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Functional Analysis 2
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Homework 7

84. Let S_∞ be the group of finite permutations of $\mathbb{N}$.

(1) Show that S_∞ is ICC. Deduce that LS_∞ is a II_1 factor.

Proof. Fix $n \geq 2$. We may naturally identify $S_n \cong \{\phi \in S_\infty : \phi(m) = m \text{ for all } m > n\} \subseteq S_\infty$. There are infinitely many subgroups conjugate to S_n in S_∞ with trivial intersection: if $\phi_{n,k} = \prod_{j=1}^{n}(j, kn+j)$, then $\phi_{n,k} S_n \phi_{n,k} \bigcap \phi_{n,i} S_n \phi_{n,i}$ is trivial whenever $i \neq j$. Since every $\phi \in S_\infty$ lies in some S_n, if ϕ is not the identity, there are infinitely many distinct conjugates of ϕ in S_∞. Therefore, S_∞ is ICC. We proved in class that this makes LS_∞ a type II_1 factor. $\square$

(2) Give an explicit description of a projection with trace k^{-n} for arbitrary $n, k \in \mathbb{N}$.

Proof. Let an integer $m = k^n$ be given. Let H be a subgroup of S_∞ of order m, for example generated by a cycle of length m. Set $p = m^{-1} \sum_{g \in H} \lambda_g$. Since $\lambda_{g^{-1}} = \lambda_g^*$, this p is self-adjoint. Also, $p^2 = m^{-2} \sum_{(g,h) \in H \times H} \lambda_g \lambda_h = m^{-1} \sum_{k \in H} \lambda_k = p$. Therefore, $p \in P(L\Gamma)$. Of course, $\mathrm{tr}(p) = \langle p\delta_e, \delta_e \rangle = m^{-1} \sum_{g \in H} \langle \delta_g, \delta_e \rangle = m^{-1}$, as desired. $\square$

(3) Find an increasing sequence $F_n \subseteq LS_\infty$ of finite dimensional von Neumann subalgebras such that $LS_\infty = (\bigcup_{n=1}^{\infty} F_n)$.

Proof. Since $S_\infty = \bigcup_{n \in \mathbb{N}} S_n$, we have $\mathbb{C}S_\infty = \bigcup_{n \in \mathbb{N}} \mathbb{C}S_n$, and $LS_\infty = (\mathbb{C}S_\infty)''$ by definition. $\square$

86. Let M be a factor. Prove that if M is finite or purely infinite, then M is algebraically simple.

Proof. We will deal with the three cases separately: type I_n for $n \in \mathbb{N}$, type II_1, and type III. First suppose M is type I_n. Then $M \cong M_n(\mathbb{C})$. Suppose $I \subseteq M_n(\mathbb{C})$ is a nonzero two-sided ideal. Let $A \in I$ be a nonzero matrix. Then taking $(e_i)_{i=1}^n$ as the standard basis vectors, we have $\alpha = \langle e_i | A | e_j \rangle \neq 0$ for some $1 \leq i, j \leq n$. Now observe that

$$p_k = |e_k\rangle\langle e_k| = \frac{1}{\alpha} |e_k\rangle\langle e_i|A|e_j\rangle\langle e_k| \in I$$

for $1 \leq k \leq n$. Summing over k, we have $1 = \sum_{k=1}^n p_k \in I$.

Now suppose M is type II_1. Let tr be a σ-WOT continuous (faithful) tracial state on M. Let $I \subseteq M$ be a nonzero two-sided ideal. By the Borel functional calculus, I contains a nonzero projection p. We claim that for any projection $q \in P(M)$, if $\mathrm{tr}(q) \leq \mathrm{tr}(p)$, then $q \in I$. Let q be such a projection. Then $q \preceq p$, so there is a partial isometry $u \in M$ such that $uu^* = q$ and $u^*u \leq p$. Since I is a two-sided ideal, we have

