Ethan Ackelsberg, Zachery Dell, Peter Huston
Functional Analysis 2
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Homework 11

103. (1) First, recall that any Banach space is weak-$*$ dense in its double dual. In particular, $\ell^1\Gamma$ is weak-$*$ dense in $(\ell^\infty\Gamma)^*$. Let $\phi \in (\ell^\infty\Gamma)^*$ be a state, and pick some $(\phi_\lambda) \in \ell^1\Gamma$ with $\phi_\lambda \to \phi$ in the weak-$*$ topology. For $\psi \in \ell^1\Gamma$, let ψ^+ be defined by $\psi^+(g) = \mathrm{Re}(\phi(g))^+$, and notice that ϕ^+ is still in ℓ^1. We will demonstrate that $||\phi_\lambda - \phi_\lambda^+||_1 \to 0$, implying that $\phi_\lambda - \phi_\lambda^+ \to 0$ weakly by Hölder's inequality. Notice that

$$\phi_\lambda - \phi_\lambda^+ = 2\phi_\lambda^- - i\,\mathrm{Im}(\phi_\lambda)$$

On the other hand, evaluating at 1 in $\ell^\infty\Gamma$ gives

$$\phi_\lambda(1) = ||\phi_\lambda^+||_1 - ||\phi_\lambda^-||_1 + i||\,\mathrm{Im}(\phi_\lambda)||_1$$

Since $\phi_\lambda \to \phi$ in the weak-$*$ topology and ϕ is a state, $\phi_\lambda(1) \to 1$, and the latter two 1-norms go to 0, proving our claim.

Therefore, we may replace each ϕ_λ with ϕ_λ^+ without changing the weak-$*$ limit, letting us assume that (ϕ_λ) is a net of positive members of $(\ell^1\Gamma)_1$. Define ψ_λ by $\psi_\lambda = \frac{\phi_\lambda}{\phi_\lambda(1)}$. Since $\phi_\lambda(1) \to \phi(1) = 1$, we have $\psi_\lambda \to \phi$ as well. Since $\ell^1\Gamma$ is a C*-algebra, the facts that each ψ_λ is positive, norm 1, and $\psi_\lambda(1) = 1$ imply that each ψ_λ is a state, so we are done.

(2) The product weak topology on $\oplus_{g\in F}\ell^1\Gamma$ is induced by the usual action of $\oplus_{g\in F}\ell^1\Gamma$ on $\oplus_{g\in F}\ell^\infty\Gamma$; similarly, since F is finite, all norms on the direct sum consistent with the original are equivalent, so we may as well take the norm $|| \oplus_{g\in F} x_g|| = \max_{g\in F} ||x_g||_1$. In particular, the norm on $\ell^\infty\Gamma$ is equivalent to the operator norm from $\ell^1\Gamma$, so this norm is equivalent to the operator norm from $(\oplus_{g\in F}\ell^\infty\Gamma)^*$.

To see that K is convex, notice that $\mathrm{Prob}(\Gamma)$ is convex, and recall that the weak closure of a convex set is convex. Since K is weakly closed and the norm on ℓ^∞ is the operator norm from ℓ^1, K is also norm closed.

(3) Let m be a left-Γ invariant state on $l^\infty\Gamma$. Let $F \subseteq \Gamma$ finite be given. Let (μ_λ) be a net in $\mathrm{Prob}(\Gamma)$ converging to m in the weak-$*$ toplogy. The action of each $g \in F$ on $(\ell^\infty\Gamma)^*$ is weak-$*$ continuous, so $\oplus_{g\in F} g\mu_\lambda - \mu_\lambda \to \oplus_{g\in F} gm - m$ weak-$*$. However, since m is left-Γ invariant, $gm - m = 0$ for each $g \in F$, so $\oplus_{g\in F} g\mu_\lambda \to 0$ weak-$*$, and hence weakly, since $0 \in \ell^1\Gamma$. Since K is weakly closed and each $\oplus_{g\in F} g\mu_\lambda - \lambda \in K$, we have $0 \in K$.

Since K is convex, K is in fact the norm closure of $\{\oplus_{g\in F} g\mu - \mu : \mu \in \mathrm{Prob}(\Gamma)\}$, so there is a sequence $\mu \in \mathrm{Prob}(\Gamma)$ with $\max_{g\in F} ||g\mu - \mu||_1 < \epsilon$, i.e. Γ has a left-invariant mean.

