Proceedings of Machine Learning Research vol tbd:1–37, 2022 35th Annual Conference on Learning Theory

On the Sample Complexity of Stabilizing LTI Systems on a Single Trajectory

Yang Hu HUY18@MAILS.TSINGHUA.EDU.CN
Institute for Interdisciplinary Information Sciences, Tsinghua University

Guannan Qu GQU@ANDREW.CMU.EDU
Department of Electrical and Computer Engineering, Carnegie Mellon University

Adam Wierman ADAMW@CALTECH.EDU
Department of Computing and Mathematical Sciences, California Institute of Technology

Abstract

Stabilizing an unknown control system is one of the central problems in control theory. In this paper, we study sample complexity of the learn-to-stabilize problem in noiseless Linear Time-Invariant (LTI) systems. State-of-the-art approaches generally require a sample complexity linear in n, the state dimension, which incurs a state norm that blows up exponentially in n. We propose a novel algorithm based on spectral decomposition that only needs to learn "a small effective part" of the dynamical matrix acting on its unstable subspace. We show that, under proper assumptions, our algorithm stabilizes an LTI system on a single trajectory with $\tilde{O}(k)$ samples, where k is the instability index of the system. This is the first sub-linear sample complexity result for the stabilization of LTI systems to our knowledge.

Keywords: system stabilization, linear time-invariant systems, sample complexity.

1. Introduction

Linear Time-Invariant (LTI) systems, namely

$$x_{t+1} = Ax_t + Bu_t,$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the control input, is one of the most fundamental dynamical systems in control theory with wide applications across engineering, economics, societal domains. For systems with known dynamical matrices (A, B), there is a well-developed theory on designing feedback controllers with guaranteed stability, robustness, and performance (Doyle et al., 2013; Dullerud and Paganini, 2013). However, these tools cannot be directly applied when the dynamical matrices are unknown.

Driven by the success of machine learning (Levine et al., 2015; Duan et al., 2016), there have been tremendous interests in learning-based (adaptive) control for LTI systems, where the learner does not know the underlying system dynamics and learns to control in an online manner, usually with low regret guarantees (Fazel et al., 2018; Bu et al., 2019; Li et al., 2019; Bradtke et al., 1994; Tu and Recht, 2017; Krauth et al., 2019; Zhou et al., 1996; Dean et al., 2019; Tu and Recht, 2018).

Despite the progress, an important limitation in most of these works is the assumption that the learner has a priori access to a known *stabilizing* controller. This assumption simplifies the learning task, since it ensures a bounded state trajectory in the learning stage, and thus enables the learner to learn with an acceptably low regret. However, assuming a known stabilizing controller is by no means practical, as *stabilization* itself is a nontrivial task and is considered equally important as any performance guarantee (e.g. LQR cost, regret). To overcome the limitation, in this paper we consider the *learn-to-stabilize* problem, i.e., learning to stabilize an unknown dynamical system without prior knowledge of any stabilizing controller. Understanding the learn-to-stabilize problem will be of great importance to the learning-based control literature, as it serves as a precursor to any learning-based control algorithms that assume knowledge of a stabilizing controller.

The learn-to-stabilize problem has attracted extensive attention recently. For example, Chen and Hazan (2021) introduces a model-based approach that first excites the open-loop system to learn dynamical matrices (A, B), and then designs a stabilizing controller, with a sample complexity scaling linearly in n, the state dimension. However, a linear scaling sample complexity is far from satisfactory, since the state trajectory still blows up exponentially when the open-loop system is unstable, incurring a $2^{\Theta(n)}$ state norm, and hence a $2^{\Theta(n)}$ regret (in LQR settings, for example). Another recent work by Perdomo et al. (2021) proposes a policy-gradient-based discount annealing method that solves a series of discounted LQR problems with increasing discount factors, and shows that the control policy converges to a near-optimal policy. However, this model-free approach only guarantees a worst-case sample complexity of $\mathrm{poly}(n)$. In fact, to the best of our knowledge, state-of-the-art learn-to-stabilize algorithms with theoretical guarantees always incur state norms exponential in n, which is prohibitively large for high-dimensional systems.

The exponential scaling in n may seem inevitable, since in the information-theoretic perspective, a complete recovery of A should take $\Theta(n)$ samples. However, our work is motivated by the observation that it is not always necessary to learn the whole matrix A to stabilize an LTI system. For example, if the system is open-loop stable, we do not need to learn anything to stabilize it. For general LTI systems, it is still intuitive that open-loop *stable* "modes" exist and need not be learned for the learn-to-stabilize problem, so we shall focus on learning a controller to inhibit those open-loop *unstable* "modes", making it possible to learn a stable controller without exponentially exploding state norms. Driven this observation, the central question of this paper is:

Can we learn to stabilize an LTI system on a single trajectory
without incurring an exponential state norm?

Contribution. In this paper, we answer the above question by designing an algorithm that stabilizes an LTI system with only $\tilde{O}(k)$ state samples along a single trajectory, where k is the *instability index* of the open-loop system defined as the number of unstable "modes" (i.e., eigenvalues with moduli larger than 1) of matrix A. Our result is significant in the sense that k can be considerably smaller than n for systems in reality, and in such cases we can stabilize the system via far fewer samples than prior work. This means we only need to incur state norm (and regret) in the order of $2^{\tilde{O}(k)}$, much smaller than $2^{O(n)}$ in the prior art when $k \ll n$.

To formalize the concept of unstable "modes" for the presentation of our algorithm and analysis, we formulate a novel framework based on the spectral decomposition of dynamical matrix A. More specifically, we focus on the *unstable subspace* E_{u} spanned by the eigenvectors corresponding to unstable eigenvalues, and in some sense consider the system dynamics "restricted" to it — states are orthogonally projected onto E_{u}, and we only have to learn the effective part of A within subspace E_{u}, which is possible with $O(k)$ samples. The formulation will be explained in details in Appendix A. We comment that this idea of decomposition is in stark contrast to prior work, which in one way or another seeks to learn the entire A (or other similar quantities) that takes at least $\Theta(n)$ samples.

1.1. Related Works

Learning for control assuming known stabilizing controller. There is a large literature on learning-based control with known stabilizing controllers. For example, one line of research utilizes model-free policy optimization approaches to learn controllers with certain desirable properties for LTI systems (Fazel et al., 2018; Bu et al., 2019; Li et al., 2019; Rautert and Sachs, 1997; Mårtensson and Rantzer, 2009; Malik et al., 2018; Mohammadi et al., 2019; Gravell et al., 2019; Yang et al., 2019; Zhang et al., 2019, 2020; Furieri et al., 2020; Jansch-Porto et al., 2020a,b; Fatkhullin and Polyak, 2020), all of which require a known stabilizing controller as the initializer for the policy search method. Another line of research is based on model-based methods, i.e., learning dynamical matrices (A, B) first before designing a controller (Dean et al., 2019), sometimes in an online setting. Works along this line includes, to name a few, Abbasi-Yadkori and Szepesvári (2011); Faradonbeh et al. (2017); Ouyang et al. (2017); Dean et al. (2018); Cohen et al. (2019); Mania et al. (2019); Simchowitz and Foster (2020); Simchowitz et al. (2020).

