

Homework 11

103. (1) First, recall that any Banach space is weak-* dense in its double dual. In particular, $\ell^1\Gamma$ is weak-* dense in $(\ell^\infty\Gamma)^*$. Let $\phi \in (\ell^\infty\Gamma)^*$ be a state, and pick some $(\phi_\lambda) \in \ell^1\Gamma$ with $\phi_\lambda \rightarrow \phi$ in the weak-* topology. For $\psi \in \ell^1\Gamma$, let ψ^+ be defined by $\psi^+(g) = \text{Re}(\phi(g))^+$, and notice that ϕ^+ is still in ℓ^1 . We will demonstrate that $\|\phi_\lambda - \phi_\lambda^+\|_1 \rightarrow 0$, implying that $\phi_\lambda - \phi_\lambda^+ \rightarrow 0$ weakly by Hölder's inequality. Notice that

$$\phi_\lambda - \phi_\lambda^+ = 2\phi_\lambda^- - i \text{Im}(\phi_\lambda)$$

On the other hand, evaluating at 1 in $\ell^\infty\Gamma$ gives

$$\phi_\lambda(1) = \|\phi_\lambda^+\|_1 - \|\phi_\lambda^-\|_1 + i \|\text{Im}(\phi_\lambda)\|_1$$

Since $\phi_\lambda \rightarrow \phi$ in the weak-* topology and ϕ is a state, $\phi_\lambda(1) \rightarrow 1$, and the latter two 1-norms go to 0, proving our claim.

Therefore, we may replace each ϕ_λ with ϕ_λ^+ without changing the weak-* limit, letting us assume that (ϕ_λ) is a net of positive members of $(\ell^1\Gamma)_1$. Define ψ_λ by $\psi_\lambda = \frac{\phi_\lambda}{\phi_\lambda(1)}$. Since $\phi_\lambda(1) \rightarrow \phi(1) = 1$, we have $\psi_\lambda \rightarrow \phi$ as well. Since $\ell^1\Gamma$ is a C^* -algebra, the facts that each ψ_λ is positive, norm 1, and $\psi_\lambda(1) = 1$ imply that each ψ_λ is a state, so we are done.

- (2) The product weak topology on $\oplus_{g \in F} \ell^1\Gamma$ is induced by the usual action of $\oplus_{g \in F} \ell^1\Gamma$ on $\oplus_{g \in F} \ell^\infty\Gamma$; similarly, since F is finite, all norms on the direct sum consistent with the original are equivalent, so we may as well take the norm $\|\oplus_{g \in F} x_g\| = \max_{g \in F} \|x_g\|_1$. In particular, the norm on $\ell^\infty\Gamma$ is equivalent to the operator norm from $\ell^1\Gamma$, so this norm is equivalent to the operator norm from $(\oplus_{g \in F} \ell^\infty\Gamma)^*$.

To see that K is convex, notice that $\text{Prob}(\Gamma)$ is convex, and recall that the weak closure of a convex set is convex. Since K is weakly closed and the norm on ℓ^∞ is the operator norm from ℓ^1 , K is also norm closed.

- (3) Let m be a left- Γ invariant state on $\ell^\infty\Gamma$. Let $F \subseteq \Gamma$ finite be given. Let (μ_λ) be a net in $\text{Prob}(\Gamma)$ converging to m in the weak-* topology. The action of each $g \in F$ on $(\ell^\infty\Gamma)^*$ is weak-* continuous, so $\oplus_{g \in F} g\mu_\lambda - \mu_\lambda \rightarrow \oplus_{g \in F} gm - m$ weak-*. However, since m is left- Γ invariant, $gm - m = 0$ for each $g \in F$, so $\oplus_{g \in F} g\mu_\lambda \rightarrow 0$ weak-*, and hence weakly, since $0 \in \ell^1\Gamma$. Since K is weakly closed and each $\oplus_{g \in F} g\mu_\lambda - \lambda \in K$, we have $0 \in K$.

