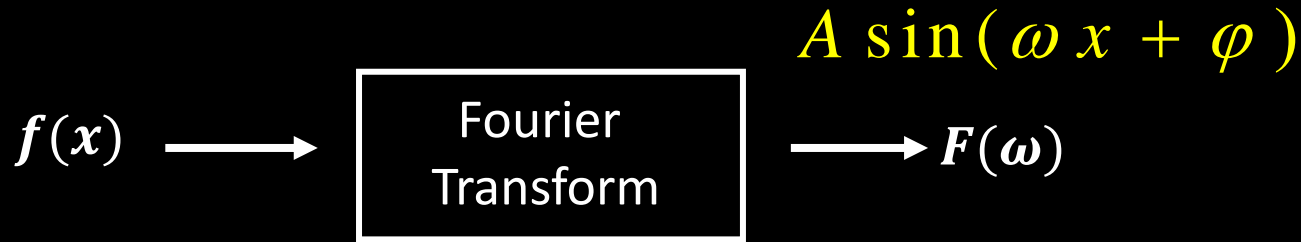
We want to understand the frequency ω of our signal.
So, let's reparametrize the signal by ω instead of x :



For every ω from 0 to ∞ (actually $-\infty$ to ∞), $F(\omega)$ holds the amplitude A and phase φ of the corresponding sinusoid

Fourier Transform

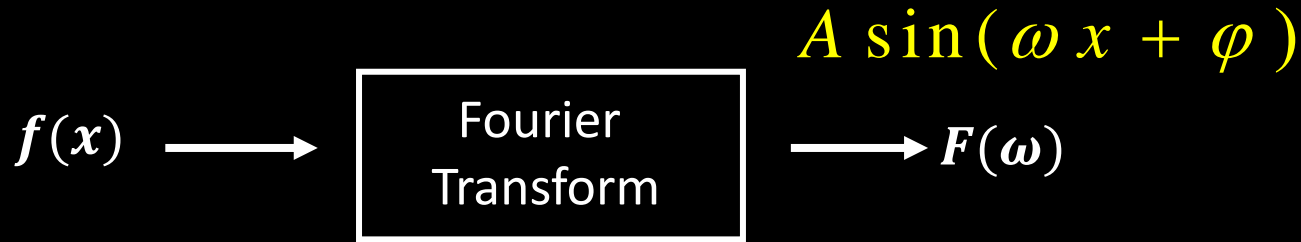
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How can F hold both amplitude and phase?
Complex number trick!

Fourier Transform

We want to understand the frequency ω of our signal.
So, let's reparametrize the signal by ω instead of x :



$$F(\omega) = \underbrace{R(\omega)}_{\text{Even}} + i \underbrace{I(\omega)}_{\text{Odd}}$$
$$A = \pm \sqrt{R(\omega)^2 + I(\omega)^2}$$
$$\varphi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$$

Computing FT: Just a basis

The infinite integral of the product of two sinusoids of *different* frequency is zero. (Why?)

$$\int_{-\infty}^{\infty} \sin(ax + \phi) \sin(bx + \varphi) dx = 0, \text{ if } a \neq b$$

And if same frequency the integral is infinite:

$$\int_{-\infty}^{\infty} \sin(ax + \phi) \sin(ax + \varphi) dx = \pm \infty$$

.... if ϕ and φ not exactly $\pi/2$ out of phase (sin and cos).

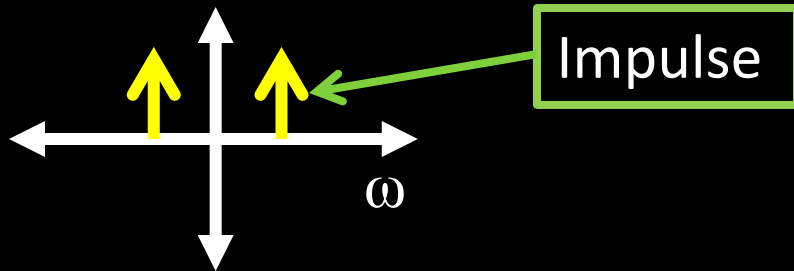
Computing FT: Just a basis

So, suppose $f(x)$ is a cosine wave of freq ω :

$$f(x) = \cos(2\pi\omega x)$$

Then: $C(u) = \int_{-\infty}^{\infty} f(x) \cos(2\pi u x) dx$

... which is infinite if u is equal to ω (or $-\omega$) and zero otherwise:



Computing FT: Just a basis

- We can do that for all frequencies u .
- But we'd have to do that for all *phases*, don't we???
- No! Any phase can be created by a weighted sum of cosine and sine. Only need each piece:

$$C(u) = \int_{-\infty}^{\infty} f(x) \cos(2\pi u x) dx$$

$$S(u) = \int_{-\infty}^{\infty} f(x) \sin(2\pi u x) dx$$

- Sinusoid demo?

Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i 2 \pi u x} dx$$

$$\text{Again: } e^{ik} = \cos k + i \sin k \quad i = \sqrt{-1}$$

Spatial Domain (x) \longrightarrow Frequency Domain (ω or u or *even* s)
(Frequency Spectrum $F(u)$ or $F(\omega)$)

Fourier Transform – more formally

Inverse Fourier Transform (IFT) – add up all the sinusoids at x :

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i 2 \pi u x} du$$

Fourier Transform - limitations

- The integral $\int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$ exists if the function f is integrable:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

- If there is a bound of width T outside of which f is zero then obviously could integrate from just $-T/2$ to $T/2$

Fourier Transform \Leftrightarrow Fourier Series

- The **Discrete FT**:

$$F(k) = \frac{1}{N} \sum_{x=0}^{x=N-1} f(x) e^{-i \frac{2\pi kx}{N}}$$

... where x is discrete and goes from the start of the signal to the end ($N-1$)

... and k is the number “cycles per period of the signal” or “cycles per image.”

- Only makes sense $k = -N/2$ to $N/2$. Why? What’s the highest frequency you can unambiguously have in a discrete image?

2D Fourier Transforms

- The two dimensional version: .

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i 2 \pi (u x + v y)} dx dy \frac{1}{2}$$

- And the 2D **Discrete FT**:

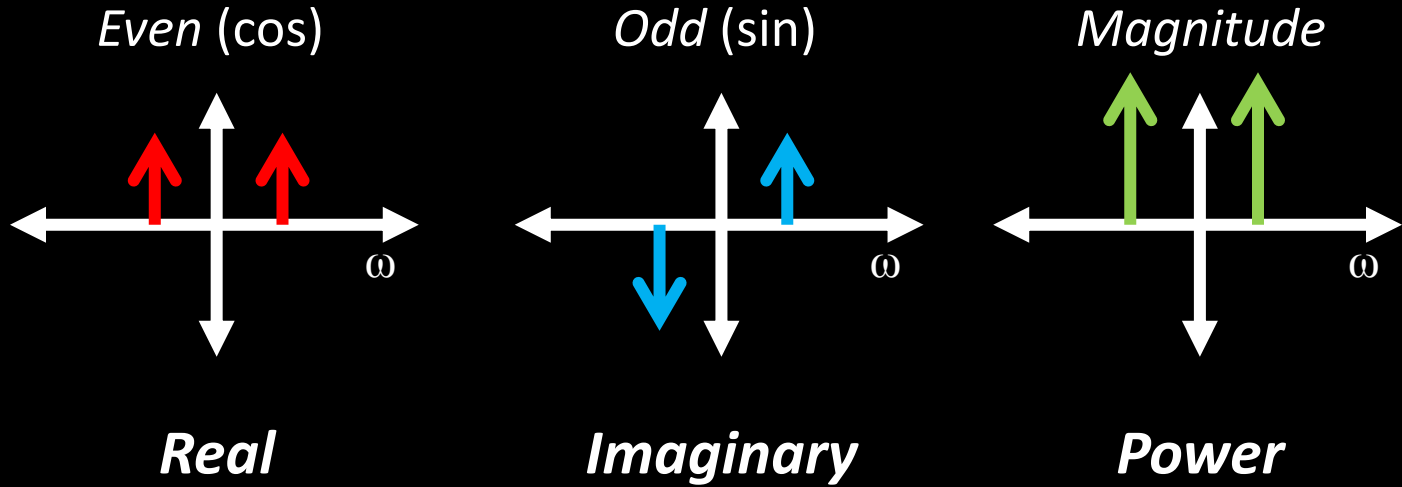
$$F(k_x, k_y) = \frac{1}{N} \sum_{x=0}^{x=N-1} \sum_{y=0}^{y=N-1} f(x, y) e^{-i \frac{2 \pi (k_x x + k_y y)}{N}}$$

- Works best when you put the origin of k in the middle....

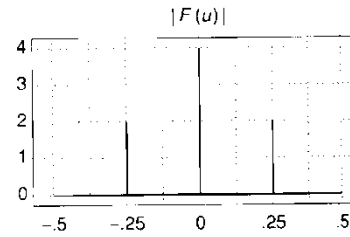
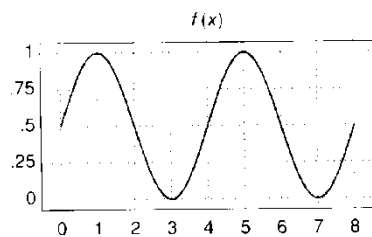
Frequency Spectra – Even/Odd

