This is a very basic problem arising in the exercise 5.1-5.3 in chapter 3 of GTM52 about generalization of arithmetic genus to abstract variety

5.2 (a). Let X be a projective. scheme over field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k. Let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in Q[z]$, such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in Z$. We call P the Hilbert polynomial of sheaf $\mathcal{F}$ with respect to the sheaf $\mathcal{O}(1)$.

(b). Now let $X = P_k^r$, and let $M = \Gamma_*(\mathcal{F})$, considered as graded $S = k[x_0, \cdots, x_r]$ module. Show that the Hilbert polynomial of sheaf $\mathcal{F}$ is exactly the Hilbert polynomial of M defined in chapter 1.

Solution: By the hint given in the book, we can construct an exact sequence suitable for induction on dimension of *Supp* $\mathcal{F}$. First of all, we can transfer the **projective scheme over k** to **projective space** P^n.

Indeed, If X is a closed subscheme of P^n, by the lemma 2.10 in [**1**], there is an isomorphism $H^i(X, \mathcal{F}) \cong H^i(P^n, j_*\mathcal{F})$ where j is the closed embedding (for the closed subset it can be seen as extension by zero). So, $\chi(\mathcal{F}) = \sum(-1)^i dim_k H^i(X, \mathcal{F}) = \chi(j_*\mathcal{F})$, and by the definition of direct image sheaf, $\Gamma_*(X, \mathcal{F}) = \Gamma_*(P^n, j_*\mathcal{F}) = M$. Here I also want to emphasize that $j_*\mathcal{F}$ is coherent so the Euler characteristic $\chi(P^n, j_*\mathcal{F})$ makes sense.

Indeed, $j_*\mathcal{F}$ is quasicoherent by proposition 5.8 in [**1**], but in the general case $j_*(\mathcal{F})$ may not be coherent when $\mathcal{F}$ is coherent. But it is assured that when morphism $j : X \to Y$ is projective and X with Y are of finite type over k this is true by corollary 5.20, chapter 2 in [**1**] (see **Remark 1** for more thoughts on this topic).

So from now on, we only consider projective space and coherent sheaf $\mathcal{F}$ on it.

In P^n, any closed subset Z as a noetherian scheme has only finite irreducible components by proposition 4.9, chapter 2 in [**2**]. Pick one closed point from each component, $f_1, \cdots, f_m$ (see **Remark 2** for choice of such closed points). Now we view the projective space as $P^n = k^{n+1}/k^*$, and the closed point has its place in such coordinate system. And the hyperplanes in P^n can be parametrized by the 'dual projective space' $(P^n)^*$, which is isomorphic to P^n. In such context, the set of hyperplanes avoiding certain point $f_i = (a_0, a_1 \cdots, a_n)$ is the open subset $U_i = \{(x_0, x_1 \cdots, x_n) | \sum a_i x_i \neq 0\}$, which is a dense subset of projective space. So $\cap U_i \neq \emptyset$, and there is always a hyperplane H avoiding all these points so that H not contains any irreducible components of closed set Z. Write H as the zero set $\sum b_i x_i = 0$ and $(b_0, \cdots, b_n)$ in $k[x_0 \cdots, x_n]_1$ actually serves as a global section s of the twisted sheaf $\mathcal{O}(1)$. It is obvious that the zero locus of the global section s - $V(s)$ contains no irreducible components of closed projective subvariety Z.

We also should see that the $M = \Gamma(\mathcal{F})$ is finitely generated module over $k\left[x_0 \cdots, x_n\right]$ so the Hilbert polynomianl of M makes sense (see **Remark 3** for more information on the local property of 'finitely generated'). For the finitely generated graded $S_k\left[x_0, \cdots, x_n\right]$-module M, by (Hilbert-Serre) theorem 7.5, chapter 1 in [1], the Hilbert function $\psi_M(n) = dim_k M_n$ can be extended to a unique numerical polynomial $P_M(z) \in Q\left[z\right]$ such that $P_M(n) = \psi_M(n) = dim_k(M_n)$ (see **Remark 4** for more information on the Hilbert regularity of finitely generated $k\left[x_0, \cdots, x_n\right]$-module M).

