Module embedding via towers of algebras

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Abstract

Jones and Penneys have shown that a finite depth subfactor planar $*$-algebra embeds in the bipartite graph planar algebra of its principal graph. By constructing a strongly Markov inclusion of finite von Neumann algebras from a given module, we extend their techniques to the case of cyclic modules over a subfactor planar algebra, relating the calculus of string diagrams in a module category and the canonical planar $*$-algebra structure on a Markov inclusion. We generalize their result, showing that a finite depth subfactor $*$-planar algebra embeds into the bipartite graph planar algebra of the fusion graph of any of its cyclic modules.

1 Introduction

2 Strongly Markov inclusions of von Neumann algebras

2.1 Strongly Markov inclusions

For the duration of this paper assume that any trace on a finite von Neumann algebra is a faithful, normal, tracial state. For finite von Neumann algebras $A_0 \subseteq A_1$ with traces tr_0 and tr_1 we write $A_0 \subseteq (A_1, \mathrm{tr}_1)$ when $\mathrm{tr}_0|_{A_0} = \mathrm{tr}_0$. Let e_1 be the jones projection onto $L^2(A_0, \mathrm{tr}_0)$ and E_{A_0} the unique trace preserving conditional expectation. Let A_2 be obtained from A_1 and A_0 via the basic construction. We will now recall some basic facts about such inclusions.

DP: do we always use $A_0 \subset (A_1, \mathrm{tr}_1)$ for a strongly Markov inclusion and $(M_n, \mathrm{tr}_n, e_n)$ for a Markov sequence? This could help alleviate some confusion!

Definition 2.1. Given an inclusion of finite von Neumann algebras with trace, $A_0 \subseteq (A_1, \mathrm{tr}_1)$, a *Pimsner-Popa basis for* A_1 *over* A_0 is any finite subset $B \subset A_1$ such that any of the following equivalent properties hold:

- $1 = \sum_{b \in B} b e_1 b*$.
- For any $x \in A_1$ we have that $x = \sum_{b \in B} b E_{A_0}(b * x)$.
- For any $x \in A_1$ we have that $x = \sum_{b \in B} E_{A_0}(bx) b*$.

Due to [?] we have the following:

Proposition 2.2. *The following are equivalent:*

- *There is a Pimsner=Popa basis for A_1 over A_0.*
- $A_2 = A_1 e_1 A_1$

We will be concerned primarily with a particular type of inclusion $A_0 \subseteq A_1$, which is given in the following definition.

Definition 2.3. Let Tr_2 be the canonical, faithful, normal, semifinite trace for A_2. The inclusion is said to be *strongly Markov* if:

- Tr_2 is finite and $\mathrm{Tr}_2(1)^{-1}Tr_2|_{A_1} = \mathrm{tr}_1$.
- There is a Pimsner-Popa basis for M_1 over M_0.

Recall that if $A_0 \subseteq (A_1, \mathrm{tr}_1)$ is a strongly Markov inclusion of finite dimensional von Neumann algebras with $[A_1 : A_0] = d^2$ then we can iterate the basic construction to obtain a tower of algebras and there is a canonical planar algebra strucutre on this tower, called the *canonical planar* $*$*-algebra of a strongly Markov inclusion* corresponding to $A_0 \subseteq (A_1, \mathrm{tr}_1)$. One important feature of this construction is that it depends on the existence of a Pimsner-Popa basis for A_1 over A_0. We refer the reader to [**?**] for more details on this construction. We will need the following:

Proposition 2.4 ([**?**, Prop 2.25])**.** *For any n, the map:*

$$\theta_n : \bigotimes_{A_0}^{n} A_1 \to A_n$$

defined by $x_1 \otimes \cdots x_n \mapsto x_1 v_1 x_2 v_2 \cdots x_n v_n$, where $v_k := d^n e_n e_{n-1} \cdots e_1$, is an $A_1 - A_1$ bimodule isomorphism.

[[towers of relative commutants]]

2.2 Markov sequences

Definition 2.5. A *Markov sequence* consists of a sequence $(M_n, \mathrm{tr}_n)_{n\geq 0}$ of finite dimensional von Neumann algebras, such that M_n is unitally included in M_{n_1}, each M_n has a faithful normal tracial states such that $\mathrm{tr}_{n+1}|_{M_n} = \mathrm{tr}_n$ for all $n \geq 0$, and there is a sequence of *Jones projections* $e_n \in M_{n+1}$ for all $n \geq 1$, such that:

- The projections (e_n) satisfy the Temperley-Lieb-Jones relations:
  (1) $e_i^2 = e_i = e_i^*$ for all i,
  (2) $e_i e_j = e_j e_i$ for $|i - j| > 1$, and
  (3) there is a fixed constant $d > 0$ such that $e_i e_{i\pm 1} e_i = d^{-2} e_i$ for all i.
- For all $x \in M_n$, $e_n x e_n = E_n(x) e_n$, where $E_n : M_n \to M_{n-1}$ is the canonical faithful trace-preserving conditional expectation.
- For all $n \geq 1$, $E_{n+1}(e_n) = d^{-2}$.
- (pull down) For all $n \geq 1$, $M_{n+1} e_n = M_n e_n$.

Remark 2.6. $M_n e_n M_n$ is a 2-sided ideal in M_{n+1} for all $n \geq 1$ if and only if the pull down condition holds. Indeed, if the pull down condition holds, then $M_{n+1} M_n e_n M_n \subseteq M_{n+1} e_n M_n = M_n e_n M_n$; the same argument holds on the right by first taking adjoints. Conversely, if $M_n e_n M_n$ is a 2-sided ideal, then $M_{n+1} e_n = (M_{n+1} e_n) e_n \subseteq (M_n e_n M_n) e_n = M_n e_n$.

Proposition 2.7. *A Markov sequence satisfies the following elementary properties for $n \geq 1$.*

(A) The map $M_n \ni y \mapsto y e_n \in M_{n+1}$ is injective.

*(B) For all $x \in M_{n+1}$, $d^2 E_{n+1}(xe_n)$ is the unique element $y \in M_n$ such that $xe_n = ye_n$ [**?**, Lem. 1.2].*

(C) The traces tr_{n+1} satisfy the following Markov property *with respect to M_n and e_n: for all $x \in M_n$, $\operatorname{tr}_{n+1}(xe_n) = d^{-2}\operatorname{tr}_n(x)$.*

(D) $e_n M_{n+1} e_n = M_{n-1} e_n$.

*(E) $X_{n+1} := M_n e_n M_n$ is a 2-sided ideal of M_{n+1}, and thus M_{n+1} splits as a direct sum of von Neumann algebras $X_{n+1} \oplus Y_{n+1}$. (In [**?**, Thm. 4.1.4 and Thm. 4.6.3], Y_{n+1} is the so-called 'new stuff'.) By convention, we define $Y_0 = M_0$ and $Y_1 = M_1$, so that $X_0 = (0)$ and $X_1 = (0)$.*

(F) The map $ae_nb \mapsto ap_nb$ gives a $*$-isomorphism from $X_{n+1} = M_n e_n M_n$ to $\langle M_n, p_n \rangle = M_n p_n M_n$, the Jones basic construction of $M_{n-1} \subseteq M_n$ acting on $L^2(M_n, \operatorname{tr}_n)$.

