# Multifractal states and anomalous mobility edges



2023.3.16

# outline

- Anomalous mobility edges
- Mechanism of multifractal states
- MBC and Ergodic breaking

# references

(1)

SciPost Phys. 12, 027 (2022)

**Anomalous mobility edges in one-dimensional quasiperiodic models**

Tong Liu[1*], Xu Xia[2*], Stefano Longhi[3,4] and Laurent Sanchez-Palencia[5]



(2)

**Exact new mobility edges between critical and localized states**

arXiv:2212.14285

# unbounded quasiperiodic potential

- a tight-binding model with nearest-neighbor hopping and quasiperiodic on-site potential

$$E\psi_n = \psi_{n+1} + \psi_{n-1} + v(2\pi\alpha n + \theta)\psi_n\,,$$

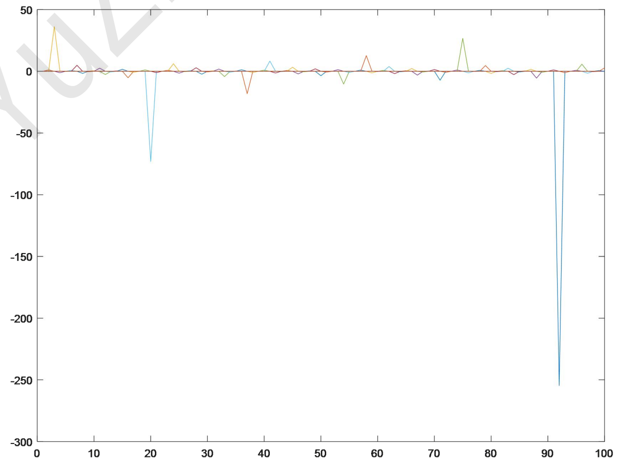
consider unbounded quasiperiodic potential but does not diverge at any lattice site n $\in$ Z

The Simon-Spencer theorem: absolutely continuous spectra/extended states are forbidden

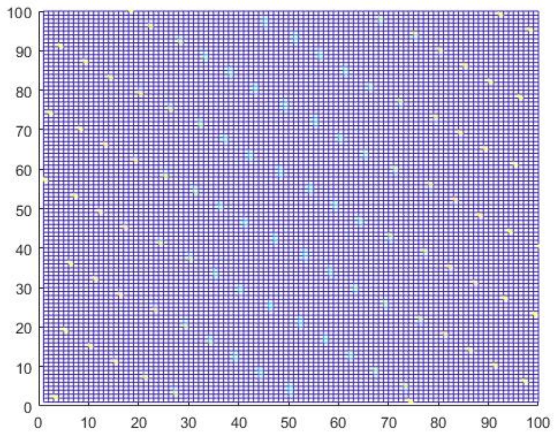
example : Maryland model

$$v(x) = V\tan(x),$$

exponentially localized states




potential




Eigenstates

Extended states are forbidden but multifractal states are allowed

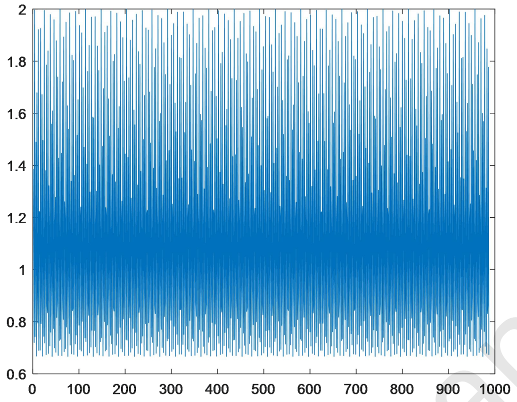
$$v(x) = \frac{V}{1 - a\cos(x)},$$

Anomalous mobility edge

multifractal states/ localized states

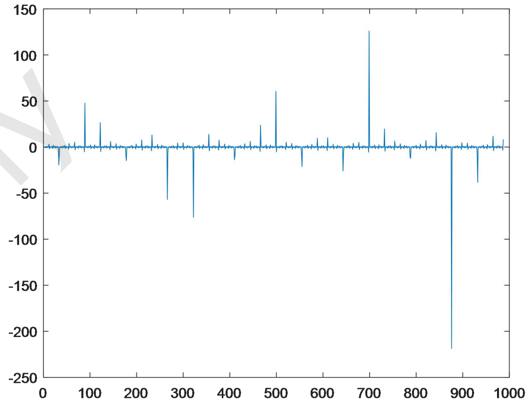
a=0.5

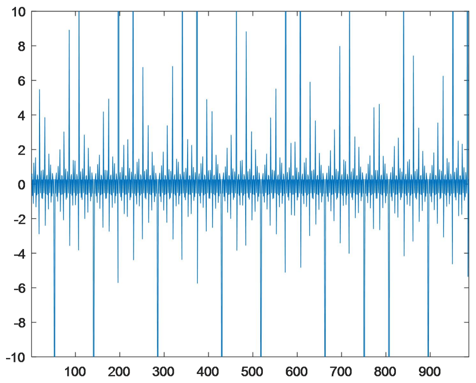
bounded potential

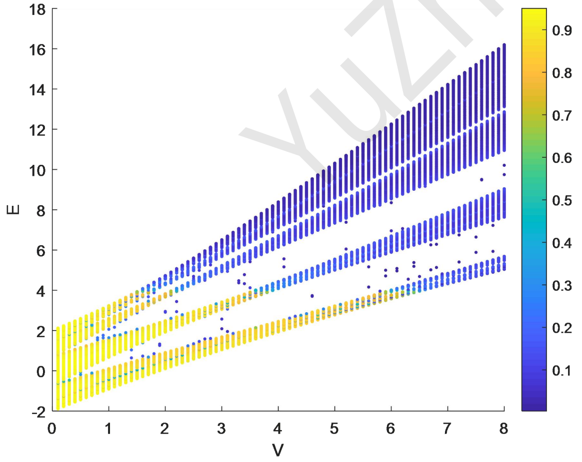


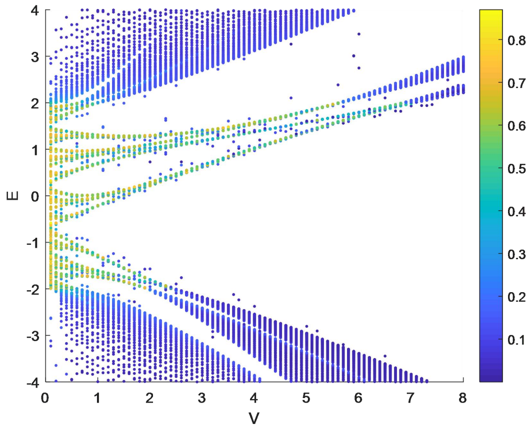
a=3

unbounded potential








Mobility edges

Anomalous mobility edges

# Mobility edges from Avila global theory

$$\psi_{n+1} + \psi_{n-1} + v_n \psi_n = E \psi_n \,,$$

$$v_n = v(x = 2\pi\alpha n + \theta), v(x) = V/[1 - a\cos(x)] \; (a > 1),$$

Lyapunov exponent:

$$\gamma_0(E) = \lim_{n\to\infty} \frac{1}{2\pi n} \int_0^{2\pi} d\theta \log ||T_n(\theta)|| \,,$$

$$T_n(\theta) = \prod_{l=0}^{n-1} \begin{pmatrix} E - v(2\pi\alpha l + \theta) & -1 \\ 1 & 0 \end{pmatrix} = \prod_{l=0}^{n-1} T(2\pi\alpha l + \theta),$$

$$T(\theta) = \begin{pmatrix} E - v(\theta) & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} E - \frac{V}{1 - a\cos\theta} & -1 \\ 1 & 0 \end{pmatrix}.$$