$$q = uu^* = u(u^*u)u^* = u(u^*up)u^* \in I.$$

This proves the claim that $I \supseteq \{q \in P(M) : \mathrm{tr}(q) \leq \mathrm{tr}(p)\}$. If $\mathrm{tr}(p) \geq \frac{1}{2}$, this implies immediately that $1 - p \in I$, whence $1 = p + (1-p) \in I$. Suppose $\mathrm{tr}(p) < \frac{1}{2}$. Then $p \preceq 1 - p$, so there is a projection $q \leq 1 - p$ with $q \approx p$. By the above claim, $q \in I$, so $p + q \in I$ and $\mathrm{tr}(p+q) = 2\,\mathrm{tr}(p)$. Thus, we may find a projection in I with trace double that of p. Repeating this if necessary, we eventually arrive at a projection of trace at least $\frac{1}{2}$, so by the previous case, $I = M$.

Finally, consider the case that M is a purely infinite factor, that is, type III, and suppose that M is countably decomposable. In this case, we adapt a proof from Jesse Peterson's notes on Von Neumann algebras. Let $p \in P(M) \setminus \{0\}$ be a non-zero projection. We will show that $p \approx 1$. First, we find $q \leq p$ with $q \approx 1 - q \approx p$. Since p must be infinite, there is some partial isometry $u \in M$ with $u^*u = p$ and $uu^* < p$ yet $uu^* \approx p$. Define $p_0 = 1 - uu^*$, and for $n \in \mathbb{N}$, define $p_n = u^n p_0 u^*$. Then p_n is a projection, and since $uu^* \leq p$ and $u^*u \leq p$, we have $p_n \leq p$.

For $n > m$, we have

$$\begin{aligned} p_n p_m &= (u^n(1 - uu^*)(u^*)^n)(u^m(1 - uu^*)(u^*)^m) \\ &= u^n(u^*)^n u^m (u^*)^m - u^n(u^*)^n u^{m+1}(u^*)^{m+1} - u^{n+1}(u^*)^{n+1} u^m (u^*)^m + u^{n+1}(u^*)^{n+1} u^{m+1}(u^*)^{m+1} \end{aligned}$$

Since $uu^* \leq p$ and $u^*u \leq p$, the presence of $p = u^*u$ in the above terms is redundant. Cancelling $m+1$ times gives

$$\begin{aligned} p_n p_m &= u^n (u^*)^{n-m} u^* m - u^n (u^*) n - (m+1)(u^*)^{m+1} - u^{n+1}(u^*)^{(n+1)-m}(u^*)^m + u^{n+1}(u^*)^{n-m}(u^*)^{m+1} \\ &= 0 \end{aligned}$$

Also notice that just as $up_nu^* = p_{n+1}$, we have $u^*p_nu = (u^*u)u^{n-1}p_0(u^*)^{n-1}(u^*u) = p_{n-1}$; for all n and m, we have $p_n \approx p_m$. Therefore, $(p_n)_{n\in\mathbb{N}}$ is a sequence of mutually orthogonal projections, each equivalent to p_0 and bounded above by p.

By Zorn's lemma, we may extend $(p_n)_{n\in\omega}$ to a maximal family of mutually orthogonal equivalent projections bounded above by p, say $(q_n)_{n\in\omega}$; we may assume the family is countable since M is countably decomposable. Let $q_\omega = 1 - \sum_{n\in\omega} q_n$. If $q_\omega \succ q_0$, then there exists v a partial isometry with $v^*v \leq q_\omega$ and $vv^* \approx p_0$, contradicting the maximality of $(q_n)_{n\in\omega}$. Therefore, $q_\omega \prec q_0$. But we know that if (a_i) and (b_i) are families of mutually orthogonal projections with $a_i \prec b_i$ for each i, then $\sum_i a_i \prec \sum_i b_i$. In particular, picking some bijection between $\omega+1$ and ω, we have

$$p = \sum_{n\in(\omega+1)} q_n \prec \sum_{n\in\omega} q_n \leq p$$

, and hence $\sum_{n\in\omega} q_n \approx p$. Similarly, $p \approx \sum_{n\in\omega} q_{2n} \approx \sum_{n\in\omega} q_{2n+1}$, so letting $q = \sum_{n\in\omega} q_{2n}$, we have $q \leq p$ and $q, p-q \approx p$.