Since K is convex, K is in fact the norm closure of $\{\oplus_{g \in F} g\mu - \mu : \mu \in \text{Prob}(\Gamma)\}$, so there is a sequence $\mu \in \text{Prob}(\Gamma)$ with $\max_{g \in F} \|g\mu - \mu\|_1 < \epsilon$, i.e. Γ has a left-invariant mean.

104. (1) Without loss of generality we may assume $a \leq b$, and then we have

$$\begin{aligned} & \int_0^1 |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)| dr \\ &= \int_0^a |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)| dr + \int_a^b |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)| dr + \int_b^1 |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)| dr \\ &= \int_a^b dr \\ &= b - a \end{aligned}$$

- (2) We have

$$\begin{aligned}
 \|h \cdot \mu - \mu\|_{\ell^1 \Gamma} &= \sum_{g \in \Gamma} |(h \cdot \mu)(g) - \mu(g)| \\
 &= \sum_{g \in \Gamma} |\mu(h^{-1}g) - \mu(g)| \\
 &= \sum_{g \in \Gamma} \int_0^1 |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))| dr, \quad \text{by part (1)} \\
 &= \int_0^1 \sum_{g \in \Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))| dr, \quad \text{by part (1), by Tonelli's theorem}
 \end{aligned}$$

(3)

$$\begin{aligned}
 hE(\mu, r) &= \{hg \mid g \in \Gamma, \mu(g) > r\} \\
 &= \{k \in \Gamma \mid \mu(h^{-1}k) > r\} \\
 &= \{g \in \Gamma \mid (h \cdot \mu)(g) > r\}.
 \end{aligned}$$

(4)

$$\begin{aligned}
 \int_0^1 |E(\mu, r)| dr &= \int_0^1 \sum_{g \in \Gamma} \chi_{(0, \mu(g)]} dr \\
 &= \sum_{g \in \Gamma} \int_0^1 \chi_{(0, \mu(g)]} dr, \text{ by Tonelli} \\
 &= \sum_{g \in \Gamma} \mu(g) \\
 &= 1
 \end{aligned}$$

(5) We know that $g \in \Gamma$ is in $hE(\mu, r) \Delta E(\mu, r)$ if and only if $\mu(h^{-1}g) \leq r < \mu(g)$ or $\mu(g) \leq r < \mu(h^{-1}g)$.

This is equivalent to $|\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))| = 1$, so we have

$$|hE(\mu, R) \Delta E(\mu, r)| = \sum_{g \in \Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|$$

(6) We have

$$\begin{aligned}
 \int_0^1 \sum_{h \in F} |hE(\mu, r) \Delta E(\mu, r)| dr &= \sum_{h \in F} \int_0^1 |hE(\mu, r) \Delta E(\mu, r)| dr \\
 &= \sum_{h \in F} \int_0^1 \sum_{g \in \Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))| dr, \quad \text{by (5)} \\
 &= \sum_{h \in F} \|h \cdot \mu - \mu\|_{\ell^1 \Gamma}, \quad \text{by (2)} \\
 &< \epsilon \\
 &= \epsilon \int_0^1 |E(\mu, r)| dr, \quad \text{by (4)}.
 \end{aligned}$$

Thus there exists $r \in [0, 1]$ such that

$$\sum_{h \in F} |hE(\mu, r) \Delta E(\mu, r)| dr < \epsilon |E(\mu, r)|.$$

and in particular for each $h \in F$ we have

$$|hE(\mu, r) \Delta E(\mu, r)| dr < \epsilon |E(\mu, r)|.$$

- (7) Now take F_n to be a sequence of finite subsets increasing to Γ , and $\epsilon_n := \frac{1}{n}$. Then there exist $\mu_n \in \text{Prob}(\Gamma)$ and $r_n \in [0, 1]$ such that

$$|hE(\mu_n, r_n) \Delta E(\mu_n, r_n)| < \frac{1}{n} |E(\mu_n, r_n)|,$$

so that

$$\lim_{n \rightarrow \infty} \frac{|hE(\mu_n, r_n) \Delta E(\mu_n, r_n)|}{|E(\mu_n, r_n)|} = 0.$$

Thus $(E(\mu_n, r_n))_{n \in \mathbb{N}}$ is a Folner sequence for Γ .