When dim $Supp(\mathcal{F}) = 0$, by the Grothendieck vanishing theorem, all cohomology group is zero except $H^0(X, \mathcal{F}(n)) = \Gamma(X, \mathcal{F}(n))$. So $\chi(\mathcal{F}(n)) = dim(\Gamma_*(\mathcal{F})_n) = P_{\Gamma_*(\mathcal{F})}(n)$ for $n >> 0$, where $P_{\Gamma_*(\mathcal{F})}(n)$ is Hilbert polynomial of module $M = \Gamma_*(\mathcal{F})$. Thus, for dimension 0 case, there is a numerical polynomial $P(z)$ restricting to $\chi(\mathcal{F}(n))$ when $z \in Z$, and this polynomial is exactly the Hilbert polynomial defined in chapter 1 in [**1**]. So (a) and (b) for case when dim Supp$(\mathcal{F}) = 0$ is done (at least for $n >> 0$).

Now all preliminary work has been done for the induction steps. For coherent sheaf $\mathcal{F}$, by the choice of $s \in \mathcal{O}(1)$ as above such that the zero locus of s will not contain any component of $Supp(\mathcal{F})$, there is a sheaf morphism $\mathcal{F}(-1) \xrightarrow{s} \mathcal{F}$ which is an isomorphism when we restrict to the open set $P^n \backslash V(s)$. And correspondingly, we have a long exact sequence of sheaf $0 \to \mathcal{K} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{C} \to 0$ where $\mathcal{K}$ and $\mathcal{C}$ are kernel and cokernel of the sheaf morphism. By the choice of s, $Supp(\mathcal{K}) \subset Supp(\mathcal{F}) \cap V(s)$ whose dimension is strictly less than $Supp(\mathcal{F})$. By induction, we can find rational coefficient polynomials $P_{\mathcal{K}}(n) = \chi(\mathcal{K}(n))$ and $Q_{\mathcal{C}}(n) = \chi(\mathcal{C}(n))$. The Euler characteristics χ is **additive** (see exercise 5.1, chapter 3 in [**1**]), i.e, for short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, there is $\sum_{i=1}^{i=3}(-1)^i\chi(F_i) = 0$. And by category theory, any additive function χ also has the same additive property for exact sequence of arbitrary length, i.e, $\chi(F_n) - \chi(F_{n-1}) = P_{\mathcal{K}}(n) - P_{\mathcal{K}}(n)$. The right hand is a polynomial $R(z) = (P_{\mathcal{K}}(z) - P_{\mathcal{C}}(z))$ in $Q\left[z\right]$ such that $R(n) \in Z$ when $n \in Z$, by proposition 7.3, chapter 1 in [**1**], we can write $R(z)$ in the form of binomial coefficient function $R(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots + c_r$, and by the equality $\binom{z+1}{r} - \binom{z}{r} = \binom{z}{r-1}$, we can construct such a numerical polynomial $P_{\mathcal{F}}(z) = c_0\binom{z}{r+1} + \cdots + c_r\binom{z}{1} + c_{r+1}$ where difference $\triangle P_{\mathcal{F}}(z) = P_{\mathcal{F}}(z+1) - P_{\mathcal{F}}(z) = R(z) = P_{\mathcal{K}}(z) - P_{\mathcal{C}}(z)$ and c_{r+1} is rightly chosen so that the constant term is $\chi(\mathcal{F})$. So far, (a) has been done because such a numerical polynomial has been found. Now we need to prove (b)—this Hilbert polynomial of sheaf $\mathcal{F}$ is exactly the Hilbert polynomial of the module $\Gamma_*(\mathcal{F})$. This is simple, because when $n >> 0$, cohomology group $H^i(P^n, \mathcal{F}(n)) = 0$ for $i > 0$, so $P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n)) = dim_k(H^0(P^n, \mathcal{F}(n))) = dim_k(\Gamma(\mathcal{F}(n))) = dim_k(\Gamma_*(\mathcal{F})_n) = P_{\Gamma_*(\mathcal{F})}(n)$ for $n >> 0$. The right hand side and left hand side are all polynomial functions so they are actually the same so (b) is done.

