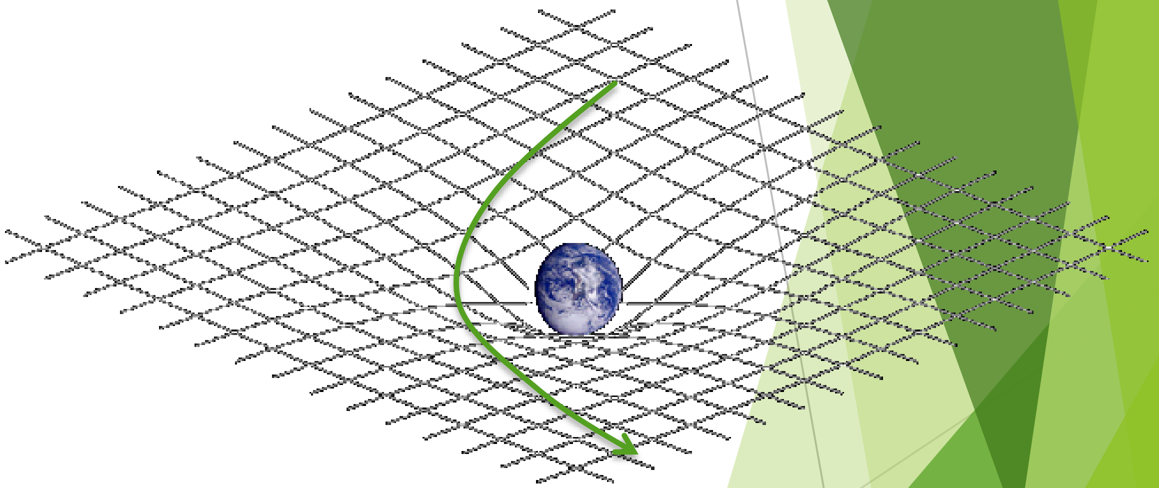
# Gravitational wave theory for data analysts

The theoretical minimum needed to understand GW data analysis

# Space-time geometry

- Gravity is not a force but the effect of a curved space-time geometry
- **Matter influences space-time geometry**
- **Space-time geometry guides the motion of matter**
- Test particles follow paths of shortest lengths (**geodesics**) in the curved geometry



# Space-time geometry

- **Intrinsic** geometry of a **manifold** is described by specifying the **distance** between every pair of points
- Distance between points in 3-D Euclidean space in cartesian coordinates $(x, y, z)$

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = (\Delta x\ \Delta y\ \Delta z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

# Space-time geometry

- **Intrinsic** geometry of a **manifold** is described by specifying the **distance** between every pair of points
- Distance between points in 3-D Euclidean space in cartesian coordinates $(x, y, z)$

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = (\Delta x \ \Delta y \ \Delta \mathrm{z}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

- Distance between points in 4-D Minkowskian space-time in coordinates $x = (x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$

$$\Delta s^2 = (\Delta x^0 \ \Delta x^1 \ \Delta x^2 \ \Delta x^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix} \equiv \sum_{\mu,\nu=0}^{3} \Delta x^\mu \Delta x^\nu \eta_{\mu\nu}$$

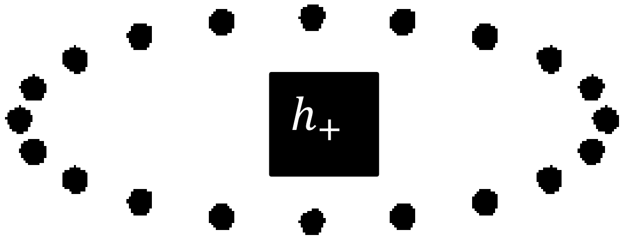
  - The matrix with elements $\eta_{\mu\nu}$ is the flat (Minkowskian) space-time **metric tensor**

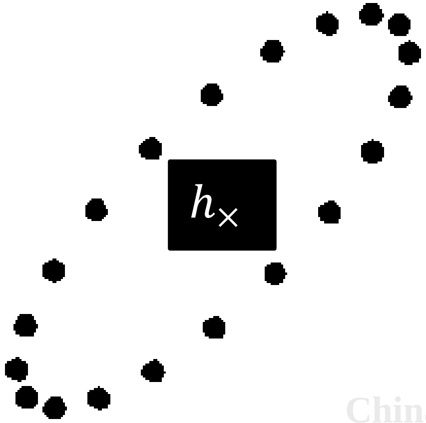
- General space-time metric tensor: $\eta_{\mu\nu} \to g_{\mu\nu}(x^0, x^1, x^2, x^3) \equiv g_{\mu\nu}(x)$

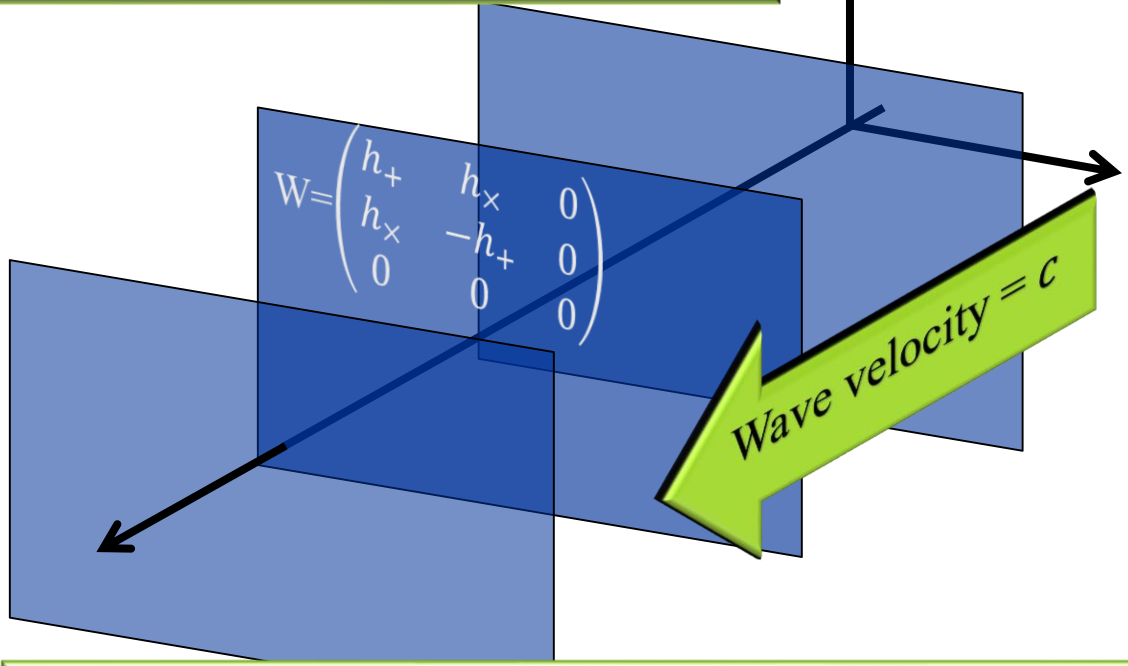
$$ds^2 = \sum_{\mu,\nu=0}^{3} \mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu}(x) \equiv dx^\mu dx^\nu g_{\mu\nu}(x)$$

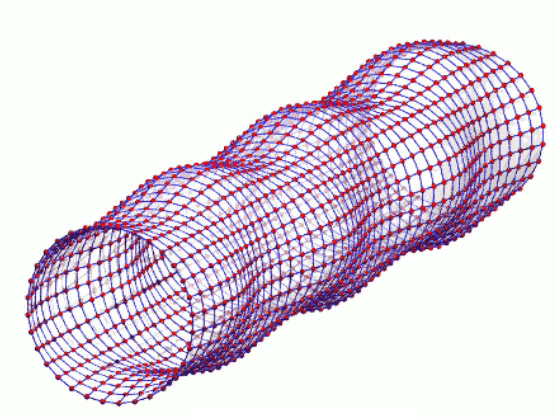
# Gravitational Wave

Perturbation in the metric tensor: $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}(x)$



Perturbation in the metric propagates as a wave

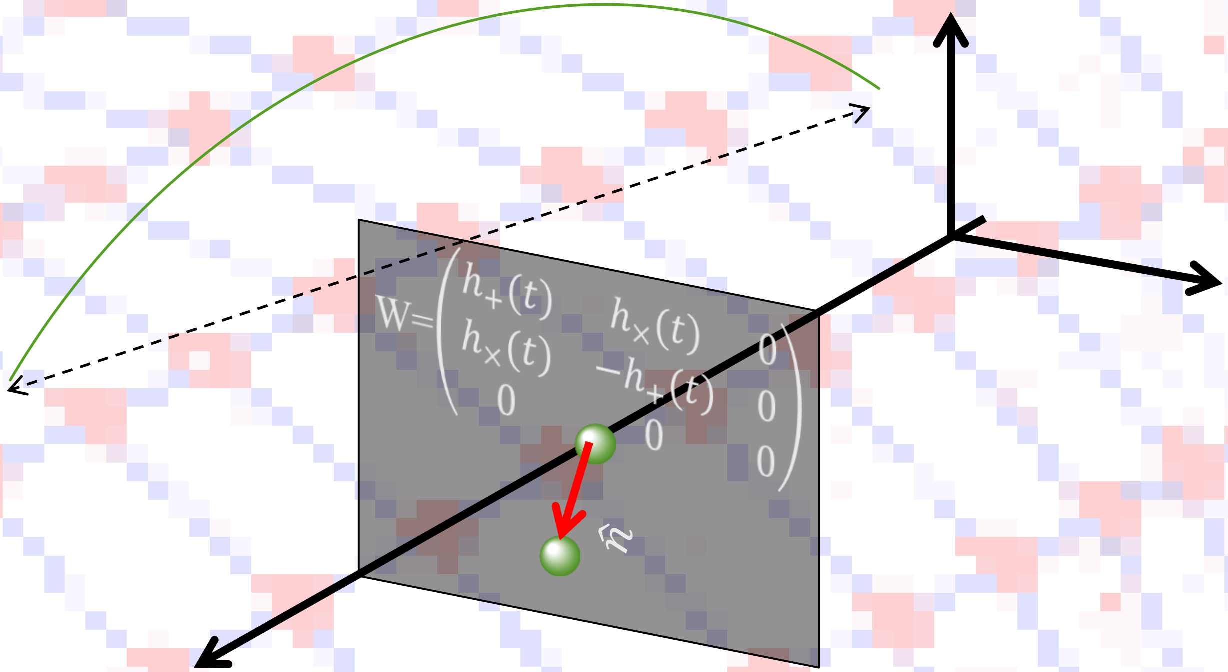
- Appropriate gauge and coordinate choices: TT Gauge