- complex extension of the LE

$$\gamma_\epsilon(E) = \lim_{n\to\infty} \frac{1}{2\pi n}\int_0^{2\pi} d\theta \log \|T_n(\theta + i\epsilon)\|.$$

- remove the singularity

$$T(\theta) = \frac{1}{1 - a\cos\theta} B(\theta),$$

$$\gamma_\epsilon(E) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \log \frac{1}{|1 - a\cos(\theta + i\epsilon)|} + \gamma_\epsilon^1(E),$$

$$\gamma_\epsilon(E) = -|\epsilon| - \log\left(\frac{a}{2}\right) + \gamma_\epsilon^1(E),$$

a>1 $\quad -|\epsilon| - \log\left(\frac{a}{2}\right)$

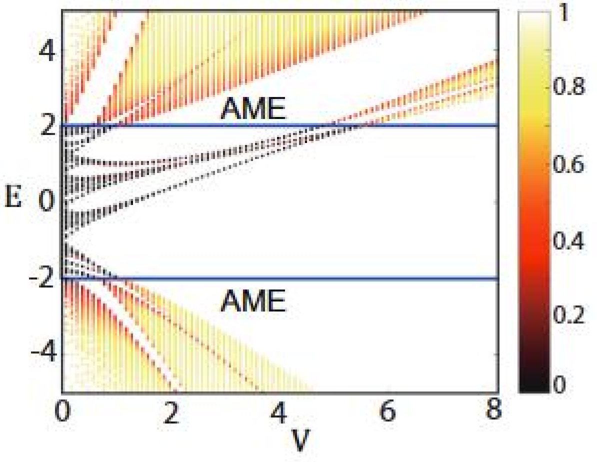
a<1 $\quad \mathrm{Log}\left[\frac{1+\sqrt{1-a^2}}{2}, E\right]$

$$\begin{aligned}\gamma_\epsilon(E) &= \lim_{n\to\infty}\frac{1}{n}\log\left\|\begin{pmatrix} E & -1 \\ 1 & 0\end{pmatrix}^n\right\| + O(1) \\ &= \log\left|\frac{E \pm \sqrt{E^2-4}}{2}\right| + O(1),\end{aligned}$$

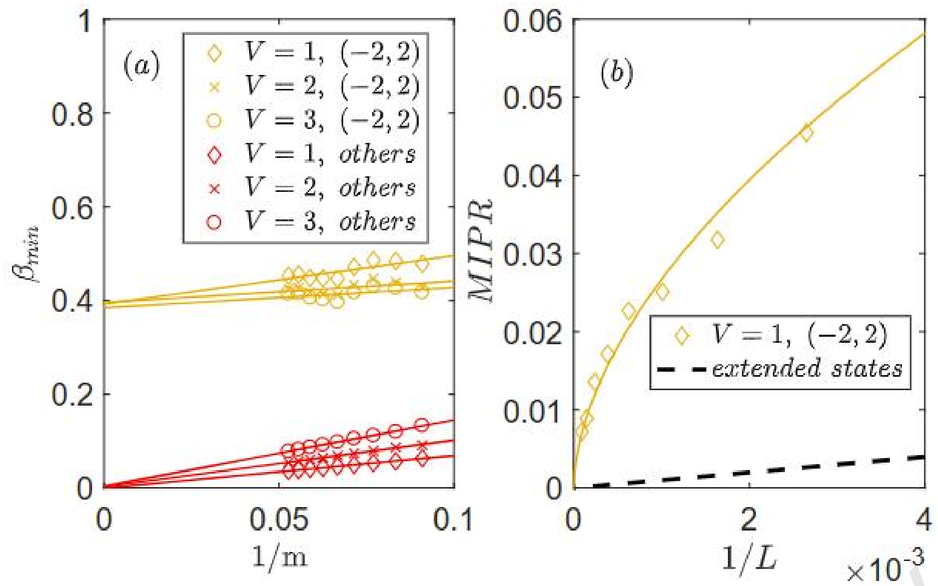
$$\gamma(E) = \max_{\pm}\left(\ln\left|\frac{E \pm \sqrt{E^2-4}}{2}\right|\right).$$

$[-2, 2]$

(b)



the energy spectrum in the interval [−2, 2] is singular continuous and the wave functions are all critical, i.e. they are neither exponentially localized nor extended, but multifractal.




$$MIPR = \frac{1}{L'}\sum IPR_n$$

$$MIPR \sim 1/L^{0.56}$$

origin:high potential

$$\overline{\lim_{n\to\infty}}\,|v(n)| = \overline{\lim_{n\to-\infty}}\,|v(n)| = \infty \ ,$$

**Trace Class Perturbations and the Absence of Absolutely Continuous Spectra**

Barry Simon[1] and Thomas Spencer[2]

multifractal analysis
another scaling behavior

for each wavefunction

F:system size

$$P_n^j = |\psi_n^j|^2 \sim (1/F_m)^{\beta_n^j}.$$

$\beta_{\min} = 1$ extended eigenstates

$\beta_{\min} = 0$ Localized eigenstates

$0 < \beta_{\min} < 1$ multifractal eigenstates

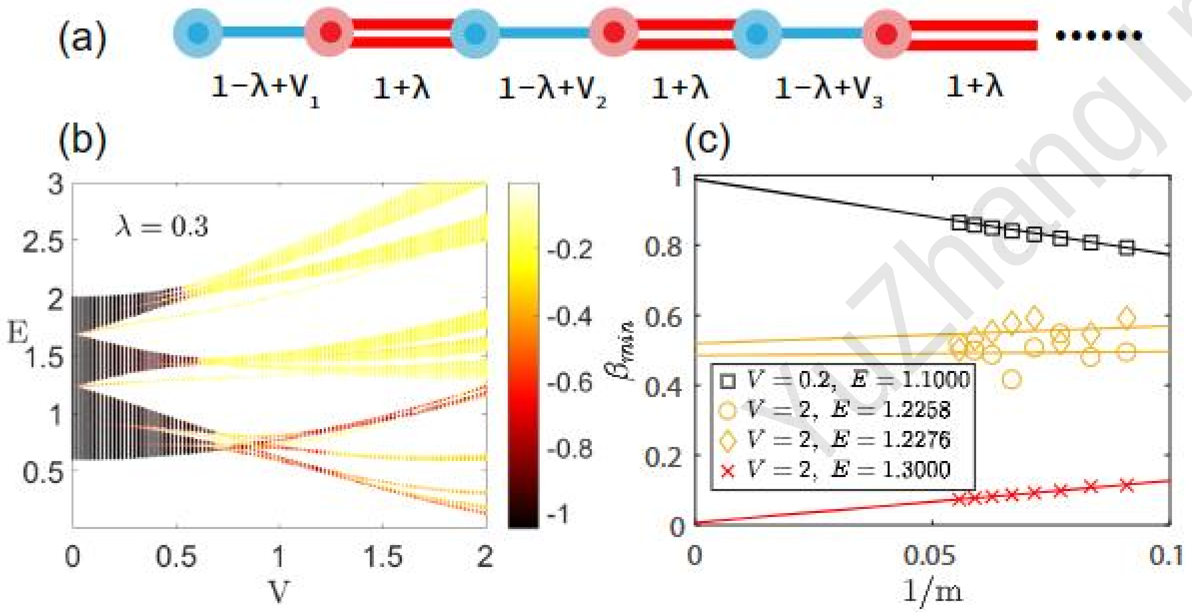
another case : $\mathrm{V} > 0$

length : $I_k \to \infty$

for E<0 discontious spectrum

Another class of models : bounded models with off-diagonal quasi-periodicity

Quasiperiodic Su-Schrieffer-Heeger model




vanishingly small but finite values of hopping amplitudes in the off-diagonal quasiperiodic SSH model play a similar role as arbitrary large but finite values of the on-site potential in the diagonal model.

