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LSA STATISTICS
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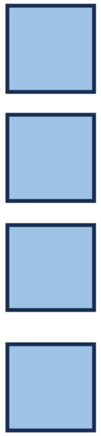
Statistical Methods for Spatio-Temporal Tensor Data

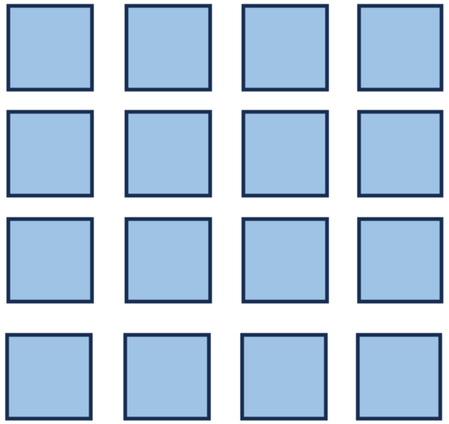
Hu Sun
Department of Statistics, University of Michigan

Doctoral Dissertation Defense
July 15, 2024
Advisor & Committee Chair: Prof. Yang Chen
Committee Members: Prof. Jian Kang, Prof. Kean Ming Tan, Prof. Ward B. Manchester

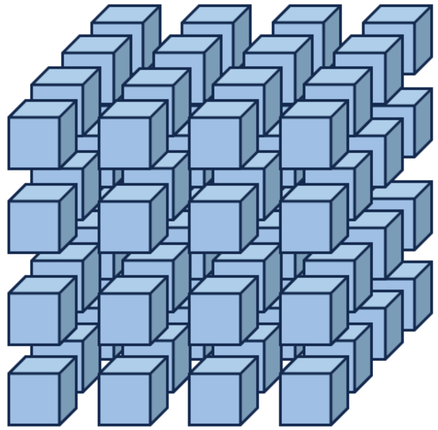
Introduction to Tensor



order-1 tensor $\mathbf{x}$
(vector)



order-2 tensor $\mathbf{X}$
(matrix)



order-3 tensor $\mathcal{X}$
(tensor)

Examples of Tensor Data

Examples of Tensor Data

rating tensor $\mathcal{X}$ in recommender system

order-3 tensor: $\mathcal{X}$ (user, item, context)

Examples of Tensor Data

rating tensor $\mathcal{X}$ in recommender system

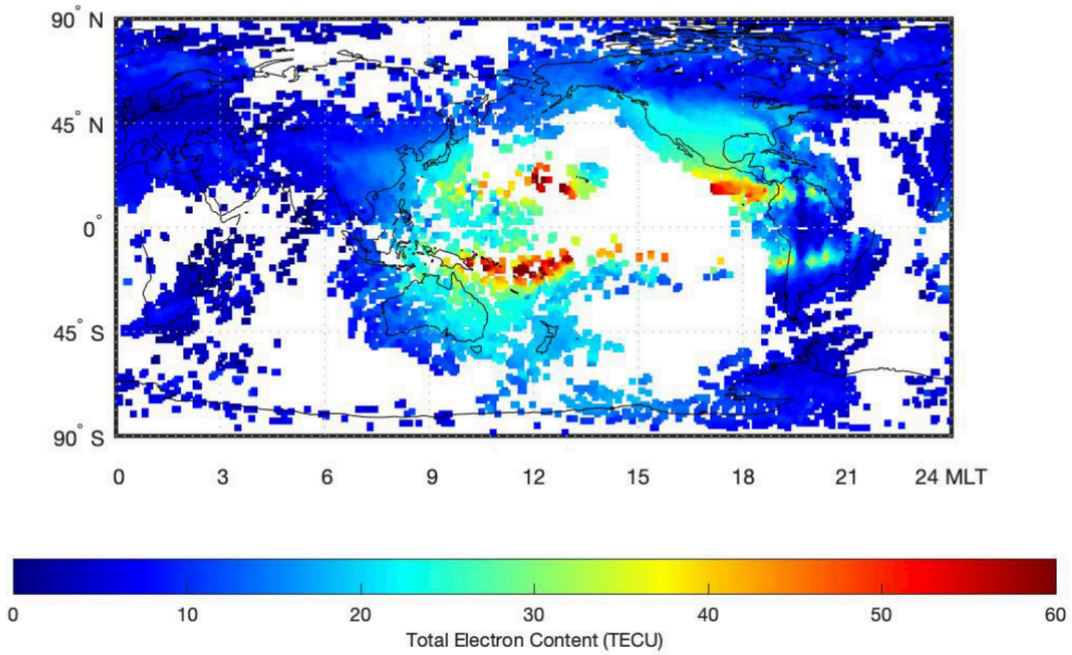
order-3 tensor: $\mathcal{X}$ (user, item, context)

functional MRI 3-D brain imaging tensor $\mathcal{Y}$

order-3 tensor: $\mathcal{Y}\ (x, y, z)$

Spatio-Temporal Tensor

Motivating Dataset: Global Total Electron Content (TEC)



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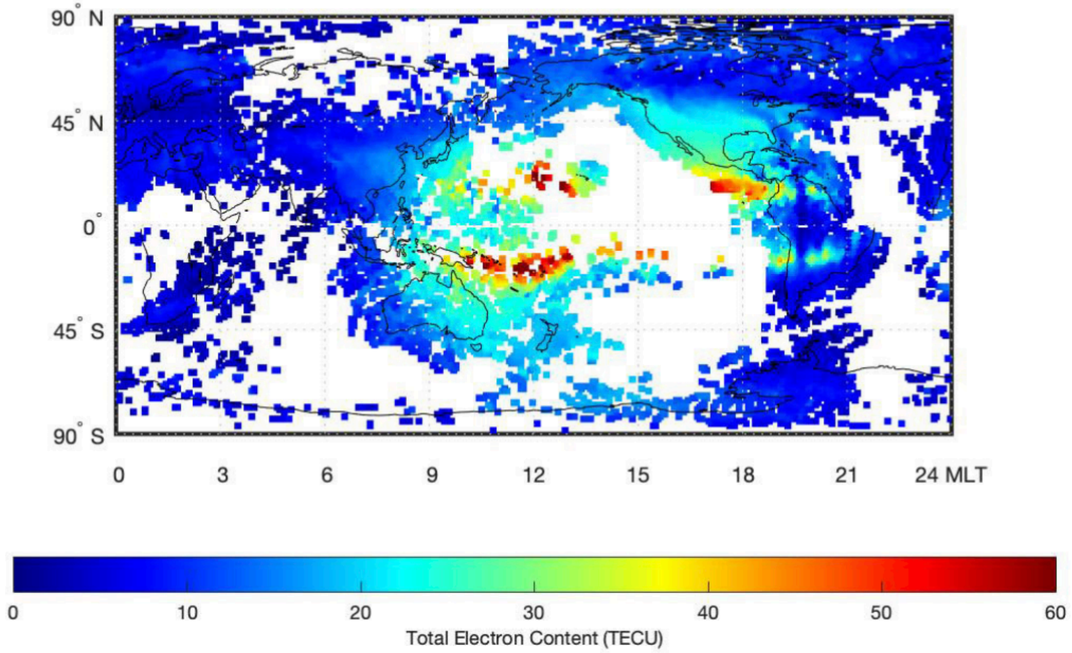

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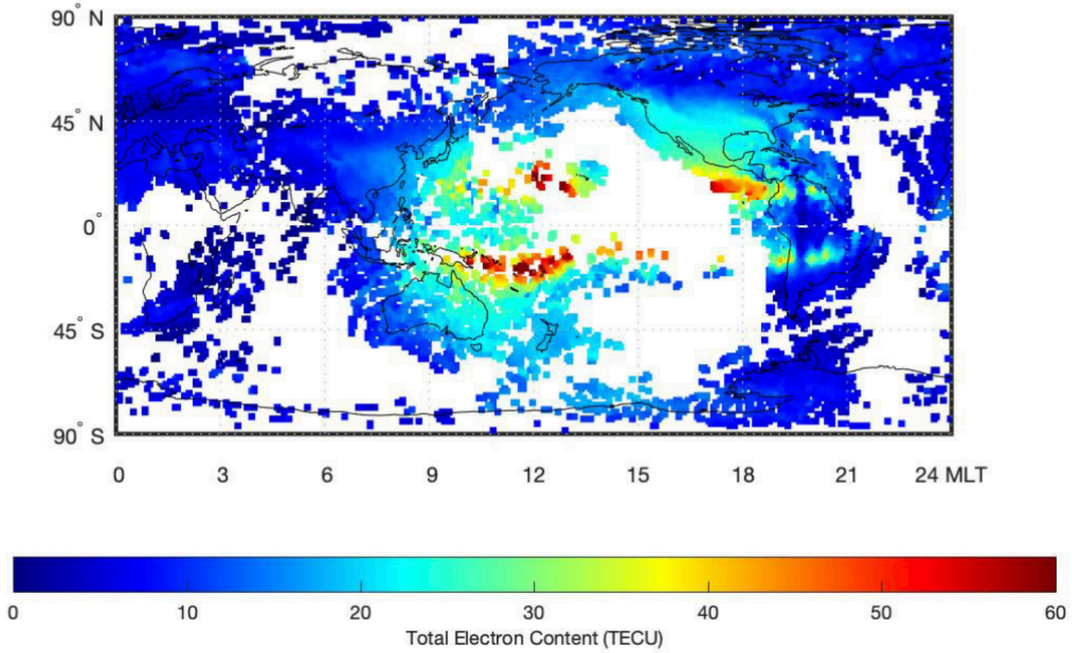

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Challenges for such spatio-temporal tensors:

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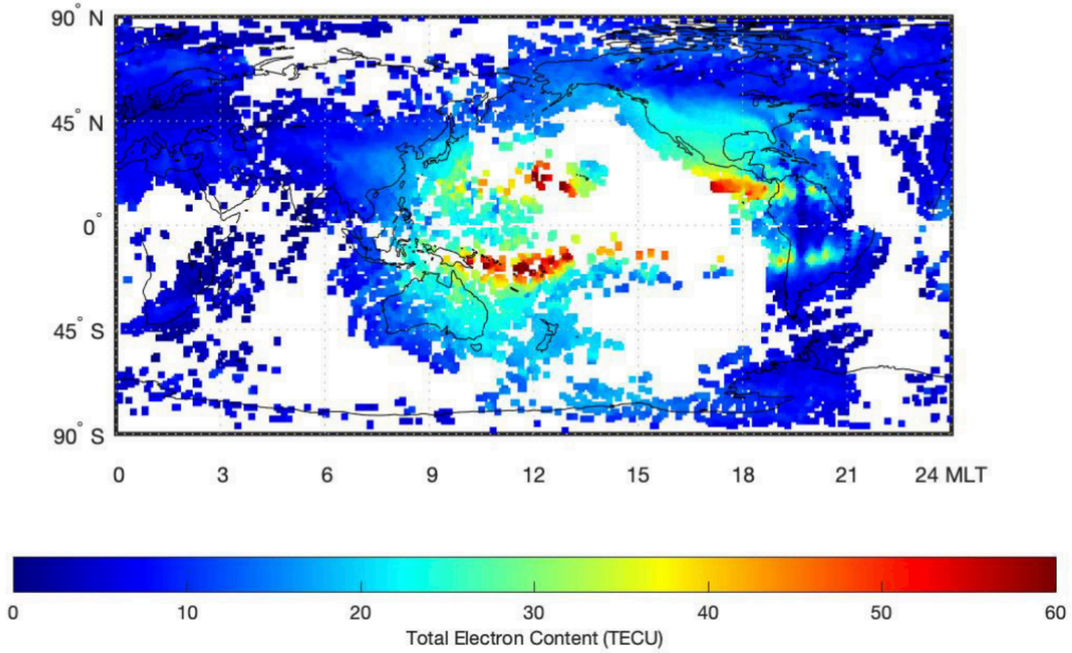

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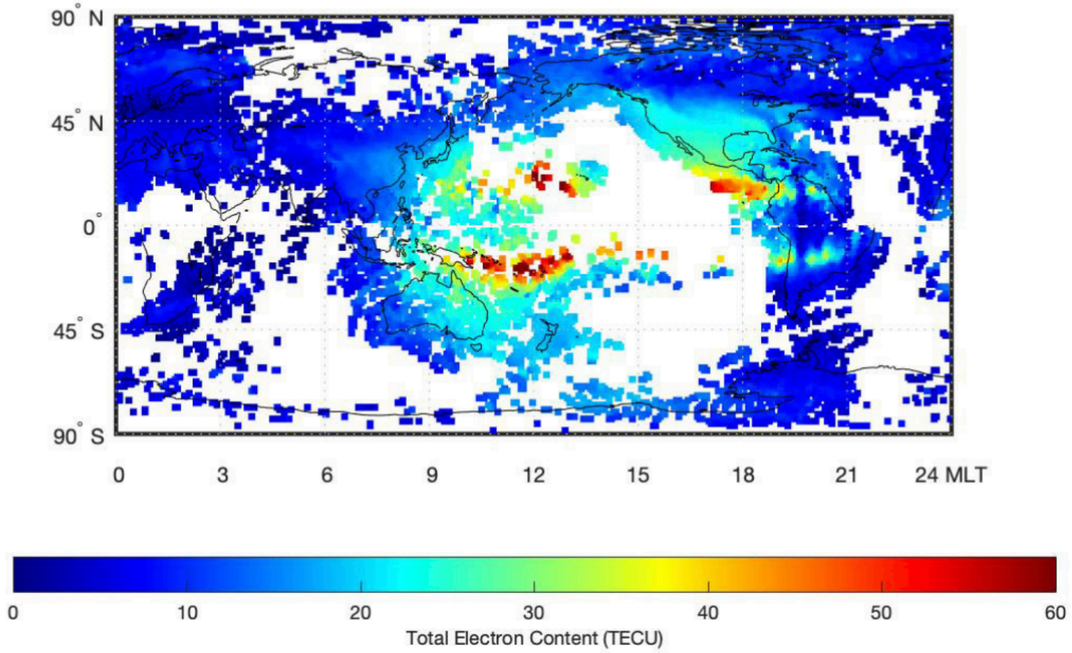

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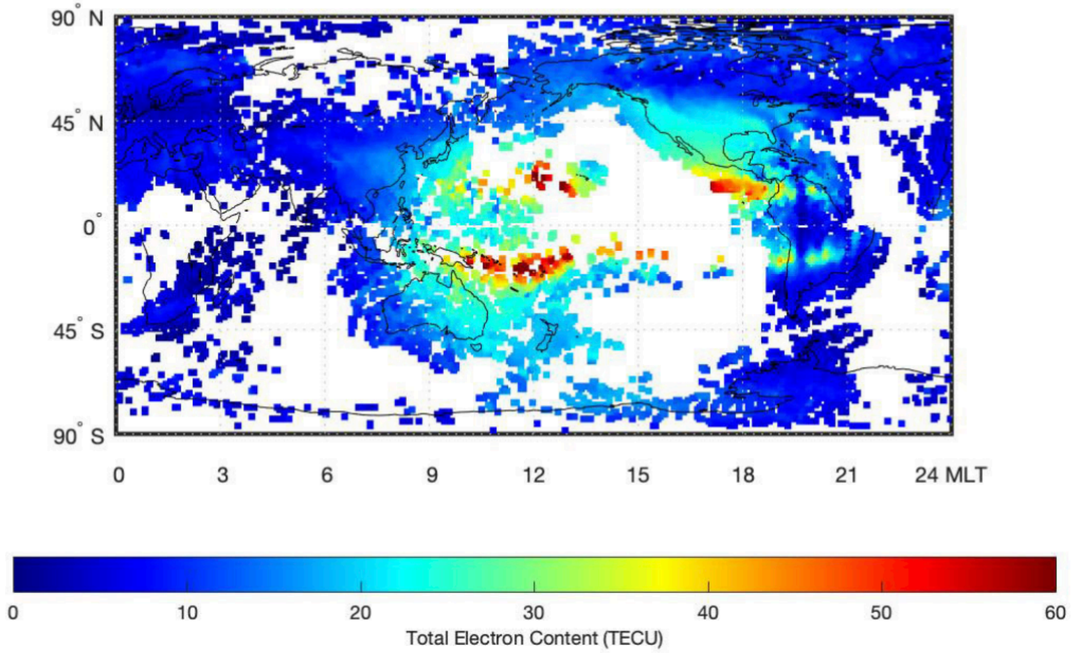

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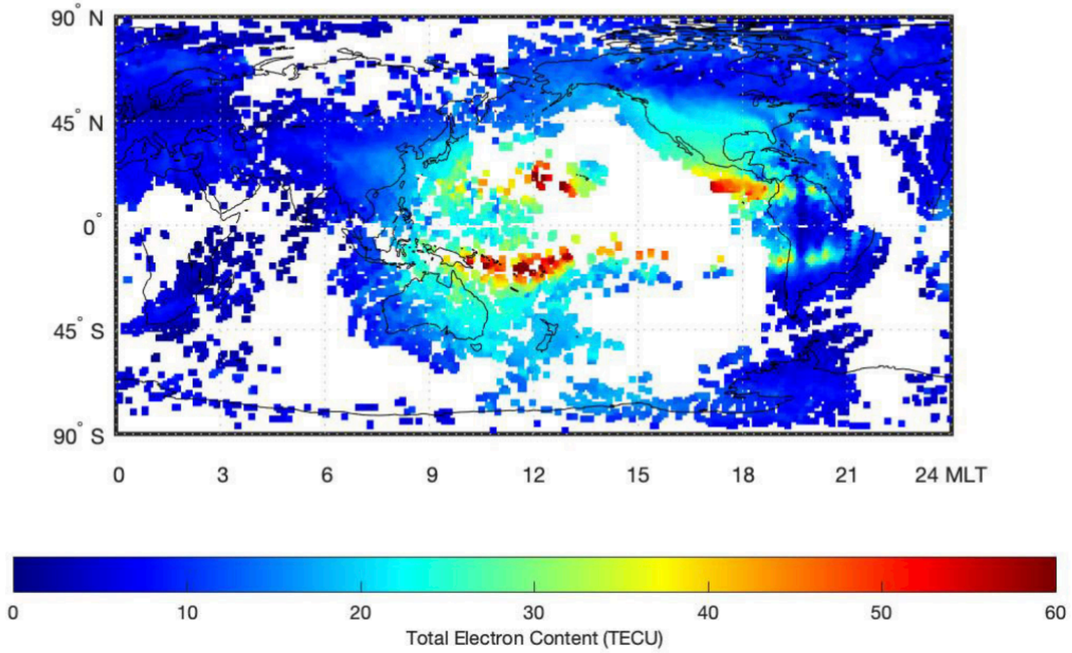

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- multi-way spatio-temporal dependency
- dependent data missingness
- multiple data modalities
- high data dimensionality

Dissertation Summary

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I. Tensor Completion with Spatio-Temporal smoothing

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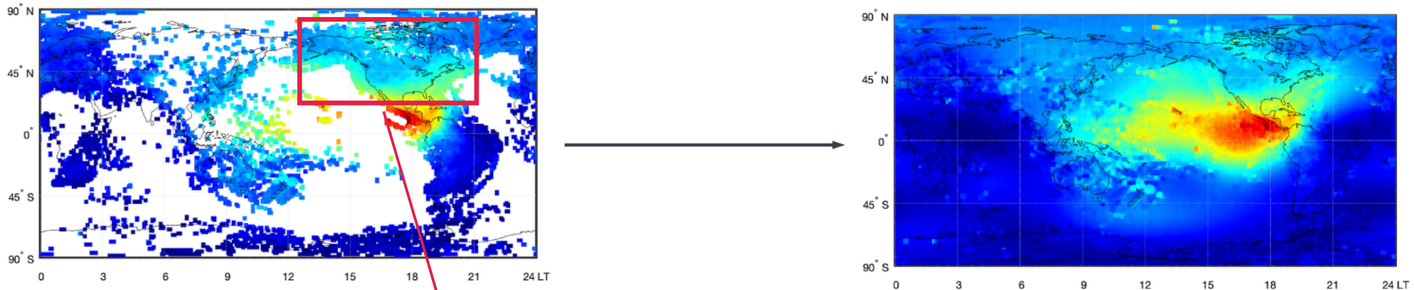
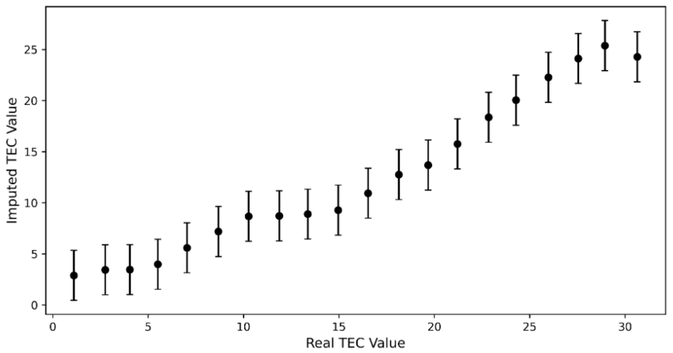
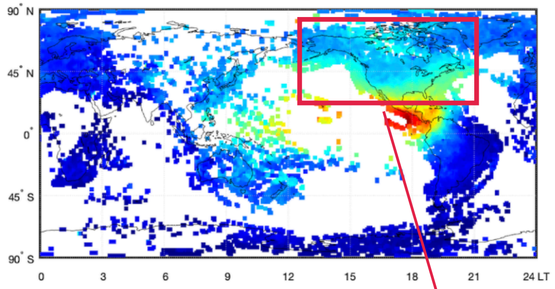


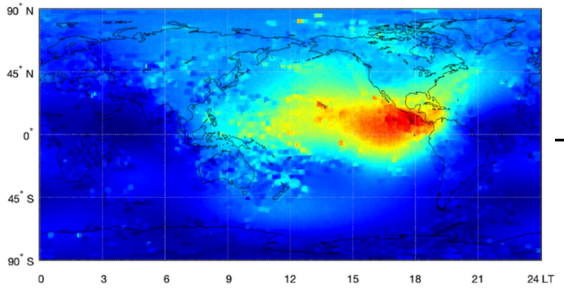


II. Uncertainty Quantification for Tensor Completion

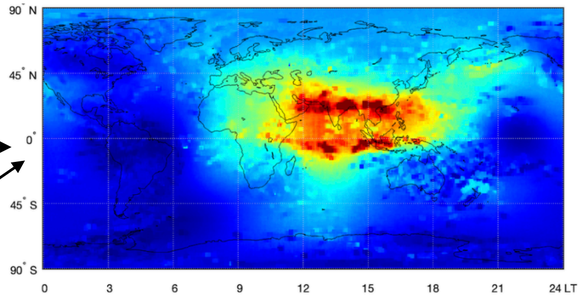
Dissertation Summary

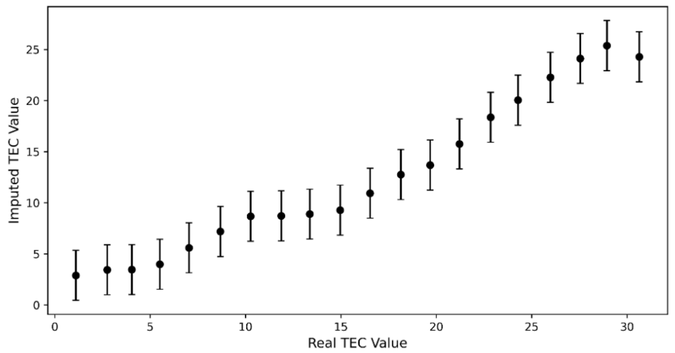
I. Tensor Completion with Spatio-Temporal smoothing

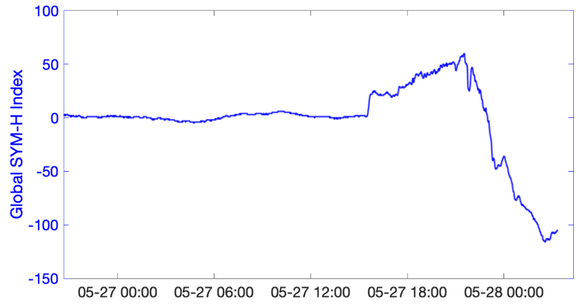




III. RKHS Method for Multi-modal Autoregression Model







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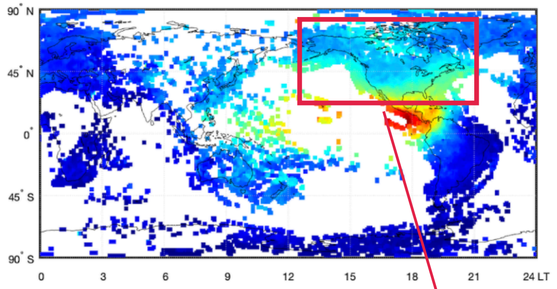

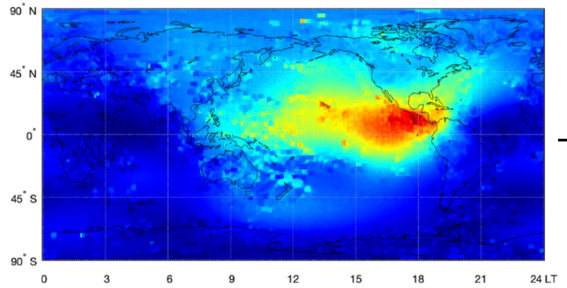

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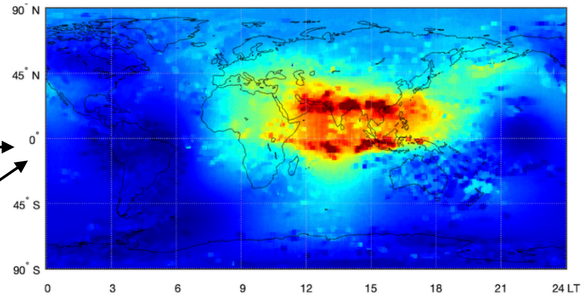