(G) Under the isomorphism $X_{n+1} \cong M_n p_n M_n$, the canonical non-normalized trace Tr_{n+1} on the Jones basic construction algebra $M_n p_n M_n$ satisfying $\operatorname{Tr}_{n+1}(ap_nb) = \operatorname{tr}_n(ab)$ for $a, b \in M_n$ equals $d^2 \operatorname{tr}_{n+1}|_{X_{n+1}}$.

*(H) If $y \in Y_{n+1}$ and $x \in X_n$, then $yx = 0$ in M_{n+1}. Hence $E_{n+1}(Y_{n+1}) \subseteq Y_n$. ("The new stuff comes only from the old new stuff" [**?**].)*

(I) If $Y_n = (0)$, then $Y_k = (0)$ for all $k \geq n$.

Proof. (A) We know that $d^2 E_{n+1}(ye_n) = y$, so the proposed map has a left inverse.

(B) This follows directly from (A)

(C) For $x \in M_n$, we have $\operatorname{tr}_{n+1}(xe_n) = \operatorname{tr}_n(E_{n+1}(xe_n)) = \operatorname{tr}_n(xE_{n+1}(e_n)) = d^{-2}\operatorname{tr}_n(x)$, since $E_{n+1}(e_n) = d^{-2}$.

(D) $M_{n+1}e_n = M_n e_n$ and $e_n x e_n = E_n(x)e_n$ for all $x \in M_n$.

(E) $M_n e_n M_n$ is a 2-sided ideal of M_{n+1} by the pull down condition, as discussed in (2.6)

(F) First, we check that the map $\phi : ae_nb \to ap_nb$ is injective. Suppose $\sum a_i p_n b_i = 0$. Then for all $a, b \in M_n$, we have $0 = p_n a \left(\sum a_i p_n b_i\right) b p_n = \sum E_n(aa_i)E_n(b_ib)p_n$, and therefore $\sum E_n(aa_i)E_n(b_ib) = 0$ as $M_n \ni x \mapsto xp_n \in \langle M_n, p_n \rangle$ is injective. Hence

$$0 = \sum E_n(aa_i)E_n(b_ib)e_n = e_n a \left(\sum a_i e_n b_i\right) b e_n$$

for all $a, b \in M_n$, and thus $\sum a_i e_n b_i = 0$, so ϕ is injective.

To show that ϕ is well-defined, we may reverse the above argument, using (A). It is clear that ϕ is surjective.

(G) For $a, b \in M_n$, we have $\operatorname{Tr}_{n+1}(ap_nb) = \operatorname{tr}_n(ab) = \operatorname{tr}_n(ba) = d^2 \operatorname{tr}_{n+1}(bae_n) = d^2 \operatorname{tr}_{n+1}(ae_nb)$.

(H) Since $X_0 = (0)$ and $X_1 = (0)$ by definition, we may assume $n \geq 2$. As in the proof of [**?**, Thm. 4.6.3.vi], we may assume y is a central projection in M_{n+1} such that $ye_n = 0$. Then for all $ae_{n-1}b \in X_n$, $yae_{n-1}b = d^2 yae_{n-1}e_ne_{n-1}b = d^2 ae_{n-1}ye_ne_{n-1}b = 0$. The final claim follows from $z_nE_{n+1}(y) = E_{n+1}(z_ny) = 0$ where z_n is the central support of e_{n-1} in M_n.

(I) This follows immediately from the previous fact.

□

Notice that by (F), the Bratteli diagram for the inclusion $M_n \subset M_{n+1}$ consists of the reflection of the Bratteli diagram for the inclusion $M_{n-1} \subset M_n$, together with possibly some new edges and vertices corresponding to simple summands of Y_{n+1}. By (H), the new vertices at level $n+1$ only connect to the vertices that were new at level n. This leads to the following definition:

Definition 2.8. The *principal graph* of the Markov sequence (M_n, tr_n) with Jones projections (e_n) consists of the *new* vertices at every level n of the Bratteli diagram, together with all the edges connecting them.

A Markov sequence is said to have *finite depth* if the principal graph is finite.

It follows that a Markov sequence has finite depth if and only if there is $n \in \mathbb{N}$ such that $Y_n = (0)$, as in 2.7 (I). Let (M_n) be a Markov sequence with finite depth, and take the minimal integer $n \in \mathbb{N}$ such that $Y_n = (0)$. Now notice that for $k < n$ the Bratteli diagram of $M_k \subseteq M_{k+1}$ is the Bratteli diagram of $M_{k-1} \subseteq M_k$ reflected upwards along with additional edges which are part of the principal graph. Because of this and the fact that the Bratteli diagram for $M_0 \subseteq M_1$ is part of the principal graph we can "unravel" the Bratteli diagram for $M_n \subseteq M_{n+1}$ to obtain the principal graph for the Markov sequence (A_n, tr_n).

Fact 2.9. *If a Markov sequence (M_n) has finite depth and $n \in \mathbb{N}$ is such that $Y_n = (0)$, then for $k \geq n$, there is a canonical graph isomorphism between the principal graph of (M_n) and the Bratteli diagram for $M_k \subset M_{k+1}$.*

Example 2.10. The Temperley-Lieb algebras of modulus $d \geq 2$ with the usual Jones projections form a Markov sequence. Their principal graph is A_∞.

2.3 Compressing inclusions of von Neumann algebras

The following lemma is well known to experts. We provide a proof for convenience and completeness.

Lemma 2.11. *Suppose $N \subset M \subset B(H)$ is an inclusion of von Neumann algebras and $p \in P(N)$.*

(1) $p(N' \cap M) = pN' \cap pMp$.

(2) Suppose the central support of p in N is $z \in Z(N)$. The map $x \mapsto px$ is an isomorphism $z(N' \cap M) \to p(N' \cap M)$.

DP: this lemma may be more at home later, but it is a purely von Neumann algebraic fact

Proof.

Proof of (1): The proof of (1) is similar to the proof of the standard fact that $(pNp)' = N'p$.

Clearly $(N' \cap M)p \subseteq (N'p) \cap pMp$. Suppose u is a unitary in $(N'p) \cap pMp$. Let K be the closure of NpH. Let $q \in B(H)$ be the projection onto K, which is clearly in $N' \cap N = Z(N)$. Define u_0 in $B(K)$ by $u_0(np\xi) := npu\xi$. One now verifies that u_0 is an isometry and thus is well-defined. Look at the operator $u_0 q \in N' \cap B(H)$, and note that $u = u_0 qp \in N'p$. We claim that $u_0 q \in M$, so that $u = u_0 qp \in (N' \cap M)p$. First, for any $m \in M'$, $n \in N$, and $\xi \in H$, we have $mu_0 np\xi = mnup\xi = nupm\xi = u_0 npm\xi = u_0 mnp\xi$. Thus $u_0 \in qMq$. Since $q \in M$, for all $m \in M'$, we have $u_0 qm\xi = u_0 mq\xi = mu_0 q\xi$. Hence $u_0 q$ commutes with M' on H, and $u_0 q \in M$.