104. (1) Without loss of generality we may assume $a \le b$, and then we have

$$\begin{aligned}
&\int_0^1 \left|\chi_{(r,1]}(a) - \chi_{(r,1]}(b)\right| dr \\
=&\int_0^a \left|\chi_{(r,1]}(a) - \chi_{(r,1]}(b)\right| dr + \int_a^b \left|\chi_{(r,1]}(a) - \chi_{(r,1]}(b)\right| dr + \int_b^1 \left|\chi_{(r,1]}(a) - \chi_{(r,1]}(b)\right| dr \\
=&\int_a^b dr \\
=&b - a
\end{aligned}$$

(2) We have

$$\begin{aligned}
\|h\cdot\mu-\mu\|_{\ell^1\Gamma} &= \sum_{g\in\Gamma}\left|\left(h\cdot\mu\right)(g)-\mu(g)\right| \\
&= \sum_{g\in\Gamma}\left|\mu\left(h^{-1}g\right)-\mu(g)\right| \\
&= \sum_{g\in\Gamma}\int_0^1\left|\chi_{(r,1]}\left(\mu\left(h^{-1}g\right)\right)-\chi_{(r,1]}\left(\mu(g)\right)\right|dr, \qquad \text{by part (1)} \\
&= \int_0^1\sum_{g\in\Gamma}\left|\chi_{(r,1]}\left(\mu\left(h^{-1}g\right)\right)-\chi_{(r,1]}\left(\mu(g)\right)\right|dr, \qquad \text{by Tonelli's theorem}
\end{aligned}$$

(3)

$$\begin{aligned}
hE(\mu,r) &= \{hg \mid g\in\Gamma, \mu(g)>r\} \\
&= \left\{k\in\Gamma \mid \mu\left(h^{-1}k\right)>r\right\} \\
&= \{g\in\Gamma \mid (h\cdot\mu)(g)>r\} \\
&= E(h\mu,r).
\end{aligned}$$

(4)

$$\begin{aligned}
\int_0^1\left|E(\mu,r)\right|dr &= \int_0^1\sum_{g\in\Gamma}\chi_{(0,\mu(g)]}dr \\
&= \sum_{g\in\Gamma}\int_0^1\chi_{(0,\mu(g)]}dr, \text{by Tonelli} \\
&= \sum_{g\in\Gamma}\mu(g) \\
&= 1
\end{aligned}$$

(5) We know that $g\in\Gamma$ is in $hE(\mu,r)\,\Delta\,E(\mu,r)$ if and only if $\mu\left(h^{-1}g\right)\le r<\mu(g)$ or $\mu(g)\le r<\mu\left(h^{-1}g\right)$.
This is equivalent to $\left|\chi_{(r,1]}\left(\mu\left(h^{-1}g\right)\right)-\chi_{(r,1]}\left(\mu(g)\right)\right|=1$, so we have

$$\left|hE(\mu,R)\,\Delta\,E(\mu,r)\right| = \sum_{g\in\Gamma}\left|\chi_{(r,1]}\left(\mu\left(h^{-1}g\right)\right)-\chi_{(r,1]}\left(\mu(g)\right)\right|$$

(6) We have

$$\begin{aligned}
\int_0^1\sum_{h\in F}\left|hE(\mu,r)\,\Delta\,E(\mu,r)\right|dr &= \sum_{h\in F}\int_0^1\left|hE(\mu,r)\,\Delta\,E(\mu,r)\right|dr \\
&= \sum_{h\in F}\int_0^1\sum_{g\in\Gamma}\left|\chi_{(r,1]}\left(\mu\left(h^{-1}g\right)\right)-\chi_{(r,1]}\left(\mu(g)\right)\right|dr, \qquad \text{by (5)} \\
&= \sum_{h\in F}\|h\cdot\mu-\mu\|_{\ell^1\Gamma}, \qquad \text{by (2)} \\
&< \epsilon \\
&= \epsilon\int_0^1\left|E(\mu,r)\right|dr, \qquad \text{by (4).}
\end{aligned}$$