## Exact new mobility edges between critical and localized states

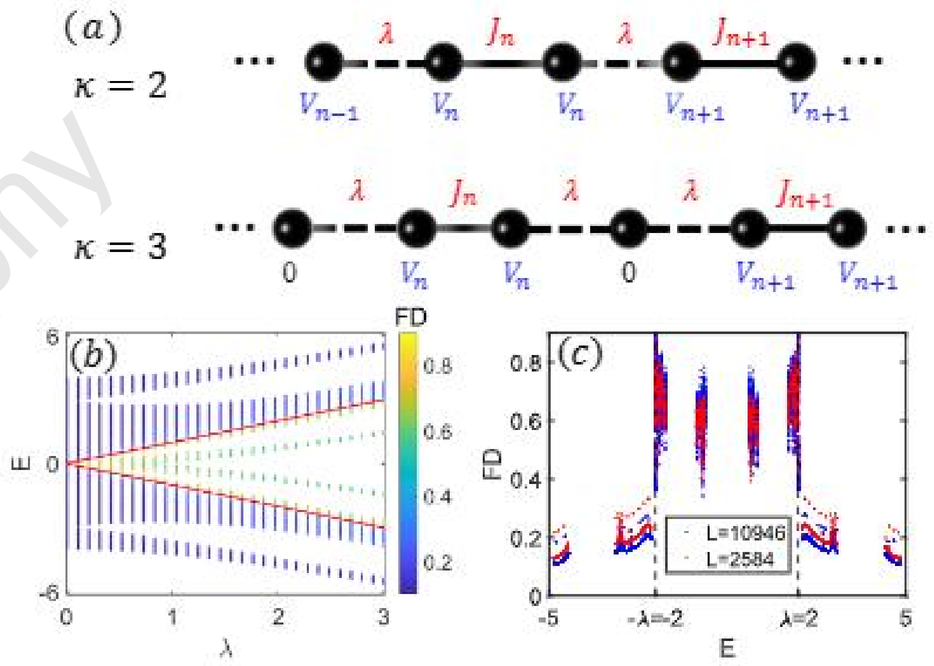
Xin-Chi Zhou, Yongjian Wang, Ting-Fung Jeffrey Poon, Qi Zhou, Xiong-Jun Liu

- exactly solvable AME in near zero hopping amplitudes
- Mechanism of critical states
- robust in the prense of perturbation and interactions

# Model

$$H = \sum_j (t_j a_j^\dagger a_{j+1} + \mathrm{h.c.}) + \sum_j V_j n_j,$$

$$\{t_j, V_j\} = \begin{cases} \{\lambda,\ 2t_0\cos[2\pi\alpha(j-1)+\theta]\}, & j = 1 \bmod \kappa, \\ 2t_0\cos(2\pi\alpha j+\theta)\{1,\ 1\}, & j = 0 \bmod \kappa, \\ \{\lambda, 0\}, & \text{else}, \end{cases} \tag{2}$$




$\lambda = 2.0$

# proof

- vanishing LEs and zeros of hopping coefficients altogether unambiguously determine the critical region
- there exists a sequence of sites $t_{2j_k} \to 0$

no AC spectrum (extended states)

If LEs=0 : multifractal states

LEs : Avila global theory

$$\gamma_\epsilon(E) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|T_n(\theta + i\epsilon)\| d\theta,$$

$\kappa = 2$

$$2\gamma_\epsilon(E) = \max\{\ln |(|E| + \sqrt{E^2 - \lambda^2})/\lambda|, 0\}.$$

$$\xi(E) = \frac{1}{\gamma_0} = \frac{2}{\ln |\frac{|E|+\sqrt{E^2-\lambda^2}}{\lambda}|}.$$

critical region for $|E| \leq |\lambda|$

localized region for $|E| > |\lambda|$.

# details of LEs

$$(H_{\lambda,\alpha,\theta}\psi)_n := V_n\psi_n + t_n\psi_{n+1} + \bar{t}_{n-1}\psi_{n-1},\ \theta\in\mathbb{T},\ \lambda\neq 0,$$

$$V_n(\theta) = \begin{cases} 2\cos[2\pi(n-1)\alpha+\theta], & n=1, \quad \bmod \kappa, \\ 2\cos(2\pi n\alpha+\theta), & n=0, \quad \bmod \kappa, \\ 0, & \text{else}, \end{cases} \qquad t_n(\theta) = \begin{cases} \lambda, & n\neq 0, \quad \bmod \kappa, \\ 2\cos(2\pi n\alpha+\theta), & n=0, \quad \bmod \kappa. \end{cases}$$

$$\gamma_\epsilon(E) = \lim_{n\to\infty}\frac{1}{n}\int \ln\|T_n(\theta+i\epsilon)\|d\theta,$$

$$\begin{aligned} T_\kappa(\theta) &= \frac{\begin{pmatrix} E/\lambda & -1 \\ 1 & 0 \end{pmatrix}^{\kappa-2}}{2\lambda\cos(2\pi\alpha+\theta)}\begin{pmatrix} E-2\cos(2\pi\alpha+\theta) & -2\cos(2\pi\alpha+\theta) \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} E-2\cos(2\pi\alpha+\theta) & -\lambda \\ 2\cos(2\pi\alpha+\theta) & 0 \end{pmatrix}, \\ &= \frac{1}{2\lambda\cos(2\pi\alpha+\theta)}\begin{pmatrix} a_\kappa & -a_{\kappa-1} \\ a_{\kappa-1} & -a_{\kappa-2} \end{pmatrix}\begin{pmatrix} E^2-4E\cos(2\pi\alpha+\theta) & -\lambda E+2\lambda\cos(2\pi\alpha+\theta) \\ \lambda E-2\lambda\cos(2\pi\alpha+\theta) & -\lambda^2 \end{pmatrix}, \end{aligned}$$

$$a_\kappa = \frac{1}{\sqrt{E^2/\lambda^2-4}}\left[\left(\frac{E/\lambda+\sqrt{E^2/\lambda^2-4}}{2}\right)^{\kappa-1} - \left(\frac{E/\lambda-\sqrt{E^2/\lambda^2-4}}{2}\right)^{\kappa-1}\right].$$

