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Functional Analysis 2
March 5, 2018

Homework 8

89. Let $\Phi : M \to N$ be a unital $*$-homomorphism between von Neumann algebras.

(1) Prove that the following two conditions are equivalent:

(a) Φ is *normal*: $x_\lambda \nearrow x$ implies $\Phi(x_\lambda) \nearrow \Phi(x)$.

(b) Φ is σ-WOT continuous.

Proof. For a bounded net of positive operators x_λ increasing to a positive operator x, we know that convergence in the WOT, SOT, σ-WOT, and σ-SOT are all equivalent. Suppose Φ is σ-WOT continuous. If $x_\lambda \nearrow x$, then in particular $x_\lambda \to x$ σ-WOT, so $\Phi(x_\lambda) \to \Phi(x)$ σ-WOT as well.

On the other hand, suppose that $\Phi : M \to N$ is normal. Let $\psi \in N_*$, the dual of N under the σ-WOT topology, be positive. Since ψ is σ-WOT continuous, ψ is normal, so $\psi \circ \Phi$ is normal and positive, and hence σ-WOT continuous by problem 88. Since M_* is spanned by positive linear functionals (Corollary 4.3.4 of Jesse Peterson's notes), this shows that $\psi \circ \Phi$ is σ-WOT continuous for every $\psi \in N_*$. Since the σ-WOT topology is the weak topology induced by the predual, this is precisely the condition for Φ to be σ-WOT continuous. □

(2) Prove that if Φ is normal, then $\Phi(M) \subseteq N$ is a von Neumann subalgebra.

Proof. If $\ker(\Phi) = 0$, then by problem 72 part 2, we know that $\Phi(M)$ is a von-Neumann subalgebra of N. Therefore, it suffices to construct unital Von-Neumann algebra $*$-homomorphism $C : M \to \operatorname{coker}(\Phi)$, with an injective factorization $\overline{\Phi} : \operatorname{coker}(\Phi) \to N$ so that $\Phi = \overline{\Phi}C$. Since $\{0\} \subseteq N$ is σ-WOT closed and Φ is σ-WOT continuous, $\ker(\Phi)$ is σ-WOT closed. Since Φ is a ring homomorphism, $\ker(\Phi)$ is a 2-sided ideal. By results proven in class, $\ker(\Phi)$ is of the form Mz for some $z \in P(Z(M))$. Since $z \in Z(M)$, we know that zH and $(1-z)H$ are M-invariant subspaces of H; if $m - n \in \ker(\Phi)$, then $(m-n)z = m - n$, so $(m-n)(1-z) = 0$. Let $C : M \to M(1-z)$ be the compression map, a σ-WOT continuous unital $*$-homomorphism for sure. If $C(x) = C(y)$, then $\Phi(x) = \Phi(y)$, so the map $\overline{\Phi} : M(1-z) \to N$ given by $\overline{\Phi}(m(1-z)) = \Phi(m)$ is a well-defined homomorphism. By definition, $\overline{\Phi}$ is unital, and since $(1-z)$ is central and self-adjoint, C and $\overline{\Phi}$ are $*$-homomorphisms. But $(m(1-z))(1-z) = m(1-z)$, so $\overline{\Phi}$ is actually a restriction of Φ, and hence still σ-WOT continuous.

For the sake of completeness, we should say why the σ-WOT topology on $M(1-z)$ is the subspace topology coming from M. When constructing the predual, we found that for a von-Neumann algebra A, we have $A_* \cong L^1(B(H))/A_\perp$, where $A_\perp$ is precisely those elements $y \in L^1(B(H))$ so that $\operatorname{tr}(ay) = 0$ for every $a \in A$. Therefore, the σ-WOT topology on any $A \subseteq B(H)$ is actually induced by the (usually redundant) family of seminorms $(a \to |\operatorname{tr}(ay)|)_{y \in L^1(B(H))}$, and every von-Neumann algebra $A \subseteq B(H)$ has the induced σ-WOT topology from $B(H)$. □

(3) Let φ be a normal state on a von Neumann algebra M, and let $(H_\varphi, \Omega_\varphi, \pi_\varphi)$ be the cyclic GNS representation of M associated to φ, i.e. $H_\varphi = L^2(M, \varphi)$, $\Omega_\varphi \in H_\varphi$ is the image of $1 \in M$ in H_φ, and $\pi_\varphi(x)m\Omega_\varphi = xm\Omega_\varphi$ for all $x, m \in M$.

(a) Show that π_φ is normal.