Frequency actually goes from $-\infty$ to $+\infty$

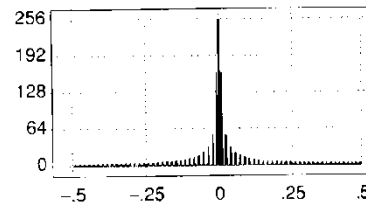
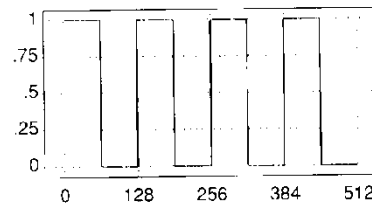
Sinusoid example:



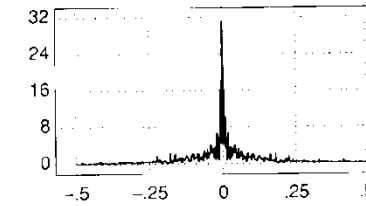
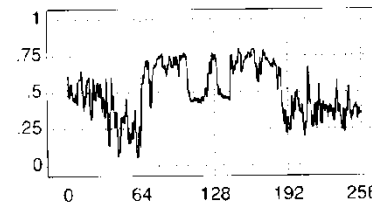
Frequency Spectra



(a)

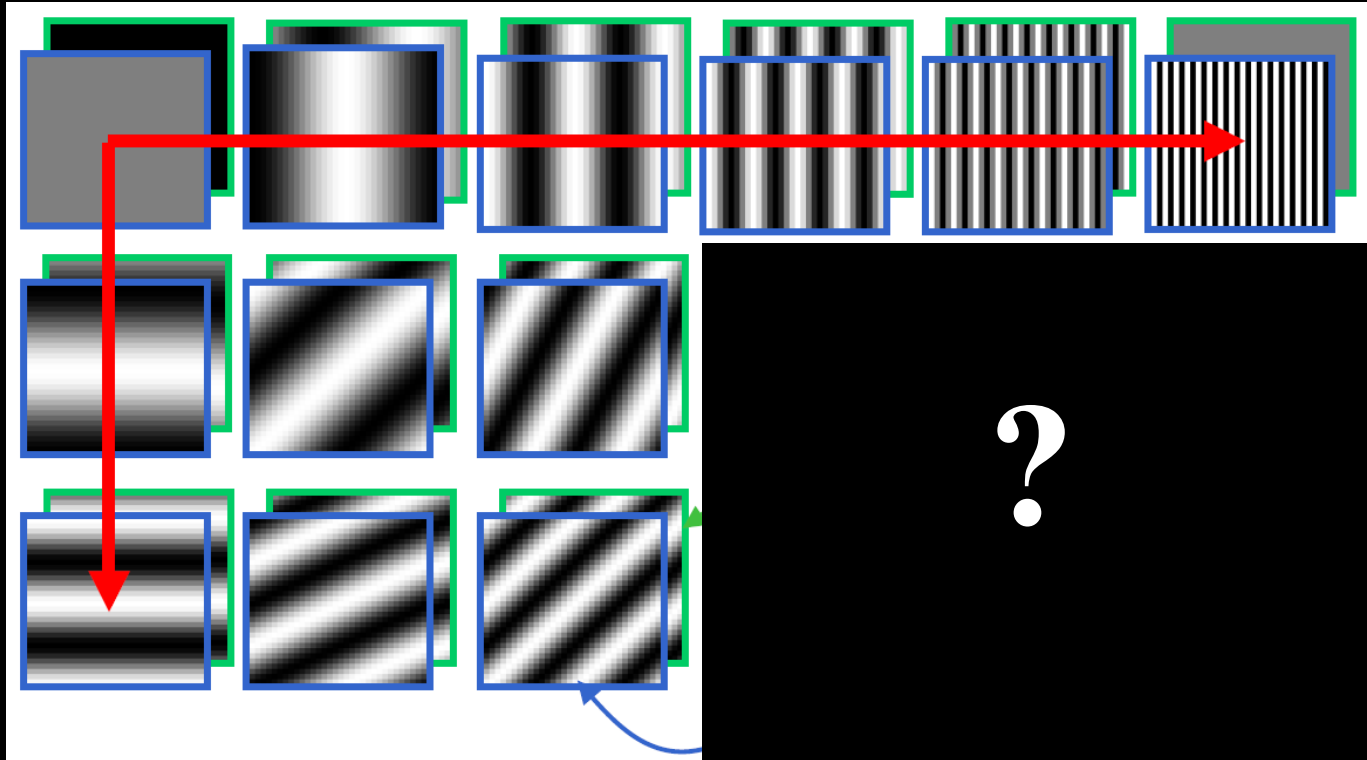


(b)

