Basic signal processing

Gravitational Wave Data Analysis School in China

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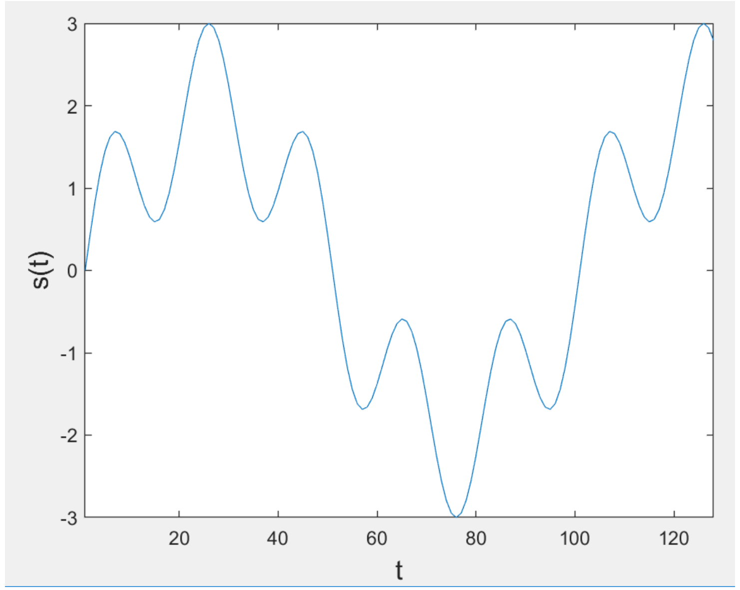
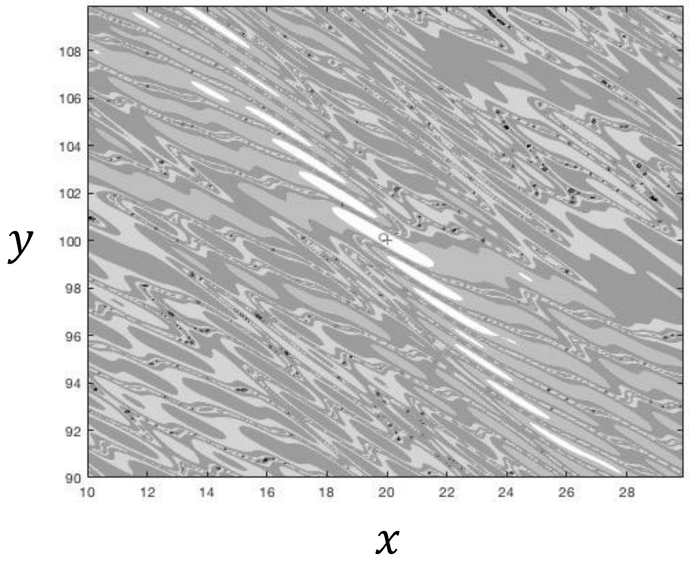
UTRGV





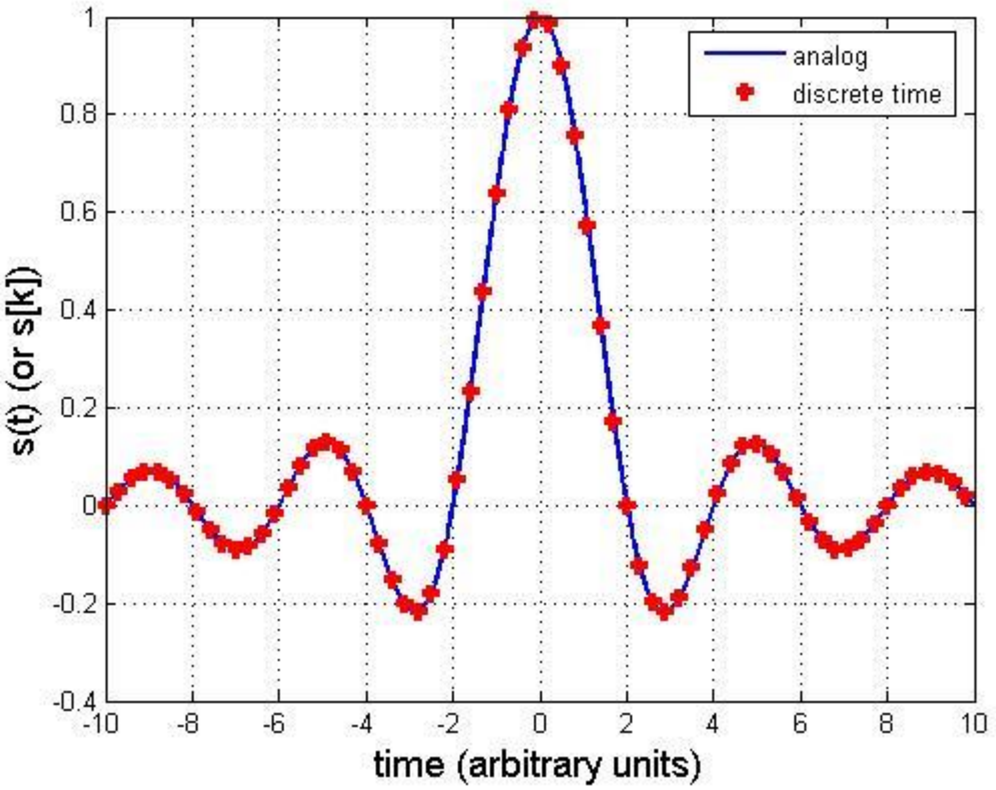
Signal

- Information carried in the form of a time or spatially varying function
  - $s(t)$: Time-domain signal
  - $s(x, y)$: Gray-scale image
- Signal processing: performing mathematical operations on a signal to transform the information contained in it
- Example:
  - De-noising: Removing noise from signal
  - Filtering: Emphasizing or de-emphasizing some information

Analog vs. Digital

- **Analog** signal: a continuous function of time $s(t)$
  - Example: The sound signal coming from your instructor to your ear
- **Discrete time** signal: sequence of values ("samples") of an analog signal

$$s[k] = s(t_k) \ \rightarrow \ \text{Notation: } \{s[k]\}, \text{ k=..., } -1, -2, 0, 1, \ldots$$



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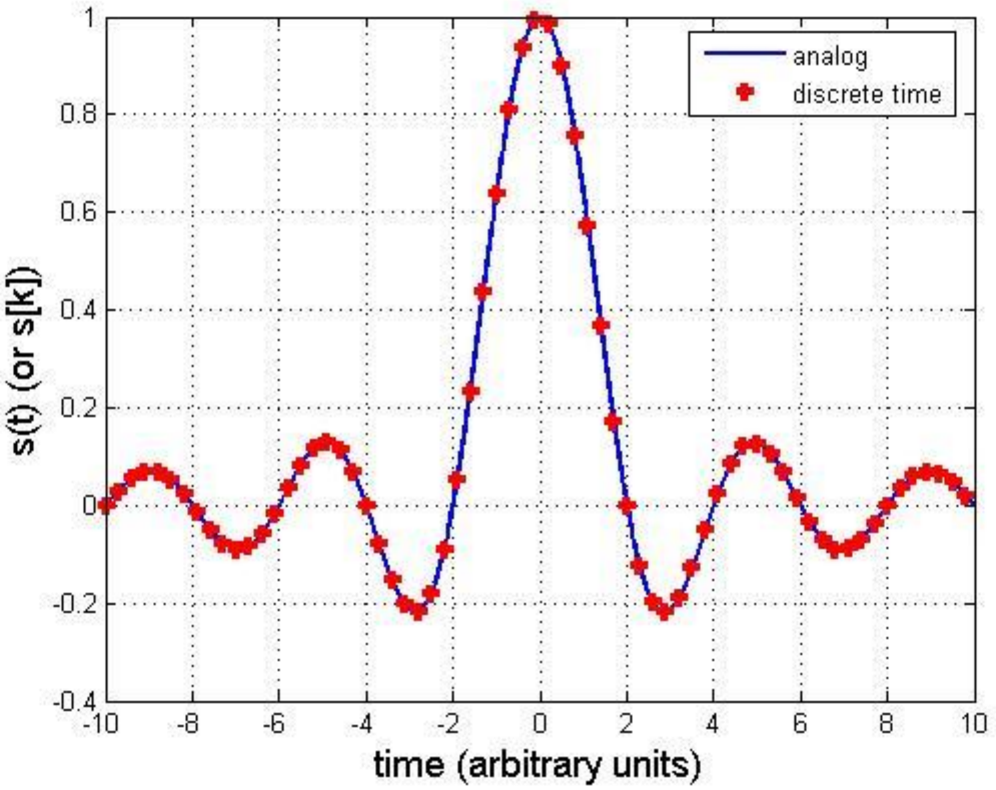
- **Digital** signal: $s[k]$ **quantized** due to representation by a finite number of bits in the binary system

$$5 \rightarrow 101 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$7 \rightarrow 111 \text{ but } 8 \rightarrow ?,\ 6.3 \rightarrow ?$$

- Example: digitized music files
- With double precision (=64 bits) representation of numbers being common in modern computers, quantization effects can be ignored for the most part
- Quantization very important in applications requiring fast data acquisiton



Discrete time signal

A discrete time signal is also called a **time series**

- Notation: $\boldsymbol{x}$ or $\bar{x} = (x[0], x[1], \cdots, x[N-1])$ or $\bar{x} = (x_0, x_1, \dots, x_{N-1})$
- $x[n] = x(t_n)$

GW data in most cases (except PTA) consists of samples spaced equally in time: **Uniformly sampled time series**

The number of samples per second: **Sampling frequency** (Hz)

- LIGO data sampling frequency is typically 16384 Hz ($= 2^{14}$ samples/sec)
- LISA sampling frequency will be ≈ 2 Hz
- PTA data sampling frequency is on average $\approx 7 \times 10^{-7}$ Hz

Fourier transform

Continuous and Discrete

Fourier transform

- A way of representing a complicated signal in terms of simpler signals
- For an analog signal

$$\tilde{s}(f) = F[s(t)] = \int_{-\infty}^{\infty} s(t)e^{-2\pi ift}\,dt$$

- Fourier transform is **invertible** (Inverse Fourier transform)

$$s(t) = F^{-1}[\tilde{s}(f)] = \int_{-\infty}^{\infty} \tilde{s}(f)e^{2\pi ift}df$$

Fourier transform

- Linearity: $F[s(t)+g(t)] = F[s(t)] + F[g(t)] = \tilde{s}(f) + \tilde{g}(f)$

- Hermiticity property: For $s(t)$ real (i.e., $s^*(t) = s(t)$)

$$s^*(t) = \int_{-\infty}^{\infty} \tilde{s}^*(f) e^{-2\pi i f t} df = \int_{-\infty}^{\infty} \tilde{s}^*(-f) e^{2\pi i f t} df = s(t) = \int_{-\infty}^{\infty} \tilde{s}(f) e^{2\pi i f t} df$$

$$\tilde{s}(-f) = \tilde{s}^*(f) \Rightarrow \begin{cases} Re\left(\tilde{s}(f)\right) \rightarrow \text{Even function} \\ \mathrm{Im}\left(\tilde{s}(f)\right) \rightarrow \text{Odd function} \end{cases}$$