Thus there exists $r \in [0,1]$ such that

$$\sum_{h \in F} |hE(\mu, r) \,\Delta\, E(\mu, r)| \, dr < \epsilon \, |E(\mu, r)| \, .$$

and in particular for each $h \in F$ we have

$$|hE(\mu, r) \,\Delta\, E(\mu, r)| \, dr < \epsilon \, |E(\mu, r)| \, .$$

(7) Now take F_n to be a sequence of finite subsets increasing to Γ, and $\epsilon_n := \frac{1}{n}$. Then there exist $\mu_n \in \mathrm{Prob}(\Gamma)$ and $r_n \in [0,1]$ such that

$$|hE(\mu_n, r_n) \,\Delta\, E(\mu_n, r_n)| < \frac{1}{n} \, |E(\mu_n, r_n)| \, ,$$

so that

$$\lim_{n \to \infty} \frac{|hE(\mu_n, r_n) \,\Delta\, E(\mu_n, r_n)|}{|E(\mu_n, r_n)|} = 0.$$

Thus $(E(\mu_n, r_n))_{n \in \mathbb{N}}$ is a Folner sequence for Γ.

105. (1) Find a bijection from the set of ultrafilters on $\mathbb{N}$ to $\beta\mathbb{N}$.

Proof. Let $\mathscr{U}$ be the set of ultrafilters on $\mathbb{N}$. Give $\mathscr{U}$ the topology generated by basic open sets of the form $\widetilde{A} := \{\omega \in \mathscr{U} : A \in \omega\}$ for $A \subseteq \mathbb{N}$. We will show that $\mathscr{U}$ with this topology satisfies the universal property for the Stone-Čech compactification of $\mathbb{N}$.

For $n \in \mathbb{N}$, let $\widehat{n} \in \mathscr{U}$ be the principal ultrafilter on n, i.e. $\widehat{n} := \{A \subseteq \mathbb{N} : n \in A\}$. The map $\iota : \mathbb{N} \to \mathscr{U}$ given by $\iota(n) := \widehat{n}$ is clearly an injection. Since $\mathbb{N}$ is discrete, ι is a continuous function. We want to show that the image $\iota(\mathbb{N})$ is dense in $\mathscr{U}$. But this is easy: given $\varnothing \neq A \subseteq \mathbb{N}$, we have $\widehat{n} \in \widetilde{A}$ for all $n \in A$.

Now we claim that $\mathscr{U}$ is a Hausdorff space. Let $\omega_1 \neq \omega_2 \in \mathscr{U}$. Then there is a set $A \subseteq \mathbb{N}$ such that $A \in \omega_1$ and $\mathbb{N} \setminus A \in \omega_2$. Hence, $\omega_1 \in \widetilde{A}$ and $\omega_2 \in \widetilde{\mathbb{N} \setminus A}$, so $\mathscr{U}$ is Hausdorff.

Next, we will show that $\mathscr{U}$ is compact. Note that every basic open set is also closed: $\widetilde{A} = \mathscr{U} \setminus \left(\widetilde{\mathbb{N} \setminus A}\right)$. Moreover, for any finite collection $A_1, \dots, A_n \subseteq \mathbb{N}$, since filters are closed under finite intersections and supersets, we have

$$\widetilde{\bigcap_{i=1}^n A_i} = \{\omega \in \mathscr{U} : \bigcap_{i=1}^n A_i \in \omega\} = \{\omega \in \mathscr{U} : A_i \in \omega \text{ for } i = 1, \dots, n\} = \bigcap_{i=1}^n \widetilde{A_i}.$$

Let $(A_i)_{i \in I}$ be a collection of nonempty subsets of $\mathbb{N}$ such that $\bigcap_{i \in F} \widetilde{A_i} \neq \varnothing$ for all $F \subseteq I$ finite. Then by the previous observation, $\widetilde{\bigcap_{i \in F} A_i} \neq \varnothing$. In particular, $\bigcap_{i \in F} A_i \neq \varnothing$. That is, $(A_i)_{i \in I}$ satisfies the finite intersection property. Therefore, by Zorn's lemma, there is an ultrafilter $\omega \in \mathscr{U}$ such that $A_i \in \omega$ for every $i \in I$. That is, $\omega \in \bigcap_{i \in I} \widetilde{A_i}$, so $\bigcap_{i \in I} \widetilde{A_i} \neq \varnothing$. This proves that $\mathscr{U}$ is compact.

