Automatic Debiased Machine Learning for Dynamic Discrete Choice

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Abstract

Numerous causal and structural effects rely on regression estimates, such as policy effects and parameter estimation in economic structural models. Such regressions may involve high-dimensional covariates, so machine learning approaches are of interest. However, putting machine learning models together with identifying equations can result in regularization and model selection bias. This paper introduces a method to automatically debias in Dynamic Discrete Choice problems. This debiasing method does not require the analytically form of the bias correction term to be given. It is applicable to all forms of regression learning techniques, including neural networks, random forests, Lasso, and other techniques available for high-dimensional data. The paper also supplies robust standard error estimations to address misspecification, rates of convergence of bias correction, and asymptotic inference conditions for basing the estimation of a range of structural effects.

1 Introduction

In Dynamic Discrete Choice models, structural parameters depend on nonparametric or high-dimensional first-steps, referred as conditional choice probabilities (CCPs). Machine Learning techniques such as Lasso and Neural Networks are applied in case of a large dimension for state variables in order to alleviate the curse of dimensionality.However, utilizing ML for estimating the first-steps introduces biases into the estimation of the structural parameters due to regularization and variable selection. In this paper, we propose the utilization of an Automatic Debiased Machine Learning method via Lasso to estimate the first-steps in Dynamic Discrete Choice problems. The approach provides an automatic method for the estimation and correction of the arising bias from the first-steps estimation. Compared to the approach presented in Chernozhukov et al. [2022a], ours does not need to specify the analytical functional forms of the bias-correction terms.

Machine learning is necessary to estimate economic and causal models, particularly when there are high-dimensional covariates or state variables, as illustrated by studies such as Robins et al. [2013], Belloni et al. [2017], Athey et al. [2018], Farrell [2015]. Certain machine learning methods such as Lasso, boosting, neural networks, random forests, etc., prove useful in such situations. Orthogonal moment functions bring relief from model selection and regularization biases, which are normally encountered at initial phases of machine learning. Applying cross-fitting to debiased generalized method of moments (GMM) decreases bias even more and removes the need for Donsker conditions, which are rarely met in a broad range of machine learning applications. The large sample theory constructed herein focuses primarily on mean-square convergence behavior and thus can be applied to a broad range of machine learning methods. The advantages Automatic debiased

GMM offers over plug-in GMM make it a better alternative in a majority of machine learning applications. Chernozhukov et al. [2023] extends automated debiased machine learning to the dynamic treatment regime and, more generally, to nested functionals with the dynamic discrete choice model estimation applications. This paper contributes to this literature by studying the asymptotic theory and Monte-Carlo exploration of the Automatic Debiased Machine Learning estimator for Dynamic Discrete Choice problems.

Additional biases can be caused by the nonlinearity of moment conditions in the first step $\hat{\gamma}$. Cattaneo and Jansson [2018] and Cattaneo et al. [2019] have derived helpful bootstrap and jackknife methods that eliminate nonlinearity bias. Newey and Robins [2018] demonstrated that one can also eliminate this bias by cross-fitting in certain environments. In this paper, we employ cross-fitting to address this issue.

The structure of this paper unfolds as follows: In Section 2, we introduce the Debiased Machine Learning estimator within a general framework. Section 3 outlines the context of the Dynamic Discrete problem and explores the application of Lasso in the estimation of first steps CCPs. Moving on to Section 4, we detail the automatic estimation of influence functions. In Section 5, we conduct a Monte Carlo study to evaluate the finite-sample properties of this estimator. Finally, in Section 6, we present the Asymptotic Theory for the Automatic Debiased Machine Learning estimator in the context of Dynamic Discrete Choice problems.

2 Debiased Machine Learning Estimator

Consider the identifying moment functions as in Chernozhukov et al. [2022a]:

$$\mathbb{E}[g(W, \gamma_0, \theta_0)] = 0$$

Where θ_0 is the true value of the parameters of interest, γ_0 is the true value of the nuisance parameters and W is a data observation. A common "plug-in" GMM estimator can be used to estimate θ:

- Estimating the first step $\hat{\gamma}$ by Lasso.
- Plugging $\hat{\gamma}$ into the sample moment $\hat{g} = \sum_{i=1}^{n} g(W_i, \hat{\gamma}, \theta)/n$.
- Estimating $\hat{\theta}$ by minimizing a quadratic form in these estimated sample moments: $\hat{\theta} = arg\min_{\theta\in\Theta} \hat{g}'W\hat{g}$

Since Lasso performs variable selection/regularization, this estimator is highly biased. Let F_0 denotes the true distribution of W, and H be some alternative distribution. Let F_τ be the combination of F_0 and H:

$$F_\tau = (1-\tau)F_0 + \tau H,\ \tau \in [0, 1]$$

Intuition: F_τ represents the model mispecification.

Key assumption: Existence of the Gateaux derivative:

$$\frac{d}{d_\tau}\mathbb{E}[g(W, \gamma(F_\tau), \theta)] = \int \phi(w, \gamma_0, \alpha_0, \theta)H(dw),\ \mathbb{E}[\phi(w, \gamma_0, \alpha_0, \theta)] = 0 \tag{1}$$

This assumption allows us to measure the local effect of first step γ on the expected identifying moment. $\phi(w, \gamma, \alpha, \theta)$ is referred as the non-parametric influence function resulted from first step estimation of nuisance parameters. Then orthogonal moment functions are constructed as follow:

$$\psi(W, \gamma, \alpha, \theta) = g(W, \theta) + \phi(W, \gamma, \alpha, \theta)$$

Then the Debiased Machine Learning Estimator can be constructed using the following steps:

- Step 1: Partitioning the data into L groups I_ℓ ($\ell = 1, ..., L$).
- Step 2: Estimating the nuisance parameters $\hat{\gamma}_\ell$ using machine learning method such as LASSO.
- Step 3: Constructing the debiased sample moments functions:

$$\hat{\psi}(\theta) = \hat{g}(\theta) + \hat{\phi}, \; \hat{g}(\theta) = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell} g(W_i, \hat{\gamma}_\ell, \theta),$$

$$\hat{\phi} = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_l} \phi(W_i, \hat{\gamma}_\ell, \hat{\alpha}_\ell, \tilde{\theta}_\ell)$$

  Where $\tilde{\theta}_\ell$ is the plug-in estimator of θ obtained by minimizing a quadratic form in these estimated sample moments g using observations in groups other than group ℓ: $\tilde{\theta} = arg \min_{\theta \in \Theta} \hat{g}' W \hat{g}$.

- Step 4: Estimating $\hat{\theta}$ using ML:

$$\hat{\theta} = arg \min_{\theta \in \Theta} \hat{\psi}(\theta)' \hat{\Upsilon} \hat{\psi}(\theta)$$

Where $\tilde{\theta}_\ell$ is the plug-in GMM estimator of the paramaters of interest.

3 General Settings of Dynamic Binary Choice Problems

In this section, we present the general framework for Dynamic Discrete Choice (DDC) problems, focusing specifically on dynamic binary choice for simplicity. However, the proposed method is extendable to more sophisticated settings. Below is a table summarizing the key notations used throughout the paper.

3.1 Model Description

Individuals make decisions between two alternatives, denoted by $j = 1$ and $j = 2$, over T time periods. The objective is to maximize the expected present discounted value of per-period utility, defined as:

$$U_{jt} = D_j(X_t)'\theta_0 + \epsilon_{jt}, \quad j = 1, 2, \quad t = 1, \ldots, T \tag{2}$$

Here, ϵ_{jt} represents an independent and identically distributed (i.i.d) error term with a known cumulative distribution function (CDF). This error term is independent of the entire history $\{X_t\}_{t=1}^{\infty}$ of the state variable vector X_t. The state vector X_t is assumed to follow a first-order Markov process and is stationary. The parameter vector of interest is θ_0. We assume that the CDF of ϵ_{jt} follows a Type I extreme value distribution, and the dimensionality of X_t is high.

Table 1: Notation Definitions

Symbol	Definition
j	Choice alternative, $j = 1, 2$
t	Time period, $t = 1, \ldots, T$
U_{jt}	Utility of choosing alternative j at time t
$D_j(X_t)$	Vector of state-dependent variables for choice j at time t
θ_0	Parameter vector to be estimated
ϵ_{jt}	Idiosyncratic shock for choice j at time t, i.i.d with known CDF
X_t	State variable vector at time t, Markov of order 1 and stationary
Y_{jt}	Indicator variable, $Y_{jt} = 1$ if choice j is made at time t
$\gamma_{10}(X_t)$	Probability of choosing alternative 2 given X_t, $\Pr(Y_{2t} = 1 \vert X_t)$
$H(p)$	Known transformation function, e.g., $H(p) = 0.5227 - \ln(1 - p)$
$\gamma_{20}(X_t)$	$\mathbb{E}\left[H(\gamma_{10}(X_{t+1})) \mid X_t, Y_{2t} = 1\right]$
γ_{30}	$\mathbb{E}\left[H(\gamma_{10}(X_{t+1})) \mid Y_{1t} = 1\right]$
$V(X_t)$	Expected value function at state X_t
$\Lambda(a)$	CDF of $\epsilon_{1t} - \epsilon_{2t}$
δ	Discount factor
$\hat{\gamma}_k$	Estimator of γ_{k0} for $k = 1, 2, 3$
$\gamma_k(F)$	Probability limit of $\hat{\gamma}_k$ under misspecification
$g(W, \theta, \gamma)$	Identifying moment function

Choice 1 is designated as a renewal choice, meaning that the conditional distribution of X_{t+1} given X_t and choice 1 is independent of X_t.

