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Functional Analysis 2
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Homework 13

109. *Beware: we could not make sense of the suggested part (1). We prove something which may or may not be the intended part (1), but which will enable a proof of (3), which seems to be the point of the exercise.*
Given $\Phi : A \to M_n(\mathbb{C})$, define $\psi : M_n(A) \to \mathbb{C}$ by $\psi(B) = n^{1/2} \sum_{i=1}^n \sum_{j=1}^n \Phi(B_{ij})_{ij}$.

(1) Suppose that Φ is completely positive. This means that if $B \in M_n(A)$ is positive, then the matrix $\Phi(B)$ with $\Phi(B)_{(i,k),(j,\ell)} = \Phi(B_{ij})_{k\ell}$ is also positive. Let $\delta \in (\mathbb{C}^n)^2$ be the vector with $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. Then $\psi(B) = n^{1/2}\langle \delta | \Phi(B)\delta\rangle$ is positive.

On the other hand, suppose that ψ is positive. Stinespring's theorem says that completely positive maps from a unital C*-algebra are precisely conjugations of *-homomorphisms by a morphism of Hilbert spaces. Therefore, we attempt to find such a decomposition for Φ. As usual, the GNS construction gives a Hilbert space $H = L^2(M_n(A), \psi)$ with a representation $\pi_\psi : M_n(A) \to B(H^n)$ with cyclic vector Ω_ψ and inner product $\langle B\Omega_\psi | C\Omega_\psi\rangle = \psi(B^*C)$. Fix $n \in \mathbb{N}$. Let $(e_i)_{i=1}^n$ be the standard basis for $\mathbb{C}^n$, i.e $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots 0)$, etc. Let $((E_{ij})_{i=1}^n)_{j=1}^n$ be the system of matrix units for $M_n(\mathbb{C})$ such that $E_{ij}[i, j] = 1$ all other entries of E_{ij} are 0. (It is possible that an arbitrary choice of basis and system of matrix units would suffice, but the definition of φ seems basis dependent, making such a generalization tedious at best. In our defense, this seems true of φ as well.) Let $\iota : M_n(\mathbb{C}) \to M_n(A)$ be induced by the structural morphism $\mathbb{C} \to A$. For brevity, we identify each M with $\iota(M)$. Finally, let $h : A \to M_n(A)$ be the map so that for every $a \in A$ and (i, j), we have $h(a)_{ij} = n^{-1/2}a$. Notice that h is a non-unital (unless $n = 1$) $*$-homomorphism. Since Φ might be non-unital as it is, this need not be too disturbing.

Define $V : \mathbb{C}^n \to H^n$ by $Ve_i = \pi_\psi(E_{i1})\Omega_\psi$. Set $\Lambda : M_n(A) \to M_n(\mathbb{C})$ by $\Lambda(B) = V^*\pi_\psi(B)V$. As we know well, Λ must be completely positive. It can happen that V is not an isometry, and indeed, this must happen if ever ψ is not unital. (It seems that this is unavoidable even by a better attack to the problem, because no matter what we do, we may simply have $\Phi = 0$, or more generally, $\mathrm{img}(\Phi)$ may not have full rank.) We claim that $\Phi = \Lambda \circ h$, making Φ the composition of completely positive maps, and hence completely positive. Observe: if $a \in A$,

$$\begin{aligned}
\Lambda(h(a))_{ij} &= \langle e_j | \Lambda(h(a))e_i\rangle \\
&= \langle e_j | V^*\pi_\psi(h(a))Ve_i\rangle_{\mathbb{C}^n} \\
&= \langle \Omega_\psi | \pi_\psi(E_{1j}h(a)E_{i1})\Omega_\psi\rangle_{H^n} \\
&= n^{1/2}\sum_{k,\ell} \Phi((E_{1j}h(a)E_{i1})[k,\ell])_{k,\ell} \\
&= n^{1/2}\Phi(h(a)_{ij})_{ij} \\
&= n^{1/2}n^{-1/2}\Phi(a)_{ij} \\
&= \Phi(a)_{ij}
\end{aligned}$$

as desired.

