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CRES Assignment 3

a) $$\frac{du}{dt} = \frac{1}{C} \cdot (-i_c) \quad \text{with} \quad i_c = \frac{u_R}{R} = \frac{V_c(0)}{R}$$

$$\frac{dv}{dt} = \underbrace{\left(-\frac{1}{RC}\right)}_{\lambda} V_c(0)$$

Direct Solution $\Rightarrow$ $\dot{v} = \lambda v \Rightarrow \boxed{V_c(t) = e^{-\frac{1}{RC}} \cdot V_c(0)}$

b) Backward-Euler Solution:

$$\frac{x(k) - x(k-1)}{\tau} = \lambda x(k)$$

$$x(k) - x(k-1) = \lambda \tau \, x(k)$$

$$x(k) = \frac{x(k-1)}{1 - \lambda \tau}$$

$$\Rightarrow x(n) = \frac{x(0)}{(1-\lambda\tau)^n} \quad \text{with} \quad n = \frac{t_{max}}{\tau}$$

$$\Rightarrow V_{c_{BE}}(n) = \frac{V_c(0)}{\left(1 + \frac{1}{RC}\tau\right)^n}$$

Chosen step size:
- tau = RC * 0.9. About 10% smaller than the forward-Euler stability threshold.

Reason:
- Backward Euler doesn't have maximum step size, so it is relatively acceptable to choose any step size
- If it is higher than the threshold, c) would not be plotted correctly

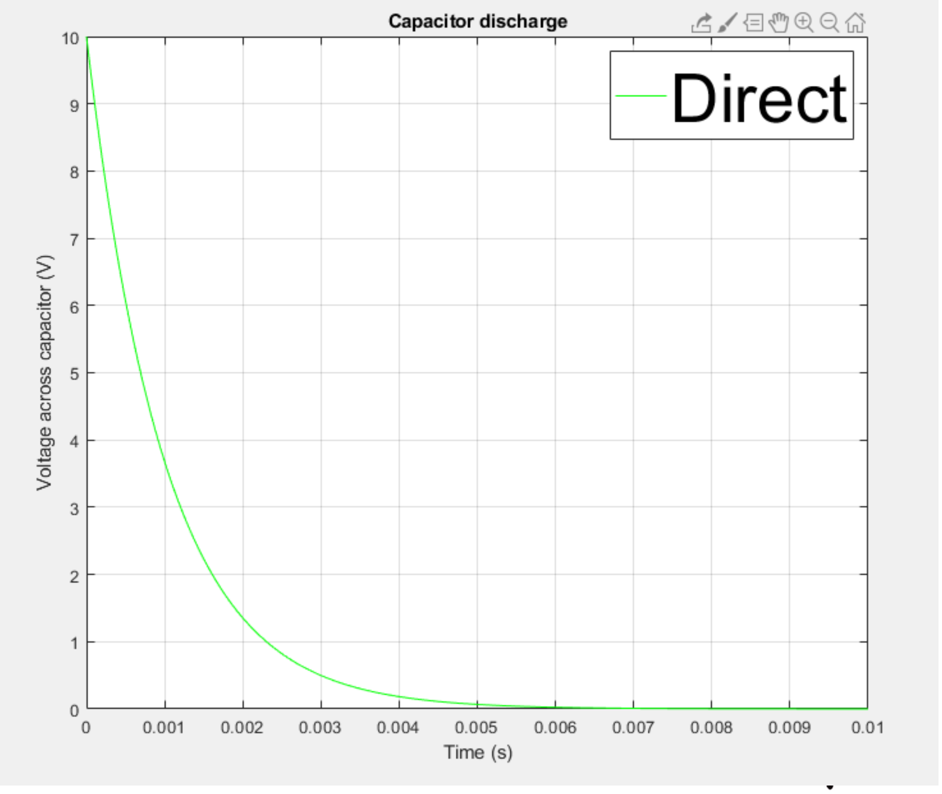

Figure 1: Curve of capacitor discharge using direct solution

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c) Forward - Euler Solution :

$$\frac{x(k) - x(k-1)}{\tau} = \lambda \, x(k-1)$$

$$\vdots$$

$$x(n) = (1 + \lambda\tau)^n \, x(0)$$

$$\Rightarrow \quad V_{CFE}(n) = \left(1 - \frac{1}{RC}\tau\right)^n \cdot V_C(0)$$