For identification purposes, we set:

$$D_1(X_t) = (-1, 0)'$$

$$D_2(X_t) = \begin{pmatrix} 0 \\ * \end{pmatrix}$$

where the first element of $D_2(X_t)$ is zero, ensuring that the first element in θ serves as a binary choice constant.

Let F represent a possible CDF for a data observation. Define the indicator variable Y_{jt} such that $Y_{jt} = 1$ when choice j is made at time t. Let $\gamma_{10}(X_t) = \Pr(Y_{2t} = 1 \vert X_t)$ and $V(X_t)$ denote the expected value function. Following Hotz and Miller [1993], there exists a known function $H(p)$ satisfying:

$$\mathbb{E}\left[V(X_{t+1})) \mid X_t, Y_{2t} = 1\right] - \mathbb{E}\left[V(X_{t+1})) \mid Y_{1t} = 1\right] = \gamma_{20}(X_t) - \gamma_{30} \tag{3}$$

As per Aguirregabiria and Mira [2002], when ϵ_{1t} and ϵ_{2t} are independent Type I extreme value, the function $H(p)$ takes the form:

$$H(p) = 0.5227 - \ln(1 - p) \tag{4}$$

The choice probabilities can be expressed using the CDF $\Lambda(a)$ of $\epsilon_{1t} - \epsilon_{2t}$, the difference in state-dependent variables $D(X_t) = D_2(X_t) - D_1(X_t)$, and the discount factor δ:

$$\Pr(Y_{2t} = 1 \vert X_t) = \Lambda\left(a(X_t, \theta_0, \gamma_{20}, \gamma_{30})\right)$$

$$a(X_t, \theta, \gamma_2, \gamma_3) = D(X_t)'\theta + \delta\left(\gamma_2(X_t) - \gamma_3\right)$$

Let $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{\gamma}_3$ denote the estimators of γ_{10}, γ_{20}, and γ_{30}, respectively. Their probability limits under general misspecification are denoted by $\gamma_1(F)$, $\gamma_2(F)$, and $\gamma_3(F)$.

The identifying moment functions are specified as the derivative of the pseudo log-likelihood associated with the binary choice probability with respect to γ:

$$g(W,\theta,\gamma) = \frac{1}{T}\sum_{t=1}^{T} D(X_t)\pi\left(a(X_t,\theta,\gamma_2,\gamma_3)\right)\left[Y_{2t} - \Lambda\left(a(X_t,\theta,\gamma_2,\gamma_3)\right)\right] \tag{5}$$

where

$$\pi\left(a(X_t,\theta,\gamma_2,\gamma_3)\right) = \frac{\Lambda_a(a)}{\Lambda(a)\left[1-\Lambda(a)\right]}, \quad \Lambda_a(a) = \frac{d\Lambda(a)}{da} \tag{6}$$

Given that ϵ_{jt} follows an independent Type I extreme value distribution, the CDF $\Lambda(a)$ is the logistic function:

$$\Lambda(a) = \frac{e^a}{1+e^a} \tag{7}$$

Consequently, the function $\pi(a)$ simplifies to:

$$\pi\left(a(X_t,\theta,\gamma_2,\gamma_3)\right) = 1 \tag{8}$$

3.2 Assumptions and Properties

- The error terms ϵ_{jt} are independent and identically distributed with a Type I extreme value distribution.
- The state variable X_t follows a first-order Markov process and is stationary.
- Choice 1 is a renewal choice, implying that the future state X_{t+1} is independent of the current state X_t given that choice 1 is made.
- The model is identified using the moment conditions derived from the derivative of the pseudo log-likelihood.

3.3 Plug-in GMM estimator of θ

The identifying moment functions 5 implies that the estimator of $\gamma_{10}(x), \gamma_{20}(x), \gamma_{30}(x)$ are needed in order to estimate the structural parameters θ. Since the dimension of X is high, we consider linear Lasso estimators for $\hat{\gamma}_{1\ell}$ and $\hat{\gamma}_{2\ell}$, in which $\hat{\gamma}_{1\ell}$ and $\hat{\gamma}_{2\ell}$ are the estimators of $\gamma_{10}(x), \gamma_{20}(x)$ from observations not in I_ℓ. Then we can plug $\hat{\gamma}_{1\ell}$ into the formula of $\hat{\gamma}_{3\ell}$ to form an estimator for γ_{30}. Let $\hat{\gamma}_{1\ell\ell'}$ be an estimator of the conditional choice probability computed from observations not in I_ℓ or $I_{\ell'}$. Let $b(x)$ denote a $p \times 1$ dictionary of functions of the state variables x. We form $\hat{\gamma}_1, \hat{\gamma}_2$

and $\hat{\gamma}_3$ as:

$$\hat{\gamma}_{2\ell}(X) = b(X)'\hat{\beta}_{2\ell},\ \hat{\beta}_{2\ell} = \arg\min_{\beta}\{-2\hat{M}'_{2\ell}\beta + \beta'\hat{Q}_{2\ell}\beta + 2r_1\sum_{j=1}^{p}|\beta_j|\},$$

$$\hat{M}'_{2\ell} = \frac{1}{(n-n_\ell)T}\sum_{\ell'\neq\ell}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} Y_{2it}b(X_{it})H(\hat{\gamma}_{1\ell\ell'}(X_{i,t+1})),$$

$$\hat{Q}_{2\ell} = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} Y_{2it}b(X_{it})b(X_{it})',$$

$$\hat{\gamma}_{3\ell} = \frac{1}{\hat{P}_1(n-n_\ell)T}\sum_{\ell'\neq\ell}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} Y_{1it}H(\hat{\gamma}_{1\ell\ell'}(X_{i,t+1})),$$

$$\hat{P}_1 = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} Y_{1it}$$

Here $\hat{\gamma}_{2\ell}$ is Lasso with LHS variable $H(\hat{\gamma}_{1\ell\ell'}(X_{i,t+1}))$ and RHS variables $b(X_{it})Y_{2it}$, and $\hat{\gamma}_{3\ell}$ is a sample mean conditional on $Y_{1it} = 1$. Then one can form a plug-in GMM estimator of θ by plugging these first-steps into the identifying moment functions 5 and minimizing the quadratic form in these estimated sample moments. However, since Lasso performs regularization/variable selection to reduce the variance of prediction, this plug-in estimator is highly biased. Note that we do not use Lasso to estimate γ_3 since it is the sample average of $H(\gamma(X_{t+1}))$ over the observations with $Y_{1t} = 1$, so it does not induce additional bias to the estimation of the structural parameters of interest.

3.4 Debiased Machine Learning estimator of θ

Following the results in Chernozhukov et al. [2022a], one can correct the biases from the first-steps by adding the influence functions (FSIF) to the identifying moment functions. The functional forms of these FSIFs are:

$$\phi_1(W,\theta,\alpha,\gamma) = \frac{1}{T}\sum_{t=1}^{T}\alpha_1(X_{t+1},\theta)[Y_{2,t+1} - \gamma_1(X_{t+1})],$$

$$\phi_2(W,\theta,\alpha,\gamma) = \frac{1}{T}\sum_{t=1}^{T}\alpha_2(X_t,Y_{2t},\theta)[H(\gamma_1(X_{t+1})) - \gamma_2(X_t)],$$

$$\phi_3(W,\theta,\alpha) = \alpha_3\frac{1}{T}\sum_{t=1}^{T}Y_{1t}[H(\gamma_1(X_{t+1})) - \gamma_3]$$

The true functions α_{10}, α_{20} and α_{30} are:

$$\alpha_{10}(x,\theta_0) = \{\mathbb{E}[\alpha_{20}(X_t,Y_{2t},\theta_0)|X_{t+1} = x] + \alpha_{30}\mathbb{E}[\alpha_{30}\cdot Y_{1t}|X_{t+1} = x]\}H_p\left(\gamma_{10}(x)\right)$$

$$\alpha_{20}(x,y_2,\theta_0) = -\delta D(x)\pi\left(a(x)\right)\frac{\Lambda_a(a(x))y_2}{\Lambda\left(a(x)\right)},\ a(x) = a(x,\theta_0,\gamma_{20},\gamma_{30}),$$

$$\alpha_{30} = -\mathbb{E}[\alpha_{20}(X_t,Y_{2t},\theta_0)]/P_{10},\ P_{10} = \mathbb{E}[Y_{1t}]$$

Then one can form the debiased identifying moment functions as follows:

$$\psi(W,\theta,\alpha,\gamma) = g(W,\theta,\gamma) + \phi_1(W,\theta,\alpha,\gamma) + \phi_2(W,\theta,\alpha,\gamma) + \phi_3(W,\gamma,\alpha) \quad (9)$$

Then one can form the Debiased Machine Learning estimator estimator of θ by plugging first-step CCPs into the Debiased identifying moment functions (4) and minimizing the quadratic form in these estimated debiased sample moments (following the procedure described in Section 2). A potential shortcoming of this approach is that the estimation of α_{20} and α_{10} require the inversion of the high-dimensional density $\Lambda\left(a(x)\right)$ and $(1-\gamma_{10})$, which can be noisy during the estimation process (when the probability gets close to 0 and 1) and increase the standard error of the estimated structural parameters of interest. To address these issues, I propose the Automatic Estimation approach in the next section.