Proof. Suppose $x_\lambda \to x$ σ-WOT. For every $z, y \in M$, we have

$$\langle \pi_\varphi(x - x_\lambda)y\Omega, z\Omega \rangle = \varphi(z^*(x - x_\lambda)y)$$

Since multiplication is σ-WOT continuous in each coordinate and φ is σ-WOT continuous by (1),

$$\langle \pi_\varphi(x - x_\lambda)y\Omega, z\Omega \rangle \to 0$$

Since z and y were arbitrary and vectors of the form $y\Omega$ are norm-dense in H_φ, this shows that $\pi_\varphi(x_\lambda) \to \pi_\varphi(x)$ WOT. Since $\pi_\varphi(x_\lambda)$ is bounded by $||\pi_\varphi(x)||$, and the σ-WOT and WOT agree on bounded sets, $\pi_\varphi(x_\lambda) \to \pi_\varphi(x)$ σ-WOT as well. By part (1), this is enough to show that π_φ is normal. □

(b) Deduce that if φ is faithful, then $M \cong \pi_\varphi(M) \subseteq B(H_\varphi)$ is a von Neumann algebra acting on H_φ.

Proof. If φ is faithful, then for $x \neq y$, we have

$$\begin{aligned} ||\pi_\varphi(x-y)||^2 &\geq ||\pi_\varphi(x-y)\Omega||_\varphi^2 \\ &= \varphi((x-y)^*(x-y)) \\ &> 0, \end{aligned}$$

since $(x-y)^*(x-y)$ is a nonzero positive operator. Hence, π_φ is a faithful representation and the claim follows from (2). $\square$

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91. (1) It's easy to see that J is a conjugate linear isometry:

$$\begin{aligned} J\left(\left(\lambda a+b\right)\Omega\right) &= \left(\lambda a+b\right)^*\Omega = \overline{\lambda} a\Omega + b\Omega, \qquad \text{and} \\ \left\|J\left(a\Omega\right)\right\| &= \left\|a^*\Omega\right\| \\ &= \left\langle a^*\Omega, a^*\Omega\right\rangle \\ &= \operatorname{tr}\left(aa^*\right) \\ &= \operatorname{tr}\left(a^*a\right) \\ &= \left\langle a\Omega, a\Omega\right\rangle \\ &= \left\|a\Omega\right\|. \end{aligned}$$

Lastly, $J\left[M\Omega\right] = M\Omega$, because M is *-closed. Since $M\Omega$ is dense in L^2M by construction, we have that $J\left[M\Omega\right]$ is dense in L^2M as well.

(2) From the previous part, we know that J extends uniquely to a conjugate linear isometry $L^2M \to L^2M$.
Now fix $\xi \in L^2M$, and take a sequence $\left(a_n\right)_{n\in\mathbb{N}}$ in M such that $a_n\Omega \to \xi$. Then

$$\begin{aligned} J^2\xi &= J^2\left(\lim_{n\to\infty} a_n\Omega\right) \\ &= \lim_{n\to\infty} J^2\left(a_n\Omega\right), \qquad \text{since } J^2 \text{ is continuous} \\ &= \lim_{n\to\infty} \left(\left(a_n^*\right)^*\Omega\right) \\ &= \lim_{n\to\infty} a_n\Omega \\ &= \xi \end{aligned}$$

so that $J^2 = 1$ on L^2M.
Next, for $\xi = b\Omega, \eta = a\Omega \in M\Omega$, we have

$$\begin{aligned} \left\langle J\eta, J\xi\right\rangle &= \left\langle a^*\Omega, b^*\Omega\right\rangle \\ &= \operatorname{tr}\left(ba^*\right) \\ &= \operatorname{tr}\left(a^*b\right) \\ &= \left\langle b\Omega, a\Omega\right\rangle \\ &= \left\langle \xi, \eta\right\rangle. \end{aligned}$$

Now if $\xi, \eta \in L^2M$, we can write $\xi = \lim_{n\to\infty} a_n\Omega$ and $\eta = \lim_{n\to\infty} b_n\Omega$ for some sequences $\left(a_n\right), \left(b_n\right) \in M$.
Then for each $k \in \mathbb{N}$,

$$\begin{aligned} \left\langle J\eta, Ja_k\Omega\right\rangle &= \left\langle J\lim_{n\to\infty} b_n\Omega, Ja_k\Omega\right\rangle \\ &= \lim_{n\to\infty} \left\langle J\left(b_n\Omega\right), J\left(a_k\Omega\right)\right\rangle \\ &= \lim_{n\to\infty} \left\langle a_k\Omega, b_n\Omega\right\rangle \\ &= \left\langle a_k\Omega, \lim_{n\to\infty} b_n\Omega\right\rangle \\ &= \left\langle a_k\Omega, \eta\right\rangle. \quad \text{Then} \\ \left\langle J\eta, J\xi\right\rangle &= \left\langle J\eta, J\lim_{k\to\infty} a_k\Omega\right\rangle \\ &= \lim_{k\to\infty} \left\langle J\eta, Ja_k\Omega\right\rangle \\ &= \lim_{k\to\infty} \left\langle a_k\Omega, \eta\right\rangle \\ &= \left\langle \lim_{k\to\infty} a_k\Omega, \eta\right\rangle \\ &= \left\langle \xi, \eta\right\rangle. \end{aligned}$$

Thus $\left\langle J\eta, J\xi\right\rangle = \left\langle \xi, \eta\right\rangle$ for all $\xi, \eta \in L^2M$.