105. (1) Find a bijection from the set of ultrafilters on \mathbb{N} to $\beta\mathbb{N}$.

Proof. Let \mathcal{U} be the set of ultrafilters on \mathbb{N} . Give \mathcal{U} the topology generated by basic open sets of the form $\tilde{A} := \{\omega \in \mathcal{U} : A \in \omega\}$ for $A \subseteq \mathbb{N}$. We will show that \mathcal{U} with this topology satisfies the universal property for the Stone-Ćech compactification of \mathbb{N} .

For $n \in \mathbb{N}$, let $\hat{n} \in \mathcal{U}$ be the principal ultrafilter on n , i.e. $\hat{n} := \{A \subseteq \mathbb{N} : n \in A\}$. The map $\iota : \mathbb{N} \rightarrow \mathcal{U}$ given by $\iota(n) := \hat{n}$ is clearly an injection. Since \mathbb{N} is discrete, ι is a continuous function. We want to show that the image $\iota(\mathbb{N})$ is dense in \mathcal{U} . But this is easy: given $\emptyset \neq A \subseteq \mathbb{N}$, we have $\hat{n} \in \tilde{A}$ for all $n \in A$.

Now we claim that \mathcal{U} is a Hausdorff space. Let $\omega_1 \neq \omega_2 \in \mathcal{U}$. Then there is a set $A \subseteq \mathbb{N}$ such that $A \in \omega_1$ and $\mathbb{N} \setminus A \in \omega_2$. Hence, $\omega_1 \in \tilde{A}$ and $\omega_2 \in \widetilde{\mathbb{N} \setminus A}$, so \mathcal{U} is Hausdorff.

Next, we will show that \mathcal{U} is compact. Note that every basic open set is also closed: $\tilde{A} = \mathcal{U} \setminus (\widetilde{\mathbb{N} \setminus A})$. Moreover, for any finite collection $A_1, \dots, A_n \subseteq \mathbb{N}$, since filters are closed under finite intersections and supersets, we have

$$\widetilde{\bigcap_{i=1}^n A_i} = \{\omega \in \mathcal{U} : \bigcap_{i=1}^n A_i \in \omega\} = \{\omega \in \mathcal{U} : A_i \in \omega \text{ for } i = 1, \dots, n\} = \bigcap_{i=1}^n \tilde{A}_i.$$

Let $(A_i)_{i \in I}$ be a collection of nonempty subsets of \mathbb{N} such that $\bigcap_{i \in F} \tilde{A}_i \neq \emptyset$ for all $F \subseteq I$ finite. Then by the previous observation, $\widetilde{\bigcap_{i \in F} A_i} \neq \emptyset$. In particular, $\bigcap_{i \in F} A_i \neq \emptyset$. That is, $(A_i)_{i \in I}$ satisfies the finite intersection property. Therefore, by Zorn's lemma, there is an ultrafilter $\omega \in \mathcal{U}$ such that $A_i \in \omega$ for every $i \in I$. That is, $\omega \in \bigcap_{i \in I} \tilde{A}_i$, so $\bigcap_{i \in I} \tilde{A}_i \neq \emptyset$. This proves that \mathcal{U} is compact.

Finally, we must show that \mathcal{U} satisfies the extension property: given a compact Hausdorff space X and a (continuous) map $f : \mathbb{N} \rightarrow X$, there is a unique continuous extension $\tilde{f} : \mathcal{U} \rightarrow X$:

$$\begin{array}{ccc} & \mathcal{U} & \\ \iota \uparrow & \searrow \tilde{f} & \\ \mathbb{N} & \xrightarrow{f} & X \end{array}$$