4 Automatic Estimation of the influence functions based on sample moment function

Chernozhukov et al. [2022a] presented an approach to correct the bias from the first-steps using the analytical form of the influence functions as described in section 3.4. In this paper, I propose the automatic framework that does not require the knowledge of the true functional forms of the influence functions. Denote Γ_1 and Γ_2 as the sets of possible directions of departures of $\gamma_1(F)$ and $\gamma_2(F)$ from γ_{10} and γ_{20}. As mentioned in the previous section, we do need to consider the misspecification of γ_3. Suppose that we can form the sample moment function by replacing the expectation in equation (4) with the sample average, and γ_{10}, γ_{20} and γ_{30} by $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$. Consider the sample average as follows:

$$\hat{\psi}_{\gamma}(\delta,\alpha)=\frac{d}{dn}\frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\psi(W_{it},\tilde{\gamma}_\ell+n\delta,\alpha,\tilde{\theta}_\ell)\Bigg|_{n=0}\qquad\text{for all }\delta\in\Gamma\tag{10}$$

Note that $\tilde{\theta}_\ell$ is the plug-in estimator (GMM) of θ. In the Dynamic Discrete Chocie problem, the sample moment functions are:

$$\hat{\psi}_{1,\gamma_1}(\delta_1,\alpha_1)=\frac{d}{dn}\frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\psi(W_{it},\hat{\gamma}_{1\ell}+n\delta_1,\alpha_1,\tilde{\theta}_\ell)\Bigg|_{n=0}=0\quad\text{for all }\delta_1\in\Gamma_1\tag{11}$$

$$\hat{\psi}_{2,\gamma_2}(\delta_2,\alpha_2)=\frac{d}{dn}\frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\psi(W_{it},\hat{\gamma}_{2\ell}+n\delta_2,\alpha_2,\tilde{\theta}_\ell)\Bigg|_{n=0}=0\quad\text{for all }\delta_2\in\Gamma_2\tag{12}$$

For the two scalar residuals $\lambda_1(W_t,\gamma_1(X_t))=Y_{2t}-\gamma_1(X_t)$ and $\lambda_2(W_t,\gamma_2(X_t))=H(\gamma_1(X_{t+1}))-\gamma_2(X_t)$, we consider $\gamma_1(F)\in\Gamma_1,\gamma_2(F)\in\Gamma_2$ satisfying:

$$\mathbb{E}_F[\alpha_1(x)\lambda_1(W,\gamma_1(F)(X))]=0\qquad\text{for all }\delta_1\in\Gamma_1\tag{13}$$

$$\mathbb{E}_F[\alpha_2(x)\lambda_2(W,\gamma_2(F)(X))]=0\qquad\text{for all }\delta_2\in\Gamma_2\tag{14}$$

In this example, the functional forms of $\gamma_1(F)(X)$ and $\gamma_2(F)(X)$ are:

$$\begin{aligned}&\gamma_1(F)(X_t)=\mathbb{E}_F[Y_{2t}|X_t]\\&\gamma_2(F)(X)=\mathbb{E}_F[H(\gamma_1(X_{t+1}))|X_t,Y_{2t}=1]=\frac{\mathbb{E}_F[H(\gamma_1(X_{t+1}))\cdot Y_{2t}|X_t]}{\mathbb{P}(Y_{2t}=1|X_t)}\end{aligned}$$

From these definitions, we can make the following inference:

- Any directions of departure of γ_1 (namely all possible α_1) can be well-approximated in mean-square by some finite linear combinations of basis functions $b_j(X)$.
- Any directions of departure of γ_2 (namely all possible α_2) can be well-approximated in mean-square by some finite linear combinations of basis functions $b_j(X)$.

Assumption 1: For all set Γ_i, there exists a set of basis $b(x) = (b_1(x), ..., b_p(x))'$, where $b_j \in \gamma_i$ $\forall j$ and $\forall \gamma \in \Gamma_i$ and $\epsilon > 0$, there are p and $\rho \in \mathbb{R}^p$ such that $\mathbb{E}[\alpha - b(x)'\rho^2] < \epsilon$.

The intuition of this assumption is that any function α belongs to the set Γ_i can be approximated in mean-squared by the linear combination of the basis functions $b_j(x)$. Then let $(b_1(X), b_2(X), ...)$ be the basis of Γ_1, and Γ_2. Therefore, we can let $\delta_1 = b_j(X)$, $\delta_2 = b_j(X)$. Let $\hat{\lambda}_{1i_{\gamma_{1\ell}}} = \frac{d\lambda_{1i}(W_i, \hat{\gamma}_{1\ell}+n\delta_1)}{dn}\Big|_{n=0} = -\delta_1$, $\hat{\lambda}_{2i_{\gamma_{2\ell}}} = \frac{d\lambda_{2i}(W_i, \hat{\gamma}_{2\ell}+n\delta_2)}{dn}\Big|_{n=0} = -\delta_2$. Let e_j be the jth column of a p-dimensional identity matrix. For $b(X) = (b_1(X), b_2(X), ..., b_p(X))'$, I have the following sample functions:

$$\hat{\psi}_{1k,\gamma_{1\ell}}(b_j, \rho' b) = \hat{M}_{jk\ell} - e_j'\hat{Q}_\ell \rho$$
$$\hat{\psi}_{2k,\gamma_2}(b_j, \beta' Y_{2i} b) = \hat{N}_{jk1} - e_j'\hat{S}_\ell \beta,$$
$$\hat{M}_{jk1} = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\frac{d}{dn}g(W_{it}, \hat{\gamma}_{1\ell}+nb_j, \tilde{\theta}_\ell),$$
$$\hat{Q} = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\hat{\lambda}_{1i_{\gamma_{1\ell}}} b(X_{it})b(X_{it})',$$
$$\hat{N}_{j1} = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\frac{d}{dn}g(W_{it}, \hat{\gamma}_{2\ell}+nb_j, \tilde{\theta}_\ell),$$
$$\hat{S} = \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\hat{\lambda}_{2i_{\gamma_{2\ell}}} b(X_{it})b(X_{it})'$$

Corresponding to the kth orthogonal moment function. The true functional form of $\hat{M}_{jk1}, \hat{N}_{jk1}$ are:

$$\begin{aligned}
\hat{M}_{jk1} &= \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\frac{d}{dn}g(W_{it}, \hat{\gamma}_{1\ell}+nb_j, \tilde{\theta}_\ell) \\
&= \frac{d\left[\frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{N} D(X_{it})[Y_{2it} - \Lambda(a(X_{it}, \tilde{\theta}_\ell, \hat{\gamma_{2\ell}}, \hat{\gamma_{3\ell}}))]\right]}{dn} \\
&= \frac{-\delta}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} D(X_{it})\Lambda_a(a(X_{it}, \hat{\gamma}_{2\ell}, \hat{\gamma}_{3\ell}, \tilde{\theta}_\ell))[\hat{\gamma}_{21\ell}(X_{it}) - \hat{\gamma}_{31\ell}(X_{it})] \\
\hat{N}_{j1} &= \frac{1}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T}\frac{d}{dn}g(W_{it}, \hat{\gamma}_{2\ell}+nb_j, \tilde{\theta}) = \frac{-\delta}{(n-n_\ell)T}\sum_{i\in I_{\ell'}}\sum_{t=1}^{T} D(X_{it})\frac{\Lambda_a(a(X_{it}, \hat{\gamma}_{2\ell}, \hat{\gamma}_{3\ell}, \tilde{\theta}_\ell))}{\Lambda(a(X_{it}))} b_j
\end{aligned}$$

Where the functional form of $\hat{\gamma}_{21\ell}(X_{it})$ and $\hat{\gamma}_{31\ell}(X_{it})$ are:

$$\hat{\gamma}_{21\ell}(X_{it}) = \hat{\mathbb{E}}[H'(\hat{\gamma}_{1\ell}(X_{i,t+1}))b_j(X_{i,t+1})|X_{it}, Y_{2it} = 1]$$
$$\hat{\gamma}_{31\ell}(X_{it}) = \hat{\mathbb{E}}[H'(\hat{\gamma}_{1\ell}(X_{i,t+1}))b_j(X_{i,t+1})|Y_{1it} = 1]$$

H' is the derivative of H w.r.t γ_1).

