A brief introduction about strong disorder RG

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outline

- brief history of strong disorder RG.
- examples

  a random spin-1/2 Heisenberg chain

  RSRG-X

Mainly focus on technical details, lacking a deeper understanding of renormalization groups

references

Strong disorder renormalization group primer and the superfluid–insulator transition

Introduction au groupe de renormalisation pour fort désordre et ses consequences pour la transition superfluide–isolant

Gil Refael [a], Ehud Altman [b,c,*]

Strong disorder RG approach of random systems

Ferenc Iglói[a, b, *], Cécile Monthus[c]

Strong disorder RG approach – a short review of recent developments

Ferenc Iglói[1,2,a] and Cécile Monthus[3,b]

Strong-randomness renormalization groups

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brief history of SDRG

- The technique was originally developed by Dasgupta and Ma to investigate the ground state and low energy behavior of the random Heisenberg spin chain.
- the SDRG scheme was extended by Bhatt and Lee and formulated rigorously by Daniel Fisher. Fisher showed in particular that the scheme gives asymptotically exact results for the low energy universal behavior of systems controlled by infinite randomness fixed points.
- richer physics is at play in the random transverse field Ising chain. A quantum phase transition between a magnetically ordered and a paramagnetic phase that is controlled by an infinite randomness fixed point.
- The analysis of the Ising model was extended to two dimensions by Motrunich who found a similar transition controlled by an infinite randomness fixed point.
- The SDRG approach has since been applied and extended to address a host of problems spanning different fields. : These include classical stochastic dynamics, Dynamic response of spin chains at low temperatures, entanglement in random spin chains and most recently non-equilibrium quantum dynamics.

Principles of strong disorder RG

- at large scale, the disorder dominates with respect to the thermal or quantum fluctuations.
- In particular, the strong disorder renormalizations are intrinsically specific to disordered systems and cannot even be defined for the pure systems which do not present space heterogeities.

Notions of infinite and strong disorder fixed points

when the scale increases, this effective disorder can either become

- (i) smaller and smaller without bound: the system is then controlled by a pure fixed point,
- (ii) larger and larger without bound: the system is then controlled by an infinite disorder fixed point,
- (iii) or it may converge towards a finite level: the system is then controlled by a finite disorder fixed point.

In conclusion, the strongdisorder renormalization methods concern:

- the infinite disorder fixed points (ii).
- the finite disorder fixed points (iii) that present strongdisorder properties, such as Griffiths phases in quantum models or localization phenomena for random walks in random media.

Example 1 : a random spin-1/2 Heisenberg chain

$$H = \sum_i J_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

The aim is to slove for the universal properties of the ground state and low energy excitations.

Main steps:

corase grain

ground state picture

how to get the master equation

solve the equation

physical properties

Dasgupta–Ma decimation in the spin-1/2 Heisenberg chain

We denote the upper energy cutoff of the chain by Ω

1.the largest exchange coupling $J_l = \Omega$

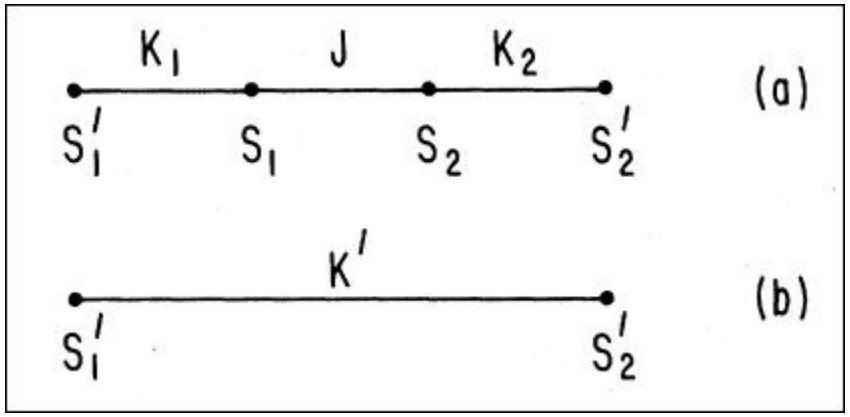
2.the largest bond is much stronger than a typical bond and in particular it is likely to be much larger than the neighboring bonds. $J_{l\pm1} \ll \Omega.$

3. in order to caluculate the ground state of the chain , we diagonalize the strongest bond with $H_0 = \Omega\, \mathbf{S}_l \cdot \mathbf{S}_{l+1}$ and treat the couplings to the rest chain as a perturbation.