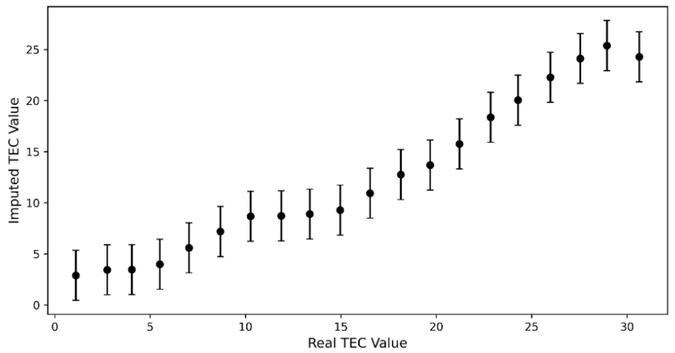

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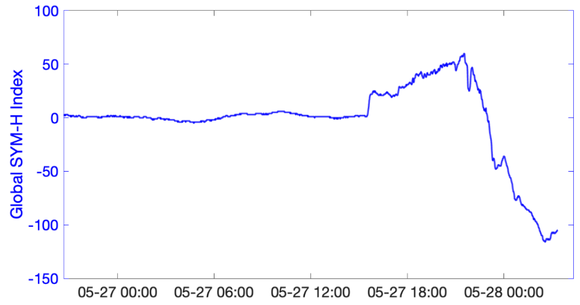
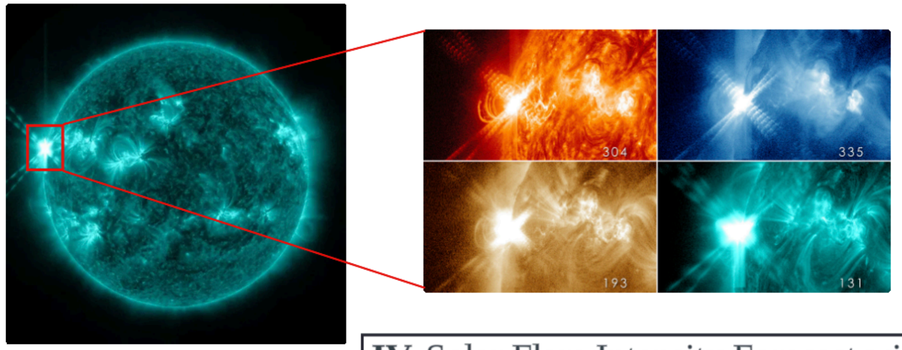
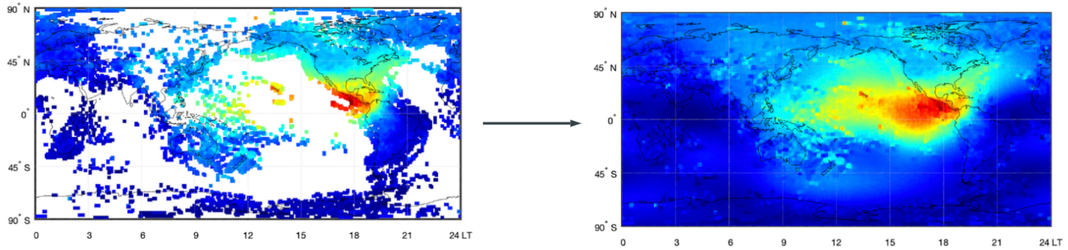


Solar Wind Parameters



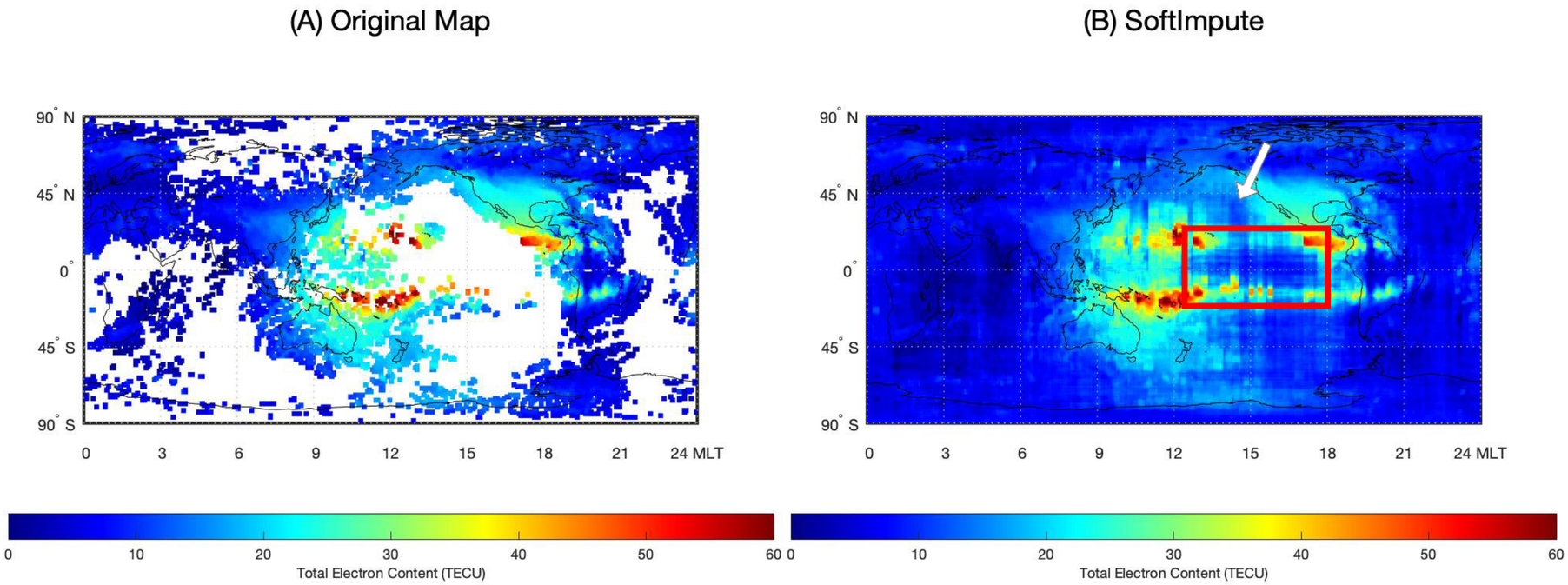
IV. Solar Flare Intensity Forecast with Scalar-on-Tensor Regression

I. Tensor Completion with Spatio-Temporal Smoothing



How to generate accurate imputations when data are missing in spatio-temporal patches?

Low-rank Completion



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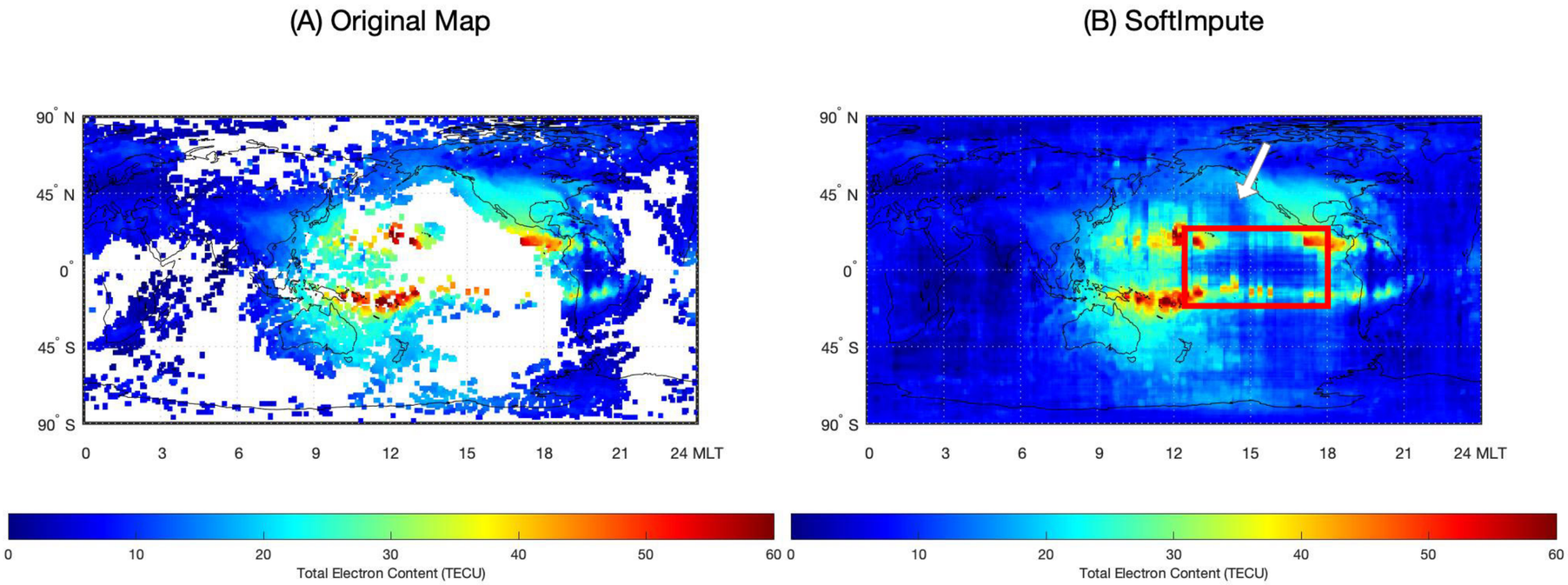
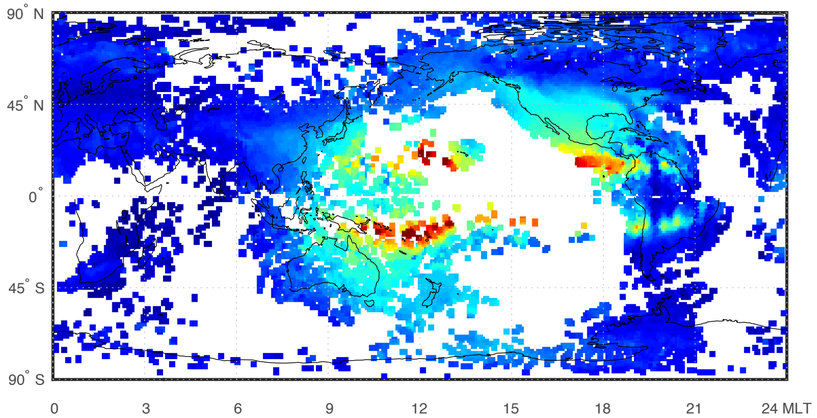


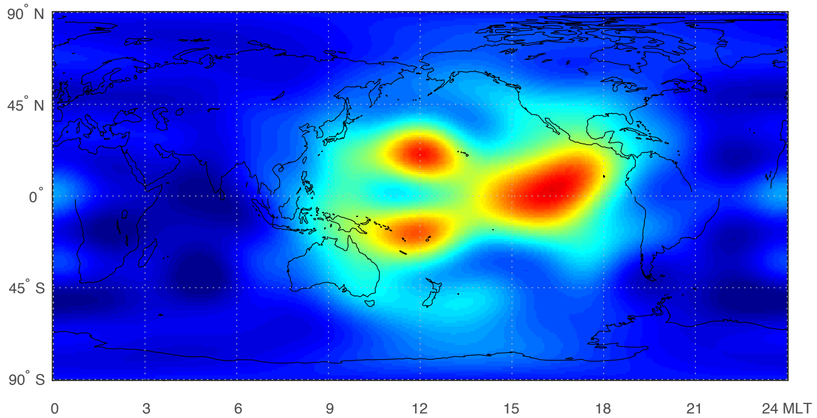
- Traditional matrix/tensor completion method [Hastie et al. (2015)] cannot impute regions with almost all entries missing (called the missing patches).

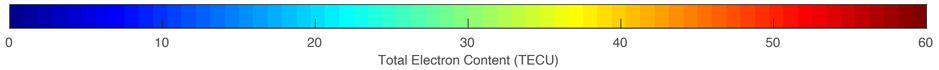
Spatio-Temporal Kernel Smoothing

(A) Madrigal TEC map



(B) Spherical Harmonics Fitting





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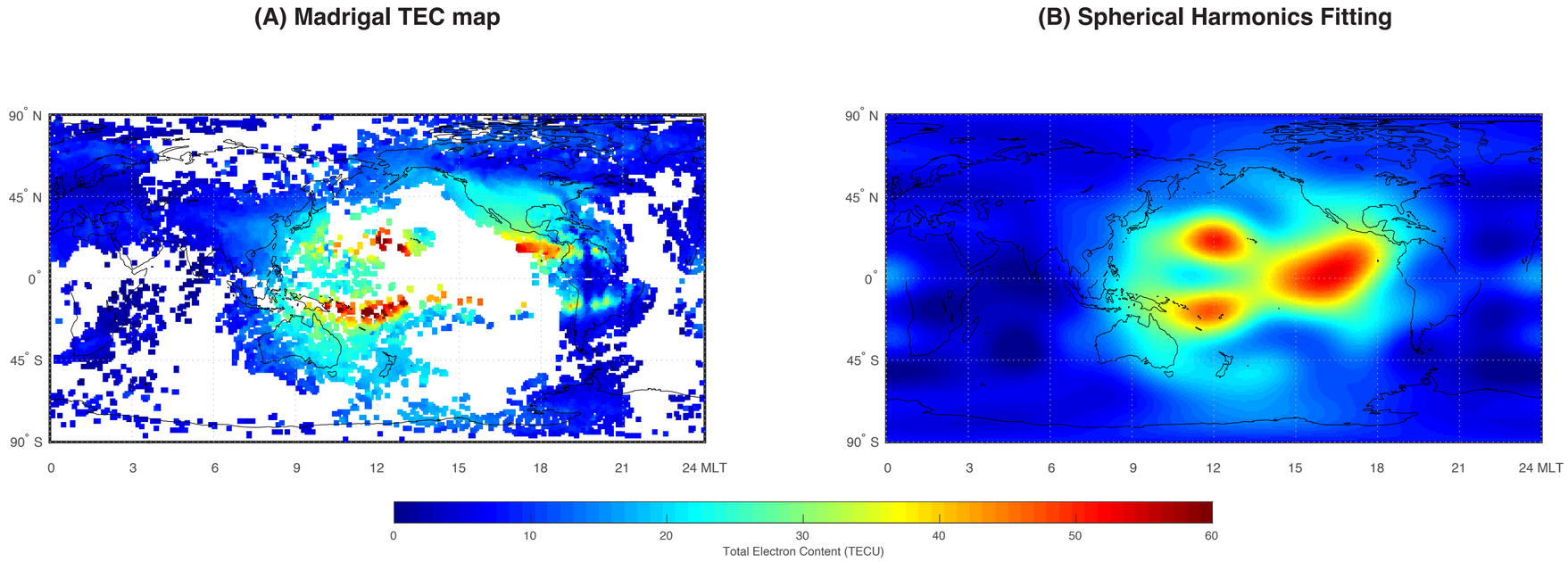

- Spatio-temporal kernel smoothing can impute the missing patches with more reasonable values, but tends to be overly-smooth.

Our Method

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- 3-D spatio-temporal tensor (as a matrix time series): $\mathbf{X}_1, \ldots, \mathbf{X}_T \in \mathbb{R}^{M \times N}$

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$$\min_{\mathbf{A}_t, \mathbf{B}_t, t \in [T]} \frac{1}{2} \sum_t \left\| \mathrm{P}_{\Omega_t} \left(\mathbf{X}_t - \mathbf{A}_t \mathbf{B}_t^\top \right) \right\|_{\mathrm{F}}^2 + \lambda_1 \sum_t \left(\left\| \mathbf{A}_t \right\|_{\mathrm{F}}^2 + \left\| \mathbf{B}_t \right\|_{\mathrm{F}}^2 \right)$$
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temporal smoothing

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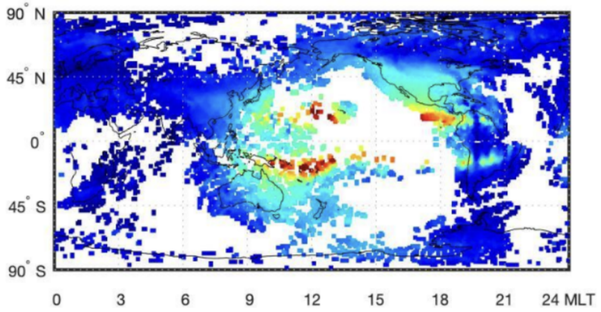
temporal smoothing

pre-imputed by kernel smoothing

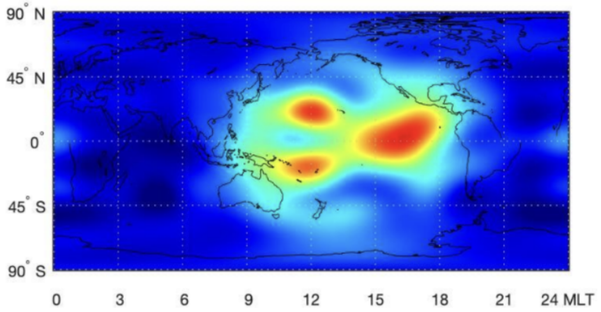
Results

Global Total Electron Content (TEC) Reconstruction

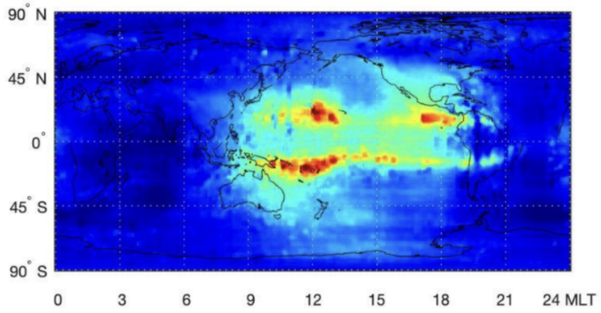
(A) Original Data



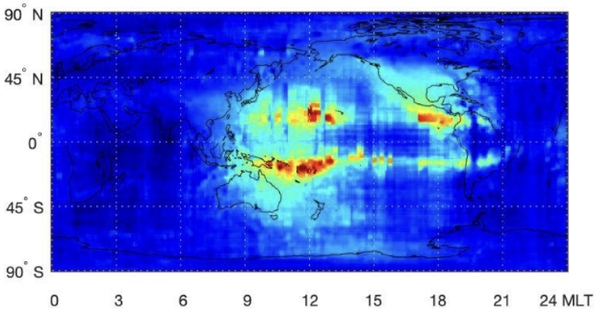
(B) Kernel Smoothing

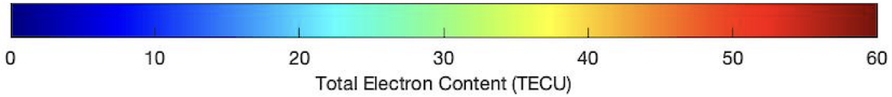


(C) Our Method

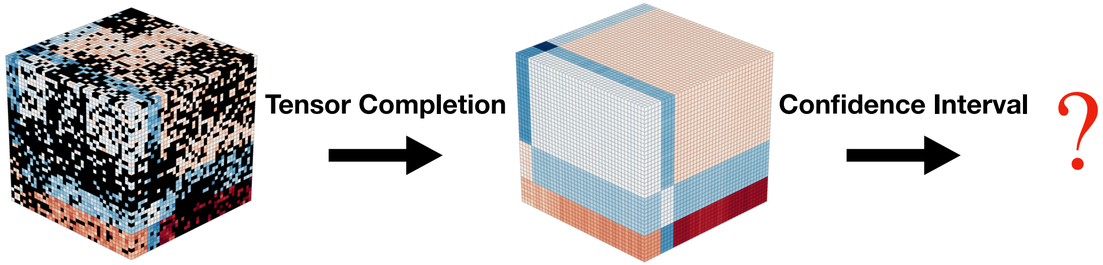


(D) Matrix Completion





II. Conformalized Tensor Completion with Riemannian Optimization



Research Question

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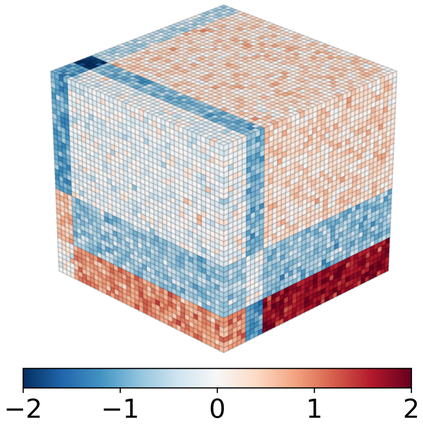
Noisy Data Tensor

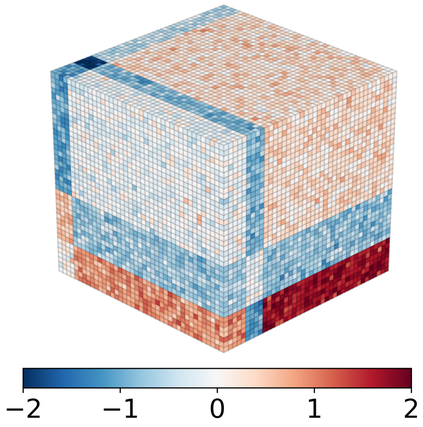
$\mathcal{X} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$



Research Question

Noisy Data Tensor

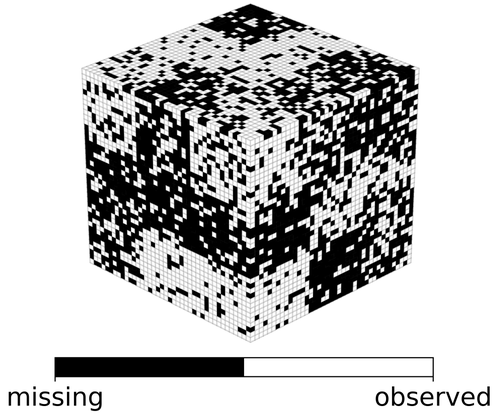
$\mathcal{X} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$



+

Missing Data

$\mathcal{W} \in \{-1,1\}^{d_1 \times \cdots \times d_K}$



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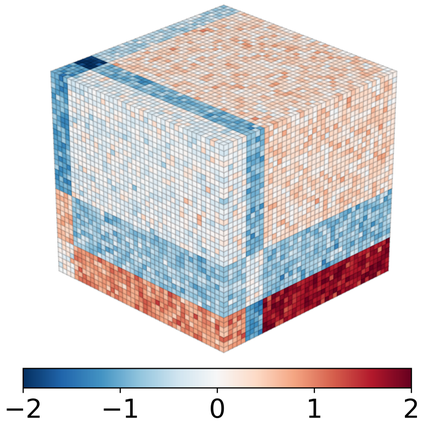

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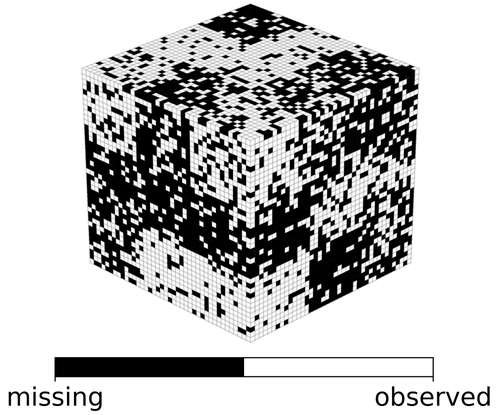

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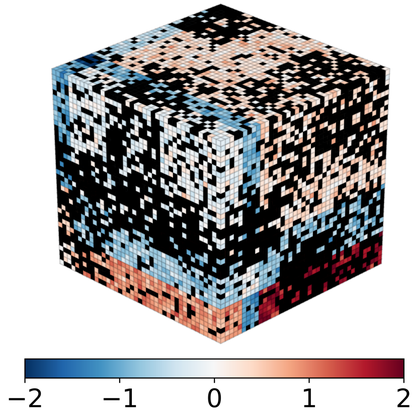
Partially-Observed Data Tensor $\mathscr{X}_{\mathbf{\Omega}}$
$\mathbf{\Omega} = \{s \in [d_1] \times \cdots [d_K] \,|\, [\mathscr{W}]_s = 1\}$



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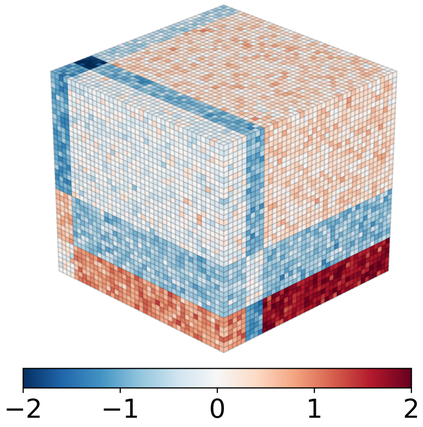