  $\Rightarrow \Box h_{\mu\nu} = 0 \Rightarrow$ Plane wave solutions

- Plane wave crossing a fixed point

$$\left.\begin{aligned} h_{11} = -h_{22} = h_+(t) \\ h_{12} = h_{21} = h_\times(t) \end{aligned}\right\} \text{two polarizations}$$

# Long wavelength approximation

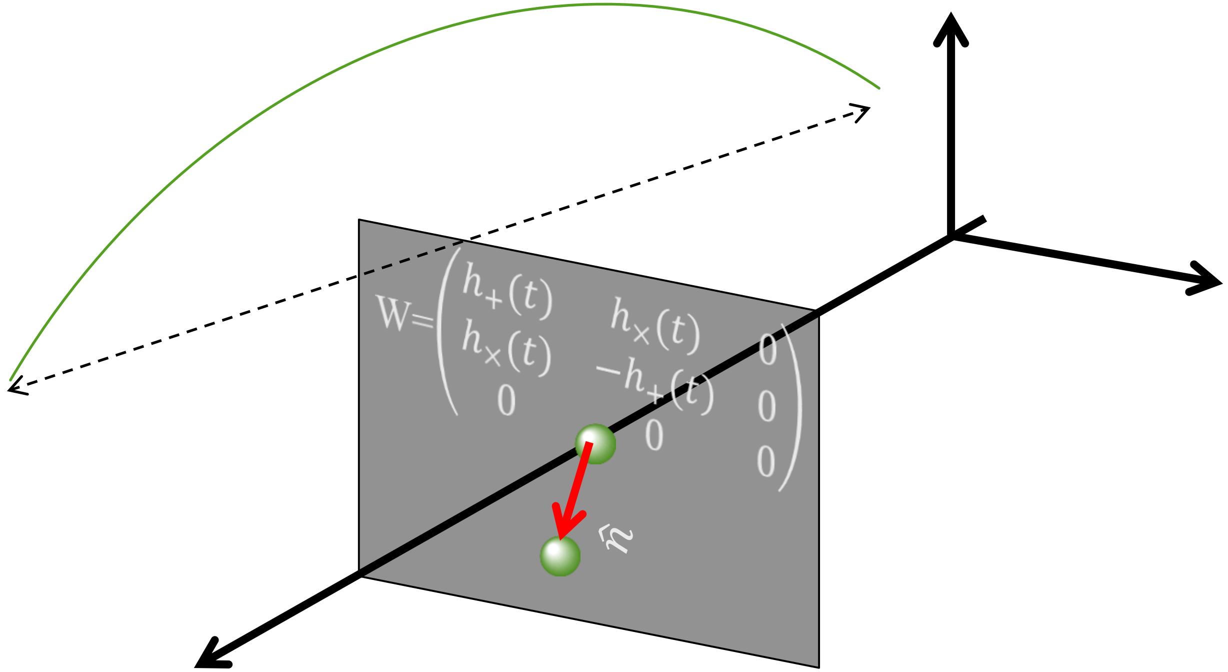



GW wavelength $\lambda \gg$ separation of the two points $\Rightarrow$ Approx. same $h_{ij}$ at the two points

$$h(t) = \frac{\Delta L(t)}{L} = \frac{\sqrt{|Ln^i g_{ij} Ln^j|} - L}{L} = \frac{1}{2} n^i W_{ij} n^j = \frac{1}{2} \hat{n}^T \, W(t) \hat{n}$$

$$= \sum_{i,j=1}^{3} n_i n_j W_{ij} = W_{ij} (\hat{n} \otimes \hat{n})^{ij}$$

# Long wavelength approximation




GW wavelength $\lambda \gg$ separation of the two points $\Rightarrow$ Approx. same $h_{ij}$ at the two points

$$h(t) = \frac{\Delta L(t)}{L} = \frac{\sqrt{|Ln^i g_{ij} Ln^j|} - L}{L} = \frac{1}{2} n^i W_{ij} n^j = \frac{1}{2} \hat{n}^T\, W(t) \hat{n}$$

$$= \sum_{i,j=1}^{3} n_i n_j W_{ij} = W_{ij} (\hat{n} \otimes \hat{n})^{ij}$$

# Strain signal

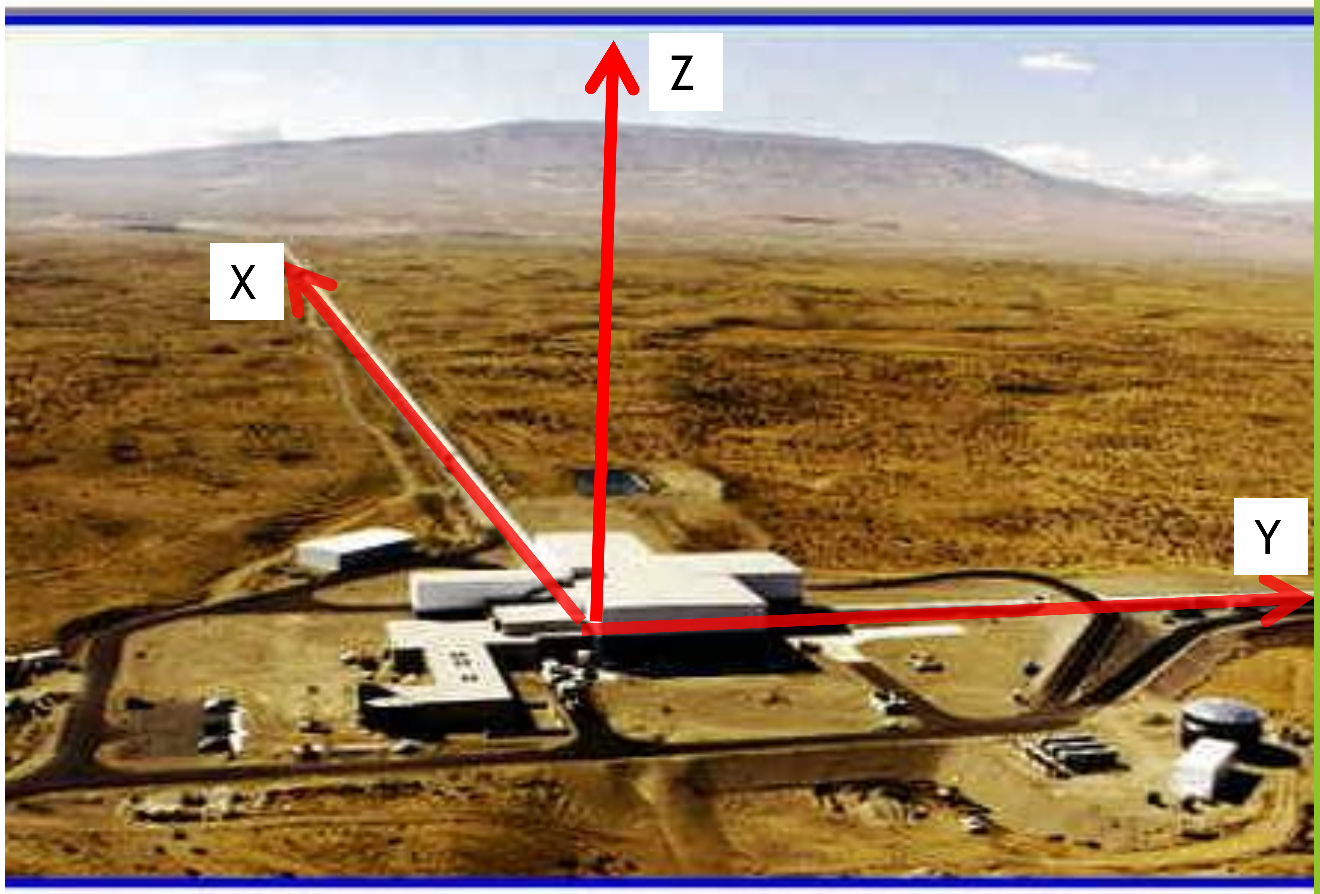
- GW interferometer: Measured quantity is the difference in arm lengths
- Strain signal:

$$s(t) = W_{ij}\left[(\hat{n}_X \otimes \hat{n}_X)^{ij} - (\hat{n}_Y \otimes \hat{n}_Y)^{ij}\right]$$