Finally, we must show that $\mathscr{U}$ satisfies the extension property: given a compact Hausdorff space X and a (continuous) map $f : \mathbb{N} \to X$, there is a unique continuous extension $\widetilde{f} : \mathscr{U} \to X$:

$$\begin{tikzcd}
\mathscr{U} \arrow[dr, dashed, "\widetilde{f}"] & \\
\mathbb{N} \arrow[u, hook, "\iota"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Let X be a compact Hausdorff space, and let $f : \mathbb{N} \to X$ be any function. Define $\widetilde{f} : \mathscr{U} \to X$ by $\widetilde{f}(\omega) := \lim_{n\to\omega} f(n)$ as in (2). By (4), $\widetilde{f}(\widehat{n}) = f(n)$ for $n \in \mathbb{N}$, so $\widetilde{f}$ is a well-defined extension of f. It remains to show that $\widetilde{f}$ is continuous. Let $U \subseteq X$ be open. If $f^{-1}(U) = \varnothing$, there is nothing to show. Suppose $f^{-1}(U) \neq \varnothing$. Since X is a compact Hausdorff space, it is normal. Therefore, we may find an open set $V \subseteq X$ (a neighborhood of a given point in $U \cap f(\mathbb{N})$) such that $\overline{V} \subseteq U$ and $f^{-1}(V) \neq \varnothing$. Let $\omega \in \widetilde{f^{-1}(V)}$. Then $f^{-1}(X \setminus \overline{V}) \notin \omega$, so $\widetilde{f}(\omega) = \lim_{n\to\omega} f(n) \in \overline{V} \subseteq U$. Thus, $\widetilde{f^{-1}(V)}$ is a nonempty open subset of $\mathscr{U}$ such that $\widetilde{f}(\widetilde{f^{-1}(V)}) \subseteq U$, so $\widetilde{f}$ is continuous. Uniqueness of f follows from the density of $\mathbb{N}$ in the Hausdorff space $\mathscr{U}$. □

Note: To see the connection with the C^*-algebra construction from last semester, we could also realize $\widehat{\ell^\infty}$ as the collection of ultralimits $\lim_{n\to\omega}$ for $\omega \in \mathscr{U}$. Linearity of these limits is shown in (5), and they are multiplicative for the same reason.

(2) Let ω be an ultrafilter on $\mathbb{N}$. Let X be a compact Hausdorff space and $f : \mathbb{N} \to X$. We say $x = \lim_{n\to\omega} f(n)$ if for every open neighborhood U of x, $f^{-1}(U) \in \omega$. Prove that $\lim_{n\to\omega} f(n)$ always exists for any function $f : \mathbb{N} \to X$.

Proof. Let $\mathscr{C} := \{K \subseteq X : K \text{ is closed and } f^{-1}(K) \in \omega\}$. Note that $\mathscr{C}$ is nonempty, since $X \in \mathscr{C}$. Moreover, $\varnothing \notin \mathscr{C}$. Given $K_1, \dots, K_n \in \mathscr{C}$, we have

$$f^{-1}\left(\bigcap_{i=1}^n K_i\right) = \bigcap_{i=1}^n f^{-1}(K_i) \in \omega,$$

since ω is a filter, so $\bigcap_{i=1}^n K_i \in \mathscr{C}$. In particular, $\bigcap_{i=1}^n K_i \neq \varnothing$. By compactness, it follows that $\bigcap_{K\in\mathscr{C}} K \neq \varnothing$.

Let $x \in \bigcap_{K\in\mathscr{C}} K$. Let U be an open neighborhood of x. We want to show $f^{-1}(U) \in \omega$. Note that $X \setminus U$ is closed and $x \notin X \setminus U$. Hence $X \setminus U \notin \mathscr{C}$ by the construction of x. But $f^{-1}(X \setminus U) = \mathbb{N} \setminus f^{-1}(U)$. Since ω is an ultrafilter, it follows that $f^{-1}(U) \in \omega$.

Finally, we claim that x is the unique limit of f along ω. Indeed, since X is Hausdorff, given any other point $y \in X$, we may find disjoint open neighborhoods $U \ni x$ and $V \ni y$ so that $f^{-1}(V) \subseteq f^{-1}(X \setminus U) = \mathbb{N} \setminus f^{-1}(U) \notin \omega$. □

(3) Show that every principal ultrafilter on $\mathbb{N}$ contains a unique singleton set, and any two ultrafilters containing the same singleton set are necessarily equal. Thus we may identify the set of principal ultrafilters on $\mathbb{N}$ with $\mathbb{N}$.