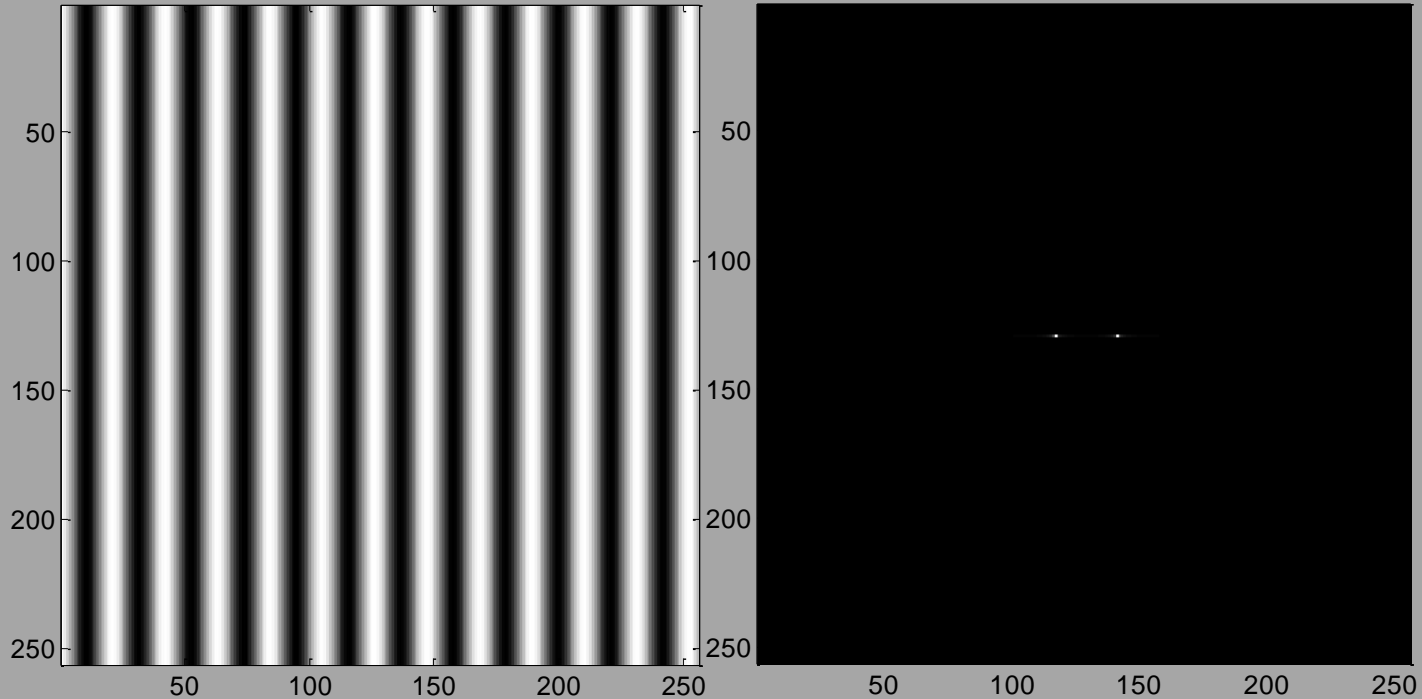


(c)

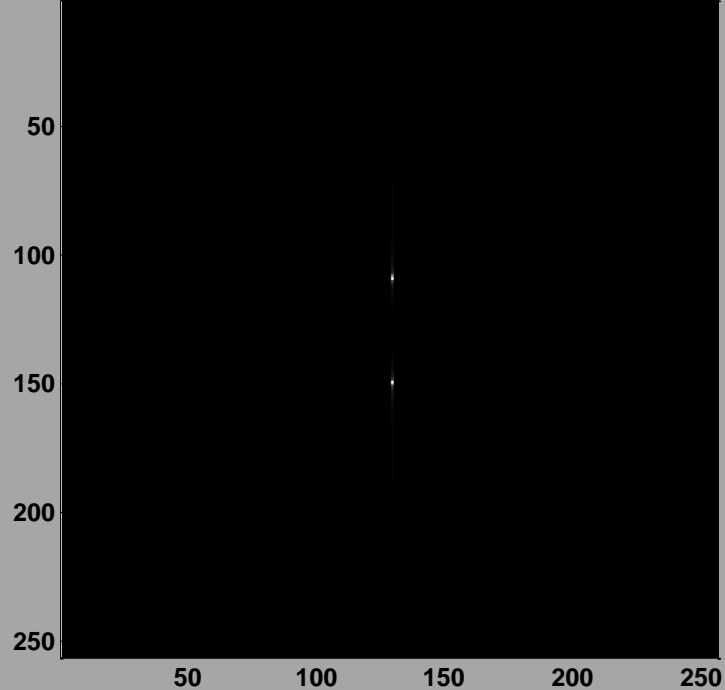
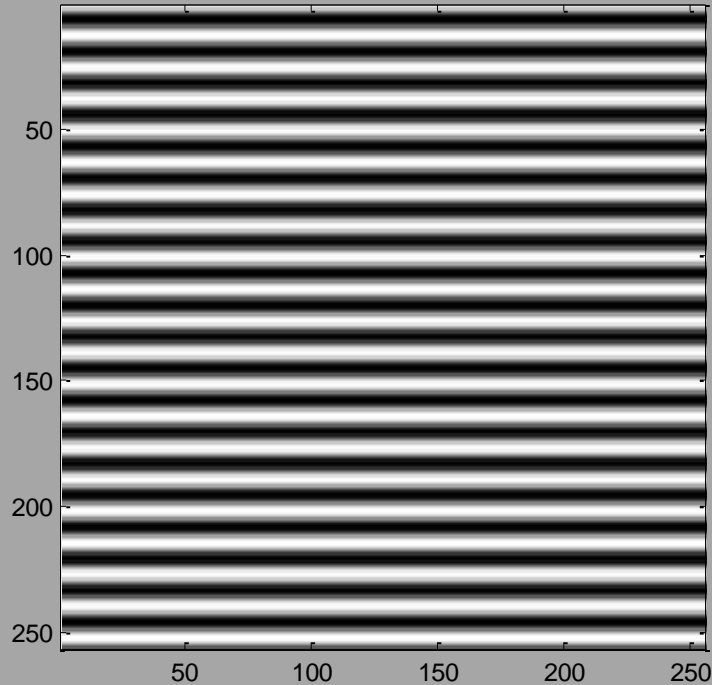
Extension to 2D