Remark 1: direct image of coherent is in general not coherent, but by corollary 5.20, chapter 2 in [**1**], we have projective morphism between schemes of finite type over k pushing forward the coherent sheaf $\mathcal{F}$ to the coherent sheaf $f_*\mathcal{F}$. Here the condition on the scheme is strong—finite type over k. However this condition on schemes can be weakened by putting a stronger condition on morphism. By proposition 1.14, chapter 5 in [**2**], for morphism $f : X \to Y$ with X noetherian and f finite, coherent sheaf $\mathcal{F}$ is pushed forward to coherent sheaf $f_*(\mathcal{F})$.
However, better version is provided in theorem 8.8, chapter 3 in [**1**] where we can see that projective morphism between noetherian schemes preserves coherence.

Remark 2: In quasi-comapct scheme, every point has a closed point in its closure. So every irreducible component as a closed irreducible subset must contain a closed point.

Remark 3: It is known that $\Gamma_*(\mathcal{F})^\sim \cong \mathcal{F}$ for quasicoherent sheaf $\mathcal{F}$, It is clear that when $\Gamma_*(\mathcal{F})^\sim$ is finitely generated, $\mathcal{F}$ is coherent, and when $\mathcal{F}$ is coherent, by lemma 34.13.1(d) in the stack project, $\Gamma_*(\mathcal{F})$ is finitely generated.

Remark 4: This is indeed very confusing: in all the textbooks, the Hilbert polynomial of graded module M is defined as a polynomial $P_M(z)$ equal to the Hilbert function $\psi_M(n) = dim_k(M_n)$ when $z = n >> 0$. And the definition of Hilbert polynomial of sheaf $\mathcal{F}$ in rising sea by Vakli also is defined as a polynomial equal to $\chi(\mathcal{F}(n))$ when $n >> 0$. However, in the Hartshone, this exercise (exercise 5.2 in chapter 3) says the polynomial should equal to the $\chi(\mathcal{F}(n))$ for all $n \in Z$. So is it a little mistake of Hartshone?

Remark 5: Following the case when $Supp(\mathcal{F}) = 0$, I am wondering the Hilbert function $\psi_{\Gamma_*(\mathcal{F})}(n)$may equal to characteristics $\chi(\mathcal{F}(n))$. Anyway, we have shown that they are the same polynomial function when $n >> 0$, but what is the relationship between the $\psi_{\Gamma_*(\mathcal{F})}(n)$ and $\chi(\mathcal{F}(n))$ as function from Z to Z? Do we have $\psi(\mathcal{F}(n)) = dim_k(\Gamma_*(\mathcal{F})_n)$?
By induction, for sheaf $\mathcal{C}$ and $\mathcal{K}$, the Hilbert polynomials of sheaf $P_\mathcal{C}$ and $P_\mathcal{K}$ equal to their corresponding Hilbert polynomials of Γ_* module. We know that in usual, global section functor Γ is not exact unless the scheme where sheaf lies on is affine. However, for quasicoherent sheaf $\mathcal{F}$ on $ProjS = Projk[x_0, \cdots, x_n]$, there is isomorphism $\Gamma_*(\mathcal{F})^\sim \cong \mathcal{F}$. So from the sheaf exact sequence: $0 \to \mathcal{K} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{C} \to 0$, we have the exact sheaf sequence $0 \to \Gamma_*(\mathcal{K})^\sim \to \Gamma_*(\mathcal{F}(-1))^\sim \to \Gamma_*(\mathcal{F})^\sim \to \Gamma_*(\mathcal{C})^\sim \to 0$, so at stalk level we have exact sequence of localization at homogeneous prime ideal not containing S_+, $\mathfrak{p}$: $0 \to \Gamma_*(\mathcal{K})_\mathfrak{p} \to \Gamma_*(\mathcal{F}(-1))_\mathfrak{p} \to \Gamma_*(\mathcal{F})_\mathfrak{p} \to \Gamma_*(\mathcal{C})_\mathfrak{p} \to 0$, so we have an exact sequence of graded S module $0 \to \Gamma_*(\mathcal{K}) \to \Gamma_*(\mathcal{F}(-1)) \to \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{C}) \to 0$. Hilbert polynomials