$$\tilde{H}_{l-1,l+2} = \frac{J_{l-1} J_{l+1}}{2\Omega} \mathbf{S}_{l-1} \cdot \mathbf{S}_{l+2}$$

$$J^{\text{eff}}_{l-1,l+2} = \frac{J_{l-1} J_{l+1}}{2\Omega} \ll J_{l-1},\, J_{l+1} \ll \Omega$$

perturbation calculation



$$H_0 = J\vec{S}_1 \cdot \vec{S}_2 \ .$$

$$\mathcal{H} = K_1\vec{S}_1' \cdot \vec{S}_1 + K_2\vec{S}_2 \cdot \vec{S}_2' \ .$$

$$E_s = -\tfrac{3}{4}J \ ,$$
$$E_t = \tfrac{1}{4}J \ ,$$

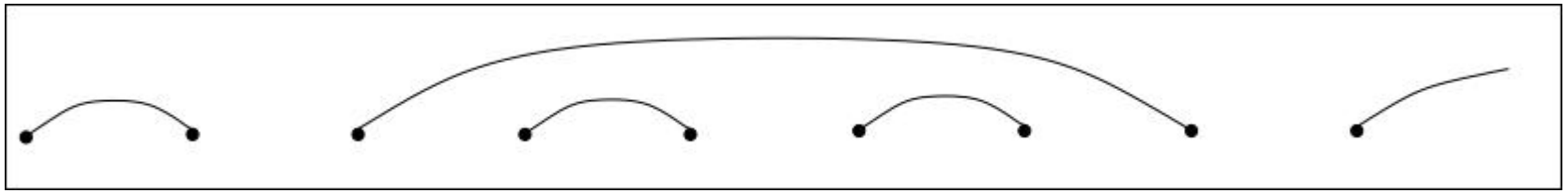
$$E_s + \langle s|\mathcal{H}|s\rangle + \sum_t |\langle s|\mathcal{H}|t\rangle|^2 \frac{1}{E_s - E_t} \equiv E' + K'\vec{S}_1' \cdot \vec{S}_2'$$

$$E' = -\tfrac{3}{4}J - \frac{3}{16J}(K_1^2 + K_2^2) \ ,$$

$$K' = \frac{K_1 K_2}{2J} \ .$$

after the decimation step we have exactly the same form of the Hamiltonian

Qualitative ground state picture: the random singlet phase



random singlet phase of a random Heisenberg model.

These singlets mostly form between nearest neighbors, but also over an arbitrarily large distance.

We focus on the correlation C_{ij} in an ensmble of chains.

The probabilty i and j survive is 1/|i-j|

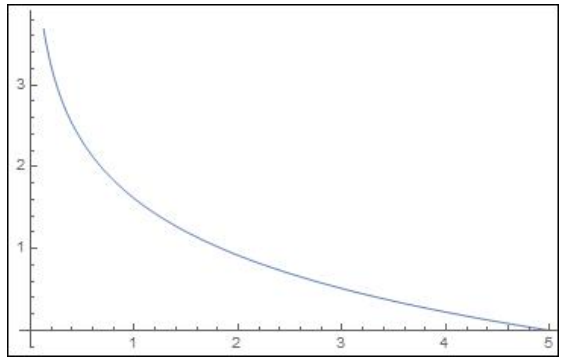
$$\bar{C}_{ij} = (-1)p_s + e^{-a\sqrt{|i-j|}}(1-p_s) \approx -p_s \sim \frac{1}{(i-j)^2}$$

Master equation for the flow of distribution functions

- how detailed and precise information about the ground state and low-energy correlations is obtained, following Fisher.

- .The key step is to translate the Dasgupta–Ma decimation rules into a master equation describing how the repeated decimations renormalize the probability distribution of exchange couplings when they operate on an ensemble of Hamiltonians.

How to get the Master equation



$\rho_0(J)$ initial distribution of the couplings

reducing the cutoff scale from Ω_0 to Ω the distribution evolves to $\rho(J,\Omega)$.

replace J and Γ by the dimensionless scaling variable $\zeta_i = \ln(\Omega/J_i)$ and $\Gamma = \ln(\Omega_0/\Omega)$.

we note $\zeta_i \in [0,\infty)$

When we take decimation, $\Gamma \to \infty$

$$\zeta_{i-1,i+2} = \zeta_i + \zeta_{i+1} + \ln 2$$

because the strong disorder assumption, we drop ln(2)

$$\zeta_{i-1,i+2} = \zeta_i + \zeta_{i+1}$$

How to get the Master equation

$P_\Gamma(\zeta).$

We integrate out the high energy shell $[\Omega - \mathrm{d}\Omega, \Omega]$

The change of distribution are divided into two parts:

1. redefine the remaining ζ' s according to the new cutoff.

$$\zeta_m \to \zeta_m - \mathrm{d}\zeta = \ln \frac{\Omega - \mathrm{d}\Omega}{J_m} = \zeta_m - \frac{\mathrm{d}\Omega}{\Omega} = \zeta_m - \mathrm{d}\Gamma$$

$$\mathrm{d}P(\zeta) = \frac{\partial P(\zeta)}{\partial \zeta}\mathrm{d}\Gamma$$

2. add the couplings generated through second-order perturbation theory across the decimated bonds.