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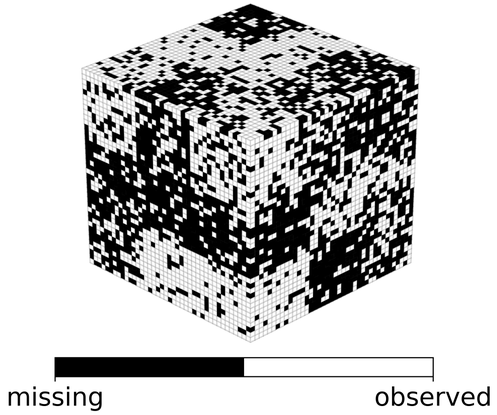

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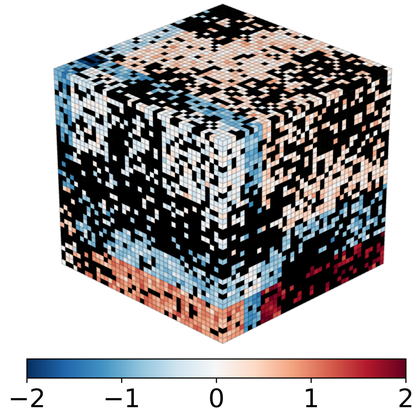
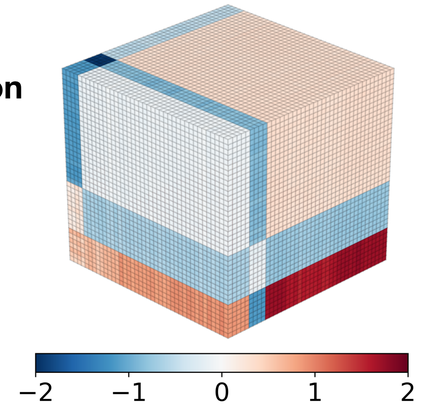


Tensor Completion

Completed Tensor $\widehat{\mathscr{X}}$



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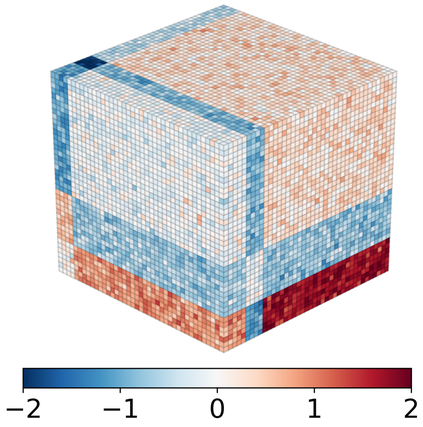

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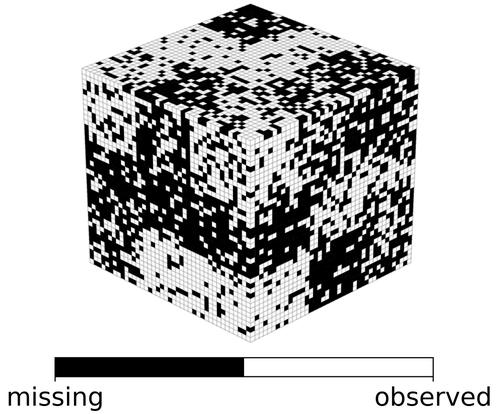

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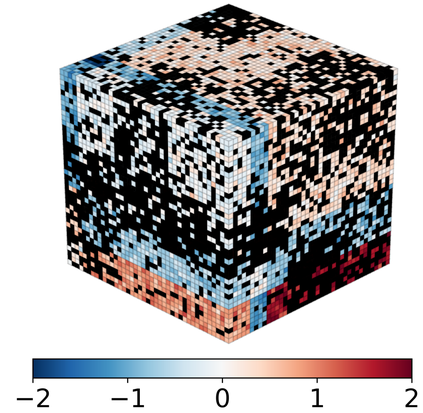

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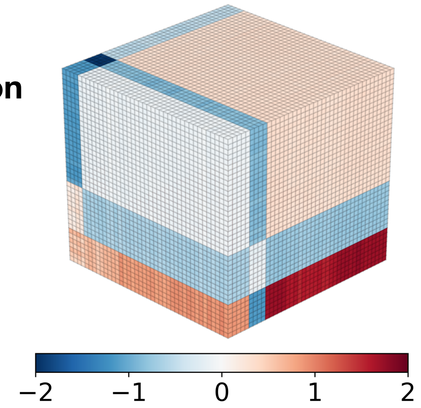

For any $s^* \in \boldsymbol{\Omega}^c = \{s \in [d_1] \times \cdots [d_K] \,|\, [\mathscr{W}]_s = -1\}$, how to construct a confidence region $\widehat{C}_{s^*}$ such that:

$$\mathrm{P}\left([\mathscr{X}]_{s^*} \in \widehat{C}_{s^*}\right) \geq 1 - \alpha,$$

with $1 - \alpha$ being a pre-specified level of confidence?

Literature Review

Two Types of Uncertainty Quantification

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- inference on true tensor: $\mathrm{P}\left([\mathcal{X}^*]_{s^*} \in \widehat{C}_{s^*}\right) \geq 1 - \alpha$
- predictive UQ: $\mathrm{P}\left([\mathcal{X}]_{s^*} \in \widehat{C}_{s^*}\right) \geq 1 - \alpha$

Literature Review

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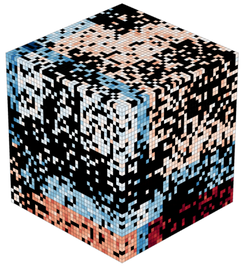
- No previous work considers a missing mechanism under the spatio-temporal context.

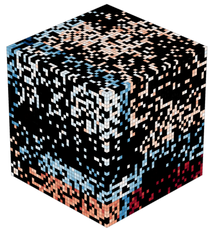
Classical Conformal Prediction (CP)

Data Tensor $\mathcal{X}_{\mathbf{\Omega}}$



Classical Conformal Prediction (CP)

Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$



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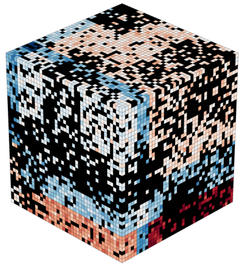
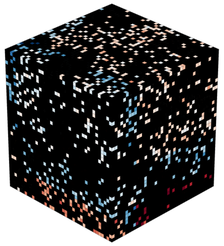
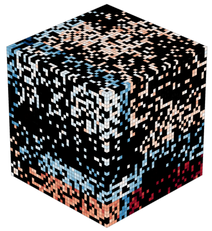


Calibration Tensor $\mathcal{X}_{\mathbf{\Omega}_{cal}}$

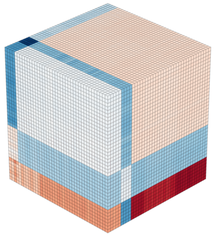


Classical Conformal Prediction (CP)

Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$

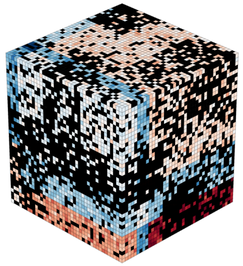


Completed Tensor $\widehat{\mathcal{X}}$



Tensor Completion

Data Tensor $\mathcal{X}_{\mathbf{\Omega}}$



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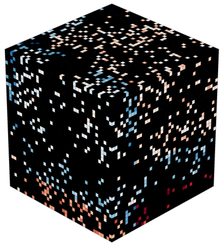

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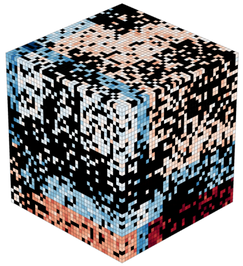
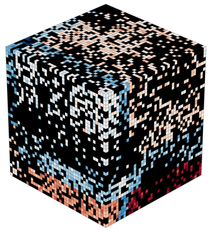


Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$



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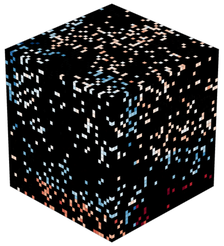
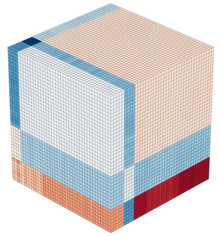


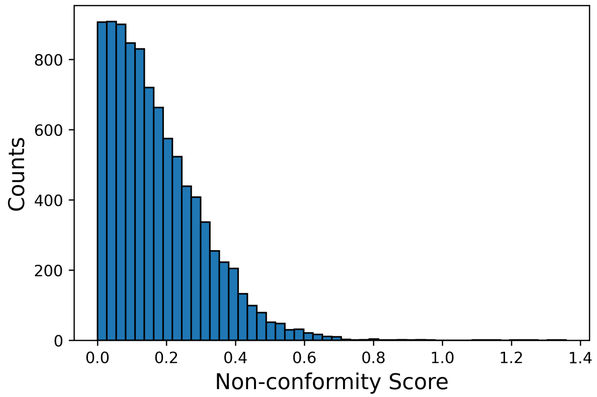
Tensor Completion

Completed Tensor $\widehat{\mathcal{X}}$



Non-conformity score

e.g. $\mathcal{S}([\mathcal{X}]_s, [\widehat{\mathcal{X}}]_s) = |[\mathcal{X}]_s - [\widehat{\mathcal{X}}]_s|$



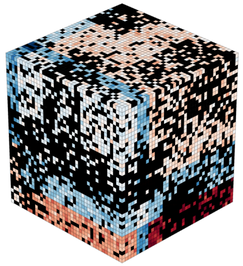
Classical Conformal Prediction (CP)

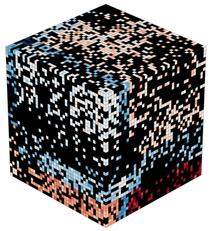
Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$

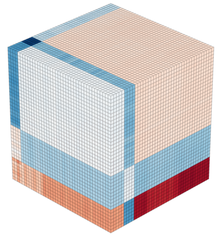
Completed Tensor $\widehat{\mathcal{X}}$

Tensor Completion

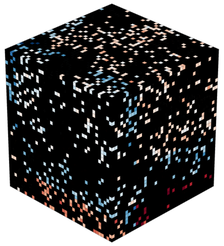
Data Tensor $\mathcal{X}_{\mathbf{\Omega}}$







Calibration Tensor $\mathcal{X}_{\mathbf{\Omega}_{cal}}$



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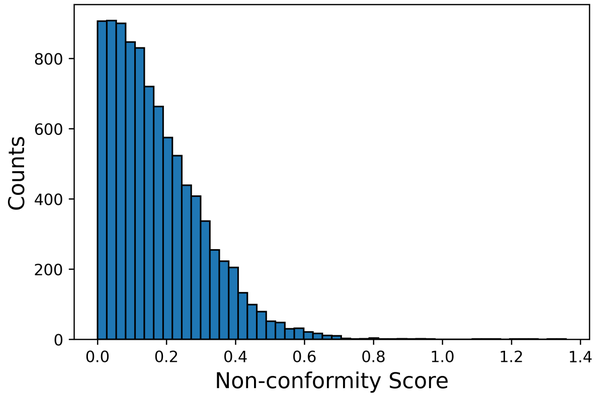
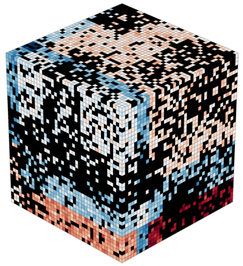


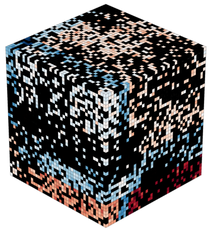
$$\widehat{C}_{1-\alpha,s*} = \left\{ x \in \mathbb{R} \,\middle|\, \mathcal{S}(x, [\widehat{\mathcal{X}}]_s) \leq \hat{q}_{1-\alpha} \left(\sum_{s \in \mathbf{\Omega}_{cal}} \frac{1}{|\mathbf{\Omega}_{cal}| + 1} \delta_{\mathcal{S}([\mathcal{X}]_s, [\widehat{\mathcal{X}}]_s)} + \frac{1}{|\mathbf{\Omega}_{cal}| + 1} \delta_{+\infty} \right) \right\}$$

Classical Conformal Prediction (CP)

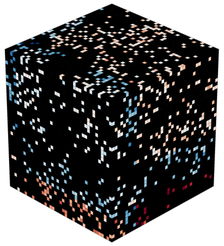
Data Tensor $\mathcal{X}_{\mathbf{\Omega}}$



Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$

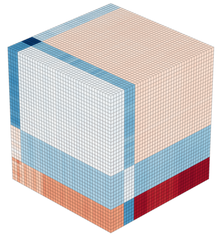


Calibration Tensor $\mathcal{X}_{\mathbf{\Omega}_{cal}}$



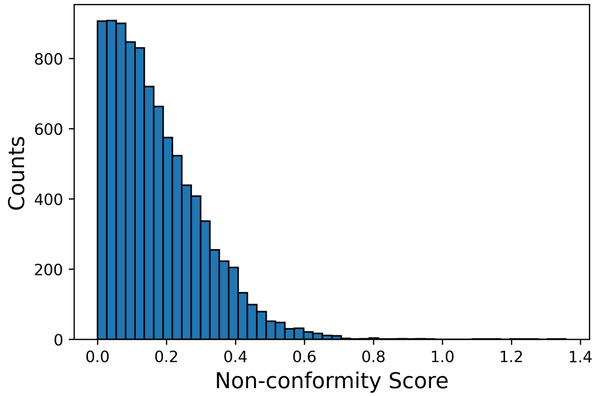
Tensor Completion

Completed Tensor $\widehat{\mathcal{X}}$



Non-conformity score

e.g. $\mathcal{S}([\mathcal{X}]_s, [\widehat{\mathcal{X}}]_s) = |[\mathcal{X}]_s - [\widehat{\mathcal{X}}]_s|$

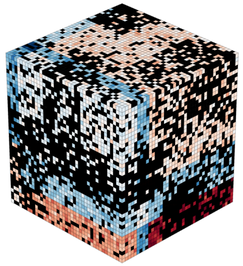


$$\widehat{C}_{1-\alpha,s^*} = \left\{ x \in \mathbb{R} \,\middle|\, \mathcal{S}(x, [\widehat{\mathcal{X}}]_s) \leq \hat{q}_{1-\alpha} \left(\sum_{s \in \mathbf{\Omega}_{cal}} \frac{1}{|\mathbf{\Omega}_{cal}| + 1} \delta_{\mathcal{S}([\mathcal{X}]_s, [\widehat{\mathcal{X}}]_s)} + \frac{1}{|\mathbf{\Omega}_{cal}| + 1} \delta_{+\infty} \right) \right\}$$

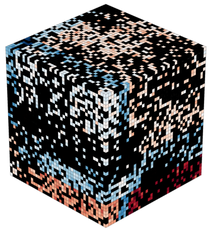
$(1-\alpha)$ level conformal interval at s^*

Classical Conformal Prediction (CP)

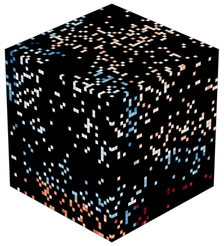
Data Tensor $\mathcal{X}_{\mathbf{\Omega}}$



Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$



Calibration Tensor $\mathcal{X}_{\mathbf{\Omega}_{cal}}$



Tensor Completion

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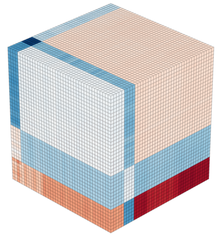

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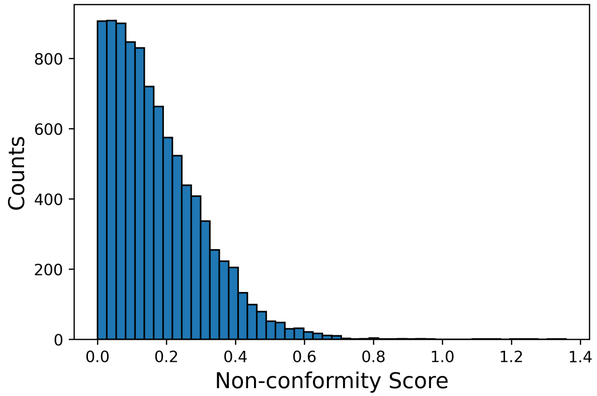

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$(1-\alpha)$ level conformal interval at $s*$

$(1-\alpha)$ quantile

Classical Conformal Prediction (CP)

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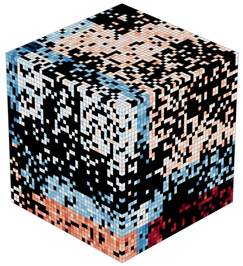
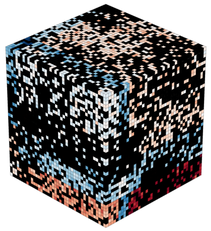
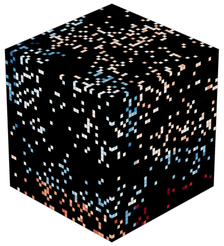


Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$



Calibration Tensor $\mathcal{X}_{\mathbf{\Omega}_{cal}}$



Tensor Completion

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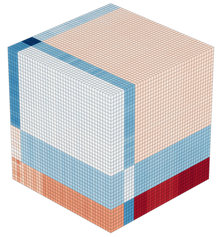

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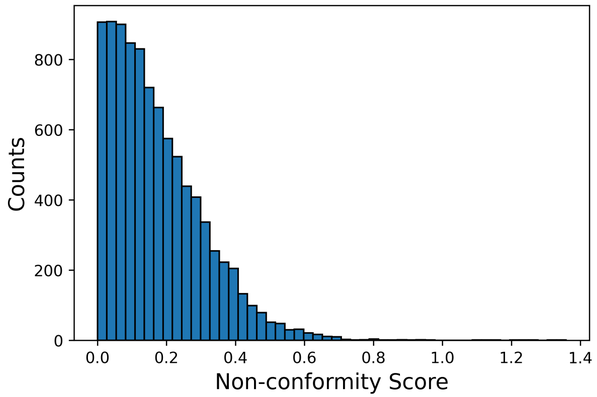

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$(1-\alpha)$ level conformal interval at $s*$

$(1-\alpha)$ quantile

eCDF of the non-conformity score

Validity of Classic Conformal Prediction

Tibshirani et al. (2019)

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- V: the random variable of the non-conformity score at an unseen entry

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$$
\begin{array}{ll}
\left(s_1, s_2, \ldots, s_n, s_{n+1}\right) & V = v_1 \\
\left(s_1, s_2, \ldots, s_n, s_{n+1}\right) & V = v_2 \\
\ldots\ldots & \ldots\ldots \\
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$$

$$
\mathrm{P}(V = v_k) = \frac{\mathrm{P}([\mathscr{W}]_s = 1, \forall s \in \mathbf{\Omega}_{tr} \cup \{s_1, \ldots, s_{n+1}\} \backslash \{s_k\}, [\mathscr{W}]_s = -1, \text{o.w.})}{\sum_{l=1}^{n+1} \mathrm{P}([\mathscr{W}]_s = 1, \forall s \in \mathbf{\Omega}_{tr} \cup \{s_1, \ldots, s_{n+1}\} \backslash \{s_l\}, [\mathscr{W}]_s = -1, \text{o.w.})} = \frac{1}{n+1}
$$

Weighted Conformal Prediction

Leave-one-out Likelihood Weighting

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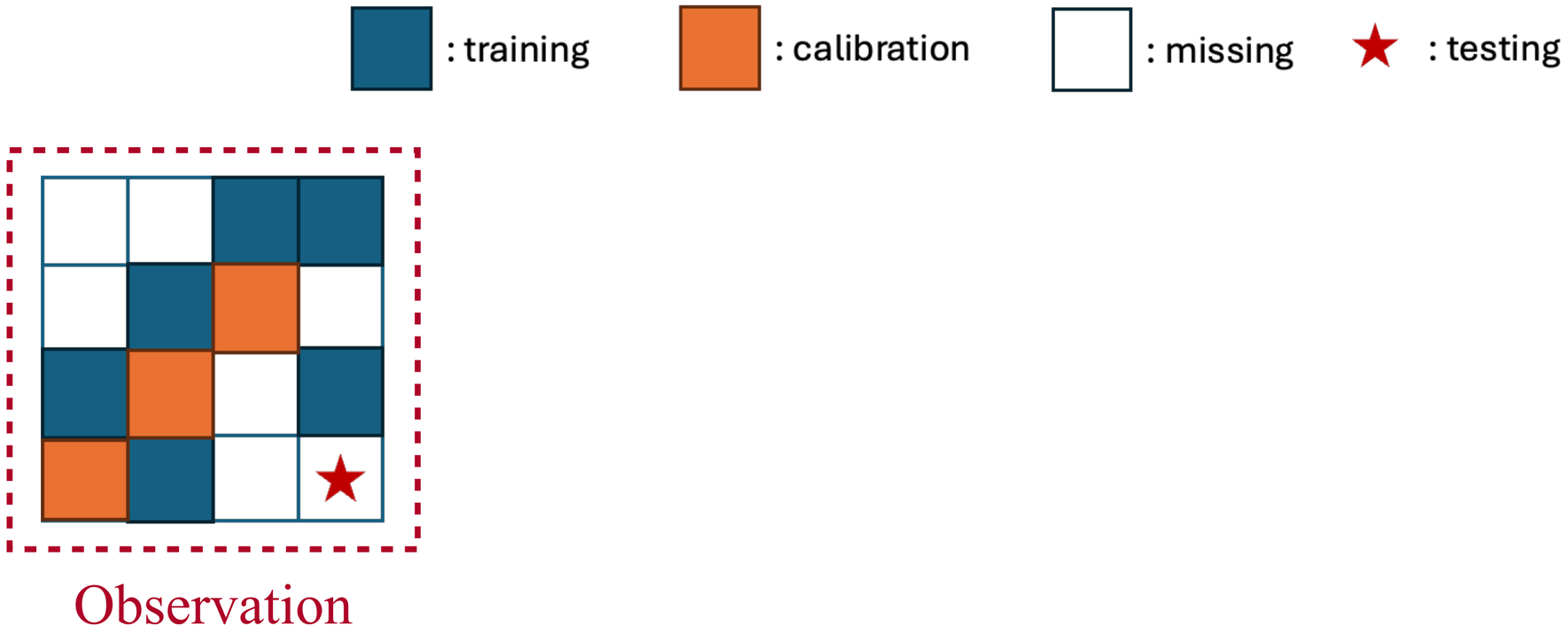
Leave-one-out Likelihood Weighting



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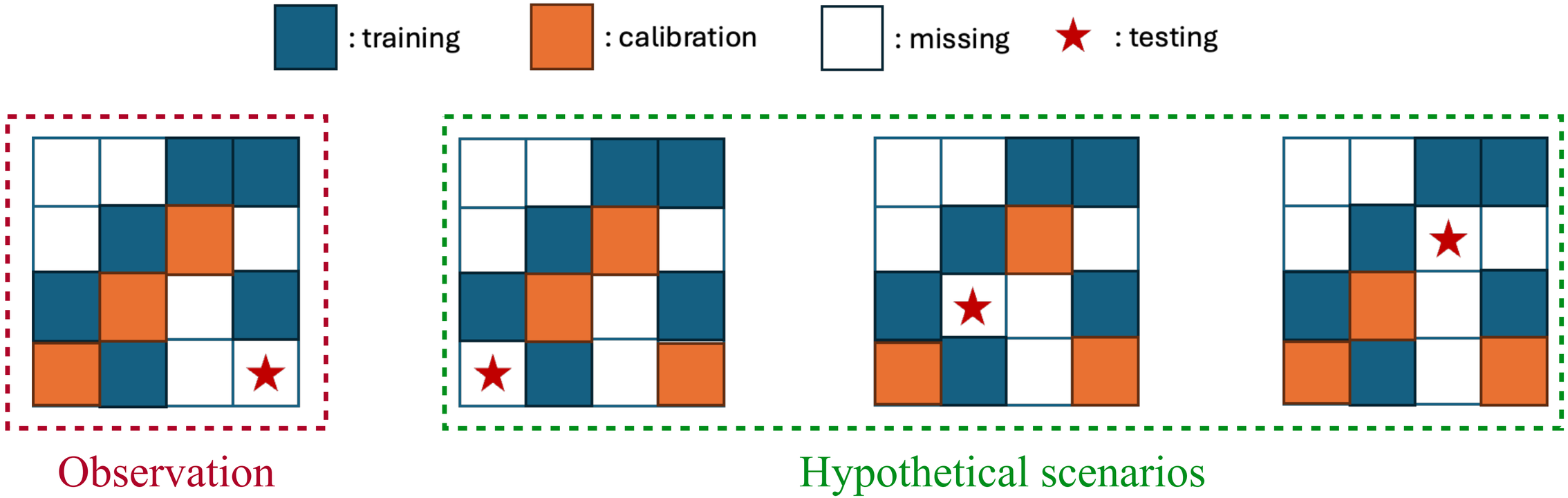