Proof of (2): For $x \in N' \cap M$, we have $p(zx) = px$. Hence the map is surjective. We now show the map is injective. Suppose $x \in z(N' \cap M)$ such that $px = 0$. By (1), $z(N' \cap M) = zN' \cap zMz$. Then for all unitaries $u \in U(N)$, $upz = z(up) \in zN$, so $0 = upxu^* = (upz)xu^* = x(upzu^*) = xupu^*$. Taking sup over $u \in U(N)$ yields $0 = xz = x$. $\square$

In the event that $A_0 \subseteq (A_1, \operatorname{tr}_1)$ is an inclusion of tracial von Neumann algebras and $p \in P(A_0)$ is a non-zero projection, we get a canonical faithful trace on the compressed inclusion $pA_0p \subset (pA_1p, \operatorname{tr}_1^p)$ given by $\operatorname{tr}_1^p(pxp) := \operatorname{tr}_1(pxp)/\operatorname{tr}_1(p)$. It is straightforward to verify that the unique trace-preserving conditional expectation $E_1^p : pA_1p \to pA_0p$ is given by $E_1^p(pxp) := E_1(pxp) = pE_1(x)p$. Notice that for all $pxp \in pA_1p$, we have $e_1p(pxp)e_1p = pe_1xe_1p = pE_1(x)e_1p = E_1(pxp)e_1p$, so the conditional expectation is implemented by e_1p.

We would like to show that pA_2p is isomorphic to the basic construction of $pA_0p \subset (pA_1p, \operatorname{tr}_1^p)$, but we will need an extra assumption on p. Toward this goal, we recall the following recognition lemma based on [**?**, Prop. 1.2], [**?**, Lem. 5.8], and [**?**, Lem. 5.3.1].

Lemma 2.12 ([**?**, Lem. 2.15])**.** *Suppose $A_0 \subset (A_1, \operatorname{tr}_1)$ is a strongly Markov inclusion of tracial von Neumann algebras, and $(B, \operatorname{tr}_B, p)$ is a tracial von Neumann algebra containing A_1 together with a projection $p \in P(B)$ such that*

(1) $pxp = E_{A_0}^{A_1}(x)p$ for all $x \in A_1$,

(2) $E_{A_1}^B(p) = [A_1 : A_0]^{-1}$, and

(3) B is algebraically spanned by A_1 and p, i.e., $B = A_1pA_1 := \operatorname{span}\{apb | a, b \in A_1\}$.

Then the map $A_2 \to B$ given by $ae_1b \mapsto apb$ is a (normal) unital $*$-isomorphism of von Neumann algebras.

Lemma 2.13. *Under the additional assumption that $A_0pA_0 = A_0$,*[1] *the inclusion $pA_0p \subset pA_1p \subset (pA_2p, \operatorname{tr}_2^p, pe_1)$ is standard.*

Proof. [[TODO:

When $A_0pA_0 = A_0$,

$$pA_1p(pe_1)pA_1p = pA_1A_0pe_1A_0A_1p = pA_1A_0pA_0e_1A_1p = pA_1A_0e_1A_1p = pA_1e_1A_1p = pA_2p.$$

]] □

Proposition 2.14. *Suppose $A_0 \subset (A_1, \operatorname{tr}_1)$ is a strongly Markov inclusion of tracial von Neumann algebras, and $p \in P(A_0)$ is a projection such that $A_0pA_0 = A_0$. Then $pA_0p \subset (pA_1p, \operatorname{tr}_1^p)$ is strongly Markov.*

Proof. [[TODO]] □

Corollary 2.15. *Assume the hypotheses of Lemma **??**, and let $(A_n, \operatorname{tr}_n, e_n)$ be the Jones tower for $A_0 \subset (A_1, \operatorname{tr}_1)$. Then $(pA_np, \operatorname{tr}_n^p, pe_n)$ is isomorphic to the Jones tower of $pA_0p \subset (pA_1p, \operatorname{tr}_1^p)$.*

Proof. [[todo]] □

Corollary 2.16. *Suppose $(M_n, \operatorname{tr}_n, e_n)$ is a Markov sequence of finite dimensional von Neumann algebras and $p \in P(M_0)$ is a nonzero projection. Then $(pM_np, \operatorname{tr}_n(p\,\cdot)/\operatorname{tr}_n(p), pe_n)$ is a Markov sequence.*

Proof. [[todo: should be OK.]] □

[1] The condition that A_0pA_0 implies that the central support of p in A_0 is 1.

3 The canonical planar algebra of a strongly Markov inclusion

3.1 Planar algebras

For the following definition, we will always assume shaded planar tangles are in *standard form*, meaning:

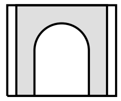
- All input/output disks are parallel rectangles.
- All strings are smooth, strings emanating from the input disks come from the top of the disk, and strings connected to the output disk connect to the top.
- The maxima and minima of any two strings occurs at different heights, and no extrema ever occurs at the same height as that of the top or bottom of a rectangle.
- The $*$ of a tangle is always on the left edge of the boundary rectangle.

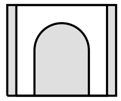
Definition 3.1. The canonical planar $*$-algebra of a strongly Markov inclusion $A_0 \subseteq (A_1, \mathrm{tr}_1)$ is defined as follows:

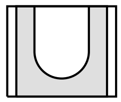
The box spaces are $P_{n,+} := \theta_n^{-1}(A_0' \cap A_n)$ and $P_{n,-} := \theta_n^{-1}(A_1' \cap A_{n+1})$. Suppose a tangle T has type $(r, \pm)$ and s input rectangles of types $(r_i, \pm_i)$. If we arrange T in standard form, the action of T can be understood by reading the tangle from top to bottom, moving a horizontal line upwards. At the bottom, we begin with $1 \in A_i$, where i is 0 if T has type $(r, +)$ and 1 if T has type $(r, -)$. By the top, we will have produced an element of A_i-invariant element of A_{r+i}, depending on a choice of pure tensor $\nu = \otimes_{i=1}^r \nu_i \in \otimes_{i=1}^r P_{r_i, \pm_i}$. Let n_y be the number of shaded regions intersected by a horizontal line at a height y. We first consider y such that the horizontal line intersects no input disks or extrema of strands. At these levels, one can read off an A_i-invariant element $\eta_y \in \otimes^{n_y} A_1$, which remains constant as long as the horizontal line intersects no disks or extrema. One should think of shaded regions as representing elements of A_1 and the unshaded regions as representing $\otimes_{A_0}$. We will now describe what happens when the line moves upwards and passes through extrema of the strings and input rectangles:

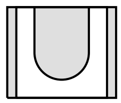
- If the horizontal line passes through the j-th input rectangle and there are t shaded regions to the left of it, and it has type $(r_j, +)$, then at vertical height y, we insert ν_j into η_y in the following way:
$$x_1 \otimes \cdots \otimes x_{n_y} \mapsto x_1 \otimes \cdots \otimes x_t \otimes \nu_j \otimes x_{t+1} \otimes \cdots \otimes x_{n_y}$$
If the rectangle has type $(r_j, +)$, then at vertical height y, we insert ν_j into η_y in the following way:
$$x_1 \otimes \cdots \otimes x_{n_y} \mapsto x_1 \otimes \cdots x_t \nu_j \otimes x_{t+1} \otimes \cdots \otimes x_{n_y} = x_1 \otimes \cdots \nu_j x_t \otimes x_{t+1} \otimes \cdots \otimes x_{n_y}$$
Both of these maps are well-defined by [**?**, Lem. 2.29].