(2) Take $s \in S$ positive with $\|s\| \leq 1$. Then $s \leq 1_S$, else there exists a unit vector $h \in H$ with $\langle h \mid sh\rangle > 1$, which is impossible. Thus $\sup\{|\psi(s)| \mid s \in (S_+)_1\} = \psi(1)$.
Since every element can be written as a linear combination of positive elements, this suffices to show that $\|\psi\| = \psi(1)$. Further, we know that for a functional ψ on a unital C*-algebra, ψ is a state if and only if $\psi(1) = \|\psi\|$. Any norm preserving extension of ψ will also have $\psi(1) = \|\psi\|$ because $S \subseteq A$ is unital, so that $\psi \geq 0$.

(3) By part (1), from Φ, we obtain some positive linear functional $\psi : M_n(S) \to \mathbb{C}$. Using Hahn-Banach, we extend to $M_n(A)$. Applying the construction of part (1), we get $\Lambda : M_n(A) \to M_n(\mathbb{C})$ completely positive. Define $\widetilde{\Phi} = \Lambda \circ h$. Following the logic of part (1), we will have $\Lambda \circ h|_S = \Phi$, so $\widetilde{\Phi}|_A = \Phi$. Since $1_A \in S$, if Φ is unital, so is $\widetilde{\Phi}$.

110. Suppose Γ is a countable discrete group, and suppose $\varphi : L\Gamma \to L\Gamma$ is a normal completely positive map. Prove that $f : \Gamma \to \mathbb{C}$ given by $f(g) := \mathrm{tr}_{L\Gamma}(\varphi(\lambda_g)\lambda_g^*)$ is a positive definite function.

Proof. Let $n \in \mathbb{N}$ and $g_1, \dots, g_n \in \Gamma$. Let $A \in M_n(\mathbb{C})$ be the matrix with entries $a_{ij} = f(g_i^{-1} g_j)$. We want to show $A \geq 0$. Note that since tr is tracial, we can rewrite the entries as follows:

$$\begin{aligned} a_{ij} &= \mathrm{tr}_{L\Gamma}(\varphi(\lambda_{g_i^{-1}g_j})\lambda^*_{g_i^{-1}g_j}) \\ &= \mathrm{tr}_{L\Gamma}(\varphi(\lambda^*_{g_i}\lambda_{g_j})\lambda^*_{g_j}\lambda_{g_i}) \\ &= \mathrm{tr}_{L\Gamma}(\lambda_{g_i}\varphi(\lambda^*_{g_i}\lambda_{g_j})\lambda^*_{g_j}). \end{aligned}$$

Let $x \in \mathbb{C}^n$. Then

$$\begin{aligned} \langle Ax, x \rangle &= \sum_{i,j=1}^n \overline{x}_i a_{ij} x_j \\ &= \sum_{i,j=1}^n \overline{x}_i \,\mathrm{tr}_{L\Gamma}(\lambda_{g_i}\varphi(\lambda^*_{g_i}\lambda_{g_j})\lambda^*_{g_j}) x_j \\ &= \mathrm{tr}_{L\Gamma}\left(\sum_{i,j=1}^n \overline{x}_i \lambda_{g_i}\varphi(\lambda^*_{g_i}\lambda_{g_j})\lambda^*_{g_j} x_j \right) \\ &= \mathrm{tr}_{L\Gamma}\left(\sum_{i,j=1}^n (\overline{x}_i x_j) \left(\lambda_{g_i}\lambda^*_{g_j}\right) \varphi(\lambda^*_{g_i}\lambda_{g_j}) \right) \end{aligned}$$

But this is just the same sort of tracial expression encountered in problem 108: $\langle A * \varphi(A^T)x|x\rangle$. Therefore, we just need to extend the result of 108, and prove that the Schur product of positive matrices valued in a von Neumann algebra is positive. Using the same argument as used last week, we can again reduce this to showing that the tensor product of positive matrices is positive.

Now, we explain why the tensor product of positive matrices valued in a von Neumann algebra is positive. Let $M \subseteq B(H)$ be a von Neumann algebra. We have already established an inclusion $M_n(B(H)) \to B(H^n)$, such that if $A, B \in M_n(B(H))$, then the Kronecker tensor product of matrices $A \otimes B$ defines the same operator as the operator tensor product $A \otimes B$. The operator tensor product is $(A \otimes I)(I \otimes B)$, so it suffices to prove that the product of two commuting positive elements of a von Neumann algebra is positive. I assume we learned this last semester, so I looked up some of the details on Stack Exchange and in Analysis Now.

