Notes on Rosenberg's Reconstruction Theorem

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Contents

1 Introduction.

There are three main theorems in the paper by Martin Brandenburg (see [2]) that I would like to give a careful explanation.

Theorem 1.1. *Let X, Y be quasi-separated schemes, if* $\mathrm{QCoh}(X) \cong \mathrm{QCoh}(Y)$*, then* $X \cong Y$.

More precisely, there is a Spec construction for the category $\mathrm{QCoh}(X)$ so that $\mathrm{Spec}(\mathrm{QCoh}(X)) \cong X$ for the quasi-separated scheme. This construction is defined for the general case when $\mathcal{A}$ is an abelian category, $\mathrm{Spec}(\mathrm{QCoh}(X))$ is a ringed space.

Besides, this correspondence between isomorphism of 'spaces' and equivalence of 'categories' is given in the following form.

Theorem 1.2. *Each equivalence of category* $\mathrm{QCoh}(X) \cong \mathrm{QCoh}(Y)$ *is isomorphic to* $f^* \otimes \mathcal{L}$ *for some isomorphism* $f : X \cong Y$ *and invertible sheaf* $\mathcal{L}$ *on* Y.

By the above equivalence of categories (actually groupoids), we have the following isomorphism of groups:

Theorem 1.3. *If X is a quasi-separated scheme, then the automorphism class group of* $\mathrm{QCoh}(X)$*, i.e. the group of isomorphism classes of auto-equivalences of* $\mathrm{QCoh}(X)$ *is equivalent to the semi-product* $\mathrm{Aut}(X) \rtimes \mathrm{Pic}(X)$*.*

In section 2, we shall define the $\mathrm{Spec}(\mathcal{A})$ as a set. In section 3, this set is equipped with a Zariski topology, as the zero set of some subcategory, and in section 4, it is equipped with a structure sheaf by the center of the category. We shall gather all these results in section 5 to prove our main theorems.

For the background materials including basic properties of abelian varieties, extension of quasi-coherent sheaf and Eilenberg-Watts theorem, see the appendix.

2 The spectrum of abelian category as a set

Recall that a scheme is a sober space, equipped with some 'specialization relation' given by the Zariski topology, and in fact, the order of prime ideals under the inclusion relation determines the topology of schemes. We shall introduce here the categorical analog of the specialization preorder.

In the following, we denote $\mathcal{A}$ as an abelian category satisfying Grothendieck axioms, i..e. it is cocomplete and directed colimits are exact.

Definition 2.1. (relation $\prec$). Let $M, N \in \mathcal{A}$ be two objects.
(1) We write $M \prec N$ when M is a subquotient of direct sum of copies of N, that is, there exists $Q \subset P \subset N^{\oplus I}$ and $M \cong P/Q$.
(2) We write $N \approx M$ and call M and N equivalent when $N \prec M \prec N$.
(3) We let $[M] = \{N \in \mathcal{A} : N \prec M\}$.

Remark 2.2.
a. The relation defined in (1) is reflexive (obviously) and transitive. Indeed, if L is a subquotient of N, and N is a subquotient of M, then $N \cong P/Q$ where $Q \subset P \subset \bigoplus_I M$, and $L = P'/Q'$ where $Q' \subset P' \subset \bigoplus_J (N = P/Q)$, $Q' \subset P' \subset \bigoplus_{I.J} M/I'$ for some $I' \subset \bigoplus_{I.J} M$, so by the analog of abelian category with the category of abelian groups, there exists $\widetilde{Q'} \subset \widetilde{P'} \subset \bigoplus_{I.J} M$ such that $\widetilde{Q'}/I' \cong Q'$, $\widetilde{P'}/I' \cong P'$ and $\widetilde{P'}/\widetilde{Q'} \cong P/Q \cong L$, thus L is also a subquotient of M.
b. Note that $\prec$ relation is preserved under the cocontinuos exact functors.
c. $\approx$ is an equivalence relation and $[M] \subset [N]$ iff $M \prec N$.
d. Obviously if $N \subset M$, then $N \prec M$.
e, Taking stalk preserves the relation $\prec$, because taking stalks is a filtered colimit, which always commutes with filtered colimits, especially, the arbitrary direct sums, and also taking stalk is exact, thus preserve the relation $\prec$.

Definition 2.3. (spectral object) We call $M \in \mathcal{A}$ if $M \neq 0$ and for all $0 \neq N \subset M$, we have $M \prec N$, so $M \approx N$. Let $\mathrm{Spec}(\mathcal{A})$ be the class of all $[M]$ where M runs over all spectral objects of $\mathcal{A}$.

We shall see some special example: the category $\mathrm{Mod}(R)$ and its spectral.

Lemma 2.4. *Let R be a commutative ring, $M \in \mathrm{Mod}(R)$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. Then $R/\mathfrak{p} \prec M$ if and only if $M_{\mathfrak{p}} \neq 0$.*

Proof. Localization preserves colimit as it is a left adjoint functor, and it is also exact, so it preserves the $\prec$ relation. Thus $0 \neq (R/\mathfrak{p})_{\mathfrak{p}} \cong \mathrm{Frac}(R/\mathfrak{p}) \prec M_{\mathfrak{p}}$, $M_{\mathfrak{p}} \neq 0$.
On the contrary, if $M_{\mathfrak{p}} \neq 0$, then $\mathrm{Ann}(M) \subset \mathfrak{p}$ and there is some $m \in M$ such that $\mathrm{Ann}(m) \subset \mathfrak{p}$. So $R/\mathfrak{p}$ is the quotient of $R/\mathrm{Ann}(m)$ and the latter admits a monomorphism to M via multiplication with m. Hence, $R/\mathfrak{p}$ is the subquotient of M. □

Proposition 2.5. *spectral object in the* $\mathrm{Mod}(R)$
Let R be a commutative ring. If $\mathfrak{p}$ is a prime ideal in R, then $R/\mathfrak{p}$ is a spectral object. For any spectral object $M \in \mathrm{Mod}(R)$, it is ***equivalent*** *to $R/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$.*
For prime ideals $\mathfrak{q}, \mathfrak{p}$, we have $R/\mathfrak{p} \prec R/\mathfrak{q}$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

Proof. For $0 \neq N \subset R/\mathfrak{p}$, here N can be viewed as a R submodule of $R/\mathfrak{p}$, and $\mathfrak{p}$ acts on N is zero, so N can also be viewed as a $R/\mathfrak{p}$ submodule of $R/\mathfrak{p}$, which is an ideal in the quotient ring $R/\mathfrak{p}$, and thus it must be in the form of $\hat{N}/\mathfrak{p}$ where $\hat{N} \subset R$ is an ideal in R. Take $0 \neq n \in N$, and multiplication with n leads to a R module monomorphism $R/\mathrm{Ann}(n) \to N$, so by composition, there is a monomorphism $R/\mathrm{Ann}(n) \to R/\mathfrak{p}$, therefore $\mathrm{Ann}(n)$ is a prime ideal and equal to $\mathfrak{p}$: $\mathrm{Ann}(n)$ must contain $\mathfrak{p}$ and this has to be monomorphism.
Now let M be a spectral object, then take an element $m \in M$, consider the submodule generated by m, we have $(m) \approx M$. So it suffices to prove that the cyclic module (m) is equivalent to some $R/\mathfrak{p}$. By definition, $(m) \cong R/\mathcal{I}$. It remains to prove that $\mathcal{I}$ is a prime ideal, which is equivalent to proving $(\mathcal{I} : r) \subset \mathcal{I}$ for any $r \in R \setminus \mathcal{I}$.
Multiplication by r gives us a monomorphism $R/(\mathcal{I} : r) \to R/\mathcal{I}$. Because $r \notin \mathcal{I}$, $R/(\mathcal{I} : r) \neq 0$ and thus $R/(\mathcal{I} : r) \approx R/\mathcal{I}$ and by lemma 2.3, this is equivalent to $\mathrm{Ann}(R/\mathcal{I}) = \mathcal{I} = \mathrm{Ann}(R/(\mathcal{I} : r)) = (\mathcal{I} : r)$.
The correspondence $R/\mathfrak{p} \prec R/\mathfrak{q}$ iff $\mathfrak{q} \subset \mathfrak{p}$ is easily deduced by the lemma 2.3.
□

Now let's try to generalize this correspondence to the quasi-separated schemes. The condition of quasi-separated here is equivalent to that any two open subsets' intersection is a quasi-compact open subset. So in essence, we are using some properties of quasi-compactness: If $i : U \to X$ is an open immersion of quasi-compact subscheme to quasi-separated scheme, then it is qcqs, so the pushforward i_* takes quasi-coherent sheaf on U to quasi-coherent sheaf on X. More often, we take U as open affine subset of X, then $U \to X$ is qcqs, and the pushforward makes sense. We shall use this property later.