Fourier transform

- Rewriting,

$$s(t) = \int_{-\infty}^{\infty} \tilde{s}(f) e^{2\pi i f t} df = \int_{-\infty}^{\infty} \Big(\underbrace{a(f)}_{even} + i \, \underbrace{b(f)}_{odd} \Big) e^{2\pi i f t} df$$

$$= \int_{-\infty}^{\infty} \left[\left(\underbrace{a(f)\cos(2\pi f t)}_{even \times even = even} \underbrace{-b(f)\sin(2\pi f t)}_{odd \times odd = even} \right) + \cancel{i \left(\underbrace{a(f)\sin(2\pi f t)}_{even \times odd = odd} + \underbrace{b(f)\cos(2\pi f t)}_{odd \times even = odd} \right)} \right] df$$

$$= 2 \int_{0}^{\infty} \sqrt{a^2(f) + b^2(f)} \cos\left(2\pi f t + \tan^{-1} \frac{b(f)}{a(f)} \right) df$$

- Thus, the meaning of the Fourier transform is that any $s(t)$ can be expressed as a sum (“superposition”) of sinusoids of the form

$$A(f)\cos(2\pi f t + \Phi(f))$$

over a range of frequencies

- The Fourier transform of a signal is called its **Spectrum** and decomposing a signal into a superposition of sinusoids is called **Spectral decomposition**

Fourier transform of a sinusoid

- If $s(t) = A\sin(2\pi f_0 t + \phi)$, its Fourier transform is

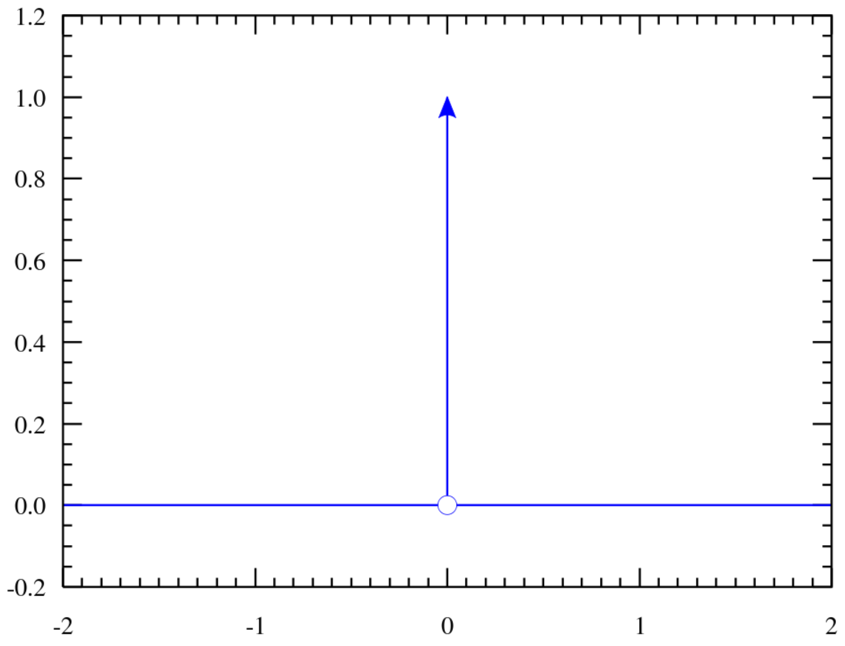
$$\int_{-\infty}^{\infty} A\sin(2\pi f_0 t + \phi)\, e^{-2\pi i f t} dt = A\delta(f - f_0)e^{i\phi} + A\delta(f + f_0)e^{-i\phi}$$

where $\delta(x)$ is the Dirac-delta function

$$\delta(x) = \int_{-\infty}^{\infty} dy\, e^{-2\pi i y x} = \begin{cases} \infty;\ x = 0 \\ 0;\ \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(x - x_0) = f(x_0)$$

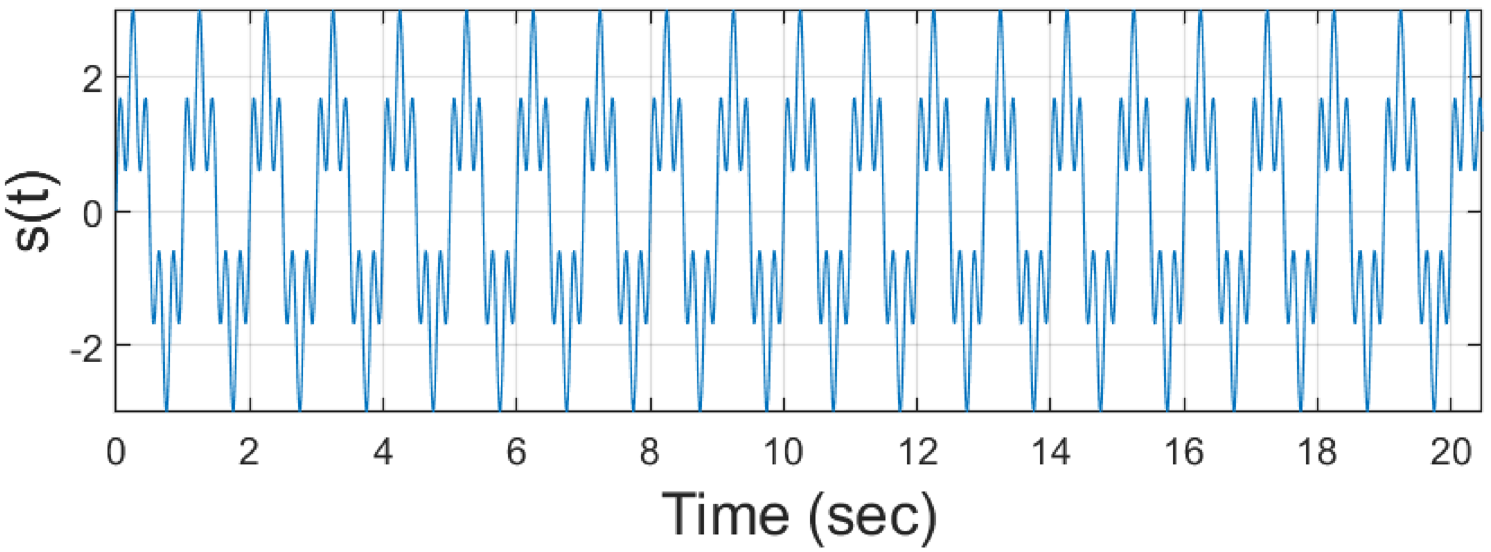
- The Fourier transform of a sinusoid exists only at the frequency of the sinusoid

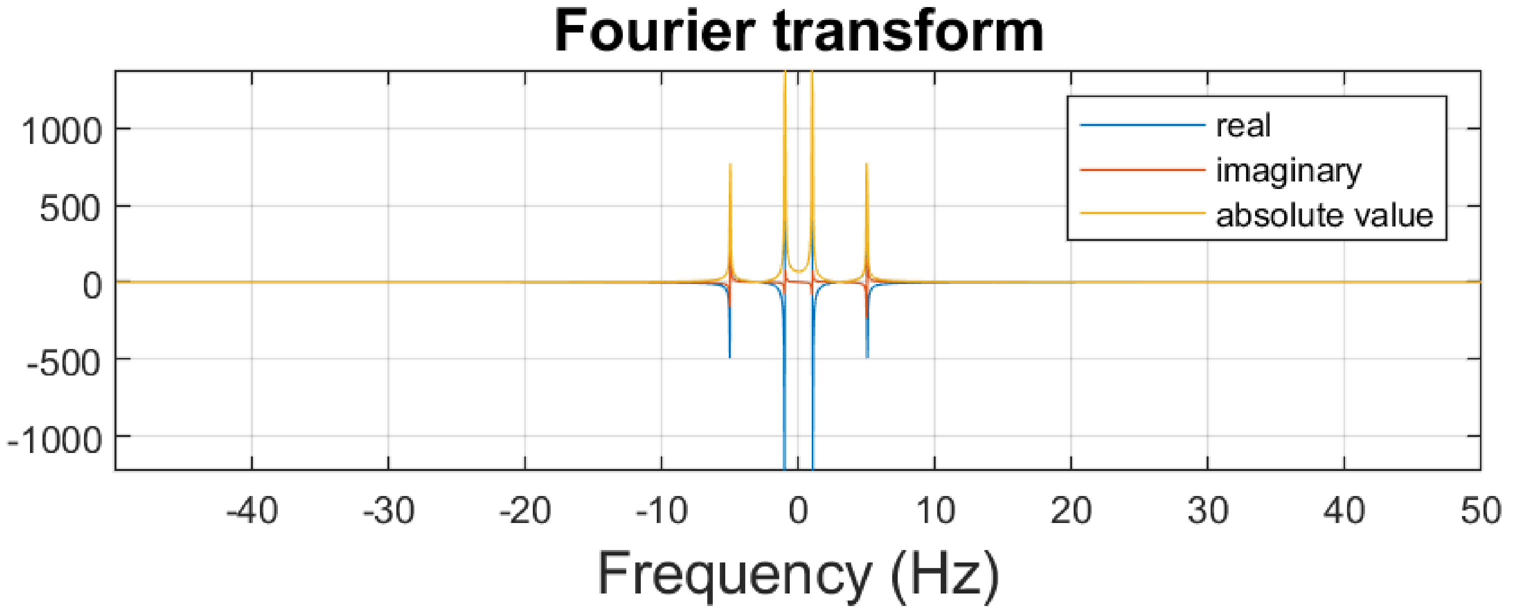


Schematic diagram of the Dirac delta function by a line surmounted by an arrow. The height of the arrow is usually used to specify the value of any multiplicative constant, which will give the area under the function. The other convention is to write the area next to the arrowhead