$$P_{\mathrm{new}}(\zeta) = \int_0^\infty \mathrm{d}\zeta_l \int_0^\infty \mathrm{d}\zeta_r \, P(\zeta_l)P(\zeta_r)\delta(\zeta - \zeta_l - \zeta_r - \ln 2)$$

$$\mathrm{d}P(\zeta) = \mathrm{d}\Gamma P(0)P_{\mathrm{new}}(\zeta).$$

Solution of the flow equation

The master equation

$$\frac{\mathrm{d}P(\zeta)}{\mathrm{d}\Gamma} = \frac{\partial P(\zeta)}{\partial \zeta} + P(0)\int_0^\infty \mathrm{d}\zeta_\ell \int_0^\infty \mathrm{d}\zeta_r \, P(\zeta_\ell)P(\zeta_r)\delta(\zeta - \zeta_\ell - \zeta_r)$$

$$P_\Gamma(\zeta) = f(\Gamma)\mathrm{e}^{-f(\Gamma)\zeta}.$$

plugging this ansatz into the master equation leads to $\partial_\Gamma f = -f^2,$ $f(\Gamma) = 1/\Gamma.$

$$P_\Gamma(\zeta) = \frac{1}{\Gamma}\mathrm{e}^{-\zeta/\Gamma}$$

$$\rho_\Omega(J) = \frac{1}{\Omega\Gamma}\left(\frac{\Omega}{J}\right)^{1-1/\Gamma}$$

infinite randomness fixed point

$$P_\Gamma(\zeta) = \frac{1}{\Gamma} e^{-\zeta/\Gamma}$$

$$\rho_\Omega(J) = \frac{1}{\Omega\Gamma}\left(\frac{\Omega}{J}\right)^{1-1/\Gamma}$$

$\Gamma \to \infty$

The system flows to infinite randomness as the width of the distribution grows without limit.

The nearest neighbor Heisenberg chain flows to the same infinite randomness fixed point regardless of the initial distribution as long as the disorder is not correlated.

Physical properties

- the relationship between energy scale and length scale

consider the distance between surviving spins at the scale Ω.

When change the RG scale by $\mathrm{d}\Gamma$,

the number changes by $\mathrm{d}N = -2P_\Gamma(0)N\,\mathrm{d}\Gamma.$

$$P(0) = f(\Gamma) = 1/\Gamma;$$

$$N(\Gamma) \sim N_0/\Gamma^2.$$

the average distance: $L(\Gamma) = l_0 N_0/N \sim \Gamma^2,$ l_0 is the original lattice spacing.

$L(\Omega) \sim l_0 \ln^2(\Omega_0/\Omega),$ → $\ln J_\ell \sim -\sqrt{\ell}$ infinite-randomness scaling.

spin susceptibility at temperature T

we consider the spin susceptibility at Temperature T

$$M = \chi H, \quad \chi = \frac{C}{T},$$

run the RG decimation from $\Omega_0 \gg T$ down to $\Omega = T$.

The decimated spins are essentially frozen into singlets and therefore do not contribute to the susceptibility.
On the other hand, surviving spins at the scale Ω are typically coupled by bonds J << T , and therefore expected to behave as free spins.

$$\chi(T) = n(T)/T$$

$$n(T) = N(T)/N_0 \sim n_0 \ln^{-2}(\Omega_0/T).$$

$$\chi(T) = \frac{n_0}{T \ln^2(\Omega_0/T)}$$

not $T^{-\alpha}$ but a log scale

Example 2 RSRG-X :deal with arbitrary-energy excitations

- Model

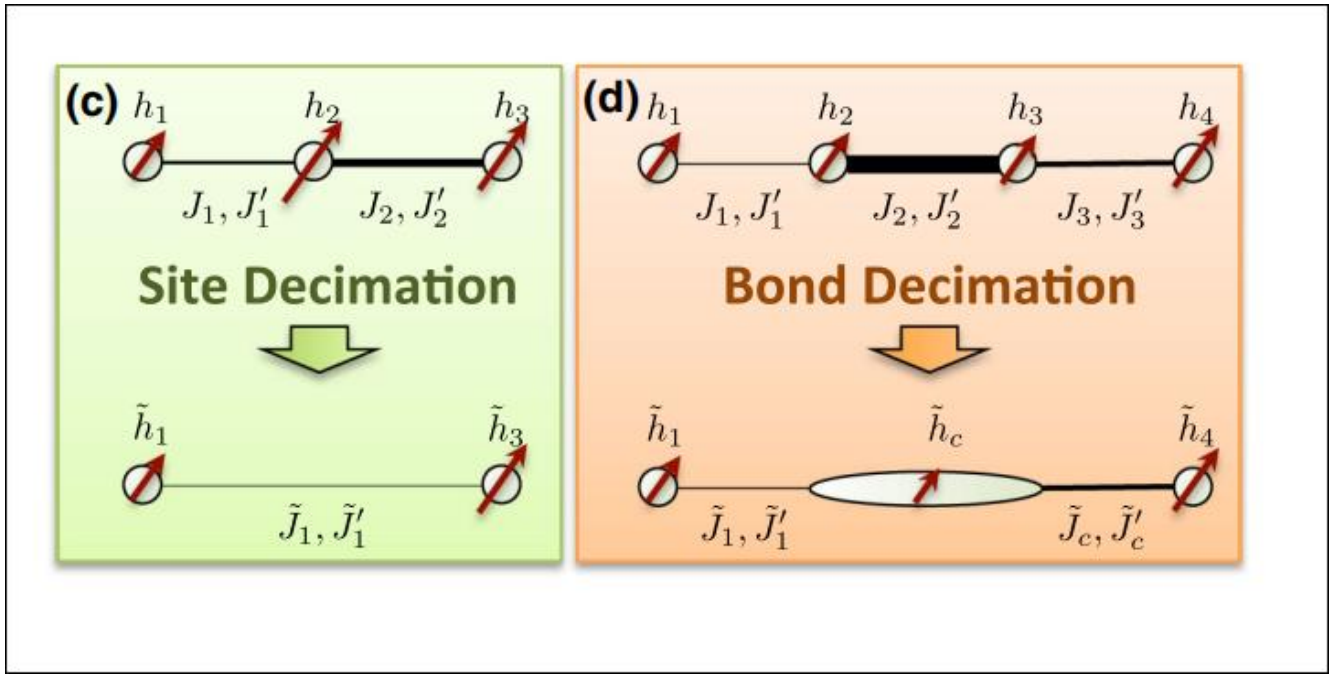
$$\mathcal{H}_{\mathrm{hJJ'}} = -\sum_i (J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z + J'_i \sigma_i^z \sigma_{i+1}^z).$$