Definition 2.6. (also some notations)
(1). For a sheaf $M \in \mathrm{QCoh}(X)$, the **annihilator sheaf** of M is defined as the kernel $\mathrm{Ann}(M) = \ker(\mathcal{O}_X \to \mathrm{End}(M) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(M, M))$, which is quasi-coherent if M is of finite type (see [4, proposition 7.34].): Indeed, here 'of finite type' makes sure that we can find a finite set of generators on each affine open subset and thus finite intersection of quasi-coherent sheaves is still quasi-coherent.
So as a quasi-coherent ideal sheaf, we have a corresponding closed subscheme $V(\mathrm{Ann}(M))$ with its underlying topological space equal to $\mathrm{Supp}(\mathcal{F})$:

$$\mathfrak{p} \in \mathrm{Supp}(\mathcal{F}) \Leftrightarrow M_{\mathfrak{p}} \neq 0 \Leftrightarrow \exists t_i : \mathrm{Ann}(t_i) \subset \mathfrak{p} \Leftrightarrow \mathfrak{p} \supset \cap_i \mathrm{Ann}(t_i) \Leftrightarrow \mathfrak{p} \in V(\mathrm{Ann}(M)).$$

(2). If X is integral with function field K, denote the constant sheaf with values on K as $\underline{K}$, then

consider the generic point $\eta \in X$: $\eta : \mathrm{Spec}(K) \to X$. We define the **torsion** module $\mathrm{Tor}(M)$ of a quasi-coherent sheaf M as the kernel of the canonical morphism $M \to \eta_*\eta^*M = M \otimes \underline{K}$, which is characterized by the property that: on every open affine open subset U, $\mathrm{Tor}(M)(U) \cong \mathrm{Tor}(\Gamma(U, M))$.
(3). When $\mathrm{Tor}(M) = 0$, we call M as torsion-free.

Observation 2.7. : If $M \approx N \in \mathrm{QCoh}(X)$, then $\mathrm{supp}(M) = \mathrm{supp}(N)$.

Proof. This is a local problem: take an open affine subset $i : U \to X$, then the pullback i^* is cocontinuous (left adjoint preserves all colimits) and exact (check on stalks), thus $M_U \approx N_U$ on U, then use the same way as shown in lemma 2.5 above. □

Observation 2.8. : $\eta_*\eta^*M = M \otimes_{\mathcal{O}_X} \underline{K}$ is a direct sum of copies of the constant sheaf $\underline{K}$.
Caution: Torsion-free is not a local property in general.

Lemma 2.9. *Let M be a spectral object in* $\mathrm{QCoh}(X)$*. Then we have:*
(1). For all quasi-compact open subset U, either $M|_U = 0$ or $M|_U$ is spectral.
(2). The ideal $Ann(M)$ is quasi-coherent and $V(Ann(M)) = supp(M)$ as sets.
(3). The subset $Z = supp(M)$ is an irreducible closed subset of X.
(4). The closed subscheme $V(Ann(M))$ is a closed subscheme.
(5). If we denote $j : Z \to X$ as the closed immersion, then j^*M is torsion-free on Z.

Proof.
(1). We can see in the above proof of observation 2.7 that pullback along any open subset (may not be quasi-compact) preserves the relation $\prec$. However, the spectral object is defined not only involved with the relation $\prec$, it also requires that all subobjects satisfy the relation '$\prec$'. The 'subobject' in $\mathrm{QCoh}(U)$ may not extends to X, so here we need to use the fact that pushforward of quasi-coherent sheaf along a quasi-compact open subset is again quasi-coherent, thus there is a natural extension.
For any $0 \neq N \subset M|_U$, denote $j : N \to M|_U$,and consider the pushforward along $i : U \to X$, then i_*N is a quasi-coherent sheaf on X, but we can't guarantee that it is the subsheaf of M. But there exists such a quasi-coherent sheaf $\tilde{N}$ on X as both a subsheaf of M and an extension of N, which is constructed using our raw extension i_*N: $\tilde{N} = \ker(M \oplus i_*N \to i_*i^*M)$ which is indeed quasi-coherent because both the canonical morphism $c : M \to i_*i^*M$ and induced morphism $i_*(j) : i_*N \to i_*M_U = i_*i^*M$ are morphisms between quasi-coherent sheaves, so the kernel of $(c, i_*(j))$ is also quasi-coherent. And also $\tilde{N}|_U = N$ is easily checked locally on affine open subset.
(2). Quasi-coherence is a local property, so it suffices to prove this theorem on an affine open subset $U = \mathrm{Spec}(A)$.
For any A module sheaf $\widetilde{M'} = M|_U$ spectral, by proposition 2.5, there exists a unique prime ideal $\mathfrak{p}$ such that $M|_U \approx R/\mathfrak{p}$, and by the observation above, $\mathrm{Ann}(\widetilde{M|_U}) = \mathrm{Ann}(\widetilde{R/\mathfrak{p}}) = \widetilde{\mathfrak{p}}$, thus the annihilator ideal sheaf is quasi-coherent even though the object $\widetilde{M'}$ may not be of finite type.[1]
If $M|_U$ is not spectral, then by (1), $M|_U = 0$, which is automatically quasi-coherent.
(3). With the above notation, we denote $x_U = \mathfrak{p}$, and by (2), $\overline{\{x_U\}} \cap U = V(\mathfrak{p})$ and $\mathrm{supp}(M) = V(\mathrm{Ann}(M))$ is closed (quasi-coherent ideal sheaf corresponds to closed subscheme), so we only

[1] Actually, when we take infinite direct sum of $R/\mathfrak{p}$, then it can never be of finite type, but they are equivalent, so they have the same annihilator sheaf.

need to consider those open subset U such that $M|_U \neq 0$. It suffices to prove that $x_U = \mathfrak{p}$ is independent of the choice of U, that is, for any affine open subset $V \subset X$ such that $\widetilde{M}_V \neq 0$, $x_U \in V$ and $x_U = x_V$.
We start with some smaller open affine subset W inside U such that $\overline{M}_W \neq 0$. If $x_U \notin W$, then $\widetilde{R/\mathfrak{p}}_W = 0$ because it is supported on the closed subset $V(\mathfrak{p})$. By definition, $V(\mathfrak{p})$ is irreducible because $\mathfrak{p}$ is prime ideal, so nonempty open subset $W \cap V(\mathfrak{p})$ is dense and again irreducible with the same generic point of $V(\mathfrak{p})$.
For the general case, any open affine subset V such that $M|_V \neq 0$, then $M|_{V\cap U} \neq 0$. Otherwise, we can find a quasi-coherent subsheaf H inside $M|_{U\cup V}$ such that $H|_U = M|_U$ and $H|_V = 0$. Here U and V are affine open subsets, thus quasi-compact, and the union of finite quasi-compact open subset is again quasi-compact, so by the same way shown in the above (2), we can find a subsheaf $\hat{H} \subset M$ on X with restriction on $U \cup V$ exactly the same as H. Because M is spectral, so $M \prec \hat{H}$, and $M|_V \prec \hat{H}|_V = H|_V = 0$, which is a contradiction because $M|_V \neq 0$.
Thus we can always find an open affine subset $W \subset U \cap V$ such that $M|_W \neq 0$, by what we have shown earlier, this leads to $x_U = x_W = x_V$.
(4). Z is irreducible by (3) and also reduced because locally it is closed subscheme determined by some prime ideals, so it is integral scheme.
(5). In our case torsion-free is a local property, and by (1), the restriction of a spectral object to any affine open subset is either zero, which is automatically torsion-free, or again a spectral object. Thus we may only consider the affine case when $X = \mathrm{Spec}(R)$ and $M = \widetilde{M'} \approx \widetilde{R/\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$, also with its annihilator ideal sheaf corresponding to an integral closed subscheme $Z = V(\mathfrak{p})$. Then pullback of M corresponds to $\widetilde{M' \otimes_R R/\mathfrak{p}}$ (which is still spectral), and it suffices to prove that $M' \otimes R/\mathfrak{p}$ is torsion free over $R/\mathfrak{p}$.
Otherwise, there exists a nontrivial torsion $R/\mathfrak{p}$ module K of $M' \otimes R/\mathfrak{p}$, then as it is spectral, so $K \approx M' \otimes R/\mathfrak{p}$. What's more, their stalks are also equivalent: $K_\mathfrak{p} \approx (M' \otimes R/\mathfrak{p}) \otimes_{R/\mathfrak{p}} \mathrm{Frac}(R/\mathfrak{p}) = (M' \otimes_R \mathrm{Frac}(R/\mathfrak{p})) \approx (R/\mathfrak{p})_\mathfrak{p} = \mathrm{Frac}(R/\mathfrak{p})$. However $K_\mathfrak{p}$ is again torsion over $(R/\mathfrak{p})_\mathfrak{p}$ which is a torsion module over a field, can only be zero, while $\mathrm{Frac}(R/\mathfrak{p}) \approx (M' \otimes \mathrm{Frac}(R/\mathfrak{p})) \approx 0$, so they can only be zero, which is impossible because $M' \neq 0$.

□

What is so special about torsion-free quasi-coherent sheaf? It turns out that on an integral scheme, torsion-free quasi-coherent sheaves are mutually equivalent, more precisely, to the constant sheaf of function field.

Lemma 2.10. *Let X be an integral scheme with function field K. Then every nontrivial torsion-free quasi-coherent module on X is equivalent to $\underline{K}$.*

Proof. Denote the generic point of X as η and the morphism to this point $\eta : \mathrm{Spec}(K) \to X$. By definition of torsion-free, the canonical morphism $M \to \eta_*\eta^*M$ is an embedding to a direct sum of copies of the constant sheaf $\underline{K}$. Thus $M \prec \underline{K}$.

