The University of Danang
University of Science and Technology

MATHEMATICS FOR COMPUTER SCIENCE

Chap 1. Number theory

Faculty of Information Technology
PHAM Cong Thang, PhD

References

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Number theory

- **Primes and factors**
- Modular arithmetic
- Solving equations

Number theory

- **Number theory:**
  - is a branch of mathematics that studies integers.
  - Number theory is a fascinating field, because many questions involving integers are very difficult to solve even if they seem simple at first glance.

Number theory

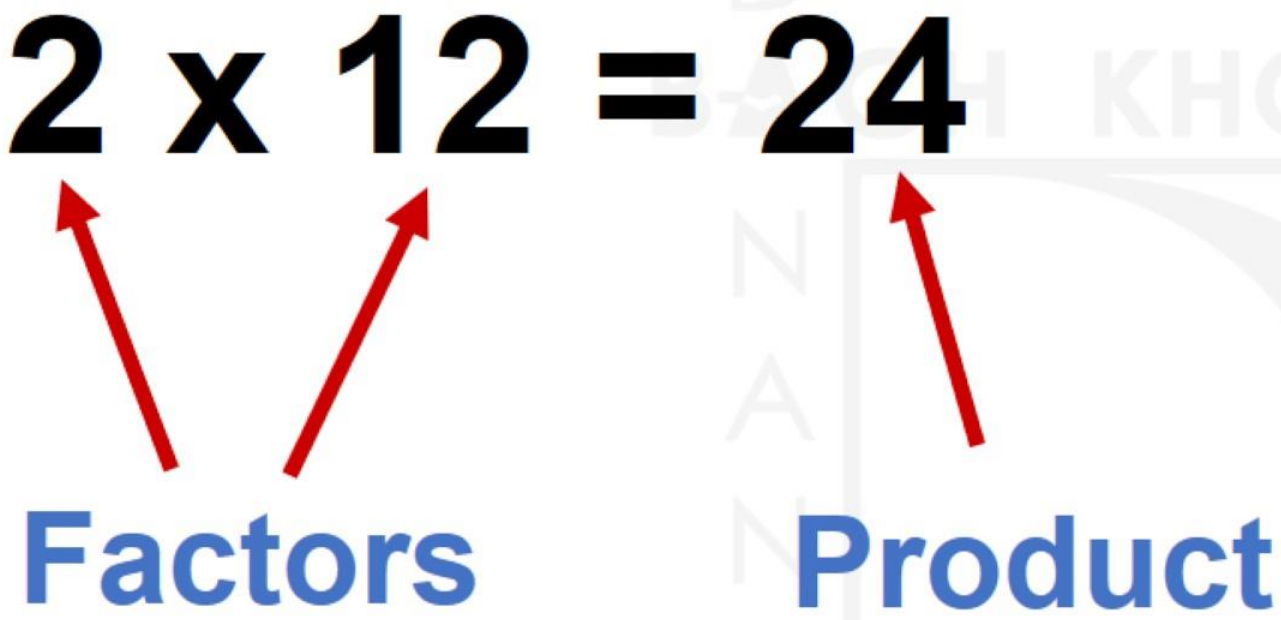
- As an example, consider the following equation:

$$x^3 + y^3 + z^3 = 33$$

  - It is easy to find three real numbers x, y and z that satisfy the equation. For example, we can choose $x = 3, y = \sqrt[3]{3}, z = \sqrt[3]{3}$
  - **However, it is an open problem in number theory if there are any three integers x, y and z that would satisfy the equation**

Primes and factors

- **Product** – An answer to a multiplication problem.
- **Factor** – a number that is multiplied by another to give a product.



Primes and factors

- A number a is called a **factor** or a **divisor** of a number **b** if **a** divides **b**.
  - If **a** is a factor of b, we write $a|b$, and
  - *otherwise* we write $a \nmid b$.
  - For example, the factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

Primes and factors

$3 \times 7 = 21$

Factors: 3 and 7
Product: 21

$5 \times 6 = 30$

Factors: 5 and 6
Product: 30

Primes and factors

- A number $n > 1$ is a prime if its only **positive factors** are **1** and **n**.
  - 2, 3, 5, 7, 11, 13, 17 are **primes**
  - 35 is not a prime, because $5 \times 7 = 35$
- **Composite number** – a number that has more than two factors.
  - 8 is Composite number: the factors of 8 are 1, 2, 4, 8.
- For every number $n > 1$, there is **a unique prime factorization**:

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1 \alpha_2, \ldots, \alpha_k$ are positive numbers

- For example, the prime factorization for 84 is

$$84 = 2^2 \cdot 3^1 \cdot 7^1$$

Primes and factors

- The number of factors of a number n is

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1)$$

because for each prime p_i, there are $(\alpha_i + 1)$ ways to choose how many times it appears in the factor.