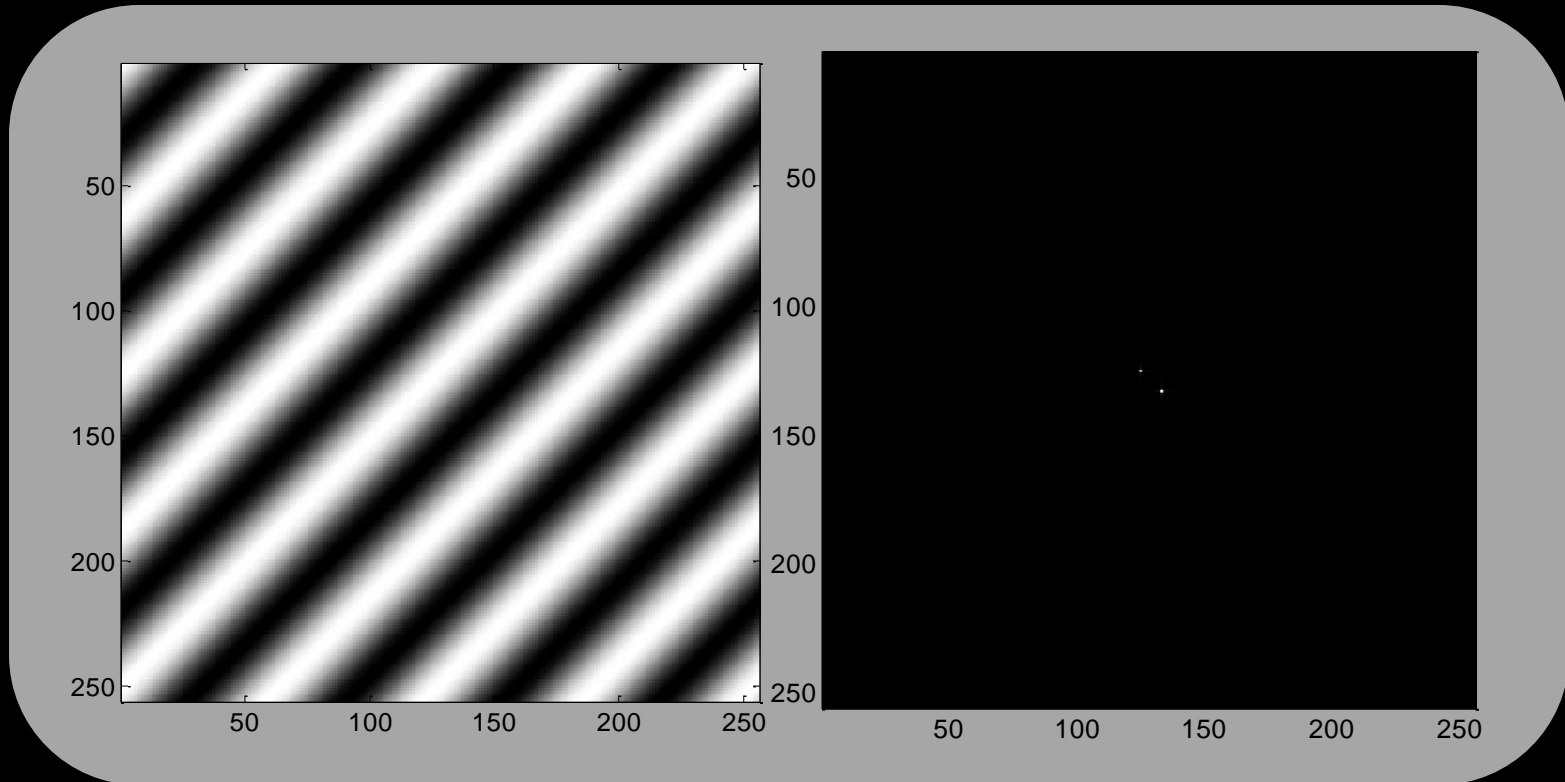
2D Examples – sinusoid magnitudes



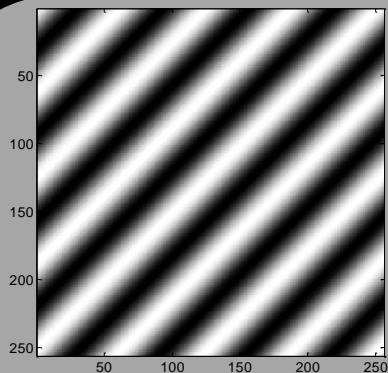
2D Examples – sinusoid magnitudes



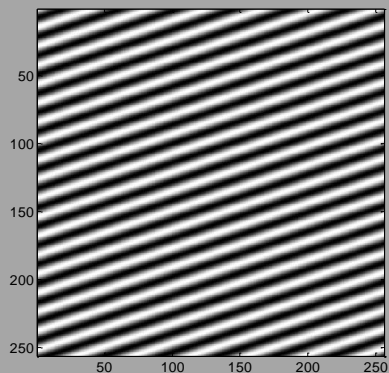