Trapezoidal :

$$\frac{x(k) - x(k-1)}{\tau} = \lambda \, \frac{x(k) + x(k-1)}{2}$$

$$\vdots$$

$$x(n) = \left(\frac{1 + \frac{\lambda\tau}{2}}{1 - \frac{\lambda\tau}{2}}\right)^n \cdot x(0)$$

$$\Rightarrow \quad V_{Ctr}(n) = \left(\frac{1 - \frac{\tau}{2RC}}{1 + \frac{\tau}{2RC}}\right)^n V_C(0)$$

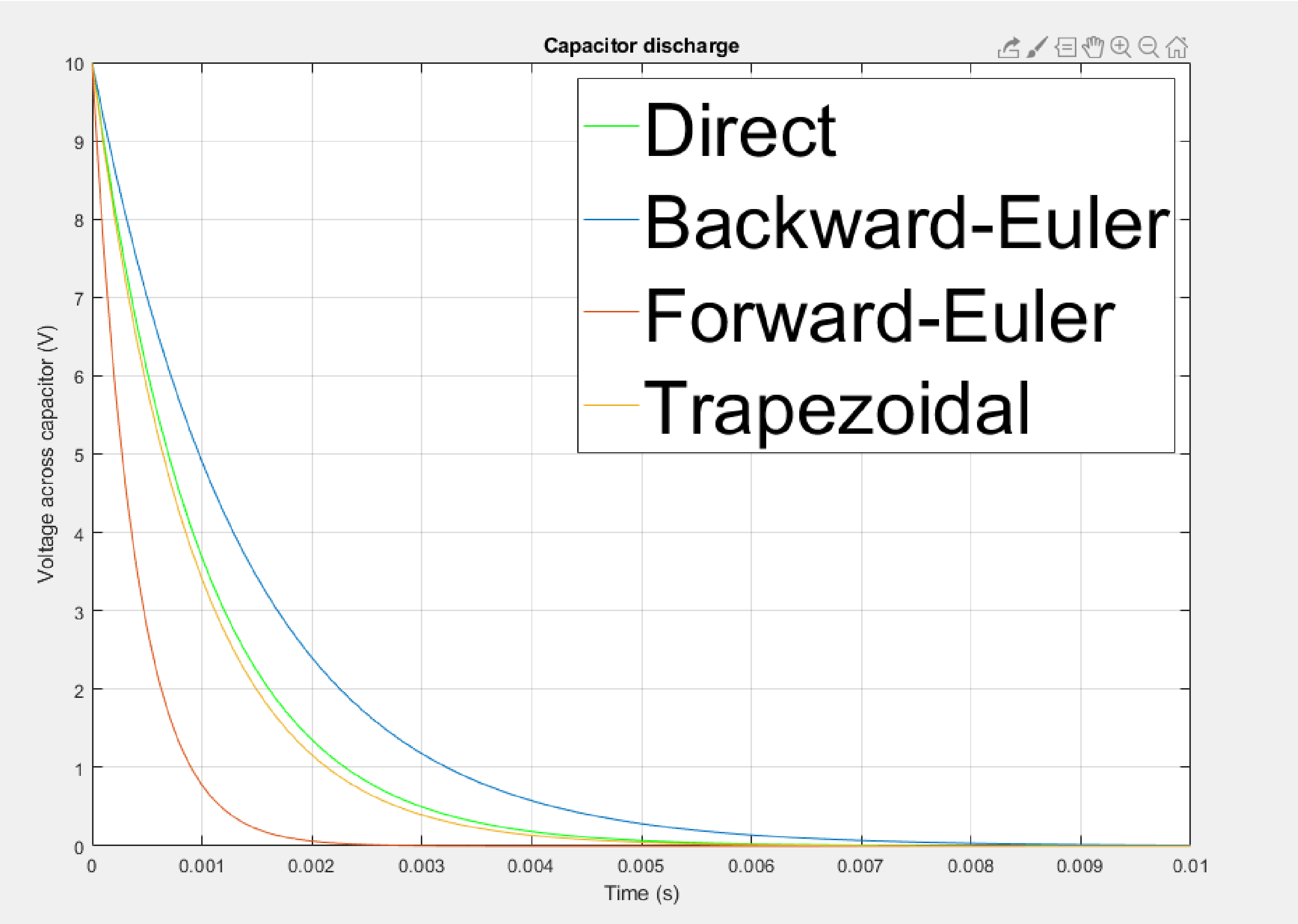

Figure 2: Curve of capacitor discharge using different methods (tau = RC * 0.9)

d)

Figure 3 shows four curves using different methods: Direct solution, backward-euler, forward-euler and trapezoidal method. As expected, the Trapezoidal method gives the most accurate result in comparison to both backward- and forward-Euler. The accuracy of backward- and forward-Euler method is relatively similar. The derivation of backward Euler and forward-Euler in Problem Sheet 1 and Chapter 2 shows that the real part of the frequency response is the same magnitude, but with different signs. This is difference of sign is also shown in the figure, backward being displaced upward and forward being displaced to the bottom. On higher step size, the forward Euler will be near to its stability threshold. In this figure, tau = 0.0005 is small enough to give similar result with backward Euler but also large enough to see the difference between trapezoidal and both backward/forward-Euler. In comparison, Figure 2 shows a less accurate forward-Euler due to tau nearing the threshold.