Let X be a compact Hausdorff space, and let $f : \mathbb{N} \rightarrow X$ be any function. Define $\tilde{f} : \mathcal{U} \rightarrow X$ by $\tilde{f}(\omega) := \lim_{n \rightarrow \omega} f(n)$ as in (2). By (4), $\tilde{f}(\hat{n}) = f(n)$ for $n \in \mathbb{N}$, so \tilde{f} is a well-defined extension of f . It remains to show that \tilde{f} is continuous. Let $U \subseteq X$ be open. If $f^{-1}(U) = \emptyset$, there is nothing to show. Suppose $f^{-1}(U) \neq \emptyset$. Since X is a compact Hausdorff space, it is normal. Therefore, we may find an open set $V \subseteq X$ (a neighborhood of a given point in $U \cap f(\mathbb{N})$) such that $\bar{V} \subseteq U$ and $f^{-1}(V) \neq \emptyset$. Let $\omega \in \widetilde{f^{-1}(V)}$. Then $f^{-1}(X \setminus \bar{V}) \notin \omega$, so $\tilde{f}(\omega) = \lim_{n \rightarrow \omega} f(n) \in \bar{V} \subseteq U$. Thus, $\widetilde{f^{-1}(V)}$ is a nonempty open subset of \mathcal{U} such that $\tilde{f}(\widetilde{f^{-1}(V)}) \subseteq U$, so \tilde{f} is continuous. Uniqueness of f follows from the density of \mathbb{N} in the Hausdorff space \mathcal{U} . \square

Note: To see the connection with the C^* -algebra construction from last semester, we could also realize $\widehat{\ell^\infty}$ as the collection of ultralimits $\lim_{n \rightarrow \omega}$ for $\omega \in \mathcal{U}$. Linearity of these limits is shown in (5), and they are multiplicative for the same reason.

- (2) Let ω be an ultrafilter on \mathbb{N} . Let X be a compact Hausdorff space and $f : \mathbb{N} \rightarrow X$. We say $x = \lim_{n \rightarrow \omega} f(n)$ if for every open neighborhood U of x , $f^{-1}(U) \in \omega$. Prove that $\lim_{n \rightarrow \omega} f(n)$ always exists for any function $f : \mathbb{N} \rightarrow X$.

Proof. Let $\mathcal{C} := \{K \subseteq X : K \text{ is closed and } f^{-1}(K) \in \omega\}$. Note that \mathcal{C} is nonempty, since $X \in \mathcal{C}$. Moreover, $\emptyset \notin \mathcal{C}$. Given $K_1, \dots, K_n \in \mathcal{C}$, we have

$$f^{-1}\left(\bigcap_{i=1}^n K_i\right) = \bigcap_{i=1}^n f^{-1}(K_i) \in \omega,$$

since ω is a filter, so $\bigcap_{i=1}^n K_i \in \mathcal{C}$. In particular, $\bigcap_{i=1}^n K_i \neq \emptyset$. By compactness, it follows that $\bigcap_{K \in \mathcal{C}} K \neq \emptyset$.

Let $x \in \bigcap_{K \in \mathcal{C}} K$. Let U be an open neighborhood of x . We want to show $f^{-1}(U) \in \omega$. Note that $X \setminus U$ is closed and $x \notin X \setminus U$. Hence $X \setminus U \notin \mathcal{C}$ by the construction of x . But $f^{-1}(X \setminus U) = \mathbb{N} \setminus f^{-1}(U)$. Since ω is an ultrafilter, it follows that $f^{-1}(U) \in \omega$.

Finally, we claim that x is the unique limit of f along ω . Indeed, since X is Hausdorff, given any other point $y \in X$, we may find disjoint open neighborhoods $U \ni x$ and $V \ni y$ so that $f^{-1}(V) \subseteq f^{-1}(X \setminus U) = \mathbb{N} \setminus f^{-1}(U) \notin \omega$. \square

- (3) Show that every principal ultrafilter on \mathbb{N} contains a unique singleton set, and any two ultrafilters containing the same singleton set are necessarily equal. Thus we may identify the set of principal ultrafilters on \mathbb{N} with \mathbb{N} .

Proof. Let ω be an ultrafilter on \mathbb{N} , and suppose $A = \{n_1, \dots, n_k\} \in \omega$ has the fewest elements of any set in ω . Suppose for contradiction that $k \geq 2$. By minimality of A , we must have $\{n_1\} \notin \omega$. But then $\mathbb{N} \setminus \{n_1\} \in \omega$, so $\{n_2, \dots, n_k\} = A \cap (\mathbb{N} \setminus \{n_1\}) \in \omega$, contradicting minimality of A .