  - The prime factorization for 84 is $84 = 2^2 \cdot 3^1 \cdot 7^1$
  - The number of factors of 84 is $\tau(84) = 3 \cdot 2 \cdot 2 = 12$.
  - The factors are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84.

Primes and factors

- The **sum of factors** of n is:

$$\sigma(n) = \prod_{i=1}^{k}(1 + p_i + \cdots + p_i^{\alpha_i}) = \prod_{i=1}^{k}\frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

where the latter formula is based on **the geometric progression formula**.

- For example, the sum of factors of **84** is:

$$\sigma(84) = \frac{2^3-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{7^2-1}{7-1} = 7 \cdot 4 \cdot 8 = 224$$

Primes and factors

- The **product of factors** of n is

$$\mu(n) = n^{\tau(n)/2}$$

because we can form $\frac{\tau(n)}{2}$ pairs from the factors, each with product n.

- For example, the factors of 84 produce the pairs $1 \cdot 84, 2 \cdot 42, 3 \cdot 28$, etc., and
- The product of the factors is $\mu(84) = 84^6 = 351298031616$

Primes and factors

- A number n is called a **perfect number** if $n = \sigma(n) - n$, i.e.,
  - n equals the sum of its factors between *1* and $n - 1$.
  - For example, 28 is a **perfect number**, because $28 = 1 + 2 + 4 + 7 + 14$.

Number of primes

- It is easy to show that there is an infinite number of primes.
  - If the number of primes would be finite, we could construct a set $P = \{p_1, p_2, \ldots, p_n\}$ that would contain all the primes.
  - For example, $p_1 = 2, p_2 = 3, p_3 = 5$, and so on. However, using P, we could form a new prime

$$p_1\, p_2 \cdots p_n + 1$$

that is larger than all elements in P. This is a contradiction, and the number of primes has to be infinite.

Density of primes

- The density of primes means how often there are primes among the numbers.
  - Let $\pi(n)$ denote the number of primes between 1 and n. For example, $\pi(10) = 4$, because there are 4 primes between 1 and 10 : 2, 3, 5 and 7.
  - It is possible to show that

$$\pi(n) \approx \frac{n}{\ln n}$$

    which means that primes are quite frequent.
  - For example, the number of primes between 1 and 10^6 is

$$\pi(10^6) = 78498 \text{ and } \frac{10^6}{\ln 10^6} \approx 72382$$

Conjectures

- There are many conjectures involving primes. Most people think that the conjectures are true, but nobody has been able to prove them.
- For example, the following conjectures are famous:
  - **Goldbach's conjecture**: Each even integer $n > 2$ can be represented as a sum $n = a + b$ so that both a and b are primes.
  - **Twin prime conjecture**: There is an infinite number of pairs of the form $\{p, p + 2\}$, where both p and $p + 2$ are primes.
  - **Legendre's conjecture**: There is always a prime between numbers n^2 and $(n + 1)^2$, where n is any positive integer.

Algorithms

- Basic algorithms
- Sieve of Eratosthenes
- Euclid's algorithm
- Euler's totient function

Basic algorithms

- If a number n is not prime, it can be represented as a product $a \cdot b$, where $a \leq \sqrt{n}$, or $b \leq \sqrt{n}$, so it certainly has a factor between 2 and $\lfloor \sqrt{n} \rfloor$
  - Using this observation, we can both test if a number is prime and find the prime factorization of a number in $O(\sqrt{n})$ time.