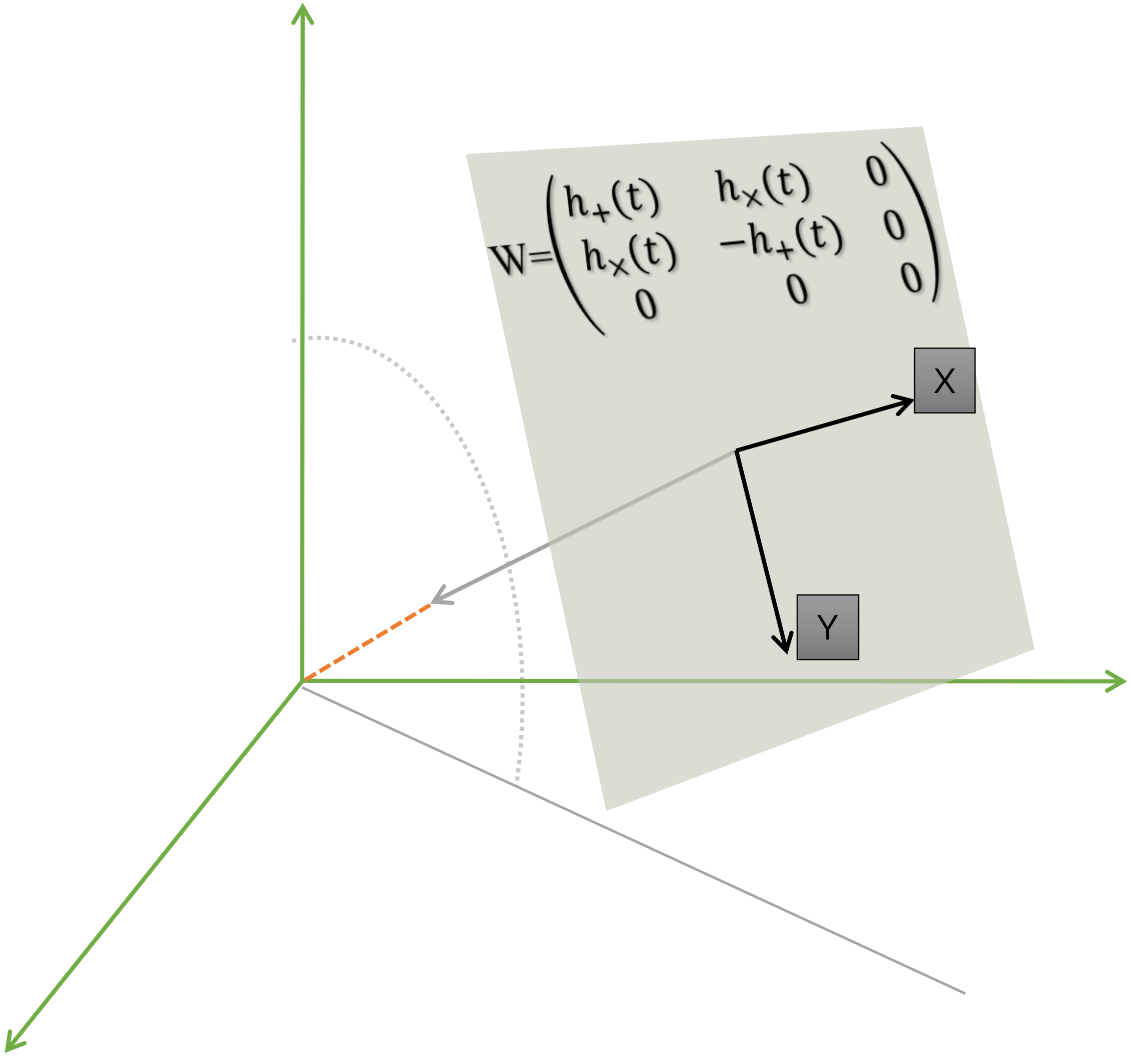
- Detector tensor (common notation in papers):

$$\overleftrightarrow{D} = \hat{n}_X \otimes \hat{n}_X - \hat{n}_Y \otimes \hat{n}_Y$$

- Defined purely by the orientation of the detector arms



# Wave tensor




- The wave tensor is expressed most simply in the "Wave frame"
- At the origin of the detector axes:

$$W = h_+(t)\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} + h_\times(t)\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$W = h_+(t)(\hat{x}\otimes\hat{x} - \hat{y}\otimes\hat{y}) + h_\times(t)(\hat{x}\otimes\hat{y} + \hat{y}\otimes\hat{x})$$

- Common notation in papers:

$$\overleftrightarrow{W} = h_+(t)\,\overleftrightarrow{e}_+ + h_\times(t)\overleftrightarrow{e}_\times$$

- $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ are called "plus" and "cross" polarization tensors
- The wave tensor is defined purely in terms of the wave frame

# Strain signal

- Detector tensor:

$$\overleftrightarrow{D} = \hat{n}_X \otimes \hat{n}_X - \hat{n}_Y \otimes \hat{n}_Y$$

- Wave tensor:

$$\overleftrightarrow{W} = h_+(t)\,\overleftrightarrow{e}_+ + h_\times(t)\overleftrightarrow{e}_\times$$

$$\overleftrightarrow{e}_+ = \hat{x}\otimes\hat{x} - \hat{y}\otimes\hat{y}; \quad \overleftrightarrow{e}_\times = \hat{x}\otimes\hat{y} + \hat{y}\otimes\hat{x}$$

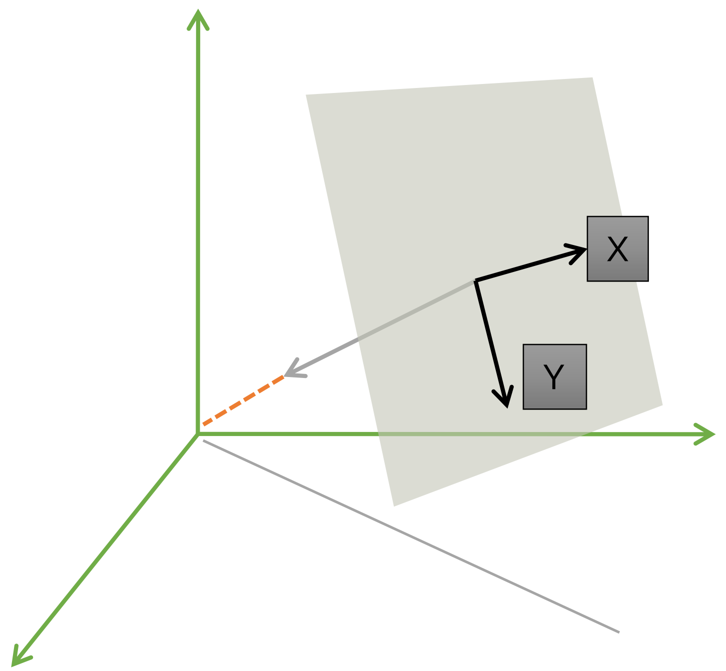
- Strain signal: “Contraction of wave and detector tensors”

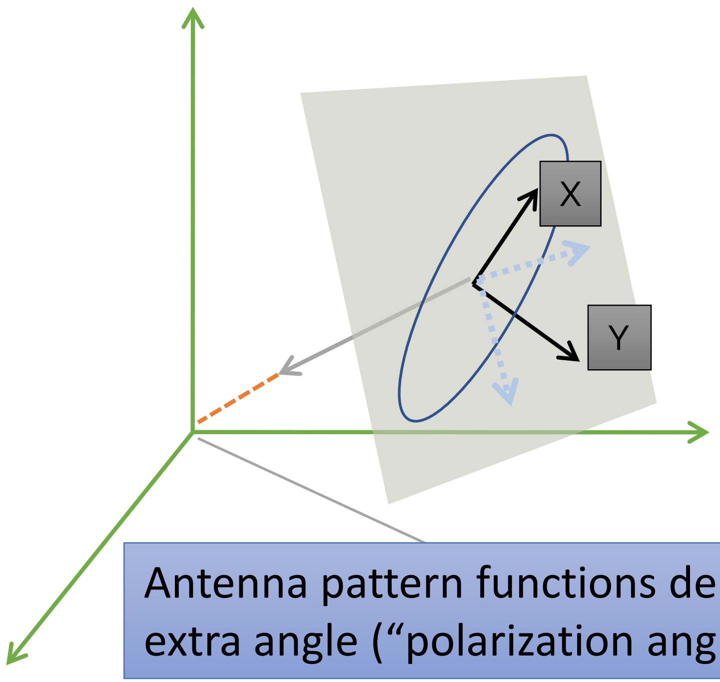
$$s(t) = \sum_{i,j=1}^{3} W_{ij} D_{ij} = W^{ij} D_{ij} = \overleftrightarrow{W} : \overleftrightarrow{D}$$

- To use the above formula, all unit vector components must be written down in the same reference frame

# Convention issues

| Burst signals | Inspiral signals |
| --- | --- |
| • Fix the wave frame XY axes by convention | • Fix the wave frame XY axes according to binary orbit projected on the sky |