- The effect of passing upwards through the extrema of a strand is as follows:


$: x \otimes y \mapsto xy$


$: x \otimes y \otimes z \mapsto dxE_{M_0}(y) \otimes z = dx \otimes E_{M_0}(y)z$


$: x \mapsto d^{-1} \sum_{b \in B} xb \otimes b^* = d^{-1} \sum_{b \in B} b \otimes b^* x$


$: x \otimes y \mapsto x \otimes 1 \otimes y$

The $*$-structure on tangles given by horizontal reflection of tangles, and the natural $*$-structure on the von Neumann algebras.

We will proceed with the same general technique for defining the embedding of planar algebras as in [**?**], in that we will also initially embed into the canonical $*$-planar algebra, and then make use of the following theorem:

Theorem 3.2 ([**?**])**.** *The canonical planar $*$-algebra associated the strongly Markov inclusion $A_0 \subseteq (A_1, \mathrm{tr}_1)$ is isomorphic to the bipartite graph planar $*$-algebra of the Bratteli diagram for the inclusion $A_0 \subseteq A_1$.*

The following two subsections will describe two isomorphisms between planar algebras constructed in this manner both of which are well known to experts and will motivate constructions detailed in the later sections of this article.

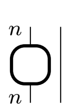
3.2 The shift isomorphism

In rest of this article we will make extensive use of the following lemma adapted from [**?**, Lem. 2.49], which provides sufficient conditions for a collection of maps to be a morphism of shaded planar $\dagger$-algebras.

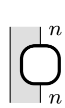
Lemma 3.3 ([**?**, Variation of Lem. 2.49])**.** *Suppose $\Phi_{n,\pm} : \mathcal{P}_{n,\pm} \to \mathcal{Q}_{n,\pm}$ is a collection of unital $\dagger$-algebra maps such that*

(1) Φ maps Jones projections in $\mathcal{P}_{n,+}$ to Jones projections in $\mathcal{Q}_{n,+}$,

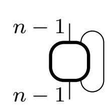
(2) Φ commutes with the action of the following tangles:



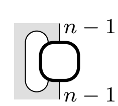
right inclusion
$\mathcal{P}_{n,+} \to \mathcal{P}_{n+1,+}$



left inclusion
$\mathcal{P}_{n,-} \to \mathcal{P}_{n+1,+}$



right capping
$\mathcal{P}_{n,+} \to \mathcal{P}_{n-1,+}$



left capping
$\mathcal{P}_{n,+} \to \mathcal{P}_{n-1,-}$.

Then Φ is a morphism of shaded planar $\dagger$-algebras.

¡¡¡¡¡¡¡ HEAD Suppose that $A_0 \subseteq (A_1, \mathrm{tr}_1)$ is a strongly Markov inclusion of von Neumann algebras and $(A_n, \mathrm{tr}_n)_{n\geq 0}$ is the tower obtained by iterating the basic construction. We know from corollary 2.18 of [?] that for any $0 \leq k \leq n$ the inclusion $A_k \subseteq (A_n, tr_n)$ is strongly markov. Thus we can find a Pimsner-Popa basis B for A_n over A_k.

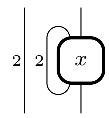
Proposition 3.4 ([?] Prop2.24)**.** *Let $0 \leq k \leq l \leq n$. Let B be a Pimsner-Popa basis for A_l over A_k. The conditonal expectation $E^{A_k'}_{A_l'} : (A_k' \cap B(L^2(A_n, tr_n)), tr_k') \to (A_k' \cap A_l' \cap B(L^2(A_n, tr_n)), tr_l')$ is given by: ======= [[TODO: the canonical planar algebra for $A_0 \subset A_1$ is canonically iso to the canonical planar algebra for $A_2 \subset A_3$ when $A_0 \subset A_1$ is a strongly Markov inclusion.]]*

Suppose that (A_n, tr_n) is a strongly Markov sequence of von Neumann algebras. We know from [?] that the inclusion $A_0 \subseteq A_2 \subseteq \big(A_4, \mathrm{tr}_4, d^{-2}(e_1e_2)(e_3e_2)\big)$ is standard. Thus we can find a Pimsner-Popa basis B for A_2 over A_0.

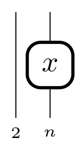
DP: false; only true if (A_n) is the Jones tower of a strongly markov inclusion.

Remark 3.5. The commutant conditonal expectation $E^{A_0'}_{A_2'}$ is given by: ¿¿¿¿¿¿¿ c9bf45fe4bd3129a6334410c295f5249

$$E^{A_l'}_{A_k'}(x) \quad = \quad d^{-2(l-k)} \sum_{b \in B} bxb^* \quad = \quad d^{-2}$$

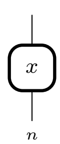


Corollary 3.6. *If x belongs to $A_2' \cap A_{n+2}$ then we have that x can represented in the following form as a string diagram in the box space $A_0' \cap A_{n+2}$ for the canonical planar $*$-algebra for $A_0 \subseteq (A_1, \mathrm{tr}_1)$:*

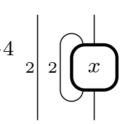


and we have a similar result for elements of $A_3' \cap A_{n+3}$ when represented in string diagrams in $A_1' \cap A_{n+3}$.

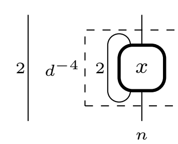
Proof. For $x \in A_2' \cap A_{n_2}$ we have:



$$= \quad E^{A'0}_{A_2'}(x) = \quad d^{-4} \sum_{b \in B} bxb^* = \quad d^{-4}$$



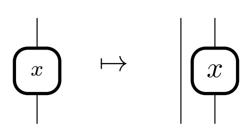
$$=$$



The proof is similar for elements1 of $A_3' \cap A_{n+3}$. $\square$

Let us denote the canonical planar $*$-algebra for the inclusion $A_0 \subseteq A_1$ by $\mathcal{A}_\bullet$ and the canonical planar $*$-algebra for $A_2 \subset A_3$ by $\mathcal{B}_\bullet$.

Theorem 3.7 (Shift by 2 Isomorphism)**.** *The map $\Phi : \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, obtained by adding two strings in front of elements of $\mathcal{A}_{n,+}$ and adding three strings in front of elements of $\mathcal{A}_{n,-}$*



defines a planar $*$-algebra isomorphism between $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$.

