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Functional Analysis 2
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Homework 9

93. (1) Since convolution is $\mathbb{C}$-linear and associative, $\ell\Gamma$ is a $\mathbb{C}$-module, closed under convolution. We know that $\delta_e \in \ell^2\Gamma$ is a unit for the convolution, with $\delta_e * f$ defined for every $f : \Gamma \to \mathbb{C}$. Therefore, $\delta_e \in \ell\Gamma$, making $\ell\Gamma$ a unital algebra. We will use part (2) to prove that $\ell\Gamma$ is $*$-closed. (Contrary to the hint, we will not use $*$-closure of $\ell\Gamma$ in proving (2)).

Let $x \in \ell\Gamma$. First, notice that the definition of x^* by $(x^*)_g = \overline{x_{g^{-1}}}$ is the only possible definition so that $(T_x)^* = T_{x^*}$, because if T_x has an adjoint at all, we must have $\langle T_x\delta_e, \delta_e\rangle = \langle \delta_e, T_x^*\delta_e\rangle$. Moreover, for every $x, \eta \in \ell^2\Gamma$, by Hölder's inequality, $x * \eta \in \ell^\infty\Gamma$ is well-defined. The only issue is checking that $||x * \eta||_2 < \infty$. Suppose $\xi \in (\ell^2\Gamma)_1$. It would be circular to assert that $\langle x * x^* * \xi, \xi\rangle = ||x^* * \xi||_2^2$, but we can approximate. For $F \subseteq \Gamma$ finite, define $\eta_F = \sum_{g\in F}(x^* * \xi)_g\delta_g$. Then we have $\langle \eta_F, x^* * \xi\rangle = ||\eta_F||_2^2 \uparrow ||x^* * \xi||_2^2$. Since each η_F is in $\ell^1\Gamma$, we can apply Fubini's theorem, obtaining

$$\begin{aligned}\langle \eta_F, x^* * \xi\rangle &= \langle x * \eta_F, \xi\rangle \\ &\le ||T_x|| \cdot ||\eta_F||2||\xi||2 \\ &\le ||T_x|| \cdot ||\xi||_2^2\end{aligned}$$

Taking limits,

$$||x^* * \xi||_2^2 \le ||T_x|| \cdot ||\xi||_2^2$$

so $||x^*|| \le ||x||$ as usual. In particular, $||x^* * \xi||_2 < \infty$, showing that $x^* \in \ell\Gamma$.

(2) Suppose $x \in \ell\Gamma$. To show that $x \in B(\ell^2\Gamma)$, we apply the closed graph theorem. It suffices to check that if $\eta_n \to \eta$ in $\ell^2\Gamma$ and $x * \eta_n \to \xi$ in $\ell^2\Gamma$, then $x * \eta = \xi$. The following calculation is basic, but in the interest of full disclosure, I read it in Jesse's notes while looking for something else. If $g \in \Gamma$, then by continuity of the inner product,

$$\begin{aligned}|(\xi - x * \eta)_g| &= \lim_{n\to\infty} |(x * \eta_n - x * \eta)_g| \\ &\le \lim_{n\to\infty} ||(x * (\eta_n - \eta))_g||_\infty \\ &\le \lim_{n\to\infty} ||x||_2||\eta_n - \eta||_2 \\ &= 0\end{aligned}$$

Therefore, ξ and $x * \eta$ agree pointwise, and hence $\xi = x * \eta$, as desired.

(3) Since $L\Gamma = R\Gamma' = \{\rho_g : g \in \Gamma\}''' = \{\rho_g : g \in \Gamma\}'$ as a subalgebra of $B(\ell^2\Gamma)$, it suffices to check that for every $x \in \ell\Gamma$ and $g \in \Gamma$, we have $x\rho_g = \rho_g x$. Let $x \in \ell\Gamma$ and $g \in \Gamma$ be given. Since x and ρ_g are norm-continuous, it suffices to check that for every $h \in \Gamma$, $x * \rho_g(\delta_h) = \rho_g(x * \delta_h)$. Let $h \in \Gamma$ be given. Then we can compute that

$$x * \rho_g(\delta_h) = x * \delta_{hg^{-1}}$$

and for every $k \in \Gamma$,

$$\begin{aligned}(x * \rho_g(\delta_h))_k &= (x * \delta_{hg^{-1}})_k \\ &= x_{kgh^{-1}} \\ &= (x * \delta_h)_{kg} \\ &= \rho_g(x * \delta_h)_k\end{aligned}$$

showing that

$$x * \rho_g(\delta_h) = \rho_g(x * \delta_h)$$

showing that

$$T_x\rho_g = \rho_g T_x$$

as desired.