Antenna pattern functions depend an extra angle (“polarization angle”): $\psi$

Geometry of wave and detector frames

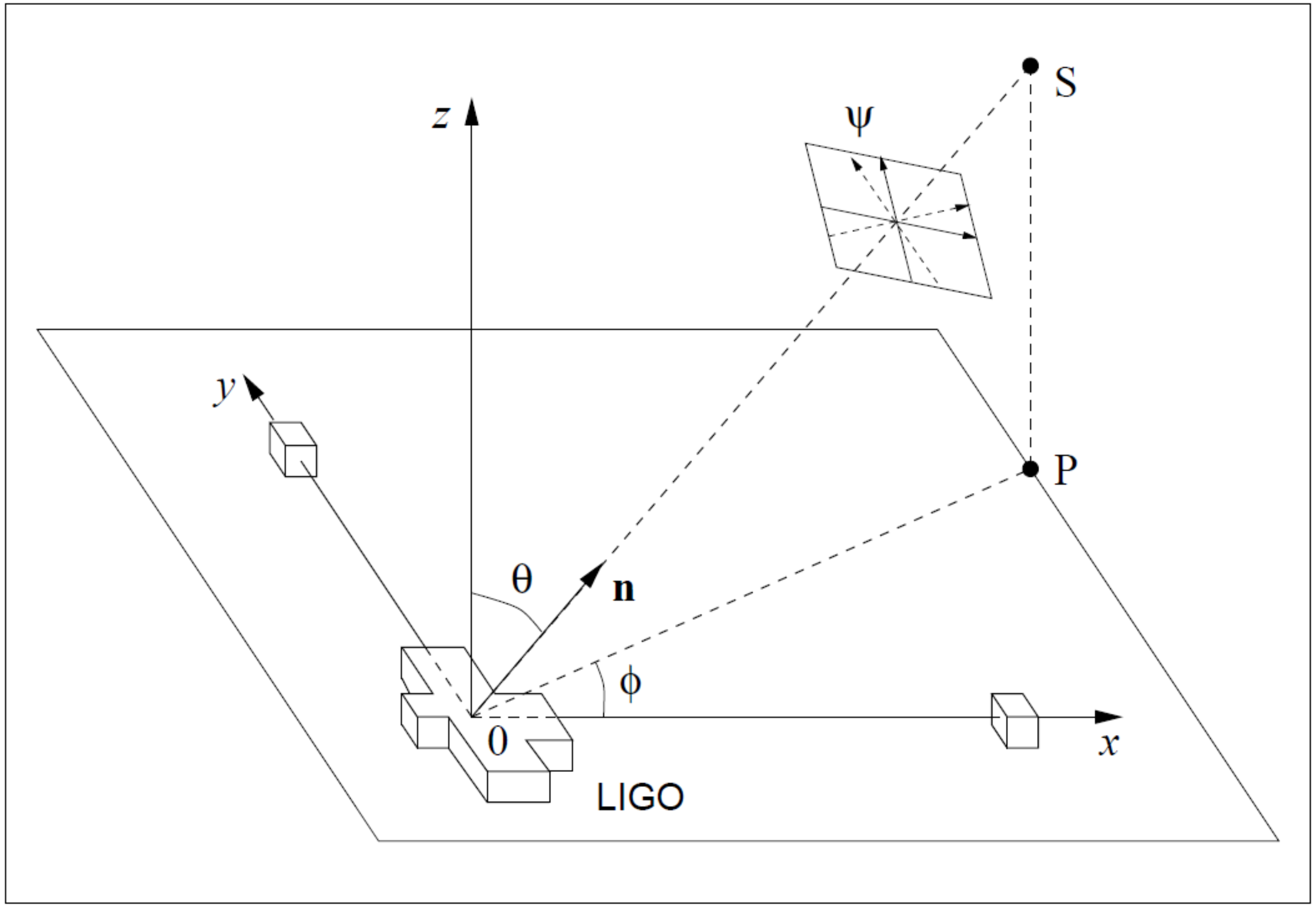



Image credit: Rakhmanov, LIGO-T060237

# Strain signal

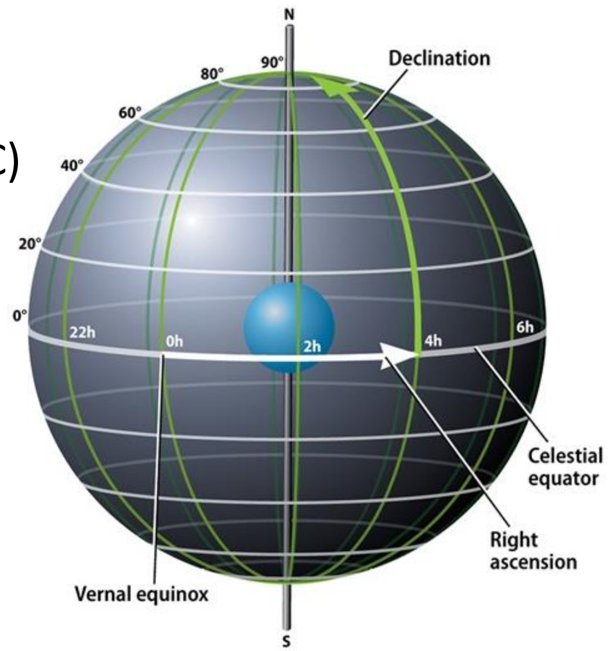
$$\overleftrightarrow{W} = h_+(t)\,\overleftrightarrow{e}_+ + h_\times(t)\overleftrightarrow{e}_\times$$

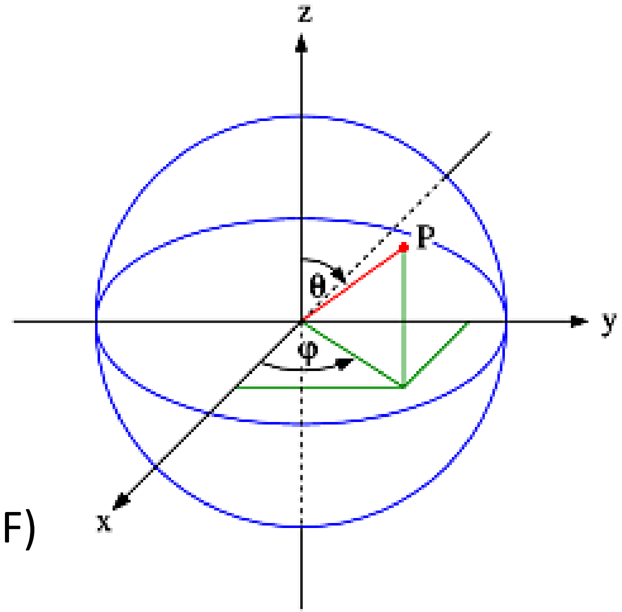
$$s(t) = \overleftrightarrow{W}:\overleftrightarrow{D} = h_+(t)\overleftrightarrow{D}:\overleftrightarrow{e}_+ + h_\times(t)\overleftrightarrow{D}:\overleftrightarrow{e}_\times$$

$$s(t) = h_+(t)F_+\left(\hat{k}\right) + h_\times(t)F_\times\left(\hat{k}\right)$$

- $F_{+,\times}$ are called the **antenna pattern functions** of the detector and depend on the direction from which the wave is coming
  - $\hat{k}$ is specified by the sky angles $\theta$ and $\phi$
  - $\Rightarrow F_{+,\times}\left(\hat{k}\right) = F_{+,\times}(\theta,\phi)$
- Note: $h_{+,\times}(t)$ are the polarization amplitudes measured at the frame **origin**

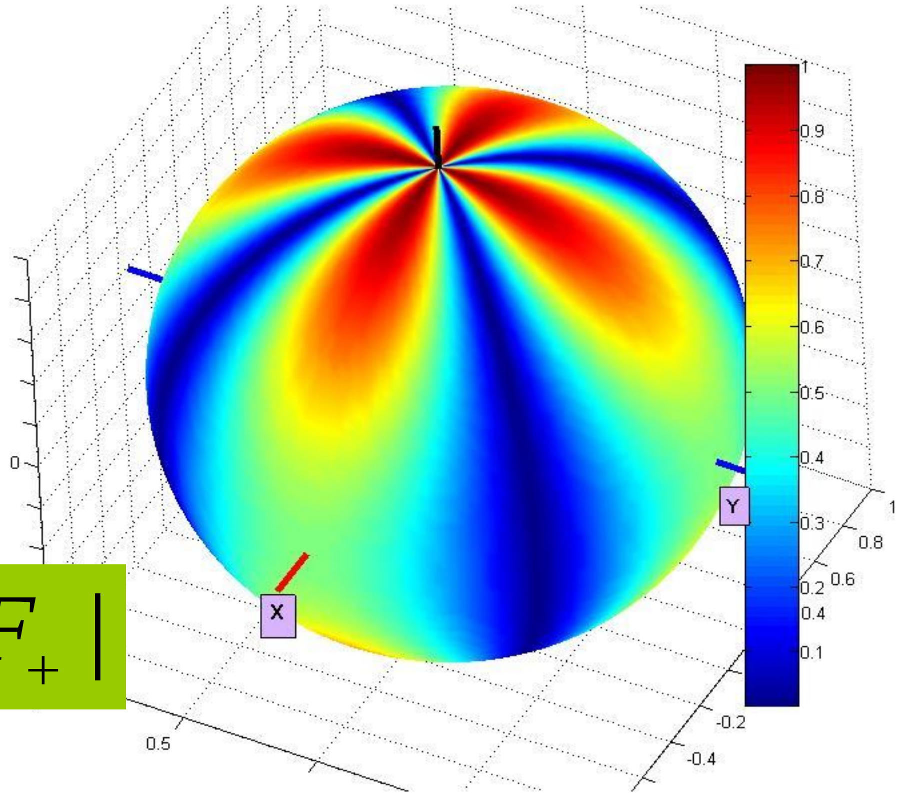
Right ascension (RA) and declination (DEC) convention



