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Functional Analysis 2
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Homework 10

100. (1) First, notice that the inclusion $\iota : N \to M$ is an isometry with respect to the trace-norm: If $a, b \in N$, then

$$\begin{aligned}\langle a, b\rangle_N &= \operatorname{tr}(b^* a)\\ &= \operatorname{tr}(\iota(b)^*\iota(a))\\ &= \langle a, b\rangle_M\end{aligned}$$

Therefore, ι extends to an isometry $N \to L^2(M, \operatorname{tr})$. Since ι is an isometry, ι extends continuously to the trace-norm closure of N, which is $L^2(N, \operatorname{tr})$. Being an isometry, $\iota : L^2(N, \operatorname{tr}) \to L^2(M, \operatorname{tr})$ must still be an inclusion.

(2) Being an orthogonal projection, e_N is norm-decreasing, and therefore continuous, with a continuous adjoint e_N^*. As for any projection onto a closed Hilbert subspace, we have $e_N^* = \iota$, and so $e_N^* e_N$ is the orthogonal projection onto $\iota(L^2(N, \operatorname{tr}))$, while $e_N e_N^* = 1_{L^2(N,\operatorname{tr})}$.

Suppose $a, b \in N$, and let Ω_N be the image of 1_N in $L^2(N, \operatorname{tr})$. Then we have

$$\begin{aligned}JaJe_N^* b\Omega_N &= JaJb\Omega_N\\ &= ba^*\Omega_N\\ &= e_N^* ba^*\Omega_N\\ &= e_N^* JaJb\Omega_N\end{aligned}$$

Since multiplication by a member of n is continuous in one component (by Cauchy-Schwarz for the trace) and e_N^* is norm-continuous, we can replace $b\Omega_N$ with any member of $L^2(N, \operatorname{tr})$, so e_N^* commutes with the right-action of N. Taking adjoints, e_N also commutes with the right action: for $a \in N$, $\eta \in L^2(M, \operatorname{tr})$, and $\xi \in L^2(N, \operatorname{tr})$,

$$\begin{aligned}\langle JaJe_N\eta, \xi\rangle &= \langle \eta, e_N^* Ja^* J\xi\rangle\\ &= \langle \eta, Ja^* Je_N^*\xi\rangle\\ &= \langle e_N JaJ\eta, \xi\rangle\end{aligned}$$

Clearly, the left action of a member of M and the right action of a member of N on $L^2(M, \operatorname{tr})$ commute. Combining these three facts, for every $x \in M$, $e_N x e_N^*$ commutes with the right action of N on $L^2(N, \operatorname{tr})$. By problem 91 part 8, we have $e_N x e_N^* \in (JNJ)' = N$, where commutant is relative to $B(L^2(N, \operatorname{tr}))$.

(3) Pick $x \in M$. Then, for any $y \in N$, we have

$$\begin{aligned}\operatorname{tr}(E(x)y)_N &= \langle e_N x^* e_N^*\Omega_N, y\Omega_N\rangle_N\\ &= \langle x^* e_N^*\Omega_N, e_N^* y\Omega_N\rangle_M\\ &= \langle x^*\Omega_M, y\Omega_M\rangle_M\\ &= \operatorname{tr}(xy)_M\end{aligned}$$

A Hilbert space is in weak duality with itself, so for any Hilbert space H, an element $\eta \in H$ is uniquely determined by a choice of $(\langle \eta, \xi\rangle)_{\xi\in H}$, provided such an η exists. In particular, this holds for $\eta = E(x)$.

101. (1) First, notice that E preserves positivity: Suppose $x \in M$ is positive. Then for any $\eta \in L^2(N, \operatorname{tr})$, we have

$$\langle E(x)\eta, \eta\rangle = \langle xe_N^*\eta, e_N^*\eta\rangle \geq 0$$

Expanding on the last argument, the sesquilinear form induced by $E(x)$ is a restriction of the sesquilinear form induced by x: for $\eta, \xi \in L^2(N, \operatorname{tr})$, we have

$$\begin{aligned}\langle E(x)\eta, \xi\rangle_N &= \langle xe_N^*\eta, e_N^*\xi\rangle_M\\ &= \langle x\eta, \xi\rangle_M\end{aligned}$$

Therefore, if $x_\lambda \nearrow x$, then $E(x_\lambda)$ is still an increasing sequence of positive operators bounded by $E(x)$, and $E(x_\lambda) \nearrow E(x)$ weakly, and hence σ-WOT. By definition, E is normal.