of module is additive, so $P_{\Gamma_*(\mathcal{K})}(z) - P_{\Gamma_*(\mathcal{C})}(z) = P_{\Gamma_*(\mathcal{F})}(z) - P_{\Gamma_*(\mathcal{F})}(z-1)$ where $P_{\mathcal{C}} = P_{\Gamma_*(\mathcal{C})}$ and $P_{\mathcal{K}} = P_{\Gamma_*(\mathcal{K})}$, so we have $P_{\mathcal{F}}(z) - P_{\mathcal{F}}(z-1) = P_{\Gamma_*(\mathcal{F})}(z) - P_{\Gamma_*(\mathcal{F})}(z-1)$.
However, this is a totally wrong way because we exclude the prime ideal containing the S_+ so we can not conclude that $0 \to \Gamma_*(\mathcal{K}) \to \Gamma_*(\mathcal{F}(-1)) \to \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{C}) \to 0$ is an exact sequence from the condition ($\mathfrak{p}$ not containg S_+): $0 \to \Gamma_*(\mathcal{K})_{\mathfrak{p}} \to \Gamma_*(\mathcal{F}(-1))_{\mathfrak{p}} \to \Gamma_*(\mathcal{F})_{\mathfrak{p}} \to \Gamma_*(\mathcal{C})_{\mathfrak{p}} \to 0$. So I think it may be wrong about the conjecture that $\psi_{\Gamma_*(\mathcal{F})}(n) = \chi(\mathcal{F}(n))$.

5.3 Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$. the $\chi(X)$ is the Euler characteristics. Note that it depends only on X, not on any projective embedding.

(a). If X is integral, and k algebraically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that $p_a(X) = \sum_{i=0}^{r-1}(-1)^i dim_k H^{r-i}(X, \mathcal{O}_X)$. In particular, if X is curve, we have $p_a(X) = dim_k H^1(X, \mathcal{O}_X)$.

(b). If X is a closed subvariety of P_r^k. Show that p_a coincides with the one defined in chapter 1, exercise 7.2 by Hilbert polynomial, which depends on the projective embedding.

(c). If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational.

