my immature thoughts on calculating hodge number of complete intersection in product of projective spaces $P^n \times P^m$

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1 generalization

From the chapter 9 of 'A Survey of the Hodge Conjecture' by James D.Lewis(all the following notation of "theorem" and "lemma" means theorem and lemma in this book by default), there is a way of calculating hodge number of complete intersection based on the Hirzeburch-Riemann-Roch theorem(HRR).

For holomorphic vector bundle V on compact complex manifold X,

$$\chi(X,V) = T(X,V) \tag{1}$$

Here the left of the equation is the Euler-Poincare characteristics of V, $\chi(X,V) = \sum_{j\geq 0}(-1)^j dimH^j(X,O_X(V))$, the right denotes the integration of the top degree component of the cup product of cohomology class $\{ch(V)\cdot td(X)\}$, where the $ch(V)$ is the total Chern character of V and $td(X)$ is the total Todd class of the tangent holomorphic bundle $T_{1,0}(X)$.

Then there is a generalization version the HRR theorem by the form of series:

$$\chi_y(X,V) = T_y(X,V) \tag{2}$$

Here is a equation of series of indeterminate y, and $\chi_y(X,V) = \sum_{p\geq 0}\chi^p(X)y^p$ where the $\chi^p(X,V) = \chi(X, V\otimes\wedge^{p,0}T^*(X))$, and if the V is exactly the trivial bundle $X\times C$, then it is the χ_y genus of X.

The right hand is defined as follows,

$$T_y(X,V) = \left\{\sum_{i\geq 1} e^{(1+y)a_i}\cdot\prod_j((1+y)v_j(1-e^{-(1+y)v_j})^{-1} - yv_j)\right\} \tag{3}$$

where the first sum is taken over the Chern roots of V,i.e, $\{a_1,...a_r\}$, and the second product is taken over the Chern roots of $T_{1,0}(X)$. For the special case when $V = X\times C$, the equation becomes $\chi_y(X) = T_y(X)$. Then the book provides an important lemma: lemma 9.6,

$$T_y(X) = \sum_{p\geq 0} T^p(X)y^p \tag{4}$$

where the $T^p(X) = T(X,\wedge^{p,0}T^*(X))$, more precisely, the series in y can be expanded as follows,

$$T_y(X) = \sum_{p\geq 0} \left\{ \sum_{i1\leq i_1 < \cdots < i_p \leq n} e^{-v_{i_1} - \cdots - v_{i_p}} \prod_i \frac{v_i}{1-e^{-v_i}} \right\}_n y^p \tag{5}$$

So the left and right are both in the form of series of indeterminate y now. And using this equation on the complete intersection of r hyperplanes in projective space P^{n+r}, we can get a simpler form.

Fix an r tuple $\mathbf{d} = (d_1, d_2, ..., d_r) \in N^r$, denoting the smooth complete intersection of type $(d_1, d_2, ..., d_r)$ as $X_n = V(f_1, f_2, ..., f_r) \subset P^{n+r}$, where $deg(F_i) = d_i$. So we get the the following theorem based on the previous theorems : theorem 9.7,

$$\sum_{n\geq 0} \chi_y(X_n) z^{n+r} = [(1+zy)(1-z)]^{-1} \prod_i \frac{(1+zy)^{d_i} - (1-z)^{d_i}}{(1+zy)^{d_i} + y(1-z)^{d_i}} \tag{6}$$

By adding another indeterminate z, the equation now is clear and easier to handle. The left side coefficients are relevant to the information of characteristics and correspondingly the hodge number, the right hand now is purely algebraic so that the calculation is much easier. And actually, the information of the Hodge number of the complete intersection $h^{p,q}(X_n)$ hides in the coefficient of term $y^p z^q$ of the two indeterminate y and z. If we take $\chi^p(X_n) = \sum_{q\geq 0} (-1)^q h^{p,q}(X_n)$ into the formula. Then we have the more explicit formula.

$$\sum_{p,q\geq 0} (h^{p,q}(X_n) - \delta_{p,q}) y^p z^q = \frac{1}{(1+y)(1+z)} \left[\left\{ \prod_i \frac{(1+y)^{d_i} - (1+z)^{d_i}}{y(1+z)^{d_i} - z(1+y)^{d_i}} \right\} - 1 \right] \tag{7}$$