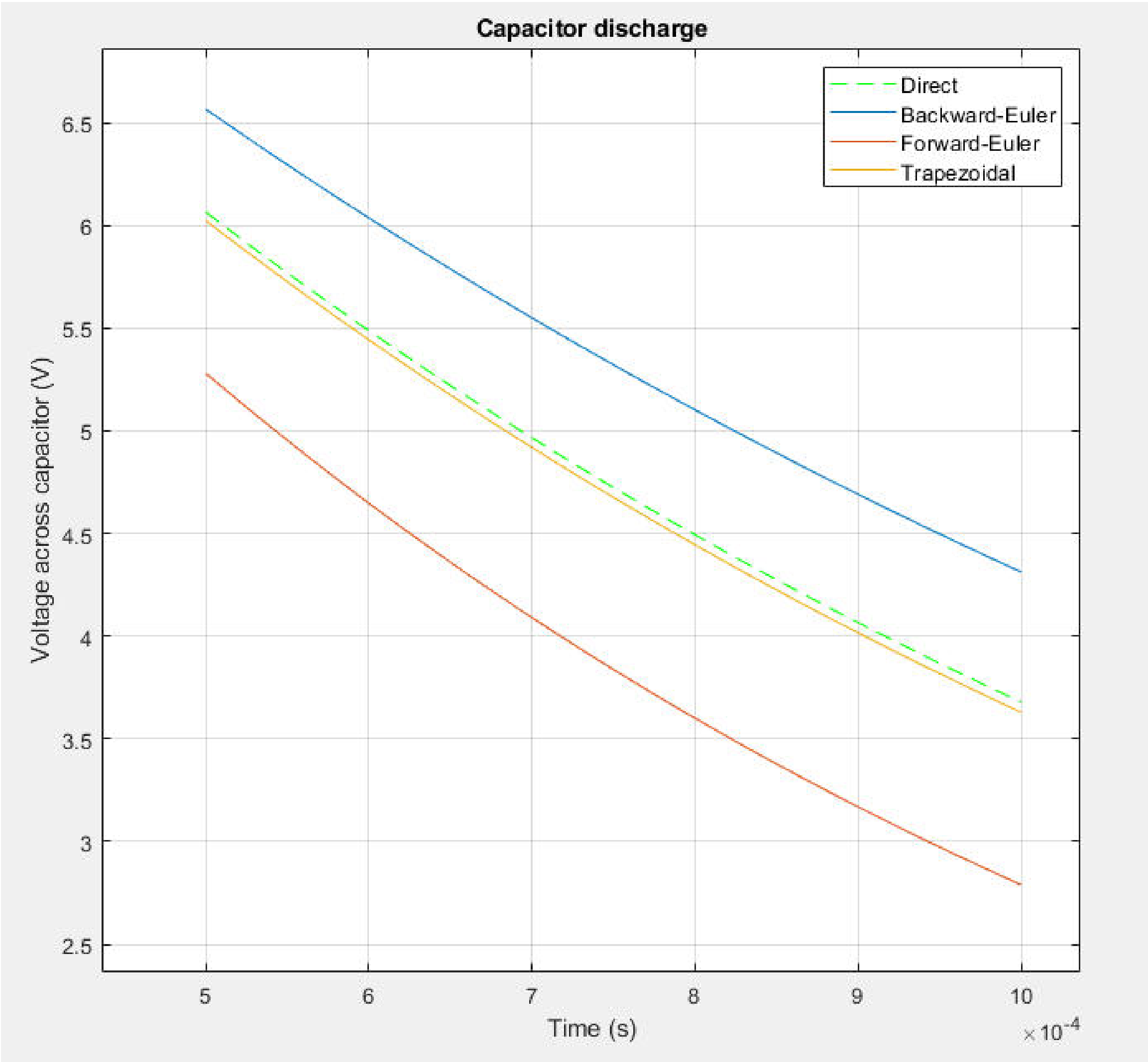

Figure 4: Plot of capacitor discharge using different methods (tau = RC * 0.5) in the region of 0.5ms to 1ms

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e) forward-Euler stability condition:

$$|1 + \lambda\tau| < 0$$

$$\tau < \left|\frac{1}{\lambda}\right|$$

The method becomes unstable for:

$$\tau \geq \frac{1}{\lambda}$$

f) trapezoid stability condition:

$$z < 0$$

$$\frac{1 + \frac{\lambda\tau}{2}}{1 - \frac{\lambda\tau}{2}} < 0$$

$$\tau < \left|\frac{2}{\lambda}\right| \Rightarrow \text{Threshold: } \tau \geq \left|\frac{2}{\lambda}\right|$$