Learning to stabilize on a single trajectory. Stabilizing systems over *infinite* horizons with asymptotic convergence guarantees is a classical problem, which has been studied extensively in a wide range of early literature like Lai (1986); Chen and Zhang (1989); Lai and Ying (1991). On the other hand, the problem of system stabilization over *finite* horizons remains partially open and has not seen significant progresses. Recently, algorithms incurring a $2^{O(n)}O(\sqrt{T})$ regret have been proposed in the setting with relatively strong assumptions of controllability and strictly stable transition matrices (Abbasi-Yadkori and Szepesvári, 2011; Ibrahimi et al., 2013), while a novel approach based on system identification that merely assumes stabilizability is introduced in Faradonbeh et al. (2019) without guarantees on regret or the number of exploration steps.

System Identification. The model-based approach of system stabilization is closely related to the system identification literature that focuses on learning the system parameters of dynamical systems. This is a line of research enjoying a long history, with early works (Ljung, 1999; Lennart, 1999) focusing on asymptotic guarantees, and more recent work (Simchowitz et al., 2018; Oymak and Ozay, 2019; Sarkar et al., 2019) focusing on finite-time guarantees.

2. Problem Formulation

We consider a noiseless LTI system $x_{t+1} = Ax_t + Bu_t$, where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the *state* and *control input* at time step t, respectively. The dynamical matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are unknown to the learner. The learner is allowed to learn about the system by interacting with it on a *single trajectory* — the initial state is sampled uniformly randomly from the unit hypersphere surface in $\mathbb{R}^n$, and then, at each time step t, the learner is allowed to observe x_t and freely determine u_t. The goal of the learner is to learn a stabilizing controller, which is defined as follows.

Definition 1 (Stabilizing controllers) *Control rule $u_t = f_t(x_t, x_{t-1}, \cdots, x_0)$ is called a **stabilizing controller** if and only if the closed-loop system $x_{t+1} = Ax_t + Bu_t$ is asymptotically stable; i.e., for any $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} \|x_t\| = 0$ is guaranteed in the controlled system.*

To achieve this goal, a simple strategy adopted in prior work (Abbasi-Yadkori and Szepesvári, 2011; Faradonbeh et al., 2019) is to let the system run open-loop and learn (A, B) (e.g., via least squares), and then design a stabilizing controller based on the learned dynamical matrices. However, from an information-theoretic perspective, it takes at least n time steps to learn (A, B), as A itself involves n^2 parameters to learn; therefore, by the time (A, B) is learned, the state norm will be in the order of $2^{\Theta(n)}$ when the system is open-loop unstable. Such an exponentially large state norm is unacceptable, and will also incur an exponentially large regret for commonly used cost functions that scale polynomially with the state norm (e.g., the quadratic costs used in LQR problems).

The reason for the exponentially large state norm is because the previous approaches seek to fully recover the system parameters (A, B) before designing stabilizing controllers. As has been discussed in the introduction, it is not necessary to learn the entire (A, B) matrices, since we only need to learn "a small effective part" of (A, B) to stabilize the LTI system, which potentially helps to avoid the exponentially large (in n) state norm. The central problem of this paper is to characterize what is the "small part" and design algorithms to learn it, which is formally stated below.

Problem Statement. What is the sample compelxity of learning a stabilizing controller for LTI systems on a single trajectory? Particularly, can we learn a stabilizing controller *without incurring an state norm exponentially large in* n?

In Section 3, we will formally introduce our algorithm, and in Section 4, we will provide sample complexity guarantees of the proposed algorithm.

Remarks *Although it is a common practice to include an additive disturbance term w_t in the dynamics, the introduction of stochasticity does not provide additional insights into our decomposition-based algorithm, but rather, merely adds to the technical complexity of the analysis. Therefore, here we omit the disturbance in theoretical results for the clarity of exposition, and will show by numerical experiments that our algorithm can also handle disturbances to a certain extent.*

Notations. The following notations are used throughout this paper. For $z \in \mathbb{C}$, $|z|$ is the modulus of z. For a matrix $A \in \mathbb{R}^{p \times q}$, $A^\top$ denotes the transpose of A; $\|A\|$ is the induced 2-norm of A (equal to its largest singular value), and $\sigma_{\min}(A)$ is the smallest singular value of A; when A is square, $\rho(A)$ denotes the spectral radius (i.e., largest norm of eigenvalue) of A. The space spanned by $\{v_1, \cdots, v_p\}$ is denoted by $\operatorname{span}(v_1, \cdots, v_p)$, and the column space of A is denoted by $\operatorname{col}(A)$. For two subspaces U, V of $\mathbb{R}^n$, $U^\perp$ is the orthogonal complement of U, and $U \oplus V$ is the direct sum of U and V. Zero matrix and identity matrix are denoted by O, I, respectively.

3. Learning to Stabilize from Zero (LTS$_0$)

The proposed algorithm, Learning to Stabilize from Zero (LTS$_0$), is based on a decomposition of the state space that characterizes the notion of unstable "modes". We will first briefly introduce the decomposition in Section 3.1, and then proceed to describe LTS$_0$ in Section 3.2.

3.1. Decomposition of the State Space

Consider the open-loop system $x_{t+1} = Ax_t$. Let $\{\lambda_1, \cdots, \lambda_n\}$ denote the spectrum of A, where

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k| > 1 > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|.$$

Suppose A is diagonalizable, and let v_i be the eigenvector corresponding to eigenvalue λ_i; if any eigenvalue appears with multiplicity, its eigenvectors are assumed be linearly independent.

As has been discussed in the introduction, we only need to learn "a small effective part" of A associated with the unstable "modes", or the unstable eigenvectors of A. This leads us to consider the orthogonal projection onto the unstable subspace $E_\mathrm{u} := \operatorname{span}(v_1, \cdots, v_k)$ as follows.

The $\boldsymbol{E_{\mathrm{u}} \oplus E_{\mathrm{u}}^{\perp}}$-decomposition. Suppose the unstable subspace E_{u} and its orthogonal complement $E_{\mathrm{u}}^{\perp}$ are given by *orthonormal* bases $P_1 \in \mathbb{R}^{n\times k}$ and $P_2 \in \mathbb{R}^{n\times(n-k)}$, respectively, namely

$$E_{\mathrm{u}} = \mathrm{col}(P_1),\ E_{\mathrm{u}}^{\perp} = \mathrm{col}(P_2).$$

Let $P = [P_1\ P_2]$, which is also orthonormal and thus $P^{-1} = P^\top = [P_1\ P_2]^\top$. For convenience, let $\mathit{\Pi}_1 := P_1 P_1^\top$ and $\mathit{\Pi}_2 = P_2 P_2^\top$ be the *orthogonal* projectors onto E_{u} and $E_{\mathrm{u}}^{\perp}$, respectively.

We proceed to decompose matrix A. Note that E_{u} is an invariant subspace with regard to A (but $E_{\mathrm{u}}^{\perp}$ not necessarily is), there exists $M_1 \in \mathbb{R}^{k\times k}$, $\mathit{\Delta} \in \mathbb{R}^{k\times(n-k)}$ and $M_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, such that

$$AP = P\begin{bmatrix} M_1 & \mathit{\Delta} \\ & M_2 \end{bmatrix} \Leftrightarrow M := \begin{bmatrix} M_1 & \mathit{\Delta} \\ & M_2 \end{bmatrix} = P^{-1}AP.$$

Intuitively, the top-right $\mathit{\Delta}$ block in M represents how much of the state is moved by A from $E_{\mathrm{u}}^{\perp}$ into E_{u} in one step. Let $y = [y_1^\top\ y_2^\top]^\top$ be the coordinate representation of x in the basis $\{v_1, \cdots, v_n\}$ (i.e., $x = Py$). The system dynamics in y-coordinates can be expressed as

$$\begin{bmatrix} y_{1,t+1} \\ y_{2,t+1} \end{bmatrix} = P^{-1}AP\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} + P^{-1}Bu_t = \begin{bmatrix} M_1 & \mathit{\Delta} \\ & M_2 \end{bmatrix}\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} + \begin{bmatrix} P_1^\top B \\ P_2^\top B \end{bmatrix} u_t,$$

which is useful in the design of stabilizing controllers.