Let $A, B \in M$ be commuting positive operators. The product of commuting normal elements is normal, so it suffices to show that $\mathrm{sp}(AB) \subseteq \mathbb{R}$. We claim that $\mathrm{Spec}(AB) \subseteq \mathrm{Spec}(A)\,\mathrm{Spec}(B)$. If we let $N = \{A, B\}''$, then N is a commuting von Neumann algebra, so by taking the Gelfand transform, $\mathrm{Spec}_N(B) \subseteq \mathrm{Spec}_N(A)\,\mathrm{Spec}_N(B)$. Now $\mathrm{Spec}_M \subseteq \mathrm{Spec}_N$, since $M \supseteq N$. However, $\mathrm{Spec}_N(A) = \mathrm{Spec}_M(A)$, because if $A - \lambda I$ is invertible in A, then by the continuous functional calculus, the inverse commutes with A and B, and is therefore in N; similarly for B. Therefore, $\mathrm{Spec}_M(AB) \subseteq \mathrm{Spec}_N(AB) \subseteq \mathrm{Spec}_M(A)\,\mathrm{Spec}_M(B) \subseteq \mathbb{R}$. □

112. Suppose that Γ is a countable discrete group such that every cocycle is inner. Suppose (H, π) is a unitary representation and $(\xi_n) \subseteq H$ is a sequence of unit vectors such that $\|\pi_g \xi_n - \xi_n\| \to 0$ as $n \to \infty$ for all $g \in \Gamma$. Follow the steps below to find a nonzero Γ-invariant vector in H.

(1) Enumerate $\Gamma = \{g_1, g_2, \dots\}$. Explain why you can pass to a subsequence of (ξ_n) to assume that for all $n \in \mathbb{N}$, $\|\pi_{g_i}\xi_n - \xi_n\| < 4^{-n}$ for all $1 \leq i \leq n$.

Proof. Since $\|\pi_{g_1}\xi_n - \xi_n\| \to 0$ as $n \to \infty$, we may choose $n_1 \in \mathbb{N}$ so that $\|\pi_{g_1}\xi_{n_1} - \xi_{n_1}\| < \frac{1}{4}$.

Suppose we have $n_1 < n_2 < \cdots < n_k$ so that $\|\pi_{g_i}\xi_{n_k} - \xi_{n_k}\| < 4^{-k}$ for all $1 \leq i \leq k$. For each $1 \leq i \leq k+1$, choose $N_i \in \mathbb{N}$ so that $\|\pi_{g_i}\xi_n - \xi_n\| < 4^{-(k+1)}$ for all $n \geq N_i$. Set $n_{k+1} := \max\{N_1, \dots, N_{k+1}, n_k + 1\}$. Then $n_{k+1} > n_k$ and $n_{k+1} \geq N_i$ for each $1 \leq i \leq k+1$, so $\|\pi_{g_i}\xi_{n_{k+1}} - \xi_{n_{k+1}}\| < 4^{-(k+1)}$ for all $1 \leq i \leq k+1$.

Replacing $(\xi_n)_{n\in\mathbb{N}}$ with $(\xi_{n_k})_{k\in\mathbb{N}}$ gives a sequence of unit vectors with the desired property. □

(2) For $n \in \mathbb{N}$, consider the inner cocycles $\beta_n(g) := \xi_n - \pi_g \xi_n$. Let $(K, \sigma) = \bigoplus_{n \in \mathbb{N}} (H, \pi)$. Define $\beta : \Gamma \to K$ by $\beta(g)_n = 2^n \beta_n(g)$. Prove that $\beta(g) \in H$ is well-defined for every $g \in \Gamma$. Then show β is a cocycle for (H, π).