The visualization of the numerical oscillations are shown in Figure 7. There the tau exceeds the threshold and thus it oscillates.

g)

Figure 5 shows a huge oscillation for forward-Euler and a straight perpendicular curve. This is due to 2RC is equal to the threshold of the trapezoidal stability shown in f). The forward-Euler is twice larger than it's threshold and therefore causes the oscillation.

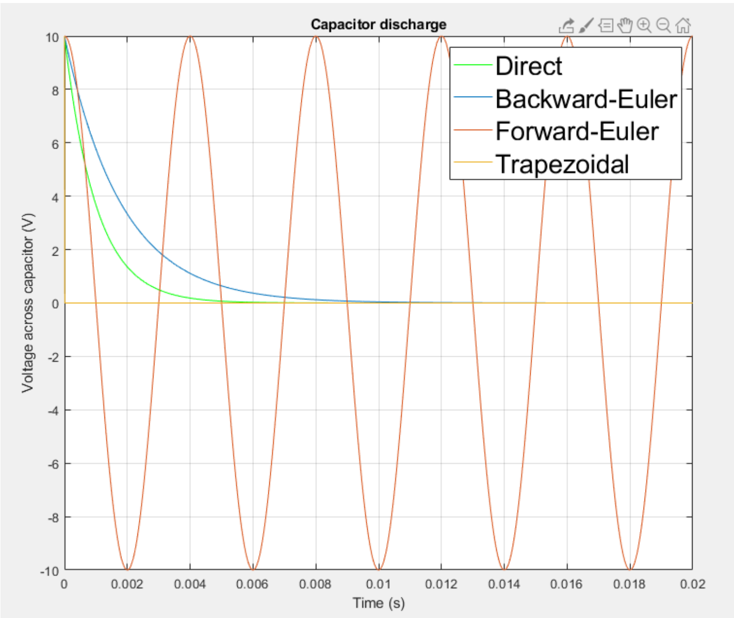

Figure 5: Plot of capacitor discharge (tau = 2RC)