The $\boldsymbol{E_{\mathrm{u}} \oplus E_{\mathrm{s}}}$-decomposition. The fact that $E_{\mathrm{u}}^{\perp}$ is not invariant introduces an extra $\mathit{\Delta}$ that adds to the difficulty of the analysis. We find it helpful to introduce the $E_{\mathrm{u}} \oplus E_{\mathrm{s}}$-decomposition to assist the analysis, where $E_{\mathrm{s}} := \mathrm{span}(v_{k+1}, \cdots, v_n)$. We also represent $E_{\mathrm{u}} = \mathrm{col}(Q_1)$ and $E_{\mathrm{s}} = \mathrm{col}(Q_2)$ by their *orthonormal* bases, and define $Q = [Q_1\ Q_2]$. Note that in general these two subspaces are not orthogonal, we additionally define $Q^{-1} =: [R_1^\top R_2^\top]^\top$. Details are deferred to Appendix A.1.

It is still left to quantify the influence of $\mathit{\Delta}$. For this purpose, note that if $E_{\mathrm{s}} = E_{\mathrm{u}}^{\perp}$, the two decompositions are identical, and $\mathit{\Delta}$ naturally vanishes. For the general case, it is intuitive that, when E_{s} is not "too oblique", i.e., when E_{s} is "close" to $E_{\mathrm{u}}^{\perp}$, the influence of $\mathit{\Delta}$ should be mild. This motivates us to define such "closeness" between subspaces as follows.

Definition 2 ($\boldsymbol{\xi}$-close subspaces) *For $\xi \in (0, 1]$, two subspaces $V_1 = \mathrm{col}(\mathit{\Gamma}_1), V_2 = \mathrm{col}(\mathit{\Gamma}_2) \subset \mathbb{R}^n$ (where $\mathit{\Gamma}_1, \mathit{\Gamma}_2$ are orthonormal) are called **ξ-close** to each other, if and only if $\sigma_{\min}(\mathit{\Gamma}_1^\top \mathit{\Gamma}_2) > 1 - \xi$.*

The above definition is well-defined, since singular values remain identical under orthonormal transformations. We point out that the definition has clear geometric interpretations and leads to connections among the bases of E_{s} and $E_{\mathrm{u}}^{\perp}$, which is technical and thus deferred to Appendix A.

3.2. Algorithm

As discussed in the introduction, we do not seek to learn the full A matrix; instead, we will only learn the restriction of the (A, B) matrix onto the unstable subspace defined in Section 3.1. In detail, our algorithm LTS_0 is divided into 4 stages: we will first learn an orthonormal basis P_1 of the unstable subspace E_{u} (Stage 1); then we will learn M_1, the restriction of A onto the subspace E_{u} (Stage 2); finally, we will learn $B_\tau = P_1^\top A^{\tau-1} B$ (Stage 3), and design a controller that seeks to cancel out the "unstable" M_1 matrix (Stage 4). This is formally described as Algorithm 1.

Now we provide detailed descriptions of the three stages in LTS_0.

(1) Learn the unstable subspace of $\boldsymbol{A}$. It suffices to learn an orthonormal basis of E_{u}. We notice that, when A is applied recursively, it will push the state closer to E_{u}. Therefore, when we let the system run open-loop (with control input $u_t \equiv 0$) for t_0 time steps, the difference between

Algorithm 1 LTS_0: learning a τ-hop stabilizing controller.

1: **Stage 1: learn the unstable subspace of A.**
2: Run the system for t_0 steps for initialization.
3: Run the system for k more steps, let $D \leftarrow [x_{t_0+1} \ \cdots \ x_{t_0+k}]$.
4: Calculate $\hat{\Pi}_1 \leftarrow D(D^\top D)^{-1}D^\top$.
5: Calculate the top k (normalized) eigenvectors $\hat{v}_1, \cdots \hat{v}_k$ of $\hat{\Pi}_1$, and let $\hat{P}_1 \leftarrow [\hat{v}_1 \ \cdots \ \hat{v}_k]$.
6: **Stage 2: approximate M_1 on the unstable subspace.**
7: Restore $\hat{M}_1 \leftarrow \hat{P}_1^\top A \hat{P}_1$ by minimizing $\mathcal{L}(M_1) := \sum_{t=t_0+1}^{t_0+k} \|\hat{y}_{1,t+1} - M_1 \hat{y}_{1,t}\|^2$
8: **Stage 3: restore B_τ for τ-hop control.**
9: For $i = 1, \cdots, k$, let the system run with 0 control input for ω time steps, and then run for τ more step with initial control input $u_{t_i} = \alpha \|x_{t_i}\| e_i$, where $t_i = t_0 + k + i\omega + (i-1)\tau$.
10: Let $\hat{B}_\tau \leftarrow [\hat{b}_1 \cdots \hat{b}_k]$, where the i^{th} column $\hat{b}_i \leftarrow \frac{1}{\alpha\|x_{t_i}\|}\left(\hat{P}_1^\top x_{t_i+\tau} - \hat{M}_1^\tau \hat{P}_1^\top x_{t_i}\right)$.
11: **Stage 4: construct a τ-hop stabilizing controller K.**
12: Construct the τ-hop stabilizing controller $\hat{K} \leftarrow -\hat{B}_\tau^{-1} \hat{M}_1^\tau \hat{P}_1^\top$.

the norms of unstable and stable components will be magnified, and the state lies "almost" in E_u. Therefore, the subspace spanned by the next k states, i.e. the column space of

$$D := [x_{t_0+1} \ \cdots \ x_{t_0+k}],$$

is very close to E_u. This motivates us to use the orthogonal projector onto $\mathrm{col}(D)$, namely

$$\hat{\Pi}_1 = D(D^\top D)^{-1} D^\top$$

as an estimation of the projector $\Pi_1 = P_1 P_1^\top$ onto E_u. Finally, calculating the top k eigenvectors of $\hat{\Pi}_1$ will give us a reasonable estimation of P_1.

(2) Learn M_1 on the unstable subspace. Recall that M_1 is the transition matrix for the E_u-component under the $E_\mathrm{u} \oplus E_\mathrm{u}^\perp$-decomposition. Therefore, to estimate M_1, we first calculate the coordinates of the states $x_{t_0+1:t_0+k}$ under basis P_1; that is, $\hat{y}_{1,t} = \hat{P}_1^\top x_t$, for $t = t_0+1, \ldots, t_0+k$. Then, we use least squares to estimate M_1, which minimizes the square loss over $\hat{M}_1$

$$\mathcal{L}(\hat{M}_1) := \sum_{t=t_0+1}^{t_0+k} \|\hat{y}_{1,t+1} - \hat{M}_1 \hat{y}_{1,t}\|^2 = \sum_{t=t_0+1}^{t_0+k} \|\hat{P}_1^\top x_{t+1} - \hat{M}_1 \hat{P}_1^\top x_t\|^2.$$

Then the columns of $\hat{P}_1$ are restored by taking the k eigenvectors of $\hat{\Pi}_1$ with k largest eigenvalues (they should be very close to 1), which form a basis of the estimated unstable subspace $\hat{E}_\mathrm{u}$. It can be shown (see Appendix B) that the unique solution to the least squares problem is $\hat{M}_1 = \hat{P}_1^\top A \hat{P}_1$.

