REPORT ON :
THE MODULE EMBEDDING THEOREM VIA TOWERS OF ALGEBRAS

General comment. The goal of the paper under review is to explore the relations between the concrete approach to subfactors given by the planar algebra formalism of Jones and the abstract categorical approach used in [2]. One of the main motivations is a new proof of the embedding of a planar algebra in the the graph planar algebra of its principal graph, when the latter graph is finite. This property, which had been already proven in a paper of Jones and Penneys [3] using loop algebras, is thus tackled here from a categorical point of view. It is now a particular case of a broader result, namely the embedding of a planar algebra into any module category on this planar algebra having good dual properties (which was already present in [2]). The main tool of the present paper is the use of Markov towers, and the important step of this version of the embedding theorem is the construction of a $\mathcal{TLJ}_d$-module category of projections from a Markov tower of modulus d, where $\mathcal{TLJ}_d$ is the Temperley-Lieb-Jones planar algebra of modulus d.

From my point of view, a very interesting new result of this paper is precisely Corollary B, and this fact should be better emphasized. Indeed, this corollary gives an equivalence between pointed Temperley-Lieb-Jones pivotal C^*-module categories and Markov towers. In [1], an equivalence had been given between Temperley-Lieb C^*-module categories (or $SU_q(2)$ homogeneous spaces) and weighted directed graphs with some conditions on the weights. In the present paper, the authors prove in a nice way that among the latter graphs, the ones corresponding to pointed pivotal C^*-module categories are the pointed bipartite graphs with a quantum dimension. From this perspective, I think it would have been better to also explain the meaning of this fact in terms of $SU_q(2)$ homogeneous spaces : namely, which homogeneous spaces correspond to pivotal C^*-module categories ?

Overall, according to me, the elegant approach of the paper, the careful writing of the manuscript and the unified proof of different results make this paper suitable for a publication in a very good journal (despite some of the results being unsurprising generalizations of known results). I particularly invite the authors to emphasize Corollary B, and if possible to develop the relation with the results of [1].

References

[1] De Commer, Kenny, and Makoto Yamashita. "Tannaka–Kreĭn duality for compact quantum homogeneous spaces II. Classification of quantum homogeneous spaces for quantum SU (2)." Journal für die reine und angewandte Mathematik 2015.708 (2015): 143-171.

[2] Grossman, Pinhas, et al. "The Extended Haagerup fusion categories." arXiv preprint arXiv:1810.06076 (2018).

[3] Jones, Vaughan FR, and David Penneys. "The embedding theorem for finite depth subfactor planar algebras." arXiv preprint arXiv:1007.3173 (2010).

List of minor comments.

(1) Introduction : you are using a lot of terms from category theory. If it is not possible to avoid them, at least provide a good reference to get the relevant definitions.
(2) p.4, l.40 : The tower of algebra $\mathcal{TLJ}_d$ is never properly defined in the manuscript. Please introduce it, since it plays a central role in the paper.
(3) p.4, l.41 : add that the graph is locally finite.
(4) p.5, l.55 : mention that the strings don't intersect the input disk.
(5) p.6, l.5 : tangle $T \to$ tangle S.
(6) p.10, l.4 : what is p_v ? The projection on loops based at v ?
(7) p.10, example 2.10 : There is nothing wrong with the definition, but I wonder why you need to add the inner disk 1, since the definition of $\mathcal{M}_v$ implies that the left white region

is already colored v (in the original spin picture of the paper of Jones, I guess this is the one you are using here). What is the map Φ ?