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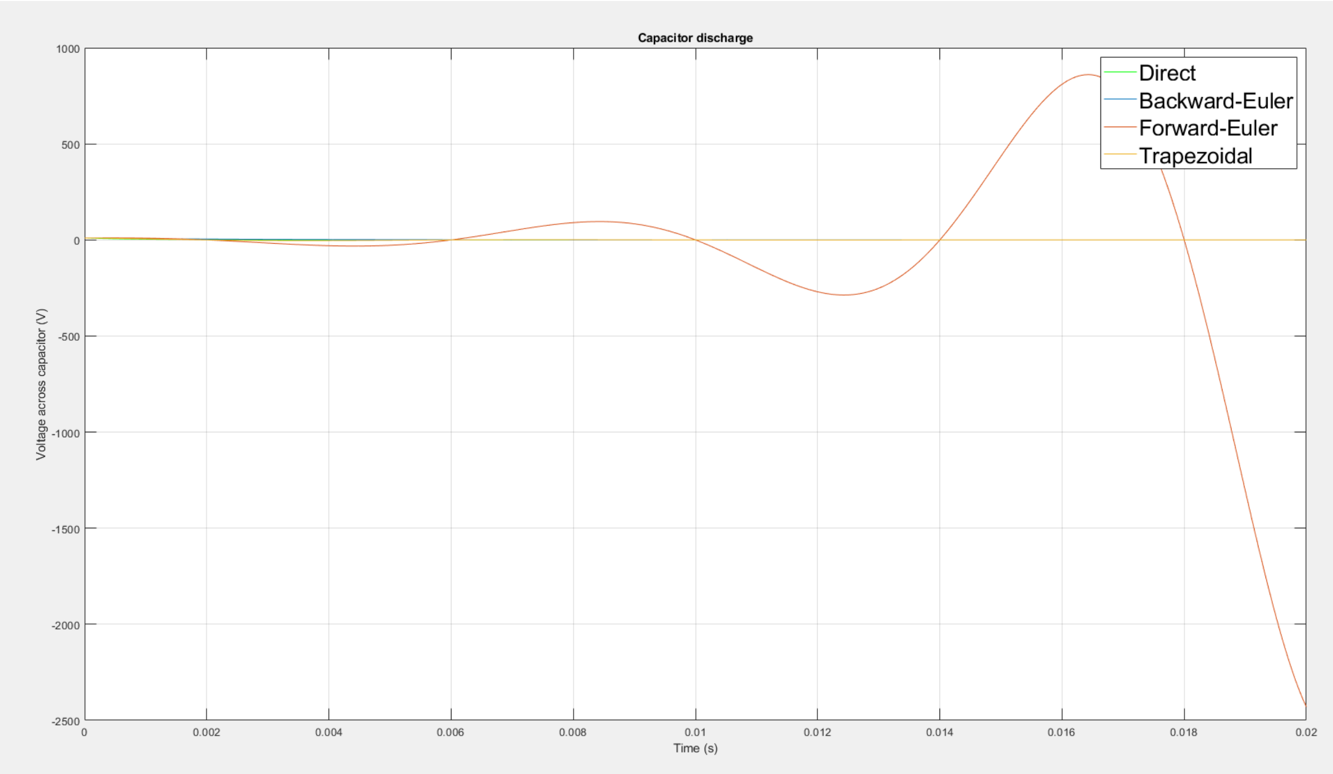
h)

Figure 6 shows a large oscillation for the forward-Euler. Trapezoidal method should also show an oscillation. But is indistinguishable due to the forward-Euler plot. This is due to 4RC being two times trapezoidal's threshold and four times forward-Euler's threshold.