Let $\hat{M}_{k\ell} = (\hat{M}_{1k\ell}, ..., \hat{M}_{pk\ell})'$, $\hat{N}_{1k} = (\hat{N}_{k\ell}, ..., \hat{N}_{pk\ell})'$ so that:

$$\hat{\psi}_{1k,\gamma_{1\ell}}(b_j, \rho' b) = \frac{\partial}{\partial \rho_j}\{\hat{M}'_{k\ell}\rho - \rho'\hat{Q}_\ell\rho/2\}$$
$$\hat{\psi}_{2k,\gamma_2}(b_j, \beta' Y_2 b) = \frac{\partial}{\partial \beta_j}\{\hat{N}'_{k\ell}\beta - \beta'\hat{S}_\ell\beta/2\}$$

The sample moment condition $-2\hat{\psi}_{1k,\gamma_{1\ell}}(b_j, \rho' b) = 0$ $(j = 1, ..., p)$ is the first order condition for minimizing $-2\hat{M}'_{k\ell}\rho + \rho'\hat{Q}\rho$ $\left(\hat{\psi}_{2k,\gamma_2}(b_j, \beta' Y_2 b) \text{ also follows the same principle}\right)$. Adding an L_1 penalty to these objective functions and minimizing leads to the Lasso minimum distance estimators:

$$\hat{\alpha}_{1k\ell} = \hat{\rho}_{k\ell} b(X), \quad \hat{\rho}_{k\ell} = arg\min_{\rho}\{-2\hat{M}'_{k\ell}\rho + \rho'\hat{Q}_\ell\rho + 2r_1\sum_{j=1}^{p}|\rho_j|\}$$
$$\hat{\alpha}_{2k\ell} = \hat{\beta}_{k\ell} b(X), \quad \hat{\beta} = arg\min_{\beta}\{-2\hat{N}_{k\ell}\beta + \beta'\hat{S}_\ell\beta + 2r_2\sum_{j=1}^{p}|\beta_j|\}$$

Intuition: First we notice that the set $\mathcal{A}_1$ of possible α_1 is Γ_1, the set $\mathcal{A}_2$ of possible α_2 is Γ_2. Since $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are linear combination of the basis of Γ_1, Γ_2 respectively, they are orthogonal to the corresponding residuals $\lambda_1(W, \gamma_{10}(X))$ and $\lambda_2(W, \gamma_{20}(X))$. Therefore, the property of the influence function is preserved:

$$\mathbb{E}[\phi(W, \gamma_{10}, \hat{\alpha}_1, \theta)] = \mathbb{E}[\hat{\alpha}_1(X, \theta)\lambda_1(W, \gamma_{10}(X))] = 0,$$
$$\mathbb{E}[\phi(W, \gamma_{20}, \hat{\alpha}_2, \theta)] = \mathbb{E}[\hat{\alpha}_2(X, \theta)\lambda_2(W, \gamma_{20}(X))] = 0$$

Second, under this approach, the first step estimations of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ have no first order effect on the sample moment functions, and hence on the estimation of the parameter of interest $\hat{\theta}$, because the moment functions (5) and (6) are satisfied. This also ensures that the double Robustness property is satisfied.

After deriving the formulas for these α, we can plug these into the formulas for the influence functions and then follow the estimation procedure in section 2 to obtain the Automatic Debiased Machine Learning estimator of θ. The functional forms of the influence functions are:

$$\phi_1(W, \theta, \alpha, \gamma) = \frac{1}{T}\sum_{t=1}^{T}\alpha_1(X_{t+1}, \theta)[Y_{2,t+1} - \gamma_1(X_{t+1})],$$
$$\phi_2(W, \theta, \alpha, \gamma) = \frac{1}{T}\sum_{t=1}^{T}\alpha_2(X_t, \theta) \cdot Y_{2t} \cdot [H(\gamma_1(X_{t+1})) - \gamma_2(X_t)]$$

After obtaining the influence functions, we can follow the four steps described in Section 2 to obtain the unbiased estimates of the structural parameters of interests θ.

5 Monte Carlo study

In this section, we carried out a Monte Carlo study for a model similar to that of Rust [1987]. The state variables consisted of a positive variable x_1 (mileage) and other variables $x_2, \ldots, x_5$ with transition

$$X_{1,t+1} = 1(Y_{2,t} = 1)X_t + S_{t+1} \tag{15}$$

$$S_{t+1} = \text{iid half-normal} \cdot (1 + z_{t+1}) \tag{16}$$

$$z_t = \sum_{k=1}^{4} c_k X_{t,k+1} \tag{17}$$

where $c = (0.1, 0.025, 0.0111, 0.0063)$. $(X_{2t}, \ldots, X_{5t})$ is i.i.d. over t, X_{2t}, X_{4t} are uniformly distributed as $U(0, \sqrt{12})$, and X_{3t} and X_{5t} are binary with $\Pr(X_{kt} = 1) = \frac{1}{2}$, $k = 3, 5$. We specified that $D(X)$ is two-dimensional with $D_1(X) = (-1, 0)'$ and $D_2(x) = (0, \sqrt{x_1 + 1})'$, and that $\epsilon_{1t}, \epsilon_{2t}$ are independent Type I extreme value, so that $\Lambda(a) = \frac{e^a}{1+e^a}$ corresponds to binary logit.

To generate the data, we solved the Bellman equation on a finite grid using the fact that the state space has a two-dimensional structure in terms of x_1 and $\sum_{k=1}^{4} c_k x_{k+1}$, with linear interpolation between grid points. We did not enforce this index structure in estimation, so that the estimation treated the state space as dimension six. We carried out 500 Monte Carlo replications for $T = 10$ and $n = 100, 300, 1000, 10000$. I specified five-fold cross-fitting, where $L = 5$.

We considered two specifications of the vector $b(x)$ used by Lasso: a) the elements of x and square root of elements of x, b) those from a) and squares of two elements of x.

The conditional choice probability estimators $\hat{\gamma}_1$ was logit Lasso trimmed to be between 0.0001 and 0.9999. We used the MATLAB Lasso and logit Lasso procedures for computation. The regularization values r_1, r_2, and r_3 for each Lasso were chosen by two-fold cross-validation. Although we do not know whether the resulting r satisfy the conditions in the asymptotic theory of Section 6, we do this so that the estimator in the Monte Carlo is based on an "off-the-shelf" machine learner of unknown functions.

The results are reported in Tables 2 and 3. The PI labels the plug-in GMM estimator based only on identifying moment functions, ADB is the Automatic debiased GMM, Bias is the absolute value of bias, SD denotes standard deviation, and Cvg denotes coverage probability of a nominal 95 percent confidence interval.

	PI Bias	ADB Bias	PI SD	ADB SD	PI Cvg	ADB Cvg
θ_1 (n=100)	0.030	0.002	0.227	0.233	0.970	0.944
θ_2 (n=100)	0.030	0.002	0.102	0.108	0.984	0.936
θ_1 (n=300)	0.012	0.002	0.134	0.137	0.950	0.924
θ_2 (n=300)	0.012	0.001	0.060	0.063	0.984	0.928
θ_1 (n=1000)	0.002	0.005	0.073	0.074	0.956	0.945
θ_2 (n=1000)	0.006	0.002	0.033	0.034	0.984	0.946
θ_1 (n=10000)	0.001	0.004	0.023	0.023	0.964	0.934
θ_2 (n=10000)	0.001	0.002	0.010	0.010	0.992	0.946

Table 2: $b(x)$ are linear and $\sqrt{x}$

	PI Bias	ADB Bias	PI SD	ADB SD	PI Cvg	ADB Cvg
θ_1 (n=100)	0.029	0.002	0.227	0.234	0.968	0.936
θ_2 (n=100)	0.028	0.002	0.101	0.109	0.980	0.934
θ_1 (n=300)	0.011	0.001	0.134	0.138	0.952	0.934
θ_2 (n=300)	0.012	0.001	0.060	0.063	0.984	0.936
θ_1 (n=1000)	0.000	0.002	0.072	0.073	0.960	0.946
θ_2 (n=1000)	0.004	0.001	0.033	0.034	0.986	0.956
θ_1 (n=10000)	0.002	0.001	0.022	0.023	0.968	0.946
θ_2 (n=10000)	0.000	0.000	0.010	0.010	0.992	0.966

Table 3: $b(x)$ are linear, $\sqrt{x}$, and $\sqrt{1+x}$

For less flexible basis functions (Table 2), the ADB estimator demonstrates a smaller bias compared to the PI estimator in smaller sample sizes. Notably, the standard deviation of the ADB estimator is comparable to that of the PI estimator as the sample size increases. For more flexible basis functions (Table 3), the ADB estimator exhibits smaller bias relative to the PI estimator in almost all sample sizes, and ADB coverage probabilities are quite close to the nominal value.

6 Asymptotic Theory

Here, we obtain mean square convergence rates of the Lasso minimum distance learner of $\hat{\alpha}$ and root-n consistency and asymptotic normality results for the learner $\hat{\theta}$ of the target parameters and for its corresponding asymptotic variance estimator $\hat{V}$. My proof extends the results in Chernozhukov et al. [2022b] to Dynamic Discrete Choice problems. Let ϵ_n be a sequence that converges to zero no faster than $ln(p)/n$ (where p is the number of basis functions of the set Γ) and for a random variable $a(W)$, let $\|a\| = \sqrt{\mathbb{E}[a(W)^2]}$.

Assumption 2: There exist $C > 1$, $\xi > 0$ such that for each positive integers $s \leq C\epsilon^{\frac{-2}{2\xi+1}}$, there is $\hat{\rho}$ with s nonzero elements such that:

$$\|\bar{\alpha} - b(X)'\bar{\rho}\| \leq C(s)^{-\xi}$$

Here $\|\bar{\alpha} - b(X)'\bar{\rho}\|$ is the mean square approximation error in approximating $\bar{\alpha}$ by the linear combination $b(X)'\bar{\rho}$. This approximate sparsity condition requires that there is a sparse $\bar{\rho}$ with only s non-zero elements such that the approximation error is less than or equal to $C(s)^{-\xi}$.

Our findings need a rate of convergence for $\hat{\alpha}$ faster than some power of n. Assumption 2 is a natural one that is useful for achieving such a rate. Sufficient conditions for Assumption 2 are familiar in the approximation literature, particularly when $\bar{\alpha}$ is in a Besov or Holder class of function and linear combinations of $b(x)$ can approximate any function of x.