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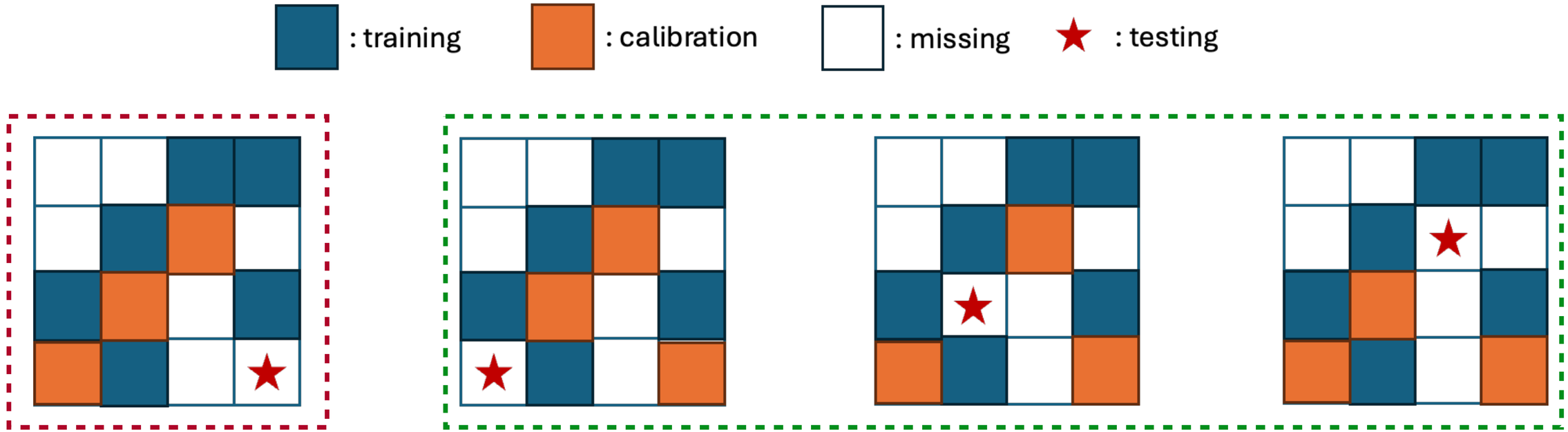

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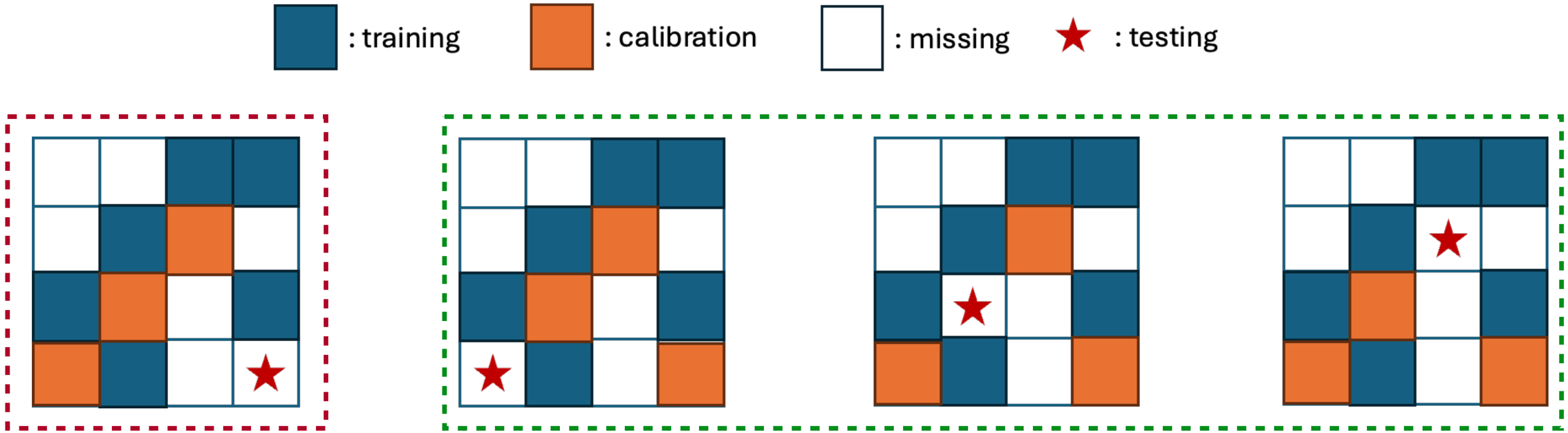

probability	p_1	p_2	p_3	p_4
weight of ★	$\frac{p_1}{p_1+p_2+p_3+p_4}$	$\frac{p_2}{p_1+p_2+p_3+p_4}$	$\frac{p_3}{p_1+p_2+p_3+p_4}$	$\frac{p_4}{p_1+p_2+p_3+p_4}$

Weighted Conformal Prediction

Leave-one-out Likelihood Weighting

Theorem 1

Let $\omega_k \propto \mathrm{P}([\mathscr{W}]_s = 1, \forall s \in \mathbf{\Omega}_{tr} \cup \{s_1, \ldots, s_{n_{cal}+1}\} \backslash \{s_k\}, [\mathscr{W}]_s = -1, \text{o.w.})$, then

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is still valid, i.e. $\mathrm{P}([\mathscr{X}]_{s*} \in \widehat{C}_{1-\alpha,s*}) \geq 1 - \alpha$.

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Now we need to estimate the likelihood $p(\mathscr{W})$ with a single sample.

Missing Propensity Model

Motivation

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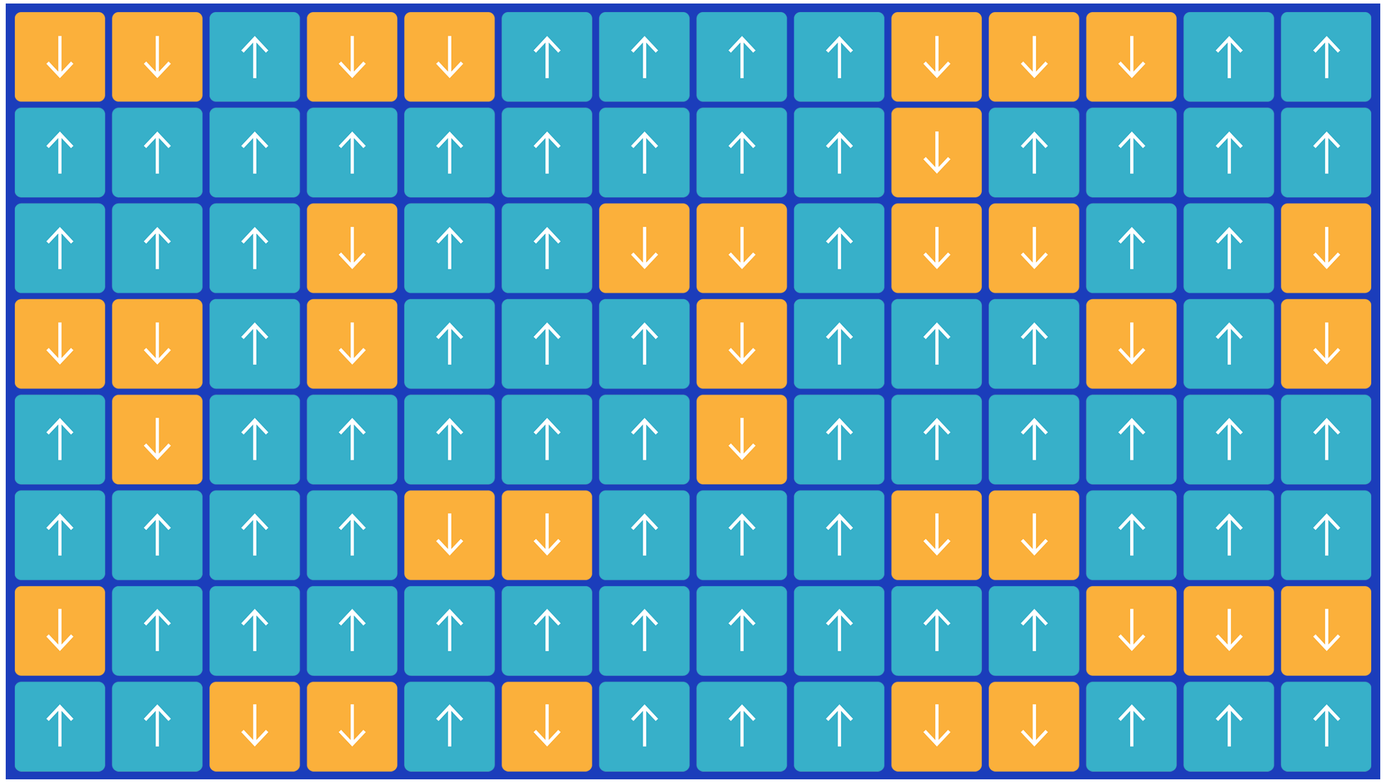
where $N(s_1) = \{s \,|\, s \sim s_1\}$, and $s \sim s_1$ means s, s_1 are neighboring tensor entries.

Missing Propensity Model

Tensor Ising Model

- We adopt the idea from the Ising model [Cipra (1987)], which models the probability distribution of atomic spins on a lattice grid.



[source]

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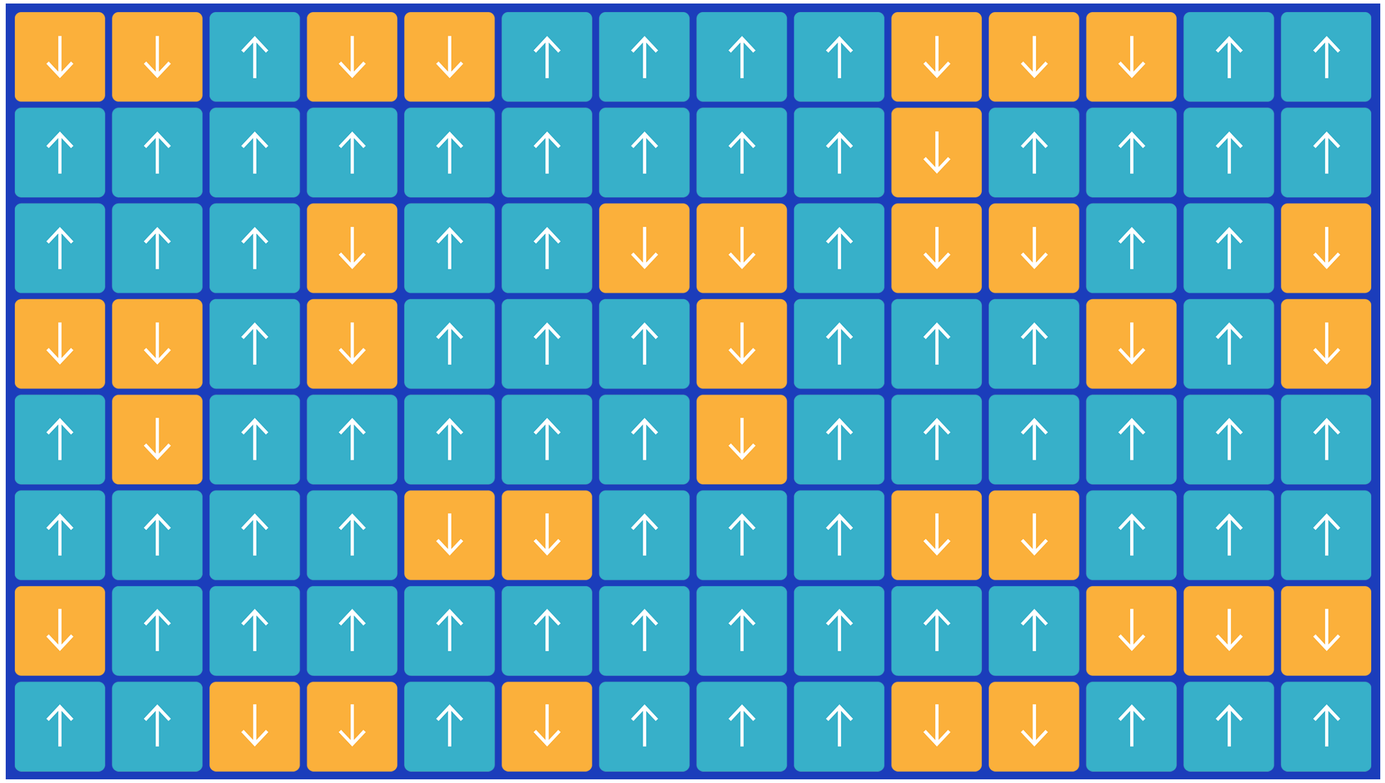

[source]

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$g_{s_1 s_2}$: interaction strength

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Missing Propensity Model

Parameter Estimation Problem of Tensor Ising Model

Missing Propensity Model

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- Given $\mathscr{W}$, we want to estimate g_{s_1,s_2} for arbitrary $s_1 \sim s_2$ and h_{s_1} for any s_1.

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- In our work, we have **a single sample** but **an order-K Ising model**.

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Latent Space Tensor Ising Model

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- The conformal weight satisfies:

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- If $g(\cdot, \cdot) = 0$, then:

$$[\mathscr{W}]_s \overset{ind.}{\sim} \text{Bern}([\mathscr{P}]_s), \quad [\mathscr{P}]_s = \frac{\exp[2h([\mathscr{B}]_s)]}{1 + \exp[2h([\mathscr{B}]_s)]}$$

- The conformal weight satisfies:

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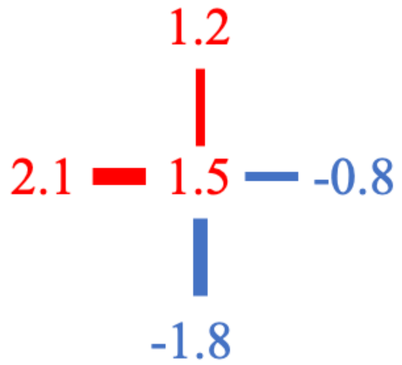

Example $\mathscr{B}$

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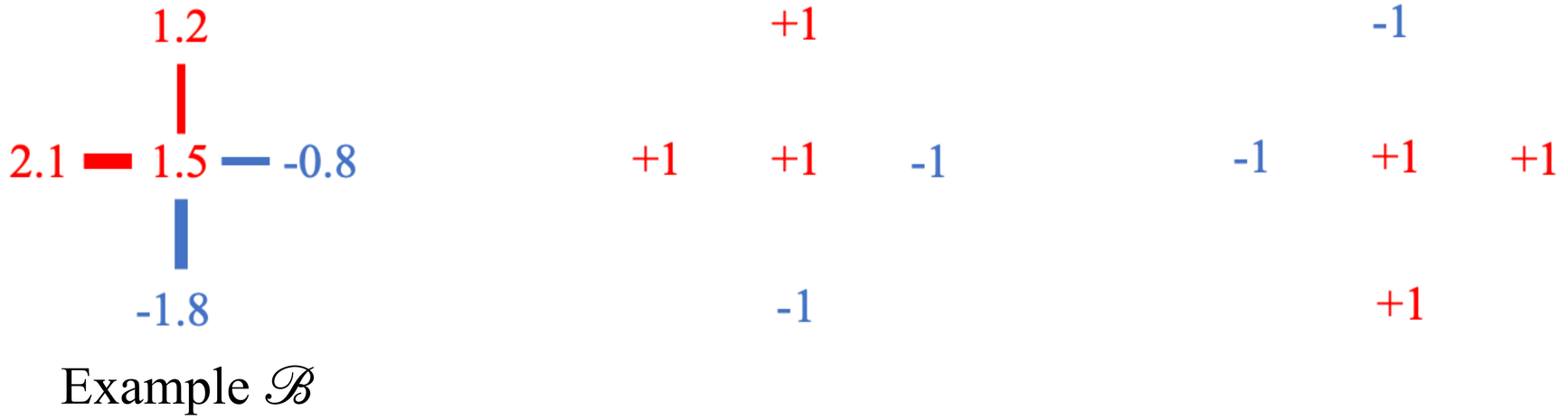

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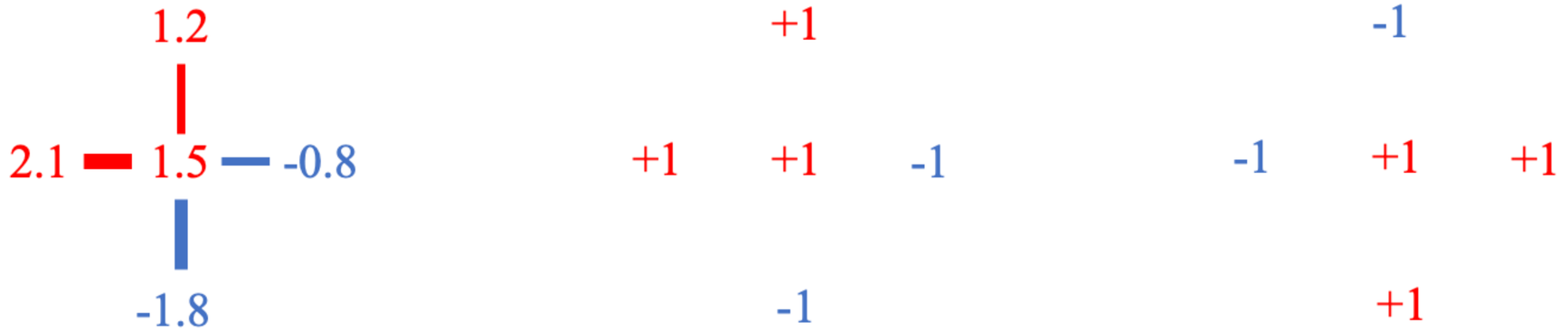

Example $\mathscr{B}$ More likely scenario Less likely scenario

Example II: Interaction-only Ising Model

- The conformal weight (almost always) satisfies:

$$\omega_s \propto \frac{1-\widetilde{p}_s}{\widetilde{p}_s} = \exp\left[-2\sum_{j\in N(s)} g([\mathscr{B}]_s, [\mathscr{B}_j]) \cdot [\mathscr{W}]_j\right]$$

Exceptions are samples in $\mathbf{\Omega}_{cal} \cap N(s_{n+1})$, where $[\mathscr{W}]_{s_{n+1}} = -1$, but the leave-one-out procedure requires $[\mathscr{W}]_{s_{n+1}} = 1$.

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where $q \in (0,1)$ is the probability of assigning data to the training set, and $\mathbb{F}$ is a feasible set of $\mathscr{B}$.

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with $r_0 = r_K = 1$, and $\mathbf{r} = (r_1, \ldots, r_{K-1})$ being the TT-rank.

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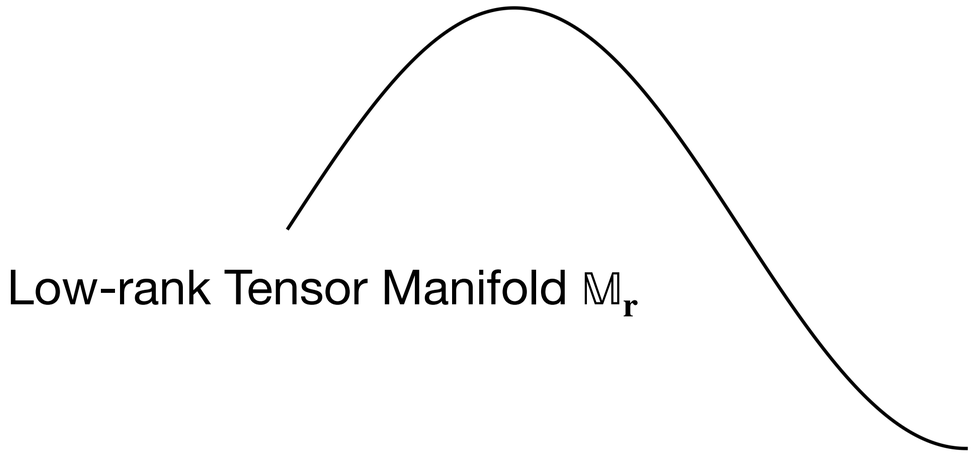

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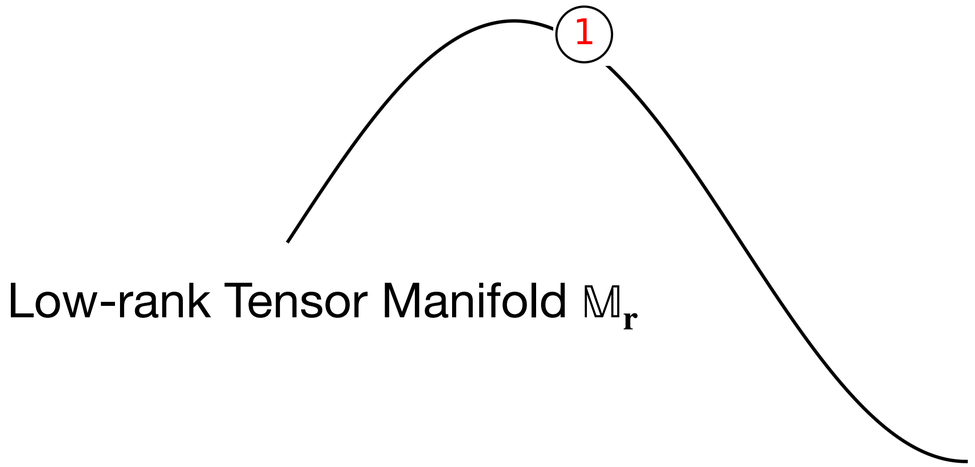

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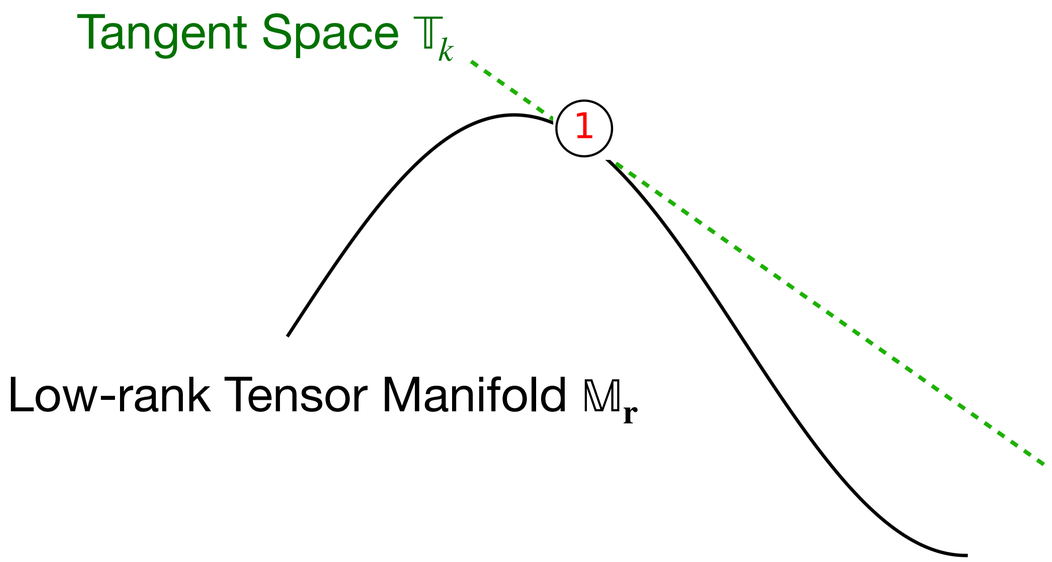

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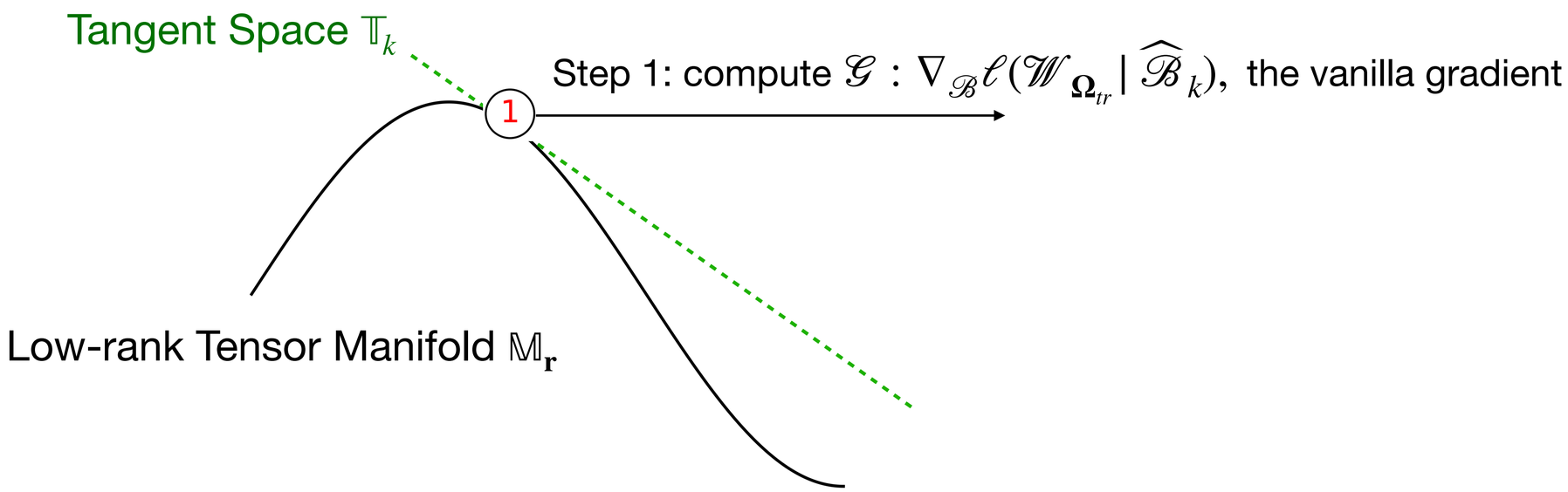