Proof. [[Have to provide surjectiveness for negative box spaces also]] $\square$

3.3 The compression isomorphism

Suppose $A_0 \subset A_1$ is a strongly Markov inclusion of tracial von Neumann algebras. Denote by $\mathcal{P}_\bullet$ the canonical planar †-algebra whose box spaces are given by

DP: I'd like to suppress the maps θ, since they are really annoying to keep track of

$$\mathcal{P}_{n,+} := \theta_n^{-1}(A_0' \cap A_n) \qquad \mathcal{P}_{n,-} := \theta_{n+1}^{-1}(A_1' \cap A_{n+1}).$$

Suppose $p \in P(A_0)$ is a projection such that $A_0 p A_0 = A_0$. By Corollary [[]], the Jones tower for $pA_0p \subset (pA_1p, \mathrm{tr}_1^p)$ is given by $(pA_np, \mathrm{tr}_n^p, e_np)$, and thus we get another canonical planar †-algebra $\mathcal{Q}_\bullet$ whose box spaces are given by

$$\mathcal{Q}_{n,+} := \theta_n^{-1}(pA_0' \cap pA_np) \qquad \mathcal{Q}_{n,-} := \theta_{n+1}^{-1}(pA_1' \cap pA_{n+1}p).$$

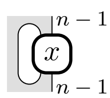
By [[]], the map $\Phi_{n,\pm} : \theta^{-1}(x) \mapsto \theta^{-1}(xp)$ gives an isomorphism of von Neumann algebras $\Phi_{n,\pm} : \mathcal{P}_{n,\pm} \to \mathcal{Q}_{n,\pm}$ for each $n \geq 0$.

Theorem 3.8. *The maps $\Phi_{n,\pm} : \mathcal{P}_{n,\pm} \to \mathcal{Q}_{n,\pm}$ constitute a planar †-algebra isomorphism.*

Proof. We prove the maps $\Phi_{n,\pm}$ satisfy the conditions of Lemma 3.3.

First, note that $\Phi_{n,\pm}(e_n) = pe_n$, so Jones projections in $\mathcal{P}_\bullet$ map to Jones projections in $\mathcal{Q}_\bullet$ by [[earlier section]]. [[TODO: edit below]]

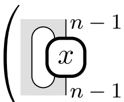
We must check that Condition (2) of Lemma 3.3 holds. The only interesting part is checking that left capping commutes with $\Phi_{n,\pm}$. First, by [?, Equation (4.1.5) and Theorem 4.2.1], the formula for capping on the left is given by



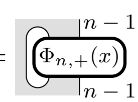
$$= [Y : X]^{-1/2} \sum_\beta \beta x \beta^*$$

where $\{\beta\}$ is a (finite) Pimsner-Popa basis for $X \subset Y$, and this map is independent of the choice of basis. This means that picking Pimsner-Popa bases $\{\beta\}$ for $X \subset Y$ and $\{b\}$ for $pXp \subset pYp$, we must show that

$$\Phi_{n-1,-}\left(\vphantom{\sum}\right.$$



$$\left.\vphantom{\sum}\right) = [Y : X]^{-1/2} p \sum_\beta \beta x \beta^* = [Y : X]^{-1/2} \sum_b bpxb^* =$$



.

Now the trick is to carefully choose a Pimsner-Popa basis for $X \subset Y$. Let $\{b\}$ be a Pimsner-Popa basis for $pXp \subset pYp$, and let $\{a\}$ be a Pimsner-Popa basis for $(1-p)X(1-p) \subset (1-p)Y(1-p)$. We claim $\{\beta\} = \{a\} \cup \{b\}$ is a Pimsner-Popa basis for $X \subset Y$. Indeed, we see

$$\sum_\beta \beta e_X \beta^* = \sum_a a e_X a^* + \sum_b b e_X b^* = \sum_a a(1-p) e_X a^* + \sum_b bp e_X b^* = (1-p) + p = 1_{\langle Y, e_X \rangle}.$$

For this special choice of $\{b\}$ and $\{\beta\}$, we immediately see that for $x \in \mathcal{P}_{n,+}$,

$$p \sum_\beta \beta x \beta^* = p \left(\sum_a a e_X a^* + \sum_b b e_X b^* \right) = \sum_b bp e_X b^*$$

and the result follows. $\square$

4 The module embedding theorem via towers of algebras

4.1 Planar algebras and the paragroup

[[find a new home for me.]]

Planar algebras were originally introduced in [?]. We will use the same definitions and conventions as [?], [?], [?], and [?]. The reader is referred to these references for further information on planar algebras. From any subfactor planar algebra, one may construct a 2-category. We refer the reader to [?] and [?] for more details. We adopt the definitions used in [?], since we wish to work with shaded planar algebras.

Definition 4.1. Let $Q_\bullet$ be a subfactor planar algebra. The *paragroup* of $Q_\bullet$ is a 2-category $\mathcal{G}$:

- The 0-morphisms of $\mathcal{G}$ are the two objects ○ and ◍.
- The 1-morphisms are projections in the box spaces of $Q_\bullet$, as follows:
  - $\mathrm{Hom}_{\mathcal{G}}($○ $\to$ ○$) = \{p \in Q_{i,+} | p$ is projection and i is even$\}$
  - $\mathrm{Hom}_{\mathcal{G}}($○ $\to$ ◍$) = \{p \in Q_{i,+} | p$ is projection and i is odd$\}$
  - $\mathrm{Hom}_{\mathcal{G}}($◍ $\to$ ○$) = \{p \in Q_{i,-} | p$ is projection and i is odd$\}$
  - $\mathrm{Hom}_{\mathcal{G}}($◍ $\to$ ◍$) = \{p \in Q_{i,-} | p$ is projection and i is even$\}$

  Composition of 1-morphisms is denoted by $\otimes$ and represented diagrammatically by horizontal concatenation of diagrams.
- The 2-morphisms of $\mathcal{G}$ are defined as follows: for $p \in P_{i,\pm}$ and $q \in P_{i,\pm}$, we set $\mathrm{Hom}_{\mathcal{G}}(p \to q) = qP_{j\to i}p$. The space $P_{j\to i}$ is just P_{i+j}, except that tangles are depicted with j with strings emanating from the top and i from the bottom. Note that by construction, $i + j$ is always even.

For a general subfactor planar algebra $Q_\bullet$, the 2-category $\mathcal{G}$ can be interpreted as a unitary multitensor category $\mathcal{C}$ with the canonical spherical unitary dual functor following [?, ?]. When $Q_\bullet$ is finite depth, $\mathcal{C}$ is a 2×2 unitary multifusion category.

