Lecture 14 **Math 20 Fall 2014, Dartmouth College**

1 Sums of independent discrete random variables

Let X, Y be independent, integer valued random variables with distributions $m_X(k)$ and $m_Y(k)$ respectively. Let $Z = X + Y$. What is the distribution $m_Z(k)$?

We want to find, for any k, the probability that $X + Y = k$. Then for any $X = x$, $Z = k$ if and only if $Y = k - z$. So:

$$P(Z = k) = \sum_{x=-\infty}^{\infty} P(X = x)P(Y = k - x)$$

Notice that we used the fact that X and Y are independent and so $P(X = x \text{ and } Y = k-x) = P(X = x)P(Y = k-x)$.

Definition 1 Let X and Y be two independent integer-valued r.v.s. Then the *convolution* of $m_X(k)$ and $m_Y(k)$ is:

$$m_Z(k) = \sum_{x=-\infty}^{\infty} m_X(x)m_Y(k - x).$$

The convolution is the distribution of the function $Z = X + Y$.

Example 1 Roll two dice, red and blue. Let X be that outcome on the red die, and Y be the outcome on the blue die. Let $Z = X + Y$. As you remember, even though X and Y are uniformly distributed, the values of Z that are close to 7 are much more likely than, say, 2 or 12. There are simply more values of X for which we CAN find a Y such that $X + Y = k$.

$$P(Z = k) = \sum_{j=1}^{6} P(X = j)P(Y = k - j)$$

$$m_Z(2) = m_X(1)m_Y(1)\frac{1}{6}\frac{1}{6} = \frac{1}{36}$$

$$m_Z(3) = m_X(1)m_Y(2) + m_X(2)m_Y(1) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$$

$$m_Z(4) = m_X(1)m_Y(4) + m_X(2)m_Y(2)m_X(3)m_Y(1) = \frac{3}{36}$$

$$\vdots$$

$$m_Z(11) = m_X(5)m_Y(6) + m_X(6)m_Y(5) = \frac{2}{36}$$

$$m_Z(12) = m_X(6)m_Y(6)\frac{1}{36}$$

Example 2 The price of a stock on a given trading day changes according to distribution:

$$m(-1) = \frac{1}{4},\ m(0) = \frac{1}{2},\ m(1) = \frac{1}{4}$$

Find the distribution for the change in stock price after two (independent) trading days.

$$m_2(-2) = m(-1)m(-1) = \frac{1}{16}$$

$$m_2(-1) = m(-1)m(0) + m(0)m(-1) == \frac{1}{4}$$

$$m_2(0) = m(-1)m(1) + m(0)m(0) + m(1)m(-1) = \frac{3}{8}$$

$$m_2(1) = m(0)m(1) + m(1)m(0) = \frac{1}{4}$$

$$m_2(2) = m(1)m(1) == \frac{1}{16}$$

Definition 2 Let $X,\ Y$ be two continuous, real-valued random variables with densities $f_X(t),\ f_Y(t)$. Then the *convolution* $(f_X \star f_Y)(t)$ of $f_X,\ f_Y$ is:

$$(f_X \star f_Y)(t) = \int_{-\infty}^{\infty} f_X(t-y) f_Y(y) dy = \int_{-\infty}^{\infty} f_Y(t-x) f_X(x) dx$$

If X and Y are independent and $Z = X + Y$, then the density of Z is the convolution of X and Y.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

Again, we are looking at probability density of pairs (x, y) such that $x + y = z$.

Example 3 (Convolution of two uniform densities)
Let $X,\ Y$ both be picked from $[0,1]$ uniformly at random. So:

$$f_X(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 \text{ if } y \in [0,1] \\ 0 \text{ otherwise} \end{cases}$$

The probability density of $Z = X + Y$ is given by the convolution of these two densities:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

The factor $f_Y(y)$ is zero unless $y \in [0,1]$ and 1 otherwise, so we can immediately chop off most of the integral:

$$f_Z(z) = \int_0^1 f_X(z-y) dy$$

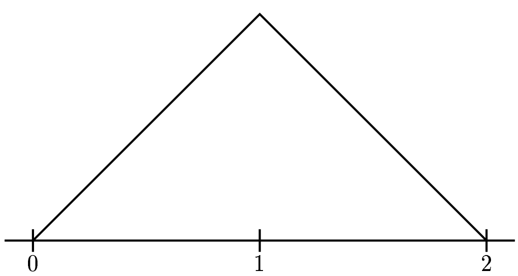
Meanwhile, the factor $f_X(z-y)$ is 1 whenever $0 \leq z - y \leq 1$ and 0 otherwise. So $y \leq z$ and $y \geq z - 1$. So for $z \in [0,1]$ we get:

$$f_Z(z) = \int_0^z 1 dy = z$$

and for $z \in (1,2]$ we get:

$$f_Z(z) = \int_{z-1}^1 1 dy = 1 - z + 1 = 2 - z$$

So the density $f_Z(z)$ loos as follows:



When you think about it, it makes sense that there are ore pairs (x, y) that add up to 1 than ones that add up to 2, right?

Example 4 (Convolution of exponential densities)
Let $X,\ Y$ be such that:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \text{ if } x \in [0, \infty] \\ 0 \text{ otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} \text{ if } y \in [0, \infty] \\ 0 \text{ otherwise} \end{cases}$$

Then for $Z = X + Y$, we have:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

Now, neither y nor $z - y$ can be negative, so $0 \leq y \leq z$. We get:

$$f_Z(z) = \int_0^z f_X(z-y) f_Y(y) dy$$

$$f_Z(z) = \int_0^z \lambda e^{-\lambda(z-y)} \lambda e^{-\lambda y} dy$$

$$f_Z(z) = \lambda^2 \int_0^z e^{-\lambda z} e^{\lambda y} e^{-\lambda y} dy$$

$$f_Z(z) = \lambda^2 e^{-\lambda z} \int_0^z 1 dy = \lambda^2 e^{-\lambda z} z$$