I am trying to do the same thing for the case of product of projective spaces, i.e, finding a similar equation as (6) for the complete intersection in product of projective space $P^n \times P^m$ and then going to (7). Taking a careful look of the process of the proof of the theorem 9.7, we can see that the key of the proof is the calculation of the $T_y(X_n)$ to transform the integration of cohomology into the coefficient of certain term in some series (the latter is an algebraic objective), and then using the HRR theorem (2) to relate the hodge number with the coefficients of the series. To calculate the $T_y(X_n)$, firstly we need to calculate the Chern class of the complete intersection of the complete intersection in the product of projective space, here we denote the $X_{n+m-r} = V(f_1, \cdots, f_r) \in P^n \times P^m$, where the$f_i$ is the biform with bidgree (d_i, e_i), then obivously the dimension of the intersection with the zero locus of hypersurface of biform will decrease by one (we can see this on some affine open subsets of $P^n \times P^m$).

2 Chern class of complete intersection

Before calculating the Chern class of the complete intersection in $P^n \times P^m$, I want to diverge to justify why the complete intersection of $P^n \times P^m$ can be written in the form of $V(f_1, \cdots, f_r)$ where f_i is biform. Complete intersection is defined as intersection of hypersurfaces while in P^n, all hypersurfaces are given as zero locus of some homogeneous polynomial, so the complete intersection in P^n is exactly in the form of zero locus of the polynomials corresponding to the hypersurfaces. So we can define the complete intersection as $V(f_1, \cdots, f_r)$. But actually in $P^n \times P^m$, this is also the case, with the notation that $p_1 : P^n \times P^m \to P^n$, $p_2 : P^n \times P^m \to P^m$, and that H_1 is the hyperplane bundle in P^n while H_2 is the hyperplane bundle in P^m,

Lemma 1. *For line bundle L of the form $p_1^* H_1^{\otimes d} \otimes p_2^* H_2^{\otimes e}$, $H^0(P^n \times P^m, L) \cong k\left[x_0, \cdots x_n, y_0, \cdots y_m\right]_{d,e}$, the right hand of the isomorphism is the bihomogeneous polynimial with bidegree (d, e), $k\left[x_0, \cdots x_n, y_0, \cdots y_m\right]_{d,e} \cong k\left[x_0, \cdots x_n\right]_d \otimes k\left[y_0, \cdots y_m\right]_e$.*

Here the homogeneous polynomial is naturally the global section of the line bundle composed of the hyperplane bundles, and the proof of the theorem is very similar to the case in P^n. By the natural morphism $H^0(P^n \times P^m, p_1^* H_1^{\otimes d}) \otimes H^0(P^n \times P^m, p_2^* H_2^{\otimes e}) \to H^0(P^n \times P^m, p_1^* H_1^{\otimes d} \otimes p_2^* H_2^{\otimes e})$, or more precisely, there is a bilinear map $H^0(P^n, H_1^{\otimes d}) \otimes H^0(P^m, H_2^{\otimes e}) \to H^0(P^n \times P^m, p_1^* H_1^{\otimes d} \otimes p_2^* H_2^{\otimes e})$ and it's already known that for P^n, $H^0(P^n, H_1^{\otimes})$ corresponds exactly to the vector space generated with the basis of $k\left[x_0 \cdots x_n\right]_e$, so the key of the proof of the lemma is showing the surjectivity of the map $k\left[x_0, \cdots, x_n, y_0, \cdots y_m\right]_{d,e} \to H^0(P^n \times P^m, p_1^* H_1^{\otimes d} \otimes p_2^* H_2^{\otimes e})$.