Hence $\omega \ni \{n\}$ for some $n \in \mathbb{N}$. This n is necessarily unique, since ω is closed under finite intersections and does not contain the empty set.

We claim that $\omega = \hat{n} := \{A : n \in A\}$. First, \hat{n} is a filter. On the other hand, if $A \in \omega$, then $A \cap \{n\} \in \omega$, so $n \in A$. Hence, $\omega \subseteq \hat{n}$. Since ω is an ultrafilter (and therefore a maximal filter), we must have $\omega = \hat{n}$. \square

- (4) Determine $\lim_{n \rightarrow \omega} f(n)$ for $f : \mathbb{N} \rightarrow X$ as in (2) when ω is principal.

Proof. Let $\omega = \hat{k}$ be a principal ultrafilter. We claim $\lim_{n \rightarrow \omega} f(n) = f(k)$. Indeed, for every open set $U \ni f(k)$, we have $k \in f^{-1}(U)$, so $f^{-1}(U) \in \hat{k} = \omega$. \square

- (5) Let ω be a free ultrafilter on \mathbb{N} . Suppose $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is a locally finite group and m_n is the uniform probability (Haar) measure on Γ_n . Define $m : 2^\Gamma \rightarrow [0, 1]$ by $m(A) = \lim_{n \rightarrow \omega} m_n(A \cap \Gamma_n)$. Prove that m is a left Γ -invariant finitely additive probability measure on Γ , i.e. Γ is amenable.

Proof. Clearly $m(\Gamma) = 1$. Let $A, B \subseteq \Gamma$ be disjoint. Then since m_n is additive for each $n \in \mathbb{N}$,

$$m(A \cup B) = \lim_{n \rightarrow \omega} (m_n(A \cap \Gamma_n) + m_n(B \cap \Gamma_n))$$

Thus, in order to show that m is finitely additive, it suffices to show that $\lim_{n \rightarrow \omega}$ is additive.

Suppose $x = \lim_{n \rightarrow \omega} x_n$ and $y = \lim_{n \rightarrow \omega} y_n$ for some sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $[0, 1]$. Let U be a neighborhood of $x + y$. Since addition is jointly continuous, we may find neighborhoods V of x and W of y such that $V + W \subseteq U$. By assumption, $\{n : x_n \in V\} \in \omega$ and $\{n : y_n \in W\} \in \omega$. Hence,

$$\{n : x_n + y_n \in U\} \supseteq \{n : x_n \in V, y_n \in W\} = \{n : x_n \in V\} \cap \{n : y_n \in W\} \in \omega.$$

Therefore, $\lim_{n \rightarrow \omega} x_n + y_n = x + y$ as desired.

It remains to show that m is left Γ -invariant. Let $g \in \Gamma$. Then there is an $N \in \mathbb{N}$ such that $g \in \Gamma_n$ for all $n \geq N$. Since ω is free, $\{n \geq N\} \in \omega$. Let $A \subseteq \Gamma$, and let U be a neighborhood of $m(A)$. Then we have

$$\begin{aligned} \{n : m_n((gA) \cap \Gamma_n) \in U\} &= \{n : m_n(g(A \cap g^{-1}\Gamma_n)) \in U\} \\ &= \{n : m_n(A \cap g^{-1}\Gamma_n) \in U\} \\ &\supseteq \{n \geq N : m_n(A \cap \Gamma_n) \in U\} \\ &= \{n \geq N\} \cap \{n : m_n(A \cap \Gamma_n) \in U\}. \end{aligned}$$

Since $m(A) = \lim_{n \rightarrow \omega} m_n(A \cap \Gamma_n)$, both of these sets in the final line are elements of ω , so their intersection is also in ω . Hence, $m(gA) = \lim_{n \rightarrow \omega} m_n((gA) \cap \Gamma_n) = m(A)$. \square