$$
\begin{aligned}
\widetilde{T}_\kappa(\theta) &:= 2\lambda\cos(2\pi\alpha+\theta)\,T_\kappa(\theta),\\
&= \begin{pmatrix} a_\kappa & -a_{\kappa-1} \\ a_{\kappa-1} & -a_{\kappa-2} \end{pmatrix} \begin{pmatrix} E^2-4E\cos(2\pi\alpha+\theta) & -\lambda E+2\lambda\cos(2\pi\alpha+\theta) \\ \lambda E-2\lambda\cos(2\pi\alpha+\theta) & -\lambda^2 \end{pmatrix},
\end{aligned}
$$

$$
\gamma_\epsilon(E) = \tilde{\gamma}_\epsilon(E) - \frac{1}{2}\int \ln|2\lambda\cos(\theta+i\epsilon)|d\theta = \tilde{\gamma}_\epsilon(E) - \frac{1}{2}\ln|\lambda| - \pi|\epsilon|,
$$

$$
T_\kappa(\theta+i\epsilon) = \frac{1}{\lambda}\begin{pmatrix} a_\kappa & -a_{\kappa-1} \\ a_{\kappa-1} & -a_{\kappa-2} \end{pmatrix}\begin{pmatrix} -2E & \lambda \\ -\lambda & 0 \end{pmatrix} + o(1).
$$

$$
\gamma_\epsilon(E) = \frac{1}{\kappa}\ln\left|\frac{|Ea_\kappa-\lambda a_{\kappa-1}|+\sqrt{(Ea_\kappa-\lambda a_{\kappa-1})^2-\lambda^2}}{\lambda}\right| + o(1).
$$

$$
\gamma_\epsilon(E) = \frac{1}{\kappa}\max\{\ln\left|\frac{|Ea_\kappa-\lambda a_{\kappa-1}|+\sqrt{(Ea_\kappa-\lambda a_{\kappa-1})^2-\lambda^2}}{\lambda}\right|, 0\}.
$$

# Mechanism of critical states

- zero hopping coefficients :

  ruling out possibility of supporting extended states

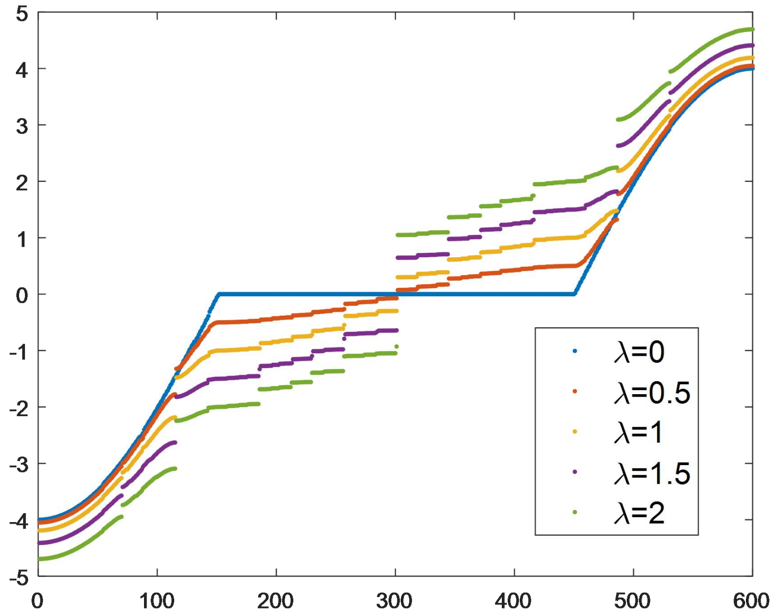
$\lambda \to 0$ a series of dimmers $J_j = V_j = 2\cos(2\pi\alpha j)$

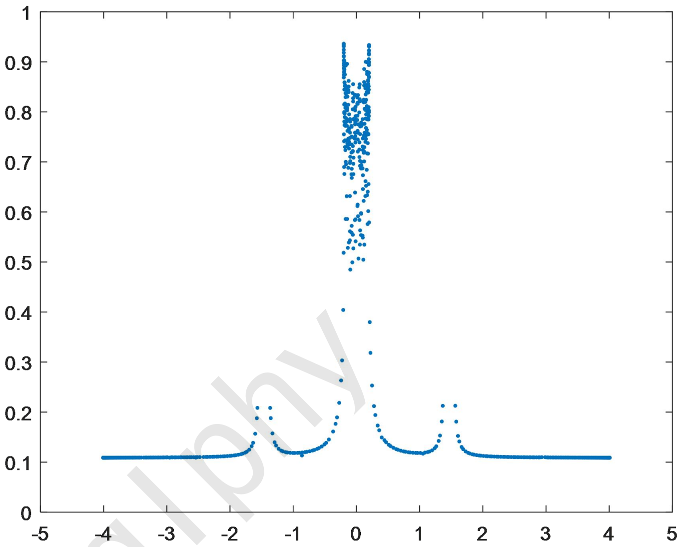
$E_1 = 2J_j$ and $E_2 = 0.$

the zero-energy flat-band modes localized modes,

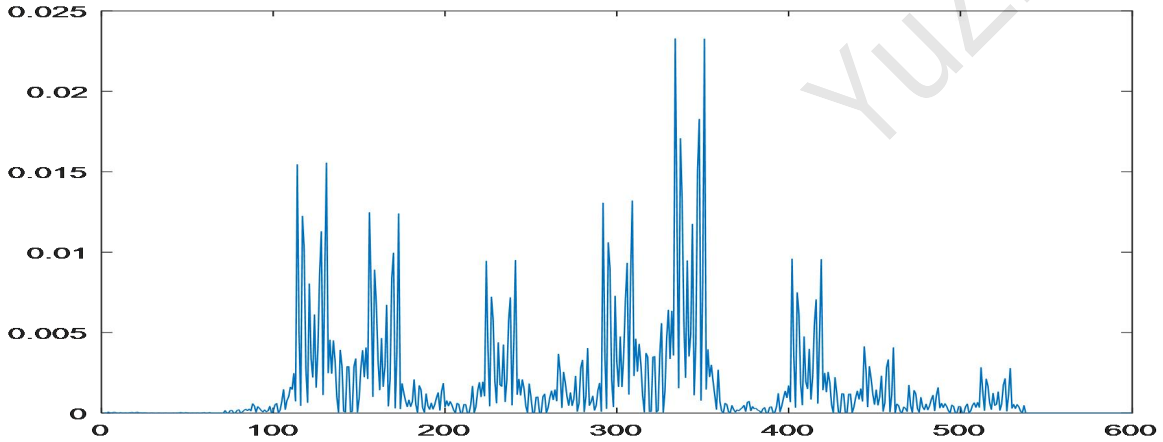
inclusion of λ hybridizes the zero-energy flat-band modes and localized modes, yielding the critical states and MEs between them and localized ones.

lambda=0.2



a typical multifractal state



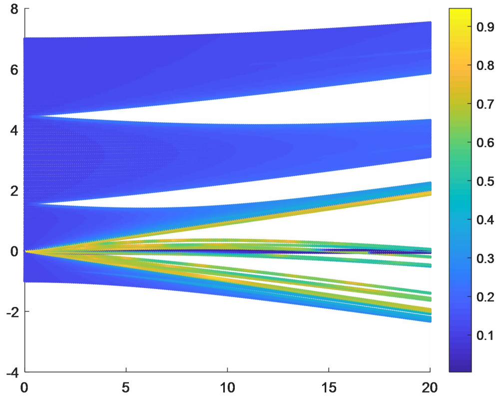
starting from band center

the number of critical states equals to that of localized states under the exactly solvable condition