There is an epimorphism $\bigoplus_{f\in K} M \xrightarrow{\otimes f} M \otimes \underline{K} = \eta_*\eta^*M$ while the target is a direct sum of copies of $\underline{K}$, so it admits an epimorphism to M. Thus $\underline{K} \prec M$. □

Now we can generalize the spectrum construction to the schemes.

Observation 2.11. In $\mathrm{Mod}(R)$, $R/\mathcal{J} \approx R/\mathfrak{p}$ if $\sqrt{\mathcal{J}} = \mathfrak{p}$(useless so far).

Observation 2.12. For a spectral object $M \in \mathrm{QCoh}(X)$, $\mathrm{supp}(M) = \overline{\{x\}} \xrightarrow{i} X$, then $i_*i^*M \cong M$.

Observation 2.13. Structure sheaf on an integral scheme is always torsion-free.

Proposition 2.14. *Let X be a quasi-separated scheme. Then the map*

$$X \to \mathrm{Spec}(\mathrm{QCoh}(X)),\ x \mapsto [\mathcal{O}_X/\mathcal{J}_x]$$

is a bijection. Here $\mathcal{J}_x$ is the vanishing ideal of $\overline{\{x\}}$.

Proof. First, let's show that $\mathcal{O}_X/\mathcal{J}_x$ is spectral. For any sub quasi-coherent sheaf $0 \neq N \subset \mathcal{O}_X/\mathcal{J}_x$. It suffices to prove that $\mathcal{O}_X/\mathcal{J}_x \prec N$. The vanishing ideal $\mathcal{J}_x$ corresponds to the reduced closed subscheme structure by definition, thus $i : Z = \overline{\{x\}} \to X$ is an integral subscheme, where the lemma 2.10 can be used.
By definition, $\mathcal{O}_Z = i^{-1}(\mathcal{O}_X/\mathcal{J}_x)$, and we have $M \cong i_*i^*M$. Indeed, at stalk level, this is equivalent to the following facts: any R submodule $M \subset R/\mathcal{I}$ is also $R/\mathcal{I}$ submodule, thus an ideal inside the quotient ring $R/\mathcal{I}$, and then R tensor product of a $R/\mathcal{I}$ ideal and $R/\mathcal{I}$ (here corresponds to the pullback i^*) is the identity, and i_* of inclusion doesn't change the stalk.
In this way, we can see that $0 \neq i^*M \subset \mathcal{O}_Z$ and also it is torsion-free: $(\eta_Z)_*(\eta_Z)^*i^*M = (\eta_Z)_*(i\circ\eta_X)^*M = (\eta_*\eta^*M)_Z$. So by lemma 2.7, $i^*M \approx \mathcal{O}_Z$ and $M \cong i_*i^*M \approx i_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{J}_x$. Thus the map $x \to \mathcal{O}_X/\mathcal{J}_x$ is well-defined.
The inverse map $[M] \to$ generic point of $\mathrm{Ann}(M)$ is well-defined by lemma 2.6. The composite $X \to \mathrm{Spec}(\mathrm{QCoh}(X)) \to X$ is identity.
For the other composition, let M be spectral in $\mathrm{QCoh}(X)$, with the same notation in lemma 2.9(5), j^*M is torsion-free on $Z = V(\mathrm{Ann}(M)) = \{x\}$. Since M is annihilated by $\mathcal{J}_x$, we have $M = j_*j^*M$, and especially $j^*M \neq 0$. So $j^*M \approx \mathcal{O}_Z$ and $j_*j^*M = M \approx j_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}_x$. □

3 Zariski topology on the spectrum

In this section we shall endow the spectrum of the abelian category with certain type of topology determined by the zero set of some subcategories.

Definition 3.1. A subcategory $\mathcal{T} \subset \mathcal{A}$ is called **topologizing** if $\mathcal{T}$ is closed under subobjects, quotients (cokernels) and direct sums. In other words, $M \prec N \in \mathcal{T}$ implies $M \in \mathcal{T}$. The preorder is closed in $\mathcal{T}$. Notice that $\mathcal{T}$ always contains the zero object, which is the zero direct sum. Thus a topologizing category is also a cocomplete abelian category.

Example 3.2. For any object $M \in \mathcal{A}$, the smallest topologizing category containing M is $[M] = \{N \prec M\}$.

Observation 3.3. : Reflector is left adjoint, thus is cocontinuous.

Lemma 3.4. *For any $M \in \mathrm{Mod}(R)$, M is a colimit of certain diagram consisting only of direct sum of copies of R.*

Proof. Take A as the M basis direct sum of R, then there is an epimorphism $f : A \to M$ with kernel $\ker(f)$. Again take B as the $\ker(f)$ basis free R module and natural epimorphism $j : B \to \ker(f)$. Then M is the cokernel of $B \xrightarrow{j} A$. □

Recall that a subcategory is called reflexive if the natural inclusion admits a left adjoint, called the reflector, there is a characterization of reflectiveness in our case:

Lemma 3.5. *Let $\mathcal{T} \subset \mathcal{A}$ be a topologizing subcategory. Then $\mathcal{T}$ is reflective if and only if for every $M \in \mathcal{A}$, there is a smallest $K \subset M$ such that $M/K \in \mathcal{T}$.*

Here in the lemma, 'the smallest' means for any $T \subset M$ such that $M/T \in \mathcal{T}$, $K \subset T$.

Lemma 3.6. *In the category* Mod(R)*, there is an inclusion-reserving bijection between ideals of R and the reflective topologizing subcategories of* Mod(R)*: it maps ideal $\mathcal{I}$ to the subcategory $\mathcal{T}_{\mathcal{I}} = \{M \in \mathrm{Mod}(R) : \mathcal{I}M = 0\}$.*

Proof. $\mathcal{T}_{\mathcal{I}}$ is obviously a topologizing category, and $\mathcal{I} \subset \mathcal{J}$ implies $\mathcal{T}_{\mathcal{I}} \subset \mathcal{T}_{\mathcal{J}}$. If $\mathcal{T}_{\mathcal{I}} \subset \mathcal{T}_{\mathcal{J}}$, then we get $R/\mathcal{I} \in \mathcal{T}_{\mathcal{J}}$, this implies $\mathcal{J} \subset \mathcal{I}$. The functor $M \to M/\mathcal{I}$ provides a reflector for $\mathcal{T}_{\mathcal{I}}$. Thus subcategory $\mathcal{T}_{\mathcal{I}}$ is topologizing reflective. It remains to prove that all reflective topologizing subcategory $\mathcal{T}$ is in this form.
Take a reflector $F : \mathrm{Mod}(R) \to \mathcal{T}$, then the unit map $u : R \to F(R)$ is epimorphism with kernel $\mathcal{I}$. Thus $F(R) \cong R/\mathcal{I}$. We claim that $\mathcal{T} = \mathcal{T}_{\mathcal{I}}$.
Since $\mathcal{I}$ annihilates $F(R)$, and F is cocontinuous, so $\mathcal{I}$ annihilates all $F(\mathrm{Mod}(R))$ by the lemma 3.4, and $\mathcal{T} \subset \mathcal{T}_{\mathcal{I}}$.
On the other hand, any element N in $\mathcal{T}_{\mathcal{I}}$ can be viewed as $R/\mathcal{I}$ module, by lemma 3.4 again, it can be expressed as quotient of direct sum of copies of $R/\mathcal{I}$, thus $N \prec R/\mathcal{I}$ and lies in $\mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}_{\mathcal{I}}$.

□

In the situation of the previous proposition we have $R/\mathfrak{p} \in \mathcal{T}_{\mathcal{I}}$ if and only if $\mathfrak{p} \supset \mathcal{I}$,i.e. $\mathfrak{p} \in V(\mathcal{I})$. Hence under the bijection in proposition 2.14: $\mathrm{Spec}(R) \xrightarrow{\cong} \mathrm{Spec}(\mathrm{Mod}(R))$, the Zaraski closed subset $V(\mathcal{I})$ corresponds to the spectral objects inside $\mathcal{T}_{\mathcal{I}}$. This motivates the following definition:

Definition 3.7. For a subcategory $\mathcal{T} \subset \mathcal{A}$, we define $V(\mathcal{T}) = \{[M] \in \mathrm{Spec}(\mathcal{A}) : M \in \mathcal{T}\}$.
In order to show the axioms of topology, we need the following analogs for the intersection and product of ideals.

Proposition 3.8. *Let $\{\mathcal{T}_i\}_{i \in I}$ be a family of topologizing subcategories, then their intersection is also topologizing. If each $\mathcal{T}_i$ is reflective, so is their intersection.*

Proof. It's obvious that the intersection of topologizing subcategories is still topologizing. It remains to check the reflectivtity. By lemma 3.5, for arbitrary M, we shall find a smallest $K \subset M$ for quotient to lie in $\cap_i \mathcal{T}_i$, which is the sum of subobjects K_i inside M where K_i is the smallest object for quotient to lie in $\mathcal{T}_i$. □

If $\mathcal{S}$ and $\mathcal{T}$ are subcategories of $\mathcal{A}$, then we define the subcategory $\mathcal{S} \cdot \mathcal{T}$ as follows: We have $M \in \mathcal{S} \cdot \mathcal{T}$ if and only if there is an exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

such that $M' \in \mathcal{T}$ and $M'' \in \mathcal{S}$. This is called the **Gabriel product** of $\mathcal{S}$ with $\mathcal{T}$.