Take any section nontrivial $t \in H^0(P^n \times P^m, p_1^* H_1^d \otimes p_2^* H_2^e)$, choose a suitable $o \neq s \in H^0(P^n \times P^m, p_1^* H_1^{\otimes d} \otimes p_2^* H_2^{\otimes e})$ induced by a homogeneous polynomial of degree (d, e),(for example ,$\sum x_i^d + \sum y_j^e$), then we can get a meromorphic function $t/s \in K(P^n \times P^m)$, and from natural projection $P : C^{n+1} \setminus \{0\} \times C^{m+1} \setminus \{0\} \to P^n \times P^m$, composing this projection we can have the map from $C^{n+1} \setminus \{0\} \times C^{m+1} \setminus \{0\}$ to C, by Riemann Extension theorem, if it is locally bounded near the $C^{n+1} \times \{0\} \cup \{0\} \times C^{m+1}$, then it can be extened to the whole $C^{n+1} \times C^{m+1}$. the local boundness can be ensured by our choice of s and thus we can have a holomorphic map $hC^{n+1} \times C^{m+1} \to C$, and it is bihomogeneous of bidegree (d, e), i.e, $h(\lambda x_0, \cdots, \lambda x_n, \mu y_0, \cdots, \mu y_m) = \lambda^{n+1} \cdot \mu^{m+1} h(x_0, \cdot \cdot \cdot, x_n, y_0, \cdot \cdot \cdot, y_m)$. Using power series expansion, we can see that this function h must be of homogeneous polynomial of bidegree (d, e).

And the injectivity is trivial so that the lemma is done. For the Picard group of product of projective spaces, there is $Pic(P^n \times P^m) = Pic(P^n) \oplus Pic(P^m)$, and the hypersurface corresponds to elements in the $Pic(P^n) \oplus Pic(P^m)$ by the natural map from divisor group to the Picard group, so that the line bundle for the hypersurface is a combination of $p_1^* H_1$ and $p_2^* H_2$, written in the form of $p_1^* H_1 \otimes p_2^* H_2$, so the hypersurface is generated as zero locus of the section of the $p_1^* H_1 \otimes p_2^* H_2$, and is of the form of zero locus of homogeneous polynomial of bidegree (d, e). In this way, the complete intersection in the product of projective spaces is the zero locus of some homogeneous polynomials.

In the chapter 8 of the book, Chern class of the complete intersection in projective space is calculated firstly by Euler sequence so that we can have the Chern class of the P^n, then we relate the normal bundle of complete intersection $N_{X(f_1, \cdots, f_r)/P^n}$ with hyperplane bundle and finally represent the Chern class of the Chern class by hyperplane bundle H. Imitating the case in P^n, the Chern class of the complete intersection in $P^n \times P^m$ can be solved.

With Euler sequence for P^n,

$$0 \to C \to H^{\oplus(n+1)} \to T_{1,0}(P^n) \to 0 \tag{8}$$

For product of projective space $P^n \times P^m$, with pullback and using the fact that tangent space of product of manifold is the direct sum of the separate tangent space $T_{1,0}(P^m \times P^m) = T_{1,0}(P^m) \oplus T_{1,0}(P^n)$,

$$0 \to C \oplus C \to p_1^* H^{\oplus n+1} \oplus p_2^* H^{\oplus m+1} \to T_{1,0}(P^n \times P^m) \to 0 \tag{9}$$

Here we don't distinguish between the vector space at a point, vector bundle and the sheaf associated with it. And the Chern class of $P^n \times P^m$ can be calculated. $c(P^n \times P^m) = p_1^* c(P^n) p_2^* c(P^m) = (1 + c_1(p_1^* H_1))^{n+1}(1 + c_1(p_2^* H_2))^{m+1}$.

If $j : V(f_1) = X \to P^n \times P^m$ is a smooth embedding of hypersurface with bidegree $deg(f_1) = (d_1, e_1)$, set $\mathbf{h_1} = j^* c_1(p_1^*(H_1))$, $\mathbf{h_2} = j^* c_1(p_2^*(H_2))$. For such embedding there exists a corresponding exact sequence of vector bundle,

$$0 \to T_{1,0}(X) \to T_{1,0}(P^n \times P^m) \to N(X/P^n \times P^m) \to 0 \tag{10}$$