By repeating the above with $p_0 = q$, we may as well assume that $p_0 \approx p - p_0 \approx p$ in the first place. Further extend $(q_n)_{n\in\omega}$ to a maximal family of mutually orthogonal projections $(r_n)_{n\in\omega}$ with each $r_n \prec p$; this family is still countable, again by countable decomposability of M. If $1 - \sum_{n\in\omega} r_n \neq 0$, then just as before, we can find $r_\omega \leq 1 - \sum_{n\in\omega} r_n$ such that $r_\omega \prec p$, contradicting maximality. Therefore, $1 = \sum_{n\in\omega} r_n$. Since each $p_n \approx p_0 \approx p$, we have

$$1 = \sum_{n\in\omega} r_n \prec \sum_{n\in\omega} p_n \leq p$$

so we have $1 \approx p$.

This means that there is a partial isometry u with $u^*u = 1$ and $uu^* = p$, so that $1 = u^*(uu^*)u$ is conjugate to p in M. If $I \subseteq M$ is a non-zero two-sided ideal, then by a previous application of the spectral theorem, there is a non-zero projection $p \in I$, and consequently, $1 \in I$ and $I = M$.

□

87. A positive linear functional $\varphi \in M^*$ is called *completely additive* if for any family of pairwise orthogonal projections (p_i), $\varphi\left(\sum p_i\right) = \sum \varphi(p_i)$, where $\sum p_i$ converges SOT.
Suppose $\varphi, \psi \in M^*$ are completely additive and $p \in P(M)$ such that $\varphi(p) < \psi(p)$. Then there is a nonzero projection $q \leq p$ such that $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$.

Proof. Let $\mathcal{E} = \{e_i \mid i \in I\}$ be a maximal family of mutually orthogonal projections such that $e_i \leq p$ and $\psi(e_i) \leq \varphi(e_i)$ for all $i \in I$.
Let $e = \bigvee_{i\in I} e_i$. Then

$$\begin{aligned}
\psi(e) &= \psi\left(\bigvee_{i\in I} e_i\right) \\
&= \psi\left(\sum_{i\in I} e_i\right), \quad \text{since the } e_i \text{ are mutually orthogonal} \\
&= \sum_{i\in I} \psi(e_i), \quad \text{since } \psi \text{ is completely additive} \\
&\leq \sum_{i\in I} \varphi(e_i) \\
&= \varphi\left(\sum_{i\in I} e_i\right), \quad \text{since } \varphi \text{ is completely additive} \\
&= \varphi\left(\bigvee_{i\in I} e_i\right) \\
&= \varphi(e),
\end{aligned}$$

so that $\psi(e) \leq \varphi(e)$. Note that this tells us that $e \neq p$.
Take $q = p - e \neq 0$. Then for all $r \leq q$, we must have $\psi(r) < \varphi(r)$, or else $\mathcal{E} \cup \{r\}$ would be a larger family satisfying the stated properties, contradicting maximality of $\mathcal{E}$.
Next, note that $\psi - \varphi \geq 0$, because on positive linear combinations of mutually orthogonal projections, we have

$$(\psi - \varphi)\left(\sum_{i=1}^{n} \alpha_i p_i\right) = \sum_{i=1}^{n} \alpha_i (\psi - \varphi)(p_i) \geq 0,$$

since $(\psi - \varphi)(p_i) \geq 0$ for all $i = 1, \ldots, n$. Since positive linear combinations of mutually orthogonal projections are dense in M_+, it follows that $\psi - \varphi \in M_+$.
Now take $x \in M_+$ such that $qxq \neq 0$.
Then we can find $\alpha > 0$ such that $\operatorname{spec}(qxq) \cap [\alpha, \infty) \neq \emptyset$.
Then since $\psi - \varphi \geq 0$, we have

$$\begin{aligned}
(\psi - \varphi)(x) &\geq (\psi - \varphi)\left(\alpha 1_{[\alpha,\infty)}(x)\right) \\
&= \alpha (\psi - \varphi)\left(1_{[\alpha,\infty)}(x)\right) \\
&> 0
\end{aligned}$$

so that $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ with $qxq \neq 0$.