Weakly interacting hJJ' model

J' is the interaction term and samller than the hopping and onsite disorder

RG decimation steps

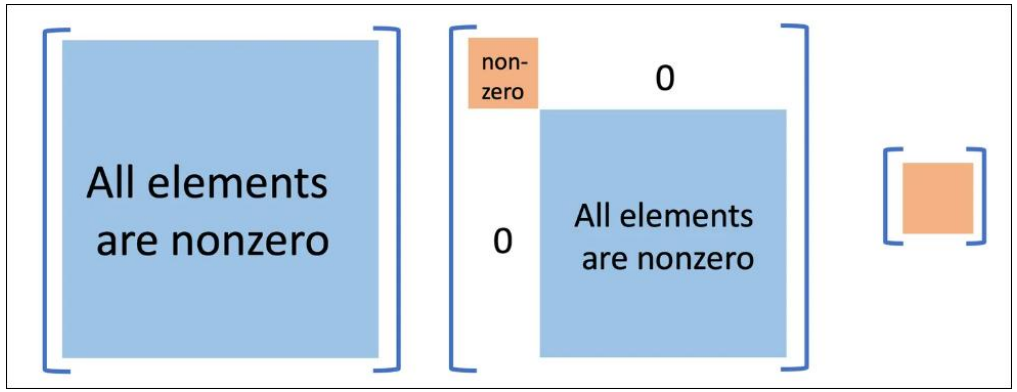
- site and bond decimations.
- site decimations:the largest gap in the system is due to a field h2.
- bond decimations: the largest gap is due to a bond J2



S-W transformation to get the new coupling terms

Consider a Hamiltonian

$$H_0 = \frac{\lambda}{2}\begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix},$$



$H_0 + V.$

$e^{iS}(H_0 + V)e^{-iS}$

a gap is needed

unitary transformation e^{iS},

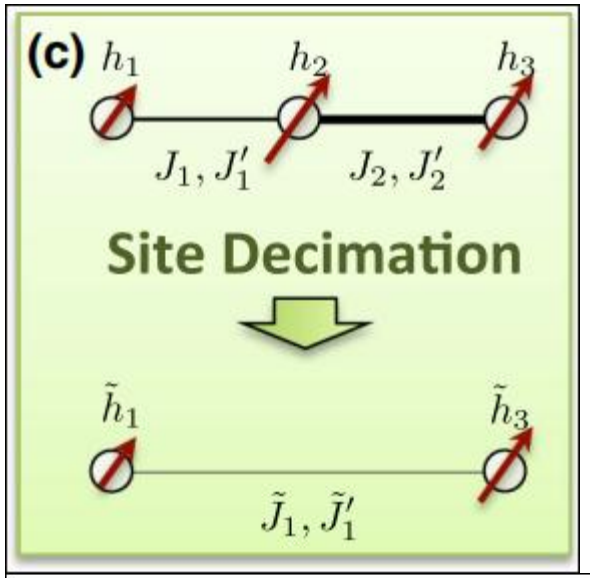
off-diagnoal terms are zero

$$P_\alpha([iS^{(1)}, H_0] + V)P_{\bar{\alpha}} = 0,$$

$$S^{(1)} = \frac{1}{i\lambda}\sum_\alpha \frac{(-1)^\alpha}{i\lambda} P_\alpha V P_{\bar{\alpha}}.$$

$$\mathcal{H}_{\text{eff}} = e^{iS}(H_0 + V)e^{-iS} - H_0 \approx \sum_{\alpha=a,b} P_\alpha(V + [iS^{(1)}, V]$$
$$-\frac{1}{2}\{(S^{(1)})^2, H_0\} + S^{(1)}H_0S^{(1)})P_\alpha, \qquad \text{(A6)}$$