Example: Fourier transform

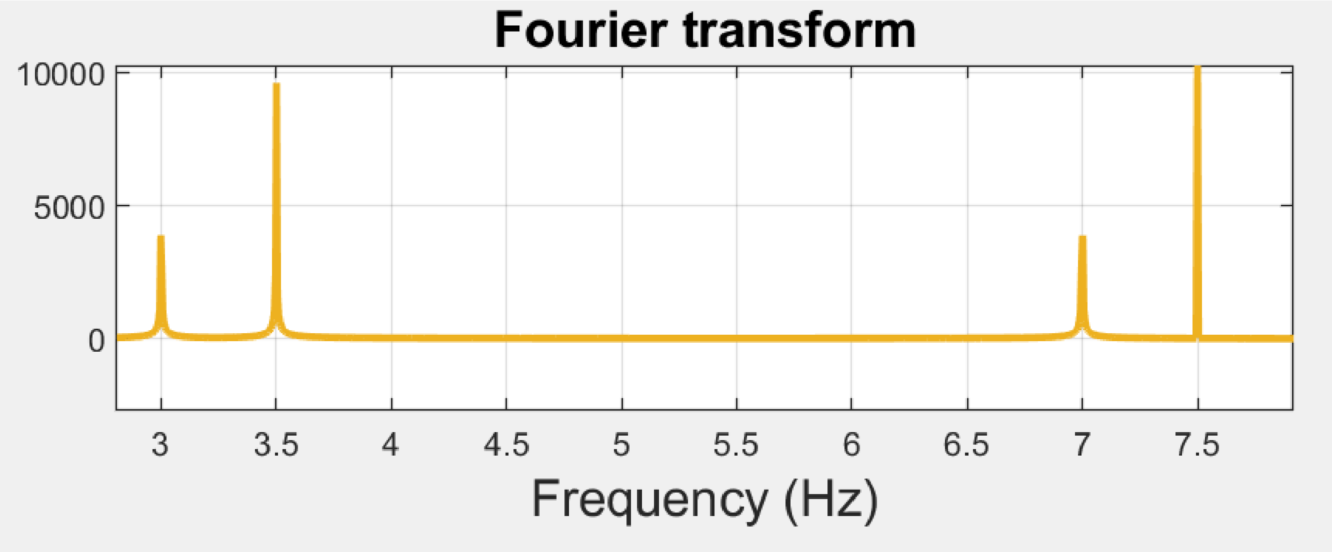
- The signal consists of the sum of two sinusoids
- The magnitude of the Fourier transform shows two peaks
- Fourier transform is useful for resolving a sum of sinusoids





Fun with signals!

- Matlab can produce and play signals as audio signals
- Signal 1: $s(t) = \sin(2\pi \times 5 \times t) + \sin(2\pi t)$
- Signal 2: $s(t) = \sin(2\pi \times 5 \times t) + \sin(2\pi \times 5.5 \times t)$
- Signal 3: Mystery signal



Convolution (“Faltung”)

How to shape the frequency content of a signal

Important operations on signals

Convolution of two analog signals,

$f(t)$ and $g(t)$:

$$z(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

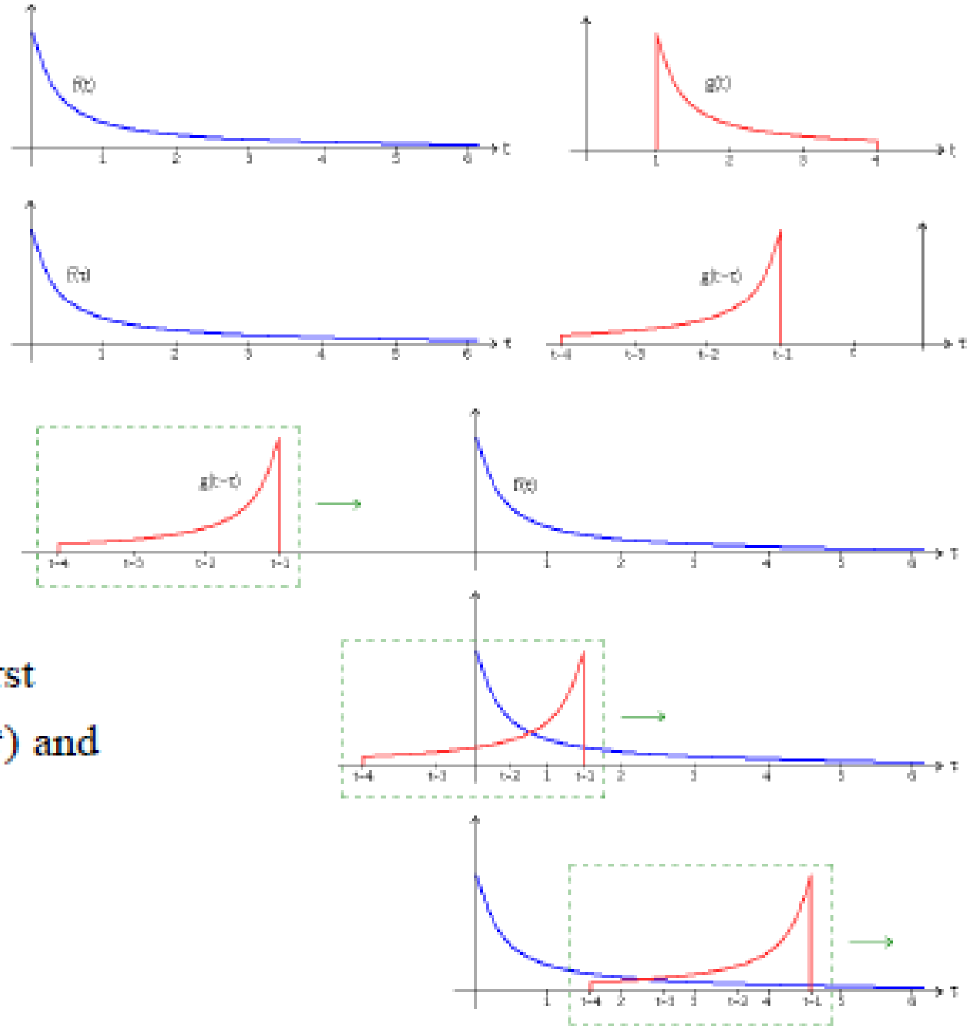
$$= \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau$$

Correlation:

$$z(t) = (f \wedge g)(t) = \int_{-\infty}^{\infty} f(\tau) g(\tau - t) d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau + t) g(\tau) d\tau$$

In convolution, one of the functions is first flipped into its mirror image $g(t) \rightarrow g(-t)$ and then a correlation is computed



Convolution (Faltung) theorem

If $\tilde{s}(f)$ is the Fourier transform of $s(t)$, and

$$z(t) = (s * g)(t) = \int_{-\infty}^{\infty} g(t-\tau)s(\tau)d\tau$$

Then, the Fourier transform of $z(t)$ is given by

$$\tilde{z}(f) = \int_{-\infty}^{\infty} z(t)e^{-2\pi i\, ft}dt = \tilde{s}(f)\tilde{g}(f)$$

Proof: Exercise (substitute for $z(t)$, substitute for $s(t)$ and $g(t)$ and do the integrals)

Convolution in the time domain is equivalent to multiplication in the Fourier domain

Convolution in the Fourier domain is equivalent to multiplication in the time domain

Windowing

- Application of convolution theorem in the reverse direction
- Example: finite length sinusoid

$$s(t) = \begin{cases} \cos(2\pi f_0 t)\,; t \in [-T/2, T/2] \\ \quad\quad 0; \quad \text{otherwise} \end{cases}$$

- **Windowing:** Box-car window

$$s(t) = \Pi(t/T)\cos(2\pi f_0 t)$$

$$\Pi(t) = \begin{cases} 1, & t \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

- Fourier transform of Box car window:

$$\tilde{\Pi}(\mathrm{f}) = T\mathrm{sinc}(fT) = T\sin(\pi f T)/(\pi f T)$$

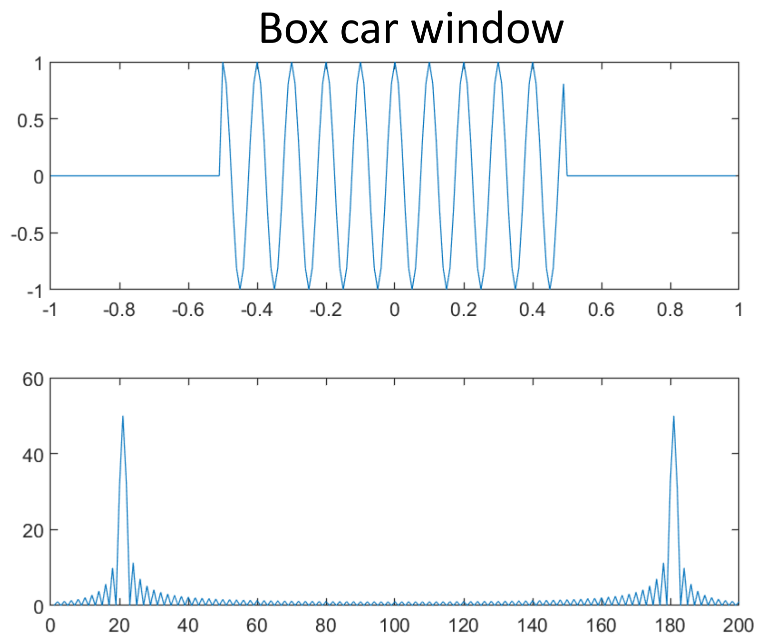
- Fourier transform of sinusoid:

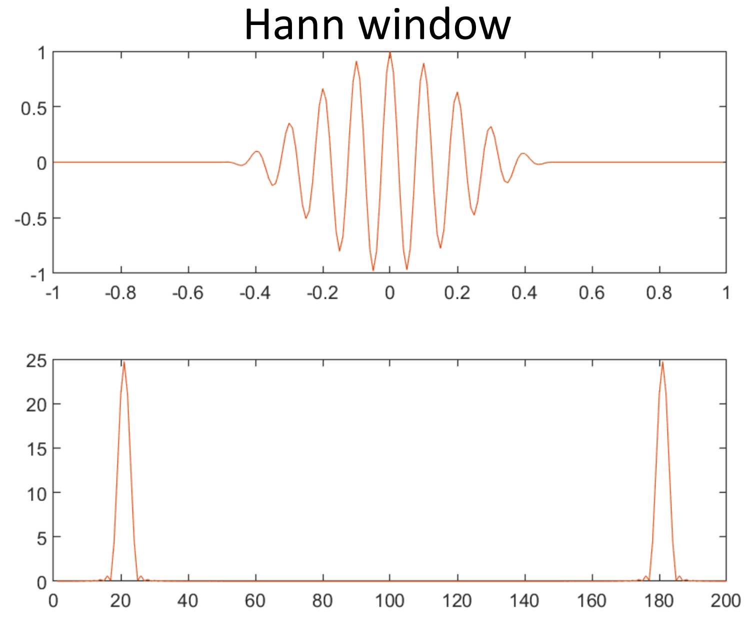
$$\frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)$$