(The function $\lfloor x \rfloor$ rounds the number x down to an integer, and the function $\lceil x \rceil$ rounds the number x up to an integer: $\left\lfloor \frac{3}{2} \right\rfloor = 1$ and $\left\lceil \frac{3}{2} \right\rceil = 2$)

Basic algorithms

- The following function *prime* checks if the given number n is prime. The function attempts to divide n by all numbers between 2 and $\lfloor\sqrt{n}\rfloor$ and if none of them divides n, then n is prime.

```
bool prime(int n) {
    if (n < 2) return false;
    for (int x = 2; x*x <= n; x++) {
        if (n%x == 0) return false;
    }
    return true;
}
```

Basic algorithms

- The following function *factors* constructs a vector that contains the prime factorization of n. The function divides n by its prime factors, and adds them to the vector. The process ends when the remaining number n has no factors between 2 and $\lfloor\sqrt{n}\rfloor$. If $n > 1$, it is prime and the last factor.

```
vector<int> factors(int n) {
    vector<int> f;
    for (int x = 2; x*x <= n; x++) {
        while (n%x == 0) {
            f.push_back(x);
            n /= x;
        }
    }
    if (n > 1) f.push_back(n);
    return f;
}
```

- Note that each prime factor appears in the vector as many times as it divides the number. For example, $24 = 2^3 \cdot 3$, so the result of the function is (2,2,2,3).

Sieve of Eratosthenes

- The **sieve of Eratosthenes** is a preprocessing algorithm that builds an array using which we can efficiently check if a given number between $2 \ldots n$ is prime and, if it is not, find one prime factor of the number.
- The algorithm builds an array sieve whose positions $2,3, \ldots, n$ are used. The value $sieve[k] = 0$ means that k is prime, and the value $sieve[k] \neq 0$ means that k is not a prime and one of its prime factors is $sieve[k]$.
- The algorithm iterates through the numbers $2 \ldots n$ one by one.
  - Always when a new prime x is found, the algorithm records that the multiples of x $(2x, 3x, 4x, \ldots)$ are not primes

Sieve of Eratosthenes

- For example, if $n = 20$, the array is as follows:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	2	0	3	0	2	3	5	0	3	0	7	5	2	0	3	0	5

- The following code implements the sieve of Eratosthenes. The code assumes that each element of sieve is initially zero.

```
for (int x = 2; x <= n; x++) {
    if (sieve[x]) continue;
    for (int u = 2*x; u <= n; u += x) {
        sieve[u] = x;
    }
}
```

Sieve of Eratosthenes

- The inner loop of the algorithm is executed n/x times for each value of x. Thus, an upper bound for the running time of the algorithm is the harmonic sum

$$\sum_{x=2}^{n}(n/x) = \frac{n}{2} + \frac{n}{3} + \frac{n}{4} + \cdots \frac{n}{n} = O(nlogn)$$

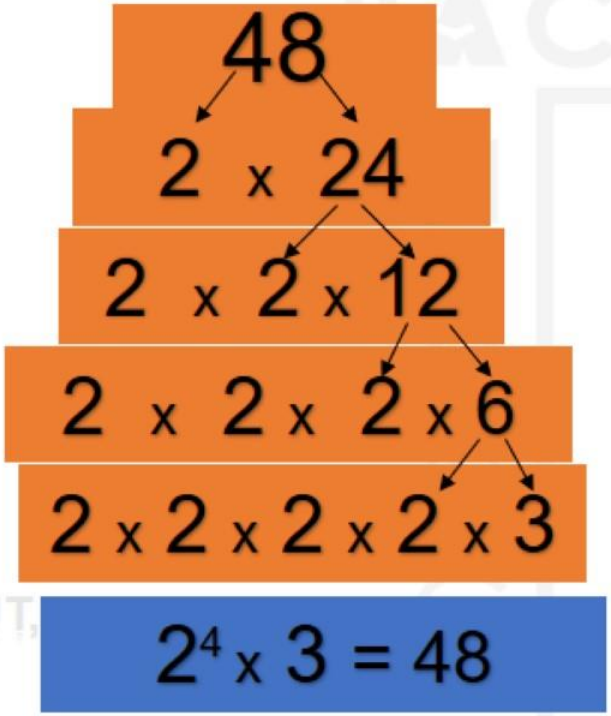
- In fact, the algorithm is more efficient, because the inner loop will be executed only if the number x is prime. It can be shown that the running time of the algorithm is only $O(nlog(logn))$, a complexity very near to $O(n)$.

How to do a Prime Factorization? Factor Tree Method

- **Step 1 – Write down any composite number.**
- **Step 2 – Start dividing by the prime #s (start with 2).**
  - If the composite number is divisible by 2, write it down and find the next factor.
  - If not, check if the factor is evenly divisible by 3, 5, 7, 9, etc.
- **Step 3 – Check the factors.**
  - If they are prime, you are done.
  - If they are not, proceed to Step 4.
- **Step 4 – Continue dividing.**
  - If one of the factors is divisible by 2, write it down and find the next factor.
  - If not, check if the factor is evenly divisible by 3, 5, 7, 9, etc.

