Data preprocessing and sampling

Part 4: Probability axioms

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Probability axioms

- A probability space consists of a sample space *S* together with a probability function *P* which takes an event $A \subseteq S$ and returns *P(A)* a number between 0 and 1.
- The function must satisfy the following axioms:
  - $\mathrm{P}(A) \geq 0$
  - $\mathrm{P}(S) = 1$
  - $\mathrm{P}(A_1 \cup A_2 \cup \cdots \cup A_k) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_k), where\ A_i \cap A_j = \emptyset\ for\ i \neq j$
  - $\mathrm{P}(A_1 \cup A_2 \cup \cdots) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots, where\ A_i \cap A_j = \emptyset\ for\ i \neq j$

Addition rule

- Let A and B be two events. Then the probability that either of them will occur is:

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Read as: P(A or B) = P(A) + P(B) - P(A and B)
- If we know that the two events are **mutually exclusive** (i.e. they cannot occur at the same time), then:

  $$P(A \cup B) = P(A) + P(B)$$

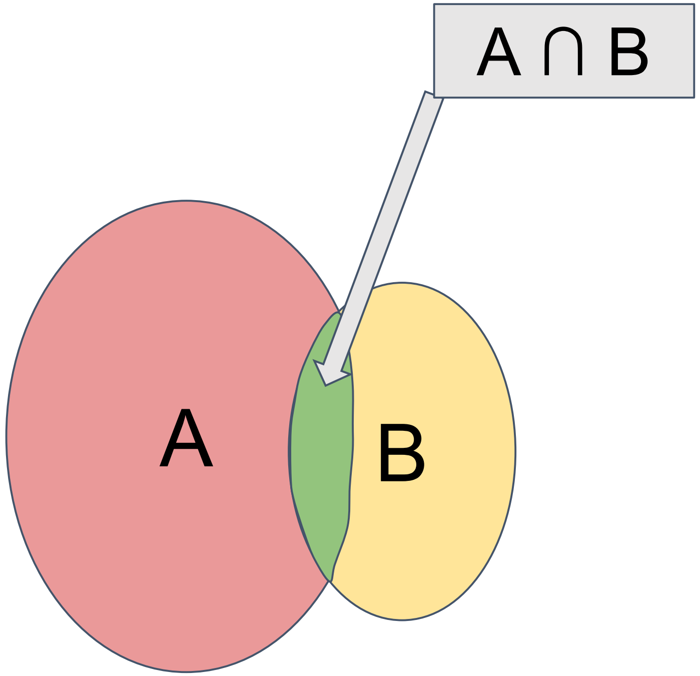
Example

- If a single card is randomly taken out from a regular pack of cards, what is probability that the card is either a jack or diamond?
- Let A be the event of picking a jack and B be the event of picking a diamond.
- $P(A) = 4/52$ $\qquad\qquad$ $P(B) = 13/52$
- Are the two events mutually exclusive? **No.**
- Because the card can be a jack and diamond at the same time (i.e. $P(A \text{ and } B) = 1/52$).

Solution:

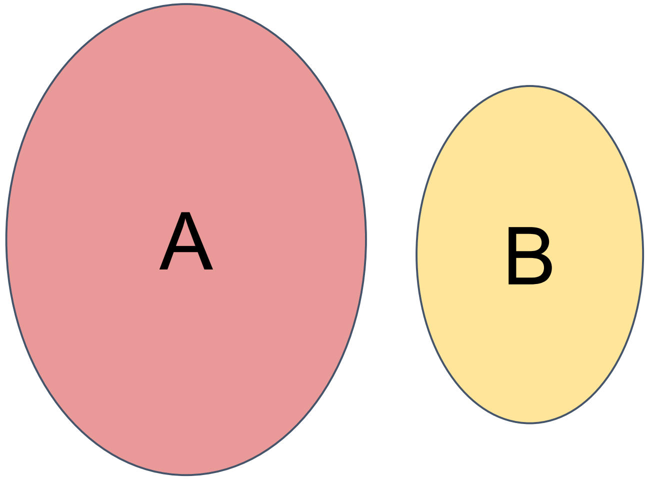
- $P(A \text{ or } B) = (4/52) + (13/52) - (1/52) = 16/52 = 4/13$

Venn diagrams



A and B are not mutually exclusive

A and B are mutually exclusive



A and B are mutually exclusive
$A \cap B = 0$

Example

Consider the experiment of tossing a six-sided die. There are six possible outcomes each of which we can assume are equally likely provided that the die is a fair die. That is, there is no physical evidence to suggest it favours one side more than the others.

The sample space is

$$S = \{1,2,3,4,5,6\}$$

and

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$$

Complementary events

- Two events A and B are said to be complementary if:

$$P(A) + P(B) = 1$$

- and

$$A \cap B = 0$$

- Event A and its complement A’ are complementary and:

$$P(A') = 1 - P(A)$$

Example

Let A and B be two events and suppose $P(A) = 0.4$, $P(B) = 0.8$ and $P(A \cup B) = 0.9$. Find:

- $P(A \cap B)$
- $P(A')$
- $P(A' \cup B)$.

Example

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

This implies:

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$0.4 + 0.8 - 0.9 = 0.3$

- $P(A') = 1 - P(A) = 1 - 0.3 = 0.7$
- $P(A' \cup B) = P(A') + P(B) - P(A' \cap B)$

  $P(A' \cap B) = P(B) - P(A \cap B) = 0.8 - 0.3 = 0.5$

  So,

  $P(A' \cup B) = 0.6 + 0.8 - 0.5 = 0.9$

Mutually exclusive events

- Two events *A* and *B* are said to be **mutually exclusive** if they cannot occur together:

$$A \cap B = 0$$

- The addition rule becomes:

$$P(A \cup B) = P(A) + P(B)$$

- Event *A* and its complement *A′* are mutually exclusive (i.e. complementary events are mutually exclusive):

$$A \cap A' = 0$$

Example

- Consider the complement of rolling a six-sided die.
- Let A be the event 'even numbers' and B be the event 'odd numbers'. Then:

$$A \cap B = 0$$

- A and B are mutually exclusive events.