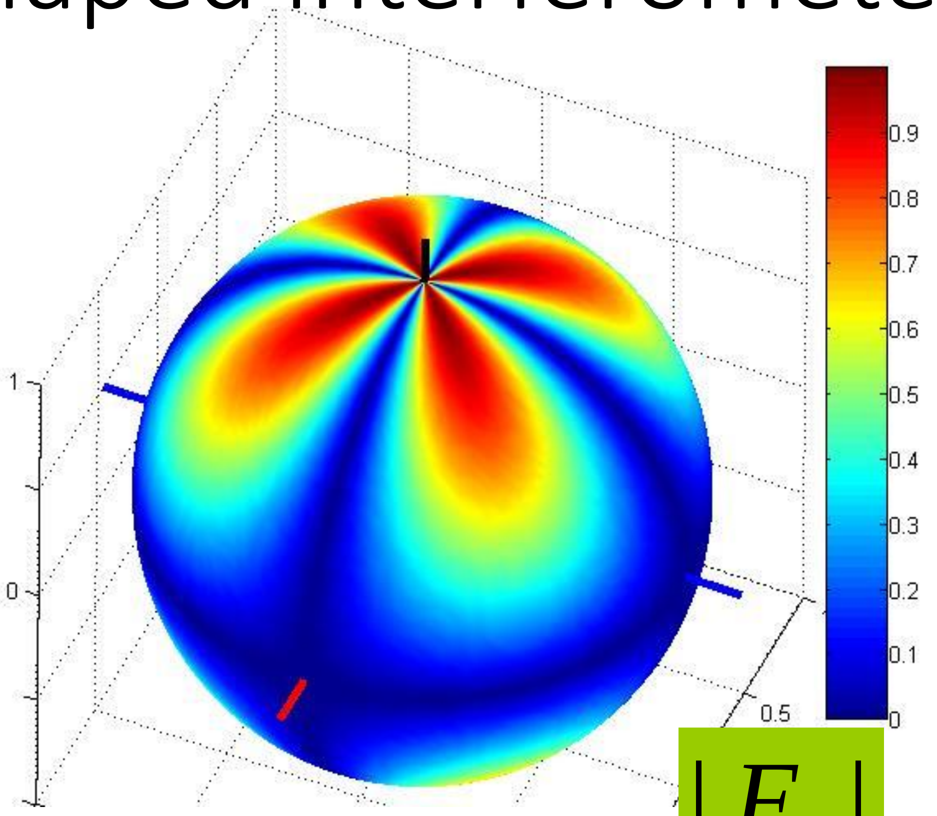




Earth-centered Earth-fixed (ECEF) frame

# Antenna patterns: L-shaped interferometer




$|F_+|$



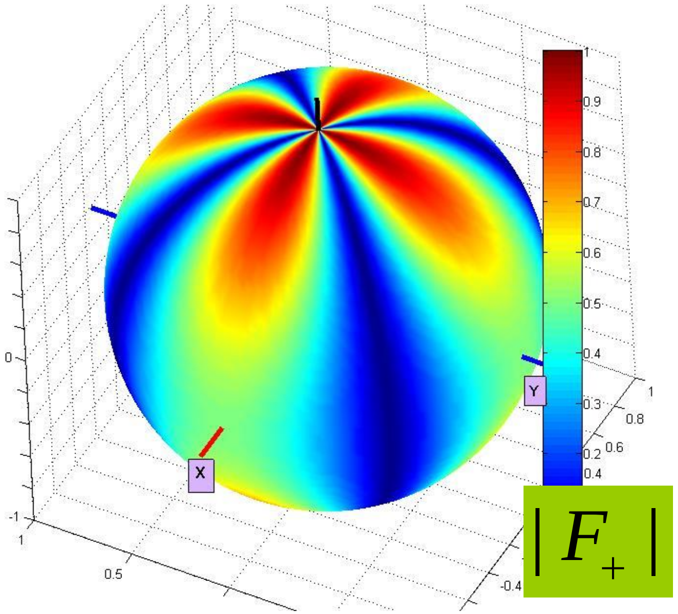

$|F_\times|$

Plotting on sphere in Matlab: github.com →LDACSchool→skyplot.m; testskyplot.m

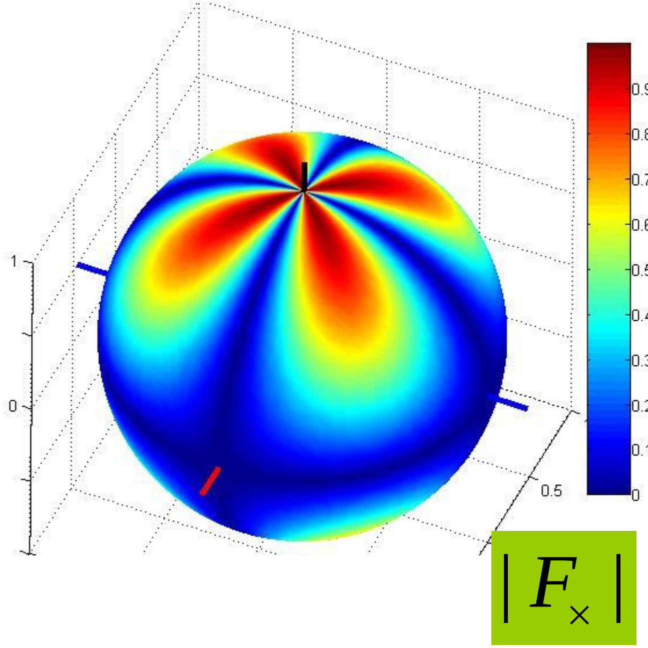
# Antenna response: L-shaped interferometer

$$s(t) = F_+(\underbrace{\theta, \phi}_{sky\ angle})h_+(t) + F_\times(\theta, \phi)h_\times(t)$$

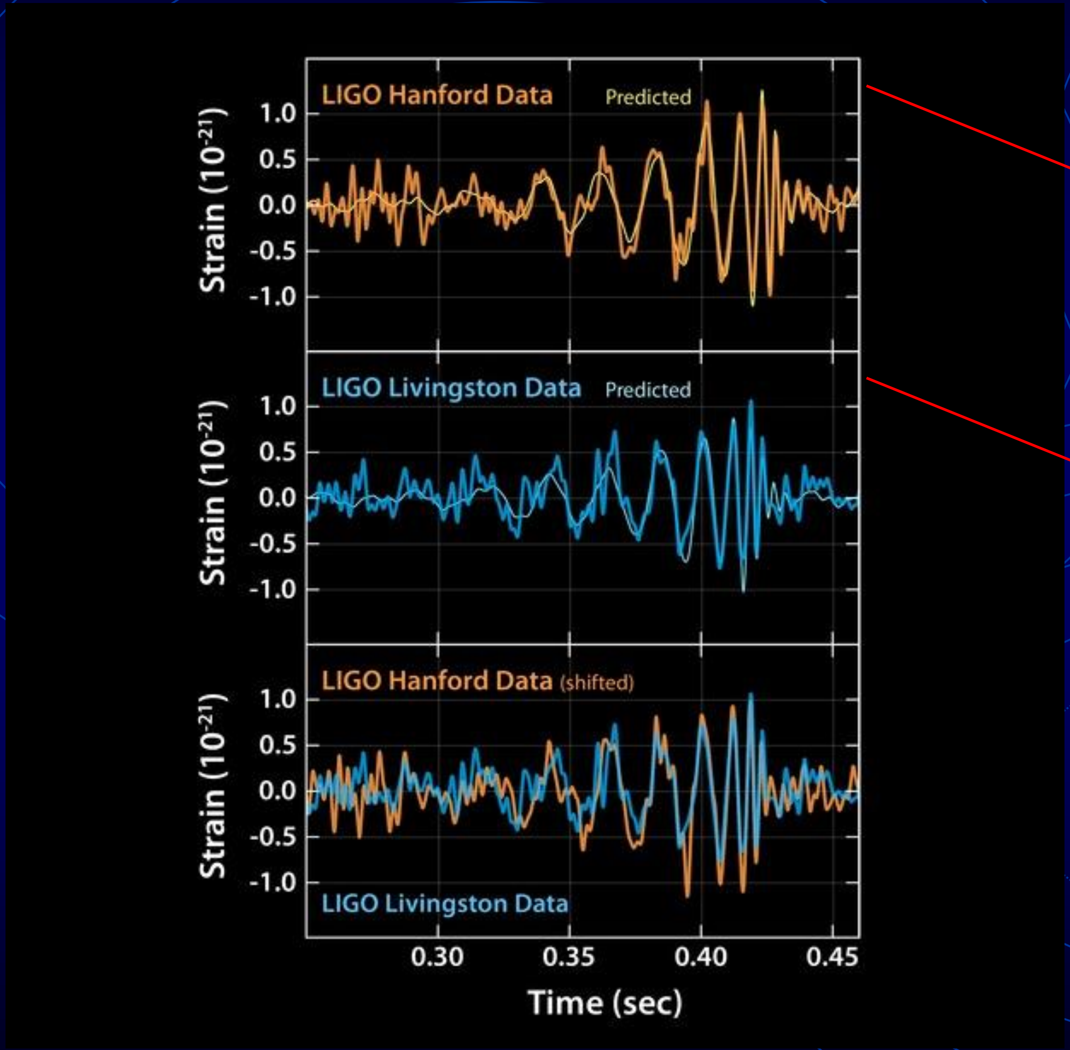
$$F_+(\theta, \phi) = \frac{1}{2}(1 + \cos^2\theta)\cos 2\phi$$




$$F_\times(\theta, \phi) = \cos\theta \sin 2\phi$$



# GW150914: Strain signal (Whitened)



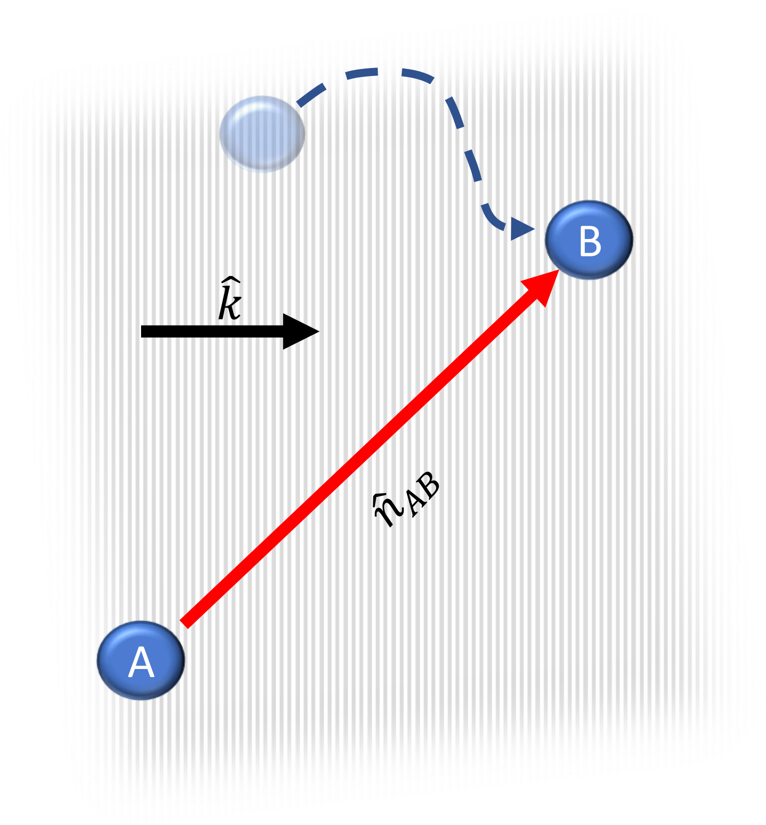

# Strain signal: General case

- $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$
- Light puls from sender (A) to point receiver (B)
  - Starting time $t_s$ and received time $t_r$
- Total distance traveled by the light pulse = add up the distance travelled over infinitesimal time intervals $\Rightarrow$

$$\int_A^B \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}} \approx |\bar{x}_B(t_r) - \bar{x}_A(t_s)| + \frac{1}{2}\hat{n}_{AB} \otimes \hat{n}_{AB}\colon \int_A^B \overleftrightarrow{W}_0\left(t - ct\hat{n}_{AB}.\hat{k}/c\right)dt$$

where $\hat{n}_{AB}$ is the unit vector pointing from A at $t_s$ (i.e., $\bar{x}_A(t_s)$ ) to the **future position** of B (i.e., $\bar{x}_B(t_r)$) and $\hat{k}$ points along the direction of the GW

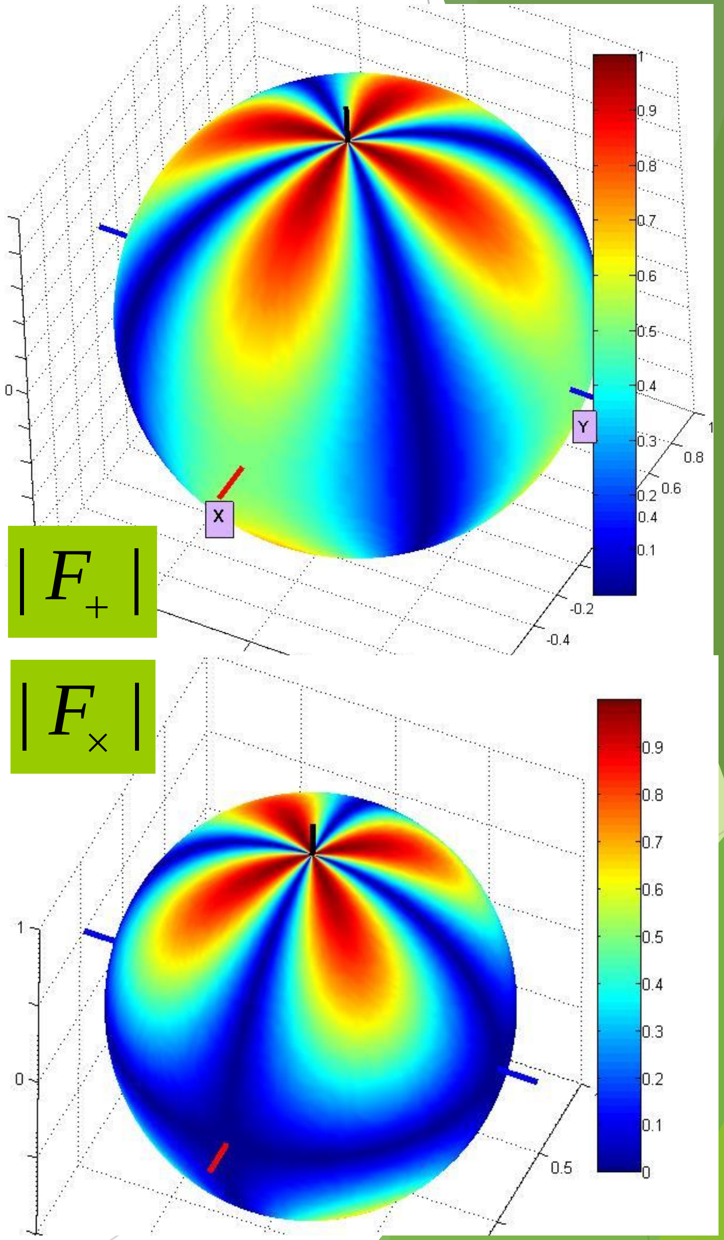
- See Cornish, Rubbo, arXiv:gr-qc/0209011v4