(4) It is clear that $x \to T_x$ is a unital homomorphism. We saw when solving part (1) that $(T_x)^* = T_{x^*}$. Because $\langle T_x\delta_e, \delta_g\rangle = x_g$, an inverse homomorphism is given by $T \to (\langle T\delta_e, \delta_g\rangle)_{g\in\Gamma}$; that the range of this homorphism is contained in $\ell\Gamma$ was proven in class.

95. Let M and N be von Neumann algebras, and $\Phi : M \to N$ a *-isomorphism.
First, Φ sends positive operators to positive operators, since $\Phi(x^*x) = \Phi(x)^* \Phi(x)$.
Let (x_λ) be an increasing net of positive operators in M, with least upper bound x.
Lastly, if $y \in N$ with $\Phi(x) \geq y \geq \Phi(x_\lambda)$ for all λ, then $x \geq \Phi^{-1}(y) \geq x_\lambda$ for all λ, and thus $y = x$, because x is the least upper bound for (x_λ).
Thus $\Phi(x_\lambda) \nearrow \Phi(x)$, so Φ is normal.
$\square$

96. (1) A well-known result in group theory says that any subgroup of a free group is free. Therefore, it suffices to prove that the conjugacy classes of the members of a generating set for $\mathbb{F}_2$ are infinite. We know that in S_n, we have $(1, 2, \dots n)^k(1, 2)(1, 2 \dots n)^{-k} = (k+1, k+2)$. Therefore, for every $n \in \mathbb{N}$, there exist groups G with elements $a, b \in G$ such that a^kba^{-k} are distinct for every $k \leq n$. If $\mathbb{F}_2 = \langle x, y\rangle$, then $|\{x^kyx^{-k}\}|$ is either infinity, or a non-zero integer multiple of every natural number. Therefore, y has infinite conjugacy class, as desired.

Since $\mathbb{F}_2$ is an ICC group, by a result proved in class, $L\mathbb{F}_2$ is a II_1 factor.

(2) Let $\Phi : \Gamma \to \Lambda$ be an isomorphism of groups. Then there is an isometric isomorphism $\Phi_* : \ell^2\Lambda \to \ell^2\Gamma$, defined by $\Phi_*(\eta)_g = \eta_{\Phi(g)}$ for $g \in \Gamma$, with $(\Phi^{-1})_* = (\Phi_*)^{-1}$. Pick $x \in \ell\Lambda$. For every $\eta \in \ell^2\Gamma$, we have $\Phi_*(x) * \eta = \Phi_*(x * \Phi_*^{-1}(\eta))$, so $\Phi_*(x) \in \ell\Gamma$. Therefore, we may consider $\Phi_*(X)$ as an algebra isomorphism $\ell\Lambda \to \ell\Gamma$. Since Φ preserves the group identity, $\Phi_*(\delta_e) = \delta_e$, showing that Φ_* is unital, and since Φ preserves inverses and Φ_* is linear, Φ_* is a $*$-homomorphism. By problem 93 part (4), we can turn Φ_* into a unital $*$-isomorphism $L\Lambda \to L\Gamma$.

In particular, σ is an automorphism of $\mathbb{F}_2$, so σ defines an automorphism α of $L\mathbb{F}_2$.