Example 3.9. In Mod(R), $\mathcal{T}_{\mathcal{I}} \cdot \mathcal{T}_{\mathcal{J}} = \mathcal{T}_{\mathcal{I}\mathcal{J}}$. In fact $\subset$ is obvious, and for $\supset$, we observe $R/\mathcal{I}\mathcal{J} \in \mathcal{T}_{\mathcal{I}} \cdot \mathcal{T}_{\mathcal{J}}$ by the following exact sequence

$$0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to R/\mathcal{I}\mathcal{J} \to R/\mathcal{I} \to 0$$

Lemma 3.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be topologizing subcategories of $\mathcal{A}$,*
(1). Then $\mathcal{S} \cdot \mathcal{T}$ is topologizing.
(2). If $\mathcal{S}$ and $\mathcal{T}$ are reflective, then so is $\mathcal{S} \cdot \mathcal{T}$.

Proof. (1). For any $M \in \mathcal{S} \cdot \mathcal{T}$, and let us choose an exact sequence for it,

$$0 \to M' \to M \xrightarrow{f} M'' \to 0$$

subobject: Take a subobject $N \subset M$, then denote $N' = N \cap M' = N \times_M M'$, $N'' = \mathrm{im}(f|_N)$. As monomorphism is preserved under pullback, so all arrows in the following diagram are monomorphisms:

$$\begin{tikzcd}
M' \times_M N \arrow[r] \arrow[d] & M' \cong \ker(f) \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

and what's more, we have the following gluing of pullback diagrams:

$$\begin{tikzcd}
M' \times_M N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M' \cong \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
N \arrow[r] & M \arrow[r, "f"] & M''
\end{tikzcd}.$$

So $\ker(f|_N) = N'$ and there is an induced exact sequence for N:

$$0 \to N' = N \times_M M' \to N \to \mathrm{im}(f|_N) \to 0.$$

Here $N' \subset M'$ by the first pullback diagram, and $\mathrm{im}(f|_N) \subset \mathrm{im}(f) = M''$, so $N' \in \mathcal{T}'$ and $N'' \in \mathcal{S}$, thus $N \in \mathcal{S} \cdot \mathcal{T}$.
quotient: For a quotient $Q = M/N$ of M for some subobject $N \subset M$, we denote the image of M' in the quotient Q as Q', which is a quotient of M': $Q' \cong M'/M' \cap N$. Then take $Q'' = Q/Q'$, which is a quotient of M'' in the same way. Thus $Q \in \mathcal{S} \cdot \mathcal{T}$.
direct sum: which is obviously in $\mathcal{S} \cdot \mathcal{T}$ because direct sum is exact.

(2) We will use lemma 3.5 here. For $M \in \mathcal{A}$, let $K_{\mathcal{S}}(M)$ be the smallest subobject of M such that $M/K_{\mathcal{S}}(M) \in \mathcal{S}$. We define $K_{\mathcal{T}}$ analogously. Let $K(M) := K_{\mathcal{T}}(K_{\mathcal{S}}(M))$, we claim that $K(M) = K_{\mathcal{S}.\mathcal{T}}(M)$.
The chain of subobjects $K_{\mathcal{T}}(K_{\mathcal{S}}(M)) = K(M) \subset K_{\mathcal{S}}(M) \subset M$ gives us an exact sequence:

$$0 \to K_{\mathcal{S}}(M)/K(M) \to M/K(M) \to M/K_{\mathcal{S}}(M) \to 0$$

$K_{\mathcal{S}}(M)/K(M) \in \mathcal{T}$ and $M/K_{\mathcal{S}}(M) \in \mathcal{S}$, so $M/K(M) \in \mathcal{S} \cdot \mathcal{T}$.
On the other hand, let L be a subobject such that $M/L \in \mathcal{S} \cdot \mathcal{T}$, then pick an exact sequence for it:

$$0 \to P \to M/L \to Q \to 0$$

where $P \in \mathcal{T}, Q \in \mathcal{S}$.
Then P is in the form of M'/L where $L \subset M' \subset M$ and $Q = M/M'$, thus $K_{\mathcal{S}}(M) \subset M'$, and $K_{\mathcal{S}}(M) + L \subset M'$: $\frac{K_{\mathcal{S}}(M)+L}{L} \cong \frac{K_{\mathcal{S}}(M)}{L \cap K_{\mathcal{S}}(M)} \subset \frac{M'}{L} \in \mathcal{T}$. Thus by the 'smallest' property for $K_{\mathcal{S}}(M)$ with respect to $\mathcal{T}$, $L \supset K_{\mathcal{S}}(M) \cap L \supset K(M)$.

□

Corollary 3.11. *The subset $V(-)$ defined in 3.7 enjoys the following properties:*

(1) $V(\{0\}) = \emptyset$, $V(\mathcal{A}) = \mathrm{Spec}(\mathcal{A})$.

(2) *For a family of topologizing reflective subcategories $\{\mathcal{T}_i\}_{i \in I}$, their intersection $\cap_{i \in I} \mathcal{T}_i$ is likewise a topologizing reflective subcategory satisfying $\cap_{i \in I} V(\mathcal{T}_i) = V(\cap_{i \in I} \mathcal{T}_i)$.*

(3) *If $\mathcal{S}, \mathcal{T}$ are reflective topologizing subcategories, the same is true for $\mathcal{S} \cdot \mathcal{T}$, and we have $V(\mathcal{S} \cdot \mathcal{T}) = V(\mathcal{S}) \cup V(\mathcal{T})$.*

Hence there is a Zariski topology on X, in which the closed subsets are those of the form $V(\mathcal{T})$ where $\mathcal{T}$ is a reflective topologizing subcategory.

Proof. (1) and (2) are by definition obvious.
(3). The inclusion follows from $\mathcal{S} \subset \mathcal{S} \cdot \mathcal{T}$ and $\mathcal{T} \subset \mathcal{S} \cdot \mathcal{T}$, and for the other inclusion let $M \in \mathcal{S} \cdot \mathcal{T}$ be spectral, let's choose an exact sequence $0 \to M' \to M \to M'' \to 0$ where $M' \in \mathcal{T}$ and $M'' \in \mathcal{S}$. If $M' = 0$, then $M \cong M'' \in \mathcal{S} \subset \mathcal{S} \cdot \mathcal{T}$. Otherwise, $M' \neq 0$ is a subobject of M, thus because M is spectral, $M' \approx M$, both contained in the topologizing subcategory $\mathcal{T} \subset \mathcal{S} \cdot \mathcal{T}$. □

Proposition 3.12. *Let X be a quasi-separated scheme, then the bijection from proposition 2.14*

$$X \to \mathrm{Spec}(\mathrm{QCoh}(X)),\ x \mapsto [\mathcal{O}_X/\mathcal{I}_x].$$

becomes a homeomorphism when $\mathrm{Spec}(\mathrm{QCoh}(X))$ *is equipped with the Zariski topology defined above.*

Proof. The closed subsets of X are the zero sets of quasi-coherent ideals $I \subset \mathcal{O}_X$: $V(I) = \{x \in X : I \subset \mathcal{I}_x\}$. The image of $V(I)$ under bijection consists of all $[\mathcal{O}_X/\mathcal{I}_x]$ such that $I \subset \mathcal{J}_x$, i.e. $I \cdot \mathcal{O}_X/\mathcal{I}_x = 0$. Then it equals to $V(\mathcal{T}_I)$ where $\mathcal{T}_I := \{M \in \mathrm{QCoh}(X) : I \cdot M = 0\}$ is a reflective topologizing subcategory as in the affine case (reflector is $M \to M/I$).
On the other hand, let $\mathcal{T}$ be a topologizing reflective subcategory of $\mathrm{QCoh}(X)$, then for $\mathcal{O}_X$, there is a smallest ideal sheaf I such that $\mathcal{O}_X/I$in $\mathcal{T}$. Then $V(\mathcal{T})$ consists of all $[\mathcal{O}_X/\mathcal{I}_x]$ such that $\mathcal{O}_X/\mathcal{I}_x \in \mathcal{T}$, so by the 'smallest' property, $I \subset \mathcal{J}_x$ and corresponds exactly to the points in $V(I) \subset X$. □

4 The structure sheaf on the spectrum

Recall that the center $\mathrm{Z}(\mathcal{C})$ of a category $\mathcal{C}$ is the monoid of all natural transformations $\eta : id_{\mathcal{C}} \to id_{\mathcal{C}}$. Thus an element $\eta \in Z(\mathcal{C})$ is a family of morphism $M \xrightarrow{\eta_M} M$ such that the following diagram commutes:
$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\eta_N} & N \end{array}$$

Observation 4.1. $\mathrm{Z}(\mathcal{C})$ is a commutative monoid, especially, when $\mathcal{C}$ is enriched over **Grp**, $\mathrm{Z}(\mathcal{C})$ is a commutative ring.

