CRES - 1st Assignment

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Problem 1

Forward-euler method:

$$\frac{o(k) - o(k-1)}{\tau} = e(k-1) \tag{1.0}$$

a) 4 Points Determine an analytic expression for the frequency response of the forward-Euler integrator.

By inserting:

$$o(k) = W_{IfE}(f) \cdot e^{j2\pi f(k)\tau} \tag{1.1}$$

to [1.0] we get:

$$\frac{W_{IfE}(f) \cdot e^{j2\pi f k\tau} - W_{IfE}(f) \cdot e^{j2\pi f(k-1)\tau}}{\tau} = e^{j2\pi f(k-1)\tau}, \tag{1.2}$$

$$e^{j2\pi f k\tau} \cdot \left(\frac{W_{IfE}(f) \cdot (1 - e^{-j2\pi f\tau})}{\tau}\right) = \frac{e^{j2\pi f k\tau}}{e^{j2\pi f\tau}}, \tag{1.3}$$

$$\frac{W_{IfE}(f) \cdot (e^{j2\pi f\tau} - 1)}{\tau} = 1, \tag{1.4}$$

$$W_{IfE}(f) = \tau \cdot \frac{1}{e^{j2\pi f\tau} - 1}, \tag{1.5}$$

$$W_{IfE}(f) = \tau \cdot \frac{1}{e^{j\pi f\tau} \cdot e^{j\pi f\tau} - 1}, \tag{1.6}$$

$$W_{IfE}(f) = \tau \cdot \frac{e^{-j\pi f\tau}}{e^{j\pi f\tau} - e^{-j\pi f\tau}}, \tag{1.7}$$

by substituting:

$$e^{j\pi f\tau} = \cos(\pi f\tau) + j\sin(\pi f\tau), \tag{1.8}$$

we get:

$$W_{IfE}(f) = \tau \cdot \frac{\cos(\pi f\tau) - j\sin(\pi f\tau)}{\cos(\pi f\tau) + j\sin(\pi f\tau) - (\cos(\pi f\tau) - j\sin(\pi f\tau))}, \tag{1.9}$$

$$W_{IfE}(f) = \tau \cdot \frac{\cos(\pi f\tau) - j\sin(\pi f\tau)}{2j\sin(\pi f\tau)}, \tag{1.10}$$

$$W_{IfE}(f) = -\frac{\tau}{2} + \frac{\tau}{2j} \cdot \frac{1}{\tan(\pi f\tau)}, \tag{1.11}$$

by substituting to Nyquist frequency:

$$W_{IfE}(f) = -\frac{1}{4f_{Ny}} + \frac{1}{4f_{Ny}} \cdot \frac{1}{\tan\left(\frac{\pi f}{2f_{Ny}}\right)}, \quad [1.12]$$

lastly, inserting the trapezoidal frequency response leads to:

$$W_{IfE}(f) = -\frac{1}{4f_{Ny}} + W_{ITr}(f). \quad [1.13]$$

b) 3 Points Given is an ideal inductance L. Determine an analytic expression for the distortion of L in a circuit simulator using the forward-Euler method.

The complex admittance of the inductor is defined as follows:

$$Y_{LfE}(f) = W_{IfE}(f)\frac{1}{L}, \quad [1.14]$$

Inserting [1.12] into [1.14] we get:

$$Y_{LfE}(f) = (-\frac{1}{4f_{Ny}} + \frac{1}{4f_{Ny}})\frac{1}{L}, \quad [1.15]$$

$$Y_{LfE}(f) = (-\frac{\frac{\pi f}{2f_{Ny}}}{j2\pi f} + \frac{\frac{\pi f}{2f_{Ny}}}{j2\pi f} \cdot \frac{1}{\tan\left(\frac{\pi f}{2f_{Ny}}\right)})\frac{1}{L}, \quad [1.16]$$

$$Y_{LfE}(f) = \frac{1}{j2\pi f} \cdot \frac{\pi f}{2f_{Ny}} \cdot (\frac{1}{\tan\left(\frac{\pi f}{2f_{Ny}}\right)} - 1) \cdot \frac{1}{L}, \quad [1.17]$$

$$Y_{LfE}(f) = \frac{1}{j2\pi f} \cdot \left(\frac{\frac{\pi f}{2f_{Ny}}}{\tan\left(\frac{\pi f}{2f_{Ny}}\right)} - \frac{\pi f}{2f_{Ny}}\right) \cdot \frac{1}{L}, \quad [1.18]$$