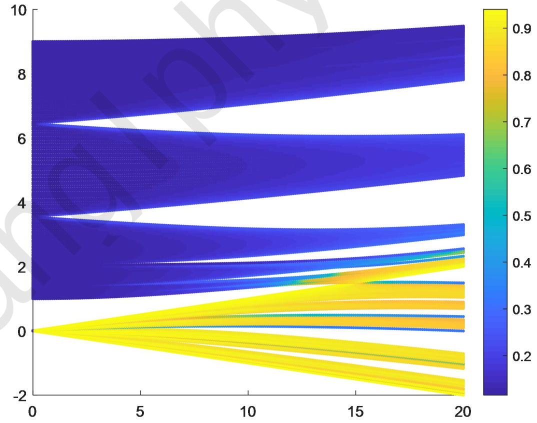
# zero hopping coefficients ?

$$J_j = V_j = J_0 + 2\cos(2\pi\alpha j)$$

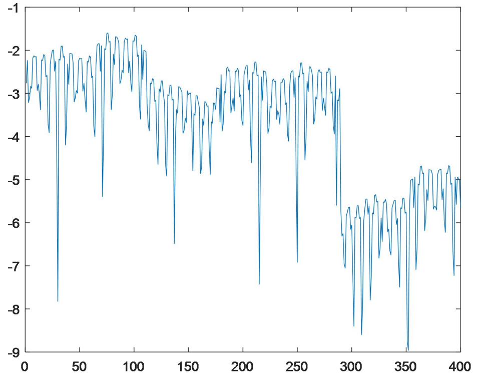
no zero hopping coefficients

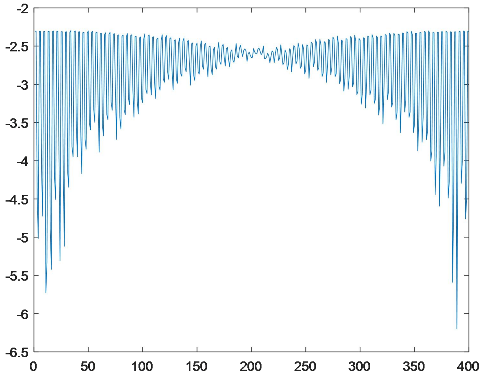



J0=1.5



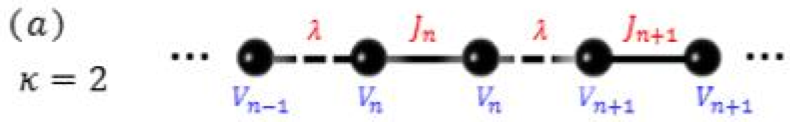

J0=2.5



# perturbations for lambda

简并微扰




H0 : zero energy states and localized eigenstates

$$J_j = V_j = J_0 + 2\cos(2\pi\alpha j)$$

$$\det(\overline{H}^{(1)} - E_\alpha^{(1)} I) = 0$$

$$\underline{U}^{(1)}_{m\beta,n\alpha} = \frac{\sum_\gamma \underline{H}^{(1)}_{m\beta,n\gamma} \overline{U}^{(0)}_{n\gamma,n\alpha}}{E_n^{(0)} - E_m^{(0)}}$$

$$E_{n\alpha} = E_n^{(0)} + \lambda E_{n\alpha}^{(1)}$$

$$\begin{aligned} |1n\alpha\rangle &= \sum_{m\beta_0} |0m\beta_0\rangle \left( \overline{U}^{(0)}_{m\beta_0,n\alpha} + \lambda \underline{U}^{(1)}_{m\beta_0,n\alpha} \right) \\ &= |0n\alpha\rangle + \lambda \sum_{m\beta_0} |0m\beta_0\rangle \underline{U}^{(1)}_{m\beta_0,n\alpha} \end{aligned}$$

# perturbations for lambda

- $\lambda$ as a perturbation parameter

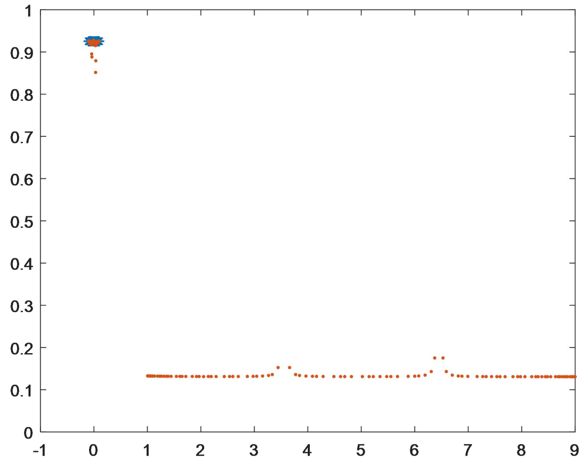
$$|1n\alpha\rangle = \sum_{m\beta_0} |0m\beta_0\rangle \left( \overline{U}^{(0)}_{m\beta_0,n\alpha} + \lambda \underline{U}^{(1)}_{m\beta_0,n\alpha} \right)$$
$$= |0n\alpha\rangle + \lambda \sum_{m\beta_0} |0m\beta_0\rangle \underline{U}^{(1)}_{m\beta_0,n\alpha}$$

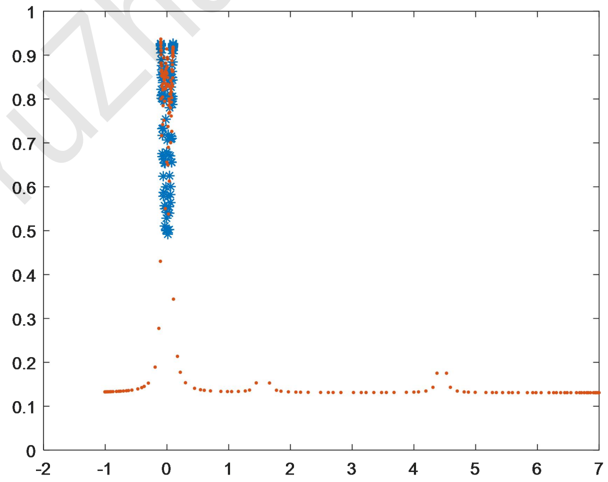
$\lambda = 0.1$

J0=2.5

J0=1.5

D




E

# Interaction

$$H = H_0 + U \sum_j n_j n_{j+1},$$

localized orbitals
extended orbitals

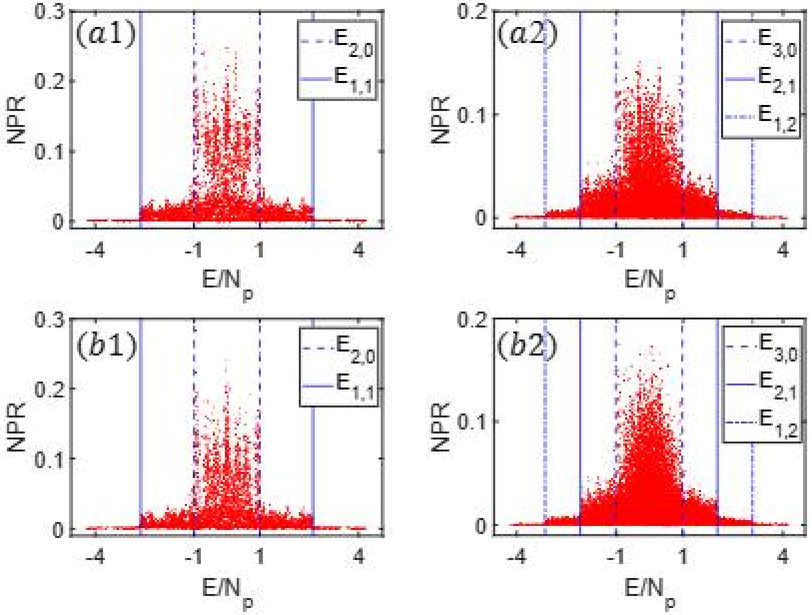
$$\mathrm{NPR} \;=\; 1/(\textstyle\sum_{\{c\}} |u_{m,c}|^4 \times V_H),$$