How to do a Prime Factorization? Factor Tree Method

- **Step 5 – Check the factors.**
  - If they are prime, proceed to Step 6.
  - If they are not, repeat Step 4.
- **Step 6 – Write the Prime Factorization in Exponential Form.**



Euclid's algorithm

- The **greatest common divisor** of numbers a and b, *gcd(a,b)*, is the greatest number that divides both a and b, and
- The **least common multiple** of a and b, **lcm(a,b)**, is the smallest number that is divisible by both a and b.
  - For example, $gcd(24,36) = 12$ and $lcm(24,36) = 72$.
- The greatest common divisor and the least common multiple are connected as follows:

$$lcm(a,b) = \frac{ab}{gcd(a,b)}$$

Euclid's algorithm

- Euclid's algorithm1 provides an efficient way to find the greatest common divisor of two numbers. The algorithm is based on the following formula:

$$\gcd(a,b) = \begin{cases} a & b = 0 \\ \gcd(b, a \bmod b) & b \neq 0 \end{cases}$$

- For example: $\gcd(24,36) = \gcd(36,24) = \gcd(24,12) = \gcd(12,0) = 12.$

Euclid's algorithm

- The algorithm can be implemented as follows:

```
int gcd(int a, int b) {
    if (b == 0) return a;
    return gcd(b, a%b);
}
```

- It can be shown that Euclid's algorithm works in $O(logn)$ time, where $n = \min(a, b)$. The worst case for the algorithm is the case when a and b are consecutive Fibonacci numbers. For example,

$$gcd(13,8) = \gcd(8,5) = \gcd(5,3) = \gcd(3,2) = \gcd(2,1) = \gcd(1,0) = 1$$

Euler's totient function

- Numbers a and b are coprime if $gcd(a, b) = 1$. Euler's totient function $\varphi(n)$ gives the number of coprime numbers to *n* between 1 and n.
  - For example, $\varphi(12) = 4$, because 1, 5, 7 and 11 are coprime to 12.
  - The value of $\varphi(n)$ can be calculated from the prime factorization of n using the formula

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$$

  - For example, $\varphi(12) = 2^1 \cdot (2-1) \cdot 3^0 \cdot (3-1) = 4$.
  - Note that $\varphi(n) = n - 1$ if n is prime.

Number theory

- Primes and factors
- **Modular arithmetic**
- Solving equations

Modular arithmetic

- In modular arithmetic, the set of numbers is limited so that only numbers $0,1,2,\ldots,m-1$ are used, where m is a constant. Each number x is represented by the number $x \bmod m$:
  - The remainder after dividing x by m.
  - For example, if $m = 17$, then 75 is represented by $75 \bmod 17 = 7$.

Modular arithmetic

- Often we can take remainders before doing calculations. In particular, the following formulas hold:

$$(x + y) mod\ m = (x\ mod\ m + y\ mod\ m)\ mod\ m$$

$$(x - y) mod\ m = (x\ mod\ m - y\ mod\ m)\ mod\ m$$

$$(x \cdot y) mod\ m = (x\ mod\ m \cdot y\ mod\ m)\ mod\ m$$

$$x^n\ mod\ m = (x\ mod\ m)^n\ mod\ m$$

Modular exponentiation

- There is often need to efficiently calculate the value of x^n mod m. This can be done in $O(logn)$ time using the following recursion:

$$x^n = \begin{cases} 1 & n = 0 \\ x^{n/2} \cdot x^{n/2} & n \text{ is even} \\ x^{n-1} \cdot x & n \text{ is odd} \end{cases}$$

- It is important that in the case of an even n, the value of $x^{n/2}$ is calculated only once. This guarantees that the time complexity of the algorithm is $O(logn)$, because n is always halved when it is even.