Following the [1.14] format leads to:

$$Y_{LfE}(f) = \frac{1}{j2\pi f} \cdot \frac{1}{L_{IfE}(f)}, \quad [1.19]$$

with:

$$L_{IfE}(f) = L \cdot \left(\frac{\frac{\pi f}{2f_{Ny}}}{\tan\left(\frac{\pi f}{2f_{Ny}}\right)} - \frac{\pi f}{2f_{Ny}}\right)^{-1}, \quad [1.20]$$

if we insert the inductance of the trapezoidal method, it becomes:

$$L_{IfE}(f) = L \cdot \left(\frac{1}{L_{Tr}(f)} - \frac{\pi f}{2f_{Ny}}\right)^{-1}, \quad [1.21]$$

If we continue [1.18] and focus on the admittance:

$$Y_{LfE}(f) = -\frac{1}{4f_{Ny}L} + W_{Itr} \cdot \frac{1}{L}. \quad [1.22]$$

c) 3 Points Compare the accuracy of the forward-Euler and backward-Euler methods.

Both methods have more or less the same accuracy. Although both methods have different signs of the real part, the magnitude remains the same. The difference in values would be later shown in **2b)** where the phase angles are different, but the magnitude of the current remains the same.

Problem 2

a) 1 Point Given an ideal circuit, calculate the amplitude $\hat{i}_L$ and the phase angle θ_i of the inductor current.

With the initial condition of $i_L = 0$ at $t = 0$. We can conclude that $\theta_i = 90^\circ$. The amplitude $\hat{i}_L$ is as follows:

$$v_L = v_S = z_L \cdot i_L, \qquad [2.0]$$

$$z_L = j \cdot 2\pi \cdot f \cdot L. \qquad [2.1]$$

The maximum inductor current occurs at $cos = 1$, which is at time intervals of $t = \frac{1}{80} \cdot n$ with $n \in \mathbf{N}$. We can simply calculate the amplitude by:

$$\hat{i}_L = \frac{\hat{v}_s}{2\pi f L} = \frac{400kV}{2\pi \cdot 60Hz \cdot 100mH} = 10.61kA. \qquad [2.2]$$

b) 3 points Using an integration step size of $\tau = 50\mu s$, calculate amplitude $\hat{i}_L$ and the phase angle θ_i in degrees when the circuit is simulated with the trapezoidal, backward-Euler & forward-Euler methods.

The general equation to solve this problem is:

$$I = U \cdot Y. \qquad [2.3]$$

Where as the Y is calculated differently for each method.

- Trapezoidal

$$Y = \frac{1}{j2\pi f} \cdot \frac{1}{L} \cdot \frac{\pi f \tau}{\tan \pi f \tau} = -j0.026526S. \qquad [2.3]$$

By transforming to Euler-form we can read the phase angle:

$$0.026526e^{-90^\circ}$$

$$\theta = -90^\circ$$

The amplitude $\hat{i}_L$ is:

$$\hat{i}_L = \hat{v}_s \cdot |Y| = 400kV \cdot 0.026526S = 10610.4kA$$

- Backward-Euler

$$Y = \frac{1}{2\tau L} + Y_{Ltr} = 0.25mS - j0.026526S$$

$$0.026526e^{-89.46^\circ}$$

$$\theta = -89.46^\circ$$

The amplitude $\hat{i}_L$ is:

$$\hat{i}_L = \hat{v}_s \cdot |Y| = 400kV \cdot \sqrt{0.25mS^2 + 0.026526S^2} = 10610.87kA$$

- Forward-Euler

$$Y = -\frac{1}{2\tau L} + Y_{Ltr} = 0.25mS - j0.026526S$$

$$0.026526e^{-89.46^\circ}$$

$$\theta = -90.539^\circ$$

The amplitude $\hat{i}_L$ is:

$$\hat{i}_L = \hat{v}_s \cdot |Y| = 400kV \cdot \sqrt{0.25mS^2 + 0.026526S^2} = 10610.87kA$$

c) 6 points For the trapezoidal and backward-Euler methods, repeat the calculations carried out in **b)** when using increasingly large time step sizes: **100** μs, **500** μs, **1** μs. Discuss the results, evaluate other times step sizes if found necessary. What is the theoretical maximum of the time step size? What is the maximal acceptable time step size in terms of accuracy?