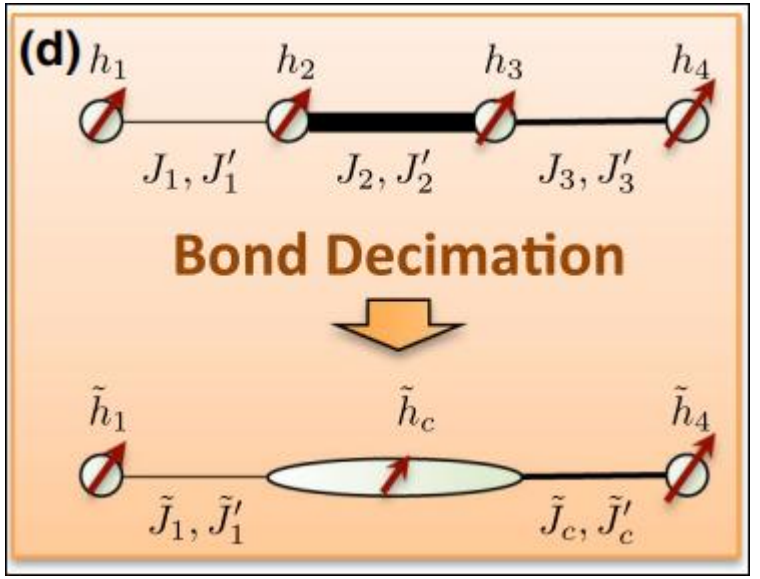
Application to the hJJ' model



$$\tilde{h}_1 \to h_1 + cJ_1', \qquad \tilde{h}_3 \to h_3 + cJ_2',$$

$$\tilde{J}_1 \to c\frac{J_1 J_2}{h_2}, \qquad \tilde{J}_1' \to 0,$$

$$\lambda \to c\left(h_2 + \frac{J_1^2 + J_2^2}{2h_2}\right),$$



$$\tilde{h}_1 \to h_1 + c\frac{h_2 J_1'}{J_2}, \qquad \tilde{h}_c \to J_2' + c\frac{h_2 h_3}{J_2},$$

$$\tilde{h}_4 \to h_4 + c\frac{h_3 J_3'}{J_2},$$

$$\tilde{J}_1 \to cJ_1, \qquad \tilde{J}_c \to J_3,$$

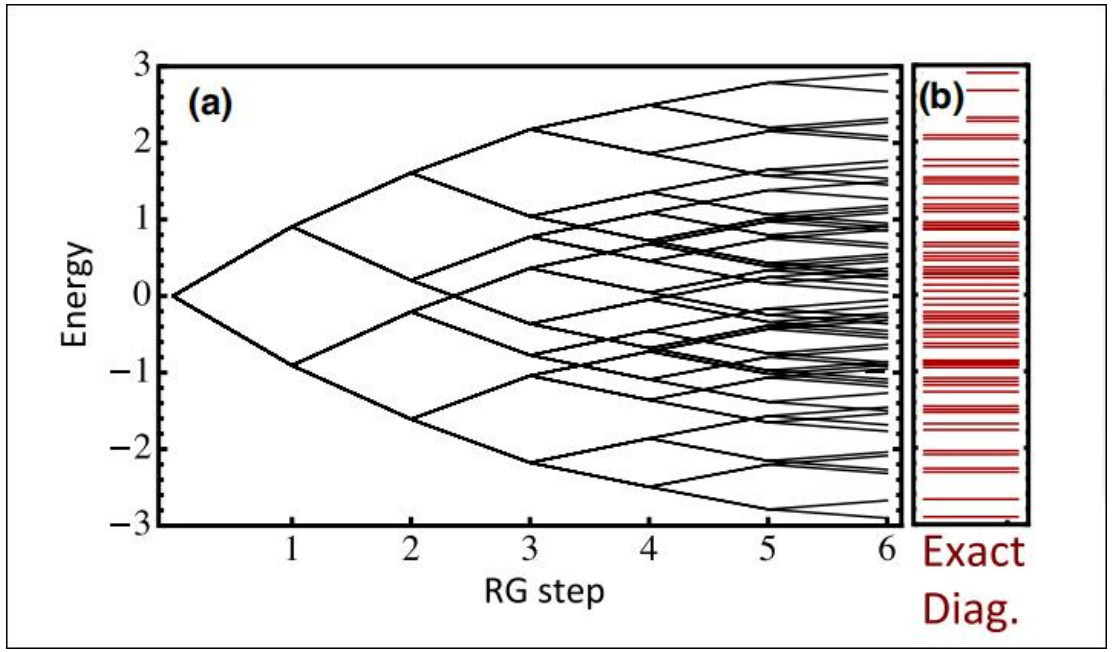
$$\tilde{J}_1' \to c\frac{h_3 J_1'}{J_2}, \qquad \tilde{J}_c' \to c\frac{h_2 J_3'}{J_2},$$

$$\lambda \to c\left(J_2 + \frac{h_2^2 + h_3^2 + (J_1')^2 + (J_3')^2}{2J_2}\right).$$

RSRG-X steps depends on c at all higher energy scales

How to construct the entire many-body spectrum?

- the construction of all branches of the tree, which is an exponentially hard task.



$$\text{branching choices } b = \{c_1, c_2, \ldots, c_L\}$$

The eigenvalues and eigenvectors are labeled by ‘b’

- In the calculation, espically the thermal average, we can use the Monte Carlo method to get typical eigenstates.