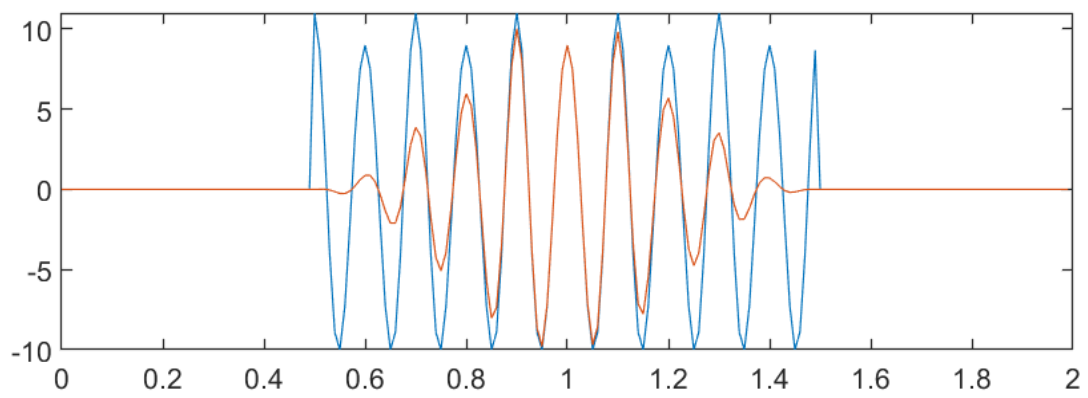
- Convolution theorem:

$$\tilde{s}(f) = \int_{-\infty}^{\infty} T\mathrm{sinc}((f - f')T)\left[\frac{1}{2}\left(\delta(f' - f_0) + \delta(f' + f_0)\right)\right]$$

$$= \frac{T}{2}\left[\mathrm{sinc}\left(\frac{f - f_0}{T}\right) + \mathrm{sinc}\left(\frac{f + f_0}{T}\right)\right]$$



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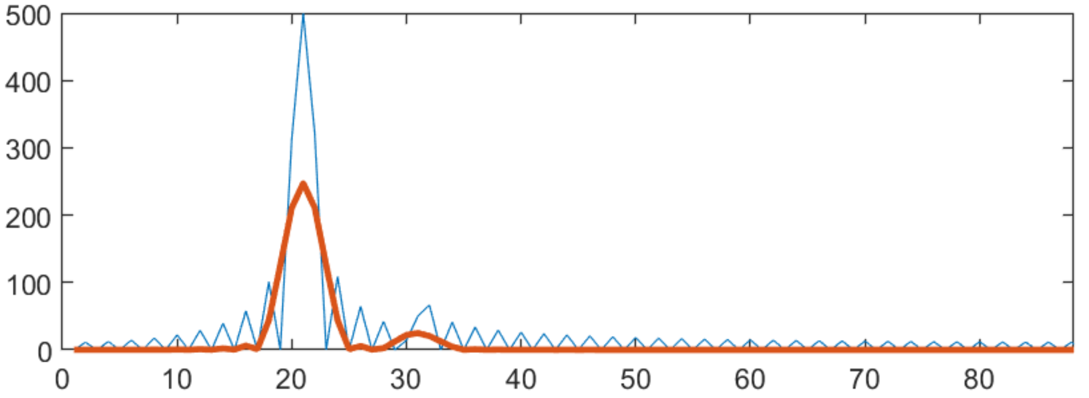
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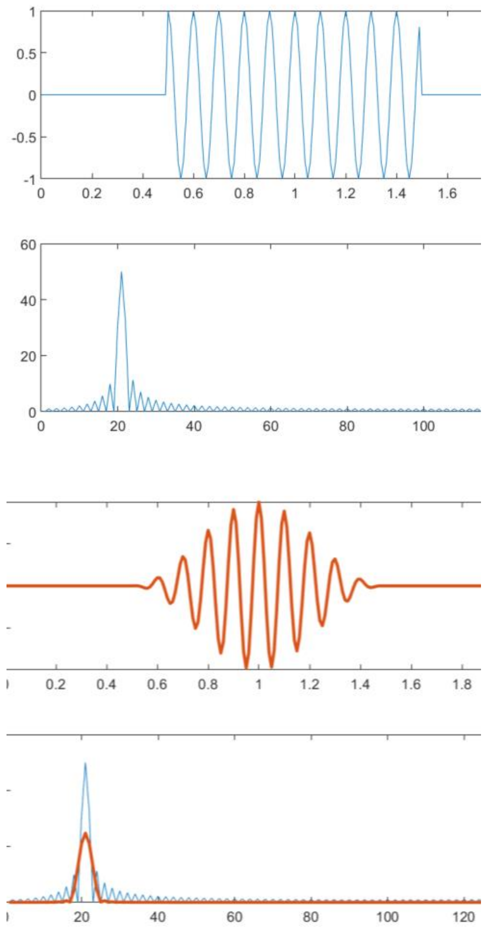
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40
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0
0 20 40 60 80 100
0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8
20 40 60 80 100 120

Windowing

Filtering

How to shape the Spectrum of a signal

Shaping the spectrum of a signal

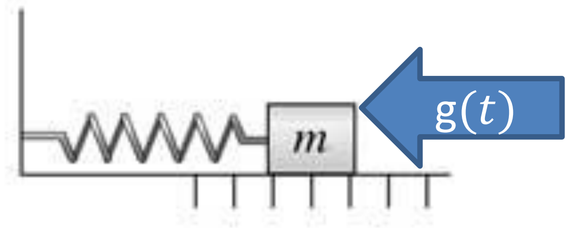
- Important signal processing task: change the spectrum of a signal
- Example: to
- Fourier transform involves an integral over all time ($-\infty < t < \infty$) $\Rightarrow$ we have to wait infinitely long to get the Fourier transform $\Rightarrow$ We can never change the spectrum of a signal directly by taking the Fourier transform
- The convolution theorem gives us a way of altering the spectrum of a signal without taking the Fourier transform

$$z(t) = (s * g)(t) \Rightarrow \tilde{z}(f) = \tilde{s}(f)\tilde{g}(f)$$

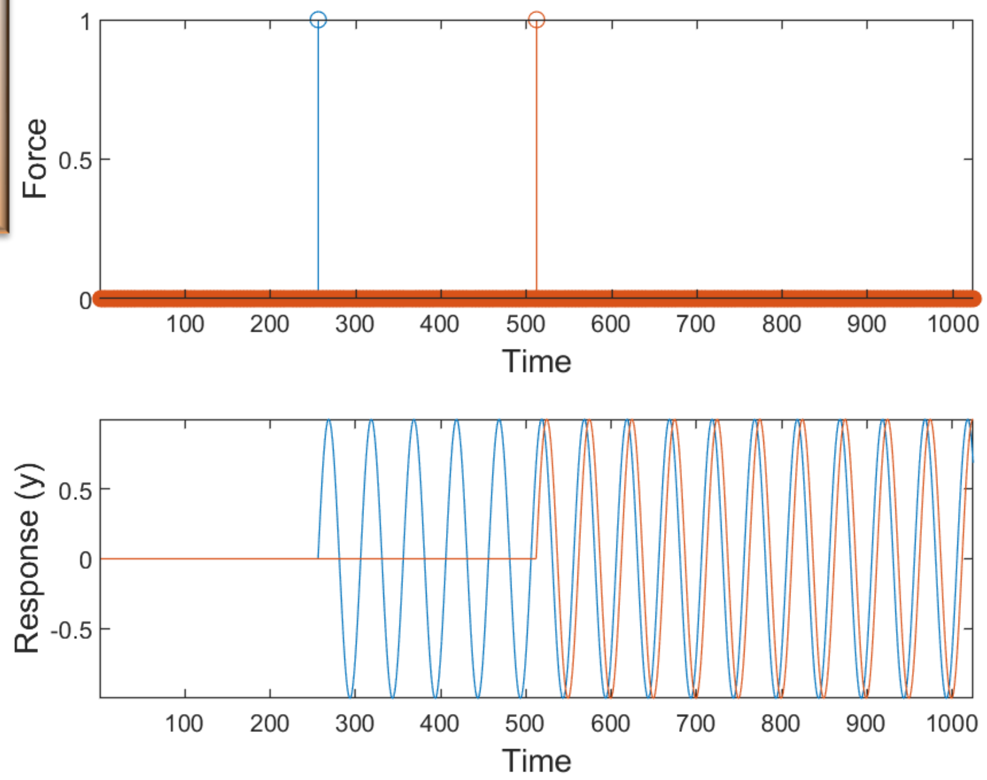
- Requirements: If $s(t)$ is the input signal,
  - We need to implement the convolution operation of $s(t)$ with any given $g(t)$: **Filtering**
  - We need to create a function $g(t)$ whose Fourier transform $\tilde{g}(f)$ is given to us: **Filter design**

Linear time invariant (LTI) system

Output of an LTI system is a convolution of its **Input** by the system **impulse response**



$$\frac{d^2y}{dt^2} = -ky + \underbrace{g(t)}_{\substack{External \\ force}}$$



- **Example:** Simple harmonic oscillator
- **Linear**: Response to external force $g(t) + b(t)$ is the sum of the responses to g(t) and $b(t)$
- **Time-invariant**: If $g(t)$ turns on at t_0, the response stays the same but starts at t_0 too

Impulse response

- **Impulse response** of an LTI system: Response to an impulsive input
- Let $y(t) = h(t)$ for impulsive force $\delta(t)$, then the response $y(t)$ to a general force g(t) is

$$y(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau = (g * h)(t); \quad \text{(Convolution)}$$

- Proof:

$$D_t = \frac{d^2}{dt^2} + k;\ D_t[h(t)] = \delta(t);\ \text{LTI} \Rightarrow D_t[h(t-t_0)] = \delta(t-t_0)$$

$$\text{Linearity} \Rightarrow D_t[y(t)] = \int_{-\infty}^{\infty} g(\tau)D_t[h(t-\tau)]d\tau = \int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau = g(t)$$

Transfer function

- Fourier transform of impulse response function
$$T(f) = F[h(t)]$$
- Response $y(t)$ to input $g(t)$
$$y(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau = (g * h)(t)$$
- Convolution theorem:
$$\tilde{y}(f) = T(f)\tilde{g}\,(f)$$
- $\Rightarrow$ We can shape the Fourier transform of a signal by passing it through an LTI system