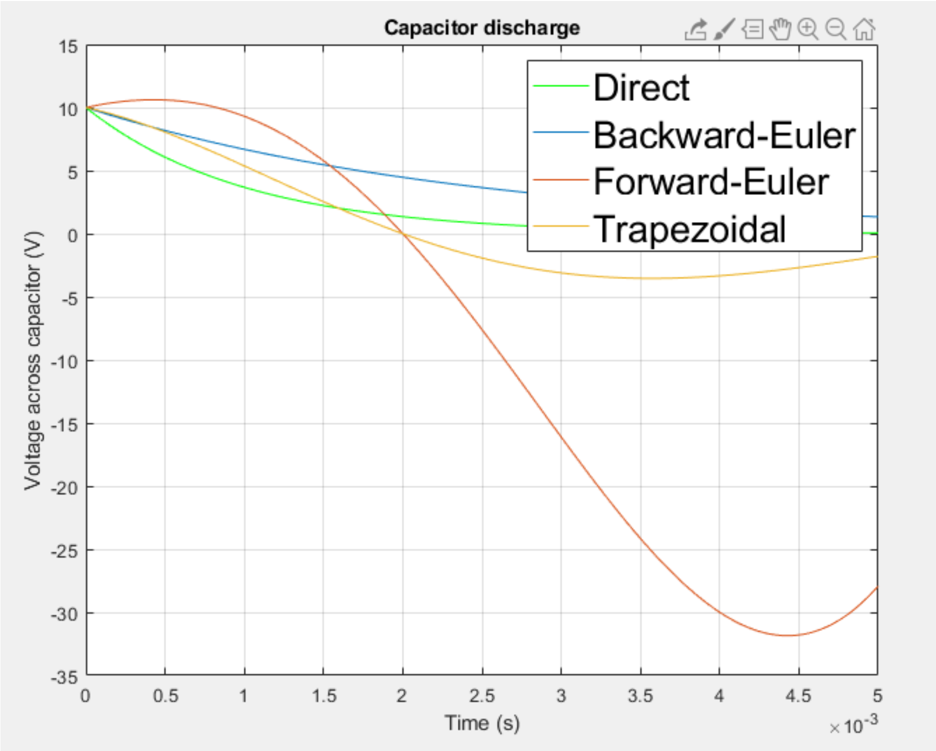


Figure 6: Plot of capacitor discharge (tau = 4RC) from 0 < t < 20ms and 0 < t < 5ms

MATLAB Code and figure 7 will be shown below.

Table of Contents


```
clc
clear all

% Submitted by Hutomo Saleh - 461980
```

Variables

```
V_C_0 = 10;
C = 1e-3;
R = 1;

t_max = 10e-3;
t = 0:1e-5:t_max;
lambda = - 1 / (R*C);
tau = 0.0008;
n = t / tau;

% Direct Solution
V_C = V_C_0 * exp(lambda * t);
% Backward Euler
V_C_be = V_C_0 * (1 - lambda * tau).^-n;
% Forward Euler
V_C_fe = V_C_0 * (1 + lambda * tau).^n;
% Trapezoidal
V_C_tr = ((1 + 0.5 * tau * lambda) ...
         / (1 - 0.5 * tau * lambda)).^n * V_C_0;
```

Oscillations

+1% to make it larger than the threshold

```
tau = abs(1 / lambda) * 1.01;
V_C_fe_oscillation = V_C_0 * (1 + lambda * tau).^n;
tau = abs(2 / lambda) * 1.01;
V_C_tr_oscillation = ((1 + 0.5 * tau * lambda) ...
                     / (1 - 0.5 * tau * lambda)).^n * V_C_0;
```

Plot

```
plot(t, V_C, '--g', 'DisplayName', 'Direct')
hold on
plot(t, V_C_be, 'DisplayName', 'Backward-Euler')
```

```
hold on
plot(t, V_C_fe, 'DisplayName', 'Forward-Euler')
hold on
plot(t, V_C_tr, 'DisplayName', 'Trapezoidal')
hold on
plot(t, V_C_fe_oscillation, 'DisplayName', 'Forward-Euler
 (Oscillation)')
hold on
plot(t, V_C_tr_oscillation, 'DisplayName', 'Trapezoidal
 (Oscillation)')
hold off
grid on
lgd = legend;
lgd.FontSize = 40;
title('Capacitor discharge')
xlabel('Time (s)')
ylabel('Voltage across capacitor (V)')
```