(3) Restore B_τ for τ-hop control. Now it is tempting to directly learn B and cancel out M_1 using the E_u-component. However, if control inputs are injected in every step, the controlled dynamics is (for simplicity, here we ignore the error introduced by the estimation error of projector)

$$y_{t+1} = \begin{bmatrix} M_1 + P_1^\top B K_1 & \Delta \\ P_2^\top B K_1 & M_2 \end{bmatrix} y_t.$$

Here rises the problem: the side-effect of control (bottom-left block) increments the $E_\mathrm{u}^\perp$-component, and Δ moves that side effect back into the E_u-component, which largely restricts the scope of application of our algorithm. To relieve this issue, we shall design *a τ-hop controller* instead, where control inputs are injected every τ steps (i.e., only $u_{s\tau}$ are non-zero, $s \in \mathbb{N}$). To write down the dynamics of the τ-hop control system, let $\tilde{x}_s := x_{s\tau}$, $\tilde{y}_s := y_{s\tau}$ and $\tilde{u}_s := u_{s\tau}$, and we have

$$\tilde{x}_{s+1} = A^\tau \tilde{x}_s + A^{\tau-1} B \tilde{u}_s,$$
$$\begin{bmatrix} \tilde{y}_{1,s+1} \\ \tilde{y}_{2,s+1} \end{bmatrix} = P^{-1} A^\tau P \begin{bmatrix} \tilde{y}_{1,s} \\ \tilde{y}_{2,s} \end{bmatrix} + P^{-1} A^{\tau-1} B \tilde{u}_s = M^\tau \begin{bmatrix} \tilde{y}_{1,s} \\ \tilde{y}_{2,s} \end{bmatrix} + \begin{bmatrix} P_1^\top A^{\tau-1} B \\ P_2^\top A^{\tau-1} B \end{bmatrix} \tilde{u}_s, \tag{1}$$

where

$$M^\tau = \left(\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix} + \begin{bmatrix} O & \Delta \\ & O \end{bmatrix} \right)^\tau = \begin{bmatrix} M_1^\tau & \sum_{i=0}^{\tau-1} M_1^i \Delta M_2^{\tau-1-i} \\ & M_2^\tau \end{bmatrix} =: \begin{bmatrix} M_1^\tau & \Delta_\tau \\ & M_2^\tau \end{bmatrix}.$$

In this case, the bottom-left block becomes $B_\tau = P_2^\top A^{\tau-1} B$, where $P_2^\top A^{\tau-1}$ becomes small by intuition when τ increases, and thus the closed-loop dynamical matrix is almost upper-triangular.

Now it suffices to restore B_τ that quantifies the "effective component" of control inputs restricted to E_{u}. Note that equation (1) shows

$$y_{1,t_i+\tau} = M^\tau y_{1,t_i} + \Delta_\tau y_{2,t_i} + B_\tau u_{t_i}.$$

Hence, for the purpose of estimation, we simply ignore the Δ_τ term, and take the i^{th} column as

$$\hat{b}_i \leftarrow \frac{1}{\|u_{t_i}\|} \big(\hat{P}_1^\top x_{t_i+\tau} - \hat{M}_1^\tau \hat{P}_1^\top x_{t_i} \big),$$

where u_{t_i} is parallel to e_i, and is multiplied by $\alpha \|x_{t_i}\|$ for normalization. Here we introduce an adjustable constant α to guarantee that the E_{u}-component still constitutes a non-negligible proportion of the state after injecting u_{t_i}, so that the iterative restoration of columns could continue.

It is evident that the ignored $\Delta_\tau P_2^\top x_{t_i}$ term will introduce an extra estimation error. Since Δ_τ contains a $M_1^{\tau-1} \Delta$ term that explodes with respect to τ, this part can only be bounded if $\frac{\|P_2^\top x_{t_i}\|}{\|x_{t_i}\|}$ is sufficiently small. For this purpose, we introduce ω heat-up steps (with 0 control input) to reduce the ratio to an acceptable level, during which time the projection of state onto $E_{\mathrm{u}}^\perp$ automatically diminishes over time since $\rho(M_2) = |\lambda_{k+1}| < 1$.

(4) Construct a $\boldsymbol{\tau}$-hop stabilizing controller $\boldsymbol{K}$. Finally, we can design a controller that cancels out M_1^τ in the τ-hop system. Under proper transformations and assumptions, we shall regard B as an n-by-k matrix, and further, B_τ as an invertible matrix. Then $\hat{B}_\tau$ is also invertible as long as it is close enough to B_τ. In this case, the τ-hop stabilizing controller can be simpliy designed as $K_1 = -\hat{B}_\tau^{-1} \hat{M}_1^\tau$ in y-coordinates, or

$$\hat{K} = -\hat{B}_\tau^{-1} \hat{M}_1^\tau \hat{P}_1^\top$$

in x-coordinates. Here $\hat{K}$ appears with a hat to emphasize the use of estimated projector $\hat{P}_1$, which introduces an extra estimation error to the final closed-loop dynamical matrix.

Remarks *Since we are only required to eliminate the unstable component via the control input, without loss of generality we shall stick to the convention that $m = k$, and that B is of full column rank. For the case where $m > k$, we shall simply select k linearly independent columns from B, and pad 0's in u_t for all unselected entries.*

It is evident that the algorithm terminates in $t_0 + (1 + \omega + \tau)k$ time steps. Therefore, it only suffices to take appropriate parameters so as to guarantee stability and sub-linear time simultaneously.

4. Stability Guarantee

We expect to find a stabilizing controller K that inhibits the growth of the unstable component. For the sake of analysis, we shall first write out the closed-loop dynamics under τ-hop controller. It must be handled with caution that everything in our algorithm is estimated, including $\hat{K}$. Note that

$$\hat{K}x = K_1 \hat{P}_1^\top P y = \begin{bmatrix} K_1 \hat{P}_1^\top P_1 \\ K_1 \hat{P}_1^\top P_2 \end{bmatrix} y$$

in y-coordinates (as opposed to $K_1 y$). Therefore, the controlled τ-hop dynamics should be

$$\tilde{y}_{s+1} = \begin{bmatrix} M_1^\tau + P_1^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_1 & \Delta_\tau + P_1^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_2 \\ P_2^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_1 & M_2^\tau + P_2^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_2 \end{bmatrix} \begin{bmatrix} \tilde{y}_{1,s} \\ \tilde{y}_{2,s} \end{bmatrix} =: \hat{L}_\tau \tilde{y}_s, \qquad (2)$$

which we will show to be asymptotically stable (i.e., $\rho(\hat{L}_\tau) < 1$).