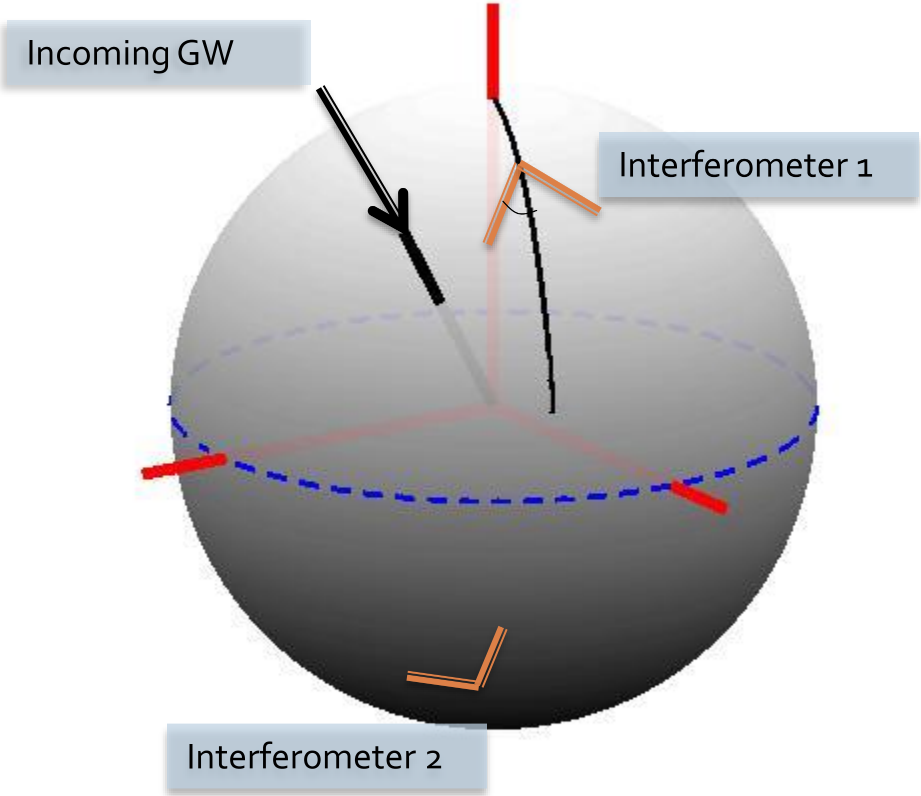
# Network of detectors

# Network data analysis

- Since interferometric detectors have **poor directionality**, a single detector cannot be used to determine the position of GW source
- A single detector cannot be used to resolve the two polarization components
- Analysis of data from a network is essential for determining direction and polarizations



# The physical setting




- Incoming GW is a plane wave

- It will hit the different detectors at different times

- time delay with respect to the arrival time at the center of the Earth is $-\bar{r}_i . \hat{n}/c$
  - $\bar{r}_i$ : position vector of the $i^{th}$ detector
  - $\hat{n}$ : unit vector pointing at the GW source

- Not only will different detectors see the signal at **different times**, they will also measure **different strain signals** because the source appears at different locations in the local detector frames

# Network data

- $N$ detectors
- Rotation and movement of detectors neglected

$$\begin{pmatrix} s_1(t) \\ \vdots \\ s_N(t) \end{pmatrix} = \begin{pmatrix} F_{+,1}(\theta,\phi)B(\tau_1(\theta,\phi)) & F_{\times,1}(\theta,\phi)B(\tau_1(\theta,\phi)) \\ \vdots & \vdots \\ F_{+,N}(\theta,\phi)B(\tau_N(\theta,\phi)) & F_{\times,N}(\theta,\phi)B(\tau_N(\theta,\phi)) \end{pmatrix} \begin{pmatrix} h_+(t) \\ h_\times(t) \end{pmatrix}$$

**Matrix notation:** $\boldsymbol{s}(t) = \mathbf{A}(\theta,\phi)\boldsymbol{h}(t)$

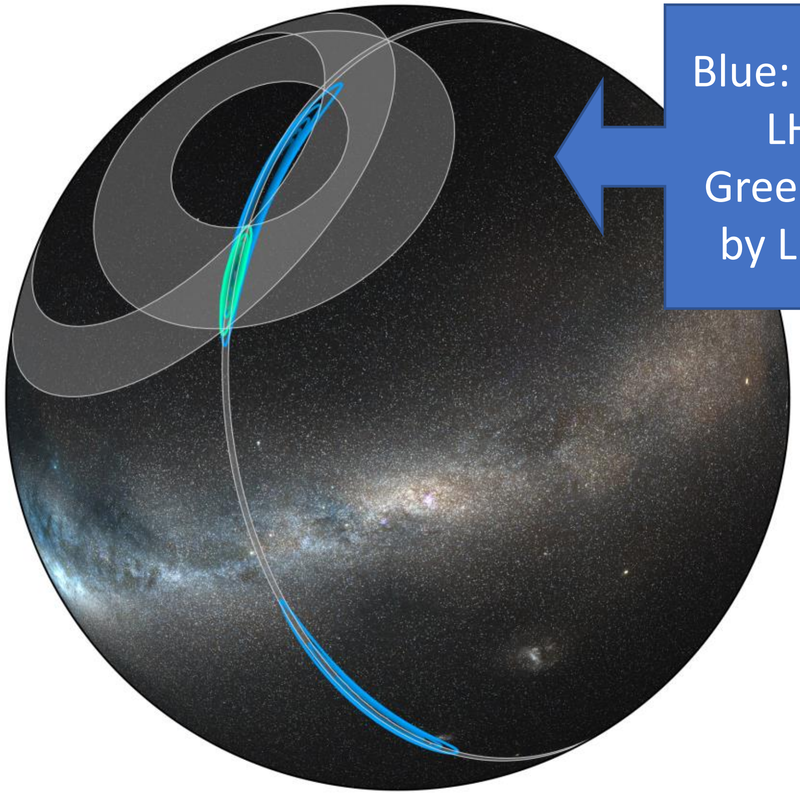
Strain signals

$(\theta, \varphi)$ : GW source position
$B(\tau)[g(t)] = g(t-\tau)$: Time shift operator

$\boldsymbol{h}(t) = \mathbf{M}\mathbf{A}^{\mathbf{T}}(\theta,\phi)\boldsymbol{s}(t)$
$\mathbf{M} = \left(\mathbf{A}^{\mathbf{T}}(\theta,\phi)\mathbf{A}(\theta,\phi)\right)^{-1}$

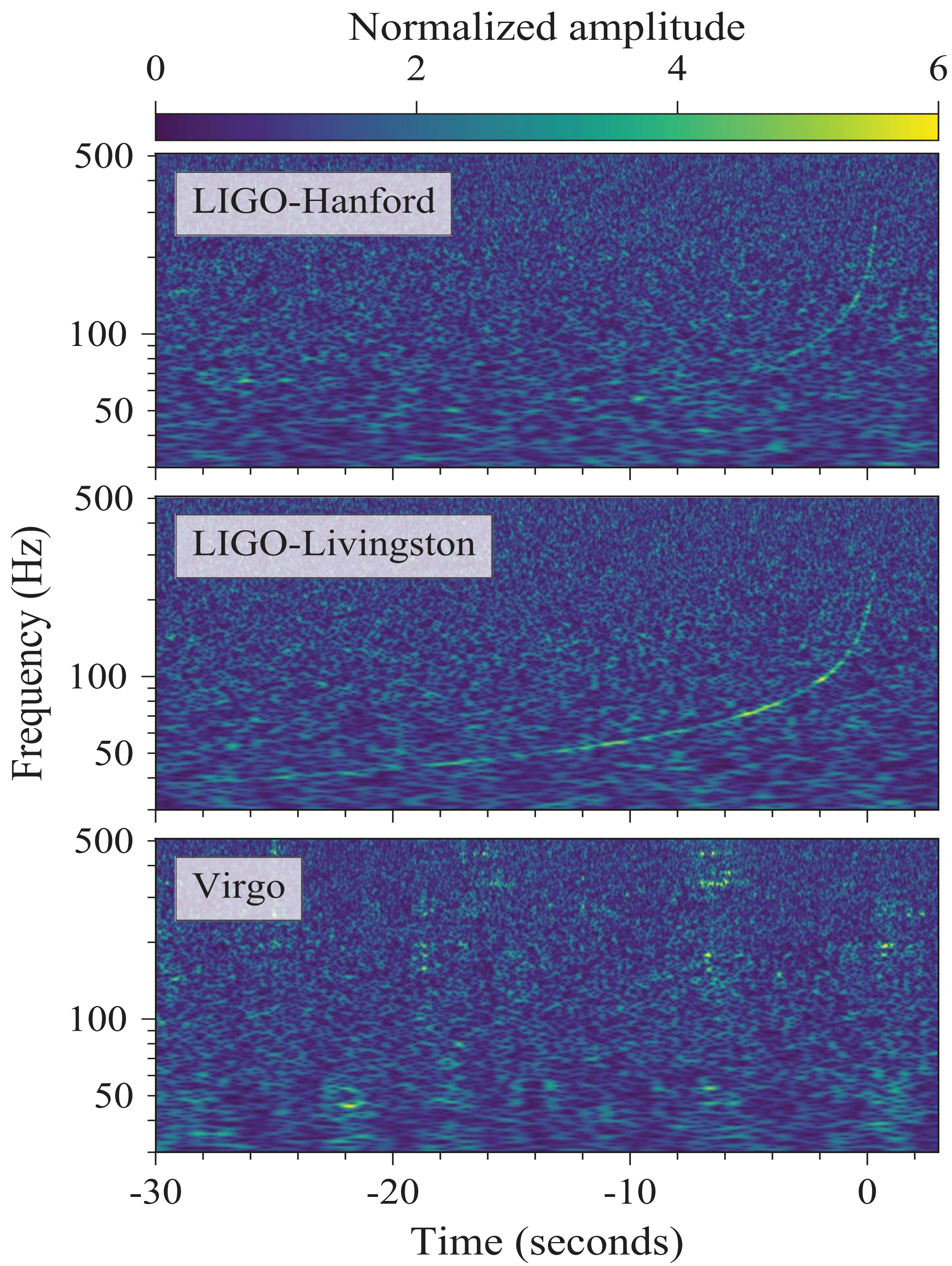
$\mathbf{A}(\theta,\varphi)$ can become rank-deficient
$\Rightarrow$ The inverse problem is **ill-posed**
$\Rightarrow$ Errors in $s(t)$ can get magnified