(3) We will show that α is free, which implies that α is outer. Happily for us, $\sigma = \sigma^{-1}$. Therefore, we can simply write, for $x \in \ell\mathbb{F}_2$, that $\alpha(x)_g = x_{\sigma(g)}$. Now for every $g, h \in \mathbb{F}_2$ and $x \in \ell\mathbb{F}_2$, we have $(x * \delta_g)_h = x_{h^{-1}g}$ and $(\delta_g * x)_h = x_{g^{-1}h}$. Suppose $x \in \ell\mathbb{F}_2$ such that for every $g \in \mathbb{F}_2$, we have $x * \alpha(\delta_{g^{-1}}) = \delta_{g^{-1}} * x$; then for every $g, h \in \mathbb{F}_2$, we get $x_{h\sigma(g)} = x_{gh}$. In particular, if $\mathbb{F}_2 \cong \langle a, b\rangle$ is a presentation, then for every $w \in \mathbb{F}_2$, we have $x_w = x_{a(a^{-1}w)} = x_{a^{-1}wb}$. Of course, counting up a's and b's via the homormohpism $\langle a, b\rangle \to \langle a, b|[a, b]\rangle \cong \mathbb{Z}^2$ shows that for any w, the set $\{a^{-n}wb^n : n \in \mathbb{Z}\}$ is infinite. This shows that x is constant on some infinite subsets which cover $\mathbb{F}_2$. Since $x \in \ell^2\mathbb{F}_2$, this means that $x = 0$. By definition, we have shown that α is free.

97. Let $\mathbb{Z}$ act on $L^\infty(S^1)$ via irrational rotation, i.e., let $T : \mathbb{T} \to \mathbb{T}$ be given by $T(x) = x + \alpha$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and define the action by $n \mapsto - \circ T^n$.
First we show that the action is ergodic. Let $S \subseteq \mathbb{T}$ be measurable and invariant under T.
Fix $\epsilon > 0$. Since $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, we can find $f \in C(\mathbb{T})$ such that

$$\|f - \mathbb{1}_S\|_1 < \frac{\epsilon}{2}.$$

Then since S is T invariant, we also have that for each $n \in \mathbb{Z}$,

$$\|f \circ T^n - \mathbb{1}_S\|_1 < \frac{\epsilon}{2}.$$

Thus for each $n \in \mathbb{Z}$, $\|f \circ T^n - f\|_1 < \epsilon$.
But since $x + n\alpha$ is dense in $\mathbb{T}$, it follows that for each $t \in \mathbb{T}$, $\|f(x + t) - f(x)\|_1 < \epsilon$, bceause f is continuous.
We claim that f is constant: (τ_x is translation by x):

$$\begin{aligned}
\left\| f - \int f \circ \tau_x (t)\, dt \right\|_1 &= \int \left| f - \int f \circ (x+t)\, dt \right| dx \\
&\leq \left| \int \int f(x) - f(x+t)\, dt dx \right| \\
&= \left| \int \int f(x) - f(x+t)\, dx dt \right|, \quad \text{by Fubini} \\
&= \int \|f - f \circ \tau_x\|_1\, dt \\
&= \|f - f \circ \tau_x\|_1 \\
&\leq \epsilon.
\end{aligned}$$

Thus f is equal a.e. to it's average value, and thus is constant. It follows that $\mu(S) = 0$ or $\mu(S) = 1$, so the action is ergodic.

Now we show that the action is free.

Since non-zero integer multiples of an irrational number are irrational, it suffices to prove that the action of T is free. Fix $f \in L^\infty(S^1, \lambda)$, where λ denotes the Haar probability measure on S^1. Suppose that for every $g \in L^\infty(S^1, \lambda)$, we have $f \cdot T(g) = g \cdot f$ almost everywhere. Suppose $f \neq 0$. Then f cannot be constant almost everywhere, since if g is the characteristic function of an open interval not of full measure, then $T(g) - g$ is non-zero on a set of positive measure. Since λ is finite and f is not constant (and in particular, not zero) almost everwhere, there is $z \in \mathbb{C} \setminus \{0\}$ and $\epsilon \in [0, |z|)$ such that $\lambda(\{|f - z| \leq \epsilon\}) \in (0, 1)$. Therefore, it suffices to consider the case where f is the characteristic function 1_A for some A with $\lambda(A) \in (0, 1)$. But the action of T is ergodic, so $T(1_A) - 1_A$ is not zero almost everywhere. Therefore, setting $g = 1_A$, we obtain a contradiction. This shows that $f = 0$ in $L^\infty(S^1, \lambda)$, and since f was arbitrary, the action of T is free.