Now we shall have a look at what actually happens to the τ-hop system in a (large) step. Note that $\tilde{x} = \tilde{x}_{\mathrm{u}} + \tilde{x}_{\mathrm{u}}^\perp = P_1 \tilde{y}_1 + P_2 \tilde{y}_2$ be the $E_{\mathrm{u}} \oplus E_{\mathrm{u}}^\perp$-decomposition. Since $\tilde{x}_{\mathrm{u}}^\perp$ still consists of stable and unstable components, these components stretch differently when A is applied, so that some part is moved from $E_{\mathrm{u}}^\perp$ into E_{u}; meanwhile, when the control input is injected into the system, it tries to inhibit the newly-moved-in unstable component, but also causes a side effect of increased $\tilde{x}_{\mathrm{u}}^\perp$. Note that $P_2^\top A^{\tau-1}$ in some sense represents the "stable part" of $A^{\tau-1}$, we shall expect that it decays exponentially as τ increases, and thus breaks the spiral of failure — $\hat{L}_\tau$ is "almost" upper-triangular, and thus the eigenvalues are largely determined by diagonal blocks.

4.1. Assumptions

Assumption 1 (spectral property) *A is diagonalizable with instability index k, with eigenvalues satisfying $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k| > 1 > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|$.*

Assumption 2 (initialization) *The initial state of the system is sampled uniformly randomly on the unit hyper-sphere surface in $\mathbb{R}^n$.*

Assumption 3 (c-effective control within unstable subspace) $\sigma_{\min}(R_1 B) > c\|B\|$.

We point out that Assumptions 1 and 2 are mild and reasonable. Assumption 3 characterizes the intuition of "effective controllability" in that it guarantees the unstable subspace to be effectively controlled in the following sense: every direction in the unstable subspace receives at least a proportion of c from the influence of any control input. This assumption is reasonable in that, if $\sigma_{\min}(R_1 B) \approx 0$, the control input u has to be very large to push the state along the direction corresponding to the smallest singular value, which could induce excessively large control cost.

4.2. Main Theorems

Here we present the main performance guarantees for our algorithm. Constants hidden in big-O notations can be found in detailed proofs in Appendices E and F, respectively.

Theorem 3 (Main Theorem) *Given a noiseless LTI system $x_{t+1} = Ax_t + Bu_t$ subject to Assumptions 1, 2 and 3, and additionally $|\lambda_1|^2 |\lambda_{k+1}| < 1$, by running Algorithm 1 with parameters*

$$\tau = O(1),\ \omega = O(\ell \log k),\ \alpha = O(1),\ \delta = O(|\lambda_1|^{-2\tau}),$$

that terminates within $O(k \log n)$ time steps, the controlled system is exponentially stable with probability $1 - O(k^{-\ell})$ over the initialization of x_0 for any $\ell \in \mathbb{N}$. Here the big-O notation hides system parameters like $|\lambda_{k+1}|$, $\|A\|$, $\|B\|$, c, α, ξ (assume $E_{\mathrm{u}}^\top$ and E_{s} are ξ-close), $\chi(\hat{L}_\tau)$ (see Lemma 10) and $\zeta_\varepsilon(\cdot)$ (see Lemma 17).

Despite its generality, the technical difficulty in proving the main theorem may probably overshadow the essence of it. To illustrate in a clearer manner why our algorithm is guaranteed to perform well, we include the special case where A is real symmetric as a warming-up example, which captures the main idea of the proof without burdensome technicality or constants.

Theorem 4 *Given a noiseless LTI system $x_{t+1} = Ax_t + Bu_t$ subject to Assumptions 1, 2 and 3, by running Algorithm 1 with parameters*

$$\tau = 1,\ \omega = 0,\ \alpha = 1,\ \delta = O\left(\frac{1 - |\lambda_{k+1}|}{c^2}\right),$$

that terminates within $O(k \log n)$ time steps, the controlled system is exponentially stable with probability 1 over the initialization of x_0. Here the big-O notation hides system parameters like $\|A\|$, $\|B\|$, c, and $\chi(\hat{L}_1)$ (see Lemma 10).

5. Proof Outline

In this section we will give a high-level overview of the key ideas in the proof. The full details of our proof can be found in Appendices D, E and F as indicated below.

Proof Structure. The proof is largely divided into two steps. In step 1, we examine how accurate the learner estimates the unstable subspace E_{u} in Stage 1 and 2. We will show that Π_1, P_1 and M_1 can be estimated up to an error of δ within $t_0 = O(\log \frac{n}{\varepsilon})$ steps. In step 2, we examine the estimation error of B_τ in Stage 3 (and thus K_1), based on which we will eventually show that the τ-hop controller output by Algorithm 1 makes the system asymptotically stable. Stability results are derived via a detailed spectral analysis of $\hat{L}_\tau$, the dynamical matrix of the closed-loop system.

Overview of Step 1. To upper bound the estimation errors in stage 1 and 2, we only have to notice that the estimation error of Π_1 completely captures how well the unstable subspace is estimated, and all other bounds should follow directly from it. The bound on $\|\Pi_1 - \hat{\Pi}_1\|$, in the first place, is shown in Theorem 6 by considering the explicit form of A under the basis of eigenvectors, taking advantage of the simple explicit coordinate representation of the system state that evolves over time without control inputs. Then we show that we can construct a specific pair of orthonormal bases for $E_{\mathrm{u}} = \mathrm{col}(\Pi_1)$ and $\hat{E}_{\mathrm{u}} = \mathrm{col}(\hat{\Pi}_1)$ that differs up to an error of δ. Finally, as in Corollary 7, we can take P_1 according to $\hat{P}_1$ so that the estimation error of P_1 and M_1 are bounded by $O(\delta)$ (note that we are still allowed to select P_1 freely).

Overview of Step 2. Recall that we have established the dynamics of the τ-hop controlled system in (2), where the dynamical matrix is given by

$$\hat{L}_\tau = \begin{bmatrix} M_1^\tau + P_1^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_1 & \Delta_\tau + P_1^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_2 \\ P_2^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_1 & M_2^\tau + P_2^\top A^{\tau-1} B K_1 \hat{P}_1^\top P_2 \end{bmatrix}.$$

To show the system is asymptotially stable, it only suffices to show $\rho(\hat{L}_\tau) < 1$. Note that $\hat{L}_\tau$ is given by a 2-by-2 block form, we can utilize the following lemma to assist the spectral analysis of block matrices, the proof of which is deferred to Appendix C.

Lemma 5 (block perturbation bound) *For 2-by-2 block matrices A and E in the form*

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix},\ E = \begin{bmatrix} O & E_{12} \\ E_{21} & O \end{bmatrix},$$

the spectral radii of A and $A + E$ differ by at most

$$|\rho(A + E) - \rho(A)| \le \chi(A + E)\|E_{12}\|\|E_{21}\|,$$

where $\chi(A + E)$ is a constant (see Appendix C).

The above lemma shows a clear roadmap of spectral analysis. First, we need to guarantee that the diagonal blocks are stable by themselves — the top-left block is stable because K_1 is designed to (approximately) eliminate it to zero (which requires the estimation error bound on B_τ), and the bottom-right block is stable because it is almost M_2^τ with a negligible error induced by inaccurate projection. Then, we need to upper-bound the norms of off-diagonal blocks via careful estimation of factors appearing in these blocks, which are different in the special and general cases.

The rest of this section just follows the above proof structure. We will first present the estimation error results in Section 5.1. Then, in Section 5.2, we proceed to analyze the stability for symmetric A's, so that we can see the big picture how the algorithm functions to inhibit the unstable component. Finally, we present the proof idea (informally) for the main theorem in Section 5.3.