Take thermal samples by Monte Carlo method

- the many-body eigenstates can be described via a sequence of branching choices

$$b = \{c_1, c_2, \ldots, c_L\}$$

L: number of sites in the chain

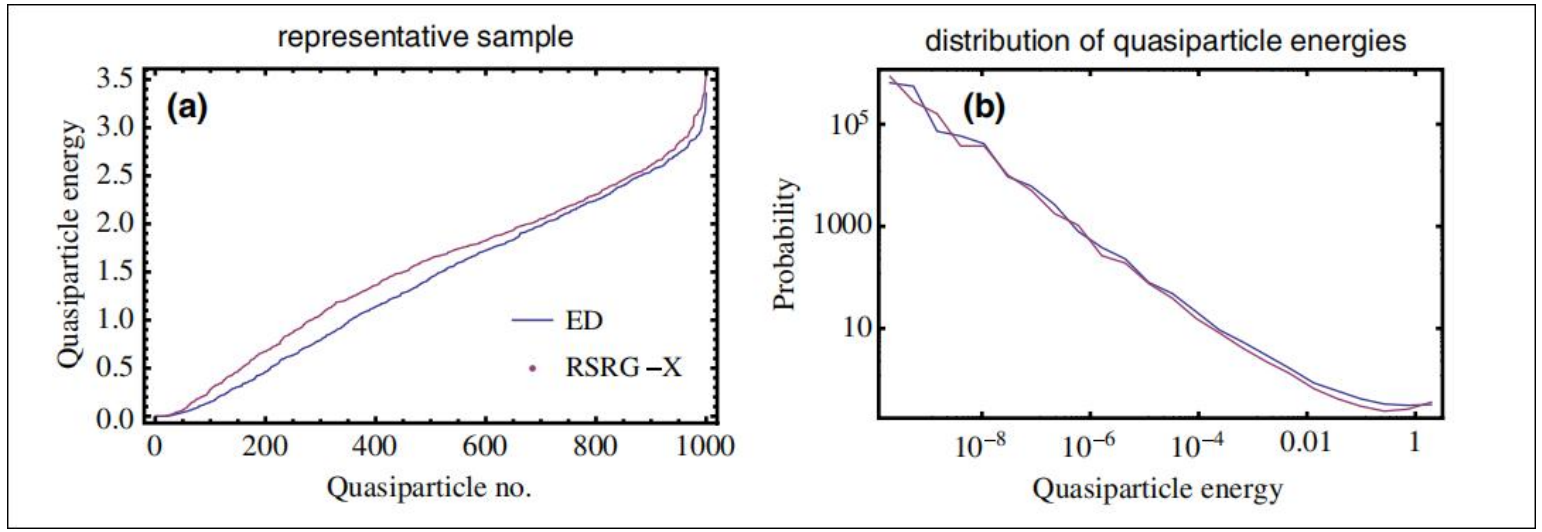
- How to get a new Monte Carlo state:

1. go up the tree a random number of nodes, e.g. m and flip $c_{L-m} \rightarrow -c_{L-m}$.
2. Then performing the RSRG-X steps from m to L
3. accept or reject the new state b' according to METROPOLIS(related to the temperature)

Benchmarking

$$H = \sum_i J_i (c_i^\dagger c_{i+1}^\dagger + c_i^\dagger c_{i+1} + \text{H.c.}) + h_i (1 - 2 c_i^\dagger c_{i+1}^\dagger)$$

- Single-particle spectrum long hJ spin chains.



The mismatch between RSRG-X and ED is a consequence of the errors made by the RSRG-X in the initial part of the flow, before strong disorder is reached.
So the mismatch is in the middle of the system

L=1000, average single-particle spectrum, for a collection of 50 disorder realizations.

For RSRG-X, in the noninteracting case, we can associate each branching choice with a quasiparticle with the energy that corresponds to the splitting between the two choices.

L=10

Benchmarking

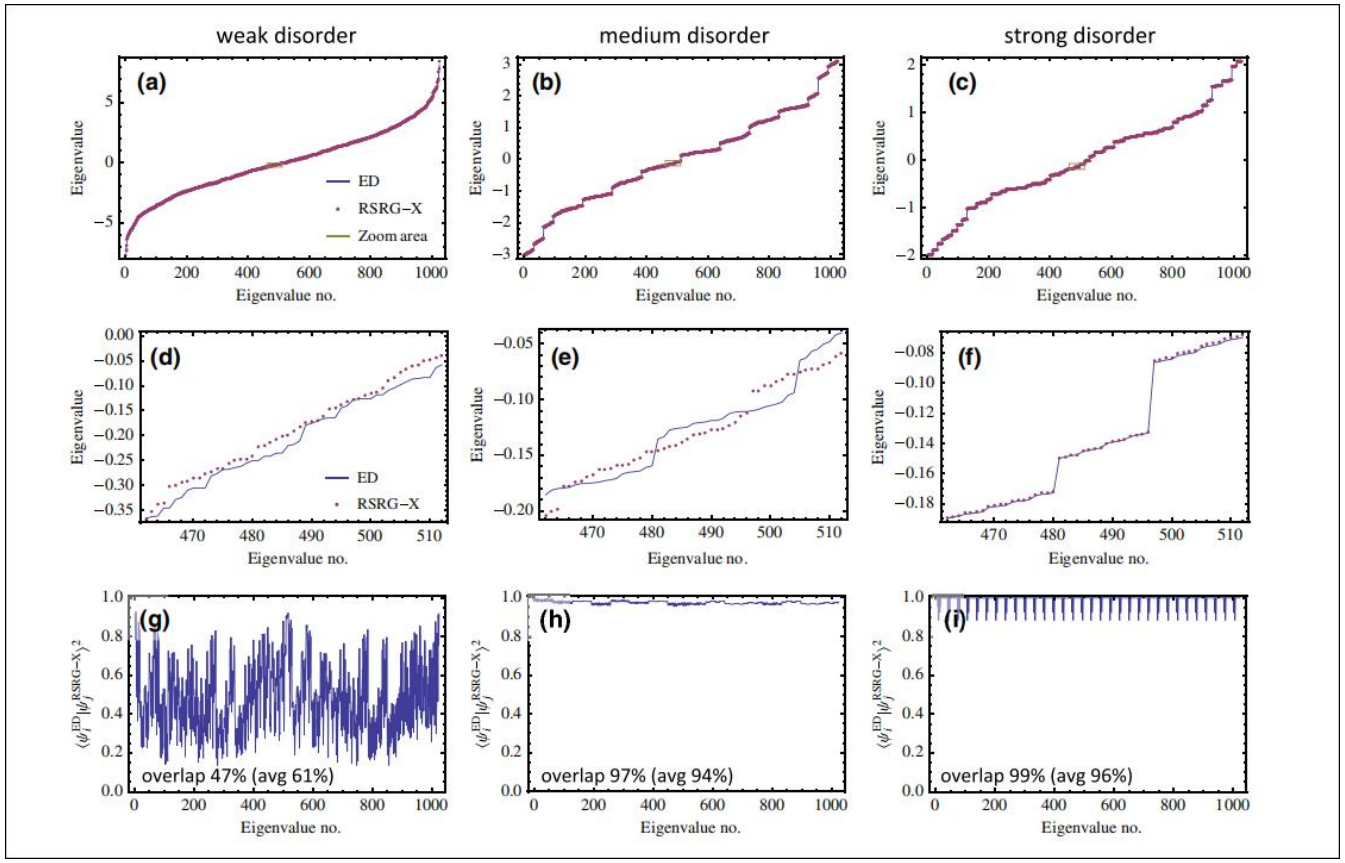
- nonintegrable spin chains

Initial distribution:

$$P(J) \sim J^{1/a-1},$$

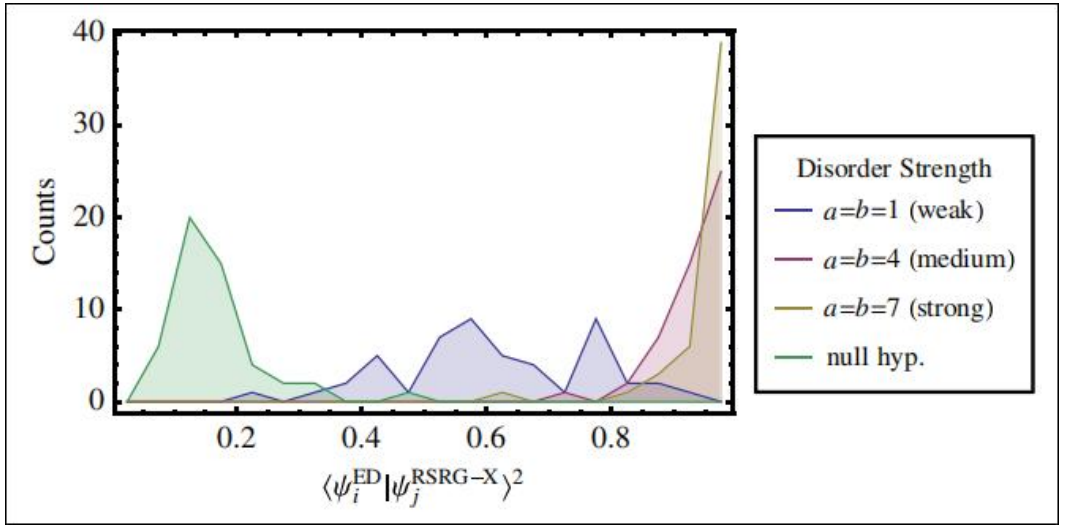
$$P(h) \sim h^{1/b-1}$$

increasing a and b corresponds to stronger disorder



Eigenvalues and Eigenstates

disorder strength: weak ($a = b = 1$), medium ($a = b = 4$), and strong ($a = b = 7$).



maximize $\langle \psi_i | \phi_j \rangle^2$ over j's

$\frac{1}{N} \sum_i \max_j [\langle \psi_i | \phi_j \rangle^2]$,

find the corresponding eigenstate

RSRG-X flows

- Accoding to the RG rules, we can see the flow of the full distributions of coupling constants.
- define two parameters to characterize the flow

$$\mathcal{C} = [\![\langle \mathrm{avg}[\ln|h|] - \mathrm{avg}[\ln|J|]\rangle_{\mathrm{Th}}]\!]$$ relative stength of h and J distributions