We will also employ a sparse eigenvalue condition, a concept well researched in Lasso literature. For a vector ρ of dimension $p \times 1$, let ρ_J be a $J \times 1$ subvector of ρ and ρ_J^c be the vector comprising elements of ρ not in ρ_J. Also, for a matrix A, define $||A||_1 = \sum_{i,j} |a_{ij}|$.

Assumption 3: $G = E[b(X)b(X)']$ has the largest eigenvalue bounded uniformly in n, and there are $C, c > 0$ such that for all $s \approx C\epsilon_n^{-2}$ with probability approaching 1,

$$\min_{J \le s} \min_{\|\rho_{J^c}\|_1 \le 3\|\rho_J\|_1} \frac{\rho' \hat{\gamma} \rho}{\rho'_J \rho_J} \ge c$$

This is a sparse eigenvalue condition that is familiar from the Lasso literature, including Belloni et al. [2017], Bickel et al. [2009] and Rudelson and Zhou [2012].

We will work with a dictionary b(X) with elements that are uniformly bounded.

Assumption 4: There is $C > 0$ such that, with probability 1, $\sup_j |b_j(X)| \le C$. This condition implies a convergence rate of $\sqrt{\ln(p)/n}$ for $\|\hat{\gamma} - G\|_\infty$, where $\|A\|_\infty = \max_{i,j} |a_{ij}|$ for a matrix $A = [a_{ij}]$.

This condition indicates a convergence rate of $\sqrt{ln(p)/n}$ for $||\hat{G} - G||_\infty$, with $||A||_\infty = \max_{i,j} |a_{ij}|$ for a matrix $A = [a_{ij}]$. Lasso mean square convergence rates are most commonly given in terms of finite sample bounds. However, as this paper's focus is root-n consistency of $\hat{\theta}$ and for some root-n convergence at certain powers of n, we simplify the convergence rate to a statement without altering the condition for $\hat{\theta}$ by allowing the Lasso regularization parameter r to decay a bit less rapidly than ϵ_n. This adjustment leads to sparseness conditions in terms of strict inequalities on the magnitude of ξ. Nevertheless, Bradic et al. [2019] have established that in fact, root-n consistency requires strict inequalities and hence no generality is lost through these conditions. We also limit the growth of p to be smaller than some power of n.

Assumption 5: $\epsilon_n = o(r)$, $r = o(n^c \epsilon_n)$ for all $c > 0$, and there exists $C > 0$ such that $p \le Cn^C$.

We also hypothesize a convergence rate for $\hat{M}$.

Assumption 6: $\|\hat{M} - M\|_\infty = O_p(\epsilon_n)$ as $\epsilon_n \to 0$.

Theorem 1: If Assumptions 1–6 are satisfied, then, for all $c > 0$, $\|\hat{\alpha} - \bar{\alpha}\| = o_p\left(\frac{nc\epsilon^2\xi}{(2\xi+1)n}\right)$.

This theorem is based on extending lemmas of Bradic et al. [2019] to allow ϵ_n to shrink slower than $\sqrt{ln(p)/n}$.

Next, We continue with conditions for the key property:

$$\sqrt{n}\hat{\psi}(\theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(W_i, \theta_0, \gamma_0, \alpha_0) + o_p(1) \tag{18}$$

Assumption D1: $\mathbb{E}[||\psi(w, \gamma_0, \theta_0, \alpha_0)||^2] < \infty$, and

i) $\int ||g(w, \hat{\gamma}, \theta_0) - g(w, \gamma_0, \theta_0)||^2 F_0(dw) \xrightarrow{p} 0$;

ii) $\int ||\phi(w, \hat{\gamma}, \alpha_0, \theta_0) - \phi(w, \gamma_0, \alpha_0, \theta_0)||^2 F_0(dw) \xrightarrow{p} 0$;

iii) $\int ||\phi(w, \gamma_0, \hat{\alpha}, \tilde{\theta}) - \phi(w, \gamma_0, \alpha_0, \theta_0)||^2 F_0(dw) \xrightarrow{p} 0$.

These are mild mean-square consistency conditions for $\hat{\gamma}$ and $(\hat{\alpha}, \tilde{\gamma})$ separately. Let $\hat{\Delta}(w) := \phi(w, \hat{\gamma}, \hat{\alpha}, \tilde{\theta}) - \phi(w, \gamma_0, \hat{\alpha}, \tilde{\theta}) - \phi(w, \hat{\gamma}, \alpha_0, \theta_0) + \phi(w, \gamma_0, \alpha_0, \theta_0)$.

Assumption D2: Either

i) $\sqrt{n} \int \hat{\Delta}(w) F_0(dw) \xrightarrow{p} 0$, $\int \hat{\Delta}(w)^2 F_0(dw) dw \xrightarrow{p} 0$;

ii) $||\hat{\Delta}(W_i)||^2/\sqrt{n} \xrightarrow{p} 0$, or

iii) $\hat{\Delta}(w_i)/\sqrt{n} \xrightarrow{p} 0$.

This condition imposes a rate condition on the interaction remainder $\hat{\Delta}(w)$, that its average must go to zero faster than $1/\sqrt{n}$.

Assumption D3:

i) $\int \phi(w, \gamma_0, \hat{\alpha}, \tilde{\theta})F_0(dw) = 0$ with probability approaching one; and either

ii) $\bar{\psi}(\gamma, \alpha_0, \theta_0)$ is affine in γ; or

iii) $|\hat{\gamma} - \gamma_0| = o_p(n^{-1/4})$ and $\bar{\psi}(\gamma, \alpha_0, \theta_0) \leq C|\gamma - \gamma_0|^2$ for all γ with $|\gamma - \gamma_0|$ small enough; or

iv) $\sqrt{n}\bar{\psi}(\hat{\gamma}, \alpha_0, \theta_0) \xrightarrow{p} 0$.

Assumption D3 marries the important characteristic of the influence function in equation 1 of Section 2 with doubly robust moment functions through condition (ii). Assumption D3 in this context imposes no additional conditions beyond Assumptions D1 and D2. Conditions (iii) and (iv) provide other minimal bias requirements, needed only for $\hat{\gamma}$, but not for $\hat{\alpha}$. Condition (iii) insists that there is a rate of convergence of $\hat{\gamma}$ that is faster than $n^{-1/4}$. Such a condition is a standard condition in semiparametric estimation literature. In most cases, condition (iii) can be satisfied by the mean-square norm $||a|| = \sqrt{E[a(W)]^2}$ thereby implying that Assumptions D1–D3 will only need mean-square rate convergence.

Lemma 1: If Assumptions D1-D3 are satisfied,then equation 18 is satisfied.

Assumption D4:

i) $\int ||g(w, \hat{\gamma}, \tilde{\theta}) - g(w, \hat{\gamma}, \theta_0)||^2 F_0(dw) \xrightarrow{p} 0$;

ii) $\int ||\hat{\Delta}(w)||^2 F_0(dw) \xrightarrow{p} 0$.

It is also important to have conditions for convergence of the Jacobian of the identifying sample moments $\frac{\partial \hat{g}(\bar{\theta})}{\partial \theta} \xrightarrow{p} G = E\left[\frac{\partial g(W,\gamma_0,\theta_0)}{\partial \theta}\right]$ for any $\bar{\theta} \xrightarrow{p} \theta_0$.

Assumption D5: G exists, and there is a neighborhood $\mathcal{N}$ of θ_0 and a norm $\|\cdot\|$ such that:

i) $||\hat{\gamma} - \gamma_0|| \xrightarrow{p} 0$;

ii) For all $||\gamma - \gamma_0||$ small enough, $g(W, \gamma, \tilde{\theta})$ is differentiable in θ on $\mathcal{N}$ with probability approaching 1, and there is $C > 0$ and $d(W, \gamma)$ such that for $\theta \in \mathcal{N}$ and $||\gamma - \gamma_0||$ small enough,

$$\left\| \frac{\partial G(W, \gamma, \theta)}{\partial \theta} - \frac{\partial G(W, \gamma_0, \theta)}{\partial \theta} \right\| \leq d(W, \gamma)||\theta - \theta_0||^{1/C}; \quad \mathbb{E}[d(W, \gamma)] < C$$

iii) For all k,

$$\int \left| \frac{\partial G(W, \hat{\gamma}, \theta_0)}{\partial \theta_k} - \frac{\partial G(W, \gamma_0, \theta_0)}{\partial \theta_k} \right| F_0(dw) \xrightarrow{p} 0.$$

Assumption D6:

i) There is $\forall \epsilon > 0$ such that $\gamma_{10}(X) \in [\epsilon, 1 - \epsilon]$, and $H(p)$ is twice continuously differentiable on $[\epsilon, 1 - \epsilon]$.

ii) $|\hat{\gamma}_1 - \gamma_0| = O_p(n^{-d_1})$, $0 < d_1 < 1/2$.

iii) Assumptions D1, D3, D4, and D5 are satisfied with $\alpha_0(x) = \gamma_{20}(x)$, $\epsilon_n = n^{-d_1}$ and sparse approximation rate $\xi_1 > 1/2$.

iv) $n^{-d_1} = o(r_1)$ and $r_1 = O(n^{-d_1} ln(n))$.