5.1. Step 1: Estimation Error of the Unstable Subspace

As stated above, it is expected that the bound of the top-left block relies heavily on the estimation error of P_1. The major concern of this section is to show that the desired estimation precision can be achieved in acceptible time — specifically, we want it to be in the order of $O(\log n)$. Following the procedure of our algorithm, we will first bound the estimation error of $\mathit{\Pi}_1$, as in Theorem 6.

Theorem 6 *For a noiseless linear dynamical system $x_{t+1} = Ax_t$, let E_u be the unstable subspace of A, $k = \dim E_\mathrm{u}$ be the instability index of the system, and $\mathit{\Pi}_1$ be the orthogonal projector onto subspace E_u. Then for any $\varepsilon > 0$, by running Stage 1 of Algorithm 1 with an arbitrary initial state that terminates in $(t_0 + k)$ time steps, where*

$$t_0 = O\left(\log \frac{n}{\varepsilon}\right),$$

with probability 1 the matrix $D^\top D$ is invertible (where $D = [x_{t_0+1} \ \cdots \ x_{t_0+k}]$), in which case we shall obtain an estimated $\hat{\mathit{\Pi}}_1 = D(D^\top D)^{-1}D^\top$ with error

$$\|\hat{\mathit{\Pi}}_1 - \mathit{\Pi}_1\| < \varepsilon.$$

To further derive a bound for $\|\hat{P}_1 - P_1\|$, one only needs to notice that norms are preserved under orthonormal coordinate transformations, so it only suffices to find a specific pair of bases that are close to each other — and the pair of bases formed by principle vectors is exactly what we want. This leads to Corollary 7 that is repeatedly used in subsequent proofs.

Corollary 7 *Under the premises of Theorem 6, for any orthonormal basis $\hat{P}_1$ of $\mathrm{col}(\hat{\mathit{\Pi}}_1)$ obtained by Algorithm 1, there exists a corresponding orthonormal basis P_1 of $\mathrm{col}(\mathit{\Pi}_1)$, such that*

$$\|\hat{P}_1 - P_1\| < \sqrt{2k}\varepsilon =: \delta,\ \|\hat{M}_1 - M_1\| < 2\|A\|\delta.$$

The proofs in this subsection are deferred to Appendix D due to limited length.

5.2. Step 2a: Stability Analysis for Symmetric A

We first consider a warming-up case where A is real symmetric. In this case, the eigenvectors of A are mutually orthogonal, which guarantees $E_{\mathrm{u}}^{\perp} = E_{\mathrm{s}}$ (i.e., they are 0-close to each other) and thus $\Delta = O$. In this case, we can simply set $\tau = 1$, $\omega = 0$ and $\alpha = 1$, so that we have

$$\hat{L}_1 = \begin{bmatrix} O(\delta) & O(\delta) \\ O(1) & |\lambda_{k+1}| + O(\delta) \end{bmatrix}.$$

The top-left block is small based on an estimation error bound for B_1, namely $\|\hat{B}_1 - B_1\| = O(\sqrt{k}\delta)$, which characterizes how well the controller can eliminate the unstable component. Meanwhile, the top-right block is also approximately zero (so that $\hat{L}_1$ is almost lower-triangular), since $\Delta = O$ and thus only projection error contributes to the top-right block.

The proofs in this subsection are deferred to Appendix E due to limited length.

5.3. Step 2b: Stability Analysis for General A

For the general case, the analysis becomes more challenging for two reasons: on the one hand, we have to apply τ-hop control, which potentially increases the norm of B_τ and $\hat{K}_1$; on the other hand, the top-right corner will no longer be $O(\delta)$ with a non-zero Δ (in fact, Δ_τ grows exponentially at rate $|\lambda_1|^\tau$). To settle these issues, we first introduce two key observations showing how key factors in the matrix could possibly be bounded:

(1) For an arbitrary matrix X, although $\|X\|$ might be significantly larger than $\rho(X)$, we always have $\|X^t\| = O(\rho(X)^t)$ when t is large enough. This is formally proven as Gelfand's Formula (see Lemma 17), and helps to establish bounds like $\|M_1\| = O(|\lambda_1|^\tau)$, $\|M_2\| = O(|\lambda_{k+1}|^\tau)$, $\|\Delta_\tau\| = O(|\lambda_1|^\tau)$, $\|P_2^\top A^{\tau-1}\| = O(|\lambda_{k+1}|^\tau)$, and $\|\hat{M}_1^\tau - M_1^\tau\| = O(|\lambda_1|^\tau\delta)$.

(2) When the system runs with 0 control inputs for a long period (specifically, for ω time steps), eventually we will see the unstable component expanding and the stable component shrinking, and consequently $\frac{\|P_2^\top A^\omega x\|}{\|A^\omega x\|} = O(|\lambda_k|^{-\omega})$. This cancels out the exponentially exploding $\|\Delta_\tau\|$, and helps to establish the estimation bound $\|\hat{B}_\tau - B_\tau\| = O(|\lambda_1|^\tau\delta)$.

Now we are ready to upper bound the norms of the blocks in $\hat{L}_\tau$:

(1) *The top-left block*: it becomes O if the estimated $\hat{P}_1$ and $\hat{M}_1$ are exact, so its norm is reasonably expected to be proportionate to δ, the estimation error. The estimation error of M_1 contributes $O(|\lambda_1|^\tau\delta)$ to the total error bound, while the estimation error of B_τ contributes $O(|\lambda_1|^{2\tau}\delta)$, so the overall bound is in the order of $O(|\lambda_1|^{2\tau}\delta)$. Hence it is stable when δ is small enough.

(2) *The bottom-right block*: the first term is in the order of $O(|\lambda_{k+1}|^\tau)$, and the second term is in the order of $O(|\lambda_1\lambda_{k+1}|^\tau\delta)$. Hence this block is also stable when τ is large and δ is small.

(3) *The bottom-left block*: $P_2^\top A^{\tau-1}$ contributes an $O(|\lambda_{k+1}|^\tau)$ factor that decays exponentially, while K_1 contributes an $O(|\lambda_1|^\tau)$ factor that explodes exponentially. The overall bound is in the order of $O(|\lambda_1\lambda_{k+1}|^\tau)$, and decays with respect to τ when $|\lambda_1\lambda_{k+1}| < 1$.

(4) *The top-right block*: the first term is in the order of $O(|\lambda_1|^\tau)$, and the second term is in the order of $O(|\lambda_1\lambda_{k+1}|^\tau\delta)$. This block is in the order of $O(|\lambda_1|^\tau)$ when δ is small enough.

Therefore, the controlled dynamical matrix is actually in the order of

$$\hat{L}_\tau = \begin{bmatrix} O(|\lambda_1|^{2\tau}\delta) & O\big(|\lambda_1|^\tau + |\lambda_1\lambda_{k+1}|^\tau\delta\big) \\ O(|\lambda_1\lambda_{k+1}|^\tau) & O\big(|\lambda_{k+1}|^\tau + |\lambda_1\lambda_{k+1}|^\tau\delta\big) \end{bmatrix}.$$

Finally, by Lemma 5, we know that asymptotic stability is guaranteed when $|\lambda_1|^2|\lambda_{k+1}| < 1$ (i.e., the norm of the bottom-left block decays faster than the norm of the top-right block grows), in which case we can set τ to be some constant determined by A and B, and δ in the order of $O(|\lambda_1|^{-2\tau})$.

The proofs in this subsection are deferred to Appendix F due to limited length.