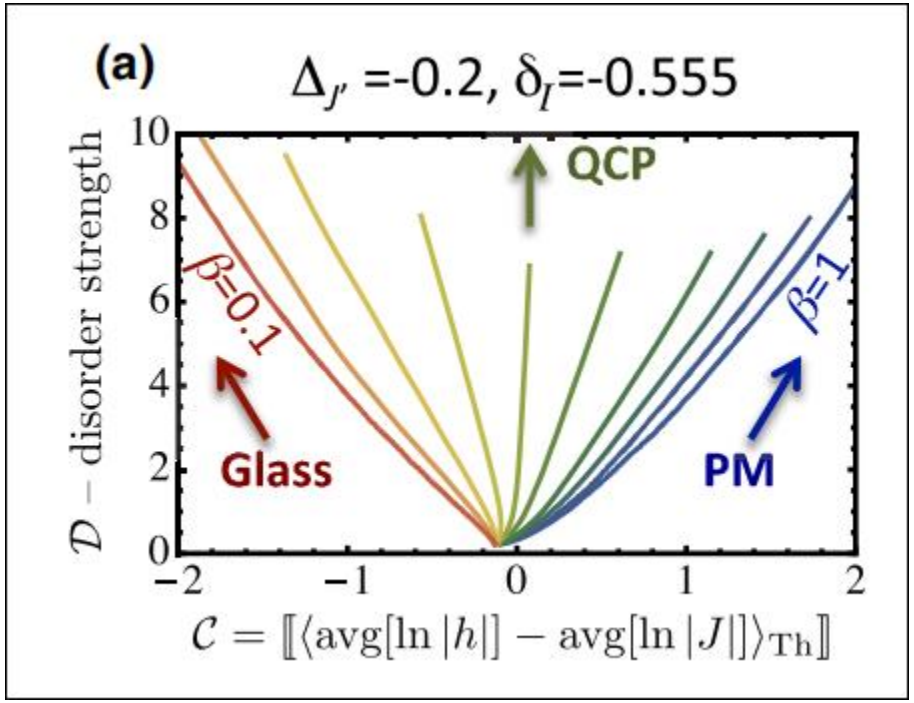
$$\mathcal{D} = [\![\langle \mathrm{var}[\log|h|] + \mathrm{var}[\log|J|]\rangle_{\mathrm{Th}}]\!],$$ disorder strength

where $\langle\cdots\rangle_{\mathrm{Th}}$ and $[\![\cdots]\!]$ represent thermal and disorder averaging.

Renormalization group flows in the C-D plane

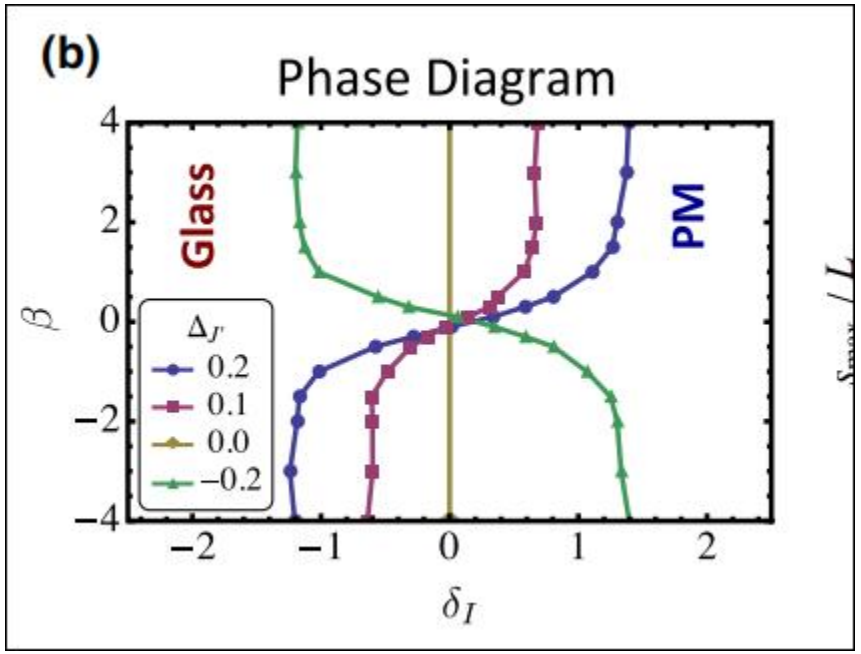
$$\beta = 1/T$$

the temperature determine the Thermal average



This leads to a temperature-driven dynamical transition, which is the focus of this work.
the inverse temperature β tunes whether the flow is towards the Hilbert glass or the paramagnet.

Dynamical phase diagram



$$\beta = 1/T$$

The shape of the phase diagram can be understood by noting the effect of the site decimation RG rule on the initial distributions:
When $\triangle$>0, J' increase h in the excited states bracnched and reduces it in low-energy branches.

Thanks