Theorem 2: If

i) Assumption D6 is satisfied,

ii) $\Lambda(a) > 0$ for all $a \in \mathbb{R}$, $\ln \Lambda_a(a)$ is concave, $\Lambda(a)$ is twice differentiable with uniformly bounded derivatives, $D(X)$ is bounded, and $\mathbb{E}[D(X)D(X)^T]$ is nonsingular,

iii) Assumptions D1 and D2 are satisfied for $\alpha_0(x)$ equal to each element of

$\mathbb{E}[D(X_t)\Lambda_a(a(X_t))Y_{2t})/\Lambda(a(X_t))|X_{t+1} = x]$ with sparse approximation rate ξ_3 and for $\mathbb{E}[Y_{1t}|X_{t+1} = x]$ with sparse approximation rate ξ_2,

iv) $d_1 > 1/4$,

v) $1 + [(2\xi_1 - 1)/(2\xi_1 + 1)] \cdot 2\xi_2/(2\xi_2 + 1) > 1/2d_1$, $n^{-d_1(2\xi_1-1)/(2\xi_1+1)} = o(r_2)$, and $r_2 = O(n^{-d_1(2\xi_1-1)/(2\xi_1+1)} \ln(n))$,

vi) $\xi_3/(2\xi_3 + 1) + d_1 > 1/2$, $\sqrt{\ln(p)/n} = o(r_3)$, and $r_3 = O(\sqrt{\ln(p)/n} \ln(n))$,

vii) $(4\xi_1 - 1)/(2\xi_1 + 1) > 1/(2d_1)$,

then for $V = G^{-1}\mathbb{E}[\psi_0(W)\psi_0(W)^T]G^{-1}$

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, V), \hat{V} \xrightarrow{p} V$$

References

V. Aguirregabiria and P. Mira. Swapping the nested fixed point algorithm: A class of estimators for discrete markov decision models. *Econometrica*, 70(4):1519–1543, 2002.

S. Athey, G. W. Imbens, and S. Wager. Approximate residual balancing: debiased inference of average treatment effects in high dimensions. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 80(4):597–623, 2018.

A. Belloni, V. Chernozhukov, I. Fernández-Val, and C. Hansen. Program evaluation and causal inference with high-dimensional data. *Econometrica*, 85(1):233–298, 2017.

P. J. Bickel, Y. Ritov, and A. B. Tsybakov. Simultaneous analysis of Lasso and Dantzig selector. *The Annals of Statistics*, 37(4):1705 – 1732, 2009. doi: 10.1214/08-AOS620. URL `https://doi.org/10.1214/08-AOS620`.

J. Bradic, V. Chernozhukov, W. K. Newey, and Y. Zhu. Minimax semiparametric learning with approximate sparsity. *arXiv preprint arXiv:1912.12213*, 2019.

M. D. Cattaneo and M. Jansson. Kernel-based semiparametric estimators: Small bandwidth asymptotics and bootstrap consistency. *Econometrica*, 86(3):955–995, 2018.

M. D. Cattaneo, M. Jansson, and X. Ma. Two-step estimation and inference with possibly many included covariates. *The Review of Economic Studies*, 86(3):1095–1122, 2019.

V. Chernozhukov, J. C. Escanciano, H. Ichimura, W. K. Newey, and J. M. Robins. Locally robust semiparametric estimation. *Econometrica*, 90(4):1501–1535, 2022a.

V. Chernozhukov, W. K. Newey, and R. Singh. Automatic debiased machine learning of causal and structural effects. *Econometrica*, 90(3):967–1027, 2022b. doi: https://doi.org/10.3982/ECTA18515. URL `https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA18515`.

V. Chernozhukov, W. Newey, R. Singh, and V. Syrgkanis. Automatic debiased machine learning for dynamic treatment effects and general nested functionals, 2023.

M. H. Farrell. Robust inference on average treatment effects with possibly more covariates than observations. *Journal of Econometrics*, 189(1):1–23, 2015.

V. J. Hotz and R. A. Miller. Conditional choice probabilities and the estimation of dynamic models. *The Review of Economic Studies*, 60(3):497–529, 1993.

W. K. Newey and J. R. Robins. Cross-fitting and fast remainder rates for semiparametric estimation. *arXiv preprint arXiv:1801.09138*, 2018.

J. M. Robins, P. Zhang, R. Ayyagari, R. Logan, E. T. Tchetgen, L. Li, T. Lumley, A. van der Vaart, H. H. R. Committee, et al. New statistical approaches to semiparametric regression with application to air pollution research. *Research report (Health Effects Institute)*, (175):3–129, 2013.

M. Rudelson and S. Zhou. Reconstruction from anisotropic random measurements. In *Conference on Learning Theory*, pages 10–1. JMLR Workshop and Conference Proceedings, 2012.

J. Rust. Optimal replacement of gmc bus engines: An empirical model of harold zurcher. *Econometrica: Journal of the Econometric Society*, pages 999–1033, 1987.

7 Appendix: Proof of Theorems 1 and 2

Proof of theorem 1 is a direct consequence from Chernozhukov et al. [2023]. In this appendix, we provide regularity conditions and convergence rates for a Lasso Minimum distance estimator of $\alpha_0(X) = \arg\min_{\alpha \in \Gamma} E[w(X)\{w(X)^{-1}v_m(X) - \alpha(X)\}^2]$, with respect to a dictionary $(b_1(X), b_2(X), \ldots)$ having elements in a mean square closed set $\mathbb{R}$, a subvector $b(X) = (b_1(X), \ldots, b_p(X))'$, an estimator $\hat{Q}$ of $Q = E[w(X)b(X)b(X)']$, and an estimator $\hat{M}$ of $M = E[v_m(X)b(X)]$, where $w(X) > 0$ is bounded and bounded away from zero. Conditions and estimates used here use estimates given in Chernozhukov et al. [2023]. For a matrix $A = [a_{ij}]$ we define $|A|_\infty = \max_{i,j} |a_{ij}|$, $|A|_1 = \sum_{i,j} |a_{ij}|$, and for a measurable function $a(X)$ of X we define $\|a\| = \sqrt{E[a(X)^2]}$. We consider an estimator of $\alpha_0(x)$ to be

$$\hat{\alpha}(x) = b(x)'\hat{\rho}, \quad \hat{\rho} = \arg\min_{\rho} \left\{ -2\hat{M}'\rho + \rho'\hat{Q}\rho + 2r \sum_{j=1}^{p} |\rho_j| \right\}.$$

Assumption A1: There is $C > 0$ such that (i) $\max_{1 \le j \le p} |b_j(X)| \le C$ and $w(X) \ge 1/C$; (ii) for every n, there is a $p \times 1$ vector ρ_n such that $|\rho_n|_1 \le C$ and $\|\alpha_0 - b'\rho_n\|^2 = O(\epsilon_n)$.

Assumption A2: (i) $|\hat{Q} - Q|_\infty = \mathrm{Op}(\epsilon_n)$, (ii) $|\hat{M} - M|_\infty = \mathrm{Op}(\epsilon_n)$; (iii) $\epsilon_n = o(r)$.

Lemma D1: If Assumptions D1 and D2 are satisfied, then $\|\hat{\alpha} - \alpha_0\| = \mathrm{Op}(\sqrt{r})$.

Proof: Assumptions D1 and D2 imply Assumptions 1–3 of Chernozhukov et al. [2022b]. The conclusion then follows by Theorem 1 of CNS. Q.E.D.

Note that we make the assumption that the moments g_t is sufficiently smooth with respect to γ, then we can also approximate ∂g_t by a finite difference:

$$\hat{\partial} g_t(W, \gamma, \theta) \approx g_t(W, \gamma + \epsilon b, \theta) - g_t(W, \gamma, \theta)$$

which for $\epsilon = o(n^{-1})$ will add negligible extra error.

This result gives a convergence rate for $\hat{\alpha}_\ell$ of $\sqrt{r}$. We can speed up this convergence rate under a stronger approximate sparsity condition and a sparse eigenvalue condition.

Assumption A3: There exist $C > 1$, $\xi > 0$ such that for all $\bar{s}$ with $\bar{s} \le C(\epsilon_n^2)^{-1/(1+2\xi)}$, there is $\bar{\rho}$ with $\bar{s}$ nonzero elements such that $\|\alpha_0 - b'\bar{\rho}\| \le C(\bar{s})^{-\xi}$.

For any $\rho = (\rho_1, \ldots, \rho_p)$, let $\mathcal{J} = \{1, 2, \ldots, p\}$, J_ρ be the subset of $\mathcal{J}$ with $\rho_j = 0$, and J_ρ^c be the complement of J_ρ in J and let $\rho_L = \arg\min_\rho \{\|\alpha_0 - b'\rho\|^2 + 2\epsilon_n |\rho|_1\}$ be the population Lasso coefficients for penalty $2\epsilon_n$. The next condition is a sparse eigenvalue condition for the population matrix $\mathcal{Q}$

Assumption A4: $\mathcal{Q}$ is nonsingular and has largest eigenvalue uniformly bounded in n. Also, there is $k > 3$ such that for $\rho = \rho_L$,

$$\inf_{\{\delta : \delta \neq 0, \sum_{j \in \mathcal{J}_{\rho_L}^c} |\delta_j| \le k \sum_{j \in \mathcal{J}_{\rho_L}} |\delta_j|\}} \frac{\delta' Q \delta}{\sum_{j \in \mathcal{J}_{\rho_L}} \delta_j^2} > 0.$$

Lemma D2: If Assumptions D1–D4 are satisfied, then $\|\hat{\alpha} - \alpha_0\| = O_p\left((\epsilon_n)^{\frac{-1}{1+2\xi}} r\right)$.