Proof. Commutativity follows from the following diagram: $\begin{array}{ccc} M & \xrightarrow{\eta_M} & M \\ {\scriptstyle \eta'_M}\downarrow & & \downarrow{\scriptstyle \eta'_M} \\ M & \xrightarrow{\eta_M} & N \end{array}$

When the category $\mathcal{C}$ is enriched over **Grp**, there is a natural inherited addition group structure, thus $\mathrm{Z}(\mathcal{C})$ is a commutative ring. □

Lemma 4.2. *If X is a ringed space with structure sheaf $\mathcal{O}_X$, and $Mod(X)$ denotes the category of $\mathcal{O}_X$ module sheaf, then there is natural isomorphism of commutative rings,*

$$\Gamma(X, \mathcal{O}_X) \cong \mathrm{Z}(Mod(X))$$

When X is a quasi-separated scheme, there is also an isomorphism:

$$\Gamma(X, \mathcal{O}_X) \cong \mathrm{Z}(\mathrm{QCoh}(X))$$

Proof. The homomorphism $\Gamma(X, \mathcal{O}_X) \to \mathrm{Z}(\mathrm{Mod}(X))$ is defined by mapping a global section to the endomorphism defined by multiplication with it. The homomorphism in the other direction is given by evaluation at $\mathcal{O}_X$: the image of $1 \in \Gamma(X, \mathcal{O}_X)$. It is clear that $\Gamma(X, \mathcal{O}_X) \to \mathrm{Z}(\mathrm{Mod}(X)) \to \mathrm{Z}(\mathrm{Mod}(X))$ is identity. This implies surjectivity of $\mathrm{Z}(\mathrm{Mod}(X)) \to \Gamma(X, \mathcal{O}_X)$. It remains to prove it is injective.
If $\eta \in \mathrm{Z}(\mathrm{Mod}(X))$ corresponds to $0 \in \Gamma(X, \mathcal{O}_X)$, then globally this is zero endomorphism for any $M \in \mathrm{Mod}(X)$: $\Gamma(X, M) \xrightarrow{\eta} 0 \in \Gamma(X, M)$. But to prove a sheaf morphism is zero, we also need to consider the local section $s \in \Gamma(U, M)$ where $j : U \to X$ is an open immersion.
We have already known that the global section is sent to zero, so for a local section, we should make it 'become' some sort of global section, using the inclusion $j : U \to X$. By the canonical immersion $f : M \to j_*j^*M$, let $s' \in \Gamma(X, j_*j^*M) = \Gamma(U, j^*M) = \Gamma(U, M) \ni s$ corresponds to the local section $s \in \Gamma(U, M)$. By checking at stalks, $\Gamma(U, f)(s) = s'|_U$, since η is natural, we obtain $\Gamma(U, f)(\Gamma(U, \eta(M)))(s) = \Gamma(U, \eta(j_*j^*(M)))(\Gamma(U, f)(s)) = \Gamma(U, \eta(j_*j^*M))(s'|_U) = \Gamma(X, \eta(j_*j^*M))(s')|_U = 0$.
The quasi-coherent sheaf is done similarly because $\mathrm{End}(\mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)$ and pushforward along open subscheme which is quasi-compact, preserves quasi-coherence. □

In this way, we have reconstructed the global section of structure sheaf by considering the center of an abelian category. To get the local section, we need to use the Gabriel's theorem on quotient of abelian category (see [3, chapter 3].).

Definition 4.3. a subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$ is called **thick** if it contains zero and closed under subquotients and extensions.
Here 'closed under extension' means that for an exact sequence in abelian category $\mathcal{A}$: $0 \to A \to C \to B \to 0$ where $A, B \in \mathcal{T}$, then $C \in \mathcal{T}$.
Thick subcategory can be used to construct the quotient category like the normal subgroup used to do quotient of group: $\mathcal{A}/\mathcal{T}$ which is an abelian category equipped with an exact functor: $P : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ such that:
(a). $P(T) \cong 0 \in \mathcal{A}/\mathcal{T}, \forall T \in \mathcal{T}$.
(b). P is bijective on objects (not contradictory to (a), where there is an isomorphism relation

between different objects).
(c). A morphism $f : M \to N$ is sent to isomorphism in $\mathcal{A}/\mathcal{T}$ by P if $\ker(f) \in \mathcal{T}$ and $\mathrm{coker}(f) \in \mathcal{T}$.

(d). Every short exact sequence in $\mathcal{A}/\mathcal{T}$ lifts to an exact sequence along P in $\mathcal{A}$.

Example 4.4. (see [3, example III.5.a]) Let X be a ringed space and $U \subset X$ be an open subset. Consider the subcategory $\mathrm{Mod}_U(X) = \{M \in \mathrm{Mod}(X) : M|_U = 0\}$ which is thick and the restriction functor $\mathrm{Mod}(X) \to \mathrm{Mod}(U)$ induces an equivalence of abelian category $\mathrm{Mod}(X)/\mathrm{Mod}_U(X) \cong \mathrm{Mod}(U)$.

When X is quasi-separated and U is quasi-compact, then define $\mathrm{QCoh}_U(X)$ in the same way, there is an equivalence and $\mathrm{QCoh}(U) \cong \mathrm{QCoh}(X)/\ \mathrm{QCoh}_U(X)$ given by pushforward along U.

Lemma 4.5. *Let $\mathcal{A}$ be an abelian category and $\mathcal{T}$ a thick subcategory, then the exact functor $P : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ induces a morphism of commutative rings $Z(\mathcal{A}) \to Z(\mathcal{A}/\mathcal{T})$.*
These morphisms are compatible in the following sense: for thick subcategory $\mathcal{S} \subset \mathcal{A}$, if $\mathcal{T} \subset \mathcal{S}$, then there is a morphism $\mathcal{A}/\mathcal{S} \to \mathcal{A}/\mathcal{T}$ induced by the isomorphism $\mathcal{A}/\mathcal{T} \cong (\mathcal{A}/\mathcal{S})/(\mathcal{A}/\mathcal{T})$, satisfying the folllowing commutative diagram of commutative rings:

$$\begin{tikzcd} Z(\mathcal{A}) \arrow[rr] \arrow[dr] & & Z(\mathcal{A}/\mathcal{S}) \\ & Z(\mathcal{A}/\mathcal{T}) \arrow[ur] & \end{tikzcd}$$

Proof. To construct a morphism of rings $Z(\mathcal{A}) \to Z(\mathcal{A}/\mathcal{T})$, the most natural way is to do the restriction: for $\eta \in Z(\mathcal{A})$, $N \in \mathcal{A}/\mathcal{T}$, by the bijection of P, there exists unique $M \in \mathcal{A}$ such that $P(M) = N$, define $\eta_N : P(\eta(M)) : P(M) = N \to P(M) = N$. Then it remains to check that the following diagram commutes:

$$\begin{tikzcd} N = P(M) \arrow[r, "\eta_N"] \arrow[d, "f"'] & N \arrow[d, "f"] \\ N' = P(M') \arrow[r, "\eta_{N'}"] & N' \end{tikzcd}$$

If morphism f in quotient category is induced by some morphism $\bar{f} : M \to M' \in \mathcal{A}$, then it's obvious commutative.
However, in general, the morphism $f : N \to N'$ in quotient category corresponds to the following diagram (see [3, III.1, proof of proposition 1]):

$$\begin{tikzcd} P(M) = N \arrow[r, "f"] & N' = P(M') \arrow[d, "P(b)"] \\ P(M) \arrow[u, "P(a)"] \arrow[r, "P(g)"] & P(M') \end{tikzcd}$$

where $a : M \to X$, $b : M' \to X'$ and $g : X \to X'$ are morphisms in the category $\mathcal{A}$ can fit into the above diagram in the category of quotient category. What's more, $P(a)$ and $P(b)$ are isomorphisms in $\mathcal{A}/\mathcal{T}$.

$$
\begin{tikzcd}
P(X) \arrow[rr, "P(g)"] \arrow[dr, "P(a)"] \arrow[dd, "P(\eta(X))"'] & & P(X') \arrow[dd, "P(\eta(X'))"] & \\
& N \arrow[rr, "f"] \arrow[dd, "\eta_N"] & & N' \arrow[ul, "P(b)"'] \arrow[dd, "\eta_{N'}"] \\
P(X) \arrow[rr, "P(g)"] \arrow[dr, "P(a)"] & & P(X') & \\
& P(M) = N \arrow[rr, "f"] & & N' = P(M') \arrow[ul, "P(b)"']
\end{tikzcd}
$$

where the back side, left side and right side are commutative by the naturality of η, thus the front side is commutative. $\square$

Example 4.6. Applying above lemma to the example 4.4, where $\mathrm{Mod}(X)/\mathrm{Mod}_U(X) \cong \mathrm{Mod}(U)$, we have the following diagram:

$$
\begin{tikzcd}
& & Z(\mathrm{Mod}(X))/\mathrm{Mod}_U(X) \arrow[drr, "\cong"] & & \\
Z(\mathrm{Mod}(X)) \arrow[urr, "res \text{ defined in lemma 4.4}"] & & & & Z(\mathrm{Mod}(U)) \arrow[dl, "\cong"] \\
& \Gamma(X, \mathcal{O}_X) \arrow[ul, "\cong"] \arrow[rr, "restriction"] & & \Gamma(U, \mathcal{O}_X) &
\end{tikzcd}
$$

To define a structure sheaf on $\mathrm{Spec}(\mathcal{A})$, we should find certain thick subcategories of $\mathcal{A}$.