Modular exponentiation

- The following function calculates the value of $x^n \ mod \ m$:

```
int modpow(int x, int n, int m) {
    if (n == 0) return 1%m;
    long long u = modpow(x,n/2,m);
    u = (u*u)%m;
    if (n%2 == 1) u = (u*x)%m;
    return u;
}
```

Fermat's theorem and Euler's theorem

- **Fermat's theorem** states that:

$$x^{m-1} \bmod m = 1$$

when m is prime and x and m are coprime. This also yields

$$x^k \bmod m = x^{k \bmod (m-1)} \bmod m$$

- **Euler's theorem** states that

$$x^{\varphi(m)} \bmod m = 1$$

when x and m are coprime. **Fermat's theorem** follows from Euler's theorem, because if m is a prime, then $\varphi(m) = m - 1$.

Modular inverse

- The inverse of $x \ modulo \ m$ is a number x^{-1} such that

$$xx^{-1} \ mod \ m = 1$$

  - For example, if $x = 6$ and $m = 17$, then $x^{-1} = 3$, because $6 \cdot 3 \ mod \ 17 = 1$.

- Using modular inverses, we can divide numbers modulo m, because division by x corresponds to multiplication by x^{-1}.
  - For example, to evaluate the value of $36/6 \ mod \ 17$, we can use the formula

$2 \cdot 3 \ mod \ 17$, because $36 \ mod \ 17 = 2$ and $6^{-1} \ mod \ 17 = 3$.

  - However, a modular inverse does not always exist. For example, if $x = 2$ and $m = 4$, the equation $xx^{-1} \ mod \ m = 1$ cannot be solved, because all multiples of 2 are even and the remainder can never be 1 $when \ m = 4$. **It turns out that the value of $x^{-1} \ mod \ m$ can be calculated exactly when x and m are coprime.**

Modular inverse

- If a modular inverse exists, it can be calculated using the formula

$$x^{-1} = x^{\varphi(m)-1}$$

  - If m is prime, the formula becomes $x^{-1} = x^{m-2}$
  - For example, $6^{-1}\ mod\ 17 = 6^{17-2} mod\ 17 = 3$.
  - **This formula allows us to efficiently calculate modular inverses using the modular exponentation algorithm.**

Modular inverse

- The formula can be derived using **Euler's theorem**.
  - First, the modular inverse should satisfy the following equation:

$$xx^{-1} \, mod \, m = 1$$

  - On the other hand, according to Euler's theorem,

$$x^{\varphi(m)} \, mod \, m = xx^{\varphi(m)-1} \, mod \, m = 1$$

so the numbers x^{-1} and $x^{\varphi(m)-1}$ are equal

Computer arithmetic

- In programming, unsigned integers are represented modulo 2^k, where k is the number of bits of the data type.
  - A usual consequence of this is that a number wraps around if it becomes too large.
  - For example, in C++, numbers of type unsigned int are represented modulo 2^{32}.
  - The following code declares an **unsigned int variable** whose value is 123456789. After this, the value will be multiplied by itself, and the result is $1234567892 \; mod \; 232 = 2537071545$.

```
unsigned int x = 123456789;
cout << x*x << "\n"; // 2537071545
```

Number theory

- Primes and factors
- Modular arithmetic
- **Solving equations**

Solving equations- Diophantine equations

- **Diophantine equations**
  - A **Diophantine equation** is an equation of the form

$$ax + by = c$$

where a, b and c are constants and the values of x and y should be found. Each number in the equation has to be an integer.

  - For example, one solution for the equation $5x + 2y = 11$ is $x = 3$ and $y = -2$.
- We can efficiently solve a Diophantine equation by using Euclid's algorithm.
  - It turns out that we can extend Euclid's algorithm so that it will find numbers x and y that satisfy the following equation: $ax + by = gcd(a, b)$
  - **A Diophantine equation** can be solved if c is divisible by *gcd(a,b)*, and otherwise it cannot be solved.

Solving equations- Diophantine equations

- As an example, let us find numbers x and y that satisfy the following equation:
  $39x + 15y = 12$
  - The equation can be solved, because $\gcd(39,15) = 3$ and 3|12.
  - When Euclid's algorithm calculates the greatest common divisor of 39 and 15, it produces the following sequence of function calls:
    $gc\,d(39,15) = gc\,d(15,9) = gc\,d(9,6) = gc\,d(6,3) = gc\,d(3,0) = 3$
  - This corresponds to the following equations:
    $$39 - 2 \cdot 15 = 9;\ 15 - 1 \cdot 9 = 6;\ 9 - 1 \cdot 6 = 3$$
  - Using these equations, we can derive: $39 \cdot 2 - 15 \cdot (-5) = 3$ and by multiplying this by 4, the result is $39 \cdot 8 + 15 \cdot (-20) = 12$, so a solution to the equation is $x = 8$ and $y = -20$.