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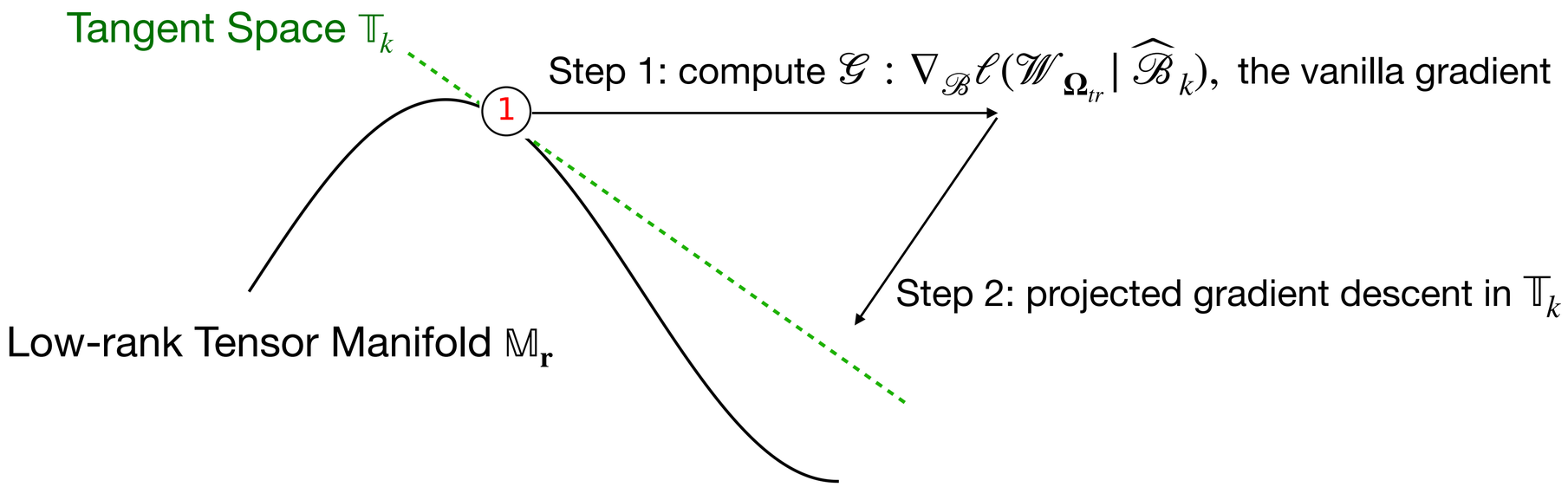

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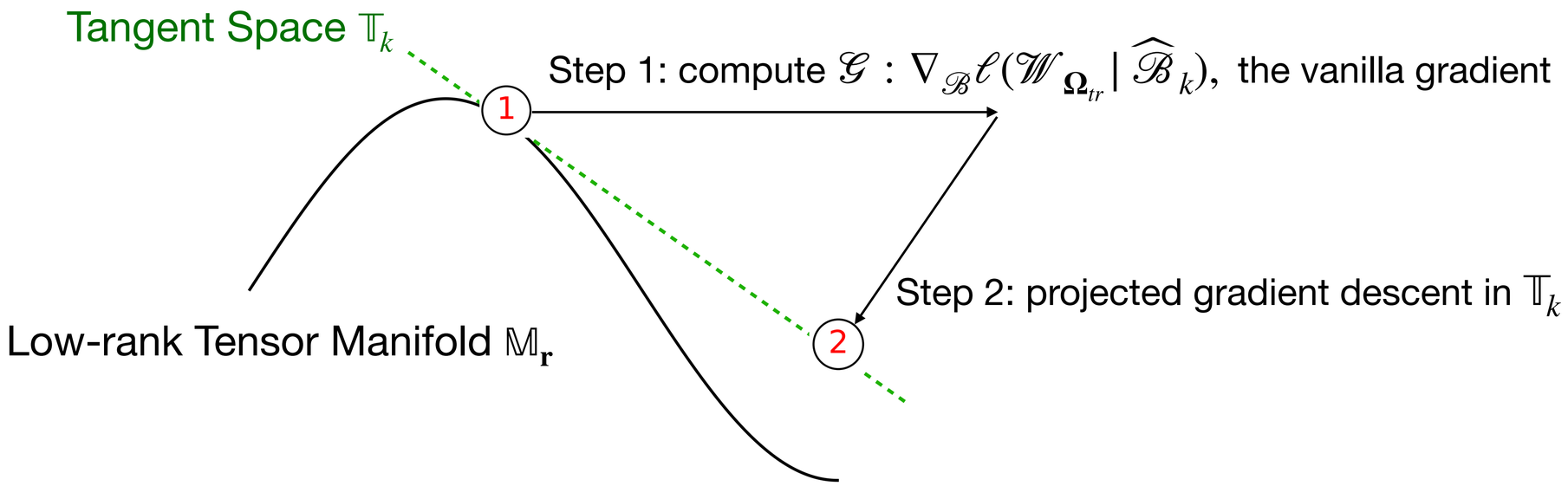

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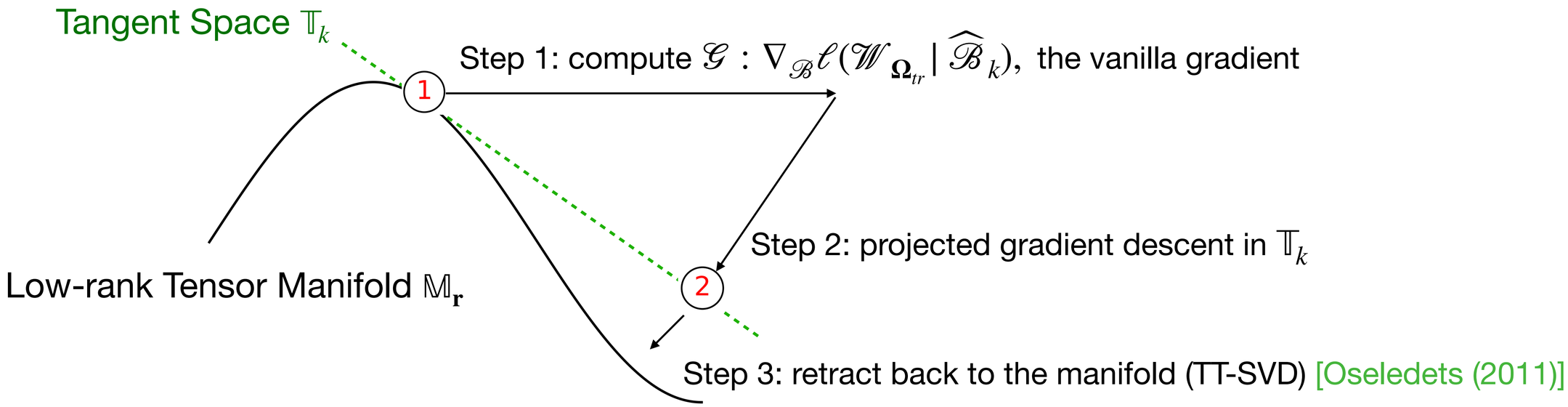

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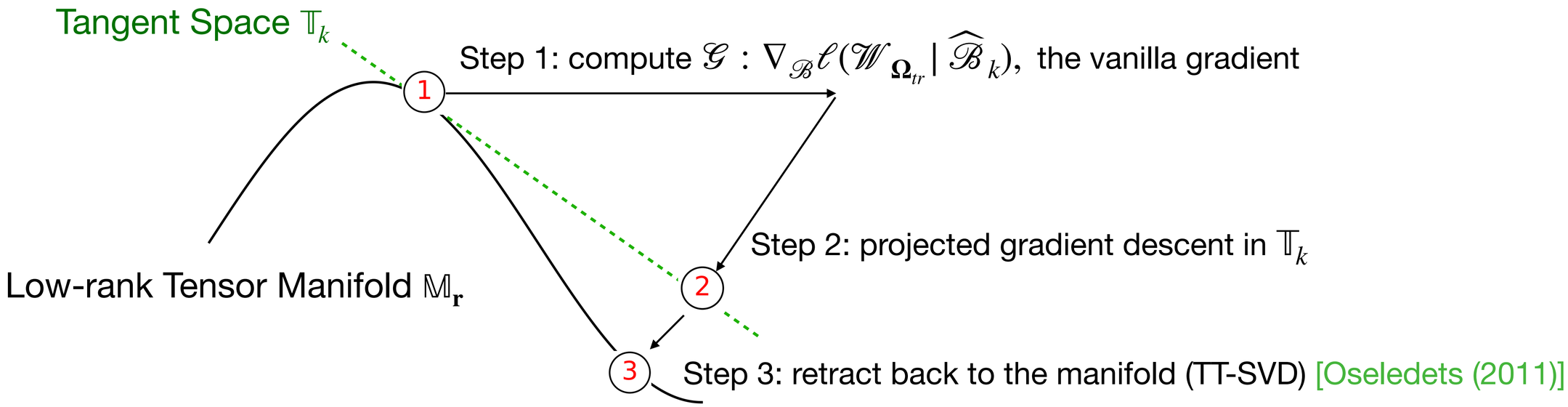
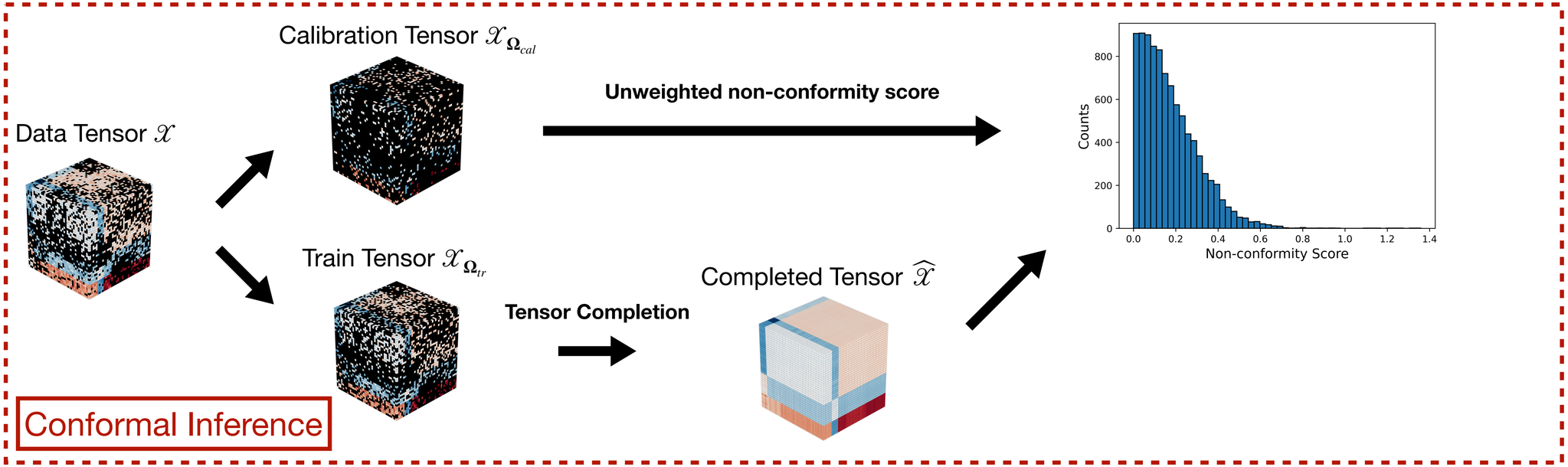
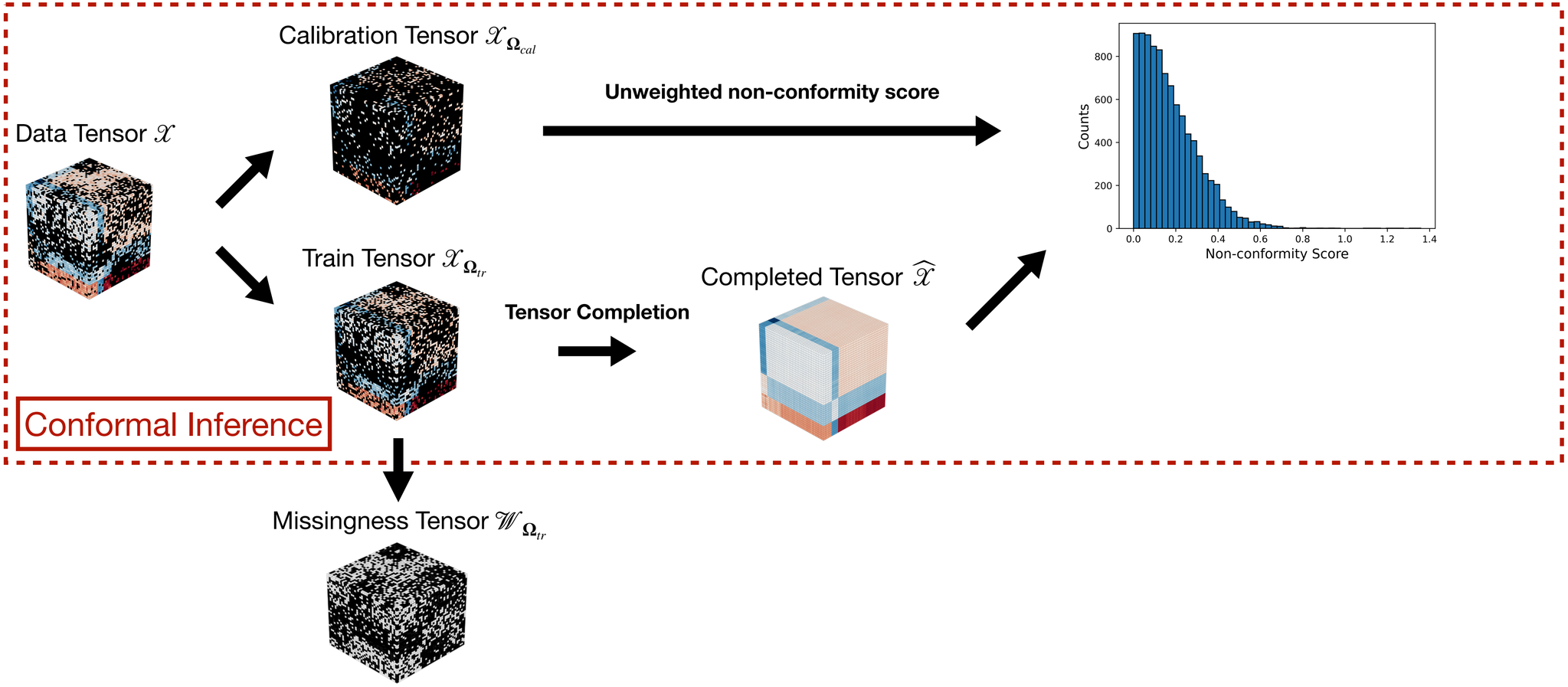



Calibration Tensor $\mathcal{X}_{\Omega_{cal}}$
Data Tensor $\mathcal{X}$
Train Tensor $\mathcal{X}_{\Omega_{tr}}$
Unweighted non-conformity score
Tensor Completion
Completed Tensor $\widehat{\mathcal{X}}$
Counts
Non-conformity Score
0
200
400
600
800
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Conformal Inference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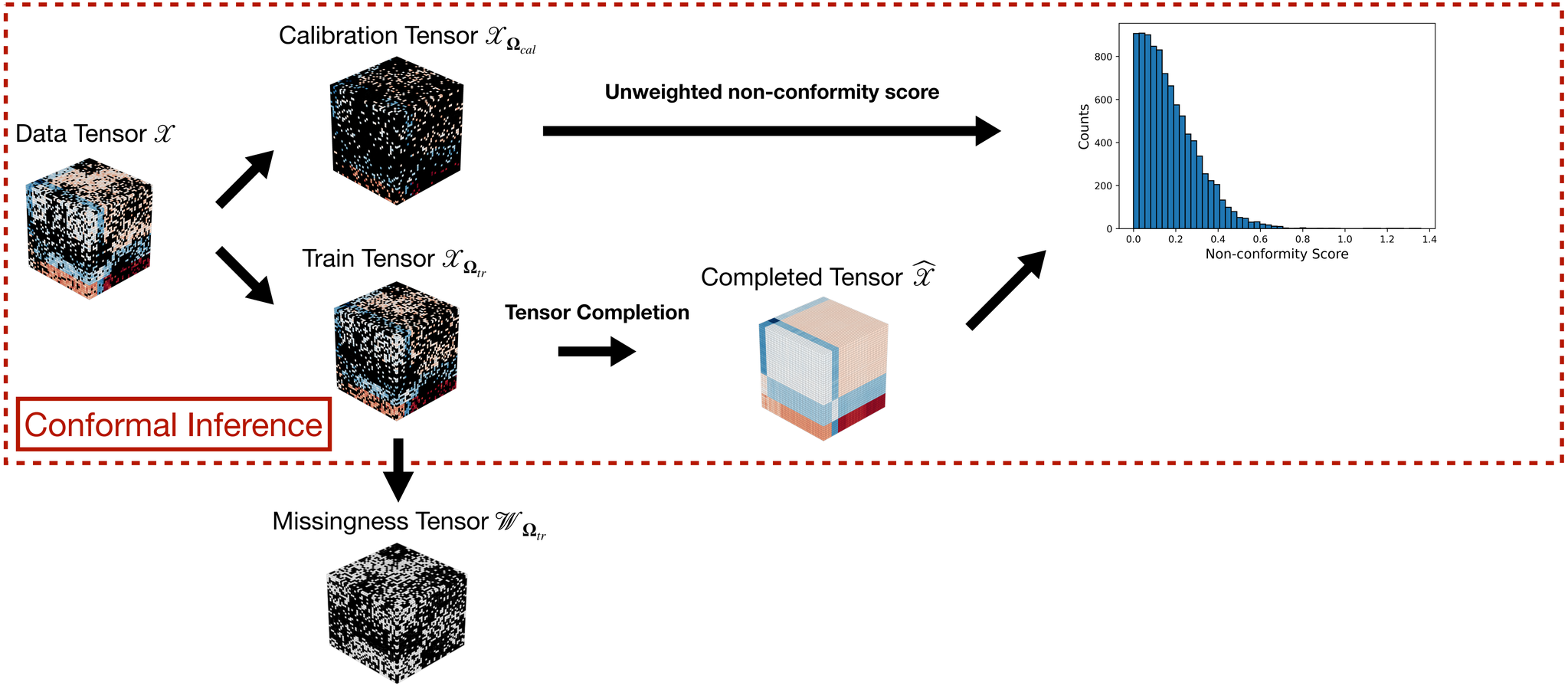

Low-rank Tensor Ising Model

$$p(\mathcal{W} \mid \mathcal{B}) \propto \exp\left[\frac{1}{2}\sum_{s_1 \sim s_2} g([\mathcal{B}]_{s_1}, [\mathcal{B}]_{s_2})[\mathcal{W}]_{s_1}[\mathcal{W}]_{s_2}\right]$$

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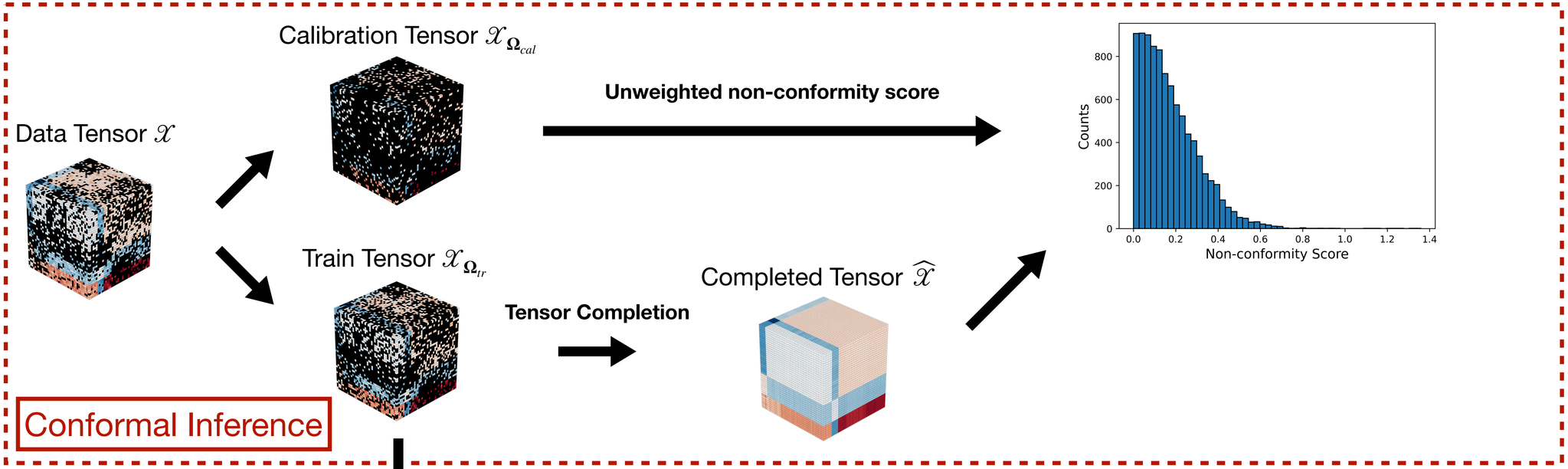

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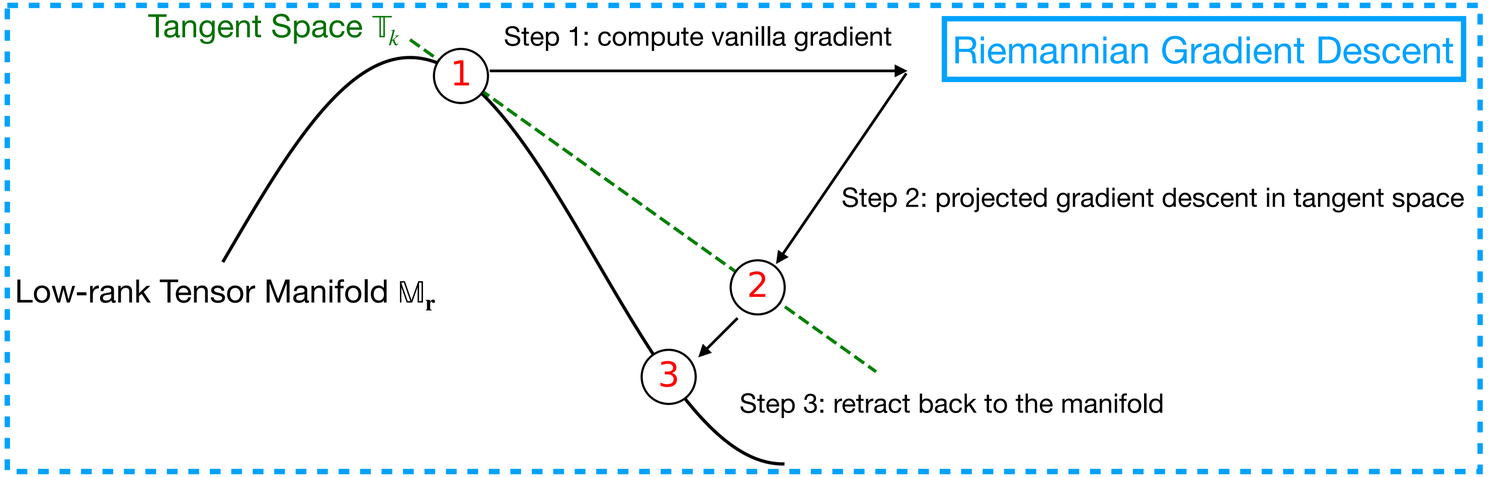

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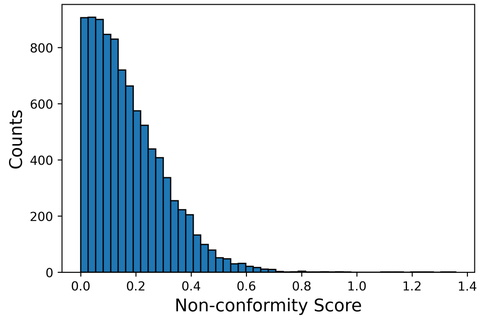

Train Tensor $\mathcal{X}_{\mathbf{\Omega}_{tr}}$

Tensor Completion

Completed Tensor $\widehat{\mathcal{X}}$

Conformal Inference

Missingness Tensor $\mathcal{W}_{\mathbf{\Omega}_{tr}}$

MPLE Estimator $\widehat{\mathcal{B}}$

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Riemannian Gradient Descent

Tangent Space $\mathbb{T}_k$

Step 1: compute vanilla gradient

Step 2: projected gradient descent in tangent space

Step 3: retract back to the manifold

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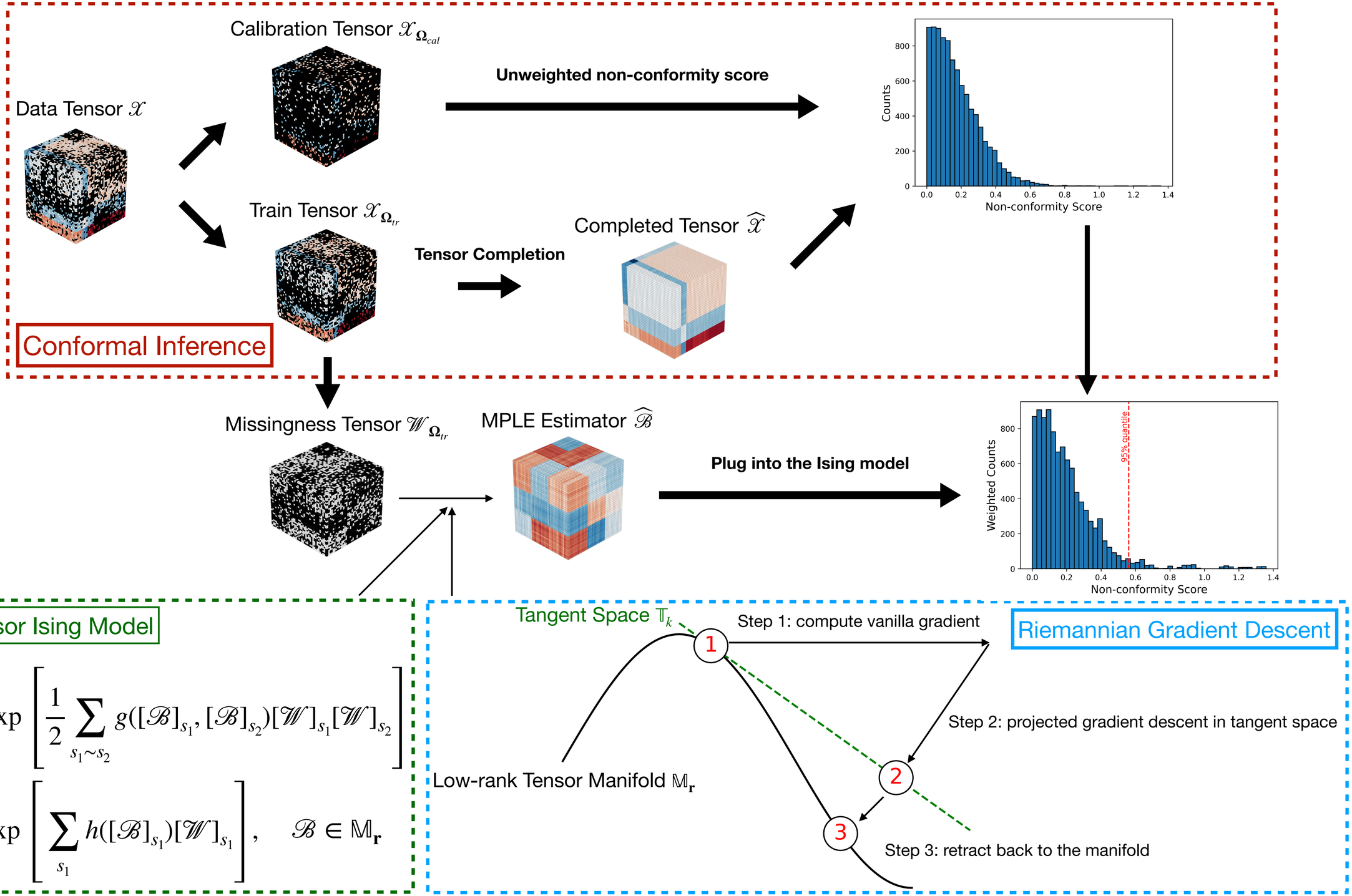
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Theoretical Results

MPLE error bound

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where $r^*, \bar{d}, d^*$ are $\prod_k r_k, \sum_k d_k, \prod_k d_k$ for $\mathcal{B}^* \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ with tensor-train rank $\mathbf{r} = (r_1, \ldots, r_{K-1})$.