2D Examples – sinusoid magnitudes



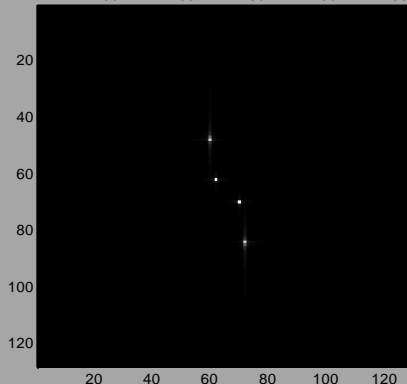
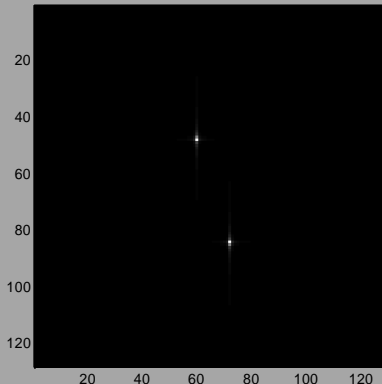
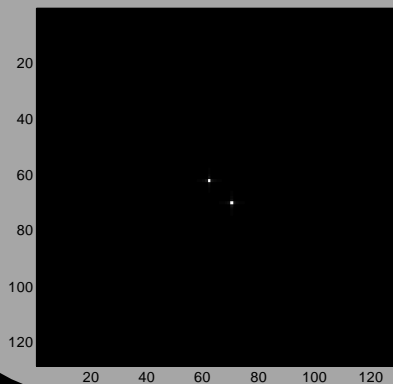
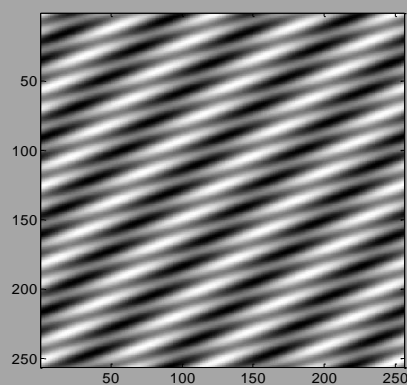
Linearity of Sum