References

John C Doyle, Bruce A Francis, and Allen R Tannenbaum. *Feedback control theory*. Courier Corporation, 2013.

Geir E Dullerud and Fernando Paganini. *A course in robust control theory: a convex approach*, volume 36. Springer Science & Business Media, 2013.

Sergey Levine, Chelsea Finn, Trevor Darrell, and Pieter Abbeel. End-to-end training of deep visuomotor policies, 2015.

Yan Duan, Xi Chen, Rein Houthooft, John Schulman, and Pieter Abbeel. Benchmarking deep reinforcement learning for continuous control. In *International Conference on Machine Learning*, pages 1329–1338, 2016.

Maryam Fazel, Rong Ge, Sham M Kakade, and Mehran Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. *arXiv preprint arXiv:1801.05039*, 2018.

Jingjing Bu, Afshin Mesbahi, Maryam Fazel, and Mehran Mesbahi. LQR through the lens of first order methods: Discrete-time case. *arXiv preprint arXiv:1907.08921*, 2019.

Yingying Li, Yujie Tang, Runyu Zhang, and Na Li. Distributed reinforcement learning for decentralized linear quadratic control: A derivative-free policy optimization approach. *arXiv preprint arXiv:1912.09135*, 2019.

Steven J Bradtke, B Erik Ydstie, and Andrew G Barto. Adaptive linear quadratic control using policy iteration. In *Proceedings of 1994 American Control Conference-ACC'94*, volume 3, pages 3475–3479. IEEE, 1994.

Stephen Tu and Benjamin Recht. Least-squares temporal difference learning for the linear quadratic regulator. *arXiv preprint arXiv:1712.08642*, 2017.

Karl Krauth, Stephen Tu, and Benjamin Recht. Finite-time analysis of approximate policy iteration for the linear quadratic regulator. In *Advances in Neural Information Processing Systems*, pages 8512–8522, 2019.

Kemin Zhou, John Comstock Doyle, Keith Glover, et al. *Robust and optimal control*, volume 40. Prentice hall New Jersey, 1996.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. On the sample complexity of the linear quadratic regulator. *Foundations of Computational Mathematics*, pages 1–47, 2019.

Stephen Tu and Benjamin Recht. The gap between model-based and model-free methods on the linear quadratic regulator: An asymptotic viewpoint. *arXiv preprint arXiv:1812.03565*, 2018.

Xinyi Chen and Elad Hazan. Black-box control for linear dynamical systems, 2021.

Juan C. Perdomo, Jack Umenberger, and Max Simchowitz. Stabilizing dynamical systems via policy gradient methods, 2021.

Tankred Rautert and Ekkehard W Sachs. Computational design of optimal output feedback controllers. *SIAM Journal on Optimization*, 7(3):837–852, 1997.

Karl Mårtensson and Anders Rantzer. Gradient methods for iterative distributed control synthesis. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, pages 549–554. IEEE, 2009.

Dhruv Malik, Ashwin Pananjady, Kush Bhatia, Koulik Khamaru, Peter L Bartlett, and Martin J Wainwright. Derivative-free methods for policy optimization: Guarantees for linear quadratic systems. *arXiv preprint arXiv:1812.08305*, 2018.

Hesameddin Mohammadi, Armin Zare, Mahdi Soltanolkotabi, and Mihailo R Jovanović. Convergence and sample complexity of gradient methods for the model-free linear quadratic regulator problem. *arXiv preprint arXiv:1912.11899*, 2019.

Benjamin Gravell, Peyman Mohajerin Esfahani, and Tyler Summers. Learning robust controllers for linear quadratic systems with multiplicative noise via policy gradient. *arXiv preprint arXiv:1905.13547*, 2019.

Zhuoran Yang, Yongxin Chen, Mingyi Hong, and Zhaoran Wang. On the global convergence of actor-critic: A case for linear quadratic regulator with ergodic cost. *arXiv preprint arXiv:1907.06246*, 2019.

Kaiqing Zhang, Zhuoran Yang, and Tamer Basar. Policy optimization provably converges to nash equilibria in zero-sum linear quadratic games. In *Advances in Neural Information Processing Systems*, pages 11602–11614, 2019.

Kaiqing Zhang, Bin Hu, and Tamer Basar. Policy optimization for H_2 linear control with H_∞ robustness guarantee: Implicit regularization and global convergence. In *Learning for Dynamics and Control*, pages 179–190, 2020.

Luca Furieri, Yang Zheng, and Maryam Kamgarpour. Learning the globally optimal distributed LQ regulator. In *Learning for Dynamics and Control*, pages 287–297, 2020.

Joao Paulo Jansch-Porto, Bin Hu, and Geir Dullerud. Convergence guarantees of policy optimization methods for markovian jump linear systems. *arXiv preprint arXiv:2002.04090*, 2020a.

Joao Paulo Jansch-Porto, Bin Hu, and Geir Dullerud. Policy learning of MDPs with mixed continuous/discrete variables: A case study on model-free control of markovian jump systems. *arXiv preprint arXiv:2006.03116*, 2020b.

Ilyas Fatkhullin and Boris Polyak. Optimizing static linear feedback: Gradient method. *arXiv preprint arXiv:2004.09875*, 2020.

Yasin Abbasi-Yadkori and Csaba Szepesvári. Regret bounds for the adaptive control of linear quadratic systems. In *Proceedings of the 24th Annual Conference on Learning Theory*, pages 1–26, 2011.

Mohamad Kazem Shirani Faradonbeh, Ambuj Tewari, and George Michailidis. Finite time analysis of optimal adaptive policies for linear-quadratic systems. *arXiv preprint arXiv:1711.07230*, 2017.

Yi Ouyang, Mukul Gagrani, and Rahul Jain. Learning-based control of unknown linear systems with thompson sampling. *arXiv preprint arXiv:1709.04047*, 2017.

Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In *Advances in Neural Information Processing Systems*, pages 4188–4197, 2018.

Alon Cohen, Tomer Koren, and Yishay Mansour. Learning linear-quadratic regulators efficiently with only $\sqrt{T}$ regret. *arXiv preprint arXiv:1902.06223*, 2019.

Horia Mania, Stephen Tu, and Benjamin Recht. Certainty equivalent control of LQR is efficient. *arXiv preprint arXiv:1902.07826*, 2019.

Max Simchowitz and Dylan J Foster. Naive exploration is optimal for online LQR. *arXiv preprint arXiv:2001.09576*, 2020.

Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. *arXiv preprint arXiv:2001.09254*, 2020.

T.L Lai. Asymptotically efficient adaptive control in stochastic regression models. *Advances in Applied Mathematics*, 7(1):23–45, 1986. ISSN 0196-8858.

Han-Fu Chen and Ji-Feng Zhang. Convergence rates in stochastic adaptive tracking. *International Journal of Control*, 49(6):1915–1935, 1989. doi: 10.1080/00207178908559752.

Tze Leung Lai and Zhiliang Ying. Parallel recursive algorithms in asymptotically efficient adaptive control of linear stochastic systems. *SIAM Journal on Control and Optimization*, 29(5):1091–1127, 1991. doi: 10.1137/0329059.

Yasin Abbasi-Yadkori and Csaba Szepesvári. Regret bounds for the adaptive control of linear quadratic systems. In Sham M. Kakade and Ulrike von Luxburg, editors, *Proceedings of the 24th Annual Conference on Learning Theory*, volume 19 of *Proceedings of Machine Learning Research*, pages 1–26, Budapest, Hungary, 2011. PMLR.