By a module over $Q_\bullet$, we mean a $*$-module category over the $*$-multitensor category $\mathcal{C}$. Let $\mathcal{M}$ be a cyclic left $\mathcal{C}$-module category with generator m. Since $\mathcal{M}$ is indecomposable, we can take m to be simple. By [?], there is a $*$-algebra A internal to $\mathcal{C}$ such that $\mathcal{M} \cong \mathsf{FreeMod}_{\mathcal{C}}(A)$, the category of free right A-modules internal to $\mathcal{C}$, namely the $\mathcal{C}$-valued internal End *of m. Since m is simple, A is in fact a Q-system, as described in [?, Rmk.2.7]. Each module in this category is of the form $X \otimes A$ for some $X \in \mathcal{C}$. Each X in $\mathcal{C}$ is trivially a bimodule over the tensor unit $1_{0,+} \oplus 1_{0,-}$, so each $X \in \mathrm{Irr}(\mathcal{C})$ is a module over either $1_{0,+}$ or $1_{0,-}$ on each side. We can therefore view $\mathcal{G}$ as the 2-category of bimodules over the algebra objects $1_{0,+}$ and $1_{0,-}$ and view $\mathcal{M}$ as the category of $1_{0,+} - A$ and $1_{0,-} - A$ bimodules. By [?, Thm.4.1], we can again obtain a 3×3 unitary multifusion category $\widetilde{\mathcal{C}}$ from the 2-category of bimodules over the algebras $1_{0,+}$, $1_{0,-}$, and A, into which both $\mathcal{C}$ and $\mathcal{M}$ have naturally monoidal embeddings. Taking the full subcategory of objects which are preserved by tensoring by the appropriate components of the tensor unit allows us to recover a 2 multifusion subcategory $\mathcal{C}$ and a cyclic $\mathcal{C}$-module $\mathcal{M}$.*

4.2 Constructing a Markov sequence

Given a finite depth subfactor planar algebra $Q_\bullet$ and a module $\mathcal{M}$ over $Q_\bullet$, let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be the multifusion categories associated to $\mathcal{G}$ and $\mathcal{M}$, as described in section 4.1. Let $1_{\widetilde{\mathcal{C}}} = 1_0 \oplus 1_1 \oplus 1_2$ be the tensor unit in $\widetilde{\mathcal{C}}$, indexed so that $\mathcal{C}$ is the non-unital 2×2 unitary multifusion subcategory of $\widetilde{\mathcal{C}}$ with $1_\mathcal{C} = 1_0 \oplus 1_1$, and $\mathcal{M} \cong \mathcal{C}_{20} \oplus \mathcal{C}_{21}$. Overall,

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}_{20} & \mathcal{C}_{21} \end{pmatrix} \qquad\qquad \mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \mathcal{C}_{10} & \mathcal{C}_{11} \end{pmatrix} \subset \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} & \mathcal{C}_{02} \\ \mathcal{C}_{10} & \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{20} & \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} = \widetilde{\mathcal{C}}$$

Let $x \in \mathcal{C}_{01}$ be the strand pictured below:

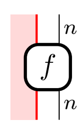
$x =$ *and,* $\overline{x} =$ *so* $x \otimes \overline{x} =$

By construction, x is a generating object for $\mathcal{C}$. If we pick a simple object $m \in \mathcal{C}_{20}$, since $\mathcal{M}$ is indecomposable, we have that any object of $\mathcal{M}$ is isomorphic to a direct summand of $m \otimes x^{alt\otimes n}$ for some nonnegative integer n, where

$$x^{alt\otimes n} := \underbrace{x \otimes \overline{x} \otimes x \otimes \cdots \otimes x^?}_{n \text{ tensorands}},$$

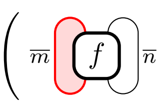
and $x^? = \overline{x}$ if n is even and x if n is odd. We define $\overline{x}^{alt\otimes n}$ similarly.

We will construct a finite depth Markov sequence of algebras from the action of $\mathcal{C}$ on $\mathcal{M}$ as follows. Set $A_n = \mathrm{End}_{\widetilde{\mathcal{C}}}(m \otimes x^{alt\otimes n})$. We can represent morphisms in A_n as



where the red strand represents m and the n represents $x^{alt\otimes n}$. We have a faithful tracial state $\mathrm{tr}_n : A_n \to \mathbb{C}$ given by

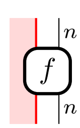
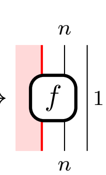
$$\mathrm{tr}_n(f) := \frac{1}{\dim_{\widetilde{\mathcal{C}}}(m)\dim_{\widetilde{\mathcal{C}}}(x)^n} \cdot \mathrm{tr}_{\widetilde{\mathcal{C}}}(f) = \frac{1}{\dim_{\widetilde{\mathcal{C}}}(m)} \cdot \frac{1}{d^n} \cdot \Bigg(\quad \Bigg) \tag{1}$$



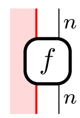
Where n and $\overline{n}$ represent $x^{alt\otimes n}$ and $\overline{x}^{alt\otimes n}$, respectively. This makes each A_n a finite dimensional von Neumann algebra.

Remark 4.2. The trace can be identified with a complex scalar because $\overline{m} \otimes m \in \mathcal{C}_{00}$, the red cup $1 \to \overline{m} \otimes m$ and cap $\overline{m} \otimes m \to 1$ factor through the simple summand 1_0. Since $1_0 \otimes 1 \cong 1_0$, the trace is a member of the 1-dimensional algebra $\mathrm{End}_{\widetilde{\mathcal{C}}}(1_0) \subseteq \mathrm{End}_{\widetilde{\mathcal{C}}}(1)$.

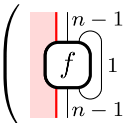
We also have a natural tracial inclusion $A_n \to A_{n+1}$

 $\mapsto$  (2)

and a trace preserving conditional expectation $E_n : A_n \to A_{n-1}$,



$$\mapsto \frac{1}{d} \cdot \Bigg(\qquad \Bigg) \tag{3}$$

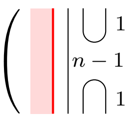


left inverse to the inclusion.

Here, $d := \dim_{\widetilde{\mathcal{C}}}(x) = \dim_{\widetilde{\mathcal{C}}}(\overline{x})$ *the value of a closed loop appearing in the diagram. Given the previously defined inclusion and that multiplication is given by vertical stacking of diagrams, it's clear that* E_n *is* A_{n-1} *bilinear. Similarly, we have that* $tr_n = tr_{n-1} \circ E_n$.

Finally, the Jones projections for each inclusion $A_n \subset A_{n+1}$ *are given by*

$$e_n := \frac{1}{d} \cdot \Bigg(\qquad \Bigg) \in A_{n+1}. \tag{4}$$



where the $n-1$ *indicates* $n-1$ *vertical strands with appropriate shading to the right of the red strand. The Temperley-Lieb- Jones relations follow immediately from the definition and the fact that closed loops count for a factor of* d. *Similarly, for any* $x \in A_n$ *we have* $e_n x e_n = E_n(x) e_n$. *Clearly,* $E_{n+1}(e_n) = d^{-2}\,\mathrm{id}_{m\otimes x^{alt\otimes n}}$. *Finally, the pull down condition holds, as for each* $x \in A_{n+1}$, *we have that* $d^2 E_{n+1}(xe_n)e_n = xe_n$. *Equivalently,* $A_n e_n A_n$ *is a two-sided ideal in* A_{n+1}. *Thus, the tower algebras given by the* A_n *is indeed a Markov sequence.*

Proposition 4.3. *The Markov sequence of von Neumann Algebras* $(A_n, tr_n)_{n\geq 0}$ *is finite depth.*

Proof. When n is even, we have:

$$m \otimes x^{\mathrm{alt}\otimes n} \cong \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C}_{20})} (n_Y Y)$$

For some nonnegative integers n_Y. Therefore,

$$\mathrm{End}_{\widetilde{\mathcal{C}}}\left(m \otimes x^{\mathrm{alt}\otimes n}\right) \cong \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C}_{20})} \mathrm{End}_{\widetilde{\mathcal{C}}}(n_Y Y)$$

The case when n is odd is similar. This means that the Bratteli diagram for $A_n \subseteq A_{n+1}$ has at most $|\,\mathrm{Irr}(\mathcal{C}_{20})| + |\,\mathrm{Irr}(\mathcal{C}_{21})|$ vertices, so the principal graph must be finite. □

Theorem 4.4. *The principal graph obtained from the Markov sequence* $(A_n, tr_n)_{n\geq 0}$ *is independent of the choice of* m, *and is in fact isomorphic to the fusion graph for the action of* x *on* $\mathcal{M}$.