Solving equations- Diophantine equations

- A solution to a Diophantine equation is not unique, because we can form an infinite number of solutions if we know one solution. If a pair (x, y) is a solution, then also all pairs:

$$(x + \frac{kb}{gcd(a,b)}, y - \frac{ka}{gcd(a,b)})$$

are solutions, where k is any integer

Solving equations-Chinese remainder theorem

- The **Chinese remainder theorem** solves a group of equations of the form

$$x \equiv a_1 \ mod \ m_1$$
$$x \equiv a_2 \ mod \ m_2$$
$$\dots$$
$$x \equiv a_n \ mod \ m_n$$

where all pairs of $m_1, m_2, \dots, m_n$ are coprime.

- Let x_m^{-1} be the inverse of $x \ modulo \ m$, and $X_k = \frac{m_1 m_2 \dots m_k}{m_k}$
- Using this notation, a solution to the equations is

$$x = a_1 X_1 X_{1_{m_1}}^{-1} + a_2 X_2 X_{2_{m_2}}^{-1} + \dots + a_n X_n X_{n_{m_n}}^{-1}$$

Solving equations-Chinese remainder theorem

- In this solution, for each $k = 1,2,\dots,n$,

$$a_k X_k X_{k_{m_k}}^{-1} \, mod \, m_k = a_k$$

because

$$X_k X_{k_{m_k}}^{-1} \, mod \, m_k = 1$$

- Since all other terms in the sum are divisible by m_k, they have no effect on the remainder, and $x \, mod \, m_k = a_k$.
  - For example, a solution for

$$x = 3 \, mod \, 5$$
$$x = 4 \, mod \, 7$$
$$x = 2 \, mod \, 3$$

is $3 \cdot 21 \cdot 1 + 4 \cdot 15 \cdot 1 + 2 \cdot 35 \cdot 2 = 263$. Once we have found a solution x, we can create an infinite number of other solutions, because all numbers of the form $x + m_1 m_2 \dots m_n$ are solutions

Solving equations-Other results

- **Lagrange's theorem**
  - Lagrange's theorem states that every positive integer can be represented as a sum of four squares, i.e., $a^2 + b^2 + c^2 + d^2$. For example, the number 123 can be represented as the sum $8^2 + 5^2 + 5^2 + 3^2$.
- **Zeckendorf's theorem**
  - Zeckendorf's theorem states that every positive integer has a unique representation as a sum of Fibonacci numbers such that no two numbers are equal or consecutive Fibonacci numbers.
  - For example, the number 74 can be represented as the sum $55 + 13 + 5 + 1$.

Solving equations-Other results

- **Pythagorean triples**
  - A Pythagorean triple is a triple (a, b, c) that satisfies the Pythagorean theorem $a^2 + b^2 = c^2$, which means that there is a right triangle with side lengths a, b and c. For example, (3,4,5) is a Pythagorean triple.
  - If (a,b, c) is a Pythagorean triple, all triples of the form (ka, kb, kc) are also Pythagorean triples where $k > 1$.
  - A Pythagorean triple is primitive if a, b and c are coprime, and all Pythagorean triples can be constructed from primitive triples using a multiplier k.

Solving equations-Other results

- **Pythagorean triples**
  - **Euclid's formula** can be used to produce all primitive Pythagorean triples. Each such triple is of the form

$$(n^2 - m^2, 2nm, n^2 + m^2),$$

where $0 < m < n$, n and m are coprime and at least one of n and m is even. For example, when $m = 1$ and $n = 2$, the formula produces the smallest Pythagorean triple:

$$(2^2 - 1^2, 2 \cdot 2 \cdot 1, 2^2 + 1^2) = (3,4,5)$$

Solving equations-Other results

- **Wilson's theorem**
  - Wilson's theorem states that a number n is prime exactly when

$$(n-1)!\, mod\, n = n-1$$

  - For example, the number 11 is prime, because

$$10!\, mod\, 11 = 10$$

  - and the number 12 is not prime, because

$$11!\, mod\, 12 = 0 \neq 11$$

  - Wilson's theorem can be used to find out whether a number is prime.
  - However, in practice, the theorem cannot be applied to large values of n, because it is difficult to calculate values of $(n-1)!$ when n is large.