(2) By the previous remark about sesquilinear forms, $E(1_M) = 1_N$. Let $y, z, w \in N$ and $x \in M$ be given; then by part (3) of 100,

$$\begin{aligned}\mathrm{tr}((yE(x)z)w)_N &= \mathrm{tr}(E(x)zwy)_N \\ &= \mathrm{tr}(xzwy)_M \\ &= \mathrm{tr}(yxzw)_M \\ &= \mathrm{tr}(E(yxz)w)_N\end{aligned}$$

and by uniqueness, $E(yxz) = \mathrm{tr}(yE(x)z)$.

(3) This follows from the remark about sesquilinear forms.

(4) Let a matrix $a = ((a_{i,j})_{i=1}^n)_{j=1}^n \in M_n(M)$ be given. As an unjustified notation, let $E(a) = (E(a_{i,j})_{i=1}^n)_{j=1}^n$. Choose a vector $\eta = (\eta_k)_{k=1}^n$ in $L^2(N, \mathrm{tr})^n$. Then we have

$$\begin{aligned}\langle a\eta, \eta\rangle &= \sum_{i=1}^n \sum_{j=1}^n \langle a_{i,j}\eta_j, \eta_i\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle E(a_{i,j})\eta_j, \eta_i\rangle \\ &= \langle E(a)\eta, \eta\rangle\end{aligned}$$

Therefore, if a is positive, then $E(a)$ is also positive, as desired.

(5) Let $x \in M$. Then for every $\eta \in L^2(N)$, we have

$$\begin{aligned}\langle E(x)^*E(x)\eta, \eta\rangle_N &= \langle E(x)\eta, E(x)\eta\rangle_N \\ &= \langle e_N^* e_N(xe_N^*\eta), xe_N^*\eta\rangle\end{aligned}$$

As explained in solving the previous problem, $e_N^* e_N$ is an orthogonal projection, so we have

$$\begin{aligned}\langle E(x)^*E(x)\eta, \eta\rangle_N &\le \langle xe_N^*\eta, xe_N^*\eta\rangle \\ &= \langle E(x^*x)\eta, \eta\rangle\end{aligned}$$

Since η was chosen arbitrarily, $E(x)^*E(x) \le E(x^*x)$.

(6) The key here is that the inclusion $N \subseteq M$ is unital. If $E(x^*x) = 0$, then in particular $\langle E(x^*x)\Omega_N, \Omega_N\rangle = \langle x^*x\Omega_M, \Omega_M\rangle = ||x||^2 = 0$, so $x = 0$.

102. Notation: Let H be a Hilbert space on which M acts as a von Neumann algebra. It seems to us that we can prove a little more: rather than just considering M with the GNS-representation, we can consider any representation of M which induces an equivalent operator norm on H. We proceed in this generality. The case $H = L^2(M, \mathrm{tr})$ is often easier.

(1) Assume first that (x_λ) is a bounded net in M with $x_\lambda \to x$ SOT. Then, for all $\xi \in H$, we have $x_\lambda\xi \to x\xi$. Also note that since

$$\|(x_\lambda - x)^*(x_\lambda - x)\xi\| \le \|(x_\lambda - x)^*\|\|(x_\lambda - x)\xi\|$$

with $x_\lambda - x$ uniformly bounded in operator norm, this implies that $(x_\lambda - x)^*(x_\lambda - x) \to 0$ SOT. Since tr is normal, this implies that $\mathrm{tr}((x_\lambda - x)^*(x_\lambda - x)) \to 0$. This is exactly the inner product on $L^2(M)$, and so we have

$$\|x_\lambda\Omega - x\Omega\|^2 = \langle (x_\lambda - x)\Omega, (x_\lambda - x)\Omega\rangle = \mathrm{tr}((x_\lambda - x)^*(x_\lambda - x)) \to 0.$$

Thus, we have shown that $x_\lambda \to x$ SOT in M with x_λ bounded implies that $\|x_\lambda\Omega - x\Omega\| \to 0$.