Transfer function of SHO

$$\frac{d^2y}{dt^2} + ky = g(t)$$

- Substitute inverse Fourier transforms on both sides

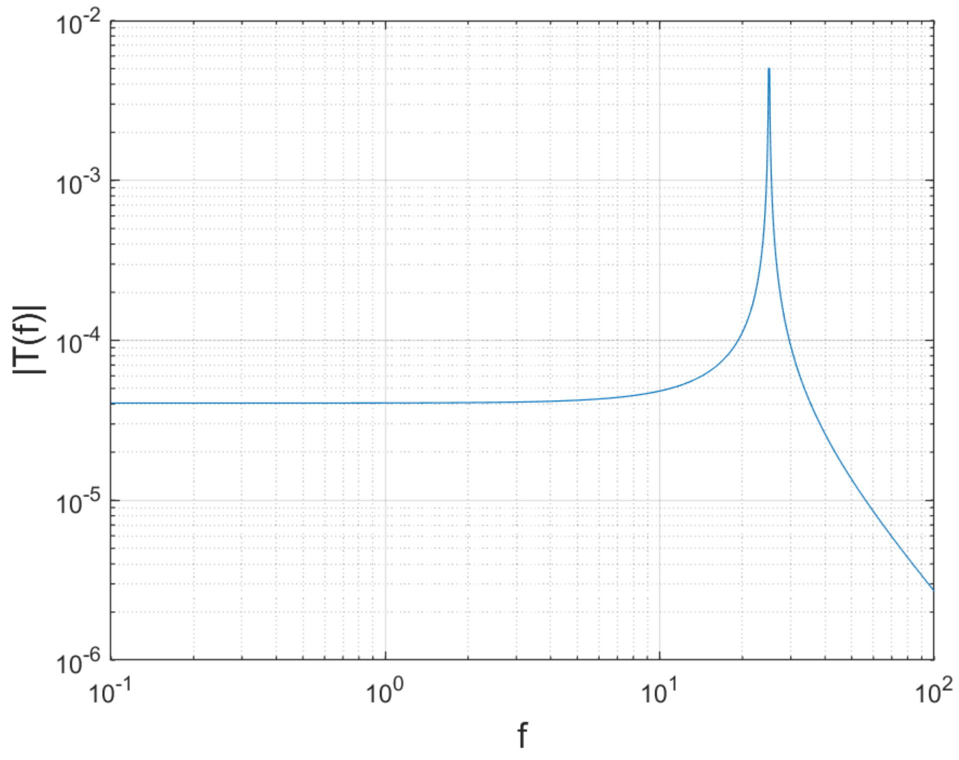
$$y(t) = \int_{-\infty}^{\infty} \tilde{y}(f) e^{2\pi i f t} df$$

$$\Rightarrow \frac{d^2y}{dt^2} = \int_{-\infty}^{\infty} (2\pi i f)^2 \tilde{y}(f) e^{2\pi i f t} df$$

$$= \int_{-\infty}^{\infty} (-4\pi^2 f^2) \tilde{y}(f) e^{2\pi i f t} df$$

$$\therefore \ \tilde{y}(f) = \underbrace{\frac{1}{k - 4\pi^2 f^2}}_{T(f) \text{ for SHO}} \tilde{g}(f)$$

- The output Fourier transform is suppressed greatly for frequencies above the resonance frequency



Only shown for $f > 0$

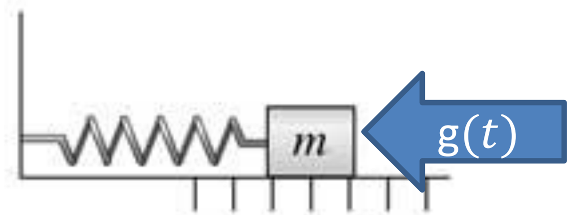
The transfer function is even in f

Filtering

- Passing a signal as input to an LTI system produces an output with a different Fourier transform: **Filtering**

$$\underbrace{\tilde{r}(f)}_{output} = T(f)\,\underbrace{\tilde{g}\,(f)}_{input}$$

- **Low-pass** filtering: Suppression of Fourier transform of input for $|f| > f_{low}$
- Example: SHO is a low pass filter for external disturbance
- **High-pass** and **band-pass** filtering also possible



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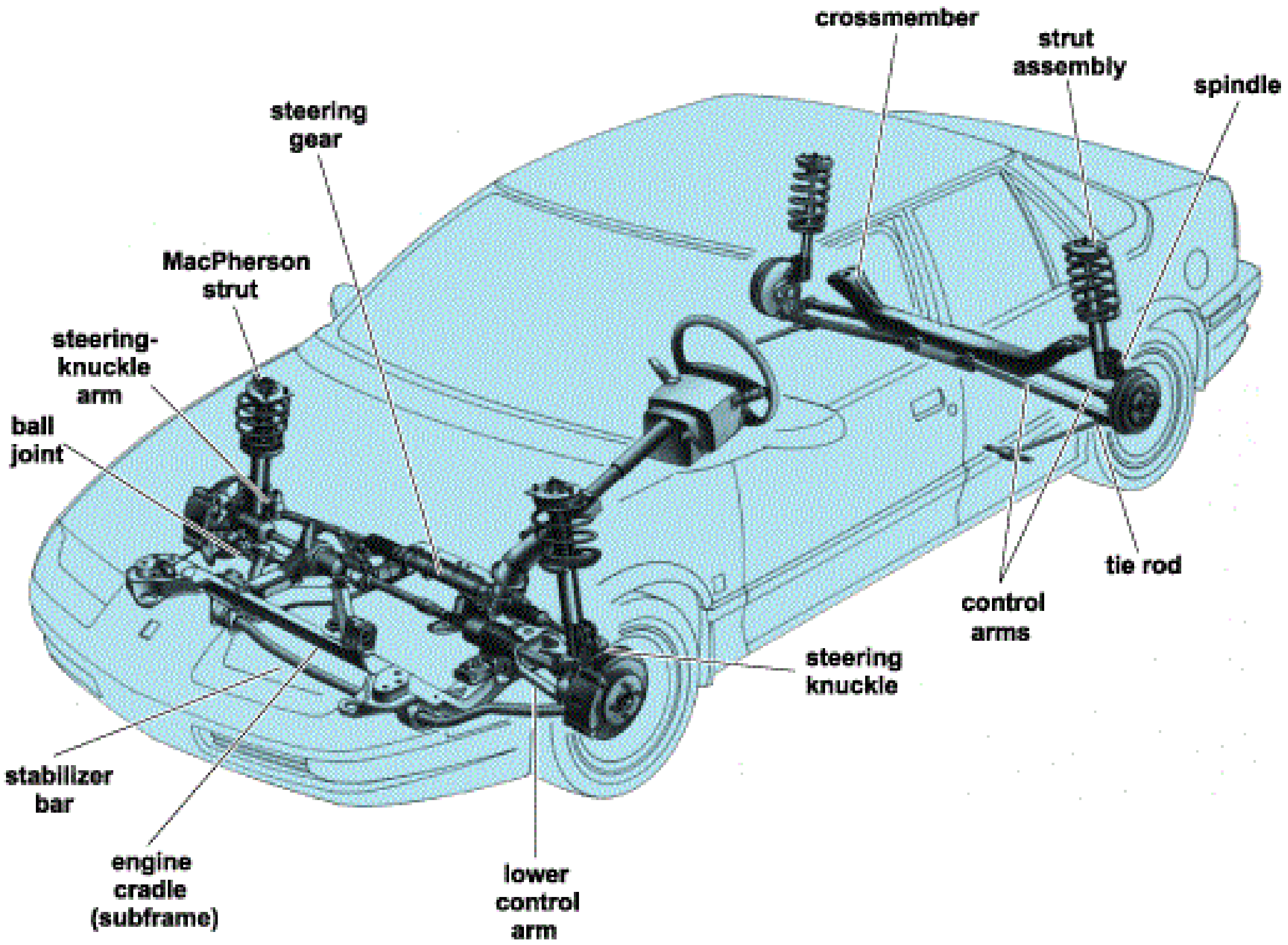
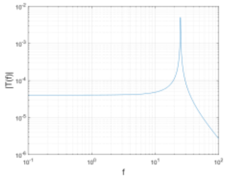
Example: Low pass filtering

- A car's suspension system is a low pass filter!
- Prevents high-frequency road noise from entering the cabin



Example: Seismic isolation in LIGO and Virgo



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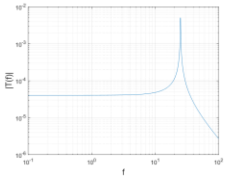

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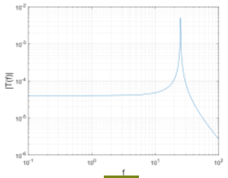
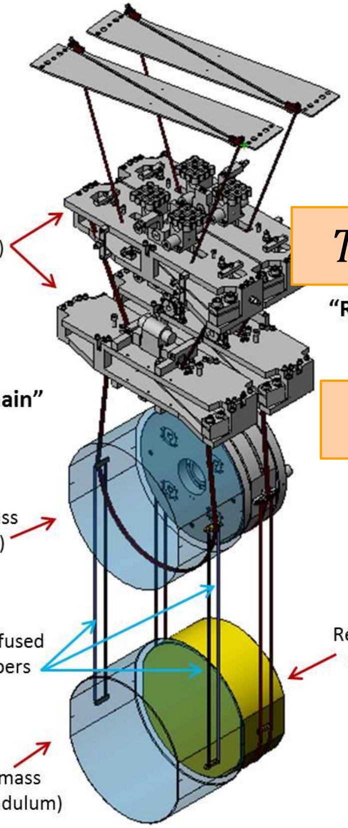


$T(f) = T_1(f)T_2(f) \ldots T_N(f)$
Frequencies above pendulum chain resonance frequencies are exponentially suppressed

Ground Motion

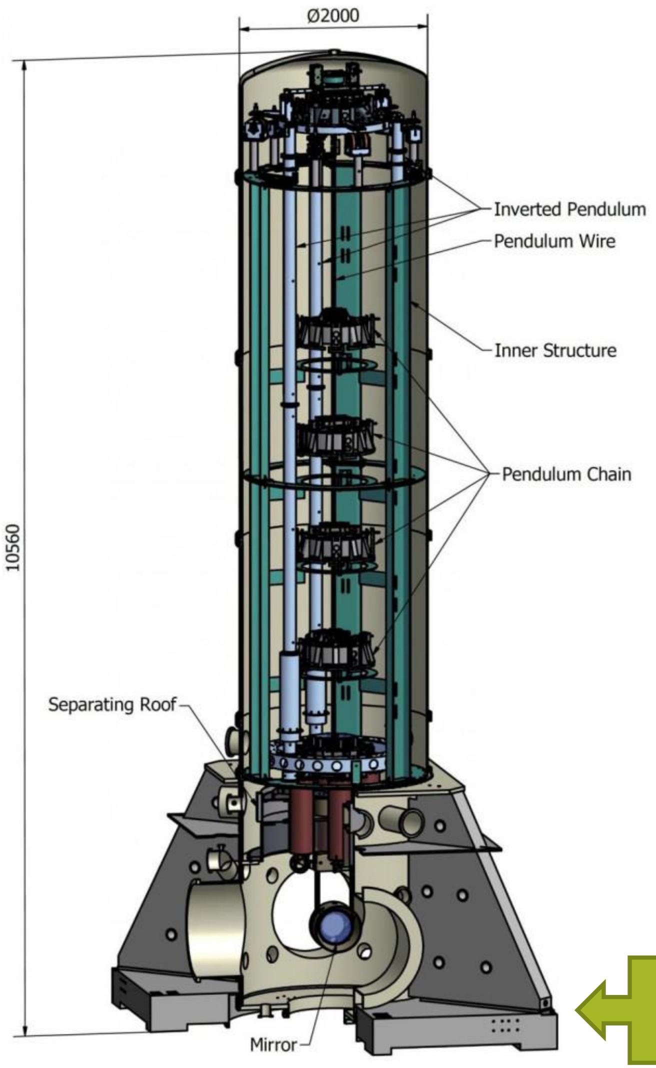
$T_1(f)$ (not shown)