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$\asymp (r/d)^{(K-1)/2}$, if $r_k \asymp r, d_k \asymp d, \forall k$

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Theorem 3 Assume the same assumptions as Theorem 2, then for any $0 < c < 1$:

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$$\mathrm{E}\left[\frac{1}{|\boldsymbol{\Omega}^c|}\sum_{s\in\boldsymbol{\Omega}^c}\mathbb{I}_{\{[\mathcal{X}]_s\in\widehat{C}_{1-\alpha,s}\}}\right] \geq 1-\alpha-\frac{2c_{K,\xi}}{(1-c)(1-q)}\sqrt{\frac{r^*\bar{d}}{d^*}} - \exp[-C_1\bar{d}\log K]-\exp\left[-\frac{c^2(1-q)d^*l_\xi}{2}\right]$$

where $q \in (0,1)$ is the train-calibration split ratio .

Theoretical Results

Coverage Guarantee

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average coverage on $\boldsymbol{\Omega}^c$

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average coverage on $\boldsymbol{\Omega}^c$ ↑ target coverage ↑

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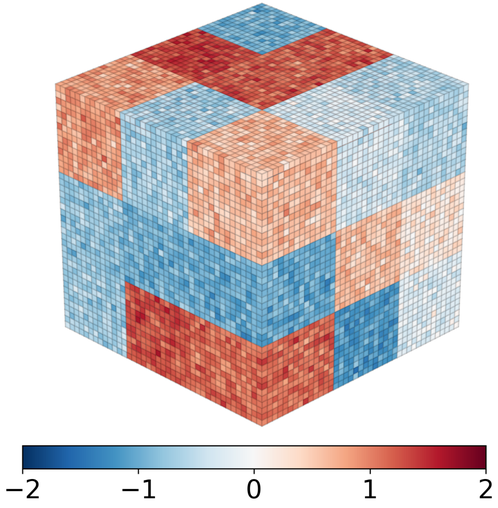
sample size requirement for calibration

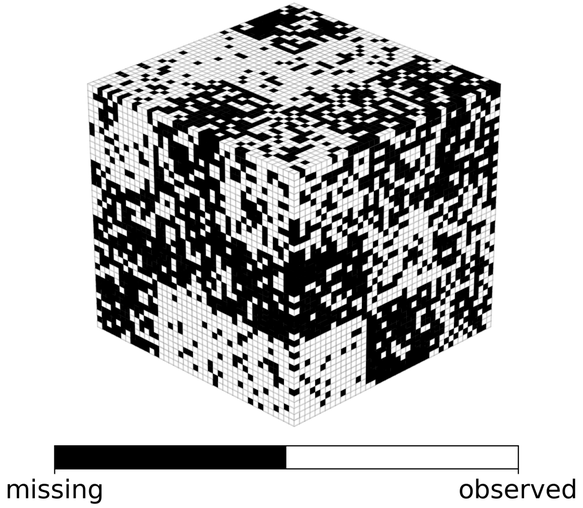
Numerical Experiments

Setup

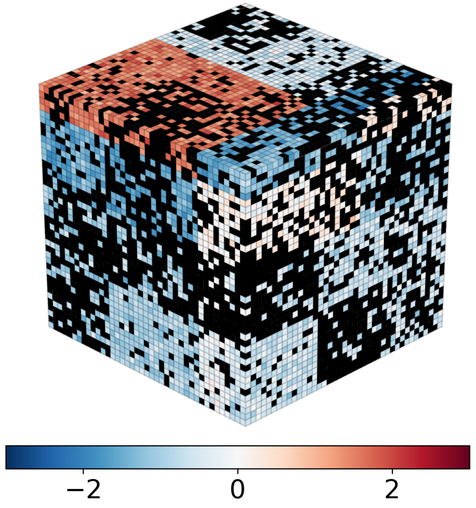
(a) Ground Truth $\mathcal{B}^*$



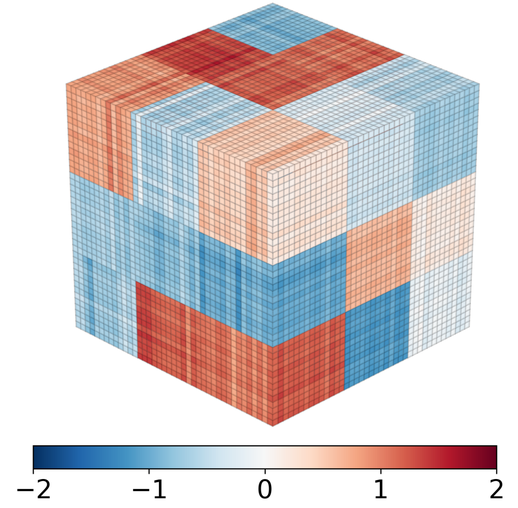
(b) Simulated Missingness $\mathcal{W}$



(c) Simulated Data Tensor $\mathcal{X}$



(d) Fitted $\widehat{\mathcal{B}}$ by RGrad



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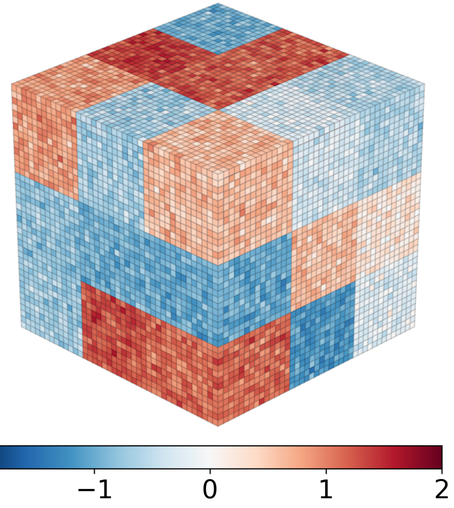

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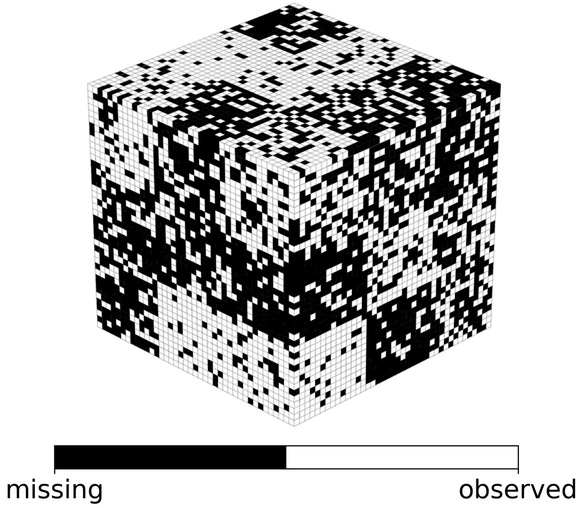

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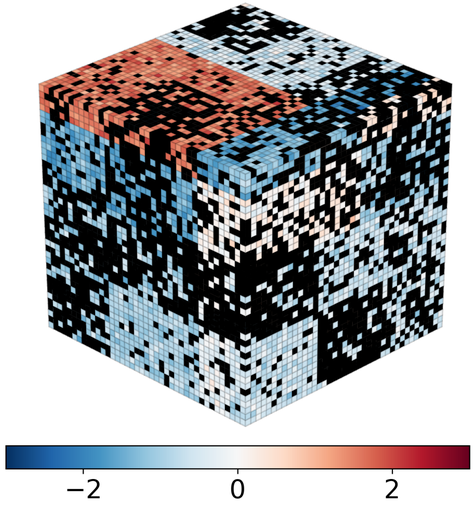

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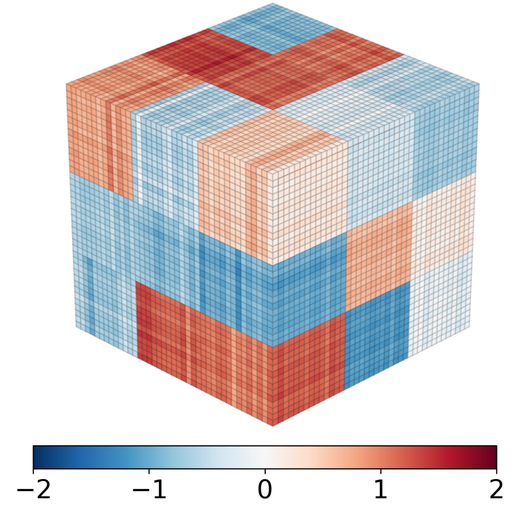

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Numerical Experiments

Setup

Simulated by MCMC with block-Gibbs sampler

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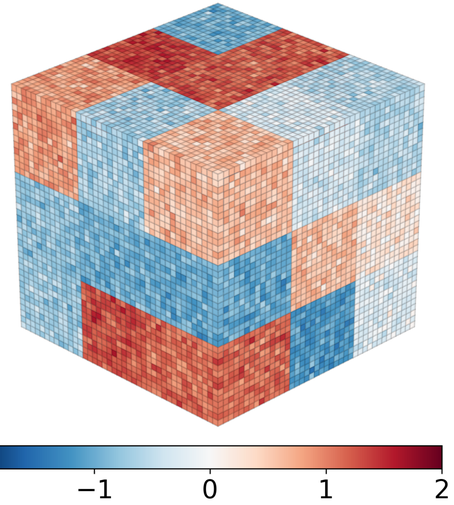

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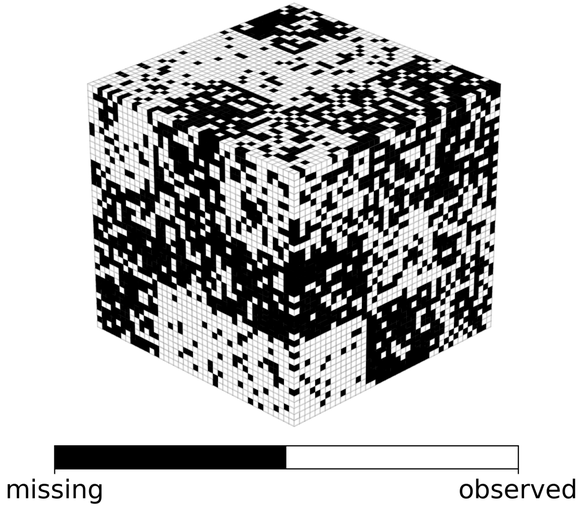

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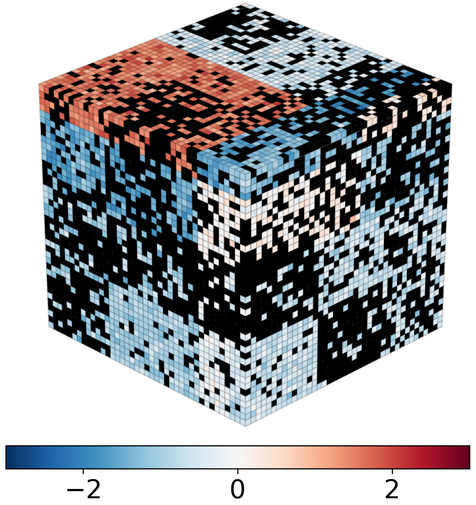

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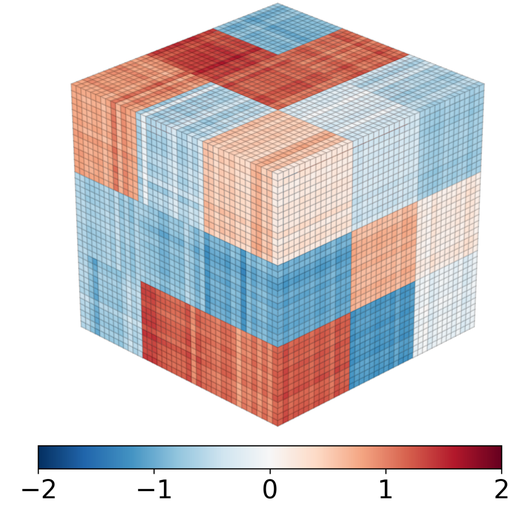

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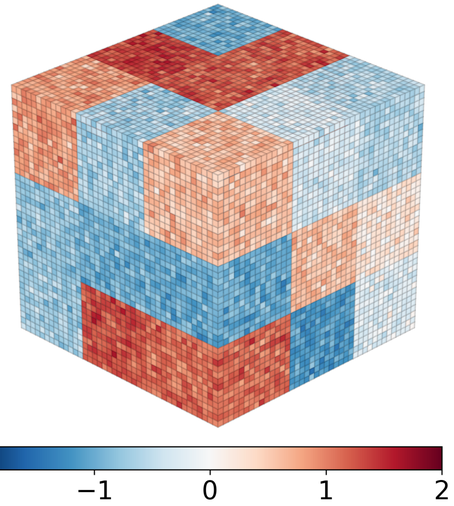

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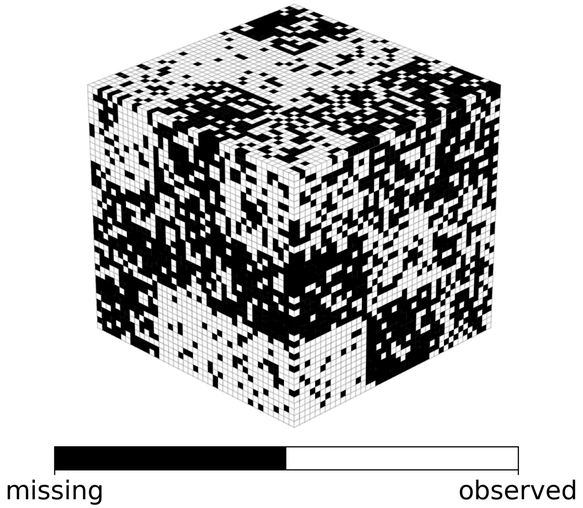

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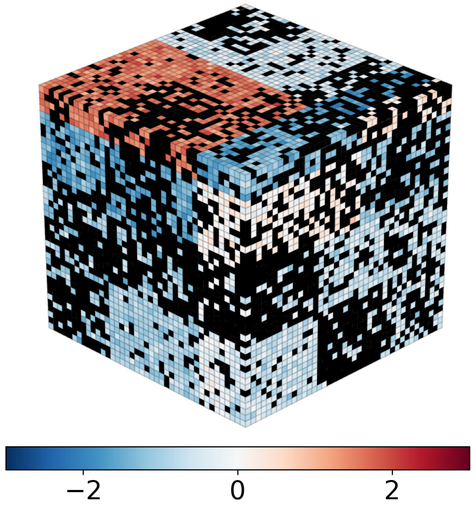

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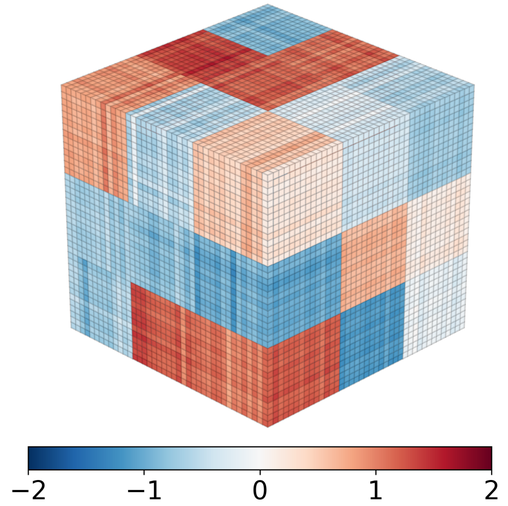

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$\mathcal{X} = \mathcal{X}^* + \mathcal{E}, \mathcal{X}^*$ **is low rank**

Numerical Experiments

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Numerical Experiments

Evaluation Metrics & Benchmarks

Numerical Experiments

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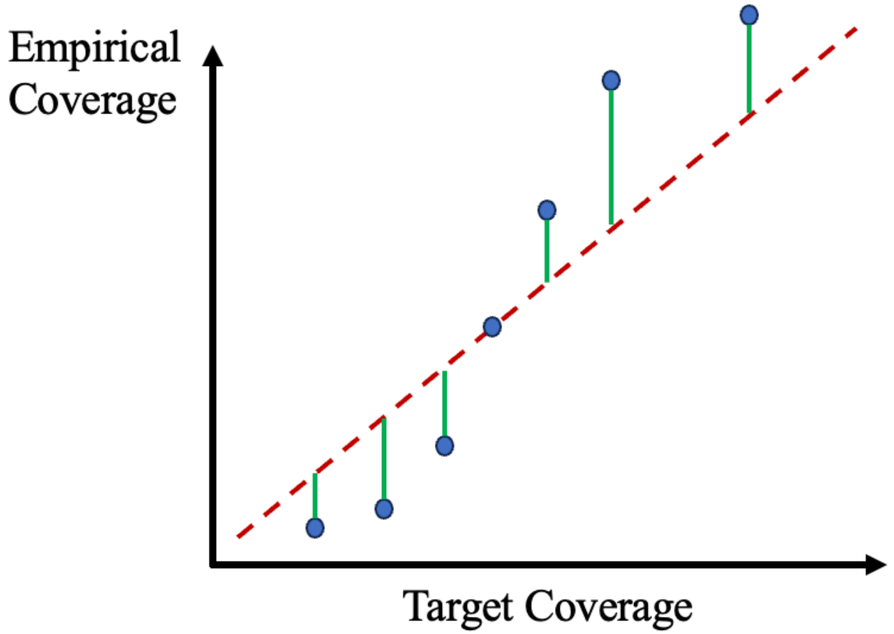

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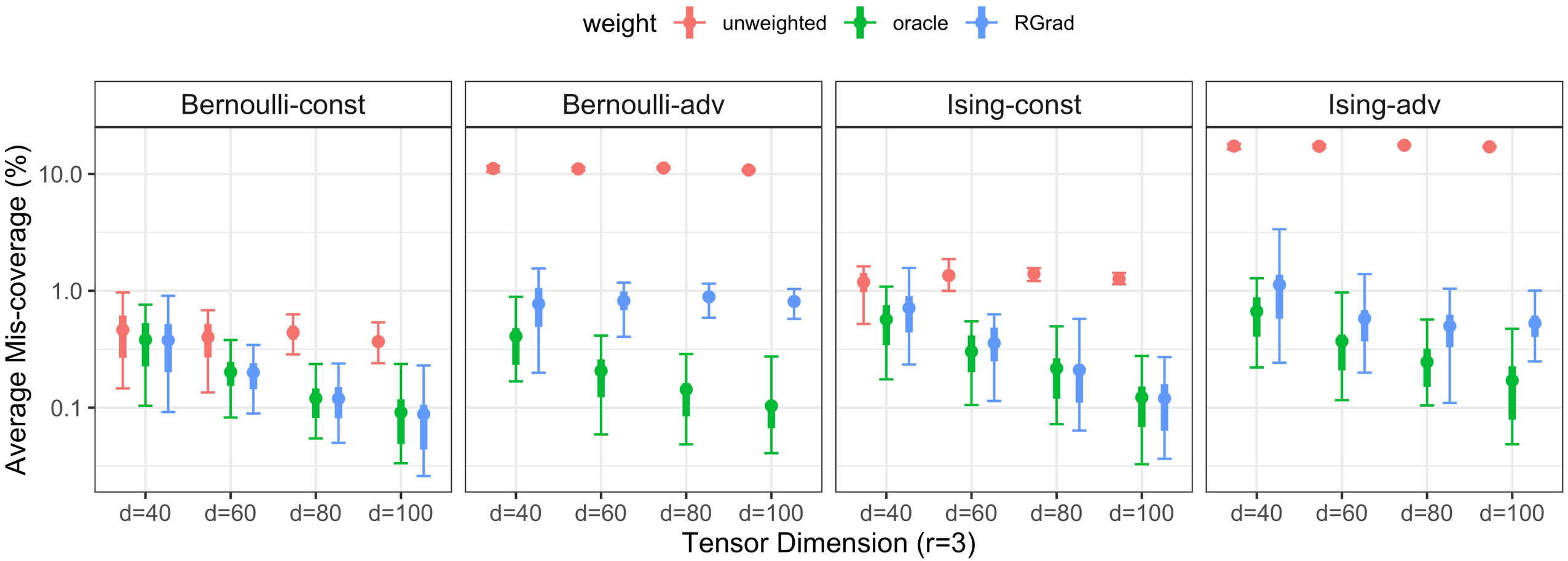
Numerical Experiments

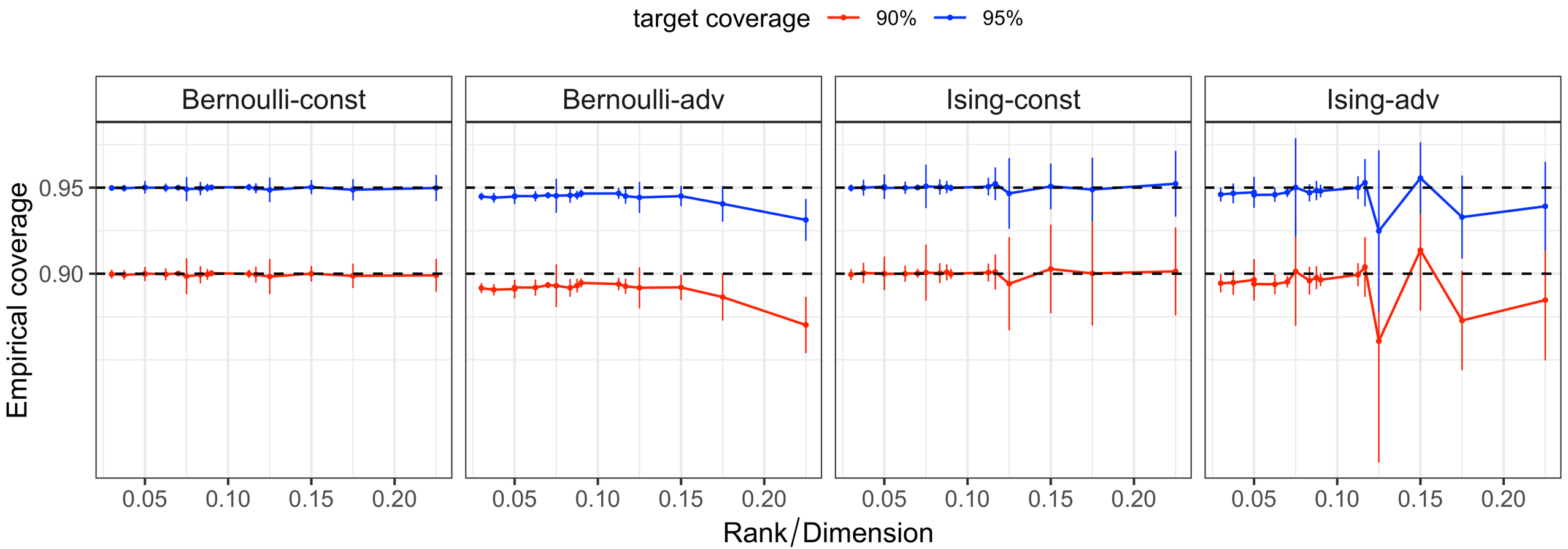
Conformal Inference Validation (rank($\mathcal{B}^*$) = (3,3))



const: $[\mathcal{E}]_s \overset{i.i.d.}{\sim} N(0,\sigma^2)$; adv: $[\mathcal{E}]_s \overset{ind.}{\sim} N(0,\sigma_s^2)$, $\sigma_s \propto [\mathcal{B}]_s^{-1}$ (higher uncertainty for missing entries)