Conversely, assume that $\|x_\lambda\Omega - x\Omega\| \to 0$ (with (x_λ) still a bounded net in M). By definition, we have $\mathrm{tr}((x_\lambda - x)^*(x_\lambda - x)) \to 0$ with $(x_\lambda - x)^*(x_\lambda - x) \ge 0$ for all λ. Since the

x_λ are uniformly bounded, we have that $(x_\lambda - x)^*(x_\lambda - x)$ is a bounded net of positive operators in M. Because the unit ball of M is σ-WOT compact, we know that there exists a subnet of the $(x_\lambda - x)^*(x_\lambda - x)$ which converges σ-WOT, say to x. Since tr is normal and $\mathrm{tr}((x_\lambda - x)^*(x_\lambda - x)) \to 0$, this implies that $\mathrm{tr}(x) = 0$. Since we are working on a bounded set, x is the WOT limit of positive operators, and therefore positive. Since $\mathrm{tr}(x) = 0$ and tr is faithful, we must have $x = 0$. We may apply the same argument to any subnet of the x_λ, so we have shown that $(x_\lambda - x)^*(x_\lambda - x)$ is a bounded, positive net in M, such that every subnet has a further subnet which converges to 0 σ-WOT; by general topology, this implies that $(x_\lambda - x)^*(x_\lambda - x) \to 0$ σ-WOT, and by boundedness, WOT. Thus, for all $\xi \in H$, we have

$$\langle (x_\lambda - x)\xi, (x_\lambda - x)\xi\rangle = \langle (x_\lambda - x)^*(x_\lambda - x)\xi, \xi\rangle \to 0$$

and so $x_\lambda - x \to 0$ SOT, as desired.

Putting these together, we have shown that for bounded nets x_λ in M, $x_\lambda \to x$ SOT if and only if $\|x_\lambda\Omega - x\Omega\| \to 0$.

(2) Fix $x \in M$. Fix $\epsilon > 0$. By assumption, we have $(\bigcup M_n)'' = M$, and note that an increasing union of unital $*$-subalgebras is still a unital $*$-subalgebra. By the bicommutant theorem, we know that $(\bigcup M_n)''$ is the SOT closure of $\bigcup M_n$. Therefore, $\bigcup M_n$ is a $*$-subalgebra which is SOT dense in the von Neumann algebra M. By the Kaplansky density theorem, there exists a bounded net (x_λ) in $\bigcup M_n$ such that $x_\lambda \to x$ SOT. By part (1), this implies that $\|x_\lambda\Omega - x\Omega\| \to 0$, so there exists some λ such that $\|x_\lambda\Omega - x\Omega\| < \epsilon$. We know that $x_\lambda \in M_n$ for some n, and so $x_\lambda\Omega \in L^2(M_n) \subseteq L^2(M)$ (in the sense of problems 100 and 101). By problems 100 and 101, we know that $E_n(x)\Omega = e_n(x\Omega)$, where e_n is the projection in $B(L^2(M))$ onto the closed subspace $L^2(M_n) \subseteq L^2(M)$. By definition of projections, we know that

$$\|E_n(x)\Omega - x\Omega\| = \inf_{y \in L^2(M_n)} \|y - x\Omega\| \leq \|x_\lambda\Omega - x\Omega\| < \epsilon.$$

So, we have that $\|E_n(x)\Omega - x\Omega\| < \epsilon$. Observe that for all $m \geq n$, $E_m(x)\Omega = e_m(x\Omega)$ is the projection onto the subspace $L^2(M_m)$ with $L^2(M_n) \subseteq L^2(M_m) \subseteq L^2(M)$. So, for all $m \geq n$, we have $e_m \geq e_n$ as projections, and thus $\|E_m(x)\Omega - x\Omega\| \leq \|E_n(x)\Omega - x\Omega\|$ whenever $m \geq n$. Therefore, we have shown that $\|E_m(x)\Omega - x\Omega\| < \epsilon$ for all $m \geq n$. Since $\epsilon > 0$ was arbitrary, we conclude that $\|E_n(x)\Omega - x\Omega\| \to 0$ as $n \to \infty$.

(3) By part (2), $\|E_n(x)\Omega - x\Omega\| \to 0$. Also, observe that $E_n(x)$ is a bounded sequence in M, since each E_n is $||\cdot||_2$-decreasing. Therefore, by part (1), this implies that $E_n(x) \to x$ SOT as $n \to \infty$.