Proof: Assumptions D1–D4 imply Assumptions 1–5 of Chernozhukov et al. [2022b], so the conclusion follows by Theorem 3 of Chernozhukov et al. [2022b]. Q.E.D.

The convergence rate here is faster than that of Lemma D1 when $\xi > 1/2$ and r is not too much larger than ϵ_n. For Theorem 2, it is useful to have a uniform convergence rate for Lasso minimum distance.

Lemma D3: If Assumptions D1 - D4 are satisfied, $\xi > 1/2$, $r = O(\ln(n)\epsilon_n)$, and there exists $C > 0$ such that $p \leq Cn^C$, then there is $C > 0$ and a $p \times 1$ vector ρ such that for $\alpha_n(x) = b(x)^T \rho$ we have $|\rho|_1 \leq C$, $\sup_x |\alpha_n(x)| \leq C$, and

$$|\hat{\rho} - \rho|_1 + \sup_x |\hat{\alpha}(x) - \alpha_n(x)| + \|\alpha_n - \alpha_0\| = \mathrm{Op}\left(\epsilon_n^{\frac{2\xi-1}{2\xi+1}} \ln(n)\right).$$

Proof: We make use of the results of Appendix C.1 of Chernozhukov et al. [2022b]. Choose ρ here to equal the ρ_L defined immediately preceding Lemma C2 of Chernozhukov et al. [2022b]. Then the first two conclusions follow by Lemma C2 of CNS and the elements of $b(x)$ uniformly bounded which give $|\alpha_n(x)| \leq C|\rho|_1$. By the last three lines on page 44 in the proof of Theorem 3 of Chernozhukov et al. [2022b], we have

$$|\hat{\rho} - \rho|_1 = \mathrm{Op}\left(\epsilon_n^{-\frac{2}{2\xi+1}} r\right).$$

So that by $r = O(\ln(n)\epsilon_n)$, $|\hat{\rho} - \rho|_1 = O_p\left(\epsilon_n^{\frac{2\xi-1}{2\xi+1}} \ln(n)\right)$. Similarly, by the triangle inequality, $\sup_x |\hat{\alpha}(x) - \alpha_n(x)| \leq C|\hat{\rho} - \rho|_1 = O_p\left(\epsilon_n^{\frac{2\xi-1}{2\xi+1}} \ln(n)\right)$. It also follows by Lemma C4 of Chernozhukov et al. [2022b] that the third inequality holds, so that the conclusion follows by the triangle inequality.

Proof of Theorem 2:

Lemma D4: If Assumption D6 and the hypotheses of Theorem 12 are satisfied, then:

$$\|\hat{\alpha}_{1\ell} - \alpha_{10}\| = O_p\left(n^{-d_1}\left(\frac{2\xi_1 - 1}{2\xi_1 + 1}\right)^{\frac{2\xi_2}{2\xi_2+1}} \ln(n)^2 + n^{-\frac{\xi_3}{2\xi_3+1}} \ln(n) + n^{-d_1}\right)$$

$$\|\hat{\alpha}_{2\ell} - \alpha_{20}\| = O_p\left(n^{-d_1}\left(\frac{2\xi_1 - 1}{2\xi_1 + 1}\right) \ln(n)\right)$$

$$\|\hat{\alpha}_{3\ell} - \alpha_{30}\| = O_p\left(n^{-d_1}\frac{2\xi_1 - 1}{2\xi_1 + 1}\right)$$

$$||H(\hat{\gamma}_{1\ell}) - H(\gamma_{10})|| = O_p(n^{-d_1})$$

Proof: Let $\gamma_{2n}(x)$ be as in the conclusion of Lemma E6 and define $a_n(x) = a(x, \theta_0, \gamma_{2n}, \gamma_{30})$. Note that $a(x)$ and $a_n(x)$ are bounded by Lemma D3 and $D(x)$ bounded, and $H(\gamma_{10}(x))$ is bounded by $\gamma_{10}(x) \in (\epsilon, 1-\epsilon)$. By Assumption 8, Lemma D3, and the fixed trimming, with $\Lambda(a) > 0$ and twice continuous differentiability of $\Lambda(a)$,

$$\begin{aligned}\sup_x |\hat{\alpha}_{2\ell k}(x, y_2) - \hat{\alpha}_{2nk}(x, y_2)| &\leq C \sup_x |\hat{a}(x) - a_n(x)| \\ &\leq C(\sup_x |\hat{\gamma}_{2\ell}(x) - \gamma_{2n}(x)| + |\hat{\gamma}_2 - \gamma_{20}|) \\ &= O_p\left(n^{-d_1} \frac{2\xi_1 - 1}{2\xi_1 + 1} \ln(n)\right)\end{aligned}$$

Similarly, $||\hat{\alpha}_{2nk}(k) - \alpha_{20}(k)|$ is the same order in probability, so the second conclusion follows by the triangle inequality. The third conclusion follows in a standard way. The last three equality follows immediately from Lemma E8 in Chernozhukov et al. [2022a].

Next, I will show that Assumption D2 is satisfied in this case. Let $\hat{\Delta}(w) := \phi(w, \hat{\gamma}, \hat{\alpha}, \tilde{\gamma}) - \phi(w, \gamma_0, \hat{\alpha}, \tilde{\gamma}) - \phi(w, \hat{\gamma}, \alpha_0, \gamma_0) + \phi(w, \gamma_0, \alpha_0, \gamma_0)$ be the estimation remainder. Note that we have:

$$\begin{aligned}
&\hat{\Delta}_\ell(w) = \hat{\Delta}_{\ell 1}(w) + \hat{\Delta}_{\ell 2}(w) + \hat{\Delta}_{\ell 3}(w) \\
&\hat{\Delta}_{\ell 1}(w) = -\frac{1}{T} \sum_{t=1}^{T} [\hat{\alpha}_{1\ell}(x_t) - \alpha_{10}(x_t)] [\hat{\gamma}_{1\ell}(x_t) - \gamma_{10}(x_t)] \\
&\hat{\Delta}_{\ell 2}(w) = \frac{1}{T} \sum_{t=1}^{T} [\hat{\alpha}_{2\ell}(x_t, y_{2t}) - \alpha_{20}(x_t, y_{2t})] \cdot [H(\hat{\gamma}_{1\ell}(x_t) - \hat{\gamma}_{2\ell}(x_t) - H(\gamma)_{10}(x_t) + \gamma_{20}(x_t)] \\
&\hat{\Delta}_3(w) = \frac{1}{T} \sum_{t=1}^{T} (\hat{\alpha}_{3\ell} - \alpha_{30})\, y_{1t} \left(H\left(\hat{\gamma}_1(x_{t+1})\right) - \hat{\gamma}_3 - H\left(\hat{\gamma}_1(x_{t+1})\right) + \gamma_{30}\right)
\end{aligned}$$

Then from the results of Theorem 1 and conditions i-iii of assumption D6, I have:

$$\begin{aligned}
&\sqrt{n} \int \sqrt{\hat{\Delta}_1(w)} F_0(dw) \\
&\quad \leq \sqrt{n} \|\hat{\alpha}_{1\ell} - \alpha_{10}\| \|\hat{\gamma}_1 - \gamma_{10}\| \\
&\quad = O_p\left(n^{-d_1} \left(\frac{2\xi_1 - 1}{2\xi_1 + 1}\right)^{\frac{2\xi_2}{2\xi_2+1}} \ln(n)^2 + n^{-\frac{\xi_3}{2\xi_3+1}} \ln(n) + n^{-d_1}\right)) \cdot O_p(n^{-d_1}) = o_p(1) \\
&\sqrt{n} \int \sqrt{\hat{\Delta}_2(w)} F_0(dw) \\
&\quad \leq \sqrt{n} \|\hat{\alpha}_{2\ell} - \alpha_{20}\| \, |H(\hat{\gamma}_1) - \hat{\gamma}_2 - H(\gamma_{10}) + \gamma_{20}| \\
&\quad = O_p\left(n^{-d_1} \left(\frac{2\xi_1 - 1}{2\xi_1 + 1}\right) \ln(n)\right) \cdot O_p(\ln(n)) = o_p(1) \\
&\sqrt{n} \int \sqrt{\hat{\Delta}_3(w)} F_0(dw) \\
&\quad \leq \sqrt{n} \|\hat{\alpha}_{3\ell} - \alpha_{30}\| \, |H(\hat{\gamma}_1) - \hat{\gamma}_3 - H(\gamma_{10}) + \gamma_{30}| = O_p\left(n^{-d_1} \frac{2\xi_1 - 1}{2\xi_1 + 1}\right) \cdot O_p(n^{-d_1}) = o_p(1)
\end{aligned}$$

This implies that Assumption D2 is verified. Next, Assumption D3(i) follows by the form of ϕ_1, ϕ_2, and ϕ_3 given in Section 2.2, and

$$E\left[Y_{2t} - \gamma_{10}(X_t) \,|\, X_t\right] = 0$$

$$E\left[Y_{2t}\left\{H\left(\gamma_{10}(X_{t+1})\right) - \gamma_{20}(X_t)\right\} \,\middle|\, X_t\right] = 0$$

$$E\left[Y_{1t}\left\{H\left(\gamma_{10}(X_{t+1})\right)-\gamma_{30}\right\}\right]=0$$