$N_{\mathrm{p}} = 2,$ $L = 120.$

$N_{\mathrm{p}} = 3,$ $L = 60.$

$U = 0$

$U = 1.4$



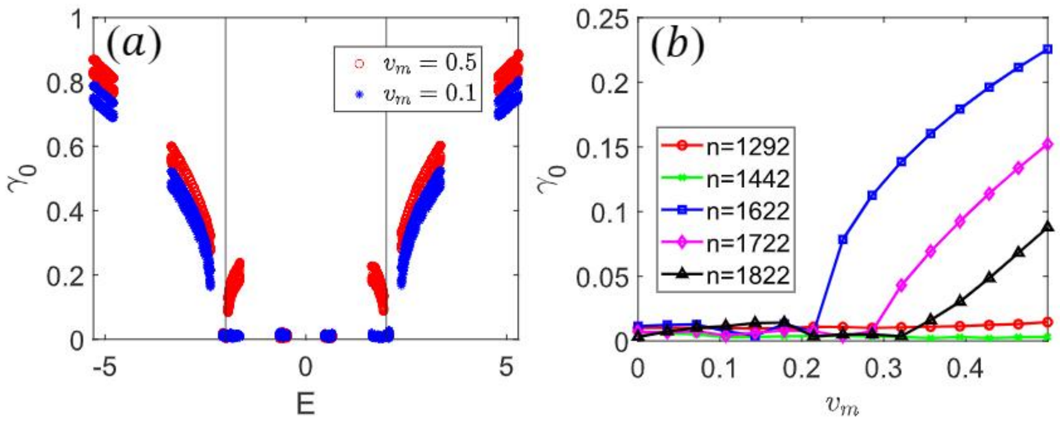
# single-particle perturbation

$$H_p = H + \sum_j V_j^p n_j,$$

$$V_j^p = v_m \cos(2\pi\alpha j + \theta) \text{ for even-}j \text{ sites}$$

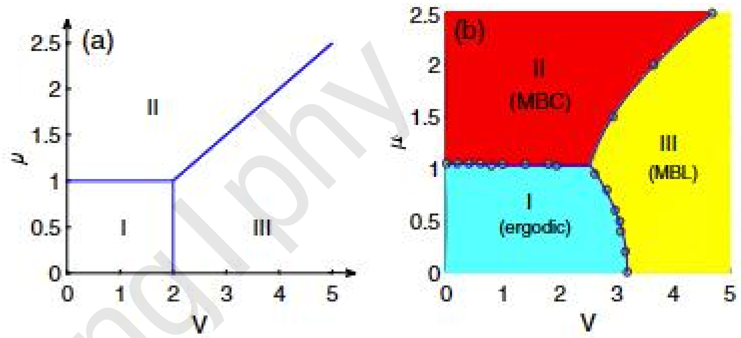
$$V_j^p = v_m \cos[2\pi\alpha(j-1) + \theta]) \text{ for odd-}j \text{ sites.}$$

$\lambda = 2.0$ and $L = 2584.$



# Many body critical region and zero hopping terms

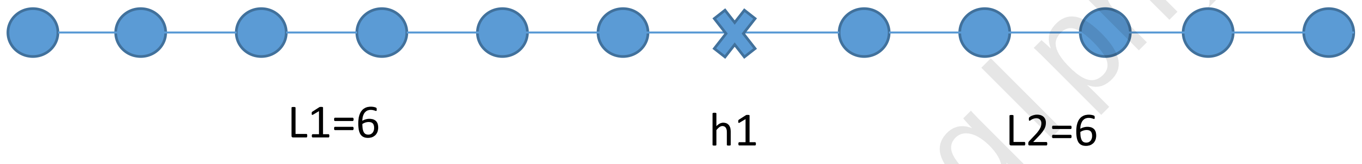
$$H_0 = \sum_j \left\{ \left( 1 + \mu \cos\left[ 2\pi \left( j + \frac{1}{2} \right) \alpha + \delta \right] \right) c_j^\dagger c_{j+1} + \text{H.c.} \right.$$
$$\left. + V \cos(2\pi j \alpha + \delta) c_j^\dagger c_j \right\}. \qquad (1)$$




$$H = H_0 + U \sum_j n_j n_{j+1},$$

- zero hopping terms ?
- divide the hilbert space into different spaces

# model



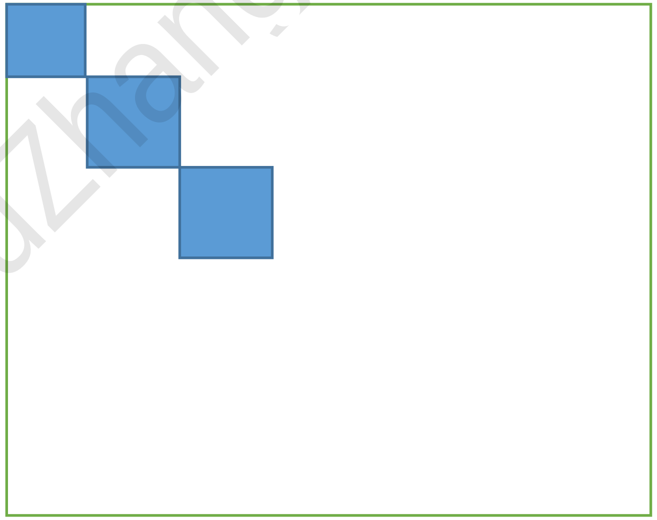

When hopping h1=0

Hilbert space are divided into many different parts

adding small quasiperiodic potential \intergal system

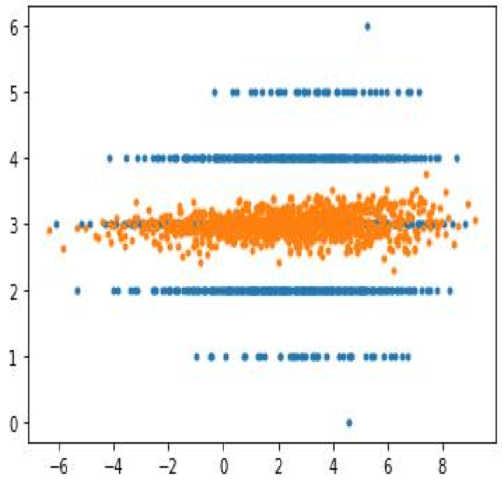
# hilbert space with h=0

- |000000111111 > 1
- |100000|111110> , |100000|111101> .... 6*6 =36
- ...
- L/2 parts ~ O(L)