# GW170817: a 3-detector event



Blue: Localization by LHO and LLO
Green: Localization by LHO-LLO-Virgo

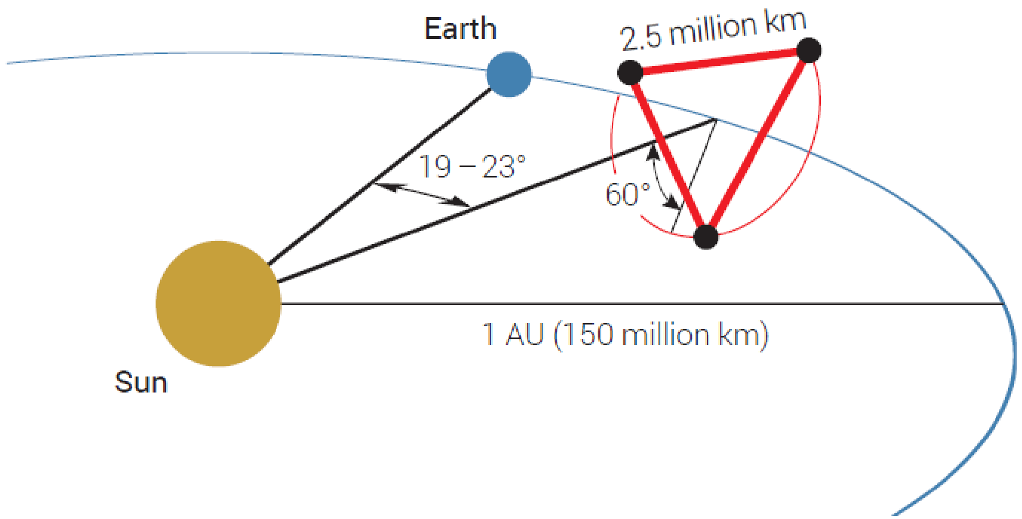
Credit: LIGO/Virgo/NASA/Leo Singer (Milky Way image: Axel Mellinger)

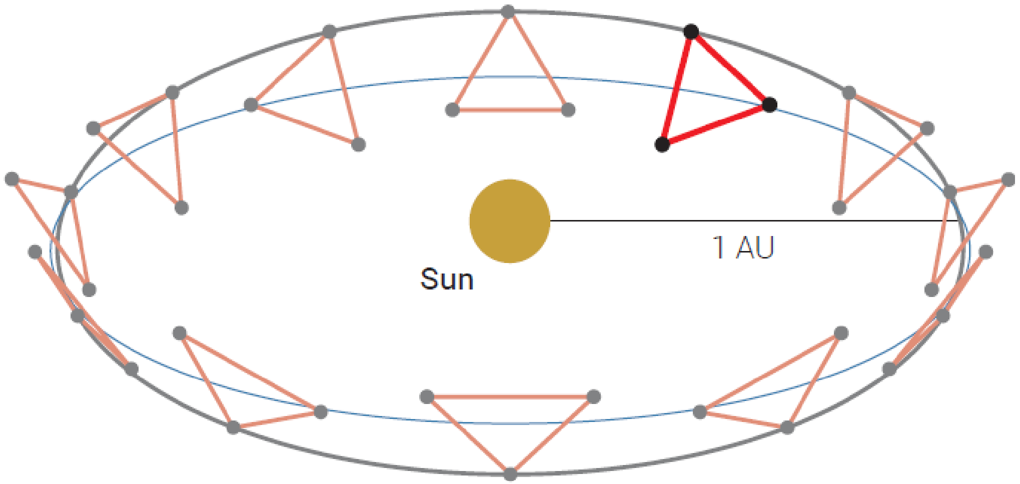


# Moving detector strain signal

# Moving and rotating detector

- LISA
  - Most signals in the frequency band of LISA will last for a year or indefinitely
- Ground-based detectors observing long-lived signals (Continuous wave sources)



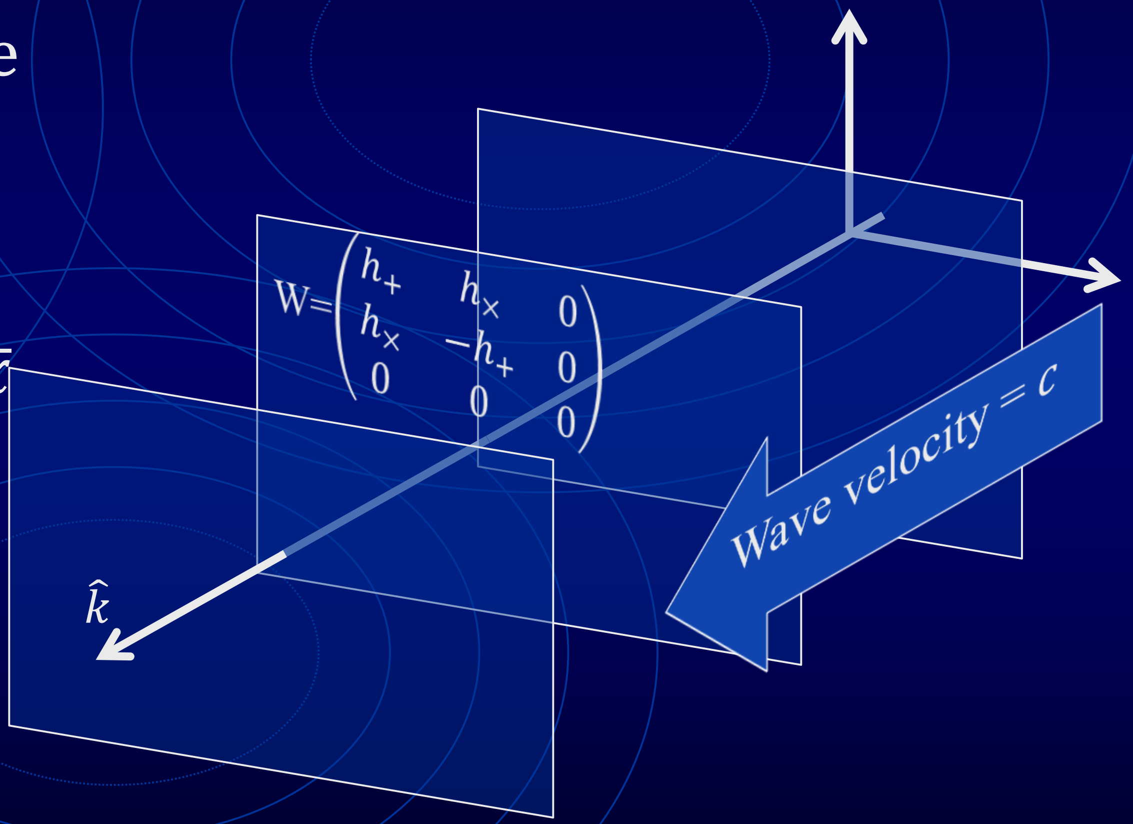
# Plane wave field

- Plane GW wave field:

$h^{(0)}_{+,\times}(t)$: Wave value at origin of TT Gauge

$$h_{+,\times}(t,\bar{x}) = h^{(0)}_{+,\times}\left(t - \hat{k}.\frac{\bar{x}}{c}\right)$$

- $h^{(0)}_{+,\times}(t)$ gets transported to a position $\bar{x}$ along $\hat{k}$ after a time interval $x/c$