(8) p.11, l.5, l.13 : $m_j \to f_j$
(9) p.11, l.12 : stands$\to$ strands and f_i instead of f_j.
(10) p.11, l.18 : $x \to x_i$.
(11) p.12, l.10 : you have to put r instead of q in the right diagram
(12) p.12, l.12 : what is $\mathcal{C}(r,s)$?
(13) p.12, l.40 : can "be" obtained
(14) p.13, l.44 : is there any difference between p_n and E_n ?
(15) p.14, l.25 : why do you claim that y commute with ae_1 ? (which is of course true assuming the claim you are proving). Another way to prove this claim is simply to use that $ae_{n-1}e_ne_{n-1}b \subset X_{n+1} = M_ne_nM_n$, which directly implies $y(ae_{n-1}e_ne_{n-1}b) = 0$ by the fact $M_{n+1} = Y_{n+1} \bigoplus M_ne_nM_n$ as direct sum of von Neumann algebras.
(16) p.14, l.53 : you did not mention how to unravel the graph. Maybe you could add a reference to explain it (or simply state the reflection formula, which is not so long to write down).
(17) p.15, l.36 : up to my knowledge, you did not define the fusion graph of a module C^*-category
(18) p.15, Lemma 3.12 : is there any other relation between the principal graph of $\mathcal{M}$ and the one of $\mathcal{P}$?
(19) p.15, l.50 : introduce the Pimsner-Popa basis before or at least refer to Definition 3.13
(20) p.16, l.7 : avoid introducing the operator J since there is no use of it later. There are already a lot of definitions.
(21) p.16, l.15 : $Tr(\sum bb^*)$
(22) p.16, l.38 : add that the graph is connected
(23) p.17, Remark 3.9 : This remark is true but a bit misleading. In general there is a general grading on the principal graph (using your notations, the degree of a vertex v being the component Y_k in which it appeared), and this grading changes after truncation. In particular if the original principal graph was pointed, this is not the case anymore after truncation. It may be a good idea to mention it.
(24) p.17, l.32 : $\mathbf{1}_{M_n} \to \mathbf{1}_{pM_np}$
(25) p.18, l.31 : how do you apply $M4$? I couldn't find a way to apply it due to the order of the multiplication $e_{nk} \dots e_{(n+1)k-1}$. Or maybe you only mean $\subset$ instead of $=$ at the beginning of $l.31$?

On line 36, once again I don't get how you get an equality. Before applying $f_{(n-1)k+1}^{(n-1)k+1+(k-1)}$, you have to apply the above diagram on $M_{(n+1)k-1}$, which is not necessarily surjective (even if it is a posteriori).

(26) p.19, l.45 : Mention that you are embedding x in M_{n+2i+j}, since otherwise the number of strings don't match.
(27) p.20, l.43 : I could not find any mention of "Linking algebra" in [GLR85]. Please precise what you mean by this term, and provide a precise reference inside [GLR85].
(28) p.21, l.49 : $\|f\|^2$
(29) p.22, l.44 : in this equality you are using the diagrammatic formulation of the conditional expectation. It may be a good idea to mention the diagrammatic picture in the definition of a planar algebra or a Markov tower.
(30) p.24, l.51 : $j \geq k$
(31) p.25 , L.20 : there is a k missing on top of a string.
(32) p.25, l.54 : could you mention the indices of both $\mathbf{1}$, in order to help the reader ?
(33) p. 26 l. 17-25-32-39 : I think there may be something wrong with the exponents of d. For example, in the second equality, adding $2j + i$ loops should turn d^{i+j} into d^{-j}, or am I wrong ?
(34) p.28, l.17 : "One now shows these two processes are mutually inverse up to dagger equivalence." It looks like there is something missing after this sentence.

(35) p.30, l.50 : I don't think that the map is surjective. Maybe you mean an isomorphism on the shifted planar algebra ? (I think this is what you meant, given the following theorem).
(36) p.31, l.55 : how do you show that $d^{-1}\sum_{b\in B} b\Phi(x)b^*$ corresponds to the right picture ?
(37) p.32, proof of (1) : I think some part of the proof could be simplified. First, since $u \in N'p$ and $p \leq q$ as projections in N and $q \in N \cap N'$, you can directly write $u = qvqp$ with $v \in N'$ (you don't need to assume u unitary and use isometries), and your u_0 corresponds to qvq. Then, you can use your reasoning "First, $\ldots = u_0 mnp\xi$" to conclude that $qvq \in qMq$. Since $q \in N \subset M$, this directly implies that $u \in Mp$.
(38) p.32, l.43 : delete "we have"
(39) p.33 l.15 : $A_0 = pA_0p$ is actually equivalent to the central support of p being 1_{A_0}
(40) p.37, l.55 (1): why is it clear that we can find k large enough such that the projection has central support $\mathbf{1}$?