Next I will show that Assumption 3(iv) is satisfied. Let $\hat{a}(x)=a(x,\theta_0,\hat{\gamma}_2,\hat{\gamma}_3)$. Then

$$\hat{\psi}(\gamma,\hat{\alpha}_0,\theta_0)=T+T1+T2+T3$$

$$T=\int D(x_t)\{y_{2t}-\Lambda(\hat{\alpha}(x_t))\}dF_0(dw)$$

$$T_1=\int \alpha_{10}(x_t)\left(y_{2t}-\hat{\gamma}_1(x_t)\right)dF_0(dw)$$

$$T_3=\alpha_{30}\int y_{1t}\left[H(\hat{\gamma}_1(x_{t+1}))-\hat{\gamma}_3\right]dF_0(dw)$$

$$T2=\int \alpha_{20}(x_t,y_{2t})[H\left(\gamma_1(x_{t+1})\right)-\hat{\gamma}_2(x_t)]dF_0(dw)$$

Note that

$$T=\bar{T}_1+\bar{T}_2+R_1+R_2$$

$$\bar{T}_1=-\delta\int D(x_t)\Lambda_a(a(x_t))\{\gamma_2(x_t)-\gamma_{20}(x_t)\}dF_0(dw)$$

$$\bar{T}_2=A(\hat{\gamma}_3-\gamma_{30}),$$

$$A=\delta\int D(x_t)\Lambda_a(a(x_t)))dF_0(dw)$$

$$R_1=-\int D(x_t)\Lambda_{aa}(\bar{a}(x_t))|\hat{a}(x_t)-a(x_t)|^2dF_0(dw)$$

$$R_2=-\int D(xt)\Lambda_a(a(x_t))[\hat{a}(x_t)-a(x_t)]dF_0(dw)$$

Also,

$$|R_1|\leq C\left(\|\hat{\gamma}_2-\gamma_{20}\|^2+|\hat{\gamma}_3-\gamma_{30}|\right)=O_p\left(n^{-4d1\zeta_1/(2\zeta_1+1)}\right)=o_p\left(n^{-1/2}\right)$$

$$|R_2|\leq C\left(\|\hat{\gamma}_2-\gamma_{20}\|^2+|\hat{\gamma}_3-\gamma_{30}|^2\right)=o_p\left(n^{-1/2}\right)$$

so that

$$T=\bar{T}_1+\bar{T}_2+o_p\left(n^{-1/2}\right)$$

Next, note that by the definition of $\alpha_{20}(x_t,y_{2t})$,

$$\bar{T}_1=\int \alpha_{20}(x_t,y_{2t})\{\gamma_2(x_t)-\gamma_{20}(x_t)\}dF_0(dw)$$

Therefore,

$$\bar{T}_1+T_2=\tilde{T}_2,$$

$$\tilde{T}_2=\int \alpha_{20}(x_t,y_{2t})\{H\left(\hat{\gamma}_1(x_{t+1})\right)-\gamma_{20}(x_t)\}dF_0(dw)$$

Note that by $\gamma_{20}(x)=E\left[H(\gamma_{10}(X_{t+1}))|X_t=x,Y_{2t}=1\right]$ and subtracting and adding the expression

$\alpha_{20}(x_t, y_{2t})H(\gamma_{10}(x_{t+1}))F_0(dw)$, I obtain

$$\tilde{T}_2 = \int \alpha_{20}(x_t, y_{2t})\{H(\hat{\gamma}_1(x_{t+1})) - H(\gamma_{10}(x_{t+1}))\}F_0(dw) + \int \alpha_{20}(x_t, y_{2t})\{H(\gamma_{10}(x_{t+1})) - \gamma_{20}(x_t)\}dF_0(dw)$$

$$= \int \alpha_{20}(x_t, y_{2t})\{H(\hat{\gamma}_1(x_{t+1})) - H(\gamma_{10}(x_{t+1}))\}F_0(dw)$$

Expanding terms give:

$$\tilde{T}_2 = \check{T}_2 + R_3$$

$$\check{T}_2 = \int \alpha_{20}(x_t, y_{2t})H_p(\gamma_{10}(x_{t+1}))\{\hat{\gamma}_1(x_{t+1}) - \gamma_{10}(x_{t+1})\}dF_0(dw)$$

$$R_3 = \int \alpha_{20}(x_t, y_{2t})H_{pp}(\bar{\gamma}_1(x_{t+1}))\{\hat{\gamma}_1(x_{t+1}) - \gamma_{10}(x_{t+1})\}^2 dF_0(dw)$$

where $\bar{\gamma}_1(x_t)$ is between $\hat{\gamma}_1(x_t)$ and $\gamma_{10}(x_t)$. It follows similarly to previous arguments that $|R_3| \leq C|\hat{\gamma}_1 - \gamma_{10}|^2 = O_p(n^{-2d1}) = o_p(n^{-1/2})$, so that $\tilde{T}_2 = \hat{T}_2 + o_p(n^{-1/2})$.

Also,

$$\hat{T}_2 = \int \zeta_{10}(x_t)H(\gamma_{10}(x_t))\{(\hat{\gamma}_1(x_t) - \gamma_{10}(x_t)\}dF_0(dw)$$

$$\zeta_{10}(x) = E[\alpha_{20}(X_t, Y_{2t})|X_{t+1} = x]$$

Note that

$$\alpha_{10}(x) = \zeta_{10}(x) + \alpha_{30}\zeta_{20}(x)$$

$$\zeta_{20}(x) = E[Y_{1t}|X_{t+1} = x]$$

Then by $\int \zeta_{10}(x_t)H(\gamma_{10}(x_t))[y_{2t} - \gamma_{10}(x_t)]dF_0(dw) = 0$, I have:

$$\tilde{T}_2 + T_1 = \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))[y_{2t} - \hat{\gamma}_1(x_t)]dF_0(dw)$$

$$= \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))[\gamma_{10}(x_t) - \hat{\gamma}_1(x_t)]dF_0(dw)$$

Next, note that by iterated expectations and $\alpha_{30} = A/P_1$,

$$T_3 = \alpha_{30}\int y_{1t}H(\hat{\gamma}_1(x_{t+1}))dF_0(dw) = \alpha_{30}\int \zeta_{20}(x_t)H(\hat{\gamma}_1(x_t))dF_0(dw) - A\hat{\gamma}_3$$

Note also that

$$A\gamma_3 = AE[y_{1t}H(\gamma_{10}(x_{t+1}))/P_1] = \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))dF_0(dw)$$

Then, by an expansion,

$$\bar{T}_2 + T_3 = \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))dF_0(dw) - A\gamma_3$$

$$= \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))dF_0(dw) - A\gamma_3$$

$$= \alpha_{30}\int \zeta_{20}(x_t)H(\gamma_{10}(x_t))(\gamma_{10}(x_t) - \hat{\gamma}_1(x_t))dF_0(dw) + R4$$

$$= -(\check{T}_2 + T_1) + R_4,$$

$$R_4 = \alpha_{30} \int H_{pp}\left(\bar{\gamma}_1(x_t)\right)\{\hat{\gamma}_1(x_t) - \gamma_{10}(x_t)\}^2 dF_0(dw)$$

where $\bar{\gamma}_1(x)$ is between $\hat{\gamma}_1(x)$ and $\gamma_{10}(x)$. It follows similarly to previous arguments that $|R_3| \leq C|\hat{\gamma}_1 - \gamma_{10}|^2 = O_p(n^{-2d_1}) = o_p(n^{-1/2})$. Therefore,

$$\bar{T}_2 + T_3 = -(\hat{T}_2 + T_1) + o_p(n^{-1/2})$$

Summarizing, it follows from what has been shown that

$$\begin{aligned}
\bar{\psi}(\hat{\gamma}, \alpha_0, \theta_0) &= T + T1 + T2 + T3 \\
&= \bar{T}_1 + \bar{T}_2 + T_1 + T_2 + T_3 + o_p(n^{-1/2}) \\
&= \bar{T}_2 + \bar{T}_2 + T_1 + T_3 + o_p(n^{-1/2}) \\
&= \check{T}_2 + \bar{T}_2 + T_1 + T_3 + o_p(n^{-1/2}) \\
&= (\check{T}_2 + T_1) + (\bar{T}_2 + T_3) + o_p(n^{-1/2}) \\
&= (\check{T}_2 + T_1) - (\bar{T}_2 + T_1) + o_p(n^{-1/2}) \\
&= o_p(n^{-1/2})
\end{aligned}$$

giving Assumption D3(iv).

Next, note that by the fixed trimming, $H(\hat{\gamma}_1(x))$ and $\hat{\gamma}_1(x)$ are uniformly bounded. Also, by Lemma E6 in Chernozhukov et al. [2022a], $\hat{\gamma}_2(x)$ and $\hat{\gamma}_3$ are uniformly bounded with probability approaching 1, so

$$\|\hat{\Delta}(w)\| \leq C\left(\|\hat{\alpha}_1(x) - \alpha_{10}(x)\| + \|\hat{\alpha}_2(x) - \alpha_{20}(x)\| + \|\hat{\alpha}_3 - \alpha_{30}\|\right)$$

The second condition of Assumption D4 then follows by Lemma E8 in Chernozhukov et al. [2022a]. The first condition of Assumption D4 also follows in a straightforward manner from uniform boundedness of $\hat{\gamma}_2(x)$ and $\hat{\gamma}_3$ with probability approaching 1. Finally, Assumption D5 follows in a straightforward manner from the same boundedness properties, so the conclusion follows by Theorem 9 in Chernozhukov et al. [2022a].