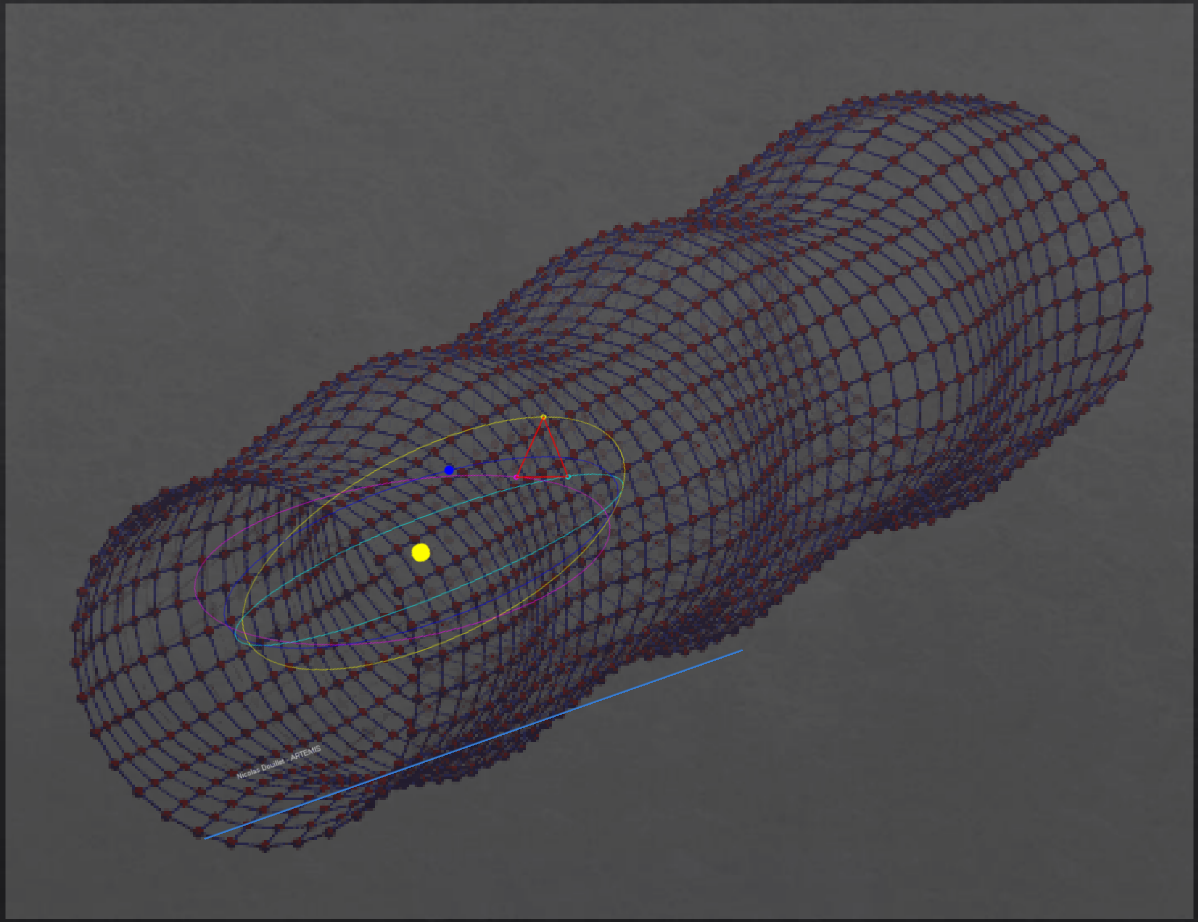


# Strain signal: moving detector

- The detector responds to the wave tensor at the detector location
- Detector located at position $\bar{x}_d(t)$ in the plane GW wave field
- Observed strain signal:

$$s(t) = F_+ h_+^{(0)}\left(t - \hat{k}.\frac{\bar{x}_d}{c}\right) + F_\times h_\times^{(0)}\left(t - \hat{k}.\frac{\bar{x}_d}{c}\right)$$

- (Assume antenna patterns are constant)



# Moving detector and monochromatic source

- Velocity of detector: $\bar{v}_d(t)$
- $\hat{n}$: Direction to GW source ($= -\hat{k}$; direction of propagation of wave)
- Source is periodic $\Rightarrow h^{(0)}_{+,\times}(t) \propto \sin(2\pi f t)$
- Strain:

$$s(t) \propto \sin\left(2\pi f\left(t + \hat{n}.\frac{\bar{x}_d(t)}{c}\right)\right)$$

- Instantaneous frequency (= time derivative of phase):

$$f(t) = f + \Delta f(= f\hat{n}.\frac{\bar{v}_d(t)}{c})$$

- $f \rightarrow f(t)$: Doppler shift

# Moving and rotating detector

- Antenna pattern functions change in time since the source moves in the detector frame
- Observed strain signal:

$$s(t) = F_+(t;\hat{k})h_+^{(0)}\left(t - \hat{k}.\frac{\bar{x}_d}{c}\right) + F_\times(t;\hat{k})h_\times^{(0)}\left(t - \hat{k}.\frac{\bar{x}_d}{c}\right)$$

- For a moving detector, the source direction is encoded in
  - The way the signal frequency gets doppler shifted
  - The way the amplitude of the signal is modulated
- $\Rightarrow$ Source direction can be obtained from even a single detector

# Detector response

# Strain signal to detector response

- Interferometric detectors have a non-flat **transfer function** to the strain signal
  - Low frequency cutoff in resonant cavities: Control systems push mirrors to counter low-frequency disturbances and keep cavities on resonance ("Locked state" of detector)
  - Intermediate frequencies: Response goes from flat to decreasing
  - High frequency: Zero response at GW frequencies that match the "free spectral range" frequency (or integer multiples)

# Interferometer transfer function

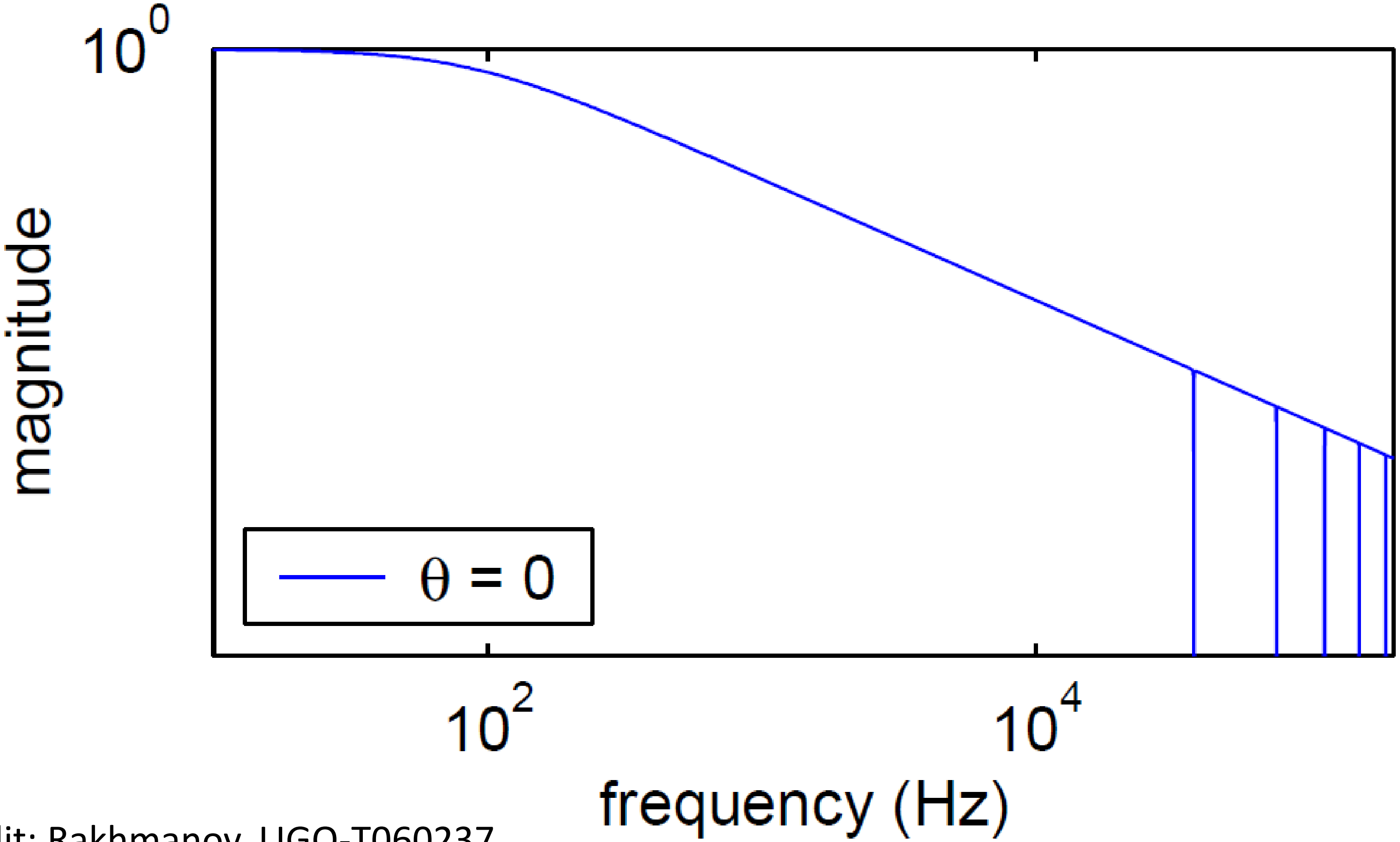
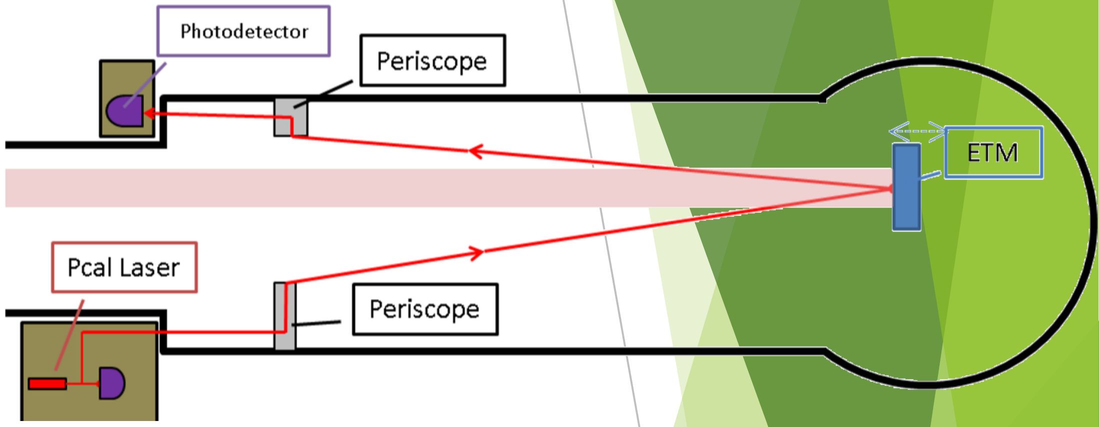



Image credit: Rakhmanov, LIGO-T060237

# Calibration

- Detector transfer function is measured by shaking the mirrors with a known displacement and measuring the detector response
  - Abbot et al, Phys. Rev. D 95, 062003 (2017)
  - The displacement signals are sinusoids
- **Calibrated data**: Data with measured transfer function removed
  - For the frequency range that is well calibrated, data analysts do not need to know the details of the detector response
- Strain signal is sufficient for analysis of calibrated data
- GW Open Science Center: Public domain calibrated data from all past science runs
- https://www.gw-openscience.org/about/



# Summary

Lab session: implementation of GW strain signal for static and moving detectors

Advanced topics not covered:

- Strain signal when the long wavelength approximation does not hold (relevant for high frequency LISA signals; relevant for all sources in PTA band)
- Non-rigid LISA
- Time-delay interferometry for LISA

Please read appendix A of textbook for refresher on probability theory