Proof. Let $g \in \Gamma$. To show that $\beta(g)$ is well-defined, it suffices to show $(\|\beta(g)_n\|)_{n \in \mathbb{N}} \in \ell^2\mathbb{N}$. Suppose $N \in \mathbb{N}$ such that $g = g_N$ is the enumeration from (1). Then

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \|\beta(g)_n\|^2 &= \sum_{n \in \mathbb{N}} \|2^n \beta_n(g)\|^2 \\
&= \sum_{n \in \mathbb{N}} 4^n \|\pi_{g_N} \xi_n - \xi_n\|^2 \\
&\leq \sum_{n=1}^{N-1} 4^n \|\pi_{g_N} \xi_n - \xi_n\|^2 + \sum_{n=N}^{\infty} 4^n (4^{-n})^2 \\
&= \sum_{n=1}^{N-1} 4^n \|\pi_{g_N} \xi_n - \xi_n\|^2 + \frac{4^{-(N-1)}}{3} \\
&< \infty.
\end{aligned}$$

Now we must check that β is a cocycle. Let $g, h \in \Gamma$. Since β_n is a cocycle for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\beta(gh) &= (2^n \beta_n(gh))_{n \in \mathbb{N}} \\
&= (2^n (\beta_n(g) + \pi_g \beta_n(h)))_{n \in \mathbb{N}} \\
&= (2^n \beta_n(g))_{n \in \mathbb{N}} + (\pi_g 2^n \beta_n(h))_{n \in \mathbb{N}} \\
&= (2^n \beta_n(g))_{n \in \mathbb{N}} + \sigma_g (2^n \beta_n(h))_{n \in \mathbb{N}} \\
&= \beta(g) + \sigma_g \beta(h).
\end{aligned}$$

That is, β is a cocycle. □

(3) Deduce β is inner and bounded. Thus there exists a $\kappa \in K \setminus \{0\}$ such that $\beta(g) = \kappa - \sigma_g \kappa$ for all $g \in \Gamma$.

Proof. This follows immediately from the assumption that every cocycle is inner. □

(4) Prove that $\|\beta_n(g)\| \to 0$ uniformly for $g \in \Gamma$. That is, show for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $\|\beta_n(g)\| < \varepsilon$ for all $g \in \Gamma$.

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > 1 + \log_2 \frac{\|\kappa\|}{\varepsilon}$. Then for $n > N$ and $g \in \Gamma$, we have

$$\begin{aligned}
\|\beta_n(g)\|_H &= 2^{-n} \|\kappa_n - \pi_g \kappa_n\|_H \\
&\leq 2^{-(n-1)} \|\kappa_n\|_H \\
&\leq 2^{-(N-1)} \|\kappa\|_K \\
&< \varepsilon.
\end{aligned}$$

□

(5) Fix $N \in \mathbb{N}$ such that $\|\beta_N(g)\| = \|\xi_N - \pi_g \xi_N\| < 1$ for all $g \in \Gamma$. Show that there is a $\xi_0 \in H \setminus \{0\}$ such that $\pi_g \xi_0 = \xi_0$ for all $g \in \Gamma$.

Proof. Let $B := \{\pi_g \xi_N : g \in \Gamma\}$. Since π is a unitary representation, $B \subseteq H_1$ is bounded. Let

$f : H \to B$ be given by $f(\eta) := \sup_{\xi \in B} \|\eta - \xi\|$. Observe that f is π-invariant:

$$\begin{aligned} f(\pi_g \eta) &= \sup_{\xi \in B} \|\pi_g \eta - \xi\| \\ &= \sup_{h \in \Gamma} \|\pi_g \eta - \pi_h \xi_N\| \\ &= \sup_{h \in \Gamma} \|\eta - \pi_{g^{-1}h} \xi_N\| \\ &= \sup_{k \in \Gamma} \|\eta - \pi_k \xi_N\| \\ &= \sup_{\xi \in B} \|\eta - \xi\| \\ &= f(\eta). \end{aligned}$$

Now by problem 106, f achieves its minimum at a unique point $\xi_0 \in H$. Since $f(\pi_g \xi_0) = f(\xi_0)$ for all $g \in \Gamma$, uniqueness implies that ξ_0 is a Γ-invariant vector.

Finally, $f(\xi_N) = \sup_{g \in \Gamma} \|\beta_N(g)\| \leq 1 = f(0)$, and $\xi_N \neq 0$, so the unique point ξ_0 at which f achieves its minimum must be a nonzero vector. $\square$