# Ergodic breaking / violate ETH

$$O(E) = \sum_{j=1}^{L/2} \langle \psi_E | n_j | \psi_E \rangle$$

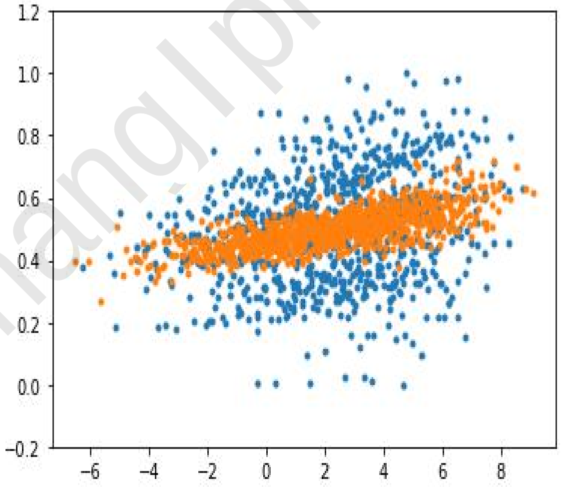


seven different parts

h1=0

# Observables

$$\langle n_3 \rangle$$



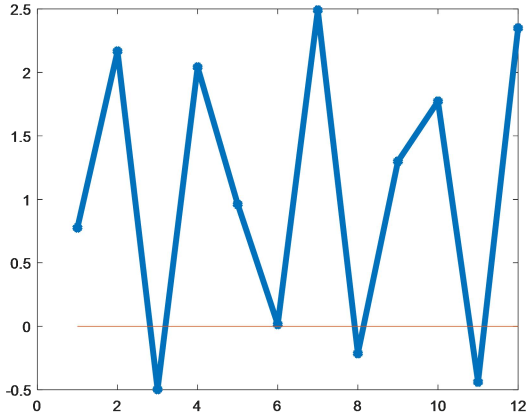
h1=0

yellow points h1=1

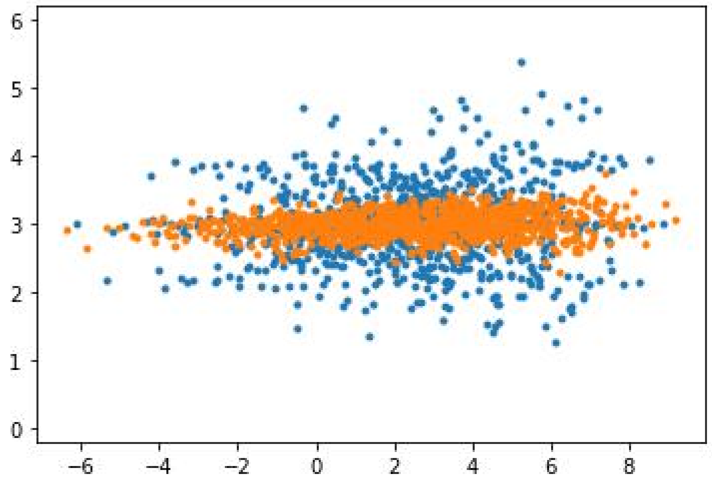
# near zero hopping terms ?

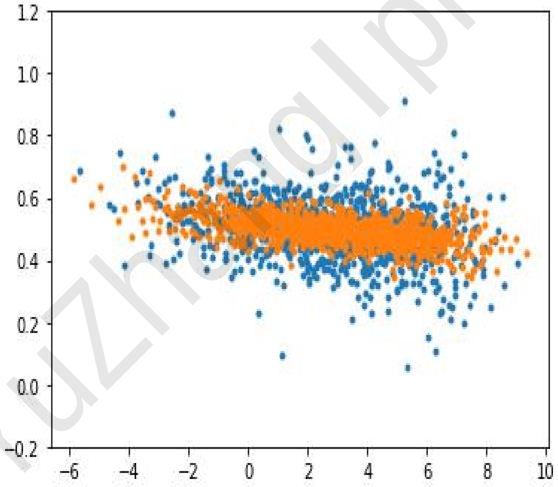
$$O(E) = \sum_{j=1}^{L/2} \langle \psi_E | n_j | \psi_E \rangle$$

$$\langle n_3 \rangle$$




\mu=1.5




h1=0.1

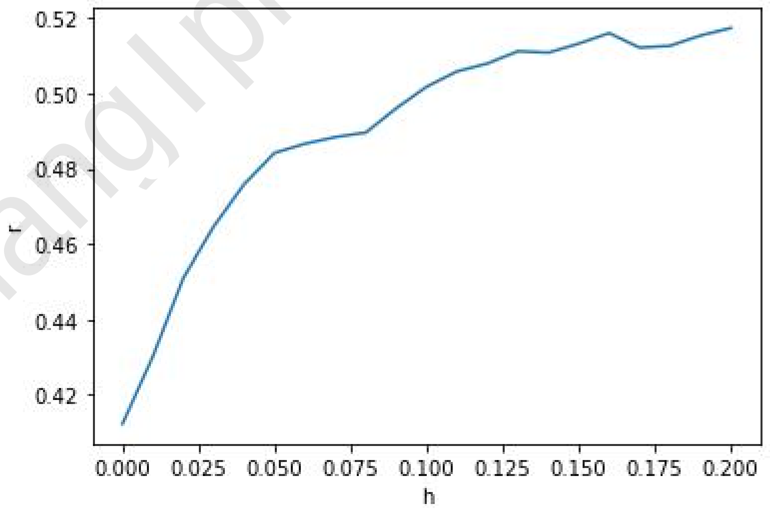
h1=0.1

yellow points h1=1

# Energy level statistics

- we consider weak disorder
- h=1 r =0.53
- h=0 r!=0.53   r=0.39?
- r=? from h=0->1

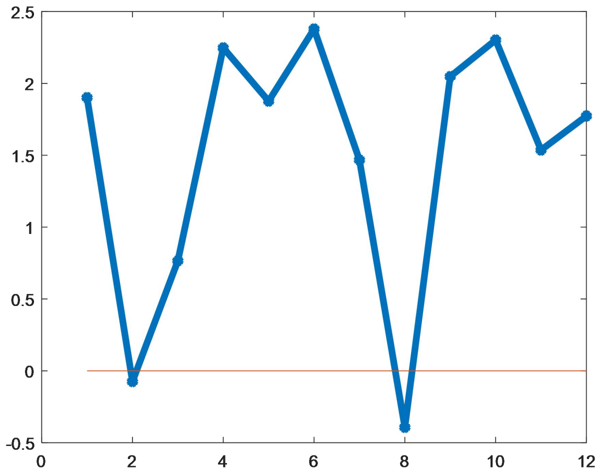
U=1;L=12, n=6;h from 0->0.2  sample =10; V=1




```
In [2]: print(data)
[0.41229179 0.43024731 0.45106207 0.46465689 0.47589313 0.48420959
 0.48664815 0.48842325 0.48961912 0.49607784 0.50182673 0.50589726
 0.50799256 0.5112459  0.51088172 0.51327693 0.51606341 0.51220575
 0.51269858 0.5154125  0.51743055]
```