Numerical Experiments

Conformal Inference Validation (90%, 95% CI)



x-axis: r/d of the tensor parameter $\mathcal{B}^*$

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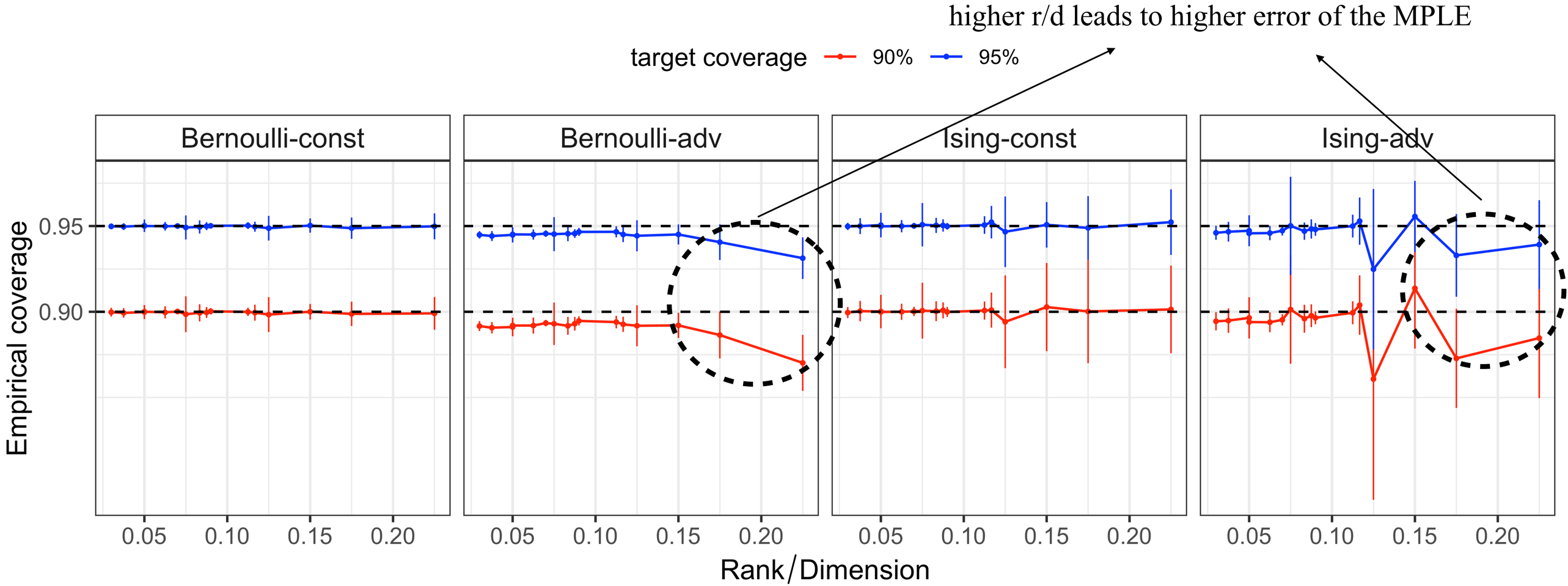

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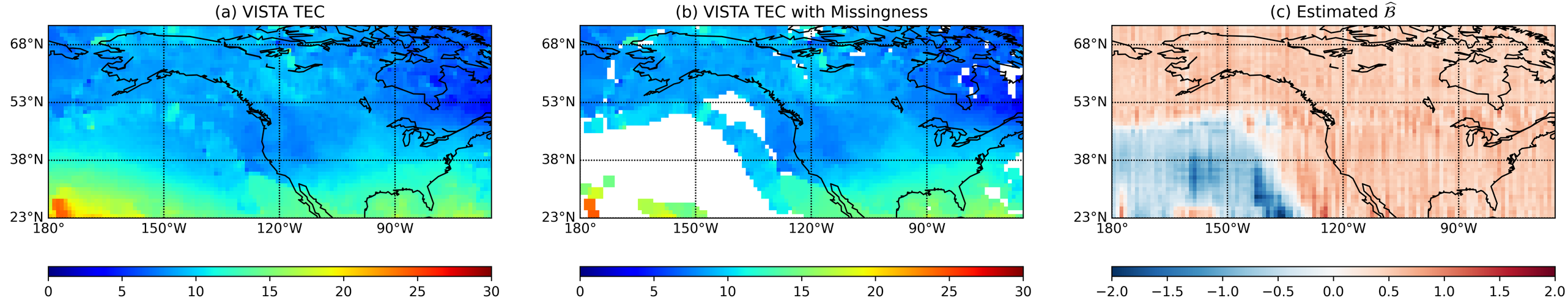
Data Application

Regional TEC Reconstruction

- We apply our method to reconstruct the total electron content (TEC) over the US and Canada for 15 days in Sept, 2017.



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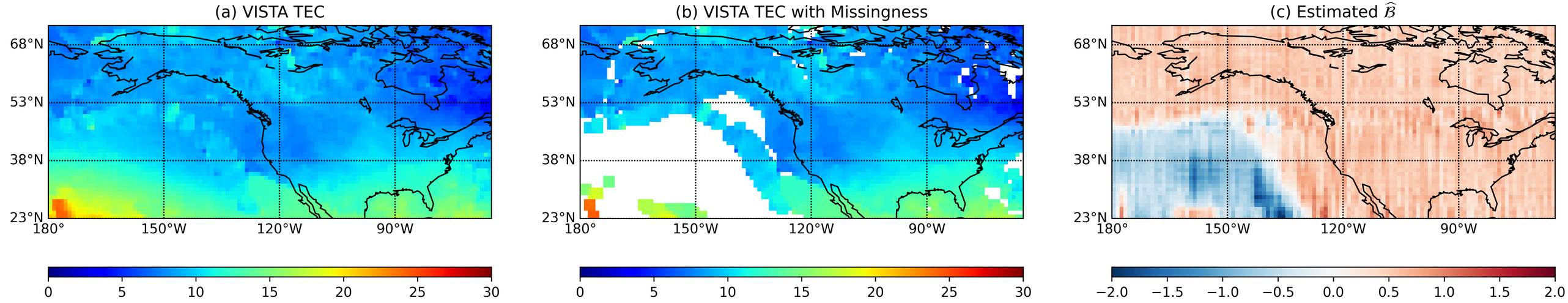

data without missingness

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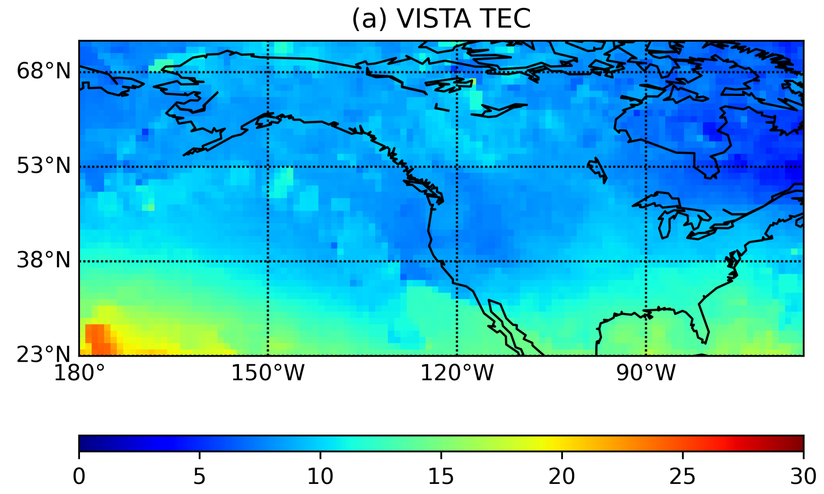

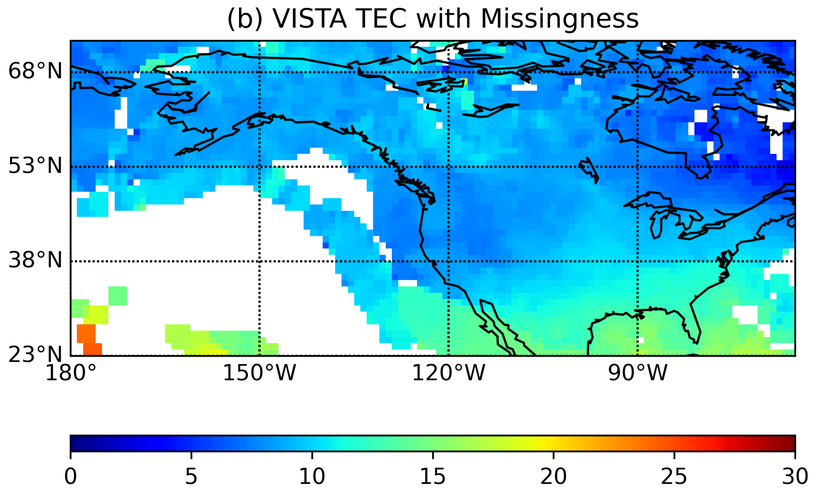

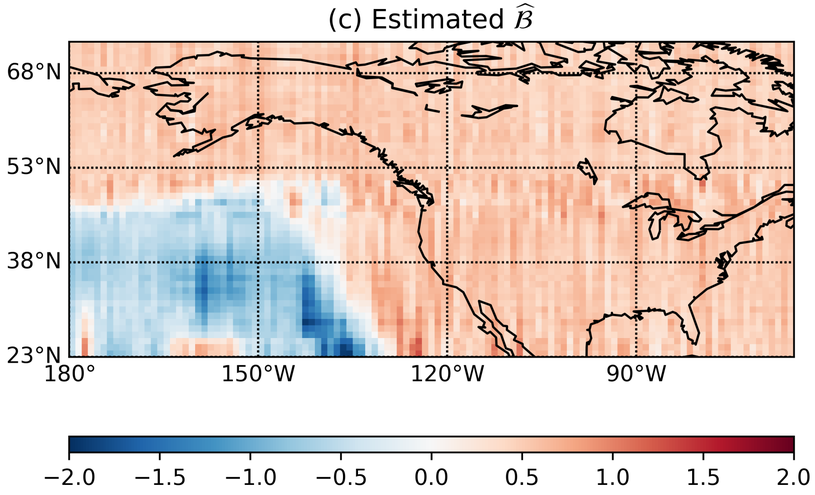

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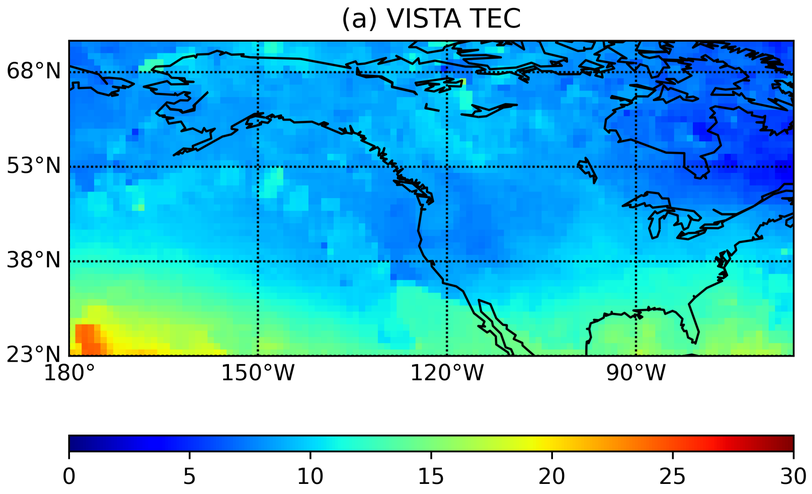

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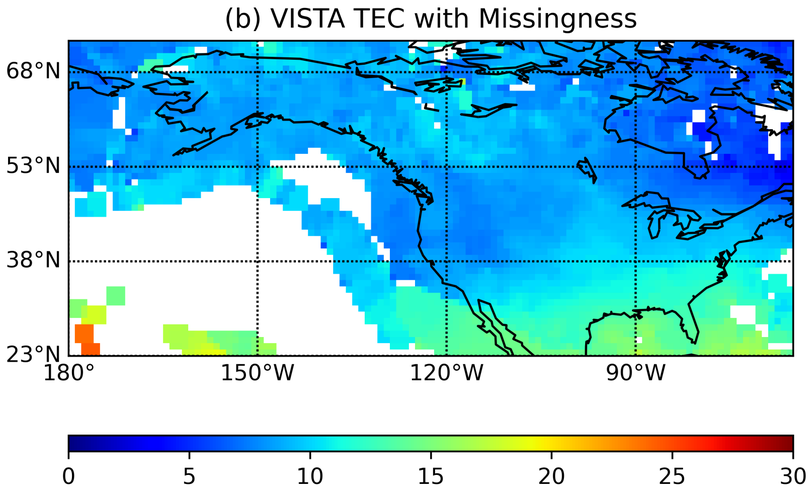

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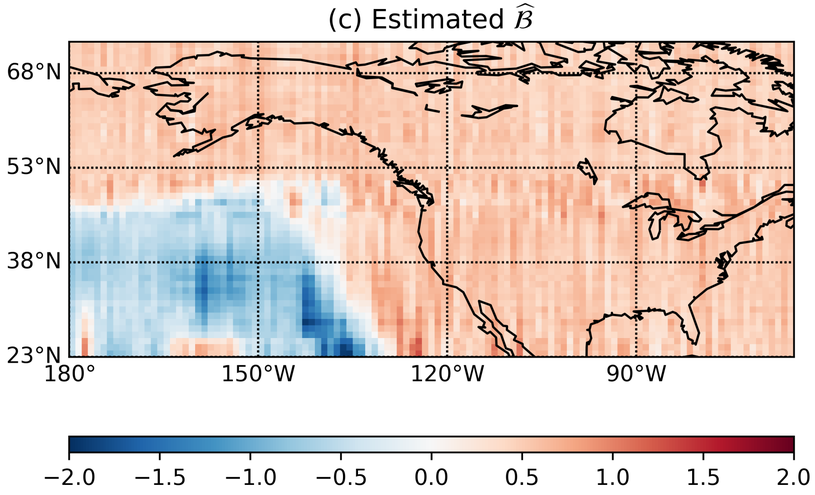

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Coverage Performance

- We **compare** our method under both Ising model and Bernoulli model with the unweighted conformal prediction:

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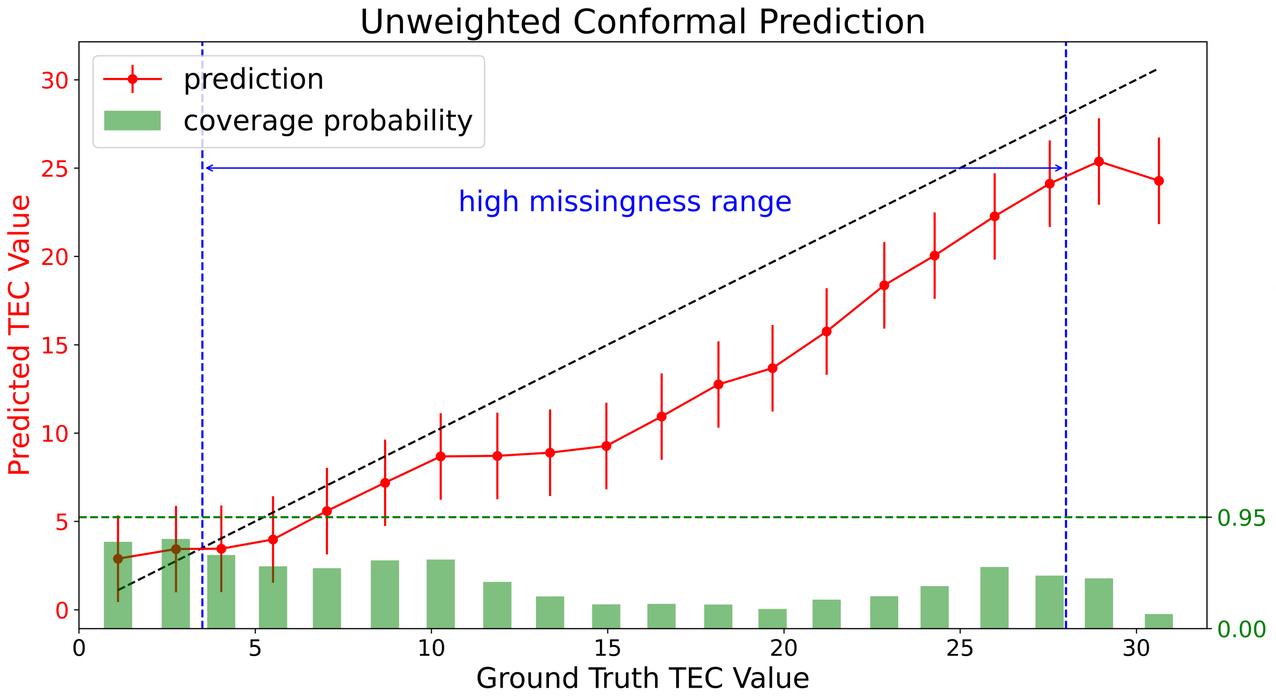
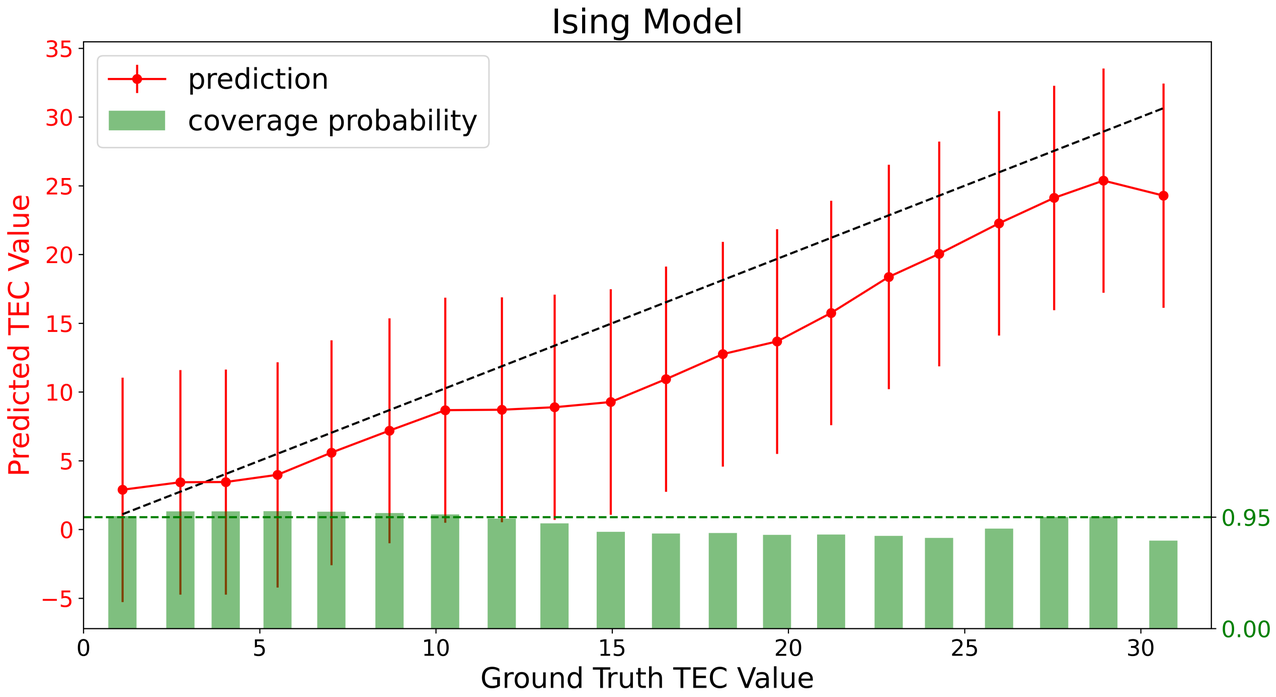
method	mis-coverage %	90% CI coverage %	95% CI coverage %
unweighted	42.1(6.49)	46.3(6.58)	52.3(7.23)
Bernoulli	23.1(5.34)	64.6(5.97)	76.8(5.03)
Ising	6.01(2.45)	90.0(6.06)	94.2(3.74)

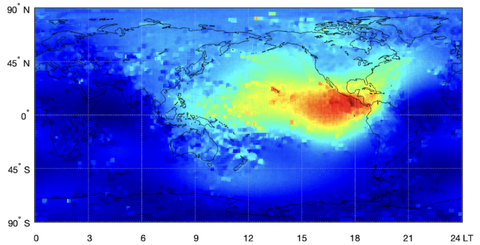
Data Application

A Single Day Example (Sept 6, 2017)

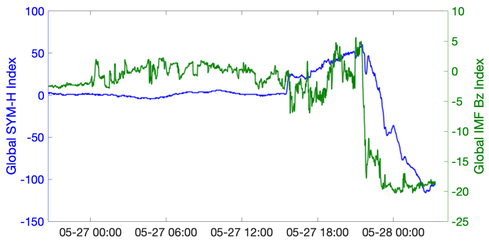




III. Matrix Autoregression with Auxiliary Data



+

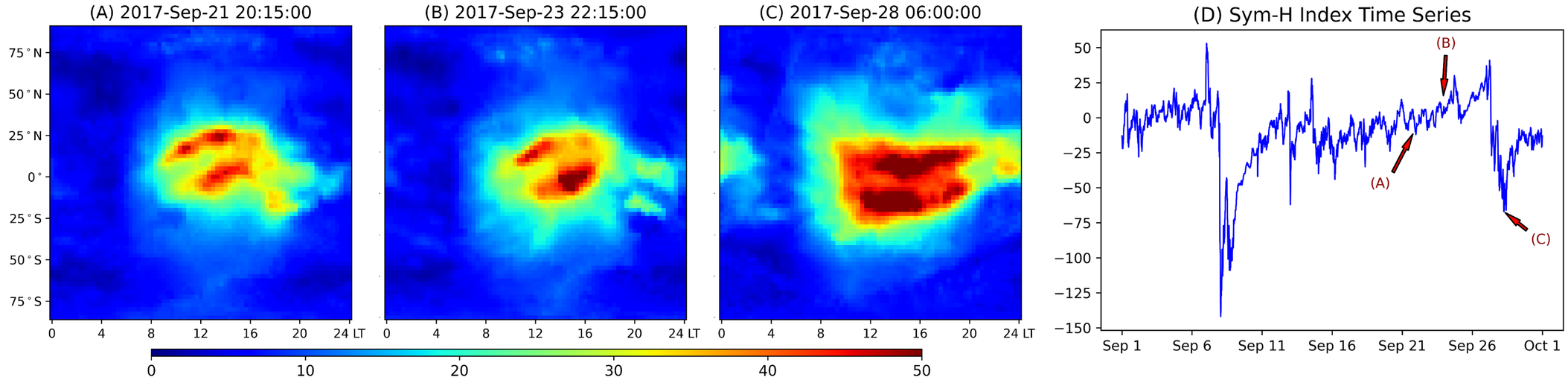


Spatial Matrix Time Series **Non-spatial Vector Time Series**

How to forecast future matrix data?