Proof. Let ω be an ultrafilter on $\mathbb{N}$, and suppose $A = \{n_1, \dots, n_k\} \in \omega$ has the fewest elements of any set in ω. Suppose for contradiction that $k \geq 2$. By minimality of A, we must have $\{n_1\} \notin \omega$. But then $\mathbb{N} \setminus \{n_1\} \in \omega$, so $\{n_2, \dots, n_k\} = A \cap (\mathbb{N} \setminus \{n_1\}) \in \omega$, contradicting minimality of A.

Hence $\omega \ni \{n\}$ for some $n \in \mathbb{N}$. This n is necessarily unique, since ω is closed under finite intersections and does not contain the empty set.

We claim that $\omega = \widehat{n} := \{A : n \in A\}$. First, $\widehat{n}$ is a filter. On the other hand, if $A \in \omega$, then $A \cap \{n\} \in \omega$, so $n \in A$. Hence, $\omega \subseteq \widehat{n}$. Since ω is an ultrafilter (and therefore a maximal filter), we must have $\omega = \widehat{n}$. □

(4) Determine $\lim_{n\to\omega} f(n)$ for $f : \mathbb{N} \to X$ as in (2) when ω is principal.

Proof. Let $\omega = \widehat{k}$ be a principal ultrafilter. We claim $\lim_{n\to\omega} f(n) = f(k)$. Indeed, for every open set $U \ni f(k)$, we have $k \in f^{-1}(U)$, so $f^{-1}(U) \in \widehat{k} = \omega$. □

(5) Let ω be a free ultrafilter on $\mathbb{N}$. Suppose $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is a locally finite group and m_n is the uniform probability (Haar) measure on Γ_n. Define $m : 2^\Gamma \to [0,1]$ by $m(A) = \lim_{n\to\omega} m_n(A \cap \Gamma_n)$. Prove that m is a left Γ-invariant finitely additive probability measure on Γ, i.e. Γ is amenable.

Proof. Clearly $m(\Gamma) = 1$. Let $A, B \subseteq \Gamma$ be disjoint. Then since m_n is additive for each $n \in \mathbb{N}$,

$$m(A \cup B) = \lim_{n \to \omega} (m_n(A \cap \Gamma_n) + m_n(B \cap \Gamma_n))$$

Thus, in order to show that m is finitely additive, it suffices to show that $\lim_{n\to\omega}$ is additive.

Suppose $x = \lim_{n\to\omega} x_n$ and $y = \lim_{n\to\omega} y_n$ for some sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $[0, 1]$. Let U be a neighborhood of $x + y$. Since addition is jointly continuous, we may find neighborhoods V of x and W of y such that $V + W \subseteq U$. By assumption, $\{n : x_n \in V\} \in \omega$ and $\{n : y_n \in W\} \in \omega$. Hence,

$$\{n : x_n + y_n \in U\} \supseteq \{n : x_n \in V, y_n \in W\} = \{n : x_n \in V\} \cap \{n : y_n \in W\} \in \omega.$$

Therefore, $\lim_{n\to\omega} x_n + y_n = x + y$ as desired.

It remains to show that m is left Γ-invariant. Let $g \in \Gamma$. Then there is an $N \in \mathbb{N}$ such that $g \in \Gamma_n$ for all $n \geq N$. Since ω is free, $\{n \geq N\} \in \omega$. Let $A \subseteq \Gamma$, and let U be a neighborhood of $m(A)$. Then we have

$$\begin{aligned}
\{n : m_n((gA) \cap \Gamma_n) \in U\} &= \{n : m_n(g(A \cap g^{-1}\Gamma_n)) \in U\} \\
&= \{n : m_n(A \cap g^{-1}\Gamma_n) \in U\} \\
&\supseteq \{n \geq N : m_n(A \cap \Gamma_n) \in U\} \\
&= \{n \geq N\} \cap \{n : m_n(A \cap \Gamma_n) \in U\}.
\end{aligned}$$

Since $m(A) = \lim_{n\to\omega} m_n(A \cap \Gamma_n)$, both of these sets in the final line are elements of ω, so their intersection is also in ω. Hence, $m(gA) = \lim_{n\to\omega} m_n((gA) \cap \Gamma_n) = m(A)$. $\square$