Now the problem is understanding the normal bundle $N(X/P^n \times P^m)$. First of all, from any analytic hypersurface Y, there is an open cover $\{U_i\}_{i \in I}$ and analytic functions $\{f_i \, U_i \to c\}$ from which we can construct a holomorphic line bundle by cocycle $\{g_{ij} = f_i/f_j\}$, and by chapter 4 proposition 4.6: $L_Y|Y \cong N(Y/P^n \times P^m)$. So we only need to calculate the transition function of the line bundle L_Y which is easy, if we restrict to open covering $\{U_i \times V_j\}$ where U_i and V_j is the standard open covering of the projective space. From $U_i \times V_j$ to $U_{i'} \times V_{j'}$, the defining function of the zero locus of V denoted by F (we have already proved that hypersurface must be of this form) is from $\frac{F}{x_i^d y_j^e}$ to $\frac{F}{x_{i'}^d y_{j'}^e}$, so the transition map is $g_{ij} = \frac{x_{i'}^d y_{j'}^e}{x_i^d y_j^e}$, which is exactly the same as line bundle $p_1^* H_1^d \otimes p_2^* H_2^e$. And for the line bundle, we have the relation of Chern class $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$ which is easily seen. Indeed, the first Chern class of line bundle corresponds to the $\delta(\frac{log(g_{ij})}{2\pi\sqrt{-1}}$ where the δ is the boundary map so that tensor operation corresponds to the $log(g_{ij}h_{ij}) = log(g_{ij}) + log(h_{ij})$. From the exact sequence (9), there is relation for Chern class $c(P^n \times P^m) = c(X)c(N(X/P^n \times P^m))$, so $c(X) = (1 + \mathbf{h_1})^{n+1}(1 + \mathbf{h_2})^{m+1})(1 + d_1\mathbf{h_1} + e_1\mathbf{h_2})^{-1}$, by induction, we can have the Chern class for the complete intersection $X_{n+m-r} = V(f_1, \cdots, f_r)$ is $(1 + \mathbf{h_1})^{n+1}(1 + \mathbf{h_2})^{m+1}(1 + d_1\mathbf{h_1} + e_1\mathbf{h_2})^{-1} \cdots (1 + d_r\mathbf{h_1} + e_r\mathbf{h_2})^{-1}$.

3 Bezout theorem in $P^n \times P^m$

Now we try to get a similar result as theorem 9.7 ,$\sum_{n\geq 0} \chi_y(X_n) z^{n+r} = [(1+zy)(1-z)]^{-1} \prod_i \frac{(1+zy)^{d_i}-(1-z)^{d_i}}{(1+zy)^{d_i}+y(1-z)^{d_i}}$. Actually, this theorem is proved by calculating the $T_y(X_n)$ and we can find that $T_y(X_n)$ is residue of certain power series, and after a change of variables, this is exactly the coefficient of the indeterminate z in the right hand of equation (6). In the process of the transformation of the $T_y(X_n)$, Bezout theorem is used as an intrinsic condition of the complete intersection to simplify the formula. But the Bezout theorem is usually applied in the intersection in the projective space instead of the product of projective spaces. But indeed the Bezout theorem can be generalized in the product case, it is shown in the book 'Basic Algebraic Algebra' by Igor R.Shafarevich. chapter 4.

If an effective divisor D is defined by a polynomial of bidegree (d, e), then if we take the generators of $Div(P^n \times P^m)/P \cong Pic(P^n \times P^m)$ as E and F, where the P denotes the principle divisor subgroup which is also the kernel of the subgroup morphism $Div(P^n \times P^m) \to Pic(P^n \times P^m)$, which is surjective for all the projective manifolds. E and F are actually defined separately by linear form in x_i and y_i, which also correspond to pullback of hyperplane bundles, $p_1^* H_1$ and $p_2^* H_2$.

Theorem 1. *Suppose that $D_i \sim k_i E + l_i F$ for i=1,...n+m. Then $D_1 D_2 \cdots D_{n+m} = \prod_{i=1}^{n+m}(k_i E + l_i F) = \sum k_{i_1} \cdots l_{j_1} \cdots l_{j_m}$, where $\{i_1, \cdot\cdot\cdot, i_n, j_1 \cdot\cdot\cdot, j_m\}$ runs over permutation of $\{1, \cdot\cdot\cdot, n+m\}$ and $i_1 < \cdots < i_n$, $k_1 < \cdots < j_m$. (Here we mean the intersection number)*

Actually, for X_{n+m-r}, which corresponds to divisor product $\prod_{i=1}^{i=r}(d_i E + e_i F)$, its intersection number with the product of a general $(r-i)$ plane Γ and a general i plane Λ is exactly the coefficient of term $E^{r-i}F^i$. If we write the fundamental class $[X]$ as sum of fundamental class $\left[P^i \times P^{k-i}\right]$(because the homology group of product of projective spaces is generated by them due to the Kunneth formula), then $[X_{n+m-r}] = \sum \tau_i \cdot [P^i \times P^{n+m-r-i}]$ where the coefficient τ_{n-r+i} is also the same as the coefficient of the term $E^{r-i}F^i$, which is very intuitive and the coefficients are actually the multidegree defined for subvariety in $P^n \times P^m$, which is defined officially in the 'Algebraic

Geometry, A First Course' by Fulton and Harris, lecture 19.