Solution: To prove (a), we only need to show that $\Gamma(X, \mathcal{O}_X) \cong k$.
As integral closed subscheme of P^n, $X = Proj(k[x_0, \cdots, x_n]/I_X)$, I_X must be prime ideal. Indeed, in the noetherian ring $k[x_0, \cdots, x_n]/I_X$, zero ideal (0) has the primary decomposition into $(0) = \cap \mathcal{Q}_i$ where $\sqrt{\mathcal{Q}_i} = P_i$ is prime ideal. From the irreducibility of X, we have there is a unique minimal ideal $\mathfrak{p}_0$ containing (0). X is reduced, so $\sqrt{I_X} = I_X = \sqrt{\cap \mathcal{Q}_i} = \cap \sqrt{\mathcal{Q}_i} = \cap \mathfrak{p}_i$, due to the minimality of $\mathfrak{p}_0$, we have $I_X = \mathfrak{p}_o$. So $S(X) = k[x_0, \cdots, x_n]/I_X$ is integral domain. So (0) is minimal homogeneous prime ideal and the function field $K(X)$ of X is $S_{(0)} = (k[x_0, \cdots, x_n]/I_X)_{(0)}$.
Thinking of $\Gamma(X, \mathcal{O}_X)$,$K(X)$ and $S(X)$ are all subrings of quotient field L of $S(X)$ (natural morphism from $\Gamma(X, \mathcal{O}_X)$ to function field $\mathcal{O}_{X,\mathfrak{p}_0}$ is injective for integral scheme), this means that $x_i^{N_i} f \in S(X)_{N_i}$ for each i and $f \in \Gamma(X, \mathcal{O}_X)$. Now chose $N \geq \sum N_i$. Then $S(X)_N$ is spanned as a k-vector space by monomials of degree N in $x_0, \cdots, x_n$, so we have $S(X)_N f \subset S(X)_N$. Iterating, we have $S(X)_N f^q \subset S(X)_N$. Thus, $x_0^N f^q \in S(X)$ and $S(X)[f]$ of L is a finitely generated $S(X)$-module, and therefore f is integral over $S(X)$ since $S(X)$ is noetherian ring. This means that there are elements $a_1, \cdots, a_m \in S(X)$ such that :$f^m + a_1 f^{m-1}, \cdots + a_m = 0$. Since $deg(f) = 0$, we can replace the a_i by their homogeneous conponents of degree 0, and still have a valid equation. But

$S(X)_0 = k$, so $a_i \in k$ and f is algebraic over k. But k is algebraically closed, so $f \in k$ which completes the proof.

(b). Back in chapter 1, the arithmetic genus of a projective variety X is defined as $(-1)^r(P_X(0)-1)$ where $P_X(z)$ is the Hilbert polynomial of $S(X)$ and r is the dimension of X. By pushing forward, $j_*(\mathcal{O}_X)$ on P^n is coherent sheaf with $dim(Supp(j_*\mathcal{F})) = dim(X)$ where $j : X \to P^n$ is closed embedding. From the above exercise, we have seen that this Hilbert polynomial P_X is exactly the Hilbert polynomial of sheaf $j_*\mathcal{F}$. So take value at 0, (b) is done .

(c). For nonsingular projective curves, birational is equivalent to isomorphism, so $p_a(X)$ is definitely a birational invariant. As for the singular curve birational to X, I have no clue proving the arithmetic genus invariance.
By exercise 7.2(b), chapter 1 in [**1**], $p_a(X) = \frac{1}{2}(d-1)(d-2)$ for plane curve of degree d in P^2, if $d \geq 3$, then $p_a \neq 0 = p_a(P^1)$, so it is not rational becasue p_a is birational invariant.

My question 1: Just like Remark 4 in the exercise 5.2 saying, What I actually prove is that when $n >> 0$, there is a polynomial $P(z) \in Q[z]$ satisfying $P(n) = \chi(\mathcal{F}(n))$. Even in the zero dimensional case, $\chi(\mathcal{F}(n)) = dim_k(\Gamma_*(\mathcal{F})_n)$, but we only have a polynomial function P equal to $dim_k(\Gamma_*(\mathcal{F})_n)$ when $n >> 0$, so this polynomial P is equal to $\chi(\mathcal{F}(n))$ when $n >> 0$. There is no evidence that when n is small, like $n = 0$ case where we need to prove the equivalence of arithmetic genus (take z=0 in Hilbert polynomial), we can't make sure that $P(0) = \chi(\mathcal{F}(0))$ so actually the solution to exercise 5.3 (b) is not solid. I don't know what's wrong with this solution?

My question 2: how to prove that arithmetic genus of curves is birational invariant when curves may have singular points like exercise 5.3(c)?

[**1**]:Harsthone, GTM 52
[**2**]: Liu Qing, algebraic geometry and arithmetic curves.
[**3**]: Vakil, the rising sea foundation of algebraic geometry.