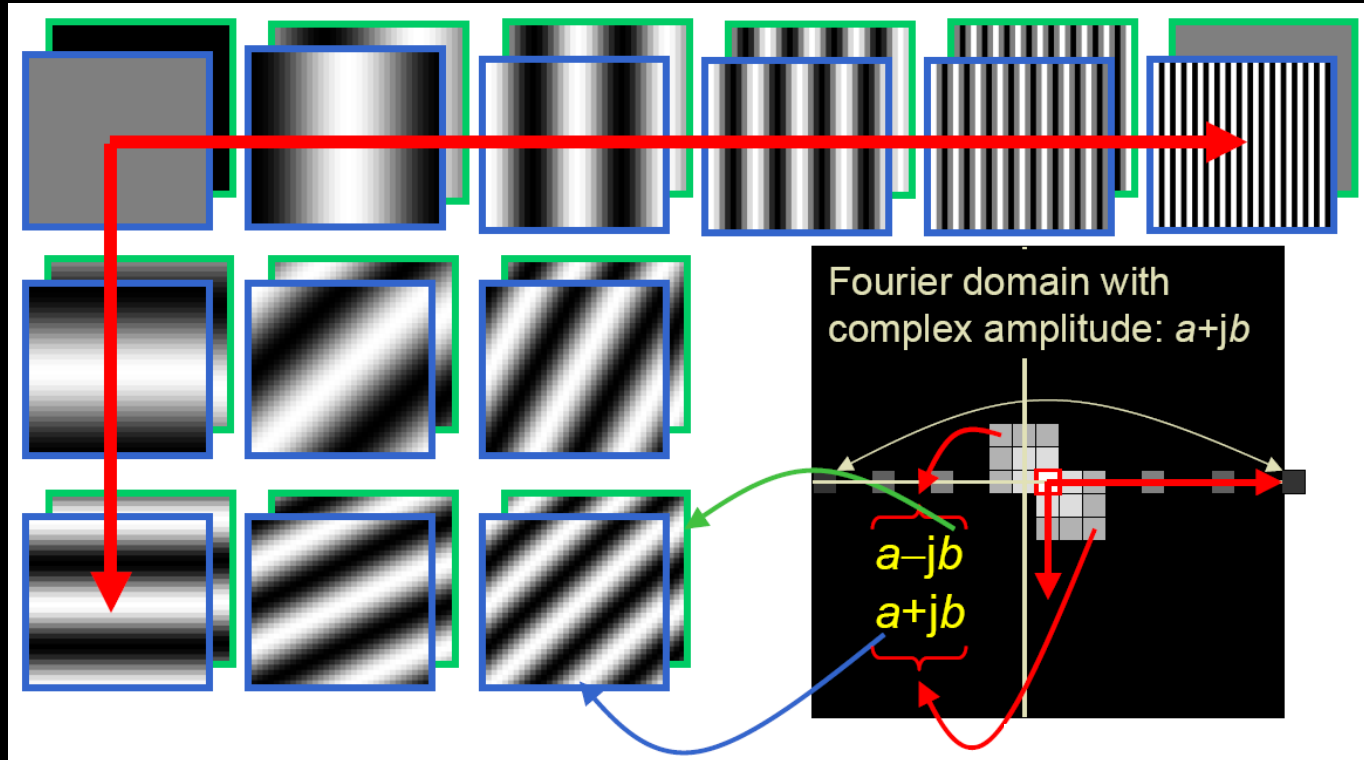
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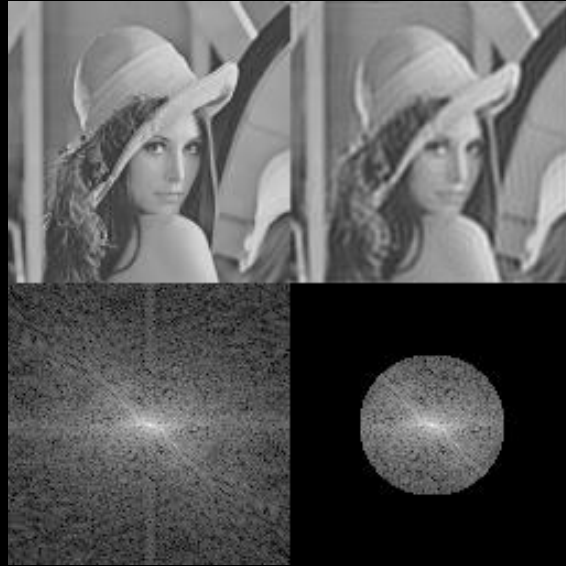
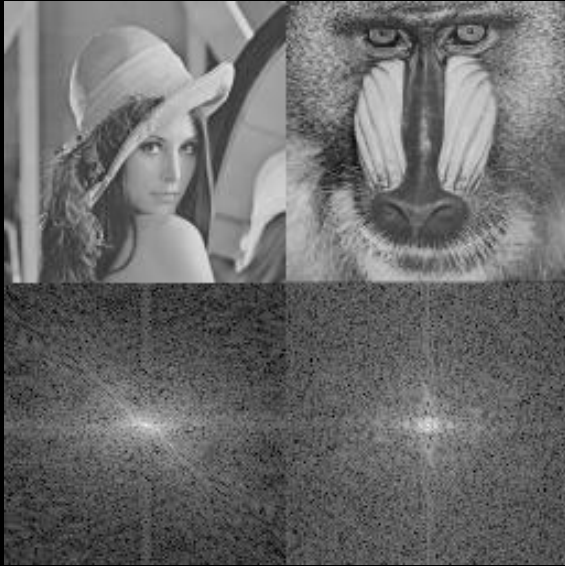


