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Functional Analysis 2
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Homework 14

114. Suppose Γ is a countable group and (H, π) is a unitary representation on a separable Hilbert space. Find a unitary $u \in B(\ell^2\Gamma \overline{\otimes} H)$ intertwining $\lambda \otimes \pi$ and $\lambda \otimes 1$, i.e. $u(\lambda_g \otimes \pi_g) = (\lambda_g \otimes 1)u$ for all $g \in \Gamma$.

   Proof. Define $u \in B(\ell^2\Gamma \overline{\otimes} H)$ by $u(\delta_g \otimes \xi) := \delta_g \otimes \pi_{g^{-1}}\xi$ for $g \in \Gamma$ and $\xi \in H$. This defines a unitary operator, since π is a unitary representation. We will check that u intertwines $\lambda \otimes \pi$ and $\lambda \otimes 1$. For $h \in \Gamma$ and $\xi \in H$, we have

$$\begin{aligned} u(\lambda_g \otimes \pi_g)(\delta_h \otimes \xi) &= u(\delta_{gh} \otimes \pi_g \xi) \\ &= \delta_{gh} \otimes \pi_{h^{-1}}\xi \\ &= (\lambda_g \otimes 1)(\delta_h \otimes \pi_{h^{-1}}\xi) \\ &= (\lambda_g \otimes 1)u(\delta_h \otimes \xi). \end{aligned}$$

   □

115. (2) In the notes, the measure on $\mathcal{R}$ induced by μ is defined to be $\nu = \theta_*(\mu \times \gamma)$. Therefore, θ is certainly a bijective isomorphism of measure spaces $X \times \gamma \to \mathcal{R}$.

   (1) For $\eta \in L^2\mathcal{R}$ and $(x, g) \in X \times \Gamma$, define $v\eta(x, g) = \eta \circ \theta$. Then for all $\eta, \xi \in L^2\mathcal{R}$, we have $\int (v\eta)(v\xi) d(\mu \times \gamma) = \int \eta \circ \theta \xi \circ \theta d(\mu \times \gamma) = \int \eta \xi d\nu$, so v is an isometry. Since θ is a bijection, $v^*\eta = \eta \circ \theta^{-1}$, and so $vv^* = 1_{L^2(X \times \Gamma)}$ and $v^*v = 1_{L^2\mathcal{R}}$.

   (3) Let $\eta \in L^2\mathcal{R}$ and $(x, g) \in X \times \Gamma$. We compute:

$$\begin{aligned} M_f v\eta(x, g) &= f(x)v\eta(x, g) \\ &= f(x)\eta(x, g^{-1}x) \\ &= \lambda(f)\eta(x, g^{-1}x) \\ &= v\lambda(f)\eta(x, g) \end{aligned}$$

   (4) Again, we have

$$\begin{aligned} u_g v\eta(x, h) &= v\eta(g^{-1}x, g^{-1}h) \\ &= \eta(g^{-1}x, h^{-1}x) \\ &= \chi_{gX}(x)\eta(\varphi_g^{-1}(x), h^{-1}x) \\ &= L_{\varphi_g}\eta(x, h^{-1}x) \\ &= uL_{\varphi_g}\eta(x, h) \end{aligned}$$

   (5) Because $L^\infty(X, \mu) \trianglelefteq \Gamma = (\{u_g\} \bigcup \{M_f\})''$ and v is unitary, we have $v^*(L^\infty(X, \mu) \trianglelefteq \Gamma)v \subseteq L\mathcal{R}''$. Since $L\mathcal{R}$ is a von Neumann algebra, it is already its own bicommutant, giving the desired containment.

   (6) The point of problems 92 and 93 was to show that the commutant of $L^\infty(X, \mu) \trianglelefteq \Gamma$ is itself a kind of right semidirect product, generated by the right multipliers $(W_f)_{f \in L^\infty}$ and $(n_g)_{g \in \Gamma}$, with $W_f v\eta(x, g) = f(g^{-1}x)\eta(x, g)$ and $n_g v\eta(x, h) = v\eta(x, hg)$. Therefore, it suffices to show that for all $f \in L^\infty$ and $g \in \Gamma$, we have $v^*W_f v, v^*n_g v \in R\mathcal{R}$. The proof is analagous to the previous two parts. Observe:

$$\begin{aligned} W_f v\eta(x, g) &= f(g^{-1}x)v\eta(x, g) \\ &= f(g^{-1}(x))\eta(x, g^{-1}(x)) \\ &= \rho(f)\eta(x, g^{-1}x) \\ &= v\rho(f)\eta(x, g) \end{aligned}$$

and

$$\begin{aligned} n_g v\eta(x,h) &= v\eta(x,hg) \\ &= \eta(x, g^{-1}h^{-1}x) \\ &= \eta(x, \varphi_g^{-1}(h^{-1}(x))) \\ &= R_{\varphi_g}\eta(x,h^{-1}x) \\ &= vR_{\varphi_g}(x,h) \end{aligned}$$

(7) Analagous to (5), we obtain $(v^*(L^\infty(X,\mu) \trianglelefteq \Gamma)v)' = v^*(L^\infty(X,\mu) \trianglelefteq \Gamma)'v \subseteq R\mathcal{R} \subseteq L\mathcal{R}'$, so $v^*(L^\infty(X,\mu) \trianglelefteq \Gamma)v \supseteq L\mathcal{R}$, completing the proof.

116. For each $f \in L^\infty(X,\mu) \subseteq L\mathcal{R}$, we have

$$(f \cdot \xi)(x,y) = f(x)\,\xi(x,y)$$

and for $g \in JAJ \subseteq L\mathcal{R}' = R\mathcal{R}$ we have

$$(g \cdot \xi)(x,y) = g(y)\,\xi(x,y).$$

Then we have $A \subseteq L^\infty\mathcal{R}$ and $JAJ \subseteq L^\infty\mathcal{R}$, by viewing $f \in A$ as $f \otimes 1 \in L^\infty\mathcal{R}$, and viewing $g \in JAJ$ as $1 \otimes g \in L^\infty\mathcal{R}$.
Thus $A \cup JAJ \subseteq L^\infty(\mathcal{R})$, and now we show the other inclusion. Since both are SOT-closed, it suffices to show that $A \cup JAJ$ is SOT-dense in $L^\infty\mathcal{R}$.
To do this we just need to show that the linear span of functions of the form $(f \otimes 1)(1 \otimes g)$ for $f, g \in L^\infty(X,\mu)$ are SOT-dense in $L^\infty\mathcal{R}$.
Linear combinations of indicators are norm-dense in L^∞, so it suffices to show that the linear span of indicators of rectangles is SOT-dense in indicators of $\mathcal{R}$-measurable sets. Since μ is a probability measure, $L^2(X,\mu) \subseteq L^1(X,\mu)$, and for any fixed $f \in L^1$, $|\int_S f d\nu|$ is uniformly continuous in S in the topology metrized by symmetric difference, so we need only show that every $S \subseteq \mathcal{R}$ measurable, the algebra (of sets) generated by rectangles is dense in ν-measure. But $fd\nu$ is a finite measure absolutely continuous with respect to $\pi_0^*\mu$, the measure induced by projection onto the first coordinate $X \times X \to X$. By definition of the product σ-algebra, and since $fd\nu$ is finite, if T_n is a sequence of indicators of sets in the algebra generated by rectangles with $\mu(\pi_0(T_n)\Delta\pi_0(S)) \to 0$, then $fd\nu(T_n) \to fd\nu(S)$. Since f was arbitrary, we are done.