Proof. Because x generates $\mathcal{C}$ and $\mathcal{M}$ is cyclic, we can assume that every simple object in $\mathcal{M}$ is a direct summand of $m \otimes x^{\mathrm{alt}\otimes n}$. Essentially, our result from the fact that subsequent in inclusions $A_k \subseteq A_{k+1}$ are given by tensoring id_x and $\mathrm{id}_{\overline{x}}$, so they are determined entirely by the fusion rules.

First, recall that for a finite depth Markov sequence, the principal graph is canonically isomorphic to the Bratteli for the inclusion $A_n \subseteq A_{n+1}$ for n large enough by 2.9. So, it suffices to show this Bratteli diagram is isomorphic to the fusion graph for $\mathcal{M}$. Furthermore if n is such that $A_n \subseteq A_{n+1} \subseteq A_{n+2}$ is standard then for $k \geq n$ the Bratteli diagram for $A_k \subseteq A_{k+1}$ is isomorphic to the Bratteli diagram for $A_n \subseteq A_{n+1}$, so without loss of generality, we may assume that n is even.

Let the non-negative integers $p_{Z,Y}$ be defined by the following equation:

$$Y \otimes x \cong \bigoplus_{\mathrm{Irr}(\mathcal{C}_{21})} (p_{Z,Y} Z)$$

These coefficients are exactly the coefficients for the adjacency matrix of the fusion graph for x acting on $\mathcal{M}$.

Recall that

$$A_n = \mathrm{End}_{\widetilde{\mathcal{C}}}\left(m \otimes x^{\mathrm{alt}\otimes n}\right) \cong \bigoplus_{\mathrm{Irr}(\mathcal{C}_{20})} \mathrm{End}_{\widetilde{\mathcal{C}}}(n_Y Y)$$

where

$$m \otimes x^{\mathrm{alt}\otimes n} \cong \bigoplus_{\mathrm{Irr}(\mathcal{C}_{20})} (n_Y Y)$$

Note that we may choose n such that each n_Y is strictly positive, as there is some n such that $\bigoplus_{\mathrm{Irr}(\mathcal{C}_{20})} Y$ is isomorphic to a direct summand of $m \otimes x^{\mathrm{alt}\otimes n}$. (In fact, the first n where this happens is exactly when the Markov sequence achieves depth.)

The inclusion of $A_n \hookrightarrow A_{n+1}$ is given by $\phi \mapsto \phi \otimes \mathrm{id}_x$. Each $\phi \in A_n$ is uniquely defined as a direct sum

$$\bigoplus_{\mathrm{Irr}(\mathcal{C}_{20})} \phi_Y$$

where each $\phi_Y \in \mathrm{End}_{\widetilde{\mathcal{C}}}(n_Y Y)$. Then, distributing tensor products over the sum, we may write ϕ as

$$\bigoplus_{\mathrm{Irr}(\mathcal{C}_{20})} (\phi_Y \otimes \mathrm{id}_x)$$

But then:

$$\phi_Y \otimes \mathrm{id}_x \in \mathrm{End}_{\widetilde{\mathcal{C}}}(n_Y Y \otimes x) \cong \bigoplus_{\mathrm{Irr}(\mathcal{C}_{21})} \mathrm{End}_{\widetilde{\mathcal{C}}}(p_{Z,Y} n_Y Z)$$

Since all $n_j \neq 0$ and the inclusion is unital, it is clear that $p_{Z,Y}$ gives the coefficients for the Bratteli diagram. □

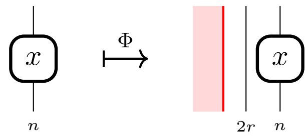
4.3 The Embedding Theorem

Claim. *A finite depth subfactor planar algebra can be embedded into the bipartite graph planar algebra of the fusion graph of the right action of the strand on a cyclic module.*

As described in [?], one can define a canonical shaded planar $*$-algebra structure on the tower of relative commutants of the base of a strongly Markov tower of algebras. This planar algebra is isomorphic to the bipartite graph planar algebra of the Bratteli diagram of the first inclusion in the tower, by Theorem 3.28 of [?]. We have shown that the Markov sequence (A_n, tr_n) defined in the previous section is finite depth. Let r be the minimal integer such that the inclusion $A_{2r} \subset A_{2r+1} \subset A_{2r+2}$ is standard, and let $B_n = A_{2r+n}$. Then the canonical planar $*$-algebra $P_\bullet$ associated to $B_0 \subseteq (B_1, tr_1)$ is isomorphic to the bipartite graph planar algebra of the principle graph of the tower (A_n); indeed, by fact 2.9, the principle graph of (A_n, tr_n) is isomorphic to the Bratteli diagram of $B_0 \subseteq B_1$. If we denote the canonical $*$-planar algebra associated to $B_0 \subseteq (B_1, tr_1)$ by $P_\bullet$, then by construction, we have the following:

$$P_{n,+} = B_0' \cap B_n$$
$$P_{n,-} = B_1' \cap B_{n+1}$$

Consider the map Φ, which places $2r$ strings to the left of elements in $Q_{n,+}$ and $2r+1$ strings to the left of elements in $Q_{n,-}$:



This map sends $Q_{n,+}$ to $P_{n,+}$ and $Q_{n,-}$ to $P_{n,-}$. In order to check that Φ is a planar-$*$ algebra map, we again use Lemma 3.3.

We denote functions in $Q_\bullet$ by $\overline{f}$ and functions in $P_\bullet$ by tr. *We denote the Jones projections in $Q_\bullet$ by e_j and the Jones projections in $P_\bullet$ by f_j.*

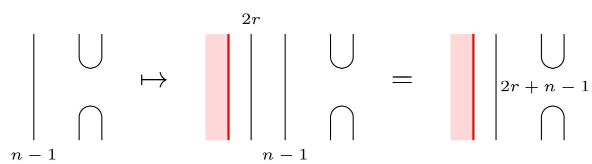
Theorem 4.5. *The image of the map $\Phi : Q_\bullet \to P_\bullet$ is a $*$-planar subalgebra of $P_\bullet$.*