(3) For $a, b \in M$, we have

$$\begin{aligned} Ja^*Jb\Omega &= Ja^*b^*\Omega = (ba)\,\Omega, \qquad \text{and thus for } m \in M, \\ (JmJb)(a\xi) &= (JmJ)(ba)\,\xi \\ &= bam^*\xi \\ &= b\,(JmJa)\,\xi \\ &= b\,(JmJ)(a\xi)\,. \end{aligned}$$

Thus JmJ commutes with b on $M\Omega$, and since $M\Omega$ is dense in L^2M, it follows that JmJ commutes with b on L^2M.
Since $b \in M$ was arbitrary, it follows that $JmJ \in M'$, so $JMJ \subseteq M'$.

(4) For $a, b, c \in M$, we compute

$$\begin{aligned} \langle Ja^*Jb\Omega, c\Omega\rangle &= \langle ba\Omega, c\Omega\rangle \\ &= \operatorname{tr}(c^*ba) \\ &= \operatorname{tr}(ac^*b) \\ &= \left\langle b\Omega, (ac^*)^*\,\Omega\right\rangle \\ &= \langle b\Omega, ca^*\Omega\rangle \\ &= \langle b\Omega, JaJc\Omega\rangle\,. \end{aligned}$$

Thus $(JaJ)^* = Ja^*J$, again because $M\Omega$ is dense in L^2M.

(5) For all $a \in M$ and $y \in M'$, we have

$$\begin{aligned} \langle Jy\Omega, a\Omega\rangle &= \langle Jy\Omega, Ja^*\Omega\rangle \\ &= \langle a^*\Omega, y\Omega\rangle \\ &= \langle y^*a^*\Omega, \Omega\rangle \\ &= \langle a^*y\Omega, \Omega\rangle \\ &= \langle y^*\Omega, a\Omega\rangle\,. \end{aligned}$$

Since $\langle Jy\Omega, a\Omega\rangle = \langle y^*\Omega, a\Omega\rangle$ for all $a \in M$, and $M\Omega$ is dense in L^2M, we have $Jy\Omega = y^*\Omega$ for all $y \in M'$.

(6) Take $x, y, z \in M'$. Then

$$\begin{aligned} Jx^*Jy\Omega &= Jx^*y^*\Omega = yx\Omega, \qquad \text{and thus} \\ \langle Jx^*Jy\Omega, z\Omega\rangle &= \langle yx\Omega, z\Omega\rangle \\ &= \operatorname{tr}(z^*yx) \\ &= \operatorname{tr}(xz^*y) \\ &= \left\langle y\Omega, (xz^*)^*\,\Omega\right\rangle \\ &= \langle y\Omega, zx^*\Omega\rangle \\ &= \langle y\Omega, JxJz\Omega\rangle\,, \end{aligned}$$

so that $(JxJ)^* = Jx^*J$.

(7) For $a, b \in M$ and $x, y \in M'$, we have

$$\begin{aligned} \langle xJyJa\Omega, b\Omega\rangle &= \langle a\Omega, Jy^*Jx^*b\Omega\rangle \\ &= \langle a\Omega, Jy^*Jbx^*\Omega\rangle \\ &= \operatorname{tr}(xb^*JyJa) \\ &= \operatorname{tr}(b^*JyJax) \\ &= \operatorname{tr}(b^*JyJxa) \\ &= \langle JyJxa\Omega, b\Omega\rangle \end{aligned}$$

(8) Thus from the above we see that each $x \in M'$ commutes with each $JyJ \in JM'J$, so that $M' \subseteq (JM'J)' = JMJ$, so that $M' = JMJ$. $\square$

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92. Let Γ be a discrete group, and let $L\Gamma = \{\lambda_g\}'' \subseteq B(\ell^2\Gamma)$. Consider the faithful σ-WOT continuous tracial state $\operatorname{tr}(x) = \langle x\delta_e, \delta_e\rangle$ on $L\Gamma$.

