Solutions to Hartshone's Algebraic Geometry

Yang

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Contents

1 Exercises in Chapter 2

1.1 Exercises in 2.6

6.3. Cones. Before solving exercises, here are several lemmas that may be of use.

lemma 1: In $\mathbf{P}^n = \mathrm{Proj}(S = k[x_0, \cdots, x_n])$, for any closed subvariety V, let $P(V)$ be its projective closure in $\mathbf{P}^{n+1}$, denote the vertex of the cone as p. Then $U = P(V) \setminus \{p\}$ is a (geometric) vector bundle of rank 1 over V.[1] .

Proof. The projective cone over $V = \mathrm{Proj}(S/I)$ is $\mathrm{Proj}(S/I[t])$[2]. Here $(S/I)[t]$ is equipped with the natural $\mathbf{Z}$ grading and the vertex p of the cone corresponds to the homogeneous ideal $(x_0, \cdots, x_n)$ in $(S/I)[t]$ (not a maximal ideal but a closed point). First, we should construct an affine morphism from $U = P(V) \setminus \{p\}$ to V:

$$\begin{tikzcd} V_i \times_k \mathbb{A}^1_k \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{A}^1_k = \mathrm{Spec}(k[t]) \arrow[d] \\ V_i \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

Here we denote the standard cover on V as $\{V_i = D_+(x_i) = \mathrm{Spec}((S/I)_{(x_i)})\}_i$. Then $V \times_k \mathbb{A}^1_k$ is covered by $\{V_i \times_k \mathbb{A}^1_k \cong \mathrm{Spec}((S/I)_{(x_i)} \otimes_k k[t])\}_i$. Here $(S/I)_{(x_i)} = ((S/I)_{x_i})_{(0)} \cong (S_{x_i}/I_{x_i})_{(0)} \cong (S_{x_i})_{(0)}/(I_{x_i})_{(0)} \cong S_{(x_i)}/I_{(x_i)}$, thus each cover can be written as $\mathrm{Spec}(S_{(x_i)}/I_{(x_i)} \otimes_k k[t]) \cong \mathrm{Spec}((S_{(x_i)}/I_{(x_i)})[t])$[3].

$U = P(V) \setminus \{p\}$ can be covered by $\{D_+(x_i)\}_i$ of $\mathrm{Proj}((S/I)[t])$, isomorphic to $\{\mathrm{Spec}(((S/I)[t])_{(x_i)})\}_i$. Indeed, consider the projective space $\mathbf{P}^{n+1} = \mathrm{Proj}(k[t_0, \cdots, t_{n+1}])$, the only homogeneous ideal not contained in the cover $\{D_+(t_i)\}_{i \neq n+1}$ is $\{(t_0, \cdots, t_n)\} = D_+(t_{n+1}) \setminus (\cup_{i \neq n+1} D_+(t_i))$. Thus $\{D_+(x_i)\}_i$ of $\mathrm{Proj}((S/I)[t])$ is indeed a covering.

To construct a morphism from U to V, it suffices to prove that there exists $\{\pi_i\}$ such that $\pi_i : D_+(x_i) \cong V_i \times_k \mathbb{A}^1_k \to V_i \to V$ satisfies: $\pi_i|_{D_+(x_i) \cap D_+(x_j)} = \pi_j|_{D_+(x_j) \cap D_+(x_j)}$, so they can patch together a global morphism $\pi : U \to V$.

Claim 1: There is an isomorphism $\psi_i : D_+(x_i) \to V_i \times_k \mathbb{A}^1_k$ in $V \times_k \mathbb{A}^1_k$.

[1]Here 'vector bundle of rank 1' means that there exists an affine projection $\pi : P(V) \to V$ and an affine open cover $\{U_i = \mathrm{Spec}(A_i)\}$ of V, such that $c_i : \pi^{-1}(U_i) \cong U_i \times_k A^1_k$ and for any affine subset $\mathrm{Spec}(T) \subset U_i \cap U_j$, $c_i \circ c_j^{-1}|_T$ is an automorphism of $\mathrm{Spec}(T[t])$ induced by a T-algebra automorphism ψ of $T[t]$ such that $\psi(t) = at$ where $a \in T$.
It's intuitively obvious that a cone without vertex can be seen as a bundle over the variety, and when the base space is not affine, or it is patched in a non-affine way: like twistingly patched together, where the projective variety serves as an example, then this bundle can never be trivial bundle.

[2]see p.262 9.3, Foundations of Algebraic Geometry, Ravi Vakil, July 31,2023 draft.

[3]Unwinding this is actually $\mathrm{Spec}(k[\bar{x}_1, \cdots, \bar{x}_n]/\bar{I}) \otimes_k k[t] \cong \mathrm{Spec}(k[\bar{x}_1, \cdots, \bar{x}_n, t]/(\bar{I}))$.

Proof. It suffices to prove that there is a k-algebra isomorphism $(S/I)_{(x_i)}[t] \longrightarrow ((S/I)[t])_{(x_i)}$.

This is obvious because $(S/I)_{(x_i)}[t] = (S_{(x_i)}/I_{(x_i)})[t]$[4] $\overset{t\to\frac{t}{x_i}}{\to} (S_{(x_i)}/I_{(x_i)})[t/x_i] = ((S/I)[t])_{(x_i)}$[5].

Indeed this is an isomorphism, because we can view everything in the large fractional field $\mathrm{Frac}((S/I)[t])$ (without any graded structure), then above isomorphism $t \to \frac{t}{x_i}$ is just the 'twist' of gradings: t is of degree zero in the first ring, and is of degree 1 in the last ring.

$\square$

Claim 2 : The isomorphisms in Claim 1 are not compatible on the intersection $D_+(x_i) \cap D_+(x_j) \cong (V_i \cap V_j) \times_k A^1_k$. So the projection cone of projective closed variety V minus the vertex can never be the trivial bundle over V.[6]

$$\begin{tikzcd}
D_+(x_i) = \mathrm{Spec}((S/I)_{(x_i)}[t]) \arrow[r, "\psi_i \text{ induced by } (t\to\frac{t}{x_i})"] & V_i \times_k \mathbb{A}^k_1 \\
D_+(x_i x_j) = \mathrm{Spec}((S/I)_{(x_i x_j)}[t]) \arrow[u, "i"] \arrow[d, "\text{induced by } (\frac{x_j}{x_i})"'] \arrow[r, "\text{restriction of } \psi_i \text{ induced by } (t\to\frac{t}{x_i})"] & (V_i \cap V_j) \times_k \mathbb{A}^k_1 \arrow[u, "i"'] \arrow[d, Rightarrow, "="] \\
D_+(x_i x_j) = \mathrm{Spec}((S/I)_{(x_i x_j)}[t]) \arrow[d, "j"'] \arrow[r, "\text{restriction of } \psi_j \text{ induced by } (t\to\frac{t}{x_j})"] & (V_i \cap V_j) \times_k \mathbb{A}^k_1 \arrow[d, "j"] \\
D_+(x_j x_i) = \mathrm{Spec}((S/I)_{(x_i x_j)}[t]) \arrow[r, "\psi_j \text{ induced by } (t\to\frac{t}{x_j})"] & V_j \times_k \mathbb{A}^k_1
\end{tikzcd}$$

By this diagram, we can see clearly that on the intersection, restriction of ψ_i and ψ_j differs from an automorphism induced by ring isomorphism $S_{(x_i x_j)} \xrightarrow{\cdot\frac{x_j}{x_i}} S_{(x_i x_j)}$.

Claim 3 :There is a natural projection $\pi : U \to V$ with restriction on $D_+(x_i)$ being exactly the composition $D_+(x_i) \xrightarrow{\psi_i} V_i \times_k \mathbb{A}^1_k \xrightarrow{p_1} V_i$.

It suffices to prove that on the intersection $U_i \cap U_j$: $(p_1 \circ \psi_i)|_{U_i \cap U_j} = (p_1 \circ \psi_j)|_{U_i \cap U_j}$. This is obvious by the above diagram because the only nontrivial 'twisted' gluing happens on variable t, while the projection corresponds to the natural inclusion $k[x_0, \cdots, x_n]_{(x_i x_j)} \to k[x_0, \cdots, x_n]_{(x_i x_j)}[t]$, where the change in t will not influence $(x_0 \cdots x_n)$.

Now we shall prove that the 'gluing' of the above projections $\pi : U \to V$ is a vector bundle of rank 1. It suffices to see how we construct a geometric vector bundle $\mathbb{V}(\mathcal{L})$ of rank one using line

[4] Here '='happens in $(\mathrm{Frac}(S/I))[t]$. This is true because here assume V is variety, so it is integral, and (S/I) is graded domain.

[5] This '=' happens in $\mathrm{Frac}((S/I)[t])$. When V is not integral, this still an isomorphism.

[6] As you can see, here the nontrivial twist of gluing happens due to the 'twist' of the projective space. So here we require V to be projective.

bundle $\mathcal{L}$, here we take $\mathcal{L} = \mathcal{O}(1)$, and it turns out that $\mathbb{V}(\mathcal{O}(1))$ is gluing in the same way as U, as shown in the following diagram:

$$\begin{tikzcd}
& \mathrm{Hom}_{V_i}(-, \mathbb{A}^1_{V_i}) \arrow[rr, Leftrightarrow, "="] & & \Gamma(-,(-)^*\mathcal{O}_{V_i}) \\
\mathrm{Hom}_{V_i}(-, \mathbb{V}(\mathcal{O}_{V_i}(1))) \arrow[ur, "\psi_i"] \arrow[rr, Rightarrow, "="'] & & \Gamma(-,(-)^*\mathcal{O}_{V_i}(-1)) \arrow[ur, "\cdot x_i"'] & \\
& \mathrm{Hom}_{V_i\cap V_j}(-, \mathbb{A}^1_{V_i\cap V_j}) \arrow[uu, "\psi_i"'] \arrow[dd, "\psi_j"'] \arrow[rr, Leftrightarrow, "="] & & \Gamma(-,(-)^*\mathcal{O}_{V_i\cap V_j}) \arrow[uu, "\hat{i}=inclusion"'] \arrow[dd, "\hat{j}=\cdot\frac{x_j}{x_i}"'] \\
\mathrm{Hom}_{V_i\cap V_j}(-, \mathbb{V}(\mathcal{O}_{V_i\cap V_j}(1))) \arrow[uu, "i"] \arrow[dd, "j"] \arrow[ur, "\psi_{ij}"] \arrow[rr, Rightarrow, "="'] & & \Gamma(-,(-)^*\mathcal{O}_{V_i\cap V_j}(-1)) \arrow[uu, "\mathbf{i}"] \arrow[dd, "\mathbf{j}"] \arrow[ur, "\cdot x_i"] & \\
& \mathrm{Hom}_{V_j}(-, \mathbb{A}^1_{V_j}) \arrow[rr, Leftrightarrow, "="] & & \Gamma(-,(-)^*\mathcal{O}_{V_j}) \\
\mathrm{Hom}_{V_j}(-, \mathbb{V}(\mathcal{O}_{V_j}(1))) \arrow[ur, "\psi_j"'] \arrow[rr, Rightarrow, "="'] & & \Gamma(-,(-)^*\mathcal{O}_{V_j}(-1)) \arrow[ur, "\cdot x_j"'] &
\end{tikzcd}$$

Remark 1: In the above diagram, $\mathrm{Hom}_{V_i\cap V_j}(-, \mathbb{V}(\mathcal{O}_{V_i\cap V_j}(1))) \hookrightarrow \mathrm{Hom}_{V_i}(-, \mathbb{V}(\mathcal{O}_{V_i}(1)))$ are all open sub-functors of $\mathrm{Hom}_{Sch/V}(-, \mathbb{V}(\mathcal{O}_V(1)))$, by definition (see [4, p.289, section(11.3)]) this is the quasi-coherent bundle defined by twisting sheaf $\mathcal{O}(1)$ on V. it corresponds exactly to the representable functor: $(-) \in \mathrm{Sch/V} \to \Gamma(-,(-)^*(\mathcal{O}(1))^\vee)$(see [4, p.290, proposition 11.3])$=\Gamma(-,(-)^*(\mathcal{O}(-1))) \in \mathrm{Set}$.

Thus here we can regard these open sub-functors as open sub-schemes of $\mathbb{V}(\mathcal{O}_V(1))$, in fact, it's trivial to see that: by base change, $\{\mathbb{V}(\mathcal{O}_V|_{V_i}) = \mathbb{V}(\mathcal{O}_V(1))|_{V_i}\}$ is an open cover of $\mathbb{V}(\mathcal{O}_V(1))$. What we want to do here is to find out how these open covers are glued together, then we shall see that it is exactly how we glue the projection cone minus vertex. Thus $P(V) \setminus \{p\}$ is exactly the quasi-coherent bundle, and more precisely, as $\mathcal{O}(1)$ is locally free of rank one, it is the vector bundle of rank one over V.

Remark 2：Now I shall explain the above diagram:
(1). The double arrows are by definition of these functors.
(2). The righthand side of the diagram is kind of 'trivialization' of the twisting sheaf.

$$\begin{tikzcd}
\Gamma(-,(-)^*\mathcal{O}_{V_i}(-1)) \arrow[r, "\cdot x_i"] & \Gamma(-,(-)^*\mathcal{O}_{V_i}) \\
\Gamma(-,(-)^*\mathcal{O}_{V_i\cap V_j}(-1)) \arrow[u, "\mathbf{i}"] \arrow[d, "\mathbf{j}"'] \arrow[r, "\cdot x_i"] & \Gamma(-,(-)^*\mathcal{O}_{V_i\cap V_j}) \arrow[u, "\hat{i}=inclusion"'] \arrow[d, "\hat{j}=\cdot\frac{x_j}{x_i}"'] \\
\Gamma(-,(-)^*\mathcal{O}_{V_j}(-1)) \arrow[r, "\cdot x_j"] & \Gamma(-,(-)^*\mathcal{O}_{V_j})
\end{tikzcd}$$

The inclusion $\mathbf{i}$ is constructed as follows: For any $T \in Sch/(V_i \cap V_j) : T \xrightarrow{\psi} V_i \cap V_j \xrightarrow{i} V_i$, we can also regard T as a V_i-scheme, and $i \circ \psi = \psi$, $\psi^* = \psi^* i^*$. So $\psi^* \mathcal{O}_{V_i \cap V_j}(-1) = \psi^* i^* \mathcal{O}_{V_i}(-1) = \psi^* \mathcal{O}_{V_i}(-1)$. So any section $s \in \Gamma(T, \psi^* \mathcal{O}_{V_i \cap V_j}(-1))$ can be naturally regarded as an element in $\Gamma(T, \psi^* \mathcal{O}_{V_i})$. And $\mathbf{j}$ is defined in the same way.

The right row is the 'trivialization' of the twisting sheaf: $\mathcal{O}_{V_i}(-1) \overset{\cdot x_i}{\cong} \mathcal{O}_{V_i}$, and as a sub-sheaf of $\mathcal{O}_{V_i}$, this also induces an isomorphism on $V_i \cap V_j$: $\mathcal{O}_{V_i \cap V_j}(-1) \overset{\cdot x_i}{\cong} \mathcal{O}_{V_i \cap V_j}$. While for $\mathcal{O}_{V_j}(-1)$, there is also an 'trivialization' $\mathcal{O}_{V_j}(-1) \overset{\cdot x_j}{\cong} \mathcal{O}_{V_j}$, to make the diagram commutes, $\hat{j}$ is defined as $\cdot \frac{x_j}{x_i}$.

Indeed, the 'trivialization' here for $V_i \cap V_j$ is not canonical, however we only consider something like $j \circ i^{-1} = \frac{x_j}{x_i}$, so it doesn't matter. Also, this uncanonical choice of 'trivialization' will not influence the naturality of the whole diagrams, so the morphisms we construct for these functors can always be transferred to the schemes by Yoneda embedding as we shall see later.
(3). For the structure $\mathcal{O}_X$, the corresponding functor $\Gamma(-, (-)^* \mathcal{O}_X)$ is represented by the fibre product $\mathbb{A}^1_X$(see [4, example 11.4]). Thus induced by 'trivialization': $\cdot x_i$ and $\cdot x_j$, there are also 'trivializations' of sub-open functors ψ_i,ψ_{ij},ψ_j, of course you can regard them as morphisms in the category of schemes.

In the same way, induced by $\hat{i}$ and $\hat{j}$, we have the inclusion ϕ_i and ϕ_j. And after adding these induced morphisms, the diagram still commutes.
(4). Now let's study the left face of the cubic diagram: It is the trivialization of $\mathbb{V}(\mathcal{O}_{V_i}(1)) \cup \mathbb{V}(\mathcal{O}_{V_j}(1))$ by gluing two pieces : $\mathbb{A}^1_{V_i}$ and $\mathbb{A}^1_{V_j}$ through the isomorphism: $im(\phi_i) \xrightarrow{\phi_i^{-1}} \mathbb{A}_{V_i \cap V_j} \xrightarrow{\phi_j} im(\phi_j)$.

$$\begin{tikzcd}
\mathbb{A}^1_{V_i} \arrow[r, "="] \arrow[dd, bend right=60, "?:\gamma"'] & \mathrm{Hom}_{V_i}(-, \mathbb{A}^1_{V_i}) \arrow[r, "="] & \Gamma(-, (-)^* \mathcal{O}_{V_i}) \arrow[dd, bend left=60, "\frac{x_j}{x_i}"] \\
\mathbb{A}^1_{V_i \cap V_j} \arrow[r, "="] \arrow[u, "inclusion"] \arrow[d, "\hat{j}"] & \mathrm{Hom}_{V_i \cap V_j}(-, \mathbb{A}^1_{V_i \cap V_j}) \arrow[r, "="] \arrow[u, "\phi_i"] \arrow[d, "\phi_j"'] & \Gamma(-, (-)^* \mathcal{O}_{V_i \cap V_j}) \arrow[u, "\hat{i}"'] \arrow[d, "\hat{j}"'] \\
\mathbb{A}^1_{V_j} \arrow[r, "="] & \mathrm{Hom}_{V_j}(-, \mathbb{A}^1_{V_j}) \arrow[r, "="] & \Gamma(-, (-)^* \mathcal{O}_{V_j})
\end{tikzcd}$$

Unwinding all the isomorphisms, we can see that $\gamma : \mathbb{A}^1_{V_i \cap V_j} \to \mathbb{A}^1_{V_i \cap V_j}$ is induced by $k[x_0, \cdots, x_n]_{(x_i, x_j)} \xrightarrow{\frac{x_j}{x_i}} k[x_0, \cdots, x_n]_{(x_i, x_j)}$.

Thus we can regard $\mathbb{V}(\mathcal{O}_V(1))$ as the gluing of $\{\mathbb{A}^1_{V_i}\}$ by the isomorphisms described above. While $\mathbb{A}^1_{V_i} = V_i \times_k \mathbb{A}^1_k$ and the gluing way is exactly the same as $U = P(V) \setminus \{p\}$ as shown in the Claim 2. Thus $U = P(V) \setminus \{p\} \cong \mathbb{V}(\mathcal{O}_V(1))$.

□

Remark 3: This conclusion makes much sense: when $k = \mathbb{C}$, then projective cone over $\mathbb{C}\mathbf{P}^1$ is $\mathbb{C}\mathbf{P}^2$. While $\mathbb{C}\mathbf{P}^2 \setminus \{p\}$ is exactly the normal bundle of $\mathbb{C}\mathbf{P}^1$ inside $\mathbb{C}\mathbf{P}^2 \setminus \{p\}$.

In this exercise we compare the class group of a projective variety V with the class group of its cone. So let V be a projective variety in $\mathbf{P}^n$, which is of dimension ≥ 1 and onosingular in codimension one. Let $X = C(V)$ be the affine cone over V in $\mathbf{A}^{n+1}$, and the projective cone $P(V)$ is the projective closure of $C(V)$ in $\mathbf{P}^n$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi : P(V) - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbf{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \mathrm{Cl}(V) \to \mathrm{Cl}(P(V) - P)$ is an isomophism. Since $\mathrm{Cl}(P(V)) \cong \mathrm{Cl}(P(V) - P)$, thus $\mathrm{Cl}(V) \cong \mathrm{Cl}(P(V)$.

Proof. By the lemma 1 above, we can see that π is vector bundle and there exists an affine open cover $\{U_i\}$ such that the projection restricted to each cover is the projection of $U_i \times_k A^1_k$. So locally $\mathrm{Cl}(U_i) \cong \mathrm{Cl}(\pi^{-1}(U_i))$, usually there is no way of patching the class group from local information, here we still need to apply the method used in (6.6).

First of all, the projection $\pi : P(V) - P \to V$ is an open morphism (as it is surjective, it's also closed) with irreducible fibre of dimension one because locally it is the proejction of the fibre product with the affine line, thus the inverse image of irreducible closed subset is again closed and irreducible (see lemma 4, 8.2). So the image $\pi(x)$ of codimension one point is still of codimension one or zero (generic point), otherwise, consider the chain of irreducible closed subsets $\overline{\pi(x)} \subset V_1 \subset V_2$ and consider the inverse image under π, which is a contradiction to the fact that x is of codimension one.

Thus we can again apply the 'type I' and 'type II' logic used in [7, proposition 6.6, 2.6]. In the same way, we can show that $\pi^* : \mathrm{Cl}(V) \to \mathrm{Cl}(P(V) - P)$ is isomorphism and $\mathrm{Cl}(P(V) - P) \cong \mathrm{Cl}(P(V))$ by [7, proposition 6.5(b)].

□

(b) We have $V \subset P(V)$ as the hyperplane section at infinity. Show that the class of the divisor V in $\mathrm{Cl}(P(V))$ is equal to the π^* (class of intersection $V \cdot H$) where H is any hyperplane of $\mathbf{P}^n$ not containing V. Thus there is an exact sequence

$$0 \to \mathbf{Z} \to \mathrm{Cl}(V) \to \mathrm{Cl}(C(V)) \to 0$$

where the first arrow sends 1 to $V \cdot H$ and the second arrow is π^* followed by restriction to $C(V) - P$ and inclusion in $C(V)$.

Proof. Here we denote $S = k[x_0, \cdots, x_n]$, $V = \mathrm{Proj}(S/I)$ and the hyperplane H is given by the homogeneous polynomial $f \in k[x_0, \cdots, x_n]$ with $\deg(f) = 1$, then $C(V) = \mathrm{Spec}(S/I)$ and $P(V) = \mathrm{Proj}((S/I)[t])$, thus the function field of the projective cone, which is the same as the function field of its minus the closed vertex point, is in the form of fraction of homogeneous polynomials f and g: $\frac{f(x_0,\cdots,x_n,t)}{g(x_0,\cdots,x_n,t)}$ with $\deg(f) = \deg(g)$.

We claim that $\pi^*(V \cdot H) - V = (\frac{f}{t})$, and thus they are the same class. This is obvious, divisor

(t) can only be the hyperplane at infinity by definition, and (f) is exactly the divisor $\pi^*(V \cdot H)$. Then the exact sequence $0 \to \mathbf{Z} \to \mathrm{Cl}(V) \to \mathrm{Cl}(C(V)) \to 0$ is by [7, proposition 6.5(c)]

$\square$

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of the affine cone, show that $S(V)$ is UFD if and only if V is projectively normal (under the given embedding, the homogeneous coordinate ring is integrally closed) and $\mathrm{Cl}(V) \cong \mathbf{Z}$ and is generated by the class $V \cdot H$.

Proof. By the exact sequence in (b), we have $\mathrm{Cl}(C(V)) = 0$ and $S(V)$ is integrally closed, by [7, proposition 6.2], $S(V)$ is UFD. $\square$

6.4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \cdots, x_n]$ be a square-free nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \cdots, x_n, z]/(z^2 - f)$, show that A is an integrally closed ring.

Proof.
claim 1: $\mathrm{Frac}\, A = k(x_1, \cdots, x_n)[z]/(z^2 - f)$.
proof of the claim 1: $k(x_1, \cdots, x_n)[z]/(z^2 - f)$ is a subfield of $\mathrm{Frac}\, A$ because field morphism is either trivial or injetcive, on the other hand, any $1/(a + bz) \in \mathrm{Frac}\, A$, we have $\frac{1}{a+bz}\frac{a-bz}{a-bz} = \frac{a-bz}{a^2+b^2 f} \in k(x_1, \cdots, x_n)[z]/(z^2 - f)$. Thus $\mathrm{Frac}\, A$ is quadratic field extension over fraction field $F = k(x_1, \cdots, x_n)$.

Claim 2: For the quadratic field extension $\mathrm{Frac}\, A$ over F, $\alpha = h + gz \in \mathrm{Frac}\, A$ where $h, g \in F$, α is integral if and only if $f, g \in k[x_1, \cdots, x_n]$.
proof of the claim 2: if $\alpha \in F$, then it is integral if and only if it is in $k[x_1, \cdots, x_n]$. Otherwise, $\alpha \in \mathrm{Frac}\, A \setminus F$, then the minimal irreducible polynomial for α is $((x - h)/g)^2 = f$, that is, $x^2 - 2hx + h^2 - g^2 f = 0$. Thus α is integral over F if and only if $h, h^2 - g^2 f \in k[x_1, \cdots, x_n]$, as f is square-free, this is equivalent to $h, g \in k[x_1, \cdots, x_n]$

Thus $A = k[x_1, \cdots, x_n][z]/(z^2 - f)$ is the integral closure of the above quadratic field extension. Thus it is integrally closed. $\square$

6.5. Quadratic Hypersurfaces. Let $\mathrm{char}(k) \neq 2$, and let X be the affine quadratic hypersurface $\mathrm{Spec}(k[x_0, \cdots, x_n]/(x_0^2 + \cdots + x_n^2)$. Here we assume k to be algebraically closed.
(a) Show that X is normal if $r \geq 2$.

Proof. If $\sum x_i^2 = f^2$ with $f \in k[x_0, \cdots, x_n]$, then as $\mathrm{char}(k) \neq 2$, this is impossible. Thus $\sum_i x_i^2$ is square-free and by exercise 6.4 above, $k[x_0, \cdots, x_n]/(x_0^2 + \cdots, x_n^2)$ is integrally closed. Thus X is normal. $\square$

(b) Show that a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \cdots + x_r^2$.
(a) If $r = 2$, then $\mathrm{Cl}(X) \cong \mathbf{Z}/2\mathbf{Z}$;

(b) If $r = 3$, then $\mathrm{Cl}(X \cong \mathbf{Z}$;
(c) If $r \geq 4$, then $\mathrm{Cl}(X) = 0$.

Proof. For any algebraically closed field k, denote one of the root of the polynomial $x^2 + 1$ as i, then $x_0 = \frac{y_0+y_1}{2}$ and $x_1 = \frac{y_0-y_1}{2i}$ with $y_j = ix_j$ for $j \geq 2$. Then $X = \mathrm{Spec}(k[y_0, \cdots, y_n]/(y_0y_1 - y_2^2 - \cdots y_r^2))$. From now on, we denote $A = k[y_0, \cdots, y_n]/(y_0y_1 - y_2^2 - \cdots - y_r^2)$.

Consider $Y = V(y_1) = \mathrm{Spec}(k[y_0, \cdots, y_n]/(y_1, y_0y_1 - y_2^2 - \cdots - y_r^2))$ which is a closed subscheme of X, and it is irreducible because $(y_1, y_0y_1 - y_2^2 - \cdots - y_r^2) = (y_1, y_2^2 + \cdots + y_r^2)$ is primary ideal because they are generated by different indeterminates.

By [7, proposition 6.5,II], there is an exact sequence $\mathbf{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U = X \setminus Y) \to 0$. While $U = D(y_1) = \mathrm{Spec}(A_{y_1})$, which is the spectrum of a UFD, thus by [7, proposition 6.2,II], $\mathrm{Cl}(U) = 0$, and $\mathrm{Cl}(X) = \mathbf{Z}/m\mathbf{Z}$.

(1) $r = 2$, then this is nearly the same as [7, example 6.5.2,II], except that n may be larger than 3. Again, $y_1 = 0 \to y_2^2 = 0$ and the uniformizer of the local ring at the generic point of Y is y_2 because $y_0^{-1}y_2^2 = y_1$ in the local ring and the maximal ideal of this local ring is generated by (y_1, y_2), thus the valuation of function y_1 is 2 and $(y_1) = 2Y = 0$ in the divisor class, it remains to prove that Y itself is not a principal ideal.

Consider the maximal (closed point 0) ideal $\mathfrak{m} = (y_0, y_1, y_2, \cdots, y_n) \subset A$, then $\mathfrak{m}/\mathfrak{m}^2$ is a k vector space of dimension $n+1$ with basis $\bar{y_0}, \cdots, \bar{y_n}$ (you can just view $\mathfrak{m}$ as some maximal ideal in $k[y_0, \cdots, y_n]$). If (y_1, y_2) is generated by one element in A, then subspace generated by $\bar{y_1}$ and $\bar{y_2}$ are of dimension one, which is a contradiction.

(2) $r = 3$, now consider the case when $r = 3$, in this case, by the same way of the above coordinate change, $x_0^2 + x_1^2 + x_2^2 + x_3^2 = xy - zw$. Thus X is the affine cone of the projective scheme $V = \mathrm{Proj}(xy - zw)$, isomorphic to the product of projective lines $\mathbf{P}^1 \times_k \mathbf{P}^1$ (see [7, example 6.6.1,II]), with its class group $\mathrm{Cl}(V) \cong \mathbf{Z} \times \mathbf{Z}$. Here is a situation, when $n \geq 3$ ($r \neq n$), this is not true???

$\square$

6.9. Singular curves. Here we give another method of calculating the Picard group of a singular curve. Let's X be a projective curve over k, let $\widetilde{X}$ be its normalization, and let $\pi : \widetilde{X} \to X$ be its projection map. For each point p, let $\mathcal{O}_p$ be its local ring, and $\widetilde{\mathcal{O}}_p$ be its integral closure. We use $*$ to denote the group of units in a ring.
(a). Show there is an exact sequence

$$0 \to \bigoplus_{p \in X} \widetilde{\mathcal{O}}_p^* / \mathcal{O}_p^* \to \mathrm{Pic}\ X \xrightarrow{\pi^*} \mathrm{Pic}\ \widetilde{X} \to 0$$

(b). Use (a) to give another proof that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbf{G}_a \to \mathrm{Pic}\ X \to \mathbf{Z} \to 0$$

and if X is a plane nodal cubic curve, there is an exact sequence

$$0 \to \mathbf{G}_m \to \text{Pic } X \to \mathbf{Z} \to 0$$

Remark 1: Here we assume that k is algebraically closed, and 'curve' means integral separated scheme of finite type over k with dimension one as a convention in Harthsone's book (see [7, Remark 4.10.1]). So as integral projective scheme over k, the global section space of X is $\Gamma(X, \mathcal{O}_X) \cong k$ (see [7, exercise 3.5.3]).

Remark 2: Normalization of k projective scheme is still projective, just check the pullback of an ample bundle along π.

Proof. (a) By the property of normalization, for any affine open subset $U = \text{Spec}(A) \subset X$, $\pi_*(\mathcal{O}_{\widetilde{X}})(U) = \widetilde{A}$, so by the inclusion of rings: $A \to \widetilde{A}$, there is an inclusion of $\mathcal{O}_X$ sheaf module inside $\mathcal{K}_X$: $0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\widetilde{X}}$, so there is a well defined quotient sheaf: $(\pi_*\mathcal{O}_{\widetilde{X}})^*/\mathcal{O}_X^*$ and exact sheaf sequence: $0 \to (\pi_*\mathcal{O}_{\widetilde{X}})^*/\mathcal{O}_X^* \to \mathcal{K_X}^*/\mathcal{O}_X^* \to \mathcal{K_X}^*/\pi_*(\mathcal{O}_{\widetilde{X}})^* \to 0$.

Because normalization is birational morphism for curves over k, the sheaf $\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^*$ is supported on a proper closed subset of X, which is a finite union of closed points. Indeed, proper closed subset of curve should be union of closed points due to dimension, and finite because it is noetherian space. So $\mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^*$ is a finite direct sum of skyscraper sheaves, which can be written in the form of $\bigoplus_{p\in X} i_p(\pi_*(\mathcal{O}_{\widetilde{X}})_p/\mathcal{O}_{X,p})$ where i_p is the abbreviation for the pushforward of $i_p : \{p\} \to X$.

Because taking integral closure commutes with localization, and π is affine morphism, so passing to the direct limit leads to the stalk isomorphism: $(\pi_*\mathcal{O}_{\widetilde{X}})_p \cong \widetilde{\mathcal{O}}_{X,p}/\mathcal{O}_{X,p}$, so is $(\pi_*\mathcal{O}_{\widetilde{X}})_p^* \cong \widetilde{\mathcal{O}}_{X,p}^*/\mathcal{O}_{X,p}^*$

So the exact sequence now becomes $0 \to \bigoplus_{p\in X} i_p(\widetilde{\mathcal{O}}_{X,p}/\mathcal{O}_{X,p}) \to \mathcal{K_X}^*/\mathcal{O}_X^* \to \mathcal{K_X}^*/\mathcal{O}_{\widetilde{X}}^* \to 0$. By exercise 1.16 in chapter 2, we can see that (1). $\widetilde{\mathcal{O}}_{X,p}/\mathcal{O}_{X,p}$ is flasque, (2). taking global section on any open subset is exact functor. So we have $0 \to \widetilde{\mathcal{O}}_{X,p}/\mathcal{O}_{X,p} \to CaDiv(X) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_{\widetilde{X}}^*) \to 0$.

lemma 1. $\Gamma(X, \mathcal{K}_X^*/\pi_*(\mathcal{O}_{\widetilde{X}})^*) \cong CaDiv(\widetilde{X})$

Proof. Because normalization here is birational morphism, so $\mathcal{K}_X \cong \mathcal{K}_{\widetilde{X}}$ which is the constant sheaf of function field $K(X) = K(\widetilde{X})$. More precisely, we can say that in this case, the morphism of structure sheaves between ringed spaces: $\mathcal{O}_X \to \pi_*(\mathcal{O}_{\widetilde{X}})$ extends to a homomorphism $\mathcal{K}_X \to \pi_*(\mathcal{K}_{\widetilde{X}}) = \mathcal{K}_X$.

Thus we can define the inverse image of Cartier divisor in a natural way: by $\mathcal{K}_X/\mathcal{O}_X \to (\pi_*\mathcal{K}_{\widetilde{X}})/(\pi_*\mathcal{O}_{\widetilde{X}}) \to \pi_*(\mathcal{K}_{\widetilde{X}}/\mathcal{O}_X)$, take the unit subgroup of them and take the global section space, then it turns into: $CaDiv(X) = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\pi_*(\mathcal{O}_{\widetilde{X}})^*) \to \Gamma(\widetilde{X}, \mathcal{K}_{\widetilde{X}}^*/\mathcal{O}_{\widetilde{X}}^*) = CaDiv(\widetilde{X})$.

Take a careful look at this sequence, we can see that the first morphism is exactly the third morphism we have in the sequence $0 \to \bigoplus_{p\in X} \widetilde{\mathcal{O}}_{X,p}^*/\mathcal{O}_{X,p}^* \to CaDiv(X) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_{\widetilde{X}}^*) \to 0$. And it suffices to prove that the second morphism $\Gamma(X, \pi_*(\mathcal{K}_{\widetilde{X}}^*)/\pi_*(\mathcal{O}_{\widetilde{X}})^*) \to \Gamma(\widetilde{X}, \mathcal{K}_{\widetilde{X}}^*/\mathcal{O}_{\widetilde{X}}^*) = CaDiv(\widetilde{X})$ is an isomorphism.

Because π is affine, so π_* takes quasi-coherent sheaf on $\widetilde{X}$ to quasi-coherent sheaf on X, and π_* admits a right adjoint $\tilde{\pi} : M \to Hom_A(B, M)$ when π is restricted to affine open subsets: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, so π_* is right exact and thus exact. So for the exact sequence: $0 \to \mathcal{O}_{\widetilde{X}} \to \mathcal{K}_{\widetilde{X}} = \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_{\widetilde{X}} \to 0$, after the action of π_*, there is again an exact sequence: $0 \to \pi_*(\mathcal{O}_{\widetilde{X}}) \to \pi_*(\mathcal{K}_{\widetilde{X}}) = \pi_*(\mathcal{K}_X) \to \pi_*(\mathcal{K}_X/\mathcal{O}_{\widetilde{X}}) \to 0$. Thus $\pi^*(\mathcal{K}_X)/\pi_*(\mathcal{O}_{\widetilde{X}}) \cong \mathcal{K}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}$ and consider

Unwinding the definition, $\psi : \Gamma(X, \mathcal{K}_X^*/\pi_*(\mathcal{O}_{\widetilde{X}})^*) \to \Gamma(\widetilde{X}, \mathcal{K}_{\widetilde{X}}^*/\mathcal{O}_{\widetilde{X}}^*) = CaDiv(\widetilde{X})$ sends $\{U_i, f_i\}$ to $\{\pi^{-1}(U_i), f_i\}$ where $f_i \in K(X)$ and $f_i/f_j \in \Gamma(\pi^{-1}(U_i \cap U_j), \mathcal{O}_{\widetilde{X}}^*)$, this is obviously injective. It now suffices to prove that it is surjective.

To prove this, we need to show that for any Carter divisor in $\widetilde{X}$, there is some Cartier divisor in X corresponding to it in the way we described above. Because $\widetilde{X}$ is normal, as a curve, it must also be nonsingular (or regular), by proposition 6.11 in chapter, Cartier divisors on $\widetilde{X}$ is equivalent to Weil divisors, where the latter ones are very simple in the curve over algebraically closed k, whose prime divisors are exactly the closed points. So it suffices to prove that for any closed point $p \in \widetilde{X}$, there is a Cartier divisor $\{\pi^{-1}(U_i), f_i\}$ corresponds to it.

For $p \in \widetilde{X}$ and its image $\pi(p) = q \in X$, take an affine neighborhood $U = \mathrm{Spec}(A)$ of q along with its pre-image $V = \pi^{-1}(U) = \mathrm{Spec}(\widetilde{A})$, and consider the restriction of $\pi : V \to U$. The pre-image of point q may be more than p, here we denote them as $p_0 = p,\ p_1, \cdots, p_n$, here as closed subscheme, the stalk at $\{p\}$ is the quotient of local ring $\mathcal{O}_{\widetilde{X},p}$. Recall that $\mathcal{O}_{\widetilde{X},p}$ is a DVR with generator of maximal ideal as u_p, so as closed subscheme, the structure sheaf (or stalk) on $\{p\}$ is in the form of $\mathcal{O}_{\widetilde{X},p}/u_p^n$ with $n \geq 1$.

We claim that there is a section $s \in \Gamma(V, \mathcal{O}_{\widetilde{X}})$ such that $s(p_i)$ is unit for $i \neq 0$, and $s(p_0) = u_{p_0}$ with valuation 1. Though s may have other zero valuation at other points, but these 'zero' points are finite and closed, so we can pick a smaller affine open neighborhood U' of q such that the only 'zero' valuation point of s in $V' = \pi^{-1}(U')$ is p_0, so $\{(U', s), (U \setminus \{q\}, 1)\}$ is the required section for $\Gamma(U, \mathcal{K}_X^*/\pi_*(\mathcal{O}_{\widetilde{X}})^*)$ mapping to divisor $\{p\}$. This procedure can easily generalize to the whole scheme (use the fact that all divisors are just closed points). Thus surjectivity is done, and we have exact sequence: $0 \to \oplus_{p\in X} \widetilde{\mathcal{O}}_{X,p}^*/\mathcal{O}_{X,p}^* \to CaDiv(X) \to CaDiv(\widetilde{X}) \to 0$. □

It now suffices to show that the image of the direct sum of skycraper sheaves $\oplus_{p\in X} \widetilde{\mathcal{O}}_{X,p}^*/\mathcal{O}_{X,p}^*$ in $CaDiv(X)$ has trivial intersection with the principal Cartier divisors, that is, the image of $\Gamma(X, \mathcal{K}_X^*)$. Otherwise, if a principal Cartier divisor (f) with $f \in K(X)$ is sent to trivial Cartier divisor on $\widetilde{X}$, so f takes value in $\mathcal{O}_{X,x}$ at every closed point x (which is also the codimension one Weil divisors on $\widetilde{X}$), then on every open affine subscheme $\mathrm{Spec}(A)$ where A is integral, by proposition 6.3A in chapter 2, $f \in A$ and thus f is a global section on $\widetilde{X}$, and by the **Remark 1** above the proof, $f \in k$, so it is also a global section on X, and it is unit at every closed point's local ring, thus (f) is not only a principal Cartier divisor on X, it is also trivial Cartier divisor on X. Thus we can modulo the principal divisors for $CaDiv(X)$ and $CaDiv(\widetilde{X})$ without changing the kernel $\oplus_{p\in X} \widetilde{\mathcal{O}}_{X,p}^*/\mathcal{O}_{X,p}^*$, this leads to the conclusion: $0 \to \oplus_{p\in X} \widetilde{\mathcal{O}}_p^*/\mathcal{O}_p^* \to CaCl\, X \xrightarrow{\pi^*} CaCl\, \widetilde{X} \to 0$, as X and $\widetilde{X}$ are both integral, so $CaCl(\cdot) \cong Pic(\cdot)$ and (a) is proved.

Remark 3: for an integral scheme X, by the above proof, we can see that there is an exact sequence $0 \to \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$

lemma 2. There is a section $s \in \Gamma(V, \mathcal{O}_{\tilde{X}})$ such that $s(p_i)$ is unit for $i \neq 0$, and $s(p_0) = u_{p_0}$ where u_{p_0} is the generator of the maximal ideal in DVR $\mathcal{O}_{p_0, \tilde{X}}$.

Proof. For an affine scheme $V = \text{Spec}(A)$, and $p_0,\ p_1, \cdots, p_n$ are all the prime ideals. Take a section as u_{p_0} (here we may assume the generator of DVR $\mathcal{O}_{p_0,X}$ is an element in Spec(A)), but this u_{p_0} may have valuations nonzero at some other points p_i. For these points p_i where u_{p_0} has valuation zero, it satisfies our requirements, so we may assume u_{p_0} has valuations zero all local rings of p_i, $0 < i \leq m$, and has nonzero valuations for $m < i \leq n$.

By the prime avoidance theorem, there is $t_i \in A$ such that $t_i \in p_i$ and $t_i \notin p_j$ for and $j \neq i$, thus consider $f = \prod_{m<j\leq n} t_j$, then $f \notin \cap_{0\leq i\leq m} p_i$ and $f \in p_j$ for $m < j \leq n$, so $s = u_{p_0} + f^2$ has valuation 1 at stalk of p_0, and its valuation at other p_i is zero, thus unit. $\square$

Remark 4: When k^* is not finitely generated, then $\mathbf{G}_m = \text{Spec}(k[t, t^{-1}])$ is not finitely generated group (even k is not algebraically closed, this is still true). Recall the exact sequence for plane cubic nodal curves in (b): because $\mathbf{Z}$ is free abelian group, thus this sequence of abelian groups splits, and $\text{Pic}(X) = k^* \times Z$ is not a finitely generated abelian group. This is an example which Picard group is not finitely generated.

Remark 5: More about Finitely Generated Picard Group
Theorem:[7] If $K = F(t_1, \cdots, t_n)$ where F is a finite field or number field, and X/K is a normal scheme of finite type, then Pic(X) is finitely generated.
This is a corollary of Mordeil-Weil theorem, and we can see that the infinitely generated example of nodal cubic curve here is not normal.

Remark 6: Other Calculations about Picard Group
There are more examples of calculation of Picard groups in section 2.7. Also, recall the important proposition (2.6.6), $\text{Cl}(X) \cong \text{Cl}(X \times A^1)$, then $\text{Pic}(X) \cong \text{Pic}(X \times A^1)$, this is the 'homotopy invariance' of Picard groups, that is, $\text{Pic}(X \times A^n) \cong \text{Pic}(X)$. Especially, $\text{Pic}(\mathbf{A}^{\mathbf{n}}_{\mathbf{R}}) \cong \text{Pic}(A)$, when $A = \mathbf{Z}$ or k, this is trivial.[8]

This property may lead to an interesting question: For which rings A is $\text{Pic}(A[t_0, \cdots, t_n]) \cong \text{Pic}(A)$? Such rings are called "seminormal' rings[9].

$\square$

6.10. The Grothendieck Group K(X). Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$, whenever there is an exact sequence $0 \to \mathcal{F}' \to$

[7] see arXiv:alg-geom/9410031 by Robert Guralnick, David Jaffe, Wayne Raskind, Roger Wiegand.
[8] Every projective module over PID is free, thus Pic(Z)=0
[9] see 'Seminormality and Picard Groups' by Carlo Traverso, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 24,$n^0$4(1970),p.585-595.

$\mathcal{F} \to \mathcal{F}'' \to 0$. If $\mathcal{F}$ is coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.
(a). If $X = \mathbf{A}^1_k$, then $K(X) \cong \mathbf{Z}$.

(b). If X is integral scheme, and $\mathcal{F}$ a coherent sheaf, we define the rank of $\mathcal{F}$ to be $dim_K \mathcal{F}_\eta$ where η is the generic point of X, and $K = \mathcal{O}_\eta$ is the function field of X. Show that the rank function defines a surjective homomorphism rank: $K(X) \to \mathbf{Z}$.
(c) If Y is a closed subscheme, show that there is an exact sequence:

$$K(Y) \to K(X) \to K(X - Y) \to 0$$

where the first map is extension by zero and the second map is restriction.

Remark 1: here 'extension by zero' is defined as pushforward along inclusion i: $i_*(\mathcal{F})$ in [7, exercise 1.19,II] and 'restriction' is the inverse image sheaf $i^{-1}(\mathcal{F})$ as shown in the end of [7, section 1,II].

Remark 2: well-defineness in (c)
(1). As X and Y are both noetherian and $i : Y \to X$ is closed immersion, by [7, exercise 5.5,II], the direct image sheaf $i_*(\mathcal{F})$ is still coherent. And the inverse image sheaf for $j^{-1}(\mathcal{F})$ is exactly the pullback $j^*(\mathcal{F})$ where j is open embedding $: X \setminus Y \to X$, so it is the restriction of coherent sheaf on an open subset, which is again coherent.

(2). *extension by zero* and *restriction* here are all exact functors. For affine morphism i, the direct image functor admits a left adjoint and is right exact, so it is exact (or just checking at stalks). While the pushback j^* is just the restriction of an exact sequence on some open subset, which is again exact by checking at stalks.

So the exact sequence in (c) is well-defined.

Remark 3: we can observe that for isomorphic sheaves $\psi : \mathcal{R} \to \mathcal{P}$, there is a short exact sequence: $0 \to 0 \to \mathcal{R} \to \mathcal{P} \to 0$, so in $K(X)$, isomorphic coherent sheaves are viewed as the same thing. Likewise, for the zero constant sheaf $\widetilde{0}$ on X, we have an exact sequence $0 \to \mathcal{F} \to \mathcal{F} \to \widetilde{0} \to 0$ for any coherent sheaf $\mathcal{F}$, so $\gamma(\widetilde{0}) = \gamma(\mathcal{F}) - \gamma(\mathcal{F}) = 0$, that is, zero sheaf class is the zero element in the abelian group $K(X)$.

Proof.
(a) For X=$\mathbf{A}^1_k$, $K(X)$ is equivalent to the category of finitely generated module M over $k[t]$ (which is a PID) module short exact sequence relations, because M is finitely generated over $A = k[t]$, so there is a surjective morphism of modules $g : A^n \to M \to 0$ where kernel of g exists :$0 \to kerg \to A^n \to M \to 0$ and $ker(g)$ is submodule of finitely generated free module over a PID, thus it is also finitely generated module free over A. So every coherent sheaf class $\gamma(\mathcal{F})$ can be expressed as $\gamma(\widetilde{A}^n) - \gamma(\widetilde{A}^m)$. By the exact sequence: $0 \to \mathcal{O}_X^n \to \mathcal{O}_X^{n+m} \to \mathcal{O}_X^m \to 0$, there is a natural relation in $K(X)$: $\gamma(\mathcal{O}_X^n) = n\gamma(\mathcal{O}_X)$.

And thus we can say that $K(X)$ is generated by the coherent sheaves in the form of $\mathcal{O}_X$. It suffices to show that $\gamma(\mathcal{O}_X^n) \neq \gamma(\mathcal{O}_X^m) \in K(X)$. This may involve lots of things like $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, so we need to find some invariants under these equivalences. So we prove (b) first.

(b) For each coherent sheaf, the function *rank* naturally extends to the free abelian group generated by coherent sheaves. To prove that the function $rank$ indeed is well defined for $K(X)$, it suffices to prove that for short exact sequences: $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$, $rank(\mathcal{F}) + rank(\mathcal{F}'') = rank(\mathcal{F}'')$.

To compute the stalk of a sheaf at the generic point, we may assume that $X = \operatorname{Spec}(A)$ is affine. In this case, coherent sheaves correspond to finitely generated module over A, we may write them in the form of :$0 \to \widetilde{F}_1 \to \widetilde{F}_2 \to \widetilde{F}_3 \to 0$ where F_i is finitely generated module over A. By proposition II 5.2A in [**1**], this is equivalent to $0 \to F_1 \to F_2 \to F_3 \to 0$. The stalk at generic point is the localization $(\widetilde{F}_i)_\eta \cong F_i \otimes_A Frac(A)$, and because localization is exact, so there is exact sequences of vector space over $Frac(A) = K$: $0 \to \mathcal{F}_\eta \to \mathcal{F}'_\eta \to \mathcal{F}''_\eta \to 0$. So $rank(\mathcal{F}) + rank(\mathcal{F}'') = rank(\mathcal{F}'')$, function $rank$ is well defined and by construction is a homomorphism to $\mathbf{Z}$. To prove that this morphism is surjective, it suffices to take the coherent sheaf in the form $\mathcal{O}_X^n$ whose rank is n.

Back to (a), $\mathbf{A}_k^1$ is integral, and the coherent sheaves $\mathcal{O}_X^n$ has rank n while $\mathcal{O}_X^m$ has rank m, so $\gamma(\mathcal{O}_X^n) \neq \gamma(\mathcal{O}_X^m)$ when $n \neq m$. Thus by $1 \to \gamma(\mathcal{O}_X)$, there is an injection $i : \mathbf{Z}_{\geq 0} \to K(X)$. And by our construction of $rank$, $rank(-\gamma(\mathcal{O}_X)) = -rank(\gamma(\mathcal{O}_X))$, so we can extend i to an injection $i : \mathbf{Z} \to K(X)$. Indeed, this is still injective, because $-\gamma(\mathcal{O}_X)^n \neq -\gamma(\mathcal{O}_X^m)$ when $n \neq m$ by symmetry, and $\gamma(\mathcal{O}_X^n) \neq -\gamma(\mathcal{O}_X^m)$ for any $n, m \geq 0$. And we have proven above that any coherent sheaf class $\gamma(\mathcal{F})$ can be generated by $\mathcal{O}_X$, so i is also surjective, thus it is isomorphism.

(c) The second exactness of the sequence: $K(Y) \to K(X) \to K(X - Y) \to 0$ is obvious by [7, exercise 5.15,II]. So it remains to prove the first exactness of the sequence. For $i : Y \to X$, and coherent sheaf $\mathcal{F}$ on Y, the restriction of $i_*(\mathcal{F})$ to $X - Y$ is the zero constant sheaf, by our **Remark 2** above, it is zero in $K(X \setminus Y)$. So the image of the extension by zero is contained in the kernel of restriction. Now it suffices to prove that the kernel of restriction is contained in the image of extension in the Grothendieck group. First of all, we give a description of the kernel of restriction.

lemma 1. if $\gamma(\mathcal{F}|_{X-Y}) = 0 \in K(X - Y)$, then there exists $\mathcal{F}'$ such that $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$ and $\mathcal{F}'$ is supported inside the closed subset Y.

So we only need to consider the coherent sheaf $\mathcal{F}$ on X such that its support is contained in Y. We need to show that for such $\mathcal{F}$, $\gamma(\mathcal{F})$ can be generated by coherent sheaves on Y by extension. By the hint given, it remains to show the following lemma,

lemma 2. There is a filtration for any coherent sheaf $\mathcal{F}$ supported inside Y

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_n = 0$$

where $\mathcal{F}_i$ are all coherent sheaves on X and $\mathcal{F}_{i+1}/\mathcal{F}_i = i_*(\mathcal{G})$ for some coherent sheaf $\mathcal{G}$ on Y.

□

Remark 4: Why do we use Grothendieck Group $K(X)$ instead of $QCoh(X)$?

6.11. Grothendieck Group of a Nonsingular Curve. Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic}(X) \otimes \mathbb{Z}$, in several steps.
(a) For any divisor $D = \sum_i P_i$ on X, let $\mathcal{L}(D) = \bigotimes_i \mathcal{O}_X(k(P_i)) \in K(X)$, where $\mathcal{O}_X(k(P_i))$ is the

skyscraper sheaf $\mathcal{O}_X$ at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{D}$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\mathcal{L}(D) = \mathcal{O}_X(D)$. Then use (6.18) to show that for any D, $\mathcal{L}(D)$ depends only on the linear equivalence class of D, so $\mathcal{L}$ defines a homomorphism $\mathcal{L} : \mathrm{Cl}(X) \to K(X)$.

(b) For any coherent sheaf $\mathcal{F}$ on X, show that there exist locally free sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$ and an exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F} \to 0.$$

Let $r_0 = \mathrm{rank}(\mathcal{F}_0)$, $r_1 = \mathrm{rank}(\mathcal{F}_1)$, and define $\det(\mathcal{F}) = (\bigwedge^{r_0} \mathcal{F}_0) \otimes (\bigwedge^{r_1} \mathcal{F}_1)^{-1} \in \mathrm{Pic}(X)$. Here $\bigwedge$ denotes the exterior power. Show that $\det(\mathcal{F})$ is independent of the resolution chosen and gives a homomorphism $\det : K(X) \to \mathrm{Pic}(X)$. Finally, show that if D is a divisor, then $\det(\mathcal{L}(D)) = \mathcal{O}_X(D)$.

(c) If $\mathcal{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence

$$0 \to \mathcal{L}(D) \otimes \mathcal{B} \to \mathcal{F} \to \mathcal{C} \to 0,$$

where $\mathcal{C}$ is a torsion sheaf. Conclude that if $\mathcal{F}$ is a sheaf of rank r, then $\mathcal{L}(\mathcal{F}) - r\mathcal{O}_X(x) \in \mathrm{Pic}(X) \otimes \mathbb{Z}$.

(d) Using the maps $\mathcal{L}$, det, rank, and $1 \mapsto \mathcal{O}_X(x)$ from $\mathbb{Z}$ to $K(X)$, show that $K(X) \cong \mathrm{Pic}(X) \otimes \mathbb{Z}$.

1.2 Exercises in 2.8

8.1. Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.
(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_k k(B) \longrightarrow \Omega_{k(B)/k} \longrightarrow 0,$$

is exact on the left also. [Hint: In copying the proof of (8.7), first pass to $\mathfrak{m}/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $\mathfrak{m}^2$.]

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free B module of rank $= \dim B + \mathrm{tr.d.}\, k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathcal{O}_{X,x}$ is a regular local ring$\}$ is an open dense subset of X.

Proof.
(a). We may assume first that B is a complete local ring. So [7, 8.25A,II] can be used here.

To prove the injection, this is equivalent to proving the dual $\delta^* : \mathrm{Hom}_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) = \mathrm{Hom}_B(\Omega_{B/k}, k(B)) \cong Der_k(B, k(B)) \to \mathrm{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$ is surjective. In the same way as (8.7), δ^* is the restriction of a derivation $d \in Der_k(B, k(B))$ to $\mathfrak{m}$, noting that $d(\mathfrak{m}^2) = 0$ because $\mathfrak{m}$ acts on $k(B)$ through quotient $B \to B/\mathfrak{m} = k(B)$, which is zero.

Now we use the condition k(B) is separably finitely generated over k, by [7, 8.25A,II], there exists a field K containing k such that $B/\mathfrak{m} \cong K \subset B$. Thus every element $b \in B$ can be written as $b = m + \lambda$ where $m \in \mathfrak{m}$ and $\lambda \in k(B)$. For any $h \in \mathrm{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$, we define a k derivation $d : B \to k(B)$ by $d(b) = h(\bar{m})$ where $\bar{m}$ is the image of m in $\mathfrak{m}/\mathfrak{m}^2$. It's trivial to check that this indeed a k derivation.

Now consider the general case when B is not complete, but its quotient $B/\mathfrak{m}^2$ is complete because the maximal ideal $\widetilde{\mathfrak{m}}$ has power zero in $B/\mathfrak{m}^2$, and the field k is still included in $B/\mathfrak{m}^2$ because k only has trivial intersection with any ideal in B and also the residue field $k(B) \cong k(B/\mathfrak{m}^2)$. So by what we have shown above, there is

$$0 \longrightarrow \frac{\mathfrak{m}/\mathfrak{m}^2}{(\mathfrak{m}/\mathfrak{m}^2)^2} \cong \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{(B/\mathfrak{m}^2)/k} \otimes_B k(B)$$

is an exact sequence. And by [7, 8.4A,II], there is an exact sequence:

$$\mathfrak{m}^2/\mathfrak{m}^4 \to \Omega_{B/k} \otimes_B B/\mathfrak{m}^2 \to \Omega_{B/\mathfrak{m}^2/k} \to 0$$

Tensoring with $k(B) = B/\mathfrak{m}$, the first term is zero and because tensor product is right exact, so there is isomorphism $\Omega_{B/k} \otimes_B B/\mathfrak{m}^2 \otimes_B B/\mathfrak{m} \cong \Omega_{(B/\mathfrak{m}^2)/k} \otimes_B B/\mathfrak{m}$. By the isomorphism of rings $R/(I+J) \cong R/I \otimes_R R/J$, the isomorphism can be written as $\Omega_{B/k} \otimes_B B/\mathfrak{m} \cong \Omega_{(B/\mathfrak{m}^2)/k} \otimes_B k(B)$. Combining the injection already shown: $0 \to \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{(B/\mathfrak{m}^2)/k} \otimes_B k(B)$, we have exact sequence: $0 \to \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B B/\mathfrak{m}$. It's trivial to check that this inclusion is exactly the first morphism described in the sequence of (8.4A) ($\bar{m}$ is always sent to $d(b) \otimes 1$).

(b). By the short exact sequence we have in (a), which is actually a $k(B)$ vector space exact sequence, there is a rank relation: $\dim_{k(B)} \Omega_{B/k} \otimes_k k(B) = \dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 + \dim_{k(B)} \Omega_{k(B)/k}$. And as $k(B)$ is separably finitely generated over k, so by [7, 8.6A,II], $\dim_{k(B)} \Omega_{k(B)/k} = \mathrm{tr.d.}\, k(B)/k$, so we have $\dim_{k(B)} \Omega_{B/k} \otimes_k k(B) = \dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 + \mathrm{tr.d.}\, k(B)/k$.

When $\dim_{k(B)} \Omega_{B/k} \otimes_k k(B) = \dim B + \mathrm{tr.d.}\, k(B)/k$, so $\dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 = \dim B$, so it is regular local ring.

When B is regular local ring, then $\dim_{k(B)} \mathfrak{m}/\mathfrak{m}^2 = \dim B$, so $\dim_{k(B)} \Omega_{B/k} \otimes_B k(B) = \dim B + \mathrm{tr.d.}\, k(B)/k$. We need to prove two things: (1) $\Omega_{B/k} \otimes_B k(B)$ is free. (2) and it is of locally rank $\dim B + \mathrm{tr.d.}\, k(B)/k$. However, these two properties are not independent, as we shall see.

Here we denote the quotient of B as K considering the fact that we assume B is regular local ring, so it is integral domain. Noting that: (1). As localization of finite type k algebra, B is noetherian. (2). $\Omega_{B/k}$ is a finitely generated module over B. (3). Here k is perfect, so any finitely generated k algebra is separably generated (see [7, 4.8A,I]), especially for the field extension here K/k, so by (8.6A), $\dim_K \Omega_{K/k} = \mathrm{tr.d.}\, K/k$. By II.8.2A, $\Omega_{B/k} \otimes_B K \cong \Omega_{K/k}$, so $\dim_K \Omega_{B/k} \otimes_k K = \mathrm{tr.d.}\, K/k$.

Then by [7, 8.9,II], to prove that $\Omega_{B/k}$ is free B module, it suffices to prove $\dim_K \Omega_{B/k} \otimes_B K = \dim_{k(B)} \Omega_{B/k} \otimes_B k(B)$ which is equivalent to proving $\operatorname{tr.d.} K/k = \dim B + \operatorname{tr.d.} k(B)/k$.

To calculate $\operatorname{tr.d.} K/k$, we need to use the condition that B is the localization of some k algebra of finite type which we denote it as A. We claim that $B = A_{\mathfrak{p}}$ where A is a finitely generated k algebra and B is its localization at a prime ideal (see **lemma 1** for the proof).

In this case, as A is k domain of finite type, by [7, 1.8A,I], $\dim(A) = \operatorname{tr.d.} K(A)/k = \operatorname{tr.d.} K(B) = \operatorname{tr.d.} K/k$, so it remains to calculate the Krull dimension of A. Again by by I.1.8A in **[1]**, $\dim A = \dim A/\mathfrak{p} + \operatorname{height} \mathfrak{p}$ where $\operatorname{height} \mathfrak{p} = \dim A_{\mathfrak{p}} = \dim B$ and $\dim A/\mathfrak{p}$ is also a k integral domain of finite type, so $\dim A/\mathfrak{p} = \operatorname{tr.d.} K(A/\mathfrak{p})/k = \operatorname{tr.d.} k(B)/k$. So $\dim_K \Omega_{B/k} \otimes_B K = \dim B + \operatorname{tr.d.} k(B)/k$.

lemma 1. If an integral domain B is the localization of a finitely generated k algebra, then there exists a finitely generated k domain A such that B is the localization of A. Moreover, if B is a local ring, then B is the localization of A at a prime ideal.

(c). If the local ring $\mathcal{O}_{x,X}$ is regular, then it is also a localization of a finitely generated k algebra. So $\Omega_{\mathcal{O}_{x,X}/k}$ is free $\mathcal{O}_{x,X}$ module of rank $\dim \mathcal{O}_{x,X} + \operatorname{tr.d.} k(x)/k$. Because X is noetherian and finite over k, $\Omega_{X/k}$ is coherent and its stalk is $\Omega_{\mathcal{O}_{x,X}/k}$ (see [7, Remark.8.9.1,III]). So by [7, exercise 5.7,II], $\Omega_{X/k}$ is finite locally free. It suffices to prove that $\dim \mathcal{O}_{x,X} + \operatorname{tr.d.} k(x)/k = \dim X$, so that $\operatorname{rank} \Omega_{X,k} = \dim X$.

As X is an irreducible scheme of finite type over k, we can take the reduced scheme structure which doesn't influence the topological information like dimension. So we may assume X is integral, so $\dim X = \operatorname{tr.d.} k(\eta)/k$ where η is the generic point. Chose an affine open subset $U = \operatorname{Spec}(C)$ where $x = \mathfrak{p} \subset C$. Then $\operatorname{height} \mathfrak{p} = \dim \mathfrak{O}_{x,X}$ and $\dim C/\mathfrak{p} = \operatorname{tr.d.} k(x)/k$ (here reduced structure doesn't change these facts too). So $\dim C = \dim C^{red} = \operatorname{height} \mathfrak{p} + \operatorname{tr.d.} k(x)/k = \dim \mathcal{O}_{x,X} + \operatorname{tr.d.} k(x)/k$ where $C^{red} = C/(\eta)$ is an integral domain and this operation changes nothing for the topological information like dimension and height.

If $\Omega_{X/k}$ is finite locally free of rank $\dim X$, so is its stalk at any point x. As we have shown, $\dim X = \dim \mathcal{O}_{x,X} + \operatorname{tr.d.} k(x)/k$, by (b), $\mathcal{O}_{x,X}$ is regular local ring.

(d). By (8.16), U is dense because it contains a dense open subset. It remains to prove that U is open. For any $x \in U$, $(\Omega_{X,k})_x$ is locally free of rank $\dim X$ by (c), so by exercise [7, 5.7,II], there is a neighborhood U' around x such that $\Omega_{X,k}|_U$ is free, so $x \in U' \subset U$. Thus U is open.

□

Remark 1: Possible ambiguity between tensor functors $\otimes_B$ and $\otimes_{k(B)}$

8.2 Let X be a variety of dimension n over k. Let $\mathcal{E}$ be a locally free sheaf of rank $r > n$ on X, and let $V \subset \Gamma(X, \mathcal{E})$ be a vector space of global sections which generate $\mathcal{E}$. Then show that there is an element $s \in V$ such that for each $x \in X$, we have $s_x \notin m_x \mathcal{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathcal{E}$ giving rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{E}'$ is also locally free. Denote $\dim V = m+1$.

Proof.
lemma 1: If there exists a section $s \in V$ such that $s_x \notin m_x \mathcal{E}_x$ for all closed point $x \in X$, then this is also true for each $x \in X$.

Proof. For each $x \in X$, consider an open affine neighborhood $U = \mathrm{Spec}(A)$ of x where A is a k algebra of finite type, and on X, $\mathcal{E} \cong \mathcal{O}_X^r$. So the section s can be written in the form of $s = (s_1, \cdots, s_r)$ where $s_i \in A$. Consider the maximal ideal $\mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n)$ (because k is algebraically closed) containing x, then there exists s_i such that $s_i \notin \mathfrak{m}$, so $s_i \notin x = \mathfrak{p}$. Thus $s_i(x) \notin \mathfrak{m}_x \mathcal{O}_X$, so $s = (s_i) \in \Gamma(U, \mathcal{E})$ satisfies $s_x \notin \mathfrak{m}_x \mathcal{E}_x$.

□

By the lemma above, It suffices to consider the maximal (closed) points on the variety X, which is the corresponding classical variety: an open subset of zero locus of a family of homogeneous polynomials. So from now on we only consider the category of classical varieties and the fibre product of classical varieties in the sense of image through the Segre embedding. Denote $|V| = \frac{V-0}{k^*}$, and construct such a product of varieties : $X \times |V|$; Consider the subset $B = \{(x,s) \in X \times |V| | s(x) = 0\}$ (here we regard the global section of $\mathcal{E}$ taking value at $x \in X$ in the vector space $\mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} k$ under the quotient $\mathcal{O}_{x,X} \to k(x) \cong k$).

lemma 2: B constructed above is closed in the variety $X \times |V|$.

Proof. The Zariski topology on product of varieties is given by the zero locus of bidegree homogeneous polynomials. It suffices to prove $X \times |V| \setminus B$ is open. For any such point $(x,s) \in X \times |V|$ such that $s_x \notin \mathfrak{m}_x \mathcal{E}_x$, again restrict to an affine neighborhood $U = \mathrm{MSpec}(A)$ of x such that $\mathcal{E}|_U \cong \mathcal{O}_X^r|_U$, so we can regard the global section s as $(s_i)_{i=1}^r$ such that $s_i \in \Gamma(U, \mathcal{O}_X)$ which is a regular function on the affine variety $U = \mathrm{MSpec}(A) = \mathrm{MSpec}(k[x_0, \cdots, x_n]/I) = Z(I)$ (here we use the notation in chapter 1 in [7]). By our choice of x, there exists s_i such that $s_i(x) \neq 0$. And for the vector space V, choose a k vector basis $e_j = (e_{j,i})_{j=1}^r$ where $j = 0, \cdots m$ for V, and any section $s \in V$ can be written as $s = \sum_j c_j e_j$, and its i component $s_i = \sum_j c_j e_{j,i}$; Here each component of a global section restricted to U can be viewed as a regular function on U, which is an element in $k[x_0, \cdots, x_n]/I$, especially, we will regard $e_{j,i}$ here as such polynomial functions.

It suffices to find an open neighborhood of (s,x) inside the open neighborhood $U \times |V|$. Consider the bihomogeneous polynomial $F = \sum_j c_j e_{j,i} \in k[x_0, \cdots, x_n, c_0, \cdots, c_m]$, so the non-vanishing open set $D(F)$ of this polynomial is an open set in the variety $U \times |V|$ which contains $(x \in U, (s_i = \sum_j c_j^0 e_{i,j}(x))_i)$ and for $(x,s) \in D(F)$, $s_i(x) \neq 0$, so $s(x) \neq 0 \in k^{m+1}$.

□

At each point $x \in X$, the fibre of x under the projection $\pi_1 : B \to X$ is exactly the $P(ker)$: where ker is the projective subspace of V under the k linear vector morphism : $V = V \xrightarrow{()_x} \mathcal{E}_x \to \mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} k \cong \mathcal{E}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X}/\mathfrak{m}_x \cong \mathcal{E}/\mathfrak{m}_x \mathcal{E}_x$. It is easily checked that this linear map is surjective and rank ker + rank $\mathcal{E}$ = rank V, so $\dim ker = m + 1 - r$ and $P(ker) \cong P^{m-r}$ (when $m - r = 0$, we regard the fibre as a singleton).

lemma 3:[10] For the variety product $P^n_k \times P^m_k$ equipped with Zaraski topology, projection map $\pi_1 : P^n_k \times P^m_k \to P^n$ is a closed map.

Proof. Consider the polynomial ring $S = A[y_0, \cdots, y_m]$ over the ring $A = k[x_0, \cdots, x_n]$. For any nonempty closed subset $V(I)$ as the zero locus of bihomogeneous polynomials $g_i \in I$, I is finitely generated because all the rings are noetherian, you can put the homogeneous ideals in the big ring $k[x_0, \cdots, x_n, y_0, \cdots, y_m]$ and find the finite generators, to make them homogeneous, just take all the homogeneous components of these finitely many polynomials, they all lie in I and serve as finite bihomogeneous generators (see **Remark 2**).

So we denote $I = \{g_i\}_{i=1}^N$ and regard them as homogeneous ideals in $S = A[x_0, \cdots, x_n]$, in essence, what we want is some description of the coefficients: Are there any homogeneous polynomials $J = \{p_i\} \subset A = k[x_0, \cdots, x_n]$ such that: the points $[(x_0, \cdots, x_n)] \in V(J)$ are exactly the points $a = [a_0, \cdots, a_n]$ in $\frac{k^n - 0}{k^*} = P^n$ when we take its points value $([a_0, \cdots, a_n])$ into polynomials in I, there is nonzero solution for these polynomials in $([y_0, \cdots, y_m]) \in P^m$.

So, viewed as polynomials over A, the coefficients of $\{g_i\}$ can be regarded as certain values in k (or more precisely, the reside field $k(\bar{x}) \cong k$), changing with the parametrization of $\bar{x} = ([x_0, \cdots, x_n]) \in P^n$. In this way, we are dealing with polynomials $\{g_i\}$ over k. For the projective space P^m over k (as classical variety), a set I of homogeneous polynomials has no solution if and only if $\sqrt{I} \supset (y_0, \cdots, y_m)$. So in contrast, we need to find the conditions for coefficients of $\{g_i\}$ so that $\sqrt{(g_1, \cdots, g_N)} \not\supset (y_0, \cdots, y_m)$. Now we denote $T = k[y_0, \cdots, y_m]$.

This is equivalent to $(y_0, \cdots, y_m)^M \not\subset (g_1, \cdots, g_N)$ for all $M \geq 0$, again equivalent to $T_M \not\subset (g_1, \cdots, g_N)$ for all $M \geq 0$ which may be written as $T_M \not\subset g_1 T_{M-\deg g_1} \oplus g_2 T_{M-\deg g_2} \cdots$. In other words, this is equivalent to the following morphism being not surjective:

$$\psi : \oplus_{i=1}^N T_{M-\deg g_i} \xrightarrow{(g_1, \cdots, g_N)} T_M \text{ for all } M \geq 0$$

If the source of morphism has rank less than T_M, then this is of course not surjective, and there is no restriction on the coefficients. Otherwise, this is equivalent to $\operatorname{rank}(\psi) < \dim T_M$, that is, all $\dim T_M$ determinants inside the matrix of ψ is zero. These determinants are actually polynomials of coefficients $(x_0, \cdots, x_n)$, and as their columns (or ranks depending on your construction of matrix for ψ) have all components homogeneous, so these determinants are actually homogeneous polynomials . Thus we can get a family of homogeneous polynomials for coefficients and thus get a Zaraski closed subset in P^n. □

There is a general fact that a closed surjective map restricted to a saturated subset is still closed. So $\pi : X \times |V| \to X$ is closed. And This is an analog of 'proper morphism' in the proof in the Bertini's theorem. So the restriction of π_1 to closed subset B is still a closed (continuous) map: $\pi : B \to X$ with fibre $\pi^{-1}(x) \cong P^{m-r}$.

***lemma 4:**[11] If $\pi : X \to Y$ is surjective closed morphism of varieties, and Y is irreducible

[10] There is a more general result called 'Fundamental Theorem of Elimination' in Vakil's Foundations of Algebraic Geometry, chapter 7.4,p.222, November 18, 2017 draft.

[11] This lemma originates from an answer on the Internet: https://math.stackexchange.com/questions/579527/use-irreducible-fibers-to-show-x-is-irreducible?lq=1 .

of dimension y, and each fibre of π is irreducible of dimension z, then X is irreducible with dimension $z + y$.

Proof. As (classical) varieties are noetherian, so if X is not irreducible, then it has finite irreducible (closed) components, denoted as $\{X_i\}$. As π is closed, so $\pi(X_x)$ is also closed in Y. As Y is irreducible, then there exists X_0 such that $\pi(X_0) = Y$.

Denote $X_y = \pi^{-1}(y)$, then $X_y = \cup_i X_i \cap X_y$ which is a union of closed subset in X_y, because X_y is irreducible, there exists some i such that $X_y = X_i \cap X_y$, so $X_y \subset X_i$. Especially, when $x \notin X_j$ for all $j \neq i$, then $X_i \supset X_{\pi(x)}$.

By applying [9, theorem 1.25, section 6.3], there is an inequality $\dim X_y \cap X_i \geq \dim X_i - \dim \pi(X_i)$ with equality on some open subset of $\pi(X_i)$ (image of irreducible set is irreducible, so $\pi(X_i)$ is irreducible). For each $y \in Y$, there exists X_i such that $X_i \cap X_y = X_y$, so $\dim X_i - \dim \pi(X_i) = \dim X_y = z$. And for each X_i, pick the point $x \in X_i \setminus \cup_{j\neq i} X_j$ (which is nonempty because they are irreducible components), then we can see that $\dim X_i - \dim \pi(X_i) = z$ for all i.

By our first inequality: $\dim X_y \cap X_i \geq \dim X_i - \dim \pi(X_i)$, we have $\dim X_y \cap X_i = \dim X_y$ for $X_i \cap X_y \neq \emptyset$, because X_y is irreducible, so $X_y \cap X_i = X_y$. So each irreducible component X_i will contain all the fibres of its image $\pi(X_i)$. As $\pi(X_0) = Y$, thus $X_0 = X$ is irreducible and the inequality $\dim X_y \cap X_0 \geq \dim X_0 - \dim \pi(X_0)$ (with equality on some open subset) becomes equality $\dim X_y = \dim X - \dim Y$.

□

By lemma 4 applied to the close map $\pi : B \to X$, there is $\dim B = \dim X + \dim P^{m-r} = n + m - r$. Because $r > n$, so $\dim B < m = \dim(P(V)) = |V|$. If $\pi' : B \to |V|$ is surjective, then because both B and $|V|$ are irreducible, then by the inequality used in lemma 4 again, $\dim B \geq \dim |V|$, contradiction. Thus $\pi' : B \to |V|$ is not surjective, and there exists $s \in V$ such that $s_x \notin \mathfrak{m}_x \mathcal{E}_x$.
□

8.3. Product Schemes.
(a). Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b). If X and Y are nonsingular varieties over field k, show that $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

(c). Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$. This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

Here k is assumed to be algebraically closed.

Proof. (a) By (8.11) and (8.10) in [7, section 2.8], there is an exact sequence:

$$p_1^*\Omega_{X/S} \xrightarrow{u_1} \Omega_{X\times_S Y/S} \xrightarrow{v_1} p_2^*\Omega_{Y/S} \cong p_2^*\Omega_{Y/S} \to 0$$

$$p_2^*\Omega_{Y/S} \xrightarrow{u_2} \Omega_{X\times_S Y/S} \xrightarrow{v_2} p_2^*\Omega_{X/S} \cong p_2^*\Omega_{X/S} \to 0$$

It remains to prove v_2 is the left inverse for u_1, this is equivalent to the first sequence has its first morphism injective and splits. To prove this sheaf morphism relations, it suffices to check locally on affine open subset. And as differential sheaf is quasi-coherent, (a) is equivalent to a pure commutative algebraic problem: for pushout of rings:

$$\begin{array}{ccc} k & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C = A\otimes_k B \end{array}$$

and the exact sequences:

$$\Omega_{B/k}\otimes_B C \xrightarrow{u_1} \Omega_{C/k} \xrightarrow{v_1} \Omega_{C/B} \overset{\psi}{\cong} \Omega_{A/k}\otimes_A C \to 0$$

$$\Omega_{A/k}\otimes_A C \xrightarrow{u_2} \Omega_{C/k} \xrightarrow{v_2} \Omega_{C/A} \overset{\phi}{\cong} \Omega_{B/k}\otimes_B C \to 0$$

To prove that v_2 is the left inverse for u_1, we need to know the exact description of u_i, v_i and ψ, ϕ, in Hartshone's AG book, this part is not detailedly explained and only provides a reference by [8], for this part, I mainly follow his idea, but even in his book, there is some proof missed, for example, the proof of 'differential module under base change', on this part, I refer to [3, p.393-395]. Here I shall give the descriptions of u_i v_i and ψ, ϕ, and then taking an element $d_{B/k}(b)\otimes c \in \Omega_{B/k}\otimes_B C$ into the morphism $v_2{\circ}u_1$ and verify that this gives back exactly the same element.

Claim 1: $u_1 : d_{B/k}(b)\otimes c \in \Omega_{B/k}\otimes_B C \to c\cdot d_{C/k}(b) \in \Omega_{C/k}$.

Claim 2: $v_2 : d_{C/k}(c) \in \Omega_{C/k} \to d_{C/A}(c) \in \Omega_{C/A}$, especially, $c'\cdot d_{C/k}(c) \to c'\cdot d_{C/A}(c)$ for arbitrary $c, c' \in C$.

Claim 3: when $C = A\otimes_k B$, $\phi : d_{C/A}(a\otimes b) \in \Omega_{C/A} \to d_{B/k}(b)\otimes(a\otimes 1) \in \Omega_{B/k}\otimes_B C$. If we regard $\Omega_{B/k}\otimes_k A\otimes_k B \cong \Omega_{B/k}\otimes_k A$, then $d_{C/A}(a\otimes b) \to d_{B/k}\otimes a$.

Then by the claims above, it's trivial that $v_2{\circ}u_1 = id$ and (a) is done.

(b) By exercise 2.5.16 in **[1]**, for a short exact sequence of locally free sheaves: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, the exterior powers $\wedge^{\dim\mathcal{F}}\mathcal{F} \cong \wedge^{\dim\mathcal{F}'}\mathcal{F}'\otimes\wedge^{\dim\mathcal{F}''}\mathcal{F}''$.

Denote $\dim X = n$ and $\dim Y = m$, then as X and Y are nonsingular, $\Omega_{X/k}$ is locally free of rank n, and $\Omega_{Y/k}$ is locally free of rank m. By (a), $\Omega_{X\times_k Y/k}$ is locally free of rank $n+m$. Thus by the exact sequence in (a) and exercise 2.5.16, $\wedge^{n+m}\Omega_{X\times_k Y/k} = \wedge^n\Omega_{X/k}\otimes\wedge^m\Omega_{Y/k}$, so $\omega_{X\times Y} \cong \omega_X\otimes\omega_Y$.

(c) The arithmetic genus of the projective variety[12] $X = Y\times_k Y$ can be calculated through **Kunneth Formula** (see [7, exercise 7.2(e),I]):

lemma 1. Let X and Y be projective varieties, then $p_a(X\times_k Y) = p_a(X)p_a(Y) + (-1)^{\dim Y}p_a(X) + (-1)^{\dim X}p_a(Y)$.

[12] see the definition of arithmetic genus in p.230, chapter 3.5, [7], or you can see the equivalent definition in the sense of classical variety, p. 54, chapter 1, [7].

Proof. There is a pure commutative algebra way of proving this lemma, see Exercise 7.2. Here I shall use a different by using cohomology groups.

theorem

□

Then $p_a(X \times_k X) = p_a(X)p_a(X) - 2p_a(X)$. Since the arithmetic genus of a projective hypersurface[13] X is determined by its degree $\deg(X) = d$: $p_a(X) = \binom{d-1}{n}$. For our cubic curve X, $p_a(X) = 1$. Thus $p_a(X \times_k X) = -1$.

For the geometric genus p_g[14] of $X = Y \times_k Y$: $p_g(X) = \dim \Gamma(X, \omega_X) = \dim \Gamma(X, p_1^*\omega_Y \otimes p_2^*\omega_Y)$ by (b). There is a complete description of canonical sheaf (see [7, Example 8.20.3,II]) on projective nonsingular hypersurface X of degree d: $\omega_Y \cong \mathcal{O}_Y(d-n-1)$. For the plane cubic curve Y, this means $\omega_Y \cong \mathcal{O}_Y$, then pullback $\pi_i^*\omega_Y = \pi_i^*\mathcal{O}_Y = \mathcal{O}_X$. Thus $p_g(Y) = \dim \Gamma(X, \pi_1^*\omega_Y \otimes \pi_2^*\omega_Y) = \dim \Gamma(X, \mathcal{O}_X) = 1$(see [7, exercise 5.3,III]).

Thus this is an example when arithmetic genus differs from geometric genus.

□

Remark 1: If X is a curve, then use the method of cohomology, we can show that $p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$, see [7, proposition 1.1, IV].

Remark 2: (c) can be generalized to higher dimensional cases:

8.4. Complete Intersection in P^n.
A closed subscheme Y of P^n is called a (strict, global) complete intersection if the homogeneous ideal I of Y in $S = k[x_0, \dots, x_n]$ can be generated by $r = \operatorname{codim}(Y, P^n)$ elements (cf. exercise 2.17)[15].

(a). Let Y be a closed subscheme of codimension r in P^n. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \dots, H_n$ such that $Y = H_1 \cap \dots \cap H_n$ as schemes, i.e., $\mathcal{I}_Y = (f_1, \dots, f_r)$
Hint: Use the fact that the unmixedness theorem holds in S (see [8, p.107]).

(b) If Y is a complete intersection of dimension ≥ 1 in P^n, and if Y is normal, then Y is projectively normal (cf. Exercise 5.14) [Hint: Apply (8.23) to the affine cone over Y.]

(c) With the same hypotheses as (b), conclude that for all $i \geq 0$, the natural map $\Gamma(P^n, \mathcal{O}_{P^n}(i)) \to \Gamma(Y, \mathcal{O}_Y(i))$ is surjective. In particular, taking $i = 0$, show that Y is connected.

(d) Now suppose given integers $d_1, \dots, d_r \geq 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \dots, H_r$ in P^n, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \dots \cap H_r$ is irreducible and nonsingular of codimension r in P^n.

[13] Here the curve may not be nonsingular, it's just projective. There are two ways of definition for arithmetic genus, one is through embedding into projective space, the other is through dimensions of cohomology groups, neither requires nonsingular conditions, only requires projectivity.

[14] As a comparison with arithmetic genus, geometric genus requires that the projective variety also be nonsingular so that there is a well defined exterior product of differential forms–canonical sheaf.

[15] However, compared with the definition in exercise $I.7.2$, here a complete intersection is not required to be irreducible.

(e) If Y is a nonsingular complete intersection as in (d), show that $\omega_Y \cong \mathcal{O}_Y(L)$ with $L = \sum_{i=1}^{r}(d_i - n - 1)$.

(f) If Y is a nonsingular hypersurface of degree d in P^n, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (cf. Exercise 7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

(g) If Y is a nonsingular curve in P^3, which is a complete intersection of nonsingular surfaces of degrees d and e, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$. Again, the geometric genus is the same as the arithmetic genus (cf. Exercise I.7.2).

Proof. **(a)***[16] Recall the definition of '(strict) complete intersection' is a variety Y with dimension $n-r$ and the homogeneous ideal $I(Y)$ is generated by r elements. There are some basic facts about complete intersection I shall claim here, where you can find details in Exercise 2.17

Claim 1 : For any irreducible closed subsecheme Y in $\mathbf{P}^n$, let $\widetilde{\mathfrak{a}}$ be the ideal sheaf for Y and supposed that $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ can be generated by q elements, then $q \geq n - \dim Y$. So complete intersection has its ideal sheaf generated by the minimal number generators.

Claim 2 : In $\mathbf{P}^n = \operatorname{Proj}(k[x_0, \cdots, x_n] = S)$, let X and Y be closed subschemes determined separately by ideal sheaves $\mathcal{I} = \widetilde{I}$ and $\mathcal{J} = \widetilde{J}$, then the scheme-theoretic intersection of X and Y is defined through fibre product, for the closed subschemes in projective scheme, we have $\operatorname{Proj}(S/I) \times_{\operatorname{Proj}(S)} \operatorname{Proj}(S/J) \cong \operatorname{Proj}(S/(I+J))$[17]. However, by Exercise 5.10, it's better to write it in the form of $\operatorname{Proj}(S/I) \cap \operatorname{Proj}(S/J) = \operatorname{Proj}(S/I) \times_{\operatorname{Proj}(S)} \operatorname{Proj}(S/J) = \operatorname{Proj}(S/\overline{I+J})$ where $\overline{I+J}$ is the saturation of a homogeneous ideal.

If Y is the scheme-theoretic intersection of $H_1, \cdots, H_n$ where hyper surface H_i is defined by a homogeneous polynomial f_i[18], by the claim 2 above, it suffices to prove that the ideal $(f_1, \cdots, f_n)$ is saturated, so the homogeneous ideal corresponding to Y is indeed generated by the minimal number r elements. In general, the saturation of sum of saturated ideals may not be saturated, so $\overline{(f_1, \cdots, f_n)}$ may have larger generators set than $(f_1, \cdots, f_n)$, then in this case, Y is not a complete intersection.

lemma 1 : Here $T = (f_1, \cdots, f_n)$ is a saturated homogeneous ideal.

Proof. By the definition of saturation (see [7, exercise 5.10, II]), then $\overline{T} = (T : \mathfrak{m}^\infty) = \cup_{d\geq 1}(T : \mathfrak{m}^d)$ where $\mathfrak{m} = (x_0, \cdots, x_n)$. Consider the minimal primary decomposition of $T = \cap_{i=0}^{j}\mathfrak{q}_j$ where $\sqrt{\mathfrak{q}_j} = \mathfrak{p}_j$, by the **unmixed theorem**[19], all associated prime ideals $\mathfrak{p}_i$ are of height r, the same as T.

[16] The solution to (a) is based on an answer on the internet: https://math.stackexchange.com/questions/140117/problem-about-complete-intersection-in-textbf-pn-from-hartshorne

[17] In $\mathbf{P}^2$, $I = (x,z)$ and $J = (y,z)$, then $(I,J) = (x,y,x)$ which is the irrelevant ideal, whose Proj is empty set. This is an example when sum of saturated ideal is not saturated ideal.

[18] homogeneous ideal in polynomial ring of height one is principal, however, here is homogeneous ideal of homogeneous height, this is still true, just need to realize that a homogeneous element in graded domain if it is reducible f=gh, then g and h are homogeneous, otherwise, take the minimal degree components of g and h separately, this leads to contradiction.

[19] see p.107, 'Commutative Algebra' by Hideyuki Matsumura, New York, 1970.

If T is not saturated, then $\overline{T} = \cup_{d\geq 1}(T : \mathfrak{m}^d) = \cup_{d\geq 1}(\cap_j \mathfrak{q}_j : \mathfrak{m}^d) = \cap_j \cup_{d\geq 1} (\mathfrak{q}_j : \mathfrak{m}^d) \supsetneqq \cap_j \mathfrak{q}_j$. Thus there exists at least one $\mathfrak{q}_j$ such that $\mathfrak{q}_j \subsetneqq (\mathfrak{q} : \mathfrak{m}^\infty)$. That is, there exists $t \in (\mathfrak{q} : \mathfrak{m}^\infty) \setminus \mathfrak{q}_j$. So for x_i for each $i = 0, \cdots, n$, there exists n_i such that $x_i^{n_i} t \in \mathfrak{q}_i$, and since $\mathfrak{q}_i$ is primary, and $t \notin \mathfrak{q}_i$, thus there exists N, $x_i^N \in \mathfrak{q}_i$, this leads to $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ containing the irreverent ideal $(x_0, \cdots, x_n)$, which is impossible, because in the Cohen-Macaulay ring $k[x_0, \cdots, x_n]$, there is no $\mathfrak{p}$-primary ideal in the minimal primary decomposition of T with height larger than T strictly. In our case height$(T) = r < n + 1 =$ height $\mathfrak{m}$, so there can't be $\mathfrak{m}$-primary ideal appearing in the associated ideals of T. Contradiction! □

If Y is a complete intersection, then the (saturated) homogeneous ideal corresponding to it by definition is generated by $r = \mathrm{codim}(Y, X)$ elements $\{f_i\}$. Thus Y is the scheme-theoretic intersection of $\{H_i\}$ corresponding to $\{f_i\}$.

(b) Here we recall some basic properties of 'normality' here.

Serre: A noetherian ring is normal if and only if it satisfies the following two conditions:
(R_1: regular in codimension one) for every prime ideal $\mathfrak{p} \subset A$ of height $\mathfrak{p} \leq 1$, $A_\mathfrak{p}$ is regular;
(S_2: Hartog's theorem) for every prime ideal $\mathfrak{p}$ of height≥ 2, we have depth $A_\mathfrak{p} \geq 2$.

lemma 2: Singular locus of a normal scheme X has codimension larger than 2: codim Sing$(X) \geq 2$.[20]

If a scheme X has its codim Sing$(X) \geq 2$, then it must be 'regular in codimension one', otherwise, if a prime ideal $\mathfrak{p}$ of (height) codimension one (or 0) has its local ring non-regular, then for any $\mathfrak{q} \subset \overline{\{\mathfrak{p}\}}$, take an affine open subset $U = \mathrm{Spec}(A)$ containing both $\mathfrak{p}$ and $\mathfrak{q}$ (this open affine subset exists, otherwise $\mathfrak{q}$ is not in $\overline{\mathfrak{p}}$), then we can see that $\mathcal{O}_{\mathfrak{p},X}$ is the localization of $\mathcal{O}_{\mathfrak{q},X}$. By Auslander-Buchsbaum-Serre's theorem[21], localization of regular local ring at prime ideal is still regular local ring, this leads to a contradiction, thus all points in $\overline{\{\mathfrak{p}\}}$ are singular, this is an irreducible closed subset of codimension 1(or 0), contradiction to our condition that codim Sing$(X)\geq 2$.

lemma 3. Affine cone and singularity: Let X be a closed subscheme in $\mathbf{P}^n = \mathrm{Proj}(k[x_0, \cdots, x_n] = S)$, if X is 'regular in codimension one', then its affine cone $C(X)$ is also 'regular in codimension one'.

Proof. Denote $X = \mathrm{Proj}(S/I)$,(then the affine cone is $C(X) = \mathrm{Spec}(S/I)$), and the vertex ($\mathfrak{m} = (x_0, \cdots, x_n)$) of $C(X)$ as p. There is a natural morphism $\pi : U = C(X) \setminus \{p\}$ which is 'locally ruled'[22]

□

□

[20]This is a classical conclusion, but I can"t find any relevant proof on any textbooks I use.

[21]see p.67, Cohen-Macaulay Rings by Bruns and Herzog, revised version, Cambridge study in advanced mathematics 39.

[22]see exercise 6.2 in 2.6 of this note: here 'locally ruled' is a temporary term, means locally on an open subset U_1 of U, $\pi : \pi^{-1}(U_1) \to U_1$ is the projection $U_1 \times_k A^1 \to U_1$.

8.6. The Infinitesimal Lifting Property. The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that Spec(A) is a nonsingular variety over k. Let $0 \to U \to B' \to B \to 0$ be an exact sequence, where B' is a K algebra and I is its ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ & I \arrow[d] \\ & B' \arrow[d] \\ A \arrow[ur, dotted, "g"] \arrow[r, "f"] & B \arrow[d] \\ & 0 \end{tikzcd}$$

We call this result the infinitesimal lifting property for A. We prove this result in several steps.
(a) First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g = g'$ is a k-derivation of A into I, which we can consider as an element $\mathrm{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of $B = B'/I$ module structure and hence also of A module. Conversely, for any $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, $g' = g+\theta$ is another k-algebra homomorphism lifting f.

Proof. If g' is another lifting, then for any $a \in A$, $\theta(a) = (g - g')(a) \overset{(-)/I}{\to} f(a) - f(a) = 0 \in B$. Thus the image of θ is contained in I, and θ is a k-module homomorphism, thus additive.

For any $c \in k$, $g(c) = c \cdot g(1) = g'(c) \in B'$, so $\theta(c) = 0$ for any $c \in k$.

For $a, a' \in A$, $\theta(aa') = (g - g')(aa') = g(a)g(a') - g'(a)g'(a')$. And it suffices to prove that $\theta(aa') = a' \cdot \theta(a) + a \cdot \theta(a')$. For any $a \in A$, the action of a on element $i \in I$ can be understood as $f(a)(+j) \cdot i$ where $j \in I$, so we can see that the action of a only depends on the class of a, thus this action can be directly understood as $g(a)i \in B'$ since $I^2 = 0$ where g is any lifting of f. So $a' \cdot \theta(a) + a \cdot \theta(a') = g(a')(g(a) - g'(a)) + g'(a)(g(a') - g'(a'))$, unwinding them leads us to $\theta(aa') = a' \cdot \theta(a) + a \cdot \theta(a')$. Thus θ is a k-derivation of A into I.

For any $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$, then $g'(a) - g(a) \in I$, so their composition with $B' \to B = B'/I$ is the same. And it's left to verify that g' is indeed a k-algebra homomorphism.

$(g+\theta)(a+a') = (g+\theta)(a)+(g+\theta)(a')$ because θ is additive. $(g+\theta)(aa') = g(a)g(a')+a\cdot\theta(a')+a'\cdot\theta(a)$ while $(g+\theta)(a)(g+\theta)(a') = g(a)g(a')+g(a)\theta(a')+\theta(a)g(a')+\theta(a')g(a) = g(a)g(a')+a' \cdot \theta(a) + cdots\theta(a')$. And for any $c \in k$, $(g + \theta)(ca) = g(ca) + \theta(ca) = cg(a) + \theta(c)a + c\theta(a)$, because $\theta(c) = 0$, so $(g+\theta)(ca) = cg(a)+c\theta(a)$, thus it is indeed a k-algebra homomorphism. $\square$

(b) Now let $P = k[x_0 \cdots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there do exists a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h induces an A-linear map $\bar{h} : J/J^2 \to I$.

Proof. By the homomorphism $P \to A = P/J \xrightarrow{f} B$, because P is free k module, for surjective morphism $B' \to B$, there is a lifting $h : P \to B'$ by freely choosing the value of monomials x^I (free generators as k-module) as any $b' \in B'$ such that $\bar{b'} = f(\overline{x^I})$. However, this is not a k-algebra homomorphism, it is only a k-module homomorphism.

This problem can be fixed, by choosing the value $h(x_i)$ for $\{x_i\}$ and then all the other monomials x^I are determined by the $h(x_i)$: $h(x^I) = h(x)^I$ and their images in the quotient ring $B = B'/I$ are still the $f(x^I)$ because quotient map $B' \to B$ is a ring homomorphism. Thus this assigning will not influence the well defines of k-module homomorphism because as free generators, their values can be freely chosen only if their images after the surjective quotient $B' \to B$ are exactly their images under f.

By the commutative diagram, the image of J under h is contained in I and by $I^2 = 0 \in B$, the image of J^2 is also zero, thus there is a well defined morphism $\bar{h} : J/J^2 \to I$. This is also an A module homomorphism by the commutative diagram. □

(c) Consider the nonsingular affine variety $Y = \mathrm{Spec}(A = P/I) \subset \mathbf{A}^n_k$, Y is defined by the ideal sheaf $\widetilde{I}$, by [7, 8.17, II], there is a $\mathcal{O}_Y$ module sheaf exact sequence:

$$0 \to \widetilde{I}/\widetilde{I}^2 \to \Omega_{\mathbf{A}^n_k/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$$

Because Y is a closed subscheme on an affine scheme, the above quasi-coherent sheaf exact sequence is equivalent to the A module exact sequence:

$$0 \to I/I^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

Recall that for the differential sheaf on Y, $\Omega_{Y/k} \cong \widetilde{\Omega_{A/k}}$ is locally free, but not free. However, locally free module is projective, so the short exact sequence above splits, and the $\mathrm{Hom}(-, I)$ is exact functor, thus there is short exact sequence:

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) \cong \mathrm{Hom}_P(\Omega_{P/k}, I)^{23} \to \mathrm{Hom}_A(J/J^2, I) \to 0$$

Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\bar{h} \in \mathrm{Hom}_A(J/J^2, I)$. Consider θ as a derivation of $P \to B'$. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

Proof. By (a), $h' = h - \theta$ is indeed a k-algebra homomorphism.

Recall how an element $\theta \in \mathrm{Hom}_p(\Omega_{P/k}, I)$ becomes a derivation of P to B': $p \in P \to d(p) \in \Omega_{P/k} \to \theta(d(b)) \in I$. And the exact sequence: $0 \to J/J^2 \to \Omega_{P/k} \otimes A$ is induced by $j \in J/J^2 \to d(j) \otimes 1$. So $\bar{h}$ induced by θ has: $\bar{h}(j) = \theta(d(j)) \in I$. Thus $h(j) - \theta(j) = 0 \in I$ and $h'(J) = 0$, and it factors through $A \xrightarrow{f} B$. □

8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathcal{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}'_X/\mathcal{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathcal{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. One such extension, the trivial one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space with the sheaf of rings $\mathcal{O}'_X$ is an infinitesimal extension of X by $\mathcal{F}$.

The general problem of classifying extensions of X by $\mathcal{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathcal{F}$ is isomorphic to the trivial one.

Proof. Denote $X = \mathrm{Spec}(A)$ is a nonsingular scheme finite over k, and let $\mathcal{F}$ be a coherent sheaf on X. For any infinitesimal lifting $X' = \mathrm{Spec}(B')$ with ideal sheaf $\widetilde{I} : \widetilde{I}^2 = 0$ on X, by the isomorphism $(\mathrm{Spec}(A), \widetilde{A}) \cong (\mathrm{Spec}(B), \widetilde{B/I})$ induced by $f : A \to B$, there is diagram:

[23] see appendix: pushforward and pullback of modules

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[ur, "\bar{h}"] \arrow[r, shift left, "f:A\to B"] \arrow[d] & B \arrow[l, shift left, "f^{-1}"] \arrow[d] \\
0 & 0
\end{tikzcd}$$

Thus $\bar{h} \circ f^{-1}$ is a section of the quotient map $B' \to B = B'/I$, thus the exact sequence of B' module $0 \to I \to B' \to B \to 0$ splits and we have isomorphism as B' modules $B' \cong I \oplus B$. So it remains to check the multiplication operation on $I \oplus B$: $(a \oplus b) \cdot (a' \oplus b') = (a+b)(a'+b') = aa' + ab' + a'b$ because $I^2 = 0$. Thus this is indeed the trivial extension.

□

1.3 Exercises in 2.9

9.1. Let X be a noetherian scheme, Y a closed subscheme, and $\widehat{X}$ the completion of X along Y. We call the ring $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the ring of formal-regular functions on X along Y. In this exercise we show that if Y is a connected, nonsingular, positive-dimensional subvariety of $X = \mathbf{P}^n_k$ over an algebraically closed field k, then $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k$.
(a). Let $\mathcal{J}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on Y: $\mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_Y(-1)^{n+1}$.

Proof. By [7, 8.13, II], for projective space $\mathbf{P}^n_k$ over k, there is an exact sequence:

$$0 \to \Omega_{\mathbf{P}^n/k} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

Then by pullback along closed subscheme $i\ :\ Y \subset X$, we have exact sequence: $0 \to i^*\Omega_{\mathbf{P}^n_k/k} \to \mathcal{O}_Y(-1)^{n+1} \to \mathcal{O}_Y$

By (8.17) **[1]**, there is an exact sequence for closed subvariety Y:

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathbf{P}^n_k/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$$

Thus there is an inclusion induced by $0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathbf{P}^n_k/k} \otimes \mathcal{O}_Y \cong i^*\Omega_{\mathbf{P}^n_k/k} \to \mathbf{O}_Y(-1)^{n+1}$, the composition of all these morphisms is injective because all of them are injective. □

(b). Show

2 Exercises in Chapter 3

2.1 Exercises in 3.2

2.1. (a). Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P,Q be distinct closed points of X, and let $U = X\{P, Q\}$. Show that $H^1(X, \mathbf{Z}_U) \neq 0$.

Proof. To calculate the cohomology group of $\mathbf{Z}|_U$, we need to find a $\Gamma(X, -)$ acyclic resolution:

$$0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_{\{P,Q\}} \to 0 \cdots$$

By this resolution, after the action of global section functor: $(0 \to \Gamma(X, \mathbf{Z}_U) \to)\Gamma(X, \mathbf{Z}) \cong \mathbf{Z} \xrightarrow{\psi} \Gamma(X, \mathbf{Z}_{\{P,Q\}}) \cong \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\phi} 0 \cdots$. The first cohomology group is $\ker\phi / \operatorname{im}(\psi)$. If it is zero, then $\operatorname{im}(\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$ which is impossible because of rank.

Remark 1: The global section space of **extension by zero** $j_!(\mathbf{Z}_U)$ behaves in an odd way: usually there is no way of directly computing it. You can only rely on the exact sequence: $0 \to \Gamma(X, \mathbf{Z}_U) = \ker\psi \to \Gamma(X, \mathbf{Z}) \xrightarrow{\psi} \Gamma(X, \mathbf{Z}_{\{P,Q\}})$ where ψ is exactly taking values on $\{P, Q\}$, thus has kernel 0, so $\Gamma(X, \mathbf{Z}_U) = 0$.[1]
Remark 2: In fact, I believe the condition here for k to be infinite field is not necessary. If k is finite with more than two points, then U is nonempty, and $\mathbb{F}_p[t]$ localized at $(t - p)$ and $(t - q)$ is again an integral domain inside $\mathbb{F}_p(t)$, which is an integral scheme and thus connected. By definition, $\Gamma(U, \mathbf{Z}_U) \cong \mathbf{Z}$, $\Gamma(X, \mathbf{Z}) \cong \mathbf{Z}$, $\Gamma(X, \mathbf{Z}_{\{P,Q\}}) \cong \mathbf{Z} \oplus \mathbf{Z}$. So the above still makes sense. □

$(b)^{*}$[2]. More generally, let $Y \subset X = \mathbf{A}_k^n$ be the union of $n + 1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathbf{Z}_U) \neq 0$. Thus Grothendieck vanishing theorem gives the best bound.

Proof. By the exact sequence of sheaf : $0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_Y \to 0$, there is a long exact sequence of cohomology groups:

$$0 \to \Gamma(X, \mathbf{Z}_U) \to \Gamma(X, \mathbf{Z}) \to \Gamma(X, \mathbf{Z}_Y) \to H^1(X, \mathbf{Z}_U) \cdots, H^{n-1}(X, \mathbf{Z}_U) \to H^{n-1}(X, \mathbf{Z}) \to$$
$$H^{n-1}(X, \mathbf{Z}_Y) \xrightarrow{\delta_n} H^n(X, \mathbf{Z}_U) \to H^n(X, \mathbf{Z}) \cdots$$

And by the fact that X is irreducible (connected), so is all the open subset of X, so $\mathbf{Z}$ is flasque on X, and $H^{i\geq 1}(-, \mathbf{Z}) = 0$, $H^n(X, \mathbf{Z}_U) \cong H^{n-1}(X, \mathbf{Z}_Y)$, and $H^*(Y, \mathbf{Z}) \cong H^*(X, \mathbf{Z}_Y)$[3], so it's left to calculate the cohomology group $H^{n-1}(Y, \mathbf{Z})$.

[1] see https://math.stackexchange.com/questions/1682362/extension-by-zero-not-quasi-coherent for an example: when X is integral, then extension of zero as $\mathcal{O}_X$ module sheaf always has global section space zero.
[2] This answer is inspired by : https://math.stackexchange.com/questions/910385/showing-grothendiecks-vanishing-theorem-provides-a-strict-bound. But my method here is slightly different from it.
[3] see lemma 2.10, p.209, Algebraic Geometry, Robin Hartshone.

Y is the union of $n+1$ hyperplanes $\{H_i\}_{i=0,\cdots,n}$ in general position, it is not irreducible, so constant sheaf $\mathbf{Z}$ is not flasque. However, on every irreducible component H_i, $\mathbf{Z}$ is flasque, this inspires us to construct the following embedding into a flasque sheaf: $0 \to \mathbf{Z}_Y \to \oplus_{i=0}^n \mathbf{Z}_{H_i}$ by j_i^{-1} where $j_i : H_i \to Y$. This is indeed injective by observing stalks.

As we shall prove the $n-1$ th cohomology nonvanishing, it makes no sense to stop here. Inspired by Céch Complex, we construct the following flasque resolution for $\mathbf{Z}_Y$:

$$0 \to \mathbf{Z}_Y \to \oplus_{i=0}^n \mathbf{Z}_{H_i} \to \oplus_{i<j} \mathbf{Z}_{H_i \cap H_j} \to \cdots \to \mathbf{Z}_{\cap_i H_i} = 0^{4}$$

This is indeed an exact sequence of sheaf: take an open subset $U \subset Y$, then $0 \to \ker(d)(U) \to \oplus_i(\Gamma(U, \mathbf{Z}_{H_i}) = \Gamma(U \cap H_i, \mathbf{Z}_{H_i})) \cong \oplus_i \mathbf{Z} \xrightarrow{d} \oplus_{i<j} \Gamma(U, \mathbf{Z}_{H_i \cap H_j}) \cong \oplus_{i<j} \mathbf{Z}$. Unwinding the sequence, this means the kernel $\ker(d)(U)$ is exactly the collection of continuous functions on H_i taking the same value in their intersection $\{H_i \cap H_j\}_{i,j}$, this is exactly the continuous function on U by definition of $function$. In the same way, we can prove this sequence of sheaf is exact.

By this resolution, we can calculate the cohomology by taking global section:

$$\oplus_i \mathbf{Z} \xrightarrow{d_0} \oplus_{i,j} \mathbf{Z} \xrightarrow{d_1} \cdots \oplus_{0,\cdots,\hat{i},\cdots,\hat{j},\cdots,n} \mathbf{Z} \xrightarrow{d_{n-2}} \oplus_{0,\cdots,\hat{i},\cdots,n} \mathbf{Z} \xrightarrow{d_{n-1}} 0$$

$H^{n-1} = \frac{\ker(d_{n-1})}{\mathrm{im}(d_{n-2})} = \frac{\mathcal{C}^{n-1} = \oplus_{0,\hat{i},n} \mathbf{Z}}{\mathrm{im}(d_{n-2})}$. If $H^{n-1}(Y, \mathbf{Z}) = 0$, then d_{n-2} is surjective, so it suffices to prove that d_{n-2} is not surjective.

lemma 1: $\mathcal{C}^{n-2} = \oplus_{\hat{i},\hat{j}}{}^{5} \Gamma(\cap_{k \neq i,j} H_k, \mathbf{Z}) \xrightarrow{d_{n-2}} \mathcal{C}^{n-1} = \oplus_{\hat{i}} \Gamma(\cap_{k \neq i} H_k, \mathbf{Z}) \cong \oplus_i \mathbf{Z}$ is not surjective, especially $(1, 0, \cdots, 0) \notin \mathrm{im}(d_{n-2})$.

Proof. If $d_{n-2}(\alpha) = (1, 0, \cdots, 0) \in \mathcal{C}^{n-1}$, then by definition of the differential operator of Céch Complex,

$$\text{0 th component: } d_{n-2}(\alpha)_{\hat{0},1,\cdots,n} = \sum_{k=1}^n (-1)^{k-1} \alpha_{1,\cdots,\hat{k},\cdots,n} = 1$$
$$j \text{ th component, } j \neq 0\text{: } d_{n-2}(\alpha)_{0,1,\cdots,\hat{j},\cdots,n} = \sum_{k=1}^{j-1} (-1)^k \alpha_{\hat{k}\hat{j}} + \sum_{k=j+1}^n (-1)^{k-1} \alpha_{\hat{j}\hat{k}} = 0$$

By multiplying the j th component with $(-1)^{j-1}$ and extract the first term, we get exactly $d_{n-2}(\alpha)_{\hat{0},1,\cdots,n}$, so:

$$-d_{n-2}(\alpha)_{\hat{0},1,\cdots,n} = -\sum_{k=1}^n (-1)^{k-1} \alpha_{1,\cdots,\hat{k},\cdots,n} = -1 =$$
$$\sum_{j=1}^n (-1)^{j-1} \left(\sum_{k=1}^{j-1} (-1)^k \alpha_{\hat{k}\hat{j}} + \sum_{k=j+1}^n (-1)^{k-1} \alpha_{\hat{j}\hat{k}}\right)$$

It's easy to see that the last term is zero, which is a contradiction. □

□

[4] $\cap_{i=0}^n H_i = \emptyset$ because they are in general positions, and all intersections $\cap H_i$ are irreducible by proper intersection condition.

[5] Here we denote $\oplus_{0,\cdots,\hat{i},\cdots,\hat{j},\cdots,n}-$ as $\oplus_{\hat{i},\hat{j}}-$.

2.2. Let $X = \mathbf{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{O}/\mathcal{K} \to 0$ is a flasque resolution of $\mathcal{O}$. Show that $H^i(X, \mathcal{O})$ for all $i \geq 1$.

Proof.
lemma 1: There is an exact sequence of $\mathcal{O}_X$ module sheaf:

$$0 \to \mathcal{O}_X \to \mathcal{K} \to \oplus_{p\in X}(i_p)_*(\mathcal{K}/\mathcal{O}_{p,X}) \to 0$$

Proof.
By the following exact sequence, it suffices to prove that $\oplus_{p\in P}\mathcal{K}/\mathcal{O}_p \cong \mathcal{K}/\mathcal{O}_X$.

$$0 \to \mathcal{O}_X \to \mathcal{K} \to \mathcal{K}/\mathcal{O}_X \to 0$$

By definition, the quotient sheaf $\mathcal{K}/\mathcal{O}_X$ is the direct product of presheaf stalks under some compatible conditions:

$$(\mathcal{K}/\mathcal{O}_X)(U) = \{(s_x) \in \prod_{x\in U} \mathcal{K}/\mathcal{O}_{x,X} | \forall x \in U, \text{there exists some neighborhood of } x \in V \subset U \text{ and } t \in K/\mathcal{O}_X(V) \text{and } s_v = t_v \text{ for all } v \in V\}$$

We shall see the compatible condition here for the quotient sheaf only makes the direct product becomes direct sum, which is important because this guarantees the flasque property of the quotient sheaf.

The quotient stalk $K/\mathcal{O}_{p,X}$ is trivial when p is the generic point, so the compatible condition automatically is correct. Apart from this point, because k is algebraically closed, the rest of X is actually the classical projective line which is an affine line plus a point at infinity.

Now we shall describe the stalk at non-generic point:

the quotient stalk is actually the 'pole' at x_i: $f \in K/\mathcal{O}_{x_i,X}$ can always be written in the form of $f = \frac{g}{(x-x_i)^{n_i}}$.

For any $f \in K$, $f = \frac{g}{h}$ where $g, h \in k[x]$ and $(g, h) = 1$, if h contains no factor $(x-x_i)$, then it is in the stalk $\mathcal{O}_{x_i,X}$, so it is equivalent to 0 in the quotient group. If $h = (x - x_i)^{n_i}\psi$ where $\deg(\psi) \neq 0$, then because k is algebraically closed, all polynomial can be decomposed into the form $\prod_j (x-x_j)$, so f must contain some other factors like $(x - x_j)^{n_j}$, and $\frac{1}{x-x_i} - \frac{1}{x-x_j} = \frac{x_i-x_j}{(x-x_i)(x-x_j)}$ where at the quotient stalk $\mathcal{O}_{x_i,X}$, $\frac{1}{x-x_j} \sim 0$, so all fractional polynomial whose denominator contains $(x - x_i)$ is always equivalent to a fractional polynomial whose denominator only contains factor $(x - x_i)$.

Under this expression, we can easily find that for all elements $(s_x) \in \oplus_{x\in U}K/\mathcal{O}_{x,X} \subset \prod_{x\in U} K/\mathcal{O}_{x,X}$, they must satisfy the compatible condition for quotient sheaf: for finite $\{x_{i_k}\}_{k=0,\cdots,N}$ where (s_{x_i}) vanishes everywhere else, we just take $t = \sum_k s_{x_{i_k}}$ which satisfies that: on every x_{i_k}, take the neighborhood $U_k = X \setminus \{\cup_{j\neq k} x_{i_j}\}$, then $t_v = s_v$ for all $v \in U$.

On the contrary, for any $(s_x) \in \prod_{x\in X} \mathcal{O}_{x,X}$ satisfying the compatible condition, recall that any open subset V on the projective line is exactly the complement of the set of finite points $\{x_i\}_{i\in I}$, so any element $t \in K/\mathcal{O}_X(V)$ is in the form of $\sum_{i\in I} \frac{f_i}{(x-x_i)^{n_i}}$, so we can see that $\oplus_{x\in U}(i_x)_*(K/\mathcal{O}_{x,X})$ is exactly the quotient sheaf.

□

A direct limit of flasque sheaf is flasque[6] (this is not true if we replace the direct sum here with direct product), so $\mathcal{K}/\mathcal{O}_X$ is flasque, and $0 \to \mathcal{O}_X \to \mathcal{K} \to \mathcal{K}/\mathcal{O}_X \to 0$ is exactly the flasque resolution.

It's left to prove that $\Gamma(X,\mathcal{K}) \to \Gamma(X,\mathcal{K}/\mathcal{O}_X) = \oplus_{x\in X} K/\mathcal{O}_{x,X}$[7] is surjective so $H^1(X,\mathcal{O}_X) = 0$. This is done in the same way as the above lemma. Thus $H^i(X,\mathcal{O}_X) = 0, \forall i \geq 1$. □

2.3. Cohomology with Supports. Let X be a topological space, and let Y be a closed subset, and $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y.
(a) Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

Proof. For a short exact sequence of sheaf $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, take the global section functor $\Gamma(X,-)$ leads to the exact sequence $0 \to \Gamma(X,\mathcal{F}_1) \to \Gamma(X,\mathcal{F}_2)\Gamma(X,\mathcal{F}_3)$, when we restrict to the subgroup of sections supported on Y, this sequence is still exact. □

We denote the right derived functors of $\Gamma_Y(X,)$ as $H^i_Y(X,)$. They are the cohomology groups with supports in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact sequence of sheaves, with $\mathcal{F}_1$ flasque, show that $0 \to \Gamma_Y(X,\mathcal{F}_1) \to \Gamma_Y(X,\mathcal{F}_2) \to \Gamma_Y(X,\mathcal{F}_3) \to 0$ is exact.

Proof. This is easily deduced by the case when $Y = X$ by restriction. □

(c) If $\mathcal{F}$ is flasque, then $H^i_Y(X,\mathcal{F}) = 0$ for all $i \geq 1$.

Proof. For a ringed space $(X,\mathcal{O}_X)$, the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$ modules has enough injectives. So we can embed $\mathcal{F}$ into an injective object (which is also flasque) $\mathcal{J}$ of $\mathfrak{Ab}(X)$ and let $\mathcal{G}$ be the quotient.

$$0 \to \mathcal{F} \to \mathcal{J} \to \mathcal{G} \to 0$$

Then the quotient of two flasque sheaves $\mathcal{G}$ is still flasque. We can show $H^i_Y(X,\mathcal{F}) = 0$ by induction: For $H^1(X,\mathcal{F})$, by the exact sequence $0 \to \Gamma_Y(X,\mathcal{F}_1) \to \Gamma_Y(X,\mathcal{F}_2) \to \Gamma_Y(X,\mathcal{F}_3) \to 0$, we can see that this is actually part of the long exact sequence of the cohomology groups with supports on Y, and so $H^1(X,\mathcal{F}) = 0$. And because injective object always has cohomology zero, so by the long exact sequence of derived right functors, $H^{i-1}(X,\mathcal{G}) \cong H^i(X,\mathcal{F})$, and $\mathcal{G}$ is also flasque, so by induction, $H^i = 0$ for all $i \geq 1$. For our case just take $\mathcal{O}_X = Z$.

□

[6]see lemma 2.8, p.209, section 3.2, Algebraic Geometry, Robin Hartshone.
[7]direct limit commutes with cohomology.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X - Y, \mathcal{F}) \to 0$$

Proof. The first exactness is by definition, and for the second exactness, for a section $s \in \Gamma(X, \mathcal{F})$, its restriction to U vanishes if and only if it is supported inside the closed subset Y by definition.

For the last exactness, we shall see that any section on open subset U can be extended to the whole space because the restriction map is surjective by definition.

□

(e) Let $U = X \setminus Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Y(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}_U) \to H^2_Y(X, \mathcal{F}) \to \cdots$$

Proof. By (d), take an injective solution $\mathcal{I}.$, and by (d), after the action of functors $\Gamma_Y(X, -)$, $\Gamma(X, -)$ and $\Gamma(X - Y, -)$, we get an exact sequence of $0 \to \Gamma_Y(X, \mathcal{I}.) \to \Gamma(X, \mathcal{I}.) \to \Gamma(X - Y, \mathcal{I}.) \to 0$, and this leads to a long exact sequence of cohomology groups.

□

(f) For any open subset V containing Y, $H^i_Y(X, \mathcal{F}) = H^i_Y(V, \mathcal{F}_V)$.

Proof. There are two δ functors here: One is $T = \{T^i = R^i\Gamma_Y(X, -)\}$, which is the unique universal δ functor with $T^0 = \Gamma_Y(X, -)$.

On the other hand, there is another δ functor $T' = \{T'^i\}_{i \geq 0} = H^i_Y(V, f^{-1}(-))\}_{i \geq 0}$ also with $T'^0 = T^0$. It suffices to prove that T' is effaceable, so it is universal. And because universal δ functor T with same T^0 is unique (up to unique isomorphism), the proof can be done.

lemma. If $\mathcal{I}$ is flasque on X, then $f^{-1}(\mathcal{I})$ is flasque on V. [8]

Proof. Any open subset of V is also open subset of X, so restriction is also flasque. □

By (c), we can see that for any flasque sheaf $\mathcal{I}$, $H^i(V, \mathcal{I}_V) = 0, \forall i \geq 1$. And for any sheaf $\mathcal{F}$, we can always embed it into a flasque sheaf (injective, more precisely) $j : \mathcal{F} \to \mathcal{I}$, and $T'^i(\mathcal{I}) = H^i(V, \mathcal{I}_V) = 0, \forall i \geq 1$. Thus $T'^i(j) = 0, \forall i \geq 1$. □

2.4. MV sequence. Let Y_1, Y_2 be closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\cdots \to H^i_{Y_1 \cap Y_2}(X, \mathcal{F}) \to H^i_{Y_i}(X, \mathcal{F}) \oplus H^i_{Y_2}(X, \mathcal{F}) \to H^i_{Y_1 \cup Y_2}(X, \mathcal{F}) \to H^{i+1}_{Y_1 \cap Y_2}(X, \mathcal{F}) \cdots$$

Proof. It suffices to prove that for an injective solution $\mathcal{I}_\bullet$ of $\mathcal{F}$, there is an exact sequence of chain complex $0 \to \Gamma_{Y_1 \cap Y_2}(X, \mathcal{I}_\bullet) \to \Gamma_{Y_1}(X, \mathcal{I}_\bullet) \oplus \Gamma_{Y_2}(X, \mathcal{I}_\bullet) \to \Gamma_{Y_1 \cup Y_2}(X, \mathcal{I}_\bullet) \to 0$ with the usual morphisms between them as a MV sequence always does.

lemma : Take the injective resolution $\mathcal{I}_\bullet$ of $\mathcal{F}$ constructed in (2.2)[9], for each $\mathcal{I}$ of this resolution,

[8] this is in general not true. Here is true because $f : V \to X$ is open subset.

[9] see proposition 2.2. p.207, section 3.2, Algebraic Geometry, Robin Hartshone.

there is an exact sequence of abelian groups:

$$0 \to \Gamma_{Y_1 \cap Y_2}(X, \mathcal{I}) \to \Gamma_{Y_1}(X, \mathcal{I}) \oplus \Gamma_{Y_2}(X, \mathcal{I}) \to \Gamma_{Y_1 \cup Y_2}(X, \mathcal{I}) \to 0$$

Proof. The first exactness is obvious by the injectiveness of $\Gamma_{Y_1 \cap Y_2}(X, \mathcal{I}) \to \Gamma_{Y_i}(X, \mathcal{I})$. The image of this natural inclusion into $\Gamma_{Y_1}(X, \mathcal{I}) \oplus \Gamma_{Y_2}(X, \mathcal{I})$ is exactly inside the kernel of morphism $\Gamma_{Y_1}(X, \mathcal{I}) \oplus \Gamma_{Y_2}(X, \mathcal{I}) \xrightarrow{s_1 - s_2} \Gamma_{Y_1 \cup Y_2}(X, \mathcal{I}) \to 0$. And for any (s_1, s_2) inside this kernel, they are exactly sections of $\Gamma(X, \mathcal{I})$ with $s_1 = s_2$ and thus their supports contained inside $Y_1 \cap Y_2$.

Recall how we construct $\mathcal{I}_\bullet$ in (2.2): $\mathcal{I}_0 = \prod_{x \in X} (j_x)_*(I_x)$ where I_x is an injective $\mathbf{Z}$ module containing $\mathcal{F}_x$ and $j_x : \{x\} \to X$. Thus it suffices to prove that $\Gamma_{Y_1}(X, \prod_{x \in X} (j_x)_*(I_x)) \oplus \Gamma_{Y_2}(X, \prod_{x \in X} j_*(I_x)) \to \Gamma_{Y_1 \cup Y_2}(X, \prod_{x \in X} j_*(I_x)) \to 0$.

For the ordinary global section space of $\mathcal{O}_X$ module of sheaf $\mathcal{I} = \prod_{x \in X} (j_x)_*(I_x)$, there is a canonical isomorphism $\Gamma(X, \mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \cong \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, (j_x)_* \mathcal{I}_x) \cong$ [10] $\prod_{x \in X} \mathcal{I}_x$. Under this isomorphism, section with support on Y is exactly the elements $s = (s_x) \in \prod_{x \in X} \mathcal{I}_x$ such that $s_x = 0$ for $x \notin X \setminus Y$. Thus $\Gamma_Y(X, \mathcal{I}) = \prod_{x \in Y} \mathcal{I}_x$.

And now it suffices to prove that $\Gamma_{Y_1}(X, \prod_{x \in X} (j_x)_*(I_x)) \oplus \Gamma_{Y_2}(X, \prod_{x \in X} j_*(I_x)) = \prod_{x \in Y_1} \mathcal{I}_x \oplus \prod_{x \in Y_2} \mathcal{I}_x \to \prod_{x \in Y_1 \cup Y_2} \mathcal{I}_x = \Gamma_{Y_1 \cup Y_2}(X, \prod_{x \in X} j_*(I_x))$. This is obvious by observing the components of the product. □

It's trivial to see that this short exact sequence above constitute an exact sequence of chain complex $\mathcal{I}_\bullet$. By basic homological algebra, this leads to the long exact sequence of cohomology groups. □

Remark. In our proof of the lemma, we only prove that the global section sequence is exact for sheaf constructed in the form of (2.2)–direct product of skyscraper sheaves. But for all the flasque sheaf $\mathcal{F}$, this conclusion is still correct: $H_Y^{i \geq 1}(X, \mathcal{F}) = 0$ and take this into the MV sequence.

2.5. Zaraski Space. Let X be a Zaraski (ringed) space[11]. Let $P \in X$ be a closed point, and let X_p be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the local space of X at P, and give it the induced topology. Let $j : X_P \to X$ be the inclusion, and for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_P = j^* \mathcal{F}$. Show that for all $i, \mathcal{F}$, we have

$$H_P^i(X, \mathcal{F}) = H_P^i(X_P, \mathcal{F}_P)$$

Proof. For any sheaf $\mathcal{F}$, take an injective (so also flasque) resolution $\mathcal{I}_\bullet$ of it, then:

lemma 1: The induced sequence $j^* \mathcal{I}_\bullet$ is a $\Gamma_P(X_P, -)$ acyclic resolution for $j^* \mathcal{F}$.

Proof. First we prove it is a resolution: The sequence of sheaf on X: $0 \to \mathcal{F} \to \mathcal{I}_\bullet$ is exact. But in general, pullback is not exact (but only right exact), but in our case, pullback along the infinitesimal neighborhood X_P of P is exact: Indeed, $j^* \mathcal{F} = \mathcal{F}_P = j^{-1} \mathcal{F} \otimes_{j^{-1} \mathcal{O}_X} j^{-1} \mathcal{O}_X = j^{-1} \mathcal{F} =$

[10] By the canonical isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X = \pi^* \mathbf{Z}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \pi_* \mathcal{F})$ where $\pi : X \to *$ is the projection to a singleton with constant sheaf $\mathbf{Z}$. Thus here we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, (j_x)_* \mathcal{I}_x) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \pi_*(j_x)_* \mathcal{I}_x \cong \mathcal{I}_x) \cong \mathcal{I}_x$.

[11] The definition of Zaraski only requires it to be a topological space, that is, ringed space equipped with $\mathbf{Z}$ constant sheaf., But here we can equip X with nontrivial structure sheaf $\mathcal{O}_X$.

$\mathcal{F}|_{X_P}$ with the same stalk as $\mathcal{F}$ on X_P, so by observing stalks, we can see that the restriction to X_P keeps the exactness of the resolution.

It's left to prove that this resolution is acyclic: $\mathcal{I}_\bullet$ is injective resolution, so it suffices to prove that for an injective sheaf $\mathcal{I}$ on X, $H_P^{i\geq 1}(X_P, \mathcal{I}|_{X_P}) = 0$.

As 'infinitesimal neighborhood', $\Gamma(X_P, \mathcal{I}|_{X_P}) = \lim_{P\in U} \Gamma(U, \mathcal{I})$ by definition that any open subset containing P contains X_P. Thus on X_P, sheaf is always flasque. So by exercise 2.2 (c) above, it is $\Gamma_P(X_P, -)$ acyclic.

□

By restriction map, there is a natural morphism of global section space $\Gamma_P(X, \mathcal{F}) \to \Gamma_P(X_P, \mathcal{F}|_{X_P})$. It suffices to prove that this morphism is isomorphism for any sheaf $\mathcal{F}$.

As $\Gamma(X_P, \mathcal{I}|_{X_P}) = \lim_{P\in U} \Gamma(U, \mathcal{I})$, so for any section $s \in \Gamma_P(X_P, \mathcal{F}|_{X_P})$, we can pick a representative $s_U \in \Gamma(U, \mathcal{F})$.

lemma 2: For any representative $s_U \in \Gamma(U, \mathcal{F})$ of $s \in \Gamma_P(X_P, \mathcal{F}|_{X_P}) \subset \Gamma(X_P, \mathcal{F}|_{X_P}) = \lim_{x\in U} \Gamma(U, \mathcal{F})$, $\mathrm{supp}(s_U) \cap X_P = P$.

Proof. It's important to tell the difference of the section $s \in \Gamma(X_P, \mathcal{F}|_{X_P})$ and section s_U in $\Gamma(U, \mathcal{F})$: they take stalks in different manners.

By definition of inverse image sheaf, $\forall Q \neq P \in X_P$, $s(Q) = \lim_{Q\in V'\cap X_P} \Gamma(V' \cap X_P, \mathcal{F}_P) = \lim_{Q\in V'\cap X_P} \lim_{V'\cap X_P \subset U'} \Gamma(U', \mathcal{F})$ while $s_U(Q) = \lim_{Q\in V} \Gamma(V, \mathcal{F})$. So in general, there is no equality between these two sections' stalks.

But there is a factorization here (for the proof of this diagram, see the lemma 3):

$$\begin{tikzcd}
s_U \in \Gamma(U, \mathcal{F}) \arrow[rr, "\text{stalk at Q of } \mathcal{F}"] \arrow[d, "\text{stalk at P}"'] & & s_U(Q) \in \lim_{Q\in V} \Gamma(V, \mathcal{F}) \\
s \in \Gamma(X_P, \mathcal{F}_P) = \lim_{P\in U} \Gamma(U, \mathcal{F}) \arrow[r, "\text{stalk at Q of } \mathcal{F}_P"] & \lim_{Q\in V'\cap X_P} \Gamma(V' \cap X_P, \mathcal{F}_P) \arrow[ur, "\psi"] &
\end{tikzcd}$$

Here especially we take $s \in \Gamma_P(X_P, \mathcal{F}|_{X_P})$, so $s(Q)$=0. Then by the above commutative diagrams, the image of s_U at the stalk of Q vanishes, which means s_U takes value zero at all $Q \in X_P$. This only works for $Q \in X_P$ because for $Q \notin X_P$, there is no such diagram.

□

By shrinking properly, we may assume that $\mathrm{supp}(s_U) = P$: this can always be done because we can find another representative s'_V such that $s'_V(y) = 0$ because the limit is only supported at P, then take $U' = U \cap V$ and this satisfies our condition.

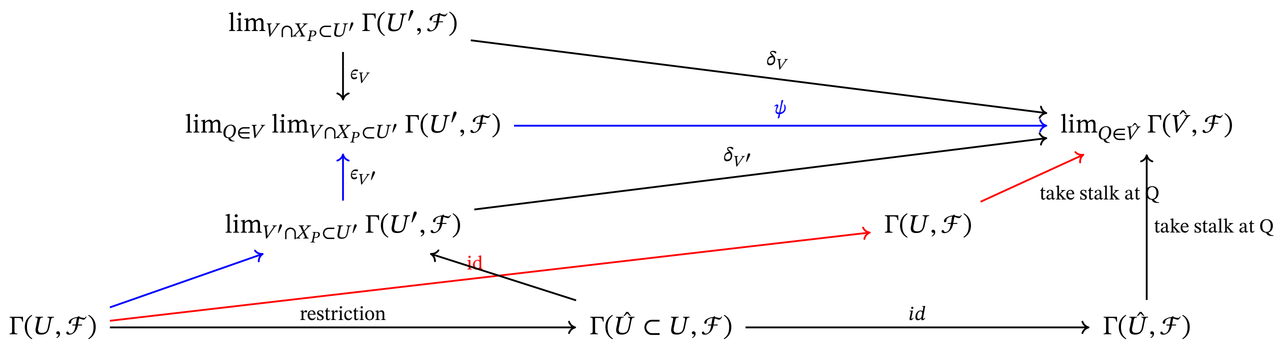
Then for any open subset $W = X \setminus P$, define $s = 0 \in \Gamma(W, \mathcal{F})$ and this is compatible with s_U, thus we get an element $\hat{s} \in \Gamma_P(X, \mathcal{F})$. This is an inverse of the natural restriction $\Gamma_P(X, \mathcal{F}) \to \Gamma_P(X_P, \mathcal{F}_P)$.

In this way, we can see that the natural restriction $\Gamma_P(X, \mathcal{F}) \to \Gamma_P(X_P, \mathcal{F}_P)$ is isomorphism. And this isomorphism is obviously natural, so we get an isomorphism of chain complex $\Gamma_P(X, \mathcal{I}_\bullet) \cong \Gamma_P(X_P, (\mathcal{I}_P)_\bullet)$, leading to isomorphism of cohomology groups.

$\square$

lemma 3: There is a morphism between limits satisfying the commutative diagram in the lemma above: $\psi : \lim_{Q \in V} \lim_{V \cap X_P \subset U'} \Gamma(U', \mathcal{F}) \to \lim_{Q \in V} \Gamma(V, \mathcal{F})$.

Proof. By the following diagram: the red line is the composition of identity and mapping to the limit, an element $s_U \in \Gamma(U, \mathcal{F})$ going through this is exactly taking stalk at Q.



$\square$

2.6. Let X be a noetherian space, and let $\{\mathcal{J}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaf of abelian group on X. Prove that $\lim_{\alpha \in A} \mathcal{J}_\alpha$ is also injective.

Proof.
lemma 1: On arbitrary space X, $\mathcal{F}$ is injective if and only if for any open subset $U \subset X$, and for any sub-sheaf $\mathcal{R} \subset \mathbf{Z}_U$, and for every map $\mathcal{R} \to \mathcal{F}$, there exists an extension of f to a map $\mathbf{Z}_U \to \mathcal{F}$.

Proof. It suffices to prove that the local extension leads to injective object, the other direction is obvious. Equivalently, we need to prove that for any injective embedding $\mathcal{R} \subset \mathcal{G}$, and any morphism $f : \mathcal{R} \to \mathcal{F}$, there is an extension of f to $\mathcal{G} \to \mathcal{F}$. Consider the maximal sheaf $\mathcal{H}$ such that $\mathcal{R} \subset \mathcal{H} \subset \mathcal{G}$ and f can be extended to $\mathcal{H}$, it suffices to prove that $\mathcal{H} = \mathcal{G}$.

Otherwise, there exists an open subset $V \subset X$ such that $s \in \mathcal{G}(V) \setminus \mathcal{H}(V)$. Thus we can define a sheaf morphism $\mathbf{Z}_V \xrightarrow{\cdot s} \mathcal{G}_V \to \mathcal{G}$, denoted as s.

$$\begin{tikzcd}
\mathcal{H} \times_{\mathcal{G}} \mathbf{Z}_V \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H} \arrow[d, "i"] \\
\mathbf{Z}_V \arrow[r, "s"] & \mathcal{G}
\end{tikzcd}$$

Consider the pullback diagram, then we claim: **monomorphism is stable under base change.** More precisely, in the category of sheaf of abelian groups, there is an equivalence between injective morphism and monomorphism. Thus, by the fact that $i : \mathcal{H} \to \mathcal{G}$ is injective, we have $p_1 : \mathcal{H} \times_{\mathcal{G}_V} \mathbf{Z}_V \to \mathbf{Z}_V$ is also injective. And by the morphism $\hat{f} : \mathcal{H} \to \mathcal{F}$ composed with p_2 induces a morphism extension $\phi_1 : \mathbf{Z}_V \to \mathcal{F}$.

By lemma 2 below, we know that in the abelian category, the pullback diagram above is equivalent to a left exact sequence: $0 \to \mathcal{H} \times_{\mathcal{G}} \mathbf{Z}_V \to \mathcal{H} \times \mathbf{Z}_V = \mathcal{H} \oplus \mathbf{Z}_V \xrightarrow{(i,s)} \mathcal{G}$.

By the extension ϕ_1 constructed above, we have the diagram:

$$\begin{tikzcd}
\mathcal{H} \times_{\mathcal{G}} \mathbf{Z}_V \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H} \arrow[d, "i"] \arrow[ddr, "\hat{f}"] & \\
\mathbf{Z}_V \arrow[r, "\cdot s"] \arrow[drr, "\phi_1"'] & \mathcal{G} & \\
& & \mathcal{F}
\end{tikzcd}$$

If this is also a pushout diagram, then there is an induced morphism from $\mathcal{G} \to \mathcal{F}$, which is beyond our expectations. However, we can't verify this at this stage. But again by the lemma 2 below, we know that in order to get a pullback diagram in the same time also a pushout diagram, the sequence corresponding to it should be both left exact and right exact. Take the cokernel of the sequence $0 \to \mathcal{H} \times_{\mathcal{G}} \mathbf{Z}_V \to \mathcal{H} \times \mathbf{Z}_V = \mathcal{H} \oplus \mathbf{Z}_V \xrightarrow{(i,s)} \mathcal{G}$, we have an exact sequence:

$$0 \to \mathcal{H} \times_{\mathcal{G}} \mathbf{Z}_V \to \mathcal{H} \times \mathbf{Z}_V = \mathcal{H} \oplus \mathbf{Z}_V \xrightarrow{(i,s)} \mathrm{im}((i,s)) \to 0$$

This exact sequence means there is a pullback in the same time a pushout diagram, also leading to an extension of f to the sub-sheaf $\mathrm{im}((i,s))$.

$$\begin{tikzcd}
\mathcal{H} \times \mathbf{Z}_V \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H} \arrow[d, "i"] \arrow[ddr, "\hat{f}"] & \\
\mathbf{Z}_V \arrow[r, "s"] \arrow[drr, "\phi_1"'] & \mathrm{im}((i,s)) \arrow[dr, dotted, "\tilde{f}"] & \\
& & \mathcal{F}
\end{tikzcd}$$

Here the image sheaf (which usually requires sheafitication), is a sub-sheaf containing both $\mathcal{H}$ and section s locally on V, which is bigger than $\mathcal{H}$, which is a contradiction! So $\mathcal{H} = \mathcal{G}$. The object $\mathcal{F}$ is injective.

$\square$

lemma 2: On a noetherian space X, any sub-sheaf $\mathcal{R} \subset \mathbf{Z}_X$ is finitely generated.

Proof. By definition, $\mathcal{Z}$ is the sheaf of locally continuous functions to $\mathbf{Z}$, which is constant sheaf on each of its connected components.

□

□

tips on injective object
Here are som properties we may use in the latter occasions about injective objects: Let Q be an injective object in $\mathcal{C}$, then
(1). For any Y and its subobject $X \subset Y$, morphism from X to Q extends (not uniquely) to the morphism from Y to X.
(2). $\mathrm{Hom}(-, Q)$ takes monomorphism to epimorphism if $\mathcal{C}$ is locally small, and $\mathrm{Hom}(-, Q)$ is and exact functor if $\mathcal{C}$ is abelian category.
(3). When $\mathcal{C}$ is abelian category, then the exact sequence $0 \to Q \to X \to Y \to 0$ splits.

2.2 Exercises in 3.3

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} is affine.

Proof. If X is affine, then $X = \mathrm{Spec}(A)$ and $X_{red} = \mathrm{Spec}(A/nil(A))$ which is again affine.
If X_{red} is affine, then denote the sheaf of nilpotent elements as $\mathcal{N}$, which is a quasi-coherent ideal sheaf on X[1]. Because X is noetherian, so all quasi-coherent ideal sheaf is coherent ideal sheaf and by picking a finite generators, there exists $m \in \mathbb{Z}$ such that $\mathcal{N}^m = 0$. Thus for any coherent sheaf $\mathcal{F}$, there exists a filtration of coherent sheaf $\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \cdots \supset \mathcal{N}^m \cdot \mathcal{F} = 0$. And for each quotient $(\mathcal{N}^{i+1} \cdot \mathcal{F})/(\mathcal{N}^i \cdot \mathcal{F}) \cong (\mathcal{N}^{i+1}/\mathcal{N}^i) \cdot \mathcal{F} = \mathcal{N}^{i+1}/\mathcal{N}^i \cdot \mathcal{F}$, it can be viewed as $\mathcal{O}_X/\mathcal{N}$ module sheaf[2] and also coherent sheaf on X_{red}.
Denote the closed immersion of X_{red} as $j : X_{red} \to X$, and consider the exact sequence $0 \to \mathcal{N}^m \cdot \mathcal{F} \to \mathcal{N}^{m-1} \cdot \mathcal{F} \to \mathcal{N}^m/\mathcal{N}^{m-1} \cdot \mathcal{F} \to 0$. As shown above, $j_* j^*(\mathcal{N}^m/\mathcal{N}^{m-1} \cdot \mathcal{F}) \cong \mathcal{F}$, so $0 = H^i(X_{red}, j^*(\mathcal{N}^m/\mathcal{N}^{m-1} \cdot \mathcal{F})) = H^i(X, \mathcal{N}^m/\mathcal{N}^{m-1} \cdot \mathcal{F})$ by Serre criterion.
Thus by the long exact sequence of cohomology groups, we have $H^1(X, \mathcal{N}^m \cdot \mathcal{F}) \to H^1(X, \mathcal{N}^{m-1} \cdot \mathcal{F}) \to 0$ is exact, while $\mathcal{N}^m = 0$, so $H^1(X, \mathcal{N}^{m-1} \cdot \mathcal{F}) = 0$. By induction, we shall have $H^1(X, \mathcal{F}) \to H^1(X, \mathcal{N} \cdot \mathcal{F}) = 0$ is surjective and thus $H^1(X, \mathcal{F}) = 0$. By Serre criterion, X is affine.

□

3.2. Let X be a noetherian reduced scheme. Show X is affine if and only if each irreducible component is affine.

Proof. If X is affine, then its irreducible component is affine corresponding to some minimal prime ideal $V(\mathfrak{p})$.
If X has all its irreducible components affine, denoted as $X = \cup_{i=1}^N Y_i$ and their corresponding ideal sheaf as $\mathcal{I}_i$. Then $0 = \mathrm{nil}(\mathcal{O}_X) = \cap_i \mathcal{I}_i$[3] because X is reduced.
For any coherent ideal sheaf $\mathcal{F}$ on X, consider the filtration of coherent sheaf $\mathcal{F} \supset \mathcal{I}_1 \cdot \mathcal{F} \supset \cdots \mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_N \mathcal{F} = 0$, denote $\mathcal{G}_i = \mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_i \mathcal{F}$.
Consider the long exact sequence of cohomology groups for $0 \to \mathcal{G}_i \to \mathcal{G}_{i-1} \to \mathcal{G}_{i-1}/\mathcal{G}_i \to 0$: $H^1(X, \mathcal{G}_i) \to H^1(X, \mathcal{G}_{i-1}) \to H^1(X, \mathcal{G}_{i-1}/\mathcal{G}_i)$.
In the same way as exercise 3.1 above, $\mathcal{G}_{i-1}/\mathcal{G}_i$ can be viewed as $\mathcal{O}_X/\mathcal{I}_i$ module sheaf. Thus

[1] Because $\mathrm{nil}(S^{-1}A) = S^{-1}\mathrm{nil}(A)$.
[2] when we say a module (sheaf) M over A can be viewed as a A/I module, this means $M \otimes A/I \cong M \otimes A \cong M$.
[3] The pullback (or more generally, limit) of quasi-coherent sheaf to some open subsets is again pullback diagram, thus we have $\mathrm{nil}(\mathcal{O}_X) = \cap_i \mathcal{I}_i$ on every open affine subset, and this leads to isomorphism as sheaf on X.

$\mathrm{supp}(\mathcal{G}_{i-1}/\mathcal{G}_i) \subset V(\mathcal{I}_i)$, and the canonical morphism $\mathcal{G}_{i-1}/\mathcal{G}_i \to j_*j^*(\mathcal{G}_{i-1}/\mathcal{G}_i)$ is isomorphism along the closed immersion $j : V(\mathcal{I}_i) \to X$. Thus $H^1(X, \mathcal{G}_{i-1}/\mathcal{G}_i) \cong H^1(V(\mathcal{I}_j), j^*(\mathcal{G}_{i-1}/\mathcal{G}_i)) = 0$ by Serre criterion. Thus $H^1(X, \mathcal{G}_i) \to H^1(X, \mathcal{G}_{i-1}) \to 0$ is surjective, by induction, $H^1(X, \mathcal{G}_N = 0) = 0 \to H^1(X, \mathcal{G}_0 = \mathcal{F})$ is surjective, thus the target is zero.

□

Remark: pullback along open subscheme
About the proof of $\mathrm{nil}(\mathcal{O}_X) = \cap \mathcal{I}_i$, see appendix D: recollement theorem.

Remark: Serre criterion for affineness
In Hartshone's book, we always assumes that the scheme is 'noetherian', but in many cases, the conclusion can be generalized to qcqs schemes. Here is a generalized version of Serre criterion:
'if' part: for any affine scheme X, and any quasi-coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$.[4]
But this uses the Céch cohomology, which is not introduced in Hartshone until section 3.4. And the noetherian case can be solved only by using definition of right derived functors.

'only if' part: let X be qcqs scheme, and the following conditions are equivalent:
(a). X is affine.
(b). $H^i(X, \mathcal{F}) = 0, \forall i \geq 1$ for all quasi-coherent sheaf $\mathcal{F}$.
(c). $H^1(X, \mathcal{F}) = 0, \forall i \geq 1$ for all quasi-coherent sheaf $\mathcal{F}$.

There is a more general result in Stacks Project, or see the appendix D: affine criterion.

Remark: Grothendieck vanishing theorem
This may be a little irrelevant here, but I just want to give another example for the unencessity of 'noetherian' condition:
theorem:[5] T be a spectral space, for $i > \dim(T)$, then $H^i(T, A) = 0$ for any sheaf of abelian groups A on T.
Especially, we have the conclusion, $\dim(T) = 0$ if and only if T is a profinite set, i.e. $T \cong \lim_i T_i$ for some direct system.

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.
(a) Show that $\Gamma_a(\cdot)$ is a left exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H_{\mathfrak{a}}(\cdot)$.

Proof. $\Gamma_{\mathfrak{a}}(M) = \{m \in M : \mathfrak{a}^n m = 0, \exists n > 0\}$ is just a submodule of M, thus it's obvious to be left exact. □

(b) Now let $X = \mathrm{Spec}(A)$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M})$$

where H^*_Y is the cohomology with supports in Y.

[4] see p.335, proposition 12.32 and its remarks, Algebraic Geometry I Schemes With Examples and Exercises, Ulrich Gortz & Torsten Wedhorn, 1st edition 2010.
[5] see https://stacks.math.columbia.edu/tag/0A3G.

Proof.
lemma : $\Gamma_V(X, \widetilde{M}) \cong \Gamma_{\mathfrak{a}}(M)$

Proof. As A is noetherian, $\mathfrak{a}$ is finitely generated, denoted as $\mathfrak{a} = (a_1, \cdots, a_n)$ where $a_i \neq 0$. Then $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_V(X, \widetilde{M}) : \forall s \in \Gamma_{\mathfrak{a}}(M)$, then for any $\mathfrak{p} \notin V(\mathfrak{a}) = (a_1, \cdots, a_n)$, there exists $a_i \notin \mathfrak{p}$, which is a unit in $A_{\mathfrak{p}}$, so $s \in \Gamma(X, \widetilde{M}) = M \to 0 \in \widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ because $a_i^n m = 0$.
On the other hand, $\forall t \in \Gamma_V(X, \widetilde{M})$, then $\operatorname{supp}(t) = V(\operatorname{Ann}(t)) \subset V$ by definition, so $\sqrt{\operatorname{Ann}(t)} \supset \sqrt{\mathfrak{a}}$. Thus there exists M such that $\mathfrak{a}^M \subset \operatorname{Ann}(t)$, and $\mathfrak{a}^M t = 0$. Thus $\Gamma_V(X, \widetilde{M}) = \Gamma_{\mathfrak{a}}(M)$. □

For $M \in \mathfrak{Mod}(A)$, the injective resolution for M is $M \to \mathcal{I}_\bullet$. Then by action of left exact functor $\Gamma_V(X, \widetilde{-}) \cong \Gamma_{\mathfrak{a}}(-)$, leading to the same cohomology groups.

□

(c) For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

Proof. This is obvious because $H^i_{\mathfrak{a}}(M)$ is the quotient of submodule of $\Gamma_{\mathfrak{a}}(-)$, which can also be annihilated by some finite powers of $\mathfrak{a}$. □

Remark: So $H^i_{\mathfrak{a}}(M)$ is injective object in $\mathfrak{Mod}(A)$
Q: Is $\Gamma_{\mathfrak{a}}(-)$ adjoint to the natural embedding $i : Inj\mathfrak{Mod}(A) \to \mathfrak{Mod}(A)$? Or more generally, how to describe the reflective subcategory of $\mathfrak{Mod}(A)$?

3.4. Cohomological interpretation of depth. If A is a ring, and $\mathfrak{a}$ is an ideal, and M and A module, then $\operatorname{depth}_{\mathfrak{a}} M$ is the maximal length of an M-regular sequence $x_1, \cdots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in II.8.
(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}}(M) \geq 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated.

Proof. If $\operatorname{depth}_{\mathfrak{a}}(M) \geq 1$, then there exists $x_1 \in \mathfrak{a}$ which is not a zero divisor for M, so is x_1^N for any N. Thus $\mathfrak{a}^N$ can never annihilate the any M except $0 \in M$.

On the other hand, if $\Gamma_{\mathfrak{a}}(M) = 0$, it remains to prove that when M is finitely generated, $\operatorname{depth}_{\mathfrak{a}} \geq 1$. Indeed, there is close relation between annihilator and zero divisors of a module, here are several lemmas illustrating it:(see [3, section 3.1, p.89-94])

lemma 1(existence theorem):[6] For a noetherian ring A and some nonzero module M over A, $\operatorname{Ass}(M) \neq \emptyset$. More generally, If I is a maximal ideal in the set of ideals of A that are annihilators of some element in M (such maximal element always exists when A is noetherian), then I is a prime ideal.

Proof. If $rs \in I = \operatorname{ann}(m)$ for some $0 \neq m \in M$, and $s \notin I$, it suffices to prove that $r \in I$. $rs \cdot m = 0$, but $s \cdot m \neq 0$, so $(I, r) \subset \operatorname{ann}(s \cdot m)$, by the maximality, $(I, r) \subset I$, thus $r \in I$. □

[6]see [3, proposition 3.4, p.91].

Attention that in lemma 1, the existence theorem doesn't require that M is finitely generated, but A should be noetherian.

lemma 2(finiteness theorem): When A is noetherian and M is finitely generated A module, $\mathrm{Ass}(M)$ is finite set.

Proof. To prove this, we need to first prove the following finite filtration theorem for A module under the same condition required above:

lemma 3(filtration): For the above A module M. there is a filtration:

$$0 = M_0 \subset M_1 \subset M_2 \cdots M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \subset A$.

Proof. By lemma 1 above, if M is nonzero, then there exists $\mathfrak{p} = \mathrm{ann}(m) \in \mathrm{Ass}(M)$ for some $m \in M$, then there is a submodule $M_1 \cong A/\mathfrak{p}_1 \subset M$ as the image of the morphism $A \xrightarrow{\cdot m} M$.

For M/M_1, if it is zero, then the proof is done, otherwise, there exists a submodule in the form of $M_2/M_1 \subset M/M_1$ for some M submodule M_2 such that $M_1 \subset M_2 \subset M$: and $(M/M_1)/(M/M_2) \cong (M_2/M_1) \cong A/\mathfrak{p}_2$ for some prime ideal $\mathfrak{p}_2$ (which may not be in $\mathrm{Ass}(M)$)

As A is noetherian, and M is finitely generated, so M is noetherian. Thus the ascending chain of submodules stabilizes in finite steps, where there is some n: $M_n = M$.

$\square$

Back to the proof of lemma 2, A module $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ is special in that it is an integral domain, so for any element $a \notin \mathfrak{p}$, $a \cdot m \neq 0 \in A/\mathfrak{p}$ unless $m = 0 \in A/\mathfrak{p}$. Thus the only prime ideal associated with $A/\mathfrak{p}$ is $\mathfrak{p}$. Consider the exact sequence: $0 \to M_{i-1} \to M_i \to (M_i/M_{i-1}) \cong A/\mathfrak{p} \to 0$, then $\mathrm{Ass}(M_i) \subset \mathrm{Ass}(M_{i-1}) \cup \mathrm{Ass}(A/\mathfrak{p}_i)$ where $\mathrm{Ass}(M_0 = 0) = \emptyset$. Thus $\mathrm{Ass}(M) \subset \{\mathfrak{p}_i\}_i$ which is a finite set because of the finiteness of the filtration. $\square$

lemma 4(zero divisor and associate prime): For noetherian ring A and finitely generated A module M, the union of prime ideals in $\mathrm{Ass}(M)$ is exactly the 0 and all the zero divisors of M. Especially, when M is finitely generated, and $\mathfrak{a}$ is an ideal consisting of 0 and zero divisors, then $\mathfrak{a}$ annihilates some element $m \in M$.

Proof. By the above lemma, any zero divisor a such that $a \cdot m = 0$ for some $m \in M$ is contained in some maximal ideal as annihilator of some element in M, which is a prime annihilator, lies in $\mathrm{Ass}(M)$, also the union of those prime ideals. On the other hand, by definition, any element in the prime ideals of $\mathrm{Ass}(M)$ is a zero divisor or 0.

When M is finitely generated, $\mathrm{Ass}(M)$ is a finite set. And If $\mathfrak{a} \subset \cup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$, then by the prime avoidance lemma (because $\mathrm{Ass}(M)$ is finite), $\mathfrak{a}$ is contained certain prime ideals in $\mathrm{Ass}(M)$, annihilating certain $0 \neq m \in M$. $\square$

By the lemma 4, if $\operatorname{depth}_{\mathfrak{a}}(M) = 0$, then there exists nonzero m annihilated by $\mathfrak{a}$, thus $\Gamma_{\mathfrak{a}}(M) \neq 0$, which is a contradiction!

When $M = (m)$ can be generated by a single element, denote $\mathfrak{a} = (a_1, \cdots, a_r)$, by the condition, there is no zero divisor for M in $\mathfrak{a}$, thus for each a_i, there exists $m_i \in M$ such that $a_i m_i = 0$. As M is cyclic, thus $M \cong R/I$ for some ideal $I \subset R$, then $\prod m_i \in \Gamma_{\mathfrak{a}}(M)$. □

Remark: Here are some further remarks on the $\operatorname{Ass}(M)$ and $\operatorname{supp}(M)$:

lemma 5(support and annihilator): If A is any ring and M is a finitely generated A module, then $\operatorname{supp}(M) = V(\operatorname{Ann}(M))$.

Proof. One direction is obvious: $\operatorname{supp}(M) \subset V(\operatorname{Ann}(M))$ is true for any A module.
On the other hand, if M is finitely generated, and $\operatorname{supp}(M) \not\supset V(\operatorname{Ann}(M))$, assume that $M = (m_1, \cdots, m_n)$: if $M_{\mathfrak{p}} = 0$ for some prime ideal $\mathfrak{p} \in V(\operatorname{Ann}(M))$, as $M_{\mathfrak{p}} = \operatorname{colimit}_{f \notin \mathfrak{p}} M_f = 0$, thus for each m_i, there exists $f_i \notin \mathfrak{p}$, such that $m_i = 0 \in M_{f_i}$ and $m_i f_i^{n_i} = 0$ for some n_i, thus there exists $F \notin \mathfrak{p}$ and $N \in \mathbb{Z}$, $F^N m_i = 0$ for all i, thus $F \in \sqrt{\operatorname{Ann}(M)} \subset \mathfrak{p}$, which is a contradiction! □

lemma 6(associate prime ideals and support): If A is noetherian and M is A module (may not be finitely generated), then $m \in M$ is zero if and only if $m \in M \to 0 \in M_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in \operatorname{Ass}(M)$.

Proof. If A is noetherian, then for any $n \neq 0$, there exists a prime maximal ideal $\mathfrak{p}$ among the annihilators of elements of M containing $\operatorname{Ann}(m)$, thus the annihilator is contained some associated prime ideal of M, especially, contained in some maximal associated prime ideal in $\operatorname{Ass}(M)$, and $m/1 \neq 0 \in M_{\mathfrak{p}}$.

Thus whether an element $m \in M$ is zero depends on the associated prime ideals of M, more precisely, the maximal associated prime ideals of M. □

(b). Prove by induction that, when M is finitely generated, that for any $n \geq 0$, the following conditions are equivalent:
(i). $\operatorname{depth}_{\mathfrak{a}}(M) \geq n$.
(ii). $H^i_{\mathfrak{a}}(M) = 0$ for $i < n$.

Proof. Recall the conclusion (c) in exercise 3.3 above: $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$, which means every element $h \in H^i_{\mathfrak{a}}(M)$ is annihilated by some power of $\mathfrak{a}$.

For $n = 1$, it is done in (a), assuming that the conclusion for n is also true, then for $\operatorname{depth}_{\mathfrak{a}}(M) \neq n + 1$, $H^i_{\mathfrak{a}}(M) = 0$ for $i \geq n - 1$ by induction, it remains to prove the n th cohomology group vanishing. Denote the regular sequence of M as $a_1, \cdots, a_{n+1} \in \mathfrak{a}$, and consider the exact sequence $0 \to (a_1 M \cong) M \xrightarrow{\cdot a_1} M \to M/a_1 M \to 0$ and the long exact sequence of cohomology groups $H^*_{\mathfrak{a}}(-)$:

$$\cdots H^{n-1}_{\mathfrak{a}}(M/a_1M) \to H^n_{\mathfrak{a}}(M) \xrightarrow{\text{induced by } \cdot a_1} H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M/a_1M) \to \cdots$$

By induction, $H^{n-1}_{\mathfrak{a}}(M/a_1M) = 0$ and $H^n_{\mathfrak{a}}(M) \xrightarrow{\text{induced by} \cdot a_1} H^n_{\mathfrak{a}}(M)$ is injective. It remains to explore what this morphism induced by $\cdot a_1$ is for $H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$?

Here it's very important to transform the exact sequence $0 \to a_1M \xrightarrow{\text{inclusion}} M \to M/a_1M \to 0$ into $0 \to M \xrightarrow{\cdot a_1} M \to M/a_1M \to 0$ because this equivalence uses the condition that a_1 is regular and more importantly, we have no ideal of the relationship between the injective resolution of an object M and the injective resolution of it subobject in general (of course there is a morphism of chain complexes induced by the natural inclusion $a_1M \to M$, but there is no explicit description of it on each component), but now we regard the a_1M as M and the inclusion as multiplication with a_1, then the injective solution for both source and target is the same and the induced morphism between chain complexes is easily described as below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathcal{I}_1 & \longrightarrow & \mathcal{I}_2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_1} & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathcal{I}_1 & \longrightarrow & \mathcal{I}_2 & \longrightarrow & \cdots \end{array}$$

Thus the morphism between the cohomology groups $H^*_{\mathfrak{a}}(M)$ is induced also by multiplication with a_1, which can't be injective unless $H^*_{\mathfrak{a}}(M) = 0$ because otherwise: $H^i_{\mathfrak{a}}(M) \neq 0$, by what is discussed above, $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$, thus for any $0 \neq h \in H^i_{\mathfrak{a}}(M)$, there exists some m such that $x_1^{m-1}h \neq 0$ but $x^m h = 0$, the kernel is nontrivial, contradiction! Thus $H^n_{\mathfrak{a}}(M) = 0$.

Now turn to the other direction, for $n = 1$, if $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M) = 0$, then by (a) again, $\mathrm{depth}_{\mathfrak{a}}(M) \geq 1$. If $H^i_{\mathfrak{a}}(M) = 0$ for $i < n+1$, by induction for the n case, there exists at least a regular sequence inside $\mathfrak{a}$ of n length for M, pick any such a_1 regular element, then by the above long exact sequence, we can see that $H^i_{\mathfrak{a}}(M/a_1M) = 0$ for all $i < n$, thus by induction, there exists regular sequence inside $\mathfrak{a}$ of length n, combining with a_1, there is a regular sequence of length $n+1$ for M. □

Remark 1: The depth defined here is the 'regular sequence of maximal length' of M, which depends on the order of the sequence, (independent when A is noetherian local ring), this is a rather rough number, thus our condition for depth is $\geq$ instead of $=$. And also, we choose x_1 to do the quotient, because in this way, $x_2, \cdots$ is again a regular sequence. But in most of the cases we care about, the regularity of a sequence doesn't depend on the order and the depth is exactly the least number i such that certain cohomology (Ext^i) stops vanishing.

Remark 2: local cohomology and depth of a coherent sheaf
If X is locally noetherian and Y is a closed subset, the following conditions are equivalent:
(a). $\underline{\mathrm{depth}_Y(\mathcal{F})} = \inf_{x \in Y} depth(\mathcal{F}_x) \geq n$
(b). $\underline{H^i_Y(X, \mathcal{F})} = 0, \forall i < n$.[7]

Remark 3: For a noetherian ring A, we have the Grothendieck vanishing theorem: $H^i(\mathrm{Spec}(A), F) = 0, \forall i > \dim(A)$, this is a global relationship between dimension of a ring and the cohomology vanishing condition. Here the depth provides a local view of this relation.

[7] see [6, chapter 3, p44]

3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:
(1). $\operatorname{depth}(\mathcal{O}_{X,p}) \geq 2$.
(2). For any open subset $U \ni p$, every section of $\mathcal{O}_X$ over $U \setminus \{p\}$ uniquely extends to U.

Proof. (1) → (2) This extension is local, thus it can be reduced to the affine case, we may assume $X = \operatorname{Spec}(A)$, $p = \mathfrak{m}$ which is a maximal ideal. By exercise 2.3 (e), there is a long exact sequence of cohomology groups:

$$0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X \setminus Y = U, \mathcal{F}|_U) \to H^1_Y(X, \mathcal{F}) \cdots$$

lemma : $\operatorname{depth}(A_{\mathfrak{p}}) \geq \operatorname{depth}_{\mathfrak{p}}(A)$ for prime ideal $\mathfrak{p}$. And this is an equality when $\mathfrak{p}$ is a maximal ideal.

By the lemma above, we have $\operatorname{depth}_{\mathfrak{m}}(A) = \operatorname{depth}(A_{\mathfrak{m}}) \geq 2$, then by the above exercises 3.3(b) and 3.4(b), $H^i_{\mathfrak{m}}(A) = H^i_{\{p\}}(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) = 0$ for $i < 2$. So by the long exact sequence of cohomology groups above with $Y = \{p\}$ and $H^1_{\{p\}}(X, \mathcal{O}_X) = 0$:

$$0 \to \Gamma_{\{p\}}(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X) \to 0 = H^1_{\{p\}}(X, \mathcal{O}_X)$$

So the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is surjective and local extension on U can be extended to the whole X.

proof of the lemma :

Proof. For the inequality $\operatorname{depth}(A_{\mathfrak{p}}) \geq \operatorname{depth}_{\mathfrak{p}}(A)$, see [2, coroallary 1.1.3, p.4].
If $\operatorname{depth}(A_{\mathfrak{p}}) = 1$ but $\operatorname{depth}_{\mathfrak{p}}(A) = 0$, then as $\mathfrak{p}$ is finitely generated, assuming the generators as $\{m_i\}_{i=1,\cdots,n}$, for each m_i, the multiplication by m_i has nontrivial kernel for A, denote a nontrivial element k_i in the kernel. Then $\prod_i k_i$ is annihilated by $\mathfrak{p}$ and as $\mathfrak{p}$ is maximal, so $\operatorname{ann}(\prod_i k_i) = \mathfrak{p}$, that is, $\mathfrak{p} \in \operatorname{Ass}(A)$. Then $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}(A_{\mathfrak{p}})$ and $\operatorname{depth}(A_{\mathfrak{p}}) = 0$. This is a contradiction, thus $\operatorname{depth}_{\mathfrak{p}}(A) \geq 1$.
If $\operatorname{depth}(A_{\mathfrak{p}}) = n \geq 1$, then by induction, $\operatorname{depth}_{\mathfrak{p}}(A) \geq n - 1$. Pick such an (n-1) regular sequence $\{t_i\}_{i=1,n-1}$ If $\operatorname{depth}_{\mathfrak{p}}(A) < n$, then there is no $(A/(t_1, \cdots, t_{n-1})$ (as A module) regular element. Thus $\operatorname{depth}_{\mathfrak{p}}(A/(t_1, \cdots, t_{n-1})) = 0$ and again $\mathfrak{p} \in \operatorname{Ass}(A/(t_1, \cdots, t_{n-1}))$ and $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}((A/(t_1, \cdots, t_{n-1}))_{\mathfrak{p}})$(see [2, p.4, coroallary 1.1.3]). As $t_i \in \mathfrak{p}$, thus $(A/(t_1, \cdots, t_{n-1}))_{\mathfrak{p}} \cong A_{\mathfrak{p}}/(t_1, \cdots, t_{n-1})$. Thus $\operatorname{depth}(A_{\mathfrak{p}}/(t_1, \cdots, t_{n-1})) = 0$, this means $(t_1, \cdots, t_{n-1})$ is a maximal $A_{\mathfrak{p}}/(t_1, \cdots, t_{n-1})$ regular sequence in $A_{\mathfrak{p}}$. This is a contradiction because every maximal regular sequence has the same length in our case (see [2, theorem 1.2.5]). □

(2) → (1) The existence of unique extension again can be veiwed as some local property around p, so is the depth of the local ring at p. In the same way above, this is done by exercise 3.4 (b). □

Remark: If the noetherian scheme is normal, then by condition S_2 of Serre, any singular point p, which must be of dimension ≥ 2 has depth of local ring at $p \geq 2$, thus the regular function around p can be extended to p uniquely.

3.6
Let X be noetherian scheme,
(a) Show that the sheaf constructed $\mathcal{G}$ consturcted in the proof of (3.6) is an injective object in the category QCoh(X) of quasi-coherent sheaves in X. Thus QCoh(X) has enough injections.

Proof. By construction, $\mathcal{G} = \bigoplus_i^n f_*(\widetilde{I_i})$ where f is the embedding of open scheme $U_i = \mathrm{Spec}(A_i)$ into X and I_i is the injective A_i module.

lemma 1: pushforward along quasi-compact and quais-separated open subscheme preserves injetcive objects.

It is trivial that the finite direct sum of injective object is again injective, thus $\mathcal{G}$ is injective.

proof of the lemma 1:

Proof. As $f : U \to X$ is quasi-compact and quasi-separated, thus the pushforward preserves quasi-coherence. And by the lemma 2 below, we learn that the right adjoint preserves injective object if the left adjoint is left exact, where pullback along open subscheme is just the restriction to this open subset, which is exact.

□

lemma 2: For adjunction (of locally small categories) $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, right adjoint $\mathcal{G}$ preserves injective object when the left adjoint $\mathcal{F}$ preserves monomorphism. The converse is true when $\mathcal{C}$ has enough injectives.

Proof. If $\mathcal{F}$ preserves monomorphism, we assume $X \in \mathcal{D}$ is an injective object, and $i : Y \to Z \in \mathcal{C}$ is a monomorphism, then we have a surjection: $\mathrm{Hom}_{\mathcal{D}}(F(Z), \mathcal{G}) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), \mathcal{G})$. Then by the by the natural isomorphism provided by the adjunction, there is a surjection: $\mathrm{Hom}_{\mathcal{C}}(Z, \mathcal{G}(X)) \to \mathrm{Hom}_{\mathcal{C}}(Y, \mathcal{G}(X)))$. Thus $\mathcal{G}(X)$ is an injective object in $\mathcal{C}$.

For a monomorphism $u : X \to Y \in \mathcal{C}$, consider the embedding i of $\mathcal{F}(X)$ into some injective object $Z \in \mathcal{D}$ (it has enough injectives), then the adjunction of i is $\widetilde{i} : X \to \mathcal{G}(Z)$, as $\mathcal{G}$ preserves injective object, then there exists a natural extension $\psi : Y \to \mathcal{G}(Z)$ such that $\widetilde{i} = \psi \circ u$, then by the naturality of the adjunction, $i = \widetilde{\psi} \circ F(u)$ is a monomorphism, thus $F(u)$ is a monomorphism

□

Thus $\mathcal{G}$ is indeed an injective object in QCoh(X) as $\widetilde{I_i}$ is an injective object in QCoh(Spec(A_i)) = Mod(A_i).

□

(b) Show that an injective object $\mathcal{F}$ in QCoh(X) is flasque.

Proof. For this part, I refer to [5, section 7, chapter II].

Let's conosider the affine case first: as the category of $Mod(A)$ is a locally noetherian category when A is noetherian, thus injective object can be decomposed into indecomposable injectives. More precisely, the indecomposable injectives is the injective hull $I = E_{A_\mathfrak{p}}(\kappa(\mathfrak{p}))$ of $\kappa(\mathfrak{p}) =$

$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal. Thus if the indecomposable injective object in Mod(A) corresponds to flasque sheaf on Spec(A), so is general injective object.

As I is an injective object in the category Mod(A), so is $\widetilde{I}$ injective in QCoh(Spec(A)). If $\widetilde{I}$ is also an injective object in the category Mod($\mathcal{O}_{\mathrm{Spec}(A)}$), then it is flasque by [7, lemma 2.7, chapter III].

lemma 3: $\widetilde{I}$ is injective in the category Mod($\mathcal{O}_{\mathrm{Spec}(A)}$).

Proof. Consider the embedding $i : \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$, then denote $I_1 = \widetilde{I}$ as the sheaf associated to the $A_{\mathfrak{p}}$ module I on Spec($A_{\mathfrak{p}}$) and $I_2 = \widetilde{I}$ as the sheaf associated to the A module I on Spec(A). We claim that $i_* I_1 = I_2$.

First of all, $(I_2)_{\mathfrak{q}} = I$ for all $\mathfrak{q} \supset \mathfrak{p}$. To prove this, it suffices to prove that $xI = I$ for all $x \in A \notin \mathfrak{p}$, multiplying by x is an injection (see (4) in [10, Tag 08Y7]), moreover, this is an essential extension and as essential extension of injective module is trivial (see [10, Tag 08XS]), $xI \subset I$ is epimorphism.

Secondly, Ass(I) = $\{\mathfrak{p}\}$. The injective hull I of $\kappa(\mathfrak{p})$ as $A_{\mathfrak{p}}$ module is the same as the injective hull of $A_{\mathfrak{p}}$ as an A module (see [10, Tag 08Y8]), thus $A/\mathfrak{p} \subset \kappa(\mathfrak{p}$ is an essential extension. If $y \in I$ has its annihilator as $\mathfrak{q}$, then there exists $a \in A$ such that $ay \in A/\mathfrak{p}$, thus its annihilator can only be $\mathfrak{p}$. Thus the support of $\widetilde{I}$ lies in $V(\mathfrak{p}$ and on Spec($A_{\mathfrak{p}}$), I_1 is supported on the unique closed point.

Thus we can see that the pushforward $i_* I_1 = I_2$, it remains to prove that $i_*(I_1)$ is an injective object in the category Mod($\mathcal{O}_{\mathrm{Spec}(A)}$). For any $\mathcal{O}_{\mathrm{Spec}(A)}$ module sheaf $\mathcal{F}$, by the adjointness, $\mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(A)}}(\mathcal{F}, i_*(I_1)) = \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(A_{\mathfrak{p}}}}(i^*\mathcal{F}, I_1) = \mathrm{Hom}_{A_{\mathfrak{p}}}(\mathcal{F}_{\mathfrak{p}}, I)$ where the last bijection is the adjointness of the closed embedding of $\{\mathfrak{p}\} \subset \mathrm{Spec}(A_{\mathfrak{p}})$ and in this case the pullback of $\mathcal{F}$ is exactly the stalk at $\mathfrak{p}$ (This only works when I_1 is supported on the closed point), and this bijection means the morphism between $\mathcal{F}$ and $I_2 = i_* I_1$ is totally determined by its induced morphism at stalk of $\mathfrak{p}$. For a monomorphism $\mathcal{F} \to \mathcal{G}$ and $\mathcal{O}_{\mathrm{Spec}(A)}$ module sheaf morphism $p : \mathcal{F} \to i_* I_1$, consider the extension at stalk level: $\mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}} \to I$, the second arrow corresponds uniquely to a sheaf morphism $\mathcal{G} \to I_2$, which is the extension of p by the naturality of the adjointness.

□

Thus we can see that the injective object in QCoh(X) is flasque locally, and flasqueness is also a local property, that is, consider the maximal open subset U of X such that $\mathcal{F}$ is flasque, then if $U \subsetneqq X$, take an affine open neighbourhood V of $x \in X \setminus U$, if $V \cap U = \emptyset$, then this can be naturally extended, otherwise, section on $V \cap U$ can be extended to $U \cup V$, thus $\mathcal{F}$ is flasque on $U \cup V \supsetneqq U$, which is contradiction.

□

Here is another try of (b):
Recall how we prove that an injective object in the category of sheaf of $\mathcal{O}_X$ module is flasque: by taking the functor $\mathrm{Hom}(-, \mathcal{F})$ on the exact sequence $0 \to \mathcal{O}_V \to \mathcal{O}_U$ for any open subsets $V \subset U$ because of the obvious isomorphism $\mathrm{Hom}(\mathcal{O}_U, \mathcal{F}) \cong \Gamma(U, \mathcal{F})$ for $\mathcal{O}_X$ module. This only works for the such 'big' category like $\mathcal{O}_X$ module sheaf, even for the affine case, injective object in the category of quasi-coherent sheaf may not be flasque unless the scheme is the spectrum of a noetherian ring, as shown in the example (see [10, Tag 0273])

Inspired by this example, we may refer to the following lemma (see [10, Tag 01PM])

lemma 3: Let A be a ring and I its finitely generated ideal, let M be an A module, then there exists a canonical injective morphism:

$$\operatorname{colim}_n \operatorname{Hom}_A(I^n, M) \longrightarrow \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}).$$

Set $M_n = \{x \in M | I^n x = 0\}$, then there is an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}).$$

What's more, the second isomorphism is compatible with the natural inclusion $J \subset I$, thus we have the following diagram:

$$\begin{tikzcd}
\operatorname{colim}_n \operatorname{Hom}_A(I^n, M/M_n) \arrow[r] \arrow[d] & \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}) \arrow[d] \\
\operatorname{colim}_n \operatorname{Hom}_A(J^n, M/M_n) \arrow[r] & \Gamma(\operatorname{Spec}(A) \setminus V(J), \widetilde{M})
\end{tikzcd}$$

To prove that the sheaf associated to the injective module M over a noetherian ring A is flasque, it suffices to prove that $\Gamma(\operatorname{Spec}(A), \widetilde{M}) \to \Gamma(U, \widetilde{M})$ is surjective for any open subset U because for any $V \subset U \subset \operatorname{Spec}(A)$, it can always be factored in the form of $\Gamma(\operatorname{Spec}(A), \widetilde{M}) \to \Gamma(U, \widetilde{M}) \to \Gamma(V, \widetilde{M})$, thus the surjection of restriction of $\operatorname{Spec}(A)$ to U implies the surjecion of restriction of U to V.

As A is noetherian, thus any open subset U can be written in the form of $U = \operatorname{Spec}(A) \setminus V(I)$ where I is a finitely generated ideal, thus the above lemma 3 can be applied. Thus it suffices to prove the arrows in the following diagram are all epimorphism.

$$\begin{tikzcd}
\operatorname{Hom}_A(A, M) = M \arrow[d] \arrow[dr, dotted] & \\
\operatorname{Hom}_A(A, M/M_n) = M/M_n \arrow[r] & \operatorname{Hom}_A(I^n, M/M_n)
\end{tikzcd}$$

It remains to prove that the natural morphism $\operatorname{Hom}_A(A, M/M_n) = M/M_n \longrightarrow \operatorname{Hom}_A(I^n, M/M_n)$ is surjective. But I was stuck here, if M/M_n is also injective, then it is trivial, but I believe this is not true, so how to show the surjection of the colimit morphism?

3 Appendix

3.1 Appendix A: basic properties of morphism

Here are several pullback diagrams that will be used frequently later:
Let $f : X \to Y$ be a morphism in the category $\mathcal{C}$. The diagonal of f: $\triangle_f$ is defined as the map:

$$\begin{tikzcd}
X \arrow[dr, dotted, "\triangle_f"] \arrow[drr, bend left, "id"] \arrow[ddr, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

For a class of morphism $\mathcal{P}$ in the category $\mathcal{C}$, we denote $\triangle(\mathcal{P}) = \{f \,:\, \triangle_f \in \mathcal{P}\}$.

The graph of f is defined as the map:

$$\begin{tikzcd}
& X \arrow[dl, equal, "id"'] \arrow[d, dotted, "gr_f"] \arrow[dr, "f"] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"] & Y
\end{tikzcd}$$

We have the pullback square:

$$\begin{tikzcd}
X \arrow[r, "gr_f"] \arrow[d, "f"'] & X \times Y \arrow[r, "p_1"] \arrow[d, "f\times id"] & X \arrow[d, "f"] \\
Y \arrow[r, "\triangle"] & Y \times Y \arrow[r, "p_2"] & Y
\end{tikzcd}$$

Given diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] & S \arrow[d, dashed, "s"] & Y \arrow[l, "g"'] \\
& S' &
\end{tikzcd}$$

There is a pullback diagram:

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] & X \times_{S'} Y \arrow[d, "f\times g"] \\
S \arrow[r, "\triangle"] & S \times_{S'} S
\end{tikzcd}$$

Here are some applications of the above pullback diagrams in proof of properties of the morphisms.

Example. Given a morphism $f \,:\, X \to Y$, if Y is affine, and X is qc(respectively, qs), then f is qc(respectively, qs):

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "qc"'] & & Y \arrow[dl, "qc"] \\
& \mathrm{Spec}(\mathbf{Z}) &
\end{tikzcd}$$

This is a special case of :
(Cancellation) lemma 1. Let $\mathcal{C}$ be a category with pullbacks,let $\mathcal{P}$ be a property of morphism stable under composition and pullback.

Given diagram: $\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$ If $p \in \mathcal{P}, q \in \triangle(\mathcal{P})$, then $f \in \mathcal{P}$.

Proof. We have the pullback diagrams: $\begin{tikzcd} X \arrow[r, "gr_f"] \arrow[d] & X \times_Z Y \arrow[d, "f \times_Z id"] \\ Y \arrow[r, "\triangle_q"] & Y \times_Z Y \end{tikzcd}$ $\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "q"] \\ X \arrow[r, "p"] & Z \end{tikzcd}$

Since $p, \triangle_q \in \mathcal{P}, gr_f, p_2 \in \mathcal{P}$, thus $f = p_2 \circ gr_f \in \mathcal{P}$. □

lemma 2: With the same setting as lemma 1, assume we have the following diagram: $\begin{tikzcd} X \arrow[r, "f"] & Z \arrow[d, "z"] & Y \arrow[l, "g"'] \\ & Z' & \end{tikzcd}$

with $\triangle_z \in \mathcal{P}$, then $X \times_Z Y \to X \times_{Z'} Y$ is in $\mathcal{P}$.

Proof. Since $\mathcal{P}$ is stable under pullback, this follows from the pullback square:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \times_{Z'} Y \arrow[d, "f \times g"] \\ Z \arrow[r, "\triangle_z"] & Z \times_{Z'} Z \end{tikzcd}$$

□

Examples.
Let $\mathcal{C}$ be a category, and let $\mathcal{P}$ be a class of morphisms in $\mathcal{C}$ that contains all isomorphisms, is stable under composition, and is stable under pullback. Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, prove the following:
(1) If $gf \in \mathcal{P}$ and g is a monomorphism, then $f \in \mathcal{P}$.

(2) If $gf \in \Delta(\mathcal{P})$, then $f \in \Delta(\mathcal{P})$.

Now, consider morphisms of schemes $f : X \to Y$ and $g : Y \to Z$, and prove the following:

(3) If gf is quasiseparated, then f is quasiseparated.

(4) If gf is qcqs and g is quasiseparated, then f is qcqs. In particular, if X is qcqs and Y is quasiseparated, then f is qcqs. Hence, morphisms between qcqs schemes are automatically qcqs.

(5) If X is locally topologically noetherian, then f is quasiseparated.

(6) If X is topologically noetherian, then f is qcqs.

(7) If gf is quasicompact and Y is locally topologically noetherian, then f is quasicompact.

(8) If gf is separated, then f is separated.

(9) If gf is a locally closed immersion, then f is a locally closed immersion.

(10) If gf is locally of finite type, then f is locally of finite type.

(11) If Y is a separated scheme (i.e., separated over Spec($\mathbb{Z}$)), then f is separated if and only if X is a separated scheme. This is quite useful when Y is affine.

Proof. (1). By the cancellation law, it suffices to prove that $gf \in \mathcal{P}$ and $\triangle_f \in \mathcal{P}$.
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

By problem 8 in problem set 2, g is monomorphism iff the diagram
$$\begin{tikzcd} Y \arrow[r, "id"] \arrow[d, "id"] & Y \arrow[d, "g"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$
is pullback, this is actually the pullback diagram for constructing diagonal map $\triangle(g)$, so $\triangle(g) : Y \to Y \times_Z Y$ is actually an isomorphism, so it is in $\mathcal{P}$, so $f \in \mathcal{P}$.

(2). it suffices to prove that the diagram
$$\begin{tikzcd} X \arrow[r, "\triangle(f)"] \arrow[d, dashed, "id"] & X \times_Y X \arrow[d, dashed, "id\times_g id"] \\ X \arrow[r, "\triangle(gf)"] & X \times_Z X \end{tikzcd}$$
is pullback.

For the morphisms pair: (ψ_1, ψ_2), we have commutative diagram:
$$\begin{tikzcd} W \arrow[drr, "\psi_2"] \arrow[ddr, "\psi_1"'] & & \\ & X \arrow[r, "\triangle(f)"] \arrow[d, dashed, "id"] & X \times_Y X \arrow[d, dashed, "id\times_g id"] \\ & X \arrow[r, "\triangle(gf)"] & X \times_Z X \end{tikzcd}$$

It suffices to prove that when $\triangle(gf)\psi_1 = (id \times_g id)\psi_2$, then $\triangle(f)\psi_1 = \psi_2$. Unwinding the definition of $\triangle(gf)$ and $\triangle(f)$., we can get the following diagram where the solid lines are commutative.

$$
\begin{tikzcd}
 & X \arrow[dr, dashed, "\triangle(f)"] \arrow[rrrd, equal, "id"] \arrow[ddrr, equal, "id"'] & & & & \\
 & & X \times_Y X \arrow[rr, "p_1"] \arrow[dr, dashed, "id \times_g id"] \arrow[dd, "p_2"'] & & X \arrow[dr, equal, "id"] \arrow[dd] & \\
 & & & X \times_Z X \arrow[rr, "p_3"] \arrow[dd, "p_4"] & & X \arrow[dd] \\
W \arrow[uuur, dotted, "\psi_1"] \arrow[uurr, dotted, "\psi_2"'] \arrow[urrr, "\triangle(gf)\psi_1=(id\times_g id)\psi_2"'] & & X \arrow[rr] \arrow[dr, equal, "id"] & & Y \arrow[dr] & \\
 & & & X \arrow[rr] & & Z
\end{tikzcd}
$$

It's by definition that ψ_2 are determined by the compositions with the two projections, $p_1\psi_2$ and $p_2\psi_2$, which is (by the commutative diagram, $p_1 = p_3(id\times_g id)$ and $p_2 = p_4(id\times_g id)$), equivalent to $p_3(id \times_g id)\psi_2 = p_3 \triangle (gf)\psi_1 = id\circ id\circ\psi_1$ and $p_4(id \times_g id)\psi_2 = p_4 \triangle (gf)\psi_1 = id\circ id\circ\psi_1$, so $p_i(\triangle(f)\psi_1) = p_i(\psi_2)$ $i = 1, 2$, so $\triangle(f)\psi_1 = \psi_2$, and $\mathcal{P}$ is stable under base change, so $\triangle(f) \in \mathcal{P}$.

(3) f is quasiseparated iff $\triangle(f)$ is quasicompact, so $\triangle(gf)$ is quasicompact, and by (2), we can see that $\triangle(f)$ is alos quasicompact. (it's trivial to check that 'quasi-compact' is indeed a property that satisfies the requirements including stability under composition and pullback, containing isomorphisms.)

(4) By (3), f is quasiseparated, and $\triangle(g)$ is quasicompact as g is quasiseparated, so by cancellation lemma, f is quasicompact, so f is qcqs.

(5) For a locally noetherian scheme, any two affine open subsets U_1, U_2 must corresponding to noetherian ring, thus noetherian spaces, so the intersection $U_1 \cap U_2$ is open subset of noetherian space, is again noetherian, which is quasicompact.

So the diagram
$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\
 & Spec(Z) &
\end{tikzcd}
$$
has gf is quasiseparated, so by (3), f is also quasiseparated.

(6). For any open subset U of Y, the preimage $f^{-1}(U)$ is open subset of a noetherian space, which is again noetherian, so it is quaiscompact, so f is qc, and by (5), it is qs, so it is qcqs.

(7). By the cancellation lemma, we only need to prove that g is quasiseparated. For the morphism $g : Y \to Z$, when Y is locally noetherian, for any affine open subset U of Z, and any point z lying in this open preimage, we can pick a small enough open affine neighborhood U_z such that it is the spectrum of a noetherian ring. So, we can pick an affine noetherian cover of $f^{-1}(U)$, for any tow subsets in this cover, their intersection is open subset of noetherian space, which is again noetherian, so it must be quasicompact, and g is quasiseparated.

(8) We know that diagonal property corresponding to separated is closed immersion, we know that property of closed immersion is stable under composition and base change, and it contains all the isomorphisms, so by (2), when $\triangle(gf)$ is quasicompact, so is $\triangle(f)$, and f is thus separated.

(9) To prove that f is locally closed immersion, we need to show that topologically, f is a homeomorphism from X to a locally closed subset of Y, and at stalk level, $\mathcal{O}_{y=f(x),Y} \to \mathcal{O}_{x,X}$ is surjective.

Topologically, we can see that there is a restriction diagram:
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, shift left, "gf"] & & g^{-1}(W) \arrow[dl, "g"] \\ & W \arrow[ul, shift left, "(gf)^{-1}"] & \end{tikzcd},$$
where W is the image of fg in Z which is locally closed, so is $f^{-1}(W)$ in Y, and we can see that $(gf)^{-1}g$ is continuous morphism inverse for $f : X \to g^{-1}(W)$, so f is a homeomorphism from X to a locally closed subset of Y.
At stalk level, we know that $(gf)^* = f^*g^* : \mathcal{O}_{gf(x),Z} \to \mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$ which is surjective, so $f^* : \mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$ is surjective.
(1). To prove that f is locally of finite type, we only need to show that for affine open subsets: $U \subset X, V \subset Y, W \subset Z$ where $f(U) \subset V, g(V) \subset W$, and this also corresponds to a ring morphism diagram,
$$\begin{tikzcd} A = \Gamma(U, \mathcal{O}_X) & & B = \Gamma(V, \mathcal{O}_Y) \arrow[ll, "f^*"'] \\ & C = \Gamma(W, \mathcal{O}_Z) \arrow[ul, "f^*g^*"] \arrow[ur, "g^*"'] & \end{tikzcd}$$

And by the condition, $f^*g^* : C \to A$ is of finite type, so A is a finitely generated C algebra, so it must also be finitely generated B algebra with the same generators. So we can construct an affine cover $\{V_i\}$ of Y and an affine cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ such that $U_ij \to V_i$ corresponds to ring morphism of finite type. (Because for affine cover $\{W_i\}$ of Z, $g^{-1}(W_i)$ is also cover for Y)

(11) When f is separated, then consider the composition: $X \xrightarrow{f} Y \to \mathrm{Spec}(Z)$ which is separated because Y is separated. So X is separated over spectrum of Z, which is separated.

When X is separated, then consider the diagram:
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & \mathrm{Spec}(Z) & \end{tikzcd}$$
we can see that gf is separated, so is f by (8).

□

3.2 Appendix B: some counterexamples

1. For integral scheme X, K(X) contains $\mathcal{O}_{\mathbf{X}}$(U), but not equal to its fraction field.[1]

Proof. When U is affine, then $\mathrm{Frac}(\mathcal{O}_X(U)) = K(X)$, however, if U is not affine, $\mathrm{Frac}(\mathcal{O}_X(U))$ can only be a subfield of $K(X)$. For instance, the global section of projective line $\mathbf{P}^1_k$ is k while its function field is $k(x)$.

Because affine open subsets constitute a basis of scheme, thus the sheafification of presheaf $U \to \mathrm{Frac}(\mathcal{O}_X(U))$ is exactly the constant sheaf $K(X)$. □

2. Localization of rings of finite type may not be of finite type[2].

Proof. Here is a very special example, $A = k[x_0, \cdots, x_n, \cdots]$ and consider the fraction field of A: $k(x_0, \cdots, x_n, \cdots)$. Then $\mathrm{Frac}(A)$ is not of finite type over A. But usually deal with finite type over

[1] see answer in https://math.stackexchange.com/questions/3559296/function-field-of-integral-scheme .
[2] this originates from exercises 8.2, chapter 2 in this note.

k and by the fact that $k(x) = k[x,y]/(xy-1)$, the localization of finite type k algebra is again of finite type over k. $\square$

3. Definition of height of an ideal[3]: $\text{height}(I) = \min_{\mathfrak{p}\supset I}(\text{height}(\mathfrak{p})) \neq \sup\{n|\mathfrak{p}_0 \subsetneqq \cdots \mathfrak{p}_n \subsetneqq I\}$.

Proof. Concept of the height of an ideal can be understood as the codimension of the closed subscheme determined by it, thus it can not be defined arbitrarily. For instance: consider the affine line $\mathbf{A}^1_k$ and its double zero point $V(x^2)$, then $\text{height}(x^2) = \min_{\mathfrak{p}\supset I}(\text{height}(\mathfrak{p})) = \text{height}((x)) = 1 \neq \sup\{n|\mathfrak{p}_0 \subsetneqq \cdots \mathfrak{p}_n \subsetneqq I\} = 0$ $\square$

4. Normal scheme classification

Proof. [10, Tag 033H]. $\square$

5. Taking global section doesn't commute with tensor product.

Give an example of a scheme X and quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X such that

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \ncong \Gamma(X, \mathcal{F}) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X, \mathcal{G}).$$

Proof. Take twisting sheaf $\mathcal{O}_X(1)$ and its dual $\mathcal{O}_X(-1)$ on projective space $X = \mathbf{P}^n$, we shall see that $\Gamma(\mathbf{P}^n, \mathcal{O}_X(-1) \otimes \mathcal{O}_X(1) = \mathcal{O}_X) = k$ while $\Gamma(\mathbf{P}^n, \mathcal{O}_X(-1)) = 0$ because there is no nontrivial global section for$\mathcal{O}_X(-1)$, thus $\Gamma(\mathbf{P}^n, \mathcal{O}_X(1)) \otimes \Gamma(\mathbf{P}^n, \mathcal{O}_X(-1)) = 0$. $\square$

6. ideal intersection commutes with localization.[4]

Proof. By the universal property of localization, $M \to S^{-1}M$ is a left adjoint functor to the forgetful functor: $\text{Mod}(S^{-1}(A)) \to \text{Mod}(A)$, thus localization commutes with all colimits, but it may not commute with limit.

However, as localization of a ring commutes with finite limits as it is flat. For arbitrary limits, this is not true: consider $\mathbf{Z}$ module $I_n = n\mathbf{Z}$, then infinite intersection $(\cap_n I_n = 0) \otimes_{\mathbf{Z}} \mathbf{Q} \neq \cap_n(I_n \otimes_{\mathbf{Z}} \mathbf{Q}) \ni 1$

Here is another example: the cannonical morphism $(\prod_{n=1}^{\infty} \mathbf{Z}) \otimes \mathbf{Q} \to \prod_{n=1}^{\infty} \mathbf{Q}$ is not isomorphic because it is not surjective (consider the element $(\frac{1}{n})_{n=1}^{\infty}$). $\square$

7. affine criterion By the affine criterion, let X be qcqs, if $\dim(X) = 0$, then X is affine. Here is a counterexample when X is not qcqs :

Proof. Let $X = \coprod_{n\in\mathbf{N}} \text{Spec}(K)$[5], X can not be quasi-compact, if it is affine, then it is quasi-compact, contradiction. $\square$

[3] this originates from an answer on the Internet: https://math.stackexchange.com/questions/2689141/definition-of-height-of-an-ideal

[4] https://math.stackexchange.com/questions/4057308/localization-does-not-commute-with-arbitrary-intersection-of-ideals.

[5] https://math.stackexchange.com/questions/1529988/spectrum-of-infinite-product-of-rings

3.3 Appendix C: extension topics

local cohomology and depth

recollement theorem
There is a natural question arising when we deal with the quasi-coherent sheaf: can all the algebra operation, whether limit or colimit, be exactly the corresponding operation for quasi-coherent sheaf restricted to affine open subscheme, or more generally, pullback to arbitrary open subscheme?

Pullback is left adjoint to pushforward, thus preserves colimit, so it may be taken for granted that colimit is preserved under the restriction (or the pullback) along any open subscheme. This is corroborated by the fact that the restriction of tensor product of quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to any affine open subset $U = \mathrm{Spec}(A)$ is exactly $\mathcal{F}(\widetilde{U) \otimes_A} \mathcal{G}(U) = \widetilde{\mathcal{F}(U)} \otimes_{\widetilde{A}} \widetilde{\mathcal{G}(U)}$ which is exactly the tensor product of $\mathcal{F}|_U$ and $\mathcal{G}|_U$.

However, the left adjointness of pullback exists in the category $\mathrm{Mod}(\mathcal{O}_X)$ instead of $\mathrm{QCoh}(X)$, if X is noetherian, or open subset U we are pulling back along is separated and quasi-compact over X, then f_* preserves quasi-coherence, and f^* is left adjoint in $\mathrm{QCoh}(X)$, but in general, this is not true.

But there is something special about the embedding of the subcategory $\mathrm{QCoh}(X) \to \mathrm{Mod}(\mathcal{O}_X)$ (see [10, Tag 077K]), it admits a right adjoint $Q : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{QCoh}(X)$ such that if $\mathcal{F} \in \mathrm{QCoh}(X)$, then $Q(\mathcal{F}) \cong \mathcal{F}$[6]. So the embedding as a left adjoint preserves all colimits, which means the colimit of quasi-coherent sheaf in the category of $\mathrm{Mod}(\mathcal{O}_X)$ is again quasi-coherent. Thus the pullback along any morphism (which may not be open immersion) preserves colimits in the category of quasi-coherent sheaf: $f^* : \mathrm{QCoh}(X) \subset \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_U) \supset \mathrm{QCoh}(X)$.

However, for the case of limit, the situation becomes subtle, all the nice preservation is gone and the only heritage from the fact that $\mathrm{QCoh}(X) \to \mathrm{Mod}(\mathcal{O}_X)$ admits a right adjoint is that $\mathrm{Qcoh}(X)$ has all limits, given by the limit in $\mathrm{Mod}(\mathcal{O}_X)$ under the functor Q.

Let $Z \xrightarrow{i} X \xleftrightarrow{j} U$ where Z is closed subset and U is open subset. There is a more general way to describe the adjointness between $Sh(Z) - Sh(X) - Sh(U)$ and $\mathrm{QCoh}(Z) - \mathrm{QCoh}(X) - \mathrm{QCoh}(U)$ called the recollement theorem.

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