Now we come back to the calculation of $T_y(X_{n+m-r})$. By definition in equation(3), $T_y(X,V) = \left\{\sum_{i\geq 1} e^{(1+y)a_i} \cdot \prod_j ((1+y)v_j(1-e^{-(1+y)v_j})^{-1} - yv_j)\right\}$, with $V = X \times C$, $T_y(X) = \left\{\prod_j ((1+y)v_j(1-e^{-(1+y)v_j})^{-1} - yv_j)\right\}_n$, if we set $Q(y;x) = x(y+1)/(1-e^{-x(y+1)}) - yx$, then $T_y(X) = \{\prod_i Q(y;v_i)\}_n$ with $\{v_i\}$ is the Chern roots of $T_{1,0}(X)$. So for X_{n+m-r}, we have known that the Chern class of it is $(1+\mathbf{h_1})^{n+1}(1+\mathbf{h_2})^{m+1}(1+d_1\mathbf{h_1}+e_1\mathbf{h_2})^{-1}\cdots$ $(1+d_r\mathbf{h_1}+e_r\mathbf{h_2})^{-1}$. In this way,

$$T_y(X_{n+m-r}) = \left\{(Q(\mathbf{h_1})^{n+1}Q(\mathbf{h_2})^{m+1}\prod_{i=1}^{i=r} Q^{-1}(d_i\mathbf{h_1}+e_i\mathbf{h_2})\right\}_{n+m-r} \tag{11}$$

here we suppress y in the $Q(y,x)$ and write it as $Q(x)$. In the case of P^{n+r} there is only one 'variable' of $\mathbf{h}$, and the n-form of this $\mathbf{h}$ is only the $\mathbf{h}^n$, with Bezout theorem in projective space, there is $\int_{X_n} \mathbf{h} = \prod d_i$. For the case of product of projective spaces, the $n+m-r$ form consists of the terms ${\mathbf{h_1}}^s{\mathbf{h_2}}^{n+m-r-s}$ with $s = 1, ..., n+m-r$, For each term ${\mathbf{h_1}}^s{\mathbf{h_2}}^{n+m-r-s}$, there is Bezout theorem in product of projective spaces, $\int_{X_{n+m-r}} {\mathbf{h_1}}^s{\mathbf{h_2}}^{n+m-r-s} = \#\prod_{i=1}^{i=r}(d_iE+e_iF)E^sF^{n+m-r-s} = \{$ the coefficient of E^nF^m in $\prod_{i=1}^{i=r}(d_iE+e_iF)E^sF^{n+m-r-s}\}$.

4 What to do next?

I am frustrated here to find that there seems to be no way using the Bezout theorem condition in equation(11) and I have no ideal of simplifying it so that the integration can be transformed into some algebraic power series and then to relate it with the characteristics $\chi_y(X_{n+m-r})$ and then finally calculating the hodge number of the complete intersection in product of projective spaces. I find that the multivariables of $\mathbf{h_1}$ and $\mathbf{h_2}$ are inevitable. In the projective space case, this $\mathbf{h}$ is used to become a new indeterminate z added to the χ_y characteristics and $T_y(X)$ so that we have a series in two indeterminate y and z in equation (6). And this two indetermiante y and z correspond to index (p,q) of $h^{p,q}$(this is clearly illustrated from theorem 9.7 to example 9.11). But now we have to add two new indeterminates originating from $\mathbf{h_1}$ and $\mathbf{h_2}$ into the original series with one indeterminate y, so this new series has three indeterminate and how can the three parameters correspond to the hodge number indexed by only two parameters (p, q) of $h^{p,q}$. I can feel that this may also have something to do with the residue of the complex function of several variables? But there isn't anything readable on this topic. Are there any mistakes in my deduction or is there any better way of solving the problem?