(1) Show that $u\delta_g = \lambda_g$ uniquely extends to a unitary $u \in B(\ell^2\Gamma, L^2L\Gamma)$ such that for all $x \in L\Gamma$ and $\xi \in \ell^2\Gamma$, $L_x u\xi = ux\xi$ where $L_x \in B(L^2L\Gamma)$ is left multiplication by x, i.e., $L_x(y\Omega) = xy\Omega$.

Proof. Since $(\delta_g)_{g\in\Gamma}$ is an orthonormal basis in $\ell^2\Gamma$, in order to show that $u\delta_g = \lambda_g$ extends uniquely to a unitary, it suffices to prove that $(\lambda_g\Omega)_{g\in\Gamma}$ is an orthonormal basis for $L^2L\Gamma$. First, we check that $(\lambda_g\Omega)_{g\in\Gamma}$ is orthonormal:

$$\begin{aligned}\langle \lambda_g\Omega, \lambda_h\Omega\rangle &= \operatorname{tr}(\lambda_h^*\lambda_g)\\ &= \langle \lambda_g\delta_e, \lambda_h\delta_e\rangle\\ &= \langle \delta_g, \delta_h\rangle\\ &= \delta_{g=h}.\end{aligned}$$

It remains to show that $\operatorname{span}\{\lambda_g\Omega : g \in \Gamma\}$ is dense in $L^2L\Gamma$. By construction, $\{x\Omega : x \in L\Gamma\}$ is dense in $L^2L\Gamma$, so it suffices to prove that $\operatorname{span}\{\lambda_g\Omega : g \in \Gamma\}$ is dense in $\{x\Omega : x \in L\Gamma\}$ is dense in $L^2L\Gamma$.

Let $x \in L\Gamma$. By the definition of $L\Gamma$, there is a net $(x_i)_{i\in I}$ so that $x_i \in \operatorname{span}\{\lambda_g : g \in \Gamma\}$ for each $i \in I$ and $x_i \to x$ SOT. By the Kaplansky density theorem, we may assume $\|x_i\| \le \|x\|$ for all $i \in I$. Then since multiplication is jointly continuous on bounded sets, $(x-x_i)^*(x-x_i) \to 0$ SOT. Therefore, $(x-x_i)^*(x-x_i) \to 0$ WOT and hence σ-WOT, since everything is bounded. Now since tr is σ-WOT continuous,

$$\|x\Omega - x_i\Omega\|_2^2 = \operatorname{tr}((x-x_i)^*(x-x_i)) \to \operatorname{tr}(0) = 0.$$

That is, $x_i\Omega \to x\Omega$ in $L^2L\Gamma$. This completes the proof that $(\lambda_g)_{g\in\Gamma}$ is an orthonormal basis for $L^2L\Gamma$.

Now for all $g, h \in \Gamma$,

$$\begin{aligned}L_{\lambda_g}u\delta_h &= L_{\lambda_g}\lambda_h\Omega\\ &= \lambda_g\lambda_h\Omega\\ &= \lambda_{gh}\Omega\\ &= u\delta_{gh}\\ &= u\lambda_g\delta_h.\end{aligned}$$

Hence by linearity and continuity of u and each λ_g, we have $L_x u\xi = ux\xi$ for all $\xi \in \ell^2\Gamma$ and all $x \in \operatorname{span}\{\lambda_g : g \in \Gamma\}$. It remains to check that if $x_i \to x$ SOT, then $L_{x_i} \to L_x$ SOT. By the Kaplansky density theorem, we may assume $\|x_i\| \le \|x\|$ for all i. Then for $y \in L\Gamma$, we have $\|x_iy\| \le \|xy\|$ and $x_iy \to xy$ SOT. Hence, by the computation in the previous paragraph, $L_{x_i}(y\Omega) \to L_x(y\Omega)$ in $L^2L\Gamma$. Since this holds for all $y \in L\Gamma$, we have $L_{x_i} \to L_x$ SOT as desired. □

(2) Deduce from Problem 91 that $L\Gamma' = R\Gamma$.

Proof. By problem 91, it suffices to show $JL\Gamma J = R\Gamma$. For $g, h \in \Gamma$, we have

$$\begin{aligned}(JL_xJ)(\lambda_h\Omega) &= JL_x(\lambda_{h^{-1}}\Omega)\\ &= J(x\lambda_{h^{-1}}\Omega)\\ &= \lambda_h x^*\Omega\\ &= R_{x^*}(\lambda_h\Omega).\end{aligned}$$

Since $(\lambda_h\Omega)_{h\in\Gamma}$ is an orthonormal basis for $L^2L\Gamma$, we have $JL_xJ = R_{x^*}$. Therefore, $JL\Gamma J = R\Gamma$. □