Motivating Example

Global TEC Forecast with Solar Wind Parameters



- Left 3 panels: Global TEC, measurement of the Earth's ionospheric activity
- Right panel: Sym-H index, measurement of the impact of solar eruption on Earth

Our Model

Matrix AutoRegression with Auxiliary Covariates (MARAC)

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- Denote $\{\mathbf{X}_t\}_{t=1}^T$ as the matrix time series, and $\{\mathbf{z}_t\}_{t=1}^T$ as the vector time series, then our MARAC(P,Q) model is:

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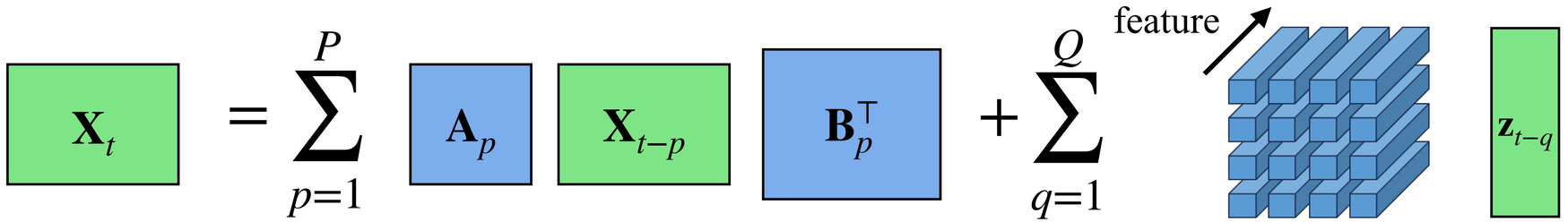

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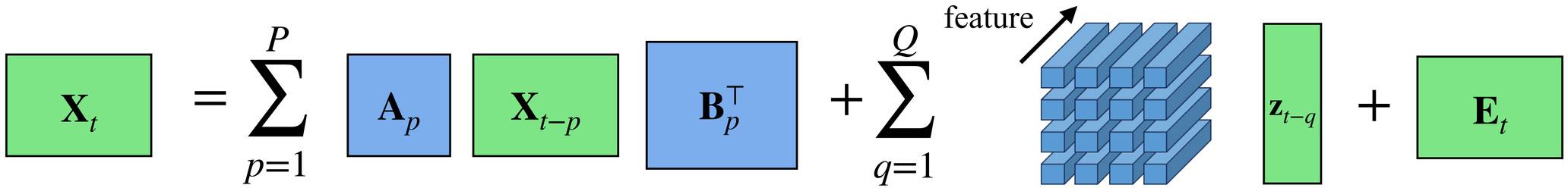
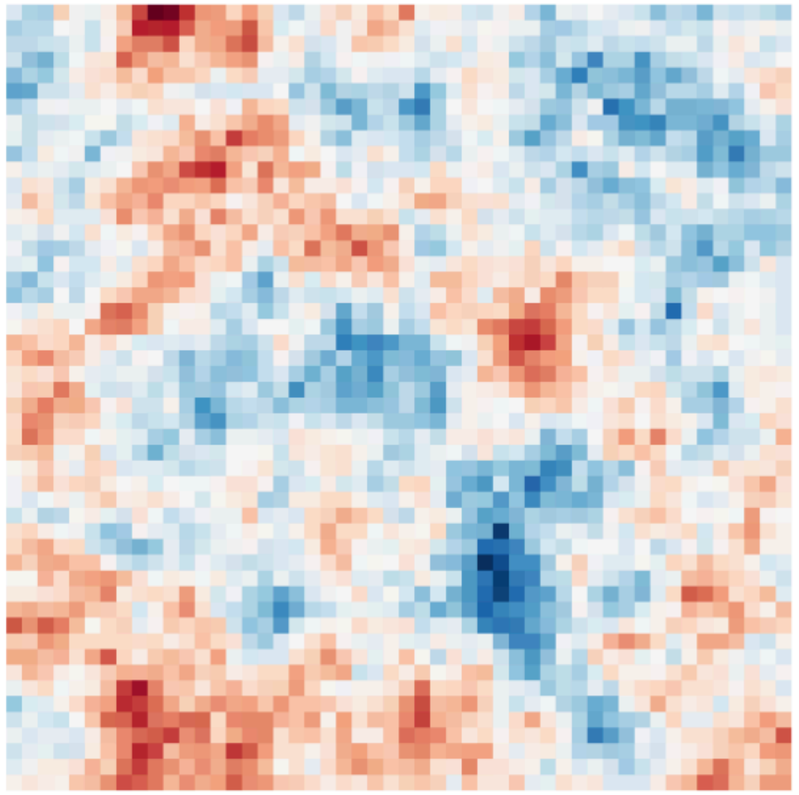


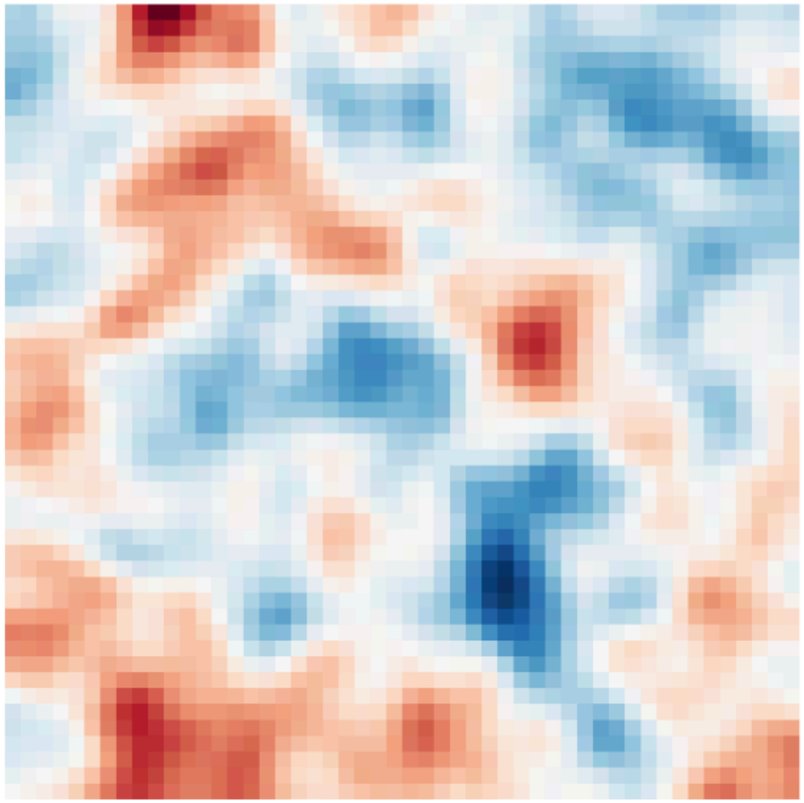
Model

Spatially-Smooth Regression Coefficient

Slice of Tensor Coefficient $\mathcal{G}_q(:,:,d)$

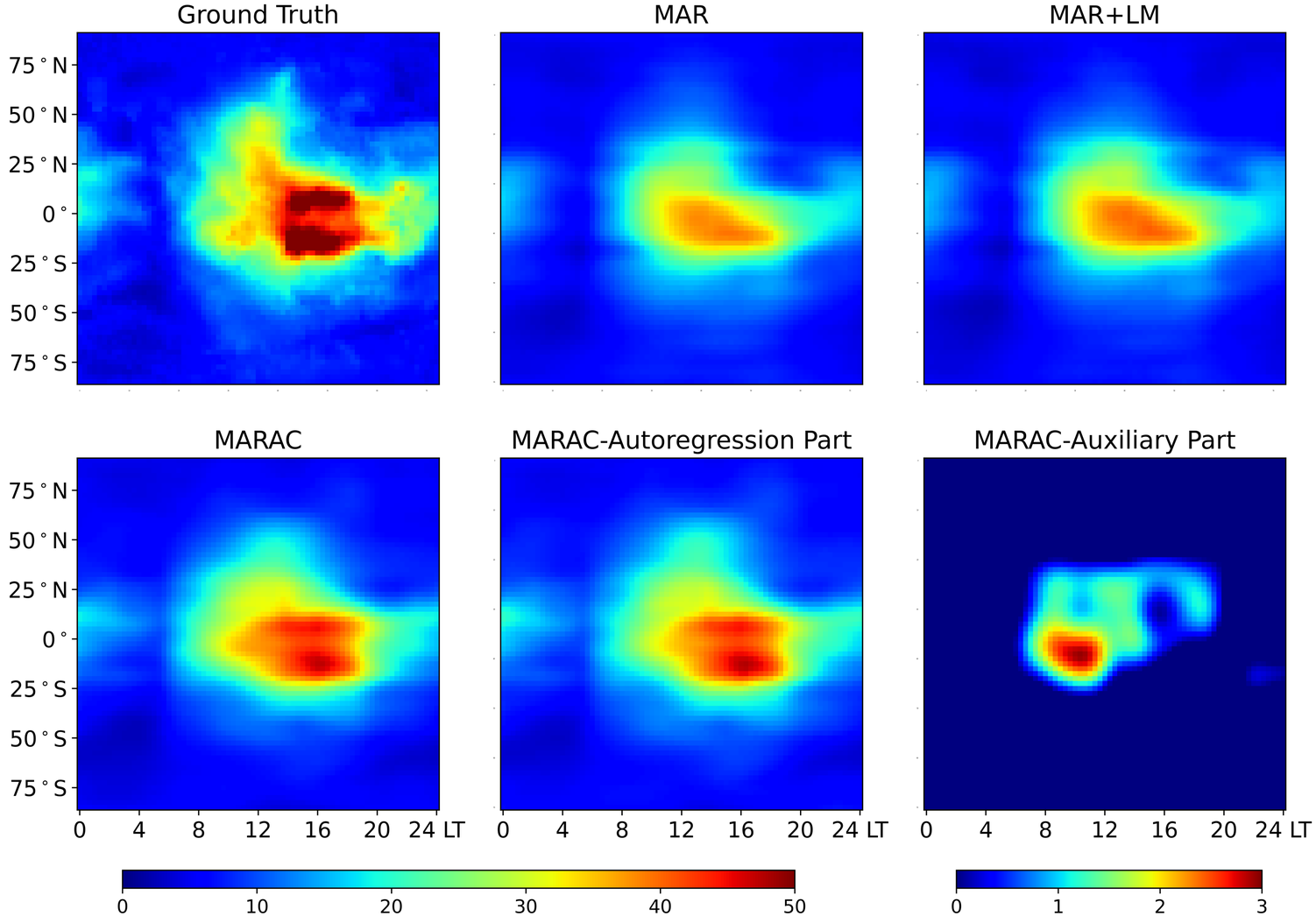


True Functional Parameter $g_{q,d}(\cdot)$



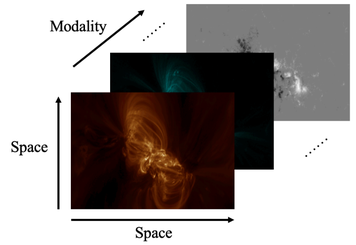
Real Data Application

Global Total Electron Content (TEC) Prediction



IV. Scalar-on-Tensor Gaussian Process with Contraction

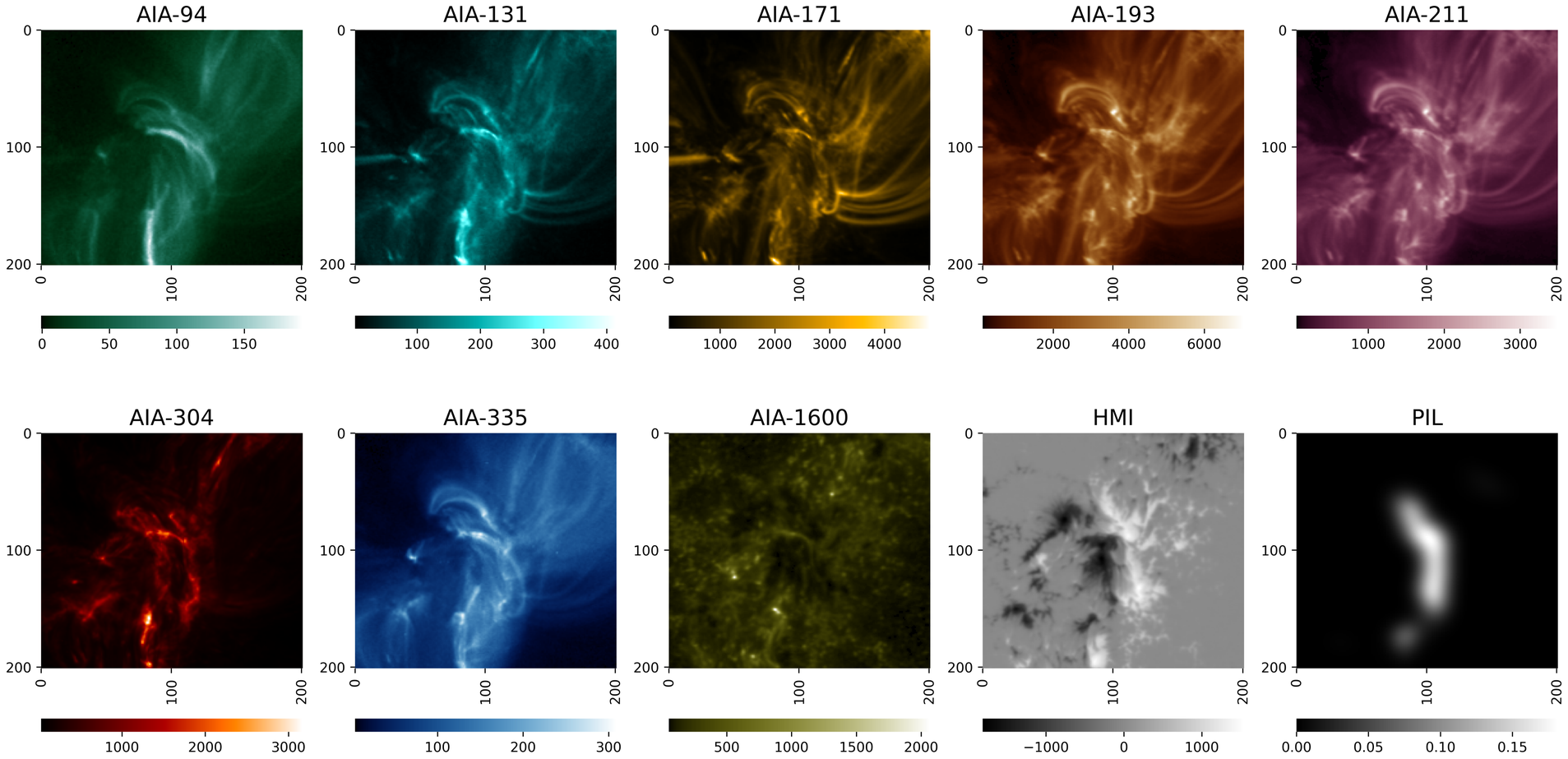
$$y = f\Big(\quad \Big) + \epsilon$$



scalar label **multi-channel imaging covariate**

Motivation

Covariate: multi-modal solar imaging data



Proposed Method

Tensor Gaussian Process with Contraction

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Let $y \in \mathbb{R}$ be the scalar label and $\mathcal{X} \in \mathbb{R}^{H \times W \times C}$ be the tensor covariate, then we have:

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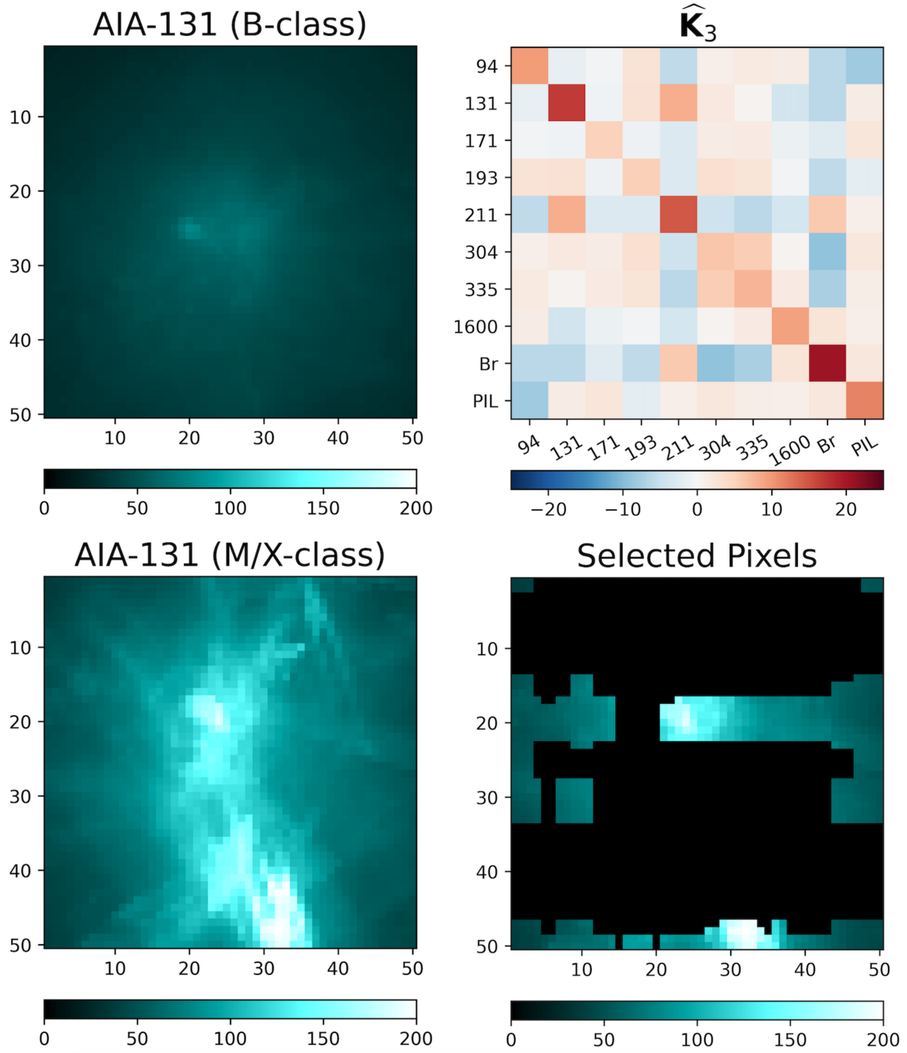
$$k(\mathcal{Z}_1, \mathcal{Z}_2) = \text{vec}\left(\mathcal{Z}_1\right)^\top \left[\bigotimes_{j=1}^{3} \mathbf{K}_j\right] \text{vec}\left(\mathcal{Z}_2\right)$$

Real Data Application

Solar Flare Intensity Forecast



Modality pairwise interaction

Feature Selection

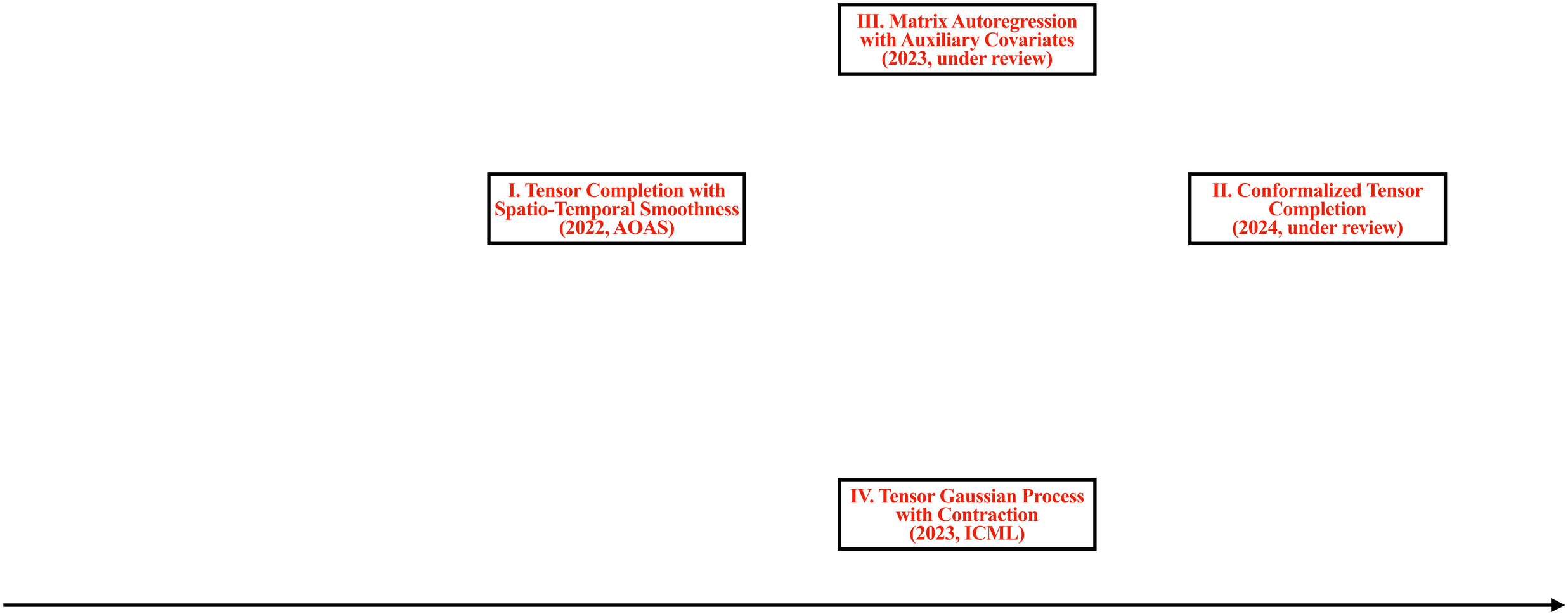
Final Remarks

Summary

Roadmaps of all works

Summary

Roadmaps of all works



Summary

Roadmaps of all works

III. Matrix Autoregression with Auxiliary Covariates (2023, under review)

I. Tensor Completion with Spatio-Temporal Smoothness (2022, AOAS)

II. Conformalized Tensor Completion (2024, under review)

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Solar Flare Forecast with Spatial Statistics (2021, Space Weather)

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Time

Summary

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III. Matrix Autoregression with Auxiliary Covariates (2023, under review)

I. Tensor Completion with Spatio-Temporal Smoothness (2022, AOAS) → Open-Source Global TEC Database (2023, Scientific Data)

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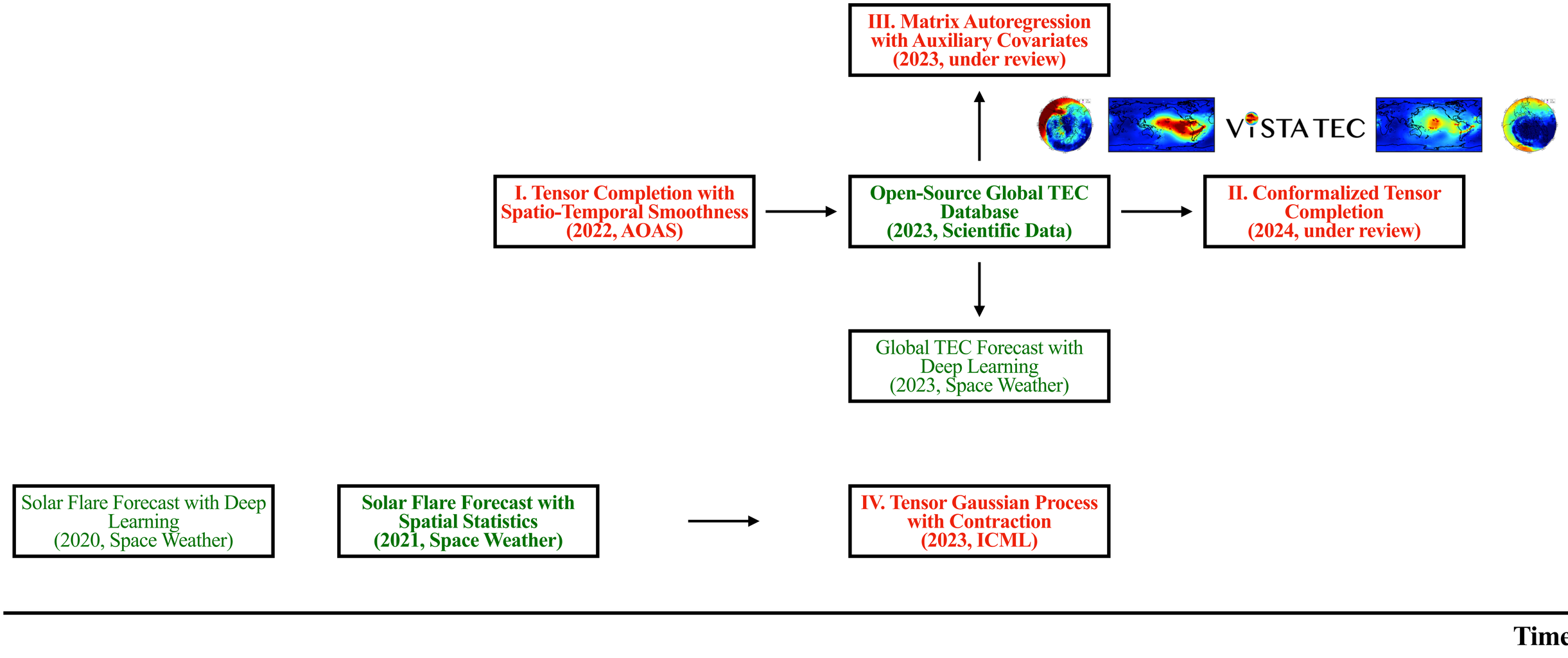
Summary

Roadmaps of all works



Future Directions

Future Directions

Theory

Future Directions

Theory

- Concentration bounds of locally-dependent tensors

Future Directions

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- Concentration bounds of locally-dependent tensors

Methodology

Future Directions

Theory

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Computation

Future Directions

Theory

- Concentration bounds of locally-dependent tensors

Methodology

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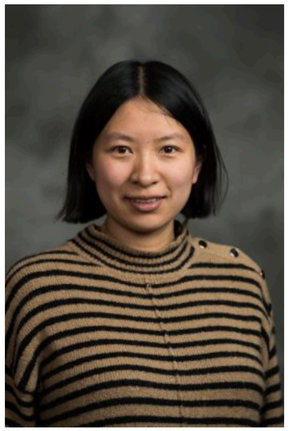
Computation

- Efficient hyper-parameter tuning

Acknowledgements

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- **Outside of UM**: Dr. Meng Jin, Dr. Yang Liu, Dr. Anthea Coster, Dr. Monica Bobra

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