Definition 4.7. For $[M] \in \mathrm{Spec}(\mathcal{A})$, define $\langle [M] \rangle = \{N \in \mathrm{Spec}(\mathcal{A}) : M \nprec N\}$
For subset $U \subset \mathrm{Spec}(\mathcal{A})$, define $\langle U \rangle = \cap_{x \in U} \langle x \rangle$.

Lemma 4.8. *For any subset U, $\langle U \rangle$ is a thick subcategory.*

Proof. It suffices to prove this for $\langle [M] \rangle$ because intersection of thick subcategories is again thick. For $N \in \langle [M] \rangle$, if some subobject $M \prec N' \subset N$, so is $M \prec N$, contradiction! So $\langle [M] \rangle$ is closed under subobjects. In the same way, $\langle [M] \rangle$ is also closed under subquotients.
It remains to prove $\langle [M] \rangle$ is closed under extension. For an exact sequence $0 \to N' \to N \to N'' \to 0$, it suffices to prove that if $N \notin \langle [M] \rangle : M \prec N$, then either $N' \notin \langle [M] \rangle$ or $N'' \notin \langle [M] \rangle$.
If $M \subset N$, then in the same way as lemma 3.7, there is an exact sequence $0 \to \bar{N}' \to M \to \bar{N}'' \to 0$ such that $\bar{N}' \subset N'$ and $\bar{N}'' \subset N''$, thus if either $M \prec \bar{N}'$ or $M \prec \bar{N}''$, then so is N' or N''.
If M is a quotient of N, then again there is an exact sequence $0 \to \widetilde{N'} \to M \to \widetilde{N''} \to 0$ where $\widetilde{N'}$is quotient of N' and $\widetilde{N''}$ is a quotient of N''. Thus if either $M \prec \widetilde{N'}$ or $M \prec \widetilde{N''}$, then so is N' or N''.
If M is an (arbitrary) direct sum of N, then the exact sequence for M is exactly the direct sum of the exact sequence for N. Thus if $M \prec N$, that is, M is a subquotient of some direct sum of N, there is an exact sequence for M induced from N preserving the order with M: $0 \to M' \to M \to M'' \to 0$ where $M' \prec N'$ and $M'' \prec N''$. If $M' \cong 0$, then $M \cong M''$, thus $M \prec M''$, else, $M' \subset M$ which is a subobject, because M is spectral, $M \approx M'$. In both cases, $M \prec M''$(or M') $\prec N''$(or N'). $\square$

Thus for arbitrary subset U, there is an associated ring $\mathcal{O}'_U := Z(\mathcal{A}/\langle U \rangle)$, to define the appropriate structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathcal{A})}$, we should pick the open subsets U and the associated rings $\mathcal{O}'_U$, they constitute the presheaf on Zariski topological space $\mathrm{Spec}(\mathcal{A})$, the structure sheaf is defined as the sheafification of $\{\mathcal{O}'_U\}$ for open subset U.

Now let's consider the special case for $\mathrm{QCoh}(X)$:

Example 4.9. (also an observation) What is $\langle U\rangle$ in $\mathrm{Spec}(\mathrm{QCoh}(X))$?
To answer this question, we shall first prove the following lemma, which is a generalization of lemma 2.4.

Lemma 4.10. *Let X be a quasi-separated scheme, $x \in X$, $M \in \mathrm{QCoh}(X)$ (which may not be spectral). Then $\mathcal{O}_X/\mathcal{J}_x \prec M$ if and only if $M_x \neq 0$.*

Proof. The lemma 2.4 is the affine case. For a general quasi-separated scheme X, if $\mathcal{O}_X/\mathcal{J}_x \prec M$, then $0 \neq \mathcal{O}_{x,X}/(\mathcal{J}_x)_x \prec M_x$ because taking stalk preserves the relation $\prec$, thus $M_x \neq 0$.
If $M_x \neq 0$, we shall prove that $\mathcal{O}_X/\mathcal{J}_x \prec M$. Pick a nonzero local section $s \in \Gamma(U, M)$ for some open affine neighborhood of x, then it generates a local quasi-coherent subsheaf $N = \widetilde{(s)} \subset M|_U$, by the extension of quasi-coherent sheaf, there exists a quasi-coherent sheaf on X: $\bar{N} \subset M$ which is an extension of N and also $\bar{N}_x \neq 0$. It suffices to prove that $\mathcal{O}_X/\mathcal{J}_x \prec \bar{N}$, form now on, we may assume $M|_U$ is generated by a single local section.
Consider the closure $Z = \overline{\{x\}}$ equipped with an integral closed subscheme structure, denote the closed immersion as $i : Z \to X$, let $N = i^*M/Tor(i^*M)$, then i_*N is a quotient of i_*i^*M where the latter one is a subsheaf of M, thus $i_*N \prec M$.
By construction, N is always torsion-free, thus if it is nontrivial, by lemma 2.9, $N \approx \mathcal{O}_Z$, then $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}_x \approx i_*N \prec M$ because pushforward along closed subscheme is exact and cocontinuous (by construction, it is exactly extension by zero outside Z). So it suffices to prove that N is nonzero.
Restrict to the open affine neighborhood $U = \mathrm{Spec}(A)$, $x = \mathfrak{p}$, and locally cyclic module sheaf $M|_U = \widetilde{A/I}$, then as $M_x \neq 0$, by lemma 2.4, $I \subset \mathfrak{p}$, thus $i^*M|_U \cong A/\mathfrak{p} \otimes A/I \cong A/\mathfrak{p}$ which is torsion free already and nonzero, thus $i^*N \cong i^*M$ is nonzero. □

Thus for $x' = [\mathcal{O}_X/\mathcal{J}_x] \in \mathrm{Spec}(\mathrm{QCoh}(X))$ corresponding to $x \in X$, $\langle x'\rangle$ is exactly set of points $\{[\mathcal{O}_X/\mathcal{J}_y]\}$ such that $(\mathcal{O}_X/\mathcal{J}_y)_x = 0$, which means $y \notin \overline{\{x\}}$, thus $\langle x\rangle = X \setminus \overline{\{x\}}$. And $\langle U\rangle = \cap_{x\in U}(X \setminus \overline{\{x\}})$ which may be not open if U is not a finite set. For any $M \in \langle U\rangle$, then $M \cong \mathcal{O}_X/\mathcal{J}_z$ where $(\mathcal{O}_X/\mathcal{J}_z)_x = 0, \forall x \in U$. Thus $z \notin \cup_{x\in U}\overline{\{x\}}$, and we know that spectral object has the same support with its equivalent object, so $M_x = 0, \forall x \in U$.

Proposition 4.11. *The homeomorphism in proposition 3.12 is a ringed space isomorphism if* Spec*(QCoh(X)) is equipped with the structure sheaf defined above.*

Proof. For any affine open subset $U = \mathrm{Spec}(A) \subset X$, the corresponding open subset $U' \subset \mathrm{Spec}(\mathrm{QCoh}(X))$ consists of all $[\mathcal{O}_X/\mathcal{J}_x]$ where $x \in U$. By the above observation, $\langle U\rangle$ is exactly those spectral objects M vanishing on U which is $\mathrm{QCoh}_U(X)$, then in the same way as example 4.4, quotient category $\mathrm{QCoh}(X)/\mathrm{QCoh}_U(X) \cong \mathrm{QCoh}(U)$, thus $Z(\mathrm{QCoh}(X)/\mathrm{QCoh}_U(X)) \cong \Gamma(U, \mathcal{O}_X)$ (which is obviously compatible with restriction by example 4.6). So the presheaf for the structure sheaf of $\mathrm{Spec}(\mathrm{QCoh}(X))$ is isomorphic to the (pre)sheaf of $\mathcal{O}_X$ on all open affine subset, which constitutes a basis for topological space $X \cong \mathrm{Spec}(\mathrm{QCoh}(X))$. We know that the sheaf is determined by its sheafification on open basis, thus after sheafification, X and $\mathrm{Spec}(\mathrm{QCoh}(X))$ have the isomorphic structure sheaf.

□

5 Proofs of the main theorems

Theorem 5.1. *Reconstruction Theorem: Let X and Y be two quasi-separated schemes, if there exists an equivalence $F : \mathrm{QCoh}(X) \cong \mathrm{QCoh}(Y)$, then X and Y are isomorphic.*

Proof. By proposition 4.10, there exists a morphism of ringed space as the composition $f : X \cong \mathrm{Spec}(\mathrm{QCoh}(X)) \xrightarrow{F} \mathrm{Spec}(\mathrm{QCoh}(Y)) \cong Y$ which is an isomorphism of ringed space. But isomorphism of ringed space is also isomorphism of locally ringed space, because ring's isomorphism between local rings is also isomorphism of local rings. Thus $X \cong Y$ as schemes. $\square$

Remark 5.2. (characterization of scheme isomorphism induced by equivalence of categories) f defined in theorem 5.1 can be characterized by the following conditions:
(1) $F(\mathcal{O}_X/\mathcal{J}_x) = \mathcal{O}_Y/\mathcal{J}_{f(x)}$.
(2) For quais-compact open subset $V \subset Y$,
$\mathrm{QCoh}(V) \xrightarrow{\sim} \mathrm{QCoh}(Y)/\mathrm{QCoh}_V(Y) \xrightarrow{\sim} \mathrm{QCoh}(X)/\mathrm{QCoh}_{f^{-1}(V)} \xrightarrow{\sim} \mathrm{QCoh}(f^{-1}(V))$ induces the ring morphism $\Gamma(V, \mathcal{O}_V) \to \Gamma(f^{-1}(V), \mathcal{O}_{f^{-1}(V)})$.
Its worth mentioning that the middle equivalence in (2)'s sequence may not be unique because the 'quasi-inverse' of the equivalence F is not unique, but this won't influence the ring morphism of the centers. (1) and (2) actually can be viewed as the definition of a ringed space isomorphism between spectrum of abelian categories.