$T_2(f)$

"Reaction Chain" Side

⋮



Ground Motion

Discrete time LTI systems

- An LTI differential equation can be approximated by a finite difference equation $\rightarrow$ **Discrete time** LTI system
- Most of the results associated with continuous time LTI systems carry over to discrete time LTI system
  - There is a discrete time analog of the faltung theorem in terms of the Discrete Fourier Transform
  - There is an impulse response and a transfer function

Example: Discrete time LTI system

Simple harmonic oscillator equation approximated by finite differences

$$\frac{dy}{dt} \approx \frac{y[t] - y[t-\Delta]}{\Delta}$$

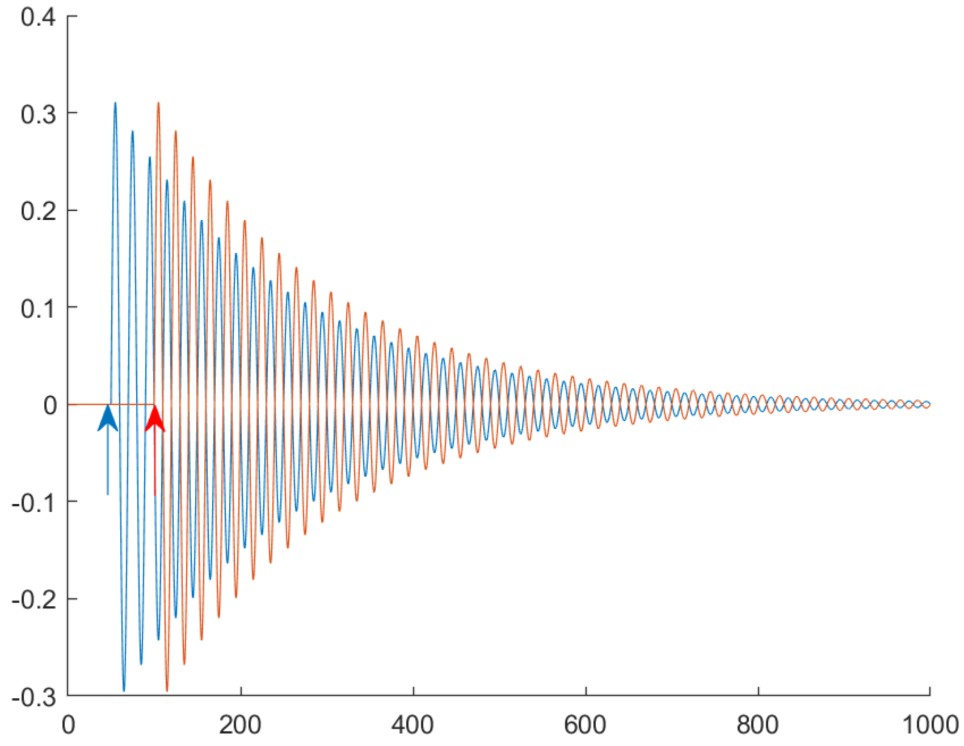
For $t_k = k\Delta \; and \; y(t_k) = y_k$,

$$\left.\frac{dy}{dt}\right|_{t_k} \approx \frac{y_k - y_{k-1}}{\Delta}$$

$$\frac{d^2y}{dt^2} + ky = f(t) \to \frac{y_k - 2y_{k-1} + y_{k-2}}{\Delta^2} + ky_k = f_k$$

$$y_k = \frac{1}{k + \Delta^{-2}}\left(f_k + \frac{2}{\Delta^2}y_{k-1} - \frac{1}{\Delta^2}y_{k-2}\right)$$

$$\Rightarrow y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_1 f_k$$



Nyquist sampling theorem

Nyquist sampling theorem: Analog to Digital

- An analog signal $s(t)$ can be sampled without any loss of information if
  - The analog signal is **band-limited**:
  $$\tilde{s}(f) = 0 \text{ for } f \notin [-f_B, f_B]$$
  - The sampling frequency $f_s \geq 2f_B$
  - The critical sampling frequency $f_s = 2f_B$ is called the **Nyquist rate**
  - f_B is called the **bandwidth**
- Intuitive explanation: $s(t)$ is a sum of sinusoids of form $A\sin(2\pi f t + \phi)$
- Period of signal $1/f \Rightarrow$to reconstruct $A\sin(2\pi f t + \phi)$ from its samples we need to reconstruct it in any one period $t \in [a, a + \frac{1}{f}]$
- Two unknowns (A, ϕ) for a given $f \Rightarrow$minimum 2 samples needed per period
- Maximum frequency $f_B \Rightarrow$ 2 samples in $1/f_B \Rightarrow$ Period of sampling $= \frac{1}{2} \times \frac{1}{f_B} \Rightarrow$ Frequency of sampling $f_s = 2f_B$

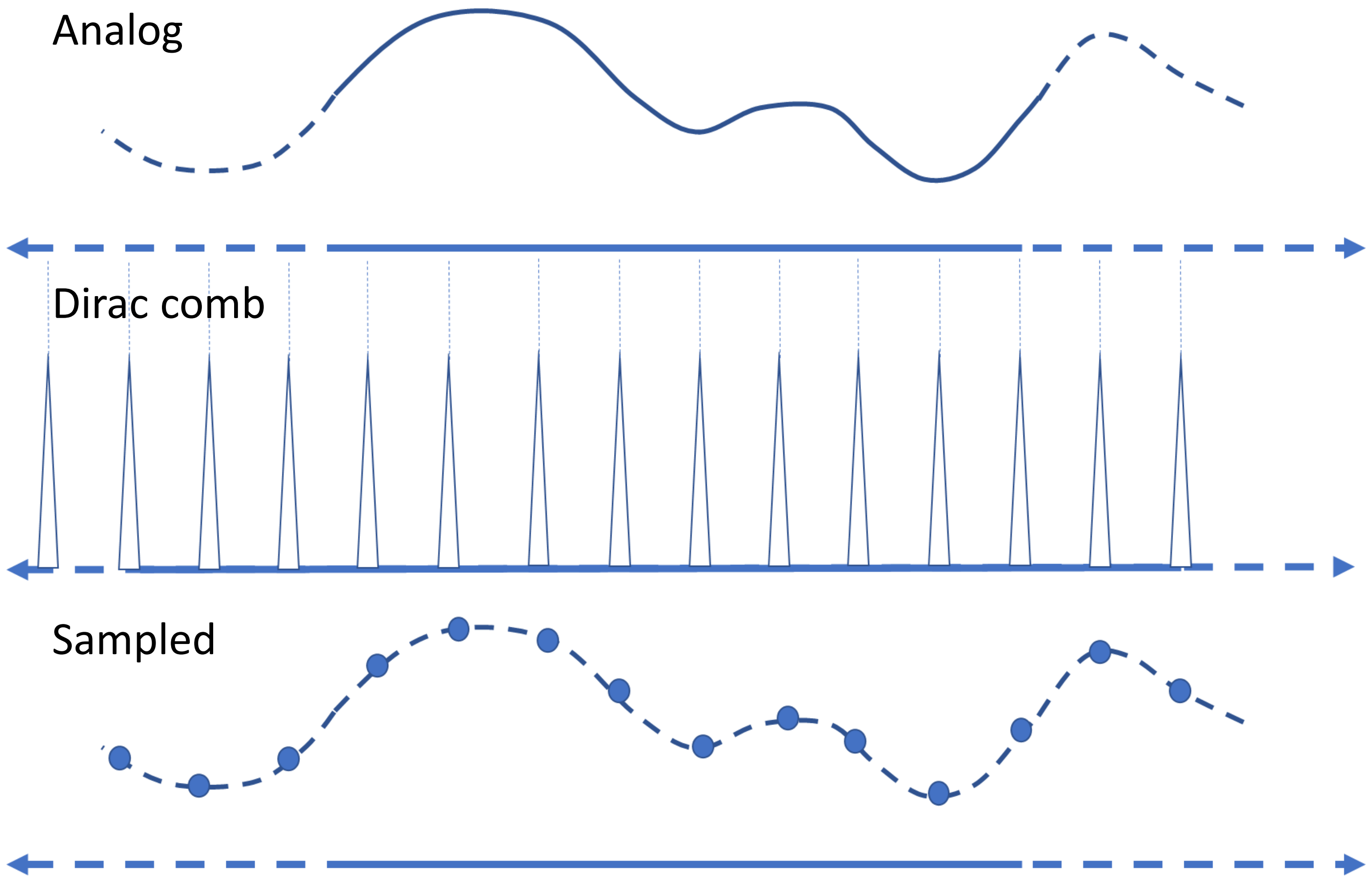
Nyquist sampling theorem: Formal proof

- **Sampled signal**: Continuous time representation of Discrete time signal

$$s_{samp}(t) = s_{anlg}(t) \sum_{k=-\infty}^{\infty} \delta(t - k\Delta)$$

- **Dirac comb function**:

$$c(t;\Delta) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta)$$



Nyquist sampling theorem: Formal proof

- Fourier transform of Dirac comb

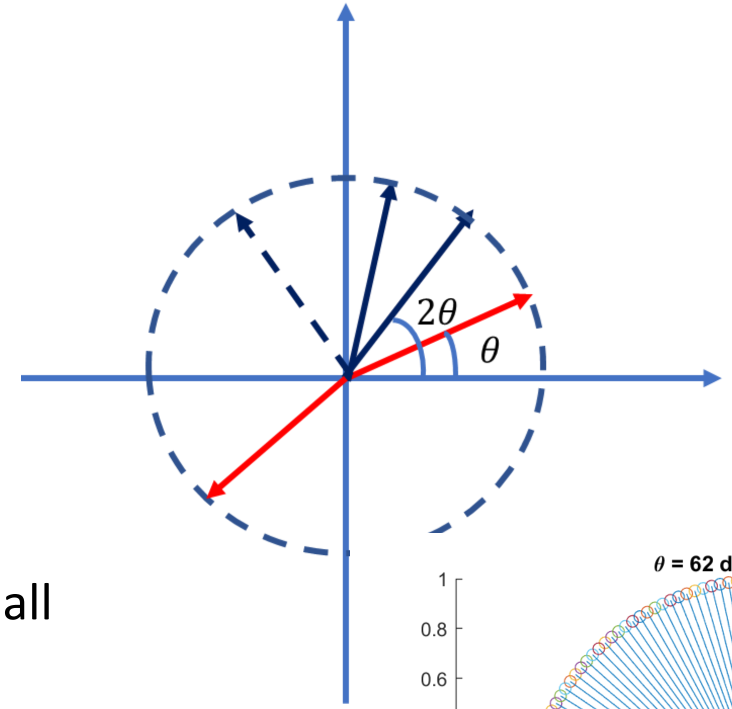
$$F[c(t)] = c(f; \frac{1}{\Delta})$$

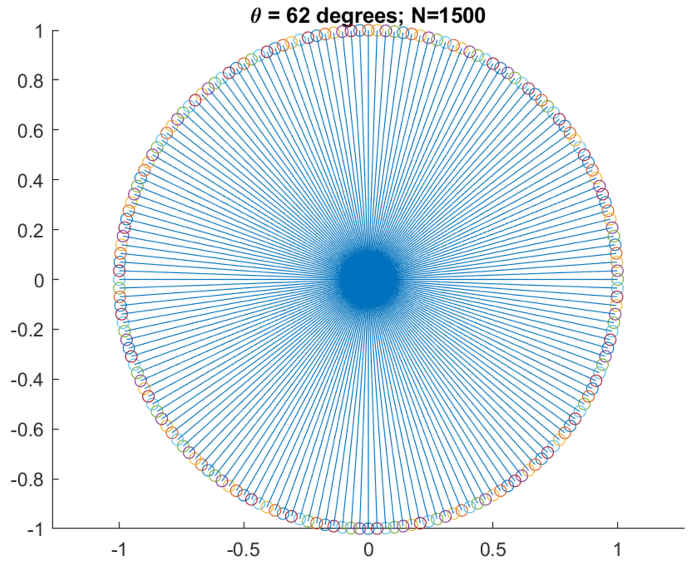
is also a Dirac comb!

- Proof:

$$\int_{-\infty}^{\infty} c(t;\Delta) e^{-2\pi i f t} dt = \sum_{k=-\infty}^{k=\infty} e^{-2\pi i f k\Delta} = \sum_{k=-\infty}^{k=\infty} e^{-ik\theta}$$

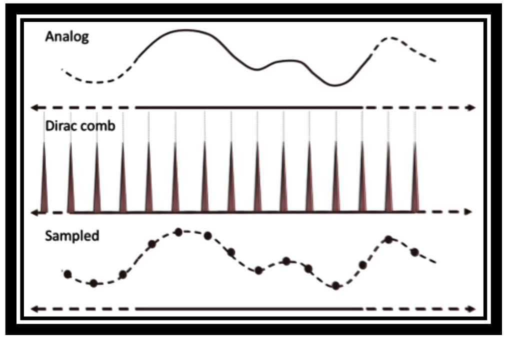
- Only when $f = p/\Delta; p = 0, \pm 1, \pm 2, \ldots$ all the phasors all line up
  - For such values $F[c(t;\Delta)] = \infty$
- For other values of f, they will approximately cancel out in pairs: $F[c(t;\Delta)] = 0$





Nyquist sampling theorem: Formal proof

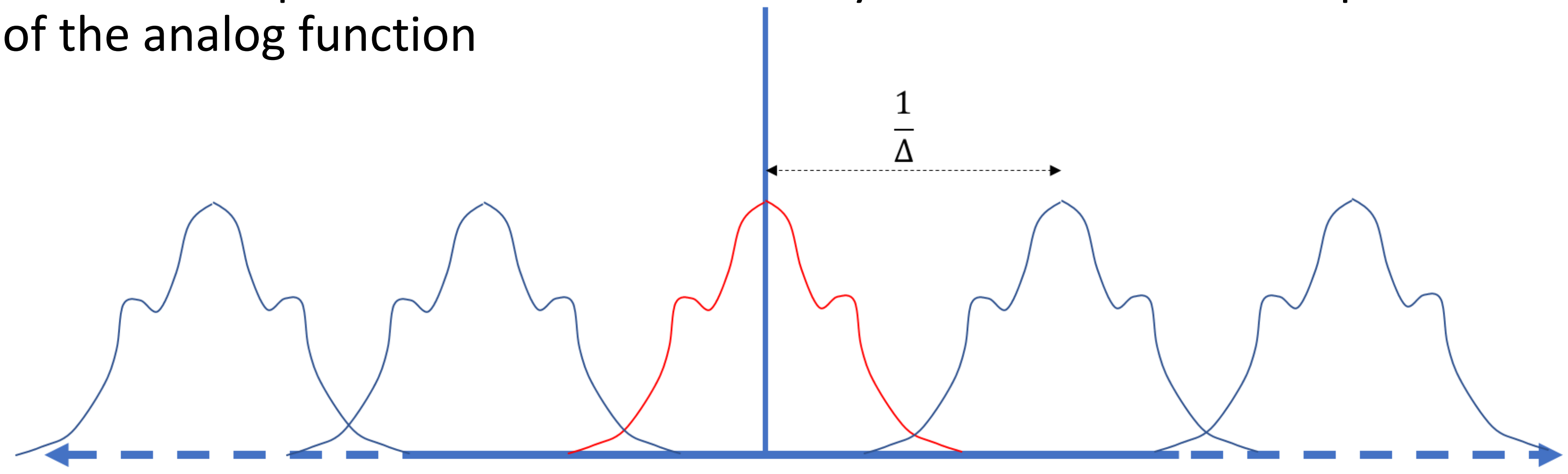
- Convolution theorem (opposite direction)$\Rightarrow$



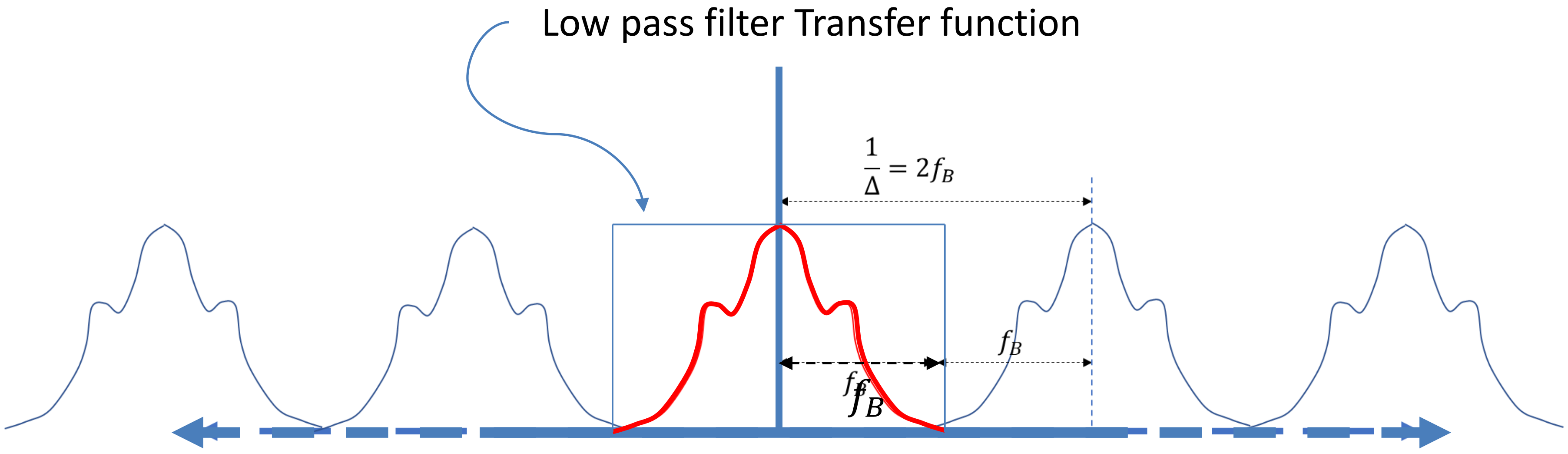
$$F\left[s_{samp}(t) = \underbrace{s_{anlg}(t) \times c(t;\Delta)}_{Product\ in\ time\ domain}\right] = \left(\underbrace{F\left[s_{anlg}(t)\right] * F[c(t;\Delta)]}_{Convoution\ in\ Fourier\ domain}\right)(f)$$

$$= \sum_{k=-\infty}^{k=\infty} \int_{-\infty}^{\infty} d\nu\, \tilde{s}_{anlg}(f-\nu)\delta\left(\nu - \frac{k}{\Delta}\right) = \sum_{k=-\infty}^{\infty} \tilde{s}_{anlg}(f - \frac{k}{\Delta})$$

- The Fourier transform of a sampled function is obtained by the sum of shifted copies of the Fourier transform of the analog function



Nyquist sampling theorem: Formal proof



- If $F[s_{anlg}(t)]$ is band limited with bandwidth f_B and

$$\frac{1}{\Delta} \geq 2f_B$$

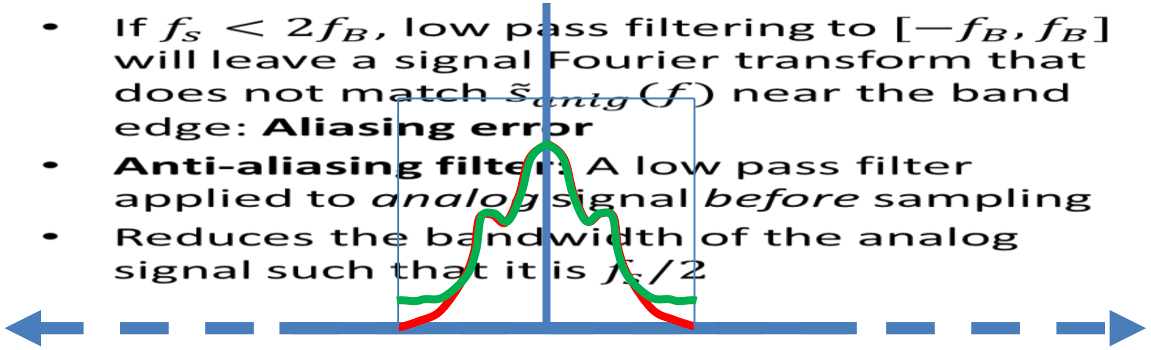
then the copies will not overlap

- Low pass filter the sampled signal to $[-f_B, f_B] \Rightarrow$ Recover $s_{anlg}(t) \Rightarrow$ No information loss!
- Nyquist rate:

$$\frac{1}{\Delta} \geq 2f_B \Rightarrow f_s \geq 2f_B$$

Aliasing error

- If $f_s < 2f_B$, low pass filtering to $[-f_B, f_B]$ will leave a signal Fourier transform that does not match $\tilde{s}_{anlg}(f)$ near the band edge: **Aliasing error**
- **Anti-aliasing filter:** A low pass filter applied to *analog* signal *before* sampling
- Reduces the bandwidth of the analog signal such that it is $f_s/2$