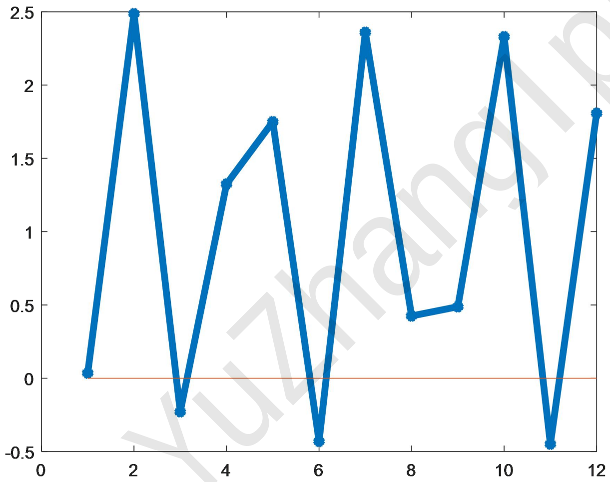
# zero hopping terms or quasiperiodic hopping ?

- we consider Anderson/quasiperiodic disorder in hopping terms.

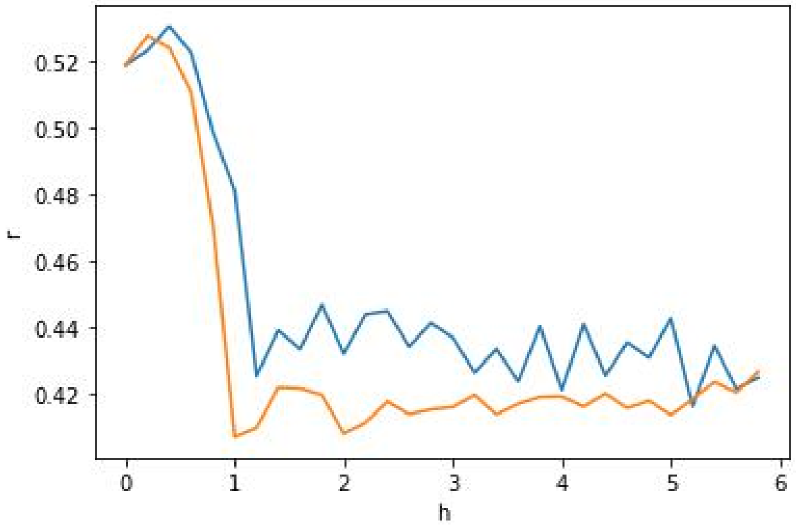



3*rand()-0.5

finite - size  L=12




1+1.5cos(2pi omega j)




```
In [19]: print(data)
[0.51879043 0.52316518 0.53039293 0.52252283 0.4986677  0.48114482
 0.4255004  0.43919631 0.43351512 0.44689997 0.43218859 0.44403378
 0.44500235 0.43433226 0.44143911 0.43700232 0.42651044 0.43362825
 0.42384912 0.4404039  0.42127096 0.441035   0.42560069 0.43558049
 0.43099058 0.44279625 0.41640298 0.43458642 0.42160907 0.42499521]

In [20]: print(data1)
[0.51879043 0.52750843 0.52398954 0.51066638 0.47094686 0.40718215
 0.40998309 0.42206203 0.42175323 0.41980382 0.40821814 0.41152673
 0.4178976  0.41405181 0.41553559 0.41621782 0.41981281 0.4139856
 0.41723629 0.41923329 0.41936474 0.41637899 0.42020651 0.41592457
 0.41813157 0.41378171 0.4186229  0.42372121 0.42056004 0.42673793]
```

# L->\infty

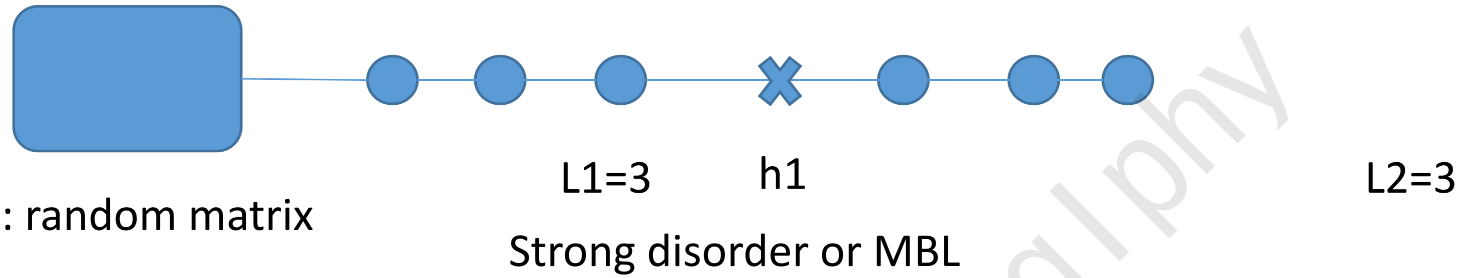
- We assume : system size L=m L_0 ,
  L_0 :mean distance between zero hopping terms
  N=L/2: particle numbers

dimension of Hilbert space： 2^L

subspace $(C_{L_0}^{L_0/2})^{L/L_0}$

Thanks

# Strong disorder regime





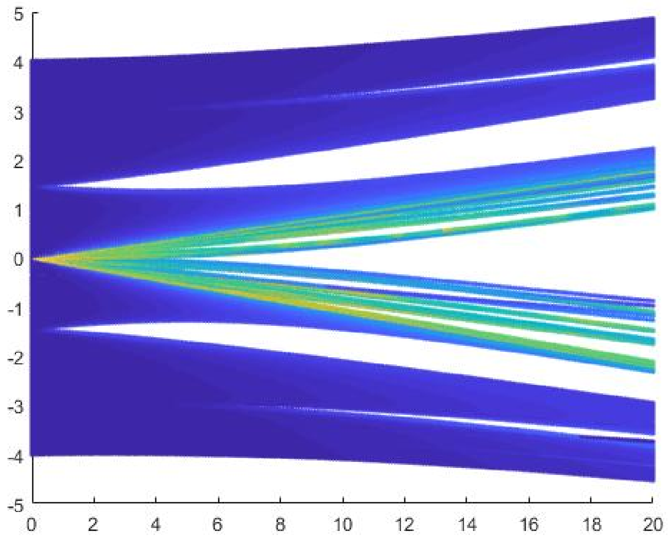

PHYSICAL REVIEW B **95**, 155129 (2017)

**Stability and instability towards delocalization in many-body localization systems**

Wojciech De Roeck[1] and François Huveneers[2]

# consider another perturbation

- [lambda/10,lambda,lambda/10,lambda,lambda/10,lambda....]




没必要 加这个微扰....

# 思想

- 多分形是扩展态和局域态耦合成的。
- MBC region / zero hopping -hilbert space shattering
  ergodic breaking...
- however， not strictly zero hopping terms... /

# global theory

complex the phase ( the effect of 2pi)

如果存在2\pi 给出的积分都是 除以2pi 。如果没有积分不需要

给出的斜率都是整数

李雅普诺夫指数取最大，代表衰减最慢的情况。

global theory 当中，什么情况下，能量不在谱上面。