Extension to 2D – Complex plane



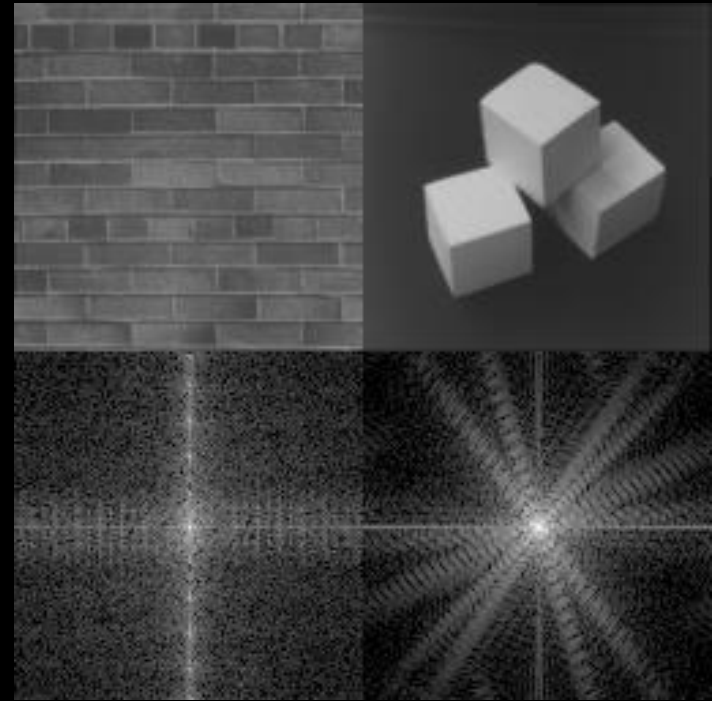
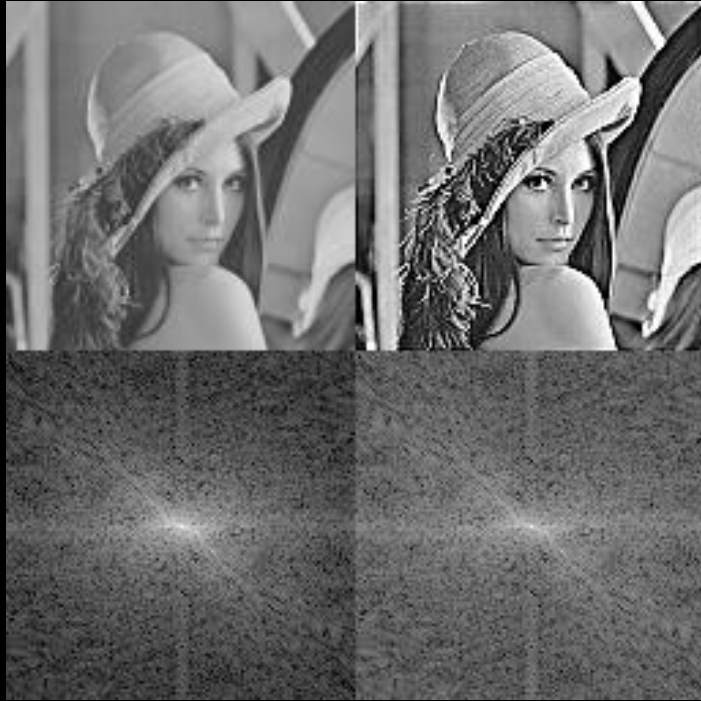
Both a Real and Im version

Examples



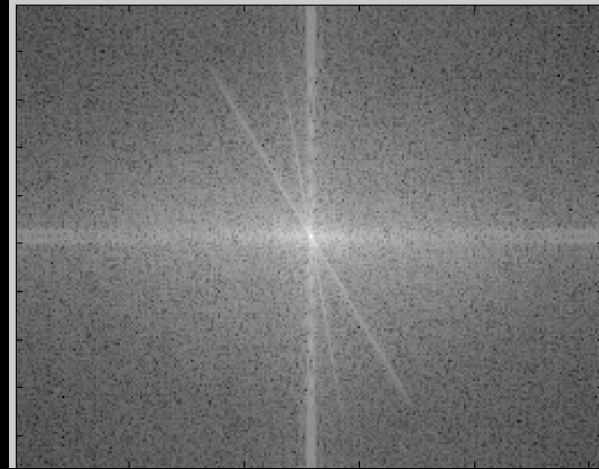
B.K. Gunturk

Examples



B.K. Gunturk

Man-made Scene



Where is this strong horizontal suggested by vertical center line?