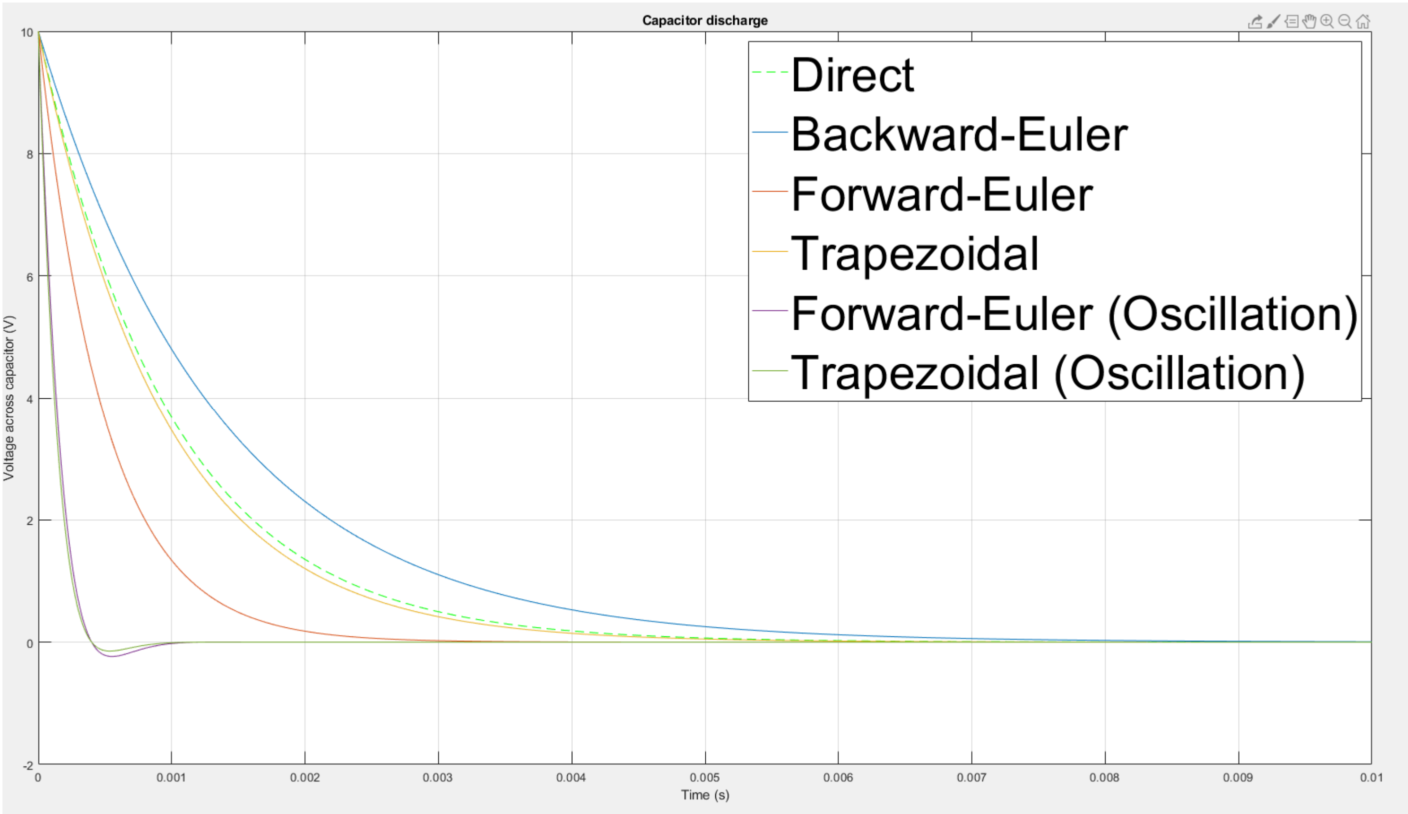


Figure 7: Plot of capacitor discharge of different methods and oscillation

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