Proof. Observe that for all $x, y \in Q_\bullet$, we have that

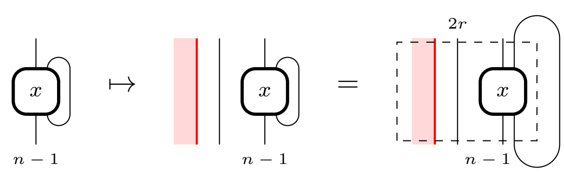
$$\Phi(x^*) = \Phi(x)^* \tag{5}$$
$$\Phi(xy) = \Phi(x)\Phi(y) \tag{6}$$
$$E_n = id_m \otimes \overline{E}_{2r+n} \tag{7}$$
$$i_n = id_m \otimes \overline{i}_{2r+n} \tag{8}$$
$$f_n = id_m \otimes e_{2r+n} \tag{9}$$

We now check that Φ is a planar algebra map:

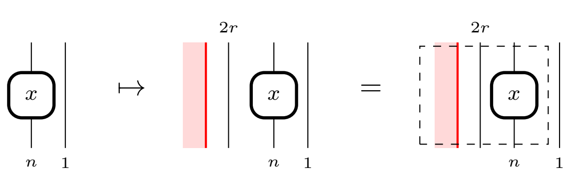
(1) (Jones Projections)



(2) (Conditional Expectation)



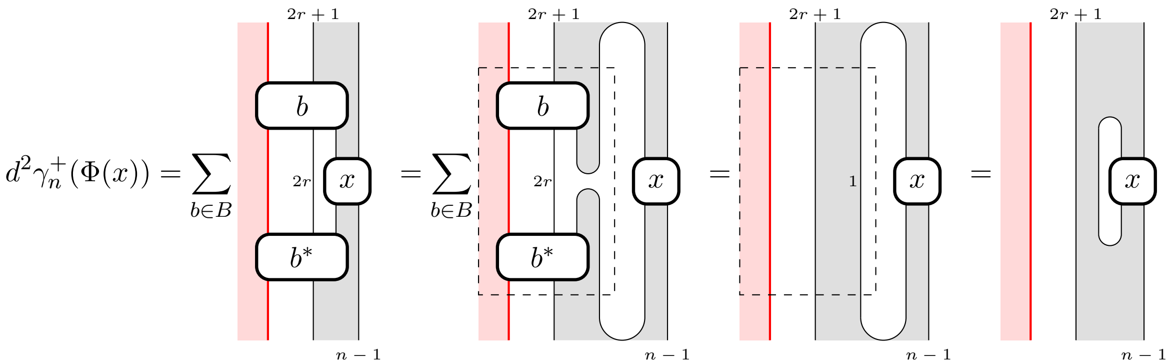
(3) (Right Inclusion)



(4) (Left Capping) In section **??**, we showed that the left-capping tangle γ_n in the canonical planar $*$-algebra is given by

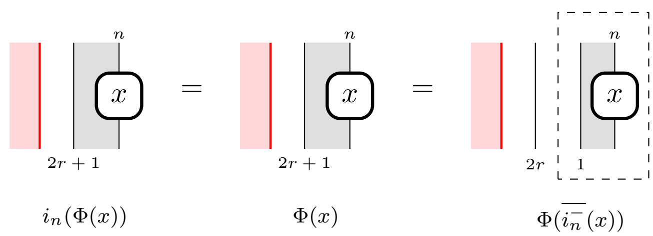
$$\gamma_n^+(x) = \frac{1}{d^2} \sum_{b \in B} bxb^*$$

where B is a Pimnser Popa basis of B_1 over B_0. We will now check that γ_n^+ preserves $\Phi(Q_\bullet)$. For any $x \in Q_{n,+}$, we have



Clearly, the last diagram depicts an element of $\Phi(Q_\bullet)$. In fact, we have $\gamma_n^+(\Phi(x)) = \Phi(\gamma_n^+(x))$.

(5) (Left Inclusion) The negative inclusion $i_n^- : P_{n,-} \to P_{n+1,+}$ is just the identity on the relative commutant planar algebra. Let $\overline{i_n^-}$ be the negative inclusion in $Q_{n,\pm}$. Graphically, this is equivalent to adding a string on the left. Thus, for x in $Q_{n,-}$ we have that:



$\square$

*We have checked that $\Phi(Q_\bullet) \subset P_\bullet$ is a planar $*$-algebra inclusion. Let $G_\bullet$ be the bipartite graph planar algebra of the fusion graph of x acting on $\mathcal{M}$. Then by Theorem 3.33 in [**?**], we know that $P_\bullet$ is a planar $*$-algebra isomorphic to $G_\bullet$. Thus, we have an embedding $Q_\bullet$ into $G_\bullet$.*

Corollary 4.6 (The Embedding Theorem)**.** *A subfactor planar algebra $Q_\bullet$ can be embedded into the bipartite graph planar algebra of the fusion graph of a cyclic $Q_\bullet$-module.*

*In particular, by considering $(Q_\bullet, 1)$ as a cyclic right module over itself, we recover the main result of [**?**] as a corollary, and by instead considering $Q_\bullet$ as a left module, we obtain an embedding into the graph planar algebra of the dual principle graph:*

Corollary 4.7. *Let $Q_\bullet$ be a finite depth subfactor planar algebra then there is an embedding of $Q_\bullet$ into the bipartite graph planar algebra of the principal graph of $Q_\bullet$, and the bipartite graph planar algebra of the dual principal graph of $Q_\bullet$.*

4.4 Invariance of the embedding

As the observant reader may have noted, for any $m \in \mathcal{M}$ one may obtain a different embedding of the subfactor planar algebra $Q_\bullet$ into $G_\bullet$ the bipartite graph planar algebra for the fusion graph given by tensoring with the strand, x. We will show that differ only by automorphisms of $G_\bullet$. Notice that show this it suffices to show that if $\mathbb{R}_\bullet$ and $S_\bullet$ are canonical planar algebras constructed through the *process in section 4, and ϕ and ϕ' are the respective embeddings of $Q_\bullet$ into each then there is some isomorphism $\Psi : R_\bullet \to R\bullet$ such that the following triangle commutes:*

$$\begin{tikzcd} Q_\bullet \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & R_\bullet \arrow[d, "\Psi"] \\ & S_\bullet \end{tikzcd}$$

It suffices to show this because by 3.2 we have that $R_\bullet$ and $S_\bullet$ are both isomorphic to $G_\bullet$ so ϕ lifts to an automorphism of $G_\bullet$. First we will show this for the special where we take two different values for r such that the inclusion $A_{2r} \subseteq A_{2r+1} \subseteq A_{2r+2}$ is standard. The following lemma is a direct consequence of the shift isomorphism discussed in section **??**.

Proposition 4.8. *Let $R_\bullet$ and $S_\bullet$ be the canonical planar algebras given by the inclusions $A_{2r} \subseteq A_{2r+1}$ and $A_{2s} \subseteq A_{2s+1}$ where $A_n := \mathrm{End}_{\widetilde{\mathcal{C}}}(m \otimes x^{alt \otimes n})$ for some $m \in \mathcal{C}_{20}$. Let ϕ and ϕ' be the embeddings of $Q_\bullet$ into $R_\bullet$ and $S_\bullet$ respectively. Then thereis some isomorphism $\Psi : R_\bullet \to R_\bullet$ such that the following triangle commutes:*

$$\begin{tikzcd} Q_\bullet \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & R_\bullet \arrow[d, "\Psi"] \\ & S_\bullet \end{tikzcd}$$