The equivalence of the category of quasi-coherent sheaf may not be induced from some scheme isomorphism, for instance, tensoring with a line bundle $\mathcal{L}$ on X also induces an auto-equivalence of $\mathrm{QCoh}(X)$.

Observation 5.3. By construction, if equivalence F and G are naturally isomorphic, then the induced morphism $f = g$.

Example 5.4. Different equivalences of category of quasi-coherent sheaf may induce the same isomorphism of schemes, obviously for those naturally isomorphic functors.
In the affine case, identity of ring morphism $R \to R$ can be induced from both id functor and $(-) \otimes_R R$ where R is equipped with the natural R module structure (instead of some nontrivial R automorphism).

Proposition 5.5. *Let X be a quasi-separated scheme, and Y be an arbitrary scheme. Let f, g be two morphisms from Y to X, if there exists an isomorphism between two functors $\alpha : f^* \cong g^*$, then $f = g$ and α is the multiplication with the scalar $\alpha(\mathcal{O}_X)(1) \in \Gamma(Y, \mathcal{O}_Y)^*$.*

Proof. First, we shall prove $f = g$ on the underlying topological space. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf associated with the closed subset $Z \subset Y$, then $f^{-1}(Z)$ is also closed, because $\mathrm{supp}(\mathcal{O}_X/\mathcal{I}) = Z$, and $f^*(\mathcal{O}_X/\mathcal{I})_z \cong (\mathcal{O}_X/\mathcal{I})_{f(z)}$ which is zero if and only if $f(z) \in Z$, so $\mathrm{supp}(\mathcal{O}_X/\mathcal{I}) = f^{-1}(Z)$. As $\alpha : f^* \cong g^*$, thus $f^{-1}(Z) = g^{-1}(Z)$ for closed subset Z, so is open subset U. As scheme is Kolomogrov space, where one point can always be separated from another point by an open subset, by lemma 6.1 in appendix, $f^{-1}(\{x\}) = g^{-1}(\{x\})$. Thus $f = g$ on underlying space.
It remains to prove that the sheaf morphism $f^\# : f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$ is the same as $g^\# : g^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$, where $g^{-1} = f^{-1}$ because they are defined only by topological information, thus we shall prove that $f^\# = g^\#$, which can be checked locally.
Assume that $Y = \mathrm{Spec}(A)$ is affine and $f(Y) = g(Y) \subset U$ where $U = \mathrm{Spec}(B)$ is affine in

X. Thus there is factorization $f = a \circ i, g = b \circ i$ where $i : U \to X$ and $a, b : Y \to U$ corresponds to ring morphisms $a^\#, b^\# : B \to A$. Because $i^* i_* = id : \mathrm{QCoh}(U) \to \mathrm{QCoh}(U)$, thus $\alpha i_* : f^* i_* = a^* i^* i_* = a^* \cong g^* i_* = b^* i^* i_* = b^*$. It suffices to prove that $a^\# = b^\#$.
a^* and b^* are actually functors from $\mathrm{Mod}_B \to \mathrm{Mod}_A$, which are additive and cocontinuous because pullback is left adjoint. Thus by Eilenberg-Watts theorem, they correspond to two $B - A$ bimodules $\widetilde{a} = A$ (with left B module through $a^\# : B \to A$ and right A module by its own ring structure) and $\widetilde{b} = A$ (with left B module through $b^\# : B \to A$ and right B module by its own ring structure), and the isomorphism $\alpha \circ i^*$ between functors turns into an isomorphism of (B, A) bimodules $h : \widetilde{a} \cong \widetilde{b}$ (just take structure sheaf, or B into the tensor product functor).
For any $m \in B$, we shall prove $a^\#(m) = b^\#(m) \in A$. $h(a^\#(m)) = h(m \cdot 1) = h(1 \cdot m) = h(b^\#(m))$, so $a^\#(m) = b^\#(m)$. Thus $f = g$ as scheme morphism.

For the affine case as above, α corresponds to an automorphism of A module $A \xrightarrow{\alpha_A} A$, which is determined by the image $\alpha_A(1) \in A$, as α is invertible, $\alpha(1)$ is invertible with $\alpha_A^{-1}(1) \cdot \alpha_A(1) = 1 \in A$. As isomorphism α between functors is natural and B module can be expressed as quotient of direct sum of B, and pullback commutes with direct sum, then α is totally determined by its component on $f^*(B) = A \xrightarrow{A} A$, that is, α for any B module M, $\alpha_M : f^*(M) \to g^*(M) = f^*(M)$ is exactly the multiplication with the invertible section $\alpha_A(1)$.

$$\begin{tikzcd}
\oplus f^*(B) \cong f^*(\oplus B) \arrow[r, "f^*(q)"] \arrow[d, "\alpha : \cong"'] & f^*(M = \oplus B/I) \arrow[d, "\alpha \cong"] \\
\oplus g^*(B) \cong g^*(\oplus B) \arrow[r, "f^*(q)"] & g^*(M = \oplus B/I)
\end{tikzcd}$$

So we get a local invertible section on any affine open subset, and obviously it is compatible with the restriction map by uniqueness, thus we can glue a global invertible section $s = \alpha(\mathcal{O}_X)(1) \in \Gamma(Y, \mathcal{O}_Y)$. $\square$

Remark 5.6. I believe here X is not necessarily assumed to be quasi-separated, because this condition is not used anywhere.

Example 5.4 revisited. Consider again the affine case, by Eilenberg-Watts theorem, any functor $F : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is naturally isomorphic to tensor product with an (R, S) bimodule N:$(-) \otimes_R N$.
More precisely, N is image of $F(R) \in \mathrm{Mod}(S)$, which is equipped with a S module structure, especially, a right S module structure. The left R module structure is determined by the functor F acting on the Hom set: $r \cdot n = F(r \cdot (-)) \cdot n$ for $n \in N, r \in R$ with $r(-) \cdot \in \mathrm{Hom}_{\mathrm{Mod}(R)}(R, R)$.
Especially, we consider the case when $R = S$, and F is an equivalence, then the automorphism $F : \mathrm{Mod}(R) \xrightarrow{\sim} \mathrm{Mod}(R)$ is naturally isomorphic to tensor product $(-) \otimes_R N$, because it is equivalence, so there exists R module M such that $M \otimes_R N \cong R$, that is, M is invertible, or $\widetilde{M}$ is a line bundle over affine scheme $\mathrm{Spec}(R)$.
Let's consider another special case, when the above N has the underlying Z module structure as R, but 'twisited' in the sense that for a given $\psi \in \mathrm{Aut}(R)$, N is the same ring R equipped with left R module structure as $r \cdot r' = \psi(r) r'$ and right R module in the natural way, denoted as ${}_\psi R$
Then we can see that $(-) \otimes_R ({}_\psi R)$ is twisting the R module structure in the same way through

automorphism ψ. This corresponds to the automorphism of affine schemes, such automorphism can only be checked by considering F on the Hom-set instead of the objects.

Theorem 5.7. *Let X and Y be two separated schemes, then there is an equivalence of groupoids,*

$$\left\{ \begin{array}{l} f : Y \xrightarrow{\cong} X \text{ isomorphism}, \\ L \text{ invertible sheaf on } Y \end{array} \right\} \approx \left\{ \text{ equivalences } Qcoh(X) \xrightarrow{\cong} Qcoh(Y) \right\}$$

given by mapping a pair $(f, \mathcal{L})$ to the equivalence $f^*(-) \otimes \mathcal{L}$.

Here we define the left side as the groupoid with the objects as $(f, \mathcal{L})$, and if $f \neq g$, then the invertible morphism set between two objects $(f, \mathcal{L})$ and $(g = f, \mathcal{L}')$ is defined to be the set of isomorphisms between line bundles: $\mathcal{L} \to \mathcal{L}'$, if $f \neq g$, then there exists no morphisms between $(f, \mathcal{L})$ and $(g, \mathcal{L}')$. The groupoid structure on the right hand set is defined as follows: the objects are the equivalences $\{F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)\}$, the morphism (invertible) between F and G is defined to be the natural isomorphisms between F and G.