Harry Nyquist

- The Nyquist Sampling theorem makes digital signal processing possible
- Digital signal processing underlies all of modern communications
- Example:
  - Anlog-to-Digital: When you speak into your cell phone, your analog speech signal is sampled, digitized and transmitted as a string of binary digits
  - Digital-to-Analog: At the other end, the digitized signal is converted back to an analog signal that is played on the speaker for your ears
- **Amazing fact:** Nyquist lived in retirement in the Rio Grande Valley! He is buried in Harlingen, TX (30 miles from Brownsville, TX)

Harry Nyquist

Harry Nyquist

Born	February 7, 1889 Stora Kil, Nilsby, Värmland, Sweden
Died	April 4, 1976 (aged 87) Harlingen, Texas, U.S.
Residence	United States
Nationality	Swedish
Citizenship	Swedish / American
Alma mater	Yale University University of North Dakota
Known for	Nyquist–Shannon sampling theorem Nyquist rate

Discrete Fourier transform

Discrete Fourier Transform (DFT)

<u>Definition of DFT and inverse DFT</u>

$$\tilde{x}_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N} \quad \text{and} \quad x_n = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{x}_k\, e^{2\pi i n k/N}$$

- DFT is the analog of Fourier transform for finite length discrete time data:

$$\int_0^{T=(N-1)\Delta} x(t)e^{-2\pi i f t}\, dt \approx \Delta \sum_{n=0}^{N-1} x_n\, e^{-2\pi i\left(\frac{k}{N\Delta}\right)(n\Delta)}$$

- $\Rightarrow$ Frequencies are $\frac{k}{N\Delta} = \frac{k}{T}$ where Δ is the sampling interval and T is the length of the data segment
- This is because the the finite data is assumed to be periodic outside $[0, T] \Rightarrow$ All Fourier sinusoids must also be periodic $\Rightarrow$ Their frequencies must be integer multiples of $1/T$ ($\Rightarrow$ Fourier series)
- Sampled data and Nyquist theorem $\Rightarrow$ maximum frequency in the data $\Rightarrow$ only a finite number of frequencies are needed

Properties of DFT

$$\tilde{x}_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N}$$

- For N even:

$$\tilde{x}_{\frac{N}{2}+p} = \sum_{n=0}^{N-1} x_n e^{-2\pi i p n/N} e^{-\pi i n} = \sum_{n=0}^{N-1} (-1)^n x_n e^{-2\pi i p n/N}$$

$$\tilde{x}_{\frac{N}{2}-p} = \sum_{n=0}^{N-1} (-1)^n x_n e^{2\pi i p n/N}$$

$$\therefore \tilde{x}_{\frac{N}{2}+p} = \tilde{x}^*{}_{\frac{N}{2}-p}$$

$\bar{x}$	DFT
1	21.0000
2	-3.0000 - 5.1962i
3	-3.0000 - 1.7321i
4	-3.0000
5	-3.0000 + 1.7321i
6	-3.0000 + 5.1962i

- Hermiticity property: DFT frequencies above $N/2$ are negative frequencies in the Fourier transform

Properties of DFT

$$\tilde{x}_k = \sum_{n=0}^{N-1} x_n e^{-2\pi ikn/N}$$

$$\tilde{x}_0 = \sum_{n=0}^{N-1} x_n e^0 = \sum_{n=0}^{N-1} x_n \Rightarrow \text{Real}$$

- For N even:

$$\tilde{x}_{\frac{N}{2}} = \sum_{n=0}^{N-1} x_n e^{-\pi in} = \sum_{n=0}^{N-1} (-1)^{np} x_n \Rightarrow \text{ Real}$$

$\overline{x}$	DFT
1	**21.0000**
2	-3.0000 - 5.1962i
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5	-3.0000 + 1.7321i
6	-3.0000 + 5.1962i

Discrete Fourier Transform: Matrix form

$$\tilde{x}_k = \sum_{n=0}^{N-1} x_n e^{-2\pi i k n/N}$$

Matrix form:

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_{N-1} \end{pmatrix} = F \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \rightarrow \tilde{x}^T = F\ \bar{x}^T$$

where $F = \begin{pmatrix} e^{2\pi i 0\times 0/N} = 1 & e^{2\pi i 0\times 1/N} = 1 & \dots & e^{2\pi i 0\times (N-1)/N} = 1 \\ e^{2\pi i 1\times 0/N} = 1 & e^{2\pi i 1\times 1/N} & \dots & e^{2\pi i 1\times (N-1)/N} \\ \vdots & \vdots & \vdots & \vdots \\ e^{2\pi i (N-1)\times 0/N} = 1 & e^{2\pi i\,(N-1)\times 1/N} & \dots & e^{2\pi i (N-1)\times (N-1)/N} \end{pmatrix}$

DFT matrix and its inverse

$$F = \begin{pmatrix} e^{2\pi i 0\times 0/N} = 1 & e^{2\pi i 0\times 1/N} = 1 & \dots & e^{2\pi i 0\times (N-1)/N} = 1 \\ e^{2\pi i 1\times 0/N} = 1 & e^{2\pi i 1\times 1/N} & \dots & e^{2\pi i 1\times (N-1)/N} \\ \vdots & \vdots & \vdots & \vdots \\ e^{2\pi i (N-1)\times 0/N} = 1 & e^{2\pi i\,(N-1)\times 1/N} & \dots & e^{2\pi i (N-1)\times (N-1)/N} \end{pmatrix}$$

$$F^\dagger = (F^T)^*$$

$$(FF^\dagger)_{mn} = \sum_{k=0}^{N-1} F_{mk} F^\dagger_{kn} = \sum_{k=0}^{N-1} e^{-2\pi i mk/N} e^{2\pi i kn/N}$$

$$= \sum_{k=0}^{N-1} e^{-2\pi i k(m-n)/N} = \frac{1 - e^{-2\pi i\,(m-n)}}{1 - e^{2\pi i\,(m-n)/N}} = \begin{cases} N \text{ for } m = n \\ 0 \text{ for } m \neq n \end{cases}$$

$$\therefore \quad FF^\dagger = N\mathbb{I}$$

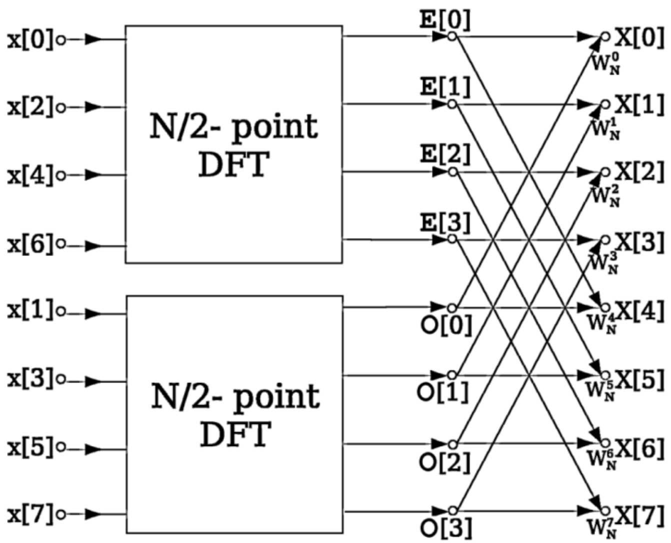
Where $\mathbb{I}$ is the identity matrix

$$\therefore \quad F^{-1} = \frac{F^\dagger}{N} \Rightarrow \bar{x} = F^{-1}\,\tilde{x} \Rightarrow x_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}_n e^{2\pi i kn/N} \quad \text{(Inverse DFT)}$$

Fast Fourier Transform

- Computing the DFT involves taking the product of an $N \times N$ matrix with an N element column vector $\Rightarrow 2N^2$ multiplications and additions (i.e., **floating point operations**)
- Fast Fourier Transform (FFT): a clever algorithm for carrying out the same matrix product with $O(N \log_2 N)$ floating point operations
- Example: For $N = 1024$, the FFT is ≈ 200 times faster!

FFT



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- Matlab function: `FFT`
- FFT was included in Top 10 Algorithms of 20th Century by the IEEE journal Computing in Science & Engineering.
- FFTs are of great importance to a wide variety of applications, from digital signal processing and solving partial differential equations to algorithms for quick multiplication of large integers.

Circular convolution theorem and cross-correlation theorem

[edit]

The convolution theorem for the continuous and discrete time Fourier transforms indicates that a convolution of two infinite sequences can be obtained as the inverse transform of the product of the individual transforms. With sequences and transforms of length N, a circularity arises:

$$\begin{aligned}\mathcal{F}^{-1}\{\mathbf{X}\cdot\mathbf{Y}\}_n &\stackrel{\text{def}}{=} \frac{1}{N}\sum_{k=0}^{N-1} X_k\cdot Y_k\cdot e^{\frac{2\pi i}{N}kn}\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} x_l e^{-\frac{2\pi i}{N}kl}\right)\cdot\left(\sum_{m=0}^{N-1} y_m e^{-\frac{2\pi i}{N}km}\right)\cdot e^{\frac{2\pi i}{N}kn}\\ &= \sum_{l=0}^{N-1} x_l \sum_{m=0}^{N-1} y_m\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}k(n-l-m)}\right).\end{aligned}$$

Inverse Fourier transform of product of Fourier transforms

$$F^{-1}[T(f)\tilde{y}(f)] = \int_{-\infty}^{\infty} h(t-\tau)y(\tau)d\tau$$

The quantity in parentheses is 0 for all values of m except those of the form $n-l-pN$, where p is any integer. At those values, it is 1. It can therefore be replaced by an infinite sum of Kronecker delta functions, and we continue accordingly. Note that we can also extend the limits of m to infinity, with the understanding that the x and y sequences are defined as 0 outside [0,N-1]:

$$\begin{aligned}\mathcal{F}^{-1}\{\mathbf{X}\cdot\mathbf{Y}\}_n &= \sum_{l=0}^{N-1} x_l \sum_{m=-\infty}^{\infty} y_m\left(\sum_{p=-\infty}^{\infty}\delta_{m,(n-l-pN)}\right)\\ &= \sum_{l=0}^{N-1} x_l \sum_{p=-\infty}^{\infty}\underbrace{\left(\sum_{m=-\infty}^{\infty} y_m\cdot\delta_{m,(n-l-pN)}\right)}_{y_{n-l-pN}}\\ &= \sum_{l=0}^{N-1} x_l\left(\sum_{p=-\infty}^{\infty} y_{n-l-pN}\right) \stackrel{\text{def}}{=} (\mathbf{x}*\mathbf{y_N})_n\ ,\end{aligned}$$

Circular convolution

which is the convolution of the $\mathbf{x}$ sequence with a periodically extended $\mathbf{y}$ sequence defined by: