Solutions to 'Introduction to Cluster Algebras' *

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Exercise 1.2.2. Show that the 2×2 minors of a 2× m matrix(equivalently, the Plucker coordinates P_{ij}) satisfy the three-term Grassmann-Plucker relations

$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$ $(1 \leq i < j < k < l \leq m)$

solution: It is easily calculated that :$P_{ik} = a_ib_k - a_kb_i, P_{jl} = a_jb_l - a_lb_j, P_{ij} = a_ib_j - b_ia_j, P_{kl} = a_kb_l - a_lb_k, P_{il} = a_ib_l - b_ia_l$. In order to prove the Grassmann-Plucker relations, we only need to check that

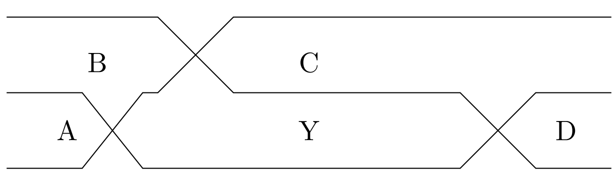
$(a_ib_k - a_kb_i)(a_jb_l - a_lb_j) = (a_ib_j - b_ia_j)(a_kb_l - a_lb_k) + (a_ib_l - b_ia_l)(a_jb_k - a_kb_j)$.

Where both sides equal to

$a_ia_jb_kb_l + a_ka_lb_ib_j - a_ka_jb_ib_l - a_ia_lb_jb_k$

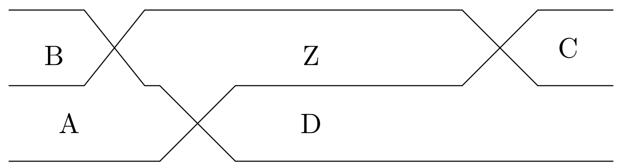
Exercise 1.3.4. Under each braid move, the corresponding collections of chamber minors are obtained from each other by exchanging the neighboring chamber minors by the identity:

$YZ = AC + BD$ **The exchange is shown in the following picture**

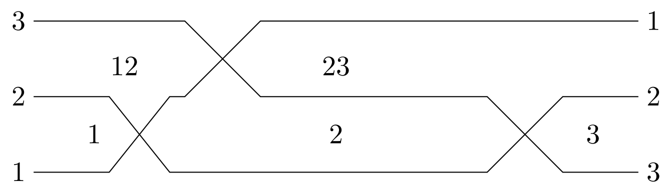


*You can read these papers on arXiv:1608.05735 and arXiv:1707.07190

$\Longleftrightarrow$

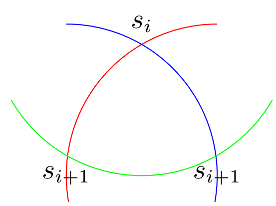


Solution: by using Muir's Law of extensible minors, the problem can be simplified to the triple case as shown in the following diagram



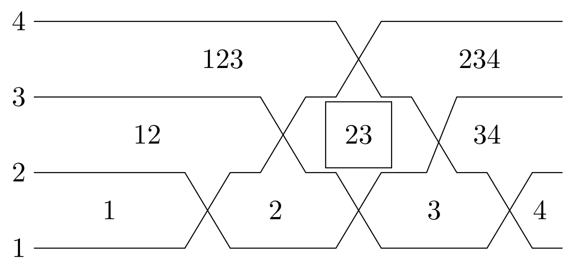
It is easily calculated that $P_2P_{13} = a_{12}(a_{11}a_{23} - a_{21}a_{13}) = a_{11}(a_{12}a_{23} - a_{22}a_{13}) + a_{13}(a_{11}a_{22} - a_{21}a_{12}) = P_1P_{23} + P_{12}P_3$, so by Muir's Law of extensible minors, the polynomial identity involving flag minors can be extended to a new identity with all flag minors P_B extending to $P_{B\cup C}$ where C is an index set disjoint from every column set appearing in flag minors in the original identity.

Rermark 1 In essence, such model can also be transformed into the following diagram.



Actually, the wiring diagram also corresponds to the decomposition of permutation in S_n into simple transpositions so that it corresponds to certain reduced words of the permutation by recording the cross of the lines as the simple transposition $s_i = (i, i+1)$ where $i-1$ is the number of lines above the crossing. So in essence, the braid move corresponds to the equality of simple transposition in the symmetric group $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$. There is a detailed description in [**1**].

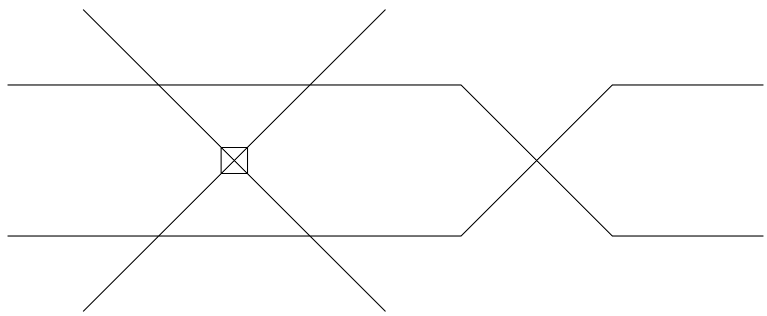
Rermark 2 I want to emphasize that such local braid move can only happen to those local intersections by exactly three different lines. There may be occasion that the shape is familiar but they are formed by more than three lines, for instance, taking $C\left[SL_4\right]^U$, the coordinate ring of affine base space as the special linear group quotient the unipotent radicals.



the mutation at the chamber P_{23} is not at all the braid move described before, actually, the braid move is exactly the mutation at the vertex of chamber with degree in the quiver as four while P_{23} is of degree six. Indeed, the mutation at P_{23} exchange this cluster variable into $\Omega = \frac{P_{234}P_{12}P_3+P_{123}P_{34}P_2}{P_{23}}$ which is exactly $P_1P_{234} + P_2P_{134}$ and is not a flag minor at all.

In conclusion, the quiver coming from certain wiring diagram can be mutated into some quiver that can never correspond to certain wiring diagram. So taking braid move of wiring diagram is not enough to display all the possible seeds mutated from the original one, or in an equivalent way, there are more ways of examination of flag totally positive(FTP) other than testing on the chambers in some wiring diagram. Besides, we can see that actually the cluster algebra generated by this seed is contained in the coordinate ring of base space $U\backslash SL_4$: $C\left[SL_4\right]^U$. Indeed, we can show that they are the same algebra.

Remark 3 There may be also doubt that the braid move in wiring diagram can only happen for the bottom line chambers if we are misled by the example in the former diagram of $C\left[SL_4\right]^U$. This is obviously wrong. The three lines intersection can also happen in the "middle" of the pseudo-lines arrangements. For example in the following local structure.

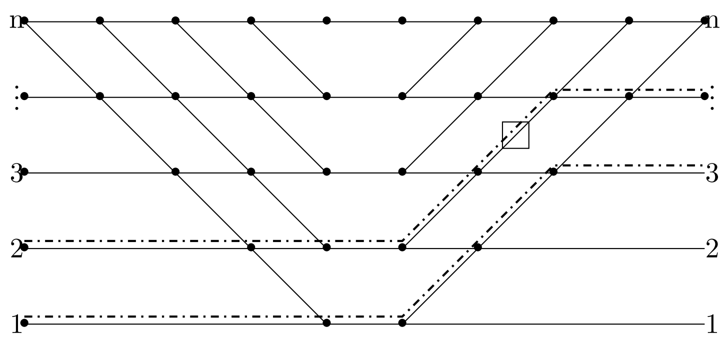


there can be many lines below the intersection of square, so the braid move can happen to the chamber directly above the intersection $\square$ which is not the bottom chambers and the index of the corresponding chambers need to be extended by the "Muri's law of extensible minors".

Exercise 1.4.2. A minor $\triangle_{I,J}$ is called solid if both I and J consist of several consecutive indices. It is easy to see that an $n \times n$ matrix z has n^2 solid minors $\triangle_{I,J}$ such that $I \cup J$ contains 1. Show that z is TP if and only if all these n^2 minors are positive

Solution : Obviously there are exactly n^2 minors described in the exercise, they correspond to each entry of $n \times n$ matrix by taking the right bottom vertex of the minor (such minors are often referred as 'initial minors'). To prove that such initial minors can test the total positivity(TP), there are two ways. One is based on the graph theory by the application of 'weighted planar networks' (see [**2**]) and its correspondence with weighted matrix. The other way is constructing a double wiring diagram such that the chambers are exactly in the form of initial minors. Here is a detailed proof of the 'weighted planar networks'.

A **planar network**(Γ, w) is an acyclic directed planar graph Γ whose edge e is given weight $w(e)$, and we assume the edges of Γ directed left to right. Also, the network will have n sources and n sinks positioned in the left (resp.right) and numbered from bottom to up.



It is a general procedure of producing totally positive matrices to construct the weight matrix $x(\Gamma, w)$ whose entry (i, j) is the sum of weights of all paths from source i to the sink j where the weights of the directed path is defined to be the products of all weights of its edges. And by the Lindstrom's Lemma, the minor $\triangle_{I,J}$ of the weighted matrix is exactly the sum of weights of all collections of vertex-disjoint paths that connect the sources labeled as I and sinks labeled as J where the weights of a collection of paths is the product of the weights of the paths. So in this way, we can see that weight matrix of the planar networks must be nonnegative if the weights are all nonnegative. If we want the matrix to be totally positive, the condition of connectedness needs to be added so that every pair of sinks and sources labeled by subsets of same cardinality of $[n]$ are connected by vertex-disjoint paths. Such planar networks are called "totally connected". In conclusion, we get the **corollary**: If a totally connected planar network has positive weights, then its weight matrix is totally positive.

The planar network shown in the figure is actually very important and can serve as a construction of certain parametrization of all totally positive matrices ([**2**]), and we denote such planar network as Γ_0. It is easy to see that Γ_0 is totally connected by observation of the figure. So, if we have a positive weighting on the Γ_0, then the weight matrix is totally positive.

Call an edge of Γ_0 **essential** if it is slanted or the horizontal edges in the middle of the network. Correspondingly there is notion of the essential **weighting** w of Γ_0 if $w(e) \neq 0$ for any essential edge and $w(e) = 1$ for all other edges.

Coming back to the **exercise 1.4.2**, we try to prove that the initial minors determine whether a matrix is totally positive. We will take two steps to prove this exercise.

Step1: initial minors uniquely determine the square matrix
If the initial minors is nonzero, then it uniquely determines the matrix. We prove that x_{ij} is uniquely determined by the initial minors. This is done by induction on $i + j$. if i or j is 1, then it is one of the initial minors. Assume $min(i, j) > 1$. Let $\triangle$ be the initial minor with right bottom vertex (i, j), i.e, the last row and the last column are i and j. Deleting the row and column, we can have a new initial minors $\triangle'$. Then, by expansion of the determination or minors, we have the equality $\triangle = \triangle' x_{ij} + P$ where P is the polynomial in the matrix entry $x_{i',j'}$ such that $i' \leq i$ and æ$' \leq j$ but $(i, j) \neq (i', j')$. If the minors are all nonzero, we have $x_{ij} = \frac{\triangle - P}{\triangle'}$ where all ingredients of P, i.e. $x_{i'j'}$ are determined by initial minors by induction on $i + j$, so is x_{ij}.

Step2: initial minors are "independent" The planar network Γ_0 has a special property that given a consecutive subset of $[1, n]$ of cardinality k, there is a unique collection of vertex-disjoint paths connecting the sources labeled by $[1, k]$ (resp. by I) and sinks labeled by I (resp. by J), (see the dashed line in the figure, it is the case when the row index including 1). In this unique collection of vertex-disjoint paths, there is a unique **remarked** essential edge (the square-marked edge in the previous figure),i.e. the uppermost essential edge. So every initial minor corresponds bijectively to one of the initial edges in the Γ_0, the left and the right correspond to cases when row index I include 1 and when column index J include 1 separately. So for every minor $\triangle$, we have a value corresponding to weight of this uppermost edge, denoted as $e(\triangle)$. Considering the Lindstrom's lemma, the minor discrimination is the product of the weights of the paths of this unique collection in the Γ_0 for the initial minors. So from the bottom of the essential edges in Γ_0, we can one by one recover the weights of initial minors from the initial minors of the weight matrix. Remember that positive weights imposed on the initial edges are rather free and independent, so correspondingly the initial minors value of discrimination is also independent from each other.

Combining these two steps, For every totally positive matrices z, taking the initial minors, if they are all positive, we can construct an essential weighting of Γ_0 by recovering the weights on the essential edges from the initial minors so that we can have a new matrix z' with the same initial minors as the z, by step 1 of uniqueness, they are the same matrix. So if we have a matrix whose initial minors are all positive, this matrix must be totally positive.

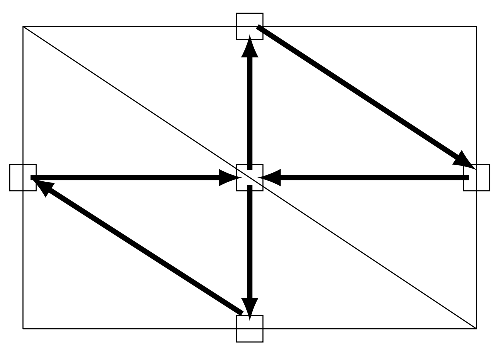
Remark 1 From the previous paragraph, we can see that every totally positive matrix must be in the form of weight matrix of Γ_0 and the n^2 minors serve as the parametrization of the totally positive matrix.

Remark 2 There is a very important theorem relating the essential positive weighting and the totally positive matrices from the above discussion.
theorem: The map $w \to x(\Gamma_0, w)$ restricts to a bijection between the set of all essential positive weightings of Γ_0 and the set of all totally positive $n \times n$ matrices.

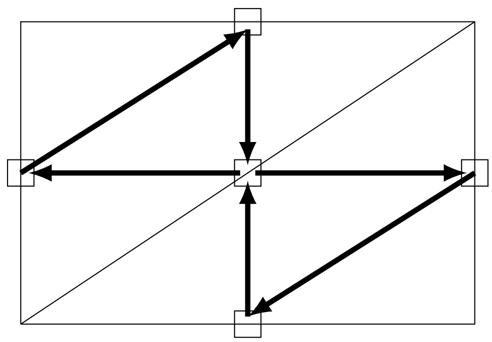
Exercise 2.1.4. Verify the following properties of quiver mutation:
(1) Mutation is involution;
(2) Mutation commutes with the simultaneous reversal of orientations of all arrows of a quiver.
(3) Let k and l be two mutable vertices which have no arrows between them (in either direction). Then mutations at k and l commute with each other: $\mu_l(\mu_k(Q))$=$\mu_k(\mu_l(Q))$
Solution: first, we take a closer look at the definition of operation of the mutation μ_k:
1. For each oriented two-arrow path $i \to k \to j$, add a new arrow $i \to j$.
2. Reverse all the arrows incident to the vertex of k
3. Remove all the oriented 2-cycles.(reversal oriented arrows pair is removed);
For the proof of the exercise, we only need to observe that the step 3 can

be 'omitted' or to be more precisely, this operation of removing oriented 2-cycle can commute with the step 1 and step 2 so that we can remove all the oriented 2-cycles until the last step when the mutation at vertex k compose with other operation like mutation at the same vertex k or reverse all the orientations of the quiver, or mutation at some other point which is irrelevant with k. However, when the two vertices k and l are relevant, i.e, there is some edge connecting the two vertices, if there are emerging edges of the two-cycle with opposite orientations between k and l after the μ_k, not removing them will cause new edges not reversal so that the 'influence' of this two-cycle can not cancel out after μ_l,. So the condition in the (3) is necessary.

Exercise 2.2.2. Flipping of triangulation and mutation of quiver
Let T be a triangulation of the polygon P_n by pairwise noncrossing diagonals. Let T^1 be the triangulation obtained from T by flipping a diagonal γ. Verify that the quiver $\mathrm{Q}(T^1)$ is obtained from $\mathrm{Q}(T)$ by mutating at the vertex labeled by γ.
Solution: Firstly we observe that we only need to consider the case of quadrilateral. In this circumstance, as the following figure shows, it is self-evident.

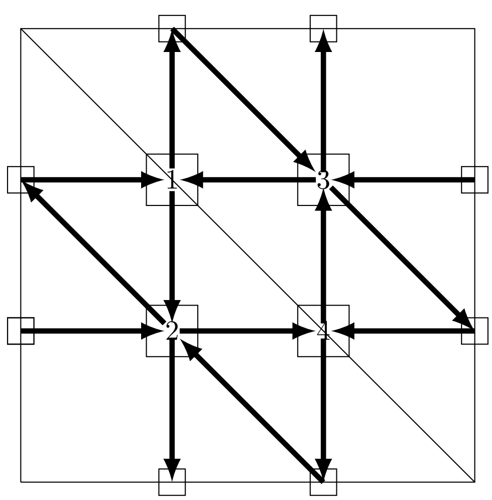


While the quiver of the triangulation after the flip is the following figure.



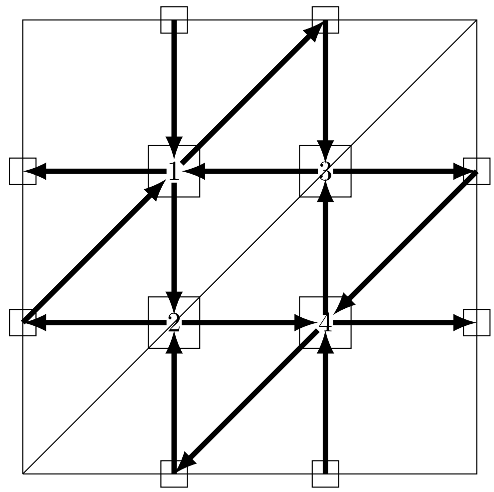
By definition, this is exactly the quiver mutation at the diagonal.

Exercise 2.2.3. Another kind of triangulation of polygon
To each triangulation T of a convex polygon P_m, we can associate a different quiver where we place two mutable vertices on the diagonal, two frozen vertices on each side of the polygon, and one mutable vertex in the interior of each triangle of the triangulation of the polygon. These vertices are connected by interior lines and and the orientations are given as parallel to the triangulation of the T. As the following figure shows.



We can see that the edge connecting the frozen vertices has been removed.

By flipping the triangulation, we have the following figure.



By the figures shown and the calculation, the second quiver $\hat{Q}$ is mutation equivalent to the first quiver Q by $\hat{Q} = \mu_3\mu_4\mu_2\mu_1 Q$.

Remark 1 This is actually a simple model of orbi-ensemble structure in the group PGL_m/SL_m, such triangulation $'partition'$ is a special case of the $m-$ *triangulation* of the bordered surface with marked points in case when $m =$ 2. Given an ideal triangulation of bordered surface with marked points $\hat{S}$, we can define a canonical coordinate system in the higher dimensional moduli space $A_{SL_m,\hat{S}}$, parametrized by the set I^m_T=(vertices of the m-triangulation of T)-(vertices at the punctures of S).

Correspondingly, we have a skew symmetric Z-valued function $\epsilon_{p,q}$ on the set of vertices of the m-triangulation of T, $\epsilon_{p,q}$=#(oriented edges from p to q)-(#oriented edges from q to p). And the Weil-Petersson form is given as $\Omega_{SL_m,\hat{S}} = \sum_{i_1,i_2} \epsilon_{i_1 i_2} \sum_{j_1,j_2} \epsilon_{j_1,j_2} \triangle_{i_1} \triangle_{i_2}$. (see p153 on [**3**] for a more details)

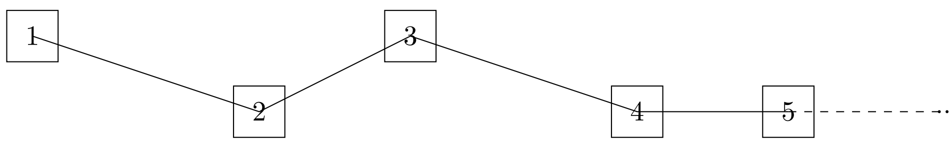
Exercise 2.6.5. Show that all orientations of a tree as quivers (without frozen vertices) are mutation equivalent to each other by mutations at sinks and sources.

Solution: as we can see, the tree graph is characterized as no cycles or uniqueness of path connecting different points. And we notice that in the vertices which are neither sources nor sinks, the mutation will cause some cycles, that's the reason why we only consider mutations at the sinks or sources.

Here we only consider quiver with finite vertices, i.e, finite trees. It is sufficient to prove that any orientation given on every edge of the tree graph can be mutated into the orientation with the one on the given edge reversed.

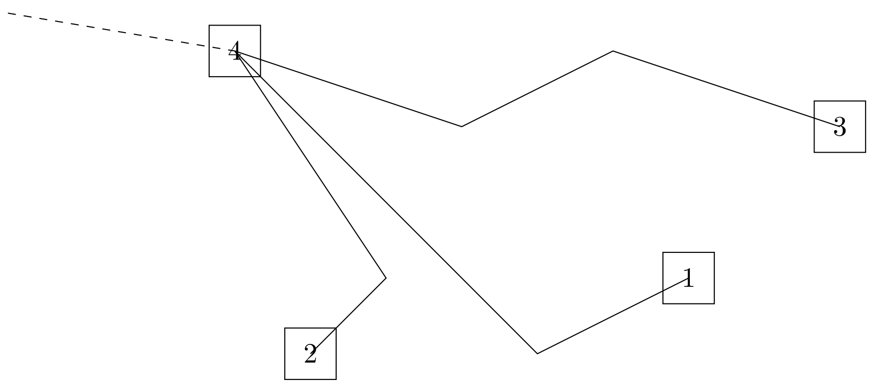
Step 1. It is trivial to observe that any edge connecting to some leaf (vertices with degree 1) satisfy the condition that mutation at the leaf will only influence the unique edge connecting this leaf.

Step 2. For the simple chain path in the tree as shown in the figure where the vertex 1 is the leaf), the situation is very easy.



No matter what the orientation of this part of the tree is, they are always mutation equivalent. The two orientations on the edge $1-2$ can be changed into each other by mutation at the leaf 1. And the two orientations on the edge $2-3$ can be reversed by mutation at 1 $\mu_1$following the mutation at 2 μ_2 and mutate at 1 again to keep the orientation of edge $1-2$ as before. By induction, all edges appearing on the figure satisfy the condition that both orientations can be transformed to each other by mutations without changing the orientation of any other edges.

Step 3. Considering that the tree is finite, so there must exist some leaf with degree 1, otherwise, we can find infinite vertices with degree bigger than 1 or some cycle, which is a contradiction. Also, by the same means, we can see that there must exist the structure of a node connecting finite $'$*simple chain paths*$'$ and only one other edge not of this form (the dashed line) near the $'end'$ of the tree as the following figure shows.



Step 4. We have seen that all the edges in the $'simple\ chain\ paths'$ in the figure connecting 1 or 2 or 3 to 4 can change their orientations freely by mutations so that all the orientations on the quiver of the figure are mutation equivalent. Even the vertex 4 is neither sink nor source, we can mutate the orientations of neighboring edges to view the vertex 4 as a sink or source. And this choice of sink or source is not free, we have to realize that this depends on the orientation on the other edge connecting the vertex 4 to the other part of the tree. When the node 4 with degree more than 1 is a sink or source, mutation at this vertex composing mutation at those vertices on the simple chains can make the dashed line orientation reversed and maintain all other edges orientations.

Step 5. Due to the finiteness of the tree graph, we can start from such simple structure near the end of the tree to inductively prove that every edge satisfy the condition that the orientation on that edge can be reversed by mutation without influencing any other edges orientation. Here we need to use the exercise 2.1.4(3) to ensure the mutation at one path is independent from the mutation at the other path.

Remark 1. By [**4**], Two acyclic quivers mutation equivalent to each other can be transformed to each other at sinks and sources. In this case, the underlying graph remain unchanged. So we can see that the orientation of the finite tree is $of\ finite\ mutation\ type$ by the finiteness of all kinds of orientations on the tree.

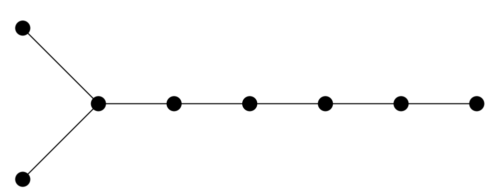
Remark 2. From the theorem in the remark 1, we can see that orientations of nonisomorphic trees are not mutation equivalent.

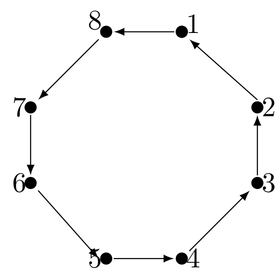
Remark 3. Here we can use the exercise to prove a property of Coxeter element: **all Coxeter elements are conjugate to each other.**
Coxeter elements are defined as products of all simple reflections corresponding to the roots in the root system Ψ. Every Coxeter element determines an orientation of the Coxeter graph in the following way: orient $j \to i$ if i precedes j in the expression of Coxeter element by simple reflections s_i.
Here we introduce an operation on Coxeter element.
Replace c by $c^{'}$ if there exists $\hat{c}$ with length $l(\hat{c}) = n - 1$ and $c = s_i\hat{c}$ and $c^{'} = \hat{c}s_i$. This operation actually corresponds mutation at sink or source in the tree graph. Every orientation of tree is mutation equivalent to each other so that all Coxeter elements are conjugate to each other.

Exercise 2.6.6. Which orientations of an n-cycle are mutation equivalent?

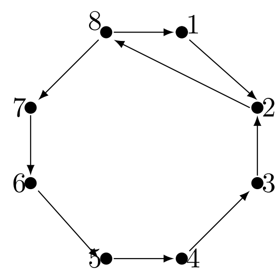
Actually, there is a relevant exercise in the chapter 5. By this exercise, it is trivial to say that orientation in order as the below figure shows is mutation equivalent to type D_n Dynkin diagram.

Exercise 5.4.1. Show that a seed pattern is of type D_n if and only if one of its exchange matrices corresponds to a quiver which is an oriented n-cycle.
Solution: this is just a very simple model of quiver of type D_n. When n=8.

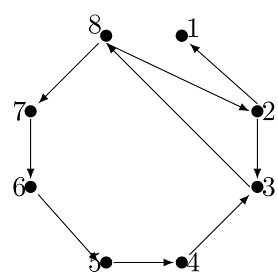




We only need to show that the above two quivers are mutation equivalent. It is easily verified that after the mutation at 1,



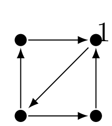
And again by mutation at 2, the edge $8-1$ is removed and a new triangle appears in the quiver.



By induction, we can see that, after the mutation series $\mu_{n-1}\mu_{n-2}...\mu_1$, we have a quiver with underlying undirected graph as type D_n Dynkin diagram.

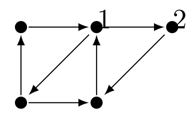
Exercise 2.6.8. Verify that each quiver in the following figure is mutation equivalent to any orientation of some Dynkin diagram.

(1).

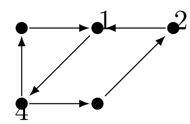


It is easily seen that mutation at 1 (μ_1) turn the quiver into an orientation of D_4.

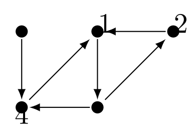
(2).



After mutation at 2, the quiver will become the following figure,

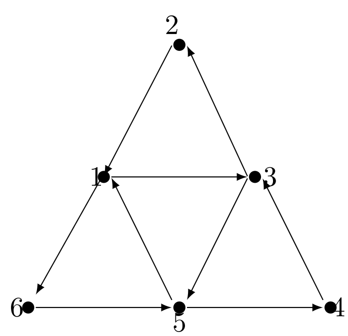


mutate at 4, we get,



Again mutate at 1 will lead to the Dynkin diagram of type D_5

Exercise 2.6.9. Show that the triangular grid quiver with three vertices on each side is mutation equivalent to an orientation of a tree.



Solution: after trials and errors, you can verify that after mutation series $\mu_6\mu_4\mu_2$, the quiver become the hexagon with the orientation in order of the vertices, by exercise 5.4.1 in the above context, we learn that this quiver is mutation equivalent to a tree, or more precisely, the Dynkin diagram D_6.

Exercise 2.7.7. Verify the following properties of the mutation of the extended *skew symmetrizable* matrix

(1) the mutated matrix $\mu_k(\widetilde{B})$ is again extended skew-symmetrizable, with the same choice of rescaling vector $(d_1, ..., d_n)$
(2) $\mu_k(\mu_k(\widetilde{B})) = \widetilde{B}$.
(3) $\mu_k(B^T) = (\mu_k(B))^T$, where B^t denotes the transpose of B. (4) $\mu_k(-\hat{B}) = -\mu_k(\hat{B})$;
(5) if $b_{ij} = b_{ji} = 0$, then $\mu_i(\mu_j(\hat{B})) = \mu_j(\mu_i(\hat{B}))$.
Solution: it is easy to see the following equations.
$b_{ij}^{'} = b_{ij} + b_{ik}b_{kj}$ if $b_{ik} > 0$ and $b_{kj} > 0$;
$b_{ij}^{'} = b_{ij} - b_{ik}b_{kj}$ if $b_{ik} < 0$ and $b_{kj} < 0$;
we have $d_ib_{ij} + d_ib_{ik}b_{kj} = d_i - d_kb_{ki}b_{kj} = d_ib_{ij} + b_{ki}d_jb_{jk} = -d_jb_{ji} + b_{ki}d_jb_{jk}$;
By similarity, we have (1), the others are by definition easily calculated.

Exercise 2.7.8. Let B be a skew-symmetrizable matrix. The skew symmetrization of B, $S(B) = (s_{ij})$ defined by $s_{ij} = sgn(b_{ij})\sqrt{|b_{ij}b_{ji}|}$ satisfies the condition that $S(\mu_k(B)) = \mu_k(S(B))$
Solution: it is important to see that as a skew symmetrizable matrix, there exists a diagonal matrix D with positive diagonal entries such that DB is skew symmetric. Take $H = D^{1/2}$, we have the equation, $S = HDBH^{-1} = H^{-1}(DB)H^{-1}$, and mutation rules are invariant under the conjugation of diagonal matrix. $b'_{ij} = \frac{k_i(b_{ij}+b_{ik}b_{kj})}{k_j} = k_ib_{ij}/k_j + k_i\frac{b_{ik}}{k_k}k_k\frac{b_{kj}}{k_j}$

Remark 1 The symmetrization $S(B)$ is equivalent to the diagram $\Gamma(B)$ in the sense that they all encode the same information of the skew symmetrizable B, this exercise tells us that they are also in the sense of mutation equivalence.
Indeed the mutation of diagram (weighted) can be defined as $\Gamma(\mu_k(S(B)))$, which we denote as $\mu_k(\Gamma(S(B)))$. The diagram mutation is similar to the mutation of the quiver with some additional information about the weight on the edges. There is an equivalent way of definition in [6].
Remark 2 Also we have the corollary that the skew symmetrizable matrix B is 2 finite if and only if the diagram $\Gamma(B)$ is 2 finite. Here I want to emphasize that the 2 finiteness depends on the whole mutation class, so the exercise 2.27 is required.

Exercise 3.2.8. Compute the cluster variables for the cluster algebra with the initial extended exchange matrix $\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ p & q \end{bmatrix}$

Solution: first of all, we have corollary 4.3.6. in [5].

Corollary 4.3.6. Consider two seed patterns, $(\sum(t))_{t\in T_n} = (x(t), y(t), B(t))_{t\in T_n}$, $(\widetilde{\sum}(t))_{t\in T_n} = (\widetilde{x}(t), \widetilde{y}(t), B(t))_{t\in T_n}$ with the same exchange matrices $B(t) = \widetilde{B}(t)$ for $t \in T_n$. Suppose that all rows of the initial extended exchange matrix $\widetilde{B}_0$ for the second seed pattern lie in the Z-span of the rows of the initial extended exchange matrix B_0. Then if two labeled or unlabeled seeds $\sum(t) = \sum(t^{'})$ coincide in the first pattern, then the corresponding seeds $\widetilde{\sum}(t) = \widetilde{\sum}(t^{'})$.

By the corollary, we can see that the recurrence of the seed pattern with zero frozen variable is the same as the seed pattern with two cluster variables and one frozen variable, for the reason that they all have exchange matrices with full Z rank. So we only need to take into the consideration of the case when there are no frozen variables and we can see that this recurrence has nothing to do with p and q.

By hand-calculation, we have $z_3 = \frac{z_2^2+1}{z_1}$, $z_4 = \frac{z_2^2+z_1+1}{z_1 z_2}$, $z_5 = \frac{z_1^2+z_2^2+2z_1+1}{z_1 z_2^2}$, $z_6 = \frac{z_1+1}{z_2}$, then comes the recurrence, $z_7 = z_1, z_8 = z_2$.

Exercise 3.2.9. Compute the cluster variables for the cluster algebra $A(1, 3)$, starting by evaluating them in the specialization z_1=z_2=1

Solution: this type is a bit hard for hand calculation, so we can use the matlab to compute it

syms x y; n = 12; I = sym('I',[1 n]) I = $\begin{pmatrix} I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{11} & I_{12} \end{pmatrix}$ I(1)=x; I(2)=y; for k=3:10 if mod(k,2)==0 I(k)=(I(k-1)+1)/I(k-2); else I(k)=(I(k-1)^3 + 1)$/I(k - 2); endendforl = 1 : 9disp(simplify(I(l))); end$

$$x \quad y \quad \frac{y^3+1}{x} \quad \frac{\frac{y^3+1}{x}+1}{y} \quad \frac{x\left(\frac{\left(\frac{y^3+1}{x}+1\right)^3}{y^3}+1\right)}{y^3+1} \quad \frac{x^2+2\,x+y^3+1}{x\,y^2}$$
$$\frac{x^3+3\,x^2+3\,x+y^3+1}{x\,y^3} \quad \frac{x+1}{y} \quad x$$

Remark 1 The above exercises serve as examples of rank 2 cluster algebra. If you try the type $A(1,4)$ with exchange matrix $\begin{bmatrix} 0 & 1 \\ -4 & 0 \\ p & q \end{bmatrix}$, then there is no recurrence (but the number series are still integer due to the Laurent phenomenon). Indeed, The rank 2 seed pattern with exchange matrix $\begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ is of finite type if and only if $bc \leq 3$ by theorem 5.1.1. in [5].

Remark 2 The seed patterns with exchange matrix $\begin{bmatrix} 0 & 1 \\ -2 & 0 \\ p & q \end{bmatrix}$ in exercise 3.2.8 generate the cluster algebra of type B_2 while the seed pattern with exchange matrix $\begin{bmatrix} 0 & 1 \\ -3 & 0 \\ p & q \end{bmatrix}$ in exercise 3.2.9 generate the cluster algebra of type G_2. And the remaining rank 2 cluster algebras are $A_1 \times A_1$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, A_2 $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $G_2^\vee$ $\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$.

Remark 3 In the rank two case, we can observe that the denominator of the Laurent polynomials appearing in the seed pattern have some relationship with their orders in the series. Different cluster variables have different denominators of the form $x_1^{d_1(m)} x_2^{d_2(m)}$. Indeed, this is the simple version of Denominator theorem. So for the rank 2 cluster algebras, the unique period is determined by the Coxeter number h with period $h+2$.

Exercise 3.4.6. Using Laurent phenomenon to show that the sequence $z_0, z_1, z_2 \cdots$ defined by initial conditions $z_0 = z_1 = z_2 = 1$ and

the recurrence $z_{m+3}z_m = z_{m+2}z_{m+1} + 1$ Solution: There are similar exercises including exercise 3.4.7 and exercise 3.4.9, but I will only solve this one.

By the initial condition, we can see that if this recurrence is given by some cluster algebra exchange relations, then every cluster variable can be expressed as Laurent polynomials whose denominators are the monomials of initial cluster variables where here they are all 1, so if we can find such cluster algebra, the problem is done.

It is easy to find that take the matrix $B = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ as the initial exchange matrix. So the initial seed (z_0, z_1, z_2, B) with zero frozen variable can generate a seed pattern.

If we follow the path $1 - 2 - 3 - 1 - 2 - 3 - \cdots$ we find that the exchange matrix sequence will be $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$
$\xrightarrow{3} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{1} \cdots$.

There is a period with 3 in this sequence and we can find that following this path gives us the recurrence in the exercise, so all numbers are integral.

Exercise 5.10.7. Show that the diagram $S^s_{p,q,r}$ is mutation equivalent to $T_{p+r-1,q,s}$

Solution: see lemma 7.3.6. in [**11**].

Exercise 5.10.8 Let Γ be a 2 finite diagram whose underlying graph is an n-cycle with some orientation of edges. Show that Γ is cyclically oriented, and moreover it must be one of the following.

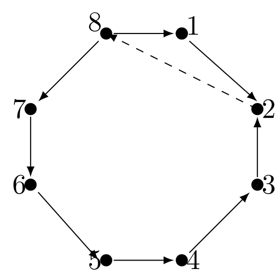
(a). an n-cycle with all weights equal to one (in this case $\Gamma \cong D_n$)

a 3-cycle with edge weights 2,2,1 (in this case, $\Gamma \cong B_3$)

(c) a 4-cycle with edge weights 2,1,2,1 (in this case, $\Gamma \cong F_4$)

Solution: it takes two steps to solve the problem.

Step 1: Γ is cyclically oriented Firstly, we observe that there must be a vertex p with one edge incoming and one edge outgoing. If not, the nearby vertex q comes into our consideration, if this vertex also fails to satisfy the condition of one side outgoing and the other incoming, then we can mutate at this vertex so that the vertex p after the mutation satisfy the condition. We can mutate at this vertex p so that we can have a diagram as follows.



So we have a subdiagram of n-1 cycle, and by the definition of mutation of diagram, the product of weights of this n-1 cycle is the same as the original n-cycle one. Now the induction comes into the play, so we need to consider the case when n=3.

Lemma 1: every triangle in the 2 finite diagram must be cyclically oriented.

Suppose on the contrary that $b_{ik}, b_{kj}, b_{ij} > 0$, then after the mutation at k, $b^{'}_{ij} = b_{ij} + b_{ik}b_{kj} \geq 2$ and $b^{'}_{ji} = b_{ji} - b_{jk}b_{ki} \leq 2$, violating the 2 finiteness. So by induction, the (n-1) cycle must be cyclically oriented, and the other triangle part is also cyclically oriented sharing the same orientation with the (n-1) cycle on their common border edge (edge 2-8) in the figure.

Step 2: Classification according to the weights restriction Again we use the induction with the following lemma.

Lemma 2: Let B be a 2-finite matrix. Then $b_{ij}b_{jk}b_{ki} = -b_{ji}b_{kj}b_{ik}$ for any distinct i, j, k. Also, in every triangle in $\Gamma(B)$, the edge weights are $\{1, 1, 1\}$ or $\{2, 2, 1\}$.

In view of lemma 1, we may assume without loss of generality that B is of the form of a 3×3 matrix $\begin{bmatrix} 0 & a_1 & -c_2 \\ -a_2 & 0 & b_1 \\ c_1 & -b_2 & 0 \end{bmatrix}$, or at least, for the triangle as a subdiagram of our concern, we can just focus on this matrix, where $a_i, b_i, c_i, i = 1, 2$ are all positive integers. Again without loss of generality, we may assume that c_2 is the maximal integer.

After mutation at 2, we will have the $\mu_2(B) = \begin{bmatrix} 0 & -a_1 & a_1b_1 - c_2 \\ a_2 & 0 & -b_1 \\ c_1 - a_2b_2 & b_2 & 0 \end{bmatrix}$

If $a_1b_1 - c_2 \neq 0$and $a_2b_2 - c_1 =\neq 0$, then from Lemma 1, we can see that $a_1b_1 - c_2$ must be positive, otherwise the right upper part of the matrix will have the same sign. So $a_1b_1 > c_2 \geq max(a_1, b_1)$ by the maximality of c_2. Therefore we have $a_1 \geq 2, b_1 \geq 2$, indeed, by the $a_1b_1 > c_2$, we have either a_1 or b_1 bigger than 1, and by the $a_1b_1 > max(a_1, b_1)$, both of a_1 and b_1 are > 1. In the common way, $a_2b_2 > c_1 \geq 1$, so $max(a_2, b_2) \geq 2$, contradicting the 2-finiteness of B.

It only remains to show that the edge weights $|b_{ij}b_{ji}|$ is either $\{1, 1, 1\}$ or $\{2, 2, 1\}$. By the restriction of 2-finiteness, the only possible choice is $c_2 = 3, c_1 = 1, \{a_1, b_1\} = \{1, 3\}$, then $|b'_{ij}b'_{ji}| = 4$

Remark 1 Actually the lemma 1 and lemma 2 serve as a version of the solution of this exercise in the case when n=3。

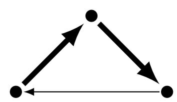
As we have seen in step 1, we can by induction reduce the n-cycle into the (n-1)-cycle without changing the product of the cycle weights. So, the weights product can only be 1 or 4, and the weights on the edges of the 2-finite diagram can only be 1,2 or 3.

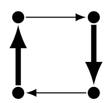
Case 1: all weights equal to 1

We have a diagram of n-cycle with cyclic orientation, by exercise 5.4.1, this must be of type D_n.

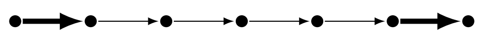
Case 2: weights product is 4

There must be two unique edges on the n-cycle with weight 2 with all other edge weight 1. In the lower dimensional case, like n=4 or 3, we have the following diagram,

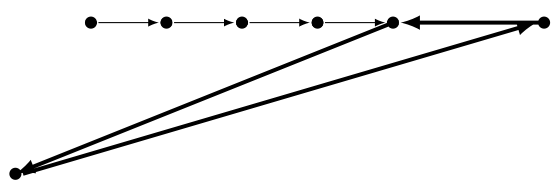




If two edges of weight 2 have one vertex in common, then they must belong to some triangle, otherwise mutation at this vertex will generate a new edge weight c satisfying $\sqrt{c} = \sqrt{2 \times 2}$, |c|=4, this is definitely not a 2-finite diagram. So the n=4 case must be of the form shown in the above figure
If n≥ 5, we can cut out a subdiagram of the type $C_m^{(1)}$, $m \geq 2$.



After mutation at the second vertex from the left, and inductively mutation from left to right, we will finally have the diagram as the figure shown.



So we have a subdiagram of a triangle with all weights (2,2,2), which is not 2-finite by lemma 2. So $C_m^{(1)}$ containing such a triangle is not 2-finite. So the exercise is done. The 2-finite n-cycle is classified.

Remark 2 The n=4 case indeed contains a path with the same weight pattern as $C_3^{(1)}$, but this is not a subdiagram, considering that a subdiagram should contain all the edges between the vertices in the subdiagram.

Exercise 5.3.10. Here is an example of cluster algebra of type A_n constructing from the **double Bruhat cells** for the special linear group SL_{n+1}, more precisely, from the cells associated with pairs of Coexter elements in the associated symmetric group S_{n+1}.
Let $L_n \in SL_{n+1}(C)$ be the subvariety of tridiagonal matrices
$\begin{bmatrix} v_1 & q_1 & 0 & \cdots & 0 \\ 1 & v_2 & q_2 & \ddots & \vdots \\ 0 & 1 & v_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & q_n \\ 0 & \cdots & 0 & 1 & v_{n+1} \end{bmatrix}$ of determinate 1 with $q_1 \cdots q_n$ nonzero. For $i, j \in \{1, ..., n+3\}$ satisfying $i+2 \leq j$, consider the solid principal minor $U_{ij} = \triangle_{[i,j-2],[i,j-2]} \in C\,[L_n]$. Prove that these functions U_{ij} satisfy the relations $U_{ik}U_{ji} = q_{j-1}q_j \cdots q_{k-2}U_{ij}U_{kl} + U_{il}U_{jk}$. Then show that there is a seed pattern of type A_n, with the frozen variables $q_1, \cdots q_n$, in which the cluster variables U_{ij} associated with the diagonals of the convex polygon P_{n+3} satisfy the above exchange relations.

Remark 1 It is very difficult to directly verify these relations by calculations. Actually, I have no clue about how to prove it in this way, but there is a better way of proof by cluster algebra, especially the technique of changing coefficients.

Solution: this solution is a special case of the general case described in the [**7**]. I just take the conditions of SL_{n+1} into the general theorem for semisimple linear alegbraic group G and add some details. For more information, please read the original paper [**7**].

Step 0: double Bruhat cell decomposition of SL_{n+1}

In general sense, the Bruhat decomposition is $G = \coprod_{u\in W} BuB = \coprod_{v\in W} B^-vB^-$. For SL_{n+1}, the Borel subgroup and the opposite Borel subgroup are the upper triangle matrices and lower triangle matrices. Indeed, different Borel subgroups correspond to different Bruhat decompositions differed by a group action, which can be understood as relabelling the rows and columns. The Weyl group is exactly the permutation group S_{n+1}, and there is an explicit description of the Bruhat cell BuB for type A group SL_{n+1}: $x \in BuB$ if and only if,

$\cdot\triangle_{w([1,i]),[1,i]}(x) \neq 0$ for i=1,$\cdots$,r

$\cdot\triangle_{w([1,i-1]\cup\{j\}),[1,i]}(x) = 0$ for $i < j, w(i) < w(j)$

Similarly, we have a description for double Bruhat cell $G^{u,v} = BuB \cap B^{-}vB^{-}$:

$\cdot\triangle_{[1,i],v^{-1}([1,i]})(x) \neq 0$, $\triangle_{w([1,i]),[1,i]}(x) \neq 0$ for i=1,$\cdots$,r

$\cdot\triangle_{[1,i],[1,i-1]\cup\{j\}}(x) = 0$ for $i < j, v^{-1}(i) < v^{-1}(j)$

$\cdot\triangle_{w([1,i-1]\cup\{j\}),[1,i]}(x) = 0$ for $i < j, w(i) < w(j)$

But actually, the subvariety of tridiagonal matrices is not one of the cells described above. We need to narrow our scope by introducing the **reduced double Bruhat cell**: $L^{u,v} = NuN \cap B^{-}vB^{-}$, in this way, there are n more equations of restriction on this subvariety by letting $\triangle_{uw_i,w_i} = 1$ for $i \in [1,n]$. Indeed, the dimension of the reduced double Bruhat cell $L^{u,v}$ $(l(u)+l(v))$ is n less than the dimension of double Bruhat cell $G^{u,v}$ $(l(u)+l(v)+n)$.

Step 1: subvariety of trigonal matrices L_n is exactly the reduced double cell $L^{c,c^{-1}}$. By the description in the Step 0, we can see that for the longest word $c = s_1 s_2 \cdots s_n$ in S_{n+1}, the equations and inequalities in the above step can be translated into the following,

A matrix $M \in SL_{n+1}$ belongs to the reduced double Bruhat cell $L^{c,c^{-1}}$ if and only if it satisfies the following conditions:

(1)$\triangle_{I,[1,k]} = \triangle_{[1,k]} = 0$ for k+1,$\cdots$,n, and all subsets $I = \{i_1 < \cdots < i_k\} \subseteq [1, n+1]$ such that $i_v > v+1$ for some $v = 1, \cdots, k$

(2)$\triangle_{[1,k],[2,k+1]} \neq 0$, and $\triangle_{[2,k+1],[1,k]} = 1$ for $k = 1, \cdots, n$

To prove $L = L^{c,c^{-1}}$, we start with the inclusion $L \subseteq L^{c,c^{-1}}$. Let $M = (m_{ij}) \in L$. For tridiagonal matrices, the matrix entry m_{ij} with $|i-j| > 1$ is zero, and for the minors in the form of (1), there must be a whole column of this minor with such m_{ij}, so (1) is satisfied for tridiagonal matrices. The minors of the form $\triangle_{[1,k],[2,k+1]}$ and $\triangle_{[2,k+1],[1,k]}$ are actually the upper and lower triangle whose determination is easily calculated as the product of the diagonal elements.

To prove the inverse inclusion $L^{c,c^{-1}} \subseteq L$, we have to apply the factorization of $L^{c,c^{-1}}$ in [**8**], every element x in $L^{c,c^{-1}}$ can be uniquely factorized as $x = x_{-1}(u_1)\cdots x_n(u_n)x_n(t_n)\cdots x_1(t_1)$ with $u_i, t_i \in C^*$ which is a parametrization of the cell $L^{c,c^{-1}}$ with index $\{[1,\cdots,n],[n,\cdots,1]\}$ for $(c,c^{-1}) \in W\times W$ when $c = (1,\cdots,n) = s_1\cdots s_n$. The $x_{-i}(u_i) = \psi_i(\begin{bmatrix} u_i^{-1} & 0 \\ 1 & u_i^{-1} \end{bmatrix})$ and $x_i(v_i) = \psi_i(\begin{bmatrix} 1 & v_i \\ 0 & 1 \end{bmatrix})$ where ψ_i is induced by the Lie algebra embedding from $\mathfrak{sl}_2$ into the semisimple Lie algebra $\mathfrak{h}$ of $G = SL_{n+1}$. The product of such forms is tridiagonal by induction. And by the condition (1) and (2), the tridiagonal matrices in $L^{c,c^{-1}}$ must also satisfy the condition that $q_i \neq 0$ above the diagonal and the elements below the diagonal is 1, which is exactly the element in the L.

Step 2: cluster algebra of $C\left[L^{c,c^{-1}}\right]$ Since we have proved the equivalence of the tridiagonal matrices L and the reduced double Bruhat cell $L^{c,c^{-1}}$, all the things can be discussed under the context of the linear algebraic group and combinatorics.

In general sense, the coordinate ring of the reduced Bruhat cell has a cluster algebra determined by its type of the simply connected semisimple group G (see theorem 1.2 in [**7**]).

Fix an element $c \in W$, considering that c can be factorized as product of simple reflections s_i, we can associate to c a $n\times n$ skew-symmetrizable matrix $B(c) = (b_{ij})_{i,j\in I}$ by setting $b_{ij} = \begin{cases} = -a_{ij} & a_{ij} \neq 0, s_i \ precedes\ s_j\ in\ c \\ = a_{ij} & a_{ij} \neq 0, s_j\ precedes\ s_i\ in\ c \\ = 0 & otherwise \end{cases}$

(We denote the condition of s_i precedes s_j in c as $i <_c j$)

In addition to the exchange matrix $B(c)$, we also need to construct the corresponding **algebraic objects** to get the cluster algebra. Here comes the notion of **generalized minors** $\triangle_{\gamma,\epsilon} \in C[G]$ labelled by weights γ, ϵ belonging to the same $W-orbit$ of the same fundamental weight (see [**9**] for detailed definition) which are special kind of regular functions on G. Restricting the **principal** minors (when $\gamma = \epsilon)\triangle_{\gamma,\gamma}$ of these functions to the reduced double cell $L^{c,c^{-1}}$, we have the cluster variables $x_{\gamma;c}$. Likewise, the coefficient tuple element $y_{j;c}$ is the restriction of $\triangle_{w_j,cw_j}\prod_{i<_cj}\triangle_{w_i,cw_i}^{a_{i,j}}$, where $w_i (i \in I)$ is the fundamental weight.

By theorem 1.2 in [**7**], we have the cluster algebra $A(c)$ with principal coefficients of the coordinate ring $C\left[L^{c,c^{-1}}\right]$ at the initial seed (x, y, B) given by $\mathbf{x}$=$(x_{w_i;c} : i \in I)$, $\mathbf{y}$=$(y_{i,c} : i \in I)$, $B = B(c)$. Furthermore, $A(c)$ is of finite type with Cartan-Killing type the same as G (this can be seen by the way of constructing matrix $B(c)$).

For the type A_n: SL_{n+1} case, the **generalized minors** are exactly the ordinary minors in the matrix (γ and ϵ in the same orbit means that they have the same length) with Weyl group naturally acting on the weights as S_{n+1} acting on $[1, \cdots, k]$.

It is easy to get the **initial seed**:$(x_0, y_0, B(c))$ in the above theorem, $x_{0,k} = \triangle_{[1,k],[1,k]}$ and the seemingly complicated $y_{0,j}$ is simply the q_j as the entry above the diagonal, considering the fact that the only index k satisfying $k <_c j$ can only be $j-1$ with all $a_{ij}(|i-j| > 1)$ vanish in the type A Cartan matrix. And the exchange matrix is the following standard type A one. $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \\ -1 & 0 & 1 & \ddots & \vdots & \\ & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & \\ 0 & \cdots & 0 & -1 & 0 & \end{bmatrix}$

Moreover, we can have all the cluster variables in this cluster algebra. By the theorem 1.5 in [**7**], they are the generalized minors indexed by a special subset $\prod(c)$ of $[1, n+1]$. In SL_n case, element in $\prod(c)$ is arbitrary consecutive subset $[m+1, m+k] \subseteq [1, n+1]$, and the total cluster variables are exactly the principal elements $\triangle_{[i,j],[i,j]}$:$1 \leq j \leq n+1, (i, j) \neq (1, n+1)$

Step 3: changing coefficient to have a cluster algebra with principal coefficient By the above steps, we have get a cluster algebra of finite type with all its cluster variables, but the coefficient semifield is still up in the air.

And back to the original aim of the exercise: $U_{ik}U_{ji} = q_{j-1}q_j \cdots q_{k-2}U_{ij}$ $U_{kl} + U_{il}U_{jk}$, obviously the tricky part of the relations is exactly the coefficient $q_{j-1}q_j \cdots q_{k-2}$. Indeed, for type A cluster algebra, there is a correspondence between the cluster variables in the $C\left[L^{c,c^{-1}}\right]$ and diagonals in the polygon by $[i,j] \leftrightarrow < i, j+2 >$ where $[i,j]$ is the label of the principal minors and $< i, j+2 >$ is the diagonal connecting vertex i and $j+2$, and the exchange relations must have the form $x_{<i,k>}x_{<j,l>} = p^{+}_{ik,jl}x_{<i,j>}x_{<k,l>} + p^{-}_{ik,jl}x_{<i,l>}x_{<j,k>}$ where p is in the coefficient semifield. In this step, we will solve the problem by projecting a 'universal' cluster algebra onto the $C\left[L^{c,c^{-1}}\right]$

In fact, this correspondence with the diagonals is especially useful after introducing the following lemma about constructing cluster algebra from an arbitrary Coxeter element c in Weyl group, see proposition 5.6 in [**7**].

Lemma 1: For every Coxeter element c, there exists a cluster algebra $\widetilde{A(c)}$ of geometric type, satisfying the following properties:

(1) The coefficient semifield of $\widetilde{A}(c)$ is $\widetilde{P}(c) = Trop(p\,[\gamma] : \gamma \in \prod(c))$.

(2) The cluster variables in $\widetilde{A}(c)$ are labeled by the set $\prod(c)$, the variable corresponding to $\gamma \in \prod(c)$ being dennoted $x\,[\gamma]$.

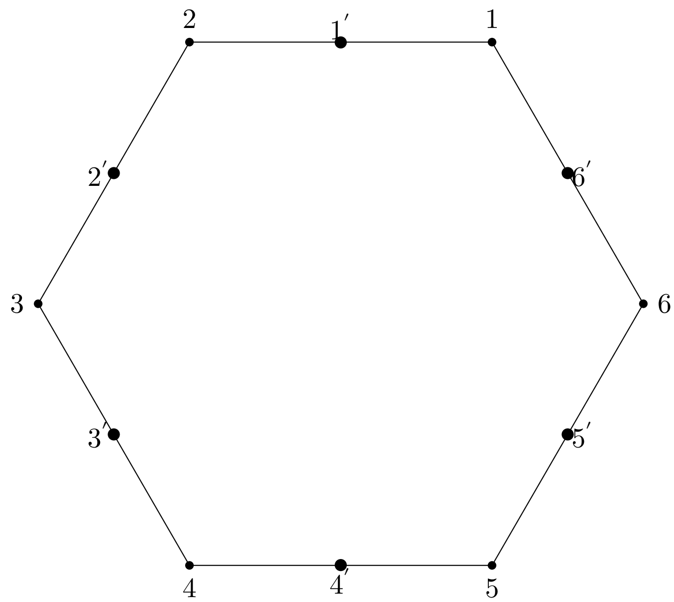
(3). The initial seed of $\widetilde{A}(c)$ is of the form (**x**,**y**,B(c)), where **x**=$(x\,[w_i] : i \in I)$ and **y**=$(y_j : j \in I)$ with

$y_j = p\,[w_j]\,p\,[cw_j]^{-1}\prod_{\gamma\in\prod(c)-\{w_j,cw_j\}} p\,[\gamma]^{(\gamma||cw_j)_c + \sum_{i<_c j} a_{i,j}(\gamma||cw_i)_c}$

Now the geometric interpretation comes into the play. By a theorem in [10], we can find a way to express the coefficients $p^{+}_{ik,jl}$ and $p^{-}_{ij,kl}$ by means of some 'geometric duality'.

Lemma 2 After relabeling the cluster variables and the generators of the coefficient semifield in the cluster algebra $\widetilde{A}(c)$ above by the diagonals with correspondence $[i,j] \leftrightarrow <i,j>$, the coefficient $p^{+}_{ik,jl}$($p^{+}_{ij,kl}$,resp) turns into the product of generators $p<a,b>$ such that the dual diagonal $<a^{'},b^{'}>$ is contained in the strip formed by $<i,j>$ and $<k,l>$(resp. by $<i,l>$ and $<j,k>$)

The dual diagonals are shown as an example in the below figure, $i^{'}$ is the midpoint of the edge connecting i and $i-1$.



However, this cluster algebra has too many 'coefficient' elements. Indeed, $\prod(c)$ is much bigger than the index set I of the generators of Weyl group, whose number is also the rank of the cluster algebra (of course the fundamental weights $w_i, i \in I$ are contained in the $\prod(c)$ by definition). But we need not use as much as the whole set of $\prod(c)$ to construct the semifield as this cluster algebra $\widetilde{A}(c)$, instead, we only need to pick the 'principal' ones. Let P°=Trop(y_i° : $i \in I$) be the coefficient semifield of the cluster algebra with principal coefficients $A^{\circ}(c)$, with $\psi : \widetilde{P}(c) \to P^{\circ}$ being a tropical semifield homomorphism acting on the generators as follows:
$\psi(p[\gamma]) = \begin{cases} = y_i^{\circ} & if\ \gamma = w_i \\ = 1 & otherwise \end{cases}$
Now we need to associate these 'abstract' cluster algebras $\widetilde{A(c)}$ and $A^{\circ}(c)$ with $C\left[L^{c.c^{-1}}\right]$ which is the $A(c)$ in step 2. In fact, the complexification of $A^{\circ}(c)$ can be identified with the coordinate ring $C\left[L^{c,c^{-1}}\right]$ and the complexification of $\widetilde{A}(c)$ is $C\left[G^{c,c^{-1}}\right]$, see chapter 4 and 5 in [**7**] for more details. So we can view $A(c)$ as the same thing as $A^{\circ}(c)$, at least for our concern of

exchange relations.

Now the last step of expressing the exchange relations in the coordinate ring of the subvariety of tridiagonal matrices $L^{c,c^{-1}}$ is projecting (or specializing) by means of the above ψ. It is easy to rewrite the $x_{<i,k>}x_{<j,l>} = p^{+}_{ik,jl}x_{<i,j>}x_{<k,l>} + p^{-}_{ik,jl}x_{<i,l>}x_{<j,k>}$ by inspecting the '**dual diagonals**' as the following one:

$x_{<i,k+2>}x_{<j,l+2>}y_{j-1}y_j\cdots y_k x_{<i,j>}x_{<k+2,l+2>} + 1 \times x_{<i,l+2>}x_{<j,k+2>}$. So the exercise is done.

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