Show that the following conditions are equivalent for a positive linear functional $\varphi \in M^*$ for a von Neumann algebra M:

(1) φ is σ-WOT continuous,

(2) φ is *normal*: $x_\lambda \nearrow x$ implies $\varphi(x_\lambda) \nearrow \varphi(x)$, and

(3) φ is completely additive.

Proof. Let $p \in P(M)$ be nonzero.
Then we can find $\xi \notin \ker(p)$ such that $\varphi(p) < \langle p\xi, \xi\rangle$ ($p\xi \neq 0$, and we can scale). Define $\psi = \langle \cdot\xi, \xi\rangle$.
Then by problem 87, there exists a nonzero projection $q \leq p$ such that

$$\varphi(qxq) < \langle xq\xi, q\xi\rangle$$

for all $x \geq 0$ with $qxq \neq 0$.
Now we claim that $x \mapsto \varphi(xq)$ is σ-WOT continuous. To see this, suppose $x_\lambda \to x$ σ-WOT, and hence $x_\lambda \to x$ WOT as well. Then

$$\begin{aligned}
|\varphi(xq) - \varphi(x_\lambda q)|^2 &= |\varphi((x - x_\lambda) q)|^2 \\
&\leq \varphi\left(q(x - x_\lambda)^*(x - x_\lambda) q\right)\varphi(1) \\
&\leq \langle (x - x0\lambda) q\xi, q\xi\rangle \varphi(1) \\
&\to 0
\end{aligned}$$

where the first inequality comes form Cauchy-Schwarz for the sesquilinear form $(x, y) :- \varphi(y^* x)$, and the second inequality comes from the definition of q. The convergence is exactly the definition of WOT convergence, so we conclude that $\varphi(\cdot q)$ is σ-WOT continuous.
Now take $(q_i)_{i\in I}$ to be a maximal family of mutually orthogonal projections such that $x \mapsto \varphi(xq_i)$ is σ-WOT continuous.
We claim that $\sum_{i\in I} q_i = 1$. Take $p = 1 - \sum_{i\in I} q_i$, and suppose $p \neq 0$. Then the construction above using this p gives us a nonzero $q \leq p$ such that $\varphi(\cdot q)$ is σ-WOT continuous, which contradicts maximality.
Define $\varphi_F(x) :- \sum_{i\in F} \varphi(xq_i)$. Then each φ_F is σ-WOT continuous, and thus each φ_F is σ-WOT continuous, so $\varphi_F \in M_*$.
It now suffices to show that $\varphi_F \to \varphi$ in norm, so that φ is also σ-WOT continuous, since M_* is norm closed.
Let $\epsilon > 0$. Choose $F \subseteq I$ finite such that $\varphi\left(\sum_{i\notin F} q_i\right) < \frac{\epsilon^2}{\|\varphi\|+1}$. Then

$$\begin{aligned}
|\varphi(x) - \varphi_F(x)|^2 &= \left|\varphi(x) - \sum_{i\in F}\varphi(xq_i)\right|^2 \\
&= \left|\varphi\left(\sum_{i\in I} xq_i - \sum_{i\in F} xq_i\right)\right|^2 \\
&= \left|\varphi\left(x\sum_{i\in F} q_i\right)\right|^2 \\
&\leq \varphi(xx^*)\varphi\left(\sum_{i\notin F} q_i\right) \\
&\leq (\epsilon\|x\|)^2,
\end{aligned}$$

and thus $\varphi_F \to \varphi$ in norm.