Proof. Denote the functor sending $(f, \mathcal{L})$ to $f^*(-) \otimes \mathcal{L}$ as F, then:
faithful: F is faithful, otherwise, $\alpha \neq \beta : \mathcal{L} \to \mathcal{L}'$ and $F(\alpha) = F(\beta) \in \mathrm{Hom}(f^*(-) \otimes \mathcal{L}, f^*(-) \otimes \mathcal{L}')$, then evaluate at $\mathcal{O}_Y$, $F(\alpha)_{\mathcal{O}_Y} = id \otimes \alpha : \mathcal{O}_X \otimes \mathcal{L} \to \mathcal{L}'$ is the same as $F(\beta)_{\mathcal{O}_Y} = id \otimes \beta : \mathcal{O}_X \otimes \mathcal{L} \to \mathcal{L}'$, so $id \otimes \alpha = id \otimes \beta$, contradiction!
fullness: If there exists a natural isomorphism between functors $\alpha : f^*(-) \otimes \mathcal{L} \to g^*(-) \otimes \mathcal{L}'$, then evaluation at $\mathcal{O}_X$ leads to isomorphism $\alpha_{\mathcal{O}} : \mathcal{L} \to \mathcal{L}'$. By composition with $id \otimes \alpha_{\mathcal{O}}$, we get an isomorphism between $f^*(-) \otimes \mathcal{L} \to g^*(-) \otimes \mathcal{L}$, by composing again with $id \otimes \mathcal{L}^\vee$, we have an isomorphism between functors $f^* \to g^*$. By proposition 5.5, this leads to $f = g$, so α is actually an isomorphism between line bundles $\hat{\alpha} : \mathcal{L} \to \mathcal{L}'$. Thus $(f, \mathcal{L}) \xrightarrow{\hat{\alpha}} (f, \mathcal{L}')$ is mapped to α under functor F.
essential surjection: Let $F : \mathrm{QCoh}(X) \xrightarrow{\sim} \mathrm{QCoh}(Y)$ be an equivalence functor, we need to construct $(f, \mathcal{L})$ such that $f^*(-) \otimes \mathcal{L}$ is natural isomorphic to F. Denote $\mathcal{L} = F(\mathcal{L})$.
By theorem 5.1, there is an induced isomorphism $f : Y \xrightarrow{\sim} X$ induced by F. Then by the observation above, for every open affine (or more generally, quasi-compact open subset) U, there is a restriction functor

$$F_U : \mathrm{QCoh}(f(U)) \xrightarrow[\sim]{i_*} \mathrm{QCoh}(X)/\,\mathrm{QCoh}(X)_{f(U)} \xrightarrow{\text{induced by } F} \mathrm{QCoh}(Y)/\,\mathrm{QCoh}(Y)_U \xrightarrow[\sim]{i_*} \mathrm{QCoh}(U)$$

, which induces (in the way of theorem 5.1) exactly the restriction map $f|_U$ of f on open subscheme $U \subset Y$.

Thus we have the following composition of equivalence functors $\mathrm{QCoh}(U) \xrightarrow{(f|_U^{-1})^*} \mathrm{QCoh}(f(U)) \xrightarrow{F_U} \mathrm{QCoh}(U)$, by remark 5.2, this functor induces the identity on centers. As U is affine: $U = \mathrm{Spec}(A)$, by Eilenberg-Watts theorem for modules over ring, there is an (A, A) bimodule L_U such that $(f|_U^{-1})^* \circ F_U$ is naturally isomorphic to $(-) \otimes_A L_U$, by example 5.4 revisited, L_U is the image $F(\mathcal{O}_U = \mathcal{O}_X|_U)$, which is exactly the restriction $\mathcal{L}|_U$ (just take the image of $\mathcal{O}_{f(U)}$ under the functor F_U), but the tensor functor $(-) \otimes_{\mathcal{O}_U} ({}_\psi\mathcal{L}|_U)$ may be twisted by some automorphism $\psi \in \mathrm{Aut}(A = \Gamma(U, \mathcal{O}_U))$, luckily, as it induces identity on centers, the left A module structure on L_U is the same as the right A module structure (that is, the A module structure of the image of A under the functor), thus $(f|_U^{-1})^* \circ F_U$ is naturally isomorphic to $(-) \otimes_{\mathcal{O}_U} \mathcal{L}|_U$. This can glue into a global naturally isomorphism $F \cong f^*(-) \otimes \mathcal{L}$ because there is no 'twist' and locally this is a canonical natural isomorphism, and $\mathcal{L}$ is invertible because locally the equivalence induces invertible module sheaf as shown in example 5.4 revisited. ∎

Corollary 5.8. *If X is quasi-separated, then the automorphism group* $\mathrm{QCoh}(X)$ *is isomorphic to the semiproduct of groups* $\mathrm{Aut}(X) \rtimes \mathrm{pic}(X)$.

Remark 5.9. The automorphism class group is defined as the set of isomorphism class of auto-equivalence of $\mathrm{QCoh}(X)$ equipped with the composition as group multiplication and quasi-inverse of equivalence as inver operation of group.

Proof. Just take $X = Y$ into theorem 5.7 and take the isomorphism class. The right hand side groupoid turns into the group described as above while correspondingly, the left hand side turns into semiproduct of $\mathrm{Aut}(X) \rtimes \mathrm{Pic}(X)$: $(g, \mathcal{L}_1) \circ (f, \mathcal{L}_0) = g^*(f^*(-) \otimes \mathcal{L}_0) \otimes \mathcal{L}_1 = (f \circ g)^*(-) \otimes g^*\mathcal{L}_0 \otimes \mathcal{L}_1$, so here $\mathrm{Aut}(X)$ is equipped with the opposite group structure of composition: $(f \cdot g) = g \circ f$ and the semiproduct structure is defined as $(g, \mathcal{L}_1) \circ (f, \mathcal{L}_0) = (f \circ g)^*(-) \otimes (g^*\mathcal{L}_0 \otimes \mathcal{L}_1)$. □

6 Appenidx

scheme as Komologrov space

Lemma 6.1. *For continuous map $f : Y \to X$ where X is Komologrov space, if $f^{-1}(U) = g^{-1}(U)$ for every open subset U, then $f = g$.*

Proof. It suffices to prove that $f^{-1}(\{x\}) = g^{-1}(\{x\})$ for every $x \in E$.
For any $x \in X$, consider the 'infinitesimal open neighborhood' T of x: $T = \cap_{x \in U_\lambda} U_\lambda$ which is the intersection of all open subset containing x. Recall for a scheme, the infinitesimal neighborhood of $x = \mathfrak{p} \subset R$ is exactly the spectrum of the local ring $\mathrm{Spec}(R_\mathfrak{p})$, so x is the closed point in this set. This can be generalized to Komologrov space, for any $t \in T \setminus \{x\}$, there is no open neighborhood of x that can separated x and t, so there must exist open neighborhood $V \ni t$ such that $x \notin V$. Thus $V \cap T$ is an open neighborhood of t in T not containing x, that is, $T \setminus \{x\}$ is open in T under the induced topology. Thus $\{x\} = T \cap B$ for some closed subset $B \subset X$, so $\{x\} = B \cap_{x \in U_\lambda} U_\lambda$, and by our condition: $f^{-1}(\{x\}) = f^{-1}(B) \cap (\cap_{x \in U_\lambda} f^{-1}(U_\lambda)) = g(B) \cap (\cap_{x \in U_\lambda} f^{-1}(U_\lambda)) = g^{-1}(\{x\})$. □

extension of quasi-coherent sheaf

Lemma 6.2. *Let $i : U \to X$ be a quasi-compact open immersion of schemes, then*
(1). Any quasi-coherent sheaf on U extends to quasi-coherent sheaf on X.
(2). For any quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}|_U$ of quasi-coherent sheaf $\mathcal{F}$ on X, there exists extension of $\mathcal{H}$ on X: $\mathcal{G}$ such that $\mathcal{G}|_U = \mathcal{H}$ and $\mathcal{G} \subset \mathcal{F}$.

Proof. (1). open immersion is separated and thus i is qcqs, whose pushforward preserves quasi-coherence.
(2). by (1), $i_*\mathcal{H}$ is quasi-coherent sheaf and there is an induced morphism of quasi-coherent sheaf $i_*\mathcal{H} \xrightarrow{i_* \text{ (inclusion)}} i_*\mathcal{F}|_U = i_*i^*\mathcal{F}$. Thus take the kernel of the quasi-coherent sheaf (which is again a quasi-coherent sheaf) $\mathcal{G} = \ker(\mathcal{F} \oplus i_*\mathcal{H} \xrightarrow{(\cdot)-(\cdot)} i_*i^*\mathcal{F})$ where the first component morphism the canonical morphism and the second is described as above. This is a subsheaf of $\mathcal{F}$ because it's obvious at the stalk level. □

Analogue of group structure in the abelian category

Lemma 6.3. *For an abelian category $\mathcal{A}$,*
(1). For a cokernel P, it can always be written in the form of M/N where $N \subset M$. Then for any subobject of the cokernel: $Q \subset P$, there exist subobject N' such that $N \subset N' \subset M$ and $N'/N \cong Q$.
(2). For subobjects $A, B \subset C$, there is a subobject defined as the sum of subobjects $A + B \subset C$ such that $A/A \cap B \cong A + B/B$.

Proof. see [1, section 1.7]. □

Eilenberg-Watts theorem

Theorem 6.4. *Given unitary rings R and S and some (R, S) bimodule N, then tensor product functor $(-) \otimes_R N : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ is additive and cocontinuous. On the contrary, if a functor between $\mathrm{Mod}(R)$ and $\mathrm{Mod}(S)$ is additive and cocontinuous, it is naturally isomorphic to $(-) \otimes_R N$ for some bimodule N.*

Proof. see [5]. □

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