For this section I used Matlab to automate the whole process. In most small step sizes there seems to be minor changes. A noticeable change is at 1 μs. The value becomes inaccurate the larger the time step becomes. The inaccuracy becomes much larger starting at 1 μs. The theoretical maximum of τ is $\frac{1}{2f_{Ny}} = \frac{1}{2 \cdot 60} = 8.333ms$. The maximal acceptable time step size should be 2x the Nyquist frequency because of aliasing.

Code:

```
cres_assignment(50e-6)
cres_assignment(500e-6)
cres_assignment(10e-6)
cres_assignment(1e-3)
cres_assignment(0.3e-2) %Very inaccurate

function cres_assignment(tau)
    disp("Tau: " + tau + 's')
    f = 60;
    v_amplitude = 4e5;
    L = 100e-3;
    y_trapez = 1 / (1j * 2 * pi * f) * 1/L * (pi*f*tau) / (tan(pi*f*tau));
    y_backward = tau / (2 * L) + y_trapez;
    y_forward = - tau / (2 * L) + y_trapez;
    disp("Admittance: ")
    disp("  Trapezoidal: " + y_trapez)
    disp("  Backward Euler: " + y_backward)
    disp("  Forward Euler: " + y_forward)

    disp("Phase angle: ")
    disp("  Trapezoidal: " + rad2deg(angle(y_trapez)))
    disp("  Backward Euler: " + rad2deg(angle(y_backward)))
    disp("  Forward Euler: " + rad2deg(angle(y_forward)))

    i_trapez = v_amplitude * y_trapez;
    i_backward = v_amplitude * y_backward;
    i_forward = v_amplitude * y_forward;
    disp("Amplitude: ")
    disp("  Trapezoidal: " + abs(i_trapez))
    disp("  Backward Euler: " + abs(i_backward))
    disp("  Forward Euler: " + abs(i_forward))
    disp(" ")
end
```

Result:

```
Tau: 5e-05s
Admittance:
  Trapezoidal: 0-0.026525i
  Backward Euler: 0.00025-0.026525i
  Forward Euler: -0.00025-0.026525i
Phase angle:
  Trapezoidal: -90
  Backward Euler: -89.46
  Forward Euler: -90.54
Amplitude:
    Trapezoidal: 10610.0154
    Backward Euler: 10610.4866
    Forward Euler: 10610.4866

Tau: 0.0005s
Admittance:
  Trapezoidal: 0-0.026447i
  Backward Euler: 0.0025-0.026447i
  Forward Euler: -0.0025-0.026447i
Phase angle:
  Trapezoidal: -90
  Backward Euler: -84.6
  Forward Euler: -95.4
Amplitude:
    Trapezoidal: 10578.895
    Backward Euler: 10626.0538
    Forward Euler: 10626.0538
```

```
Tau: 1e-05s
Admittance:
  Trapezoidal: 0-0.026526i
  Backward Euler: 5e-05-0.026526i
  Forward Euler: -5e-05-0.026526i
Phase angle:
  Trapezoidal: -90
  Backward Euler: -89.892
  Forward Euler: -90.108
Amplitude:
    Trapezoidal: 10610.317
    Backward Euler: 10610.3358
    Forward Euler: 10610.3358

Tau: 0.001s
Admittance:
  Trapezoidal: 0-0.026211i
  Backward Euler: 0.005-0.026211i
  Forward Euler: -0.005-0.026211i
Phase angle:
  Trapezoidal: -90
  Backward Euler: -79.2
  Forward Euler: -100.8
Amplitude:
    Trapezoidal: 10484.3672
    Backward Euler: 10673.4228
    Forward Euler: 10673.4228
```

```
Tau: 0.003s
Admittance:
  Trapezoidal: 0-0.023636i
  Backward Euler: 0.015-0.023636i
  Forward Euler: -0.015-0.023636i
Phase angle:
  Trapezoidal: -90
  Backward Euler: -57.6
  Forward Euler: -122.4
Amplitude:
    Trapezoidal: 9454.4872
    Backward Euler: 11197.6483
    Forward Euler: 11197.6483
```