Data preprocessing and sampling

Part 5: Conditional probability

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Conditional probability

- Let A be an event with probability $P(A)$. In this section we explore how the information '**an event B has occurred**' affects the probability of A.
- We use the notation $P(A / B)$ to denote the conditional probability of event A given that event B has occurred.
- Event B is the **conditioning event**.

Definition of conditional probability

- For any two events A and B, the conditional probability of A given that B has occurred is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} provided\ P(B) > 0. P(A|B) = \frac{P(A \cap B)}{P(B)} provided\ P(B) > 0.$$

- In terms of frequency, this expression can be viewed as:

$$\frac{number\ of\ outcomes\ in\ both\ A\ and\ B}{number\ of\ outcomes\ in\ B}\ \frac{number\ of\ outcomes\ in\ both\ A\ and\ B}{number\ of\ outcomes\ in\ B}$$

Example

- In a high school, 50% of the female students have black hair, 30% have blue eyes and 15% have both black hair and blue eyes. A female student is selected at random.

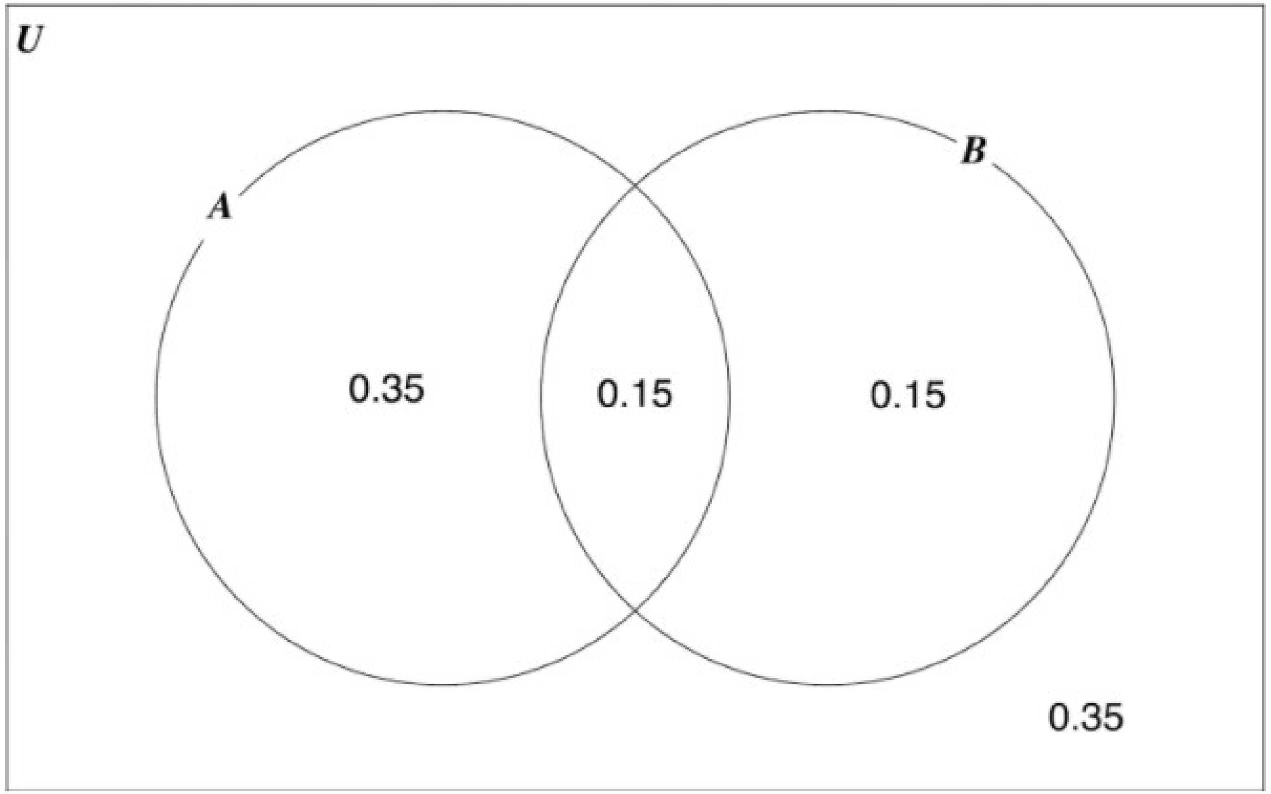
  a) Given that the student has black hair, find the probability that she has blue eyes.

  b) Given that the student has blue eyes, find the probability that she has black hair.

  c) Find the probability that the student has either black hair or blue eyes or both.

Example

- $P(A) = 50\% = 0.5$
- $P(B) = 30\% = 0.3$
- $P(A \cap B) = 15\% = 0.15$



Solution

a) $P(B / A) = P(A \cap B) / P(A) = 0.15/0.5 = 0.3$

a) $P(A / B) = P(A \cap B) / P(B) = 0.15/0.3 = 0.5$

a) $P(A \cup B) = 0.35 + 0.15 + 0.15 = 0.65$

The multiplicative rule for probabilities

- The multiplication rule is a by-product of the definition of conditional probability.
- Let A and B be two events. Then:

$$P(A \cap B) = P(A / B)P(B)$$

- When A and B are swapped, we can also have:

$$P(A \cap B) = P(B / A)P(A)$$

Example

- Suppose C and D are two events such that $P(C) = 0.3$, $P(D) = 0.5$ and $P(C / D) = 0.4$. *Find:*

  a) $P(C \cap D)$

  b) $P(D / C)$

The multiplicative rule for probabilities

- Multiple stages in succession.
- The multiplicative rule is most useful when an experiment consists of several stages in succession.
- We can easily extend the rule to experiments involving more than two stages.
- $P(A_1 \cap A_2 \cap A_3) = P(A_3 / A_1 \cap A_2) P(A_1 \cap A_2)$

  $= P(A_3 / A_1 \cap A_2) P(A_2 / A_1) P(A_1)$

  Where $P(A_1 \cap A_2) = P(A_2 / A_1) P(A_1)$

  Here, A_1 occurs first, then A_2 then A_3

Independent events

- Two events *A* and *B* are independent if

$$P(A \cap B) = P(A) \times P(B)$$

- This means:

$$P(A / B) = P(A) \quad \text{and} \quad P(B / A) = P(B)$$

In other words, knowing that event *B* has occurred does not change the probability of *A* and vice versa.

Example

- For example, landing on heads after tossing a coin and rolling a 4 on a single 6-sided die are independent events.

$$P(Heads \mid 4) = 0.5$$
$$P(4 \mid Heads) = 1/6$$

Example

- A and B are independent such that $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{5}$. Find:

  a) $P(A \cap B) = P(A) \times P(B) = \frac{1}{4} \times \frac{1}{5} = 1/20 = 0.05$

  b) $P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - 1/20 = 0.2$