Morteza Ibrahimi, Adel Javanmard, and Benjamin Van Roy. Efficient reinforcement learning for high dimensional linear quadratic systems, 2013.

Mohamad Kazem Shirani Faradonbeh, Ambuj Tewari, and George Michailidis. Finite-time adaptive stabilization of linear systems. *IEEE Transactions on Automatic Control*, 64(8):3498–3505, 2019. doi: 10.1109/TAC.2018.2883241.

Lennart Ljung. System identification. *Wiley Encyclopedia of Electrical and Electronics Engineering*, pages 1–19, 1999.

Ljung Lennart. System identification: theory for the user. *PTR Prentice Hall, Upper Saddle River, NJ*, pages 1–14, 1999.

Max Simchowitz, Horia Mania, Stephen Tu, Michael I Jordan, and Benjamin Recht. Learning without mixing: Towards a sharp analysis of linear system identification. *arXiv preprint arXiv:1802.08334*, 2018.

Samet Oymak and Necmiye Ozay. Non-asymptotic identification of LTI systems from a single trajectory. In *2019 American Control Conference (ACC)*, pages 5655–5661. IEEE, 2019.

Tuhin Sarkar, Alexander Rakhlin, and Munther A Dahleh. Finite-time system identification for partially observed LTI systems of unknown order. *arXiv preprint arXiv:1902.01848*, 2019.

Yuji Nakatsukasa. *Off-Diagonal Perturbation, First-Order Approximation and Quadratic Residual Bounds for Matrix Eigenvalue Problems*, pages 233–249. Lecture Notes in Computer Science. Springer, 2015.

E. A. Rawashdeh. A simple method for finding the inverse matrix of vandermonde matrix, 2019. URL http://www.vesnik.math.rs/vol/mv19303.pdf.

L. Elsner. An optimal bound for the spectral variation of two matrices. *Linear Algebra and its Applications*, 71:77–80, 1985.

R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2nd edition edition, 2013.

Appendices

Appendix A. Decomposition of the State Space

A.1. The $E_u \oplus E_s$-decomposition

It is evident that the following two subspaces of $\mathbb{R}^n$ are invariant with respect to A, namely

$$E_{\mathrm{u}} := \operatorname{span}(v_1, \cdots, v_k), \; E_{\mathrm{s}} := \operatorname{span}(v_{k+1}, \cdots, v_n),$$

which we refer to as the *unstable subspace* and the *stable subspace*, respectively. Since $v_1, \cdots, v_n$ are linearly independent and span the whole $\mathbb{R}^n$ space, one natural decomposition is $\mathbb{R}^n = E_{\mathrm{u}} \oplus E_{\mathrm{s}}$, so that each state can be uniquely decomposed as $x = x_{\mathrm{u}} + x_{\mathrm{s}}$. Here $x_{\mathrm{u}} \in E_{\mathrm{u}}$ is called the *unstable component*, and $x_{\mathrm{s}} \in E_{\mathrm{s}}$ is called the *stable component*.

We also decompose A based on the $E_{\mathrm{u}} \oplus E_{\mathrm{s}}$-decomposition. Suppose E_{u} and E_{s} are represented by their *orthonormal* bases $Q_1 \in \mathbb{R}^{n\times k}$ and $Q_2 \in \mathbb{R}^{n\times(n-k)}$, respectively, namely

$$E_{\mathrm{u}} = \operatorname{col}(Q_1), \; E_{\mathrm{s}} = \operatorname{col}(Q_2).$$

Let $Q = [Q_1 \; Q_2]$ (which is invertible as long as A is diagonalizable), and let $R = [R_1^\top \; R_2^\top]^\top := Q^{-1}$. Further, let $\mathit{\Pi}_{\mathrm{u}} := Q_1 R_1$ and $\mathit{\Pi}_{\mathrm{s}} = Q_2 R_2$ be the *oblique* projectors onto E_{u} and E_{s} (along the other subspace), respectively. Since E_{u} and E_{s} are both invariant with regard to A, we know there exists $N_1 \in \mathbb{R}^{k\times k}$, $N_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, such that

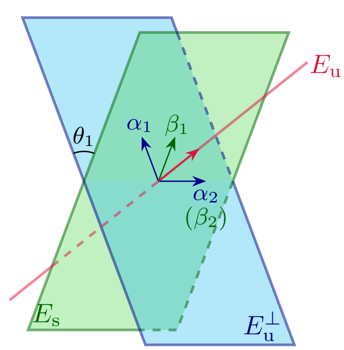
$$AQ = Q \begin{bmatrix} N_1 & \\ & N_2 \end{bmatrix} \; \Leftrightarrow \; N := \begin{bmatrix} N_1 & \\ & N_2 \end{bmatrix} = RAQ.$$

Let $z = [z_1^\top \; z_2^\top]^\top$ be the coordinate representation of x in the basis Q (i.e., $x = Qz$). The system dynamics in z-coordinates can be expressed as

$$\begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \end{bmatrix} = RAQ \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + RBu_t = \begin{bmatrix} N_1 & \\ & N_2 \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} R_1 B \\ R_2 B \end{bmatrix} u_t.$$

The major advantage of this decomposition is that the dynamical matrix in z-coordinate is block diagonal, so it would be simpler to study the behaviour of the system with zero control inputs.

A.2. Geometric Interpretation: Principle Angles



It might seem unintuitive to interpret $\sigma_{\min}(\mathit{\Gamma}_1^\top \mathit{\Gamma}_2)$ in Definition 2 as a measure of "closeness". However, this is closely related to the *principle angles* between subspaces that generalize the standard angle measures in lower dimensional cases. More specifically, suppose $\dim V_1 = d_1 \leq d_2 =$

$\dim V_2$, define recursively the i^{th} principle angle θ_i $(i = 1, \cdots, d_1)$ as

$$\theta_i := \min \left\{ \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right) \;\middle|\; \begin{array}{l} x \in V_1,\ x \perp \operatorname{span}(x_1, \cdots, x_{i-1}); \\ y \in V_2,\ y \perp \operatorname{span}(y_1, \cdots, y_{i-1}). \end{array} \right\} =: \angle(x_i, y_i), \tag{3}$$

where x_i and y_i $(i = 1, \cdots, d_1)$ are referred to as the i^{th} principle vectors accordingly. Meanwhile, let $\Gamma_1^\top \Gamma_2 = U \Sigma V^\top$ be the singular value decomposition (SVD), where $\Sigma = \operatorname{diag}(\sigma_1, \cdots, \sigma_{d_1})$ and $\sigma_1 \geq \cdots \geq \sigma_{d_1}$. Then by an equivalent recursive characterization of singular values, we have

$$\sigma_i = \max_{\substack{\|x\|=\|y\|=1 \\ \forall j<i:\ x \perp x_j,\ y \perp y_j}} x^\top \Gamma_1^\top \Gamma_2 y =: \bar{x}_i^\top \Gamma_1^\top \Gamma_2 \bar{y}_i.$$

Since Γ_1 and Γ_2 are orthonormal, $\bar{x}_i$ and $\bar{y}_i$ can be regarded as coordinate representations of $x_i = \Gamma_1 \bar{x}_i$ and $y_i = \Gamma_2 \bar{y}_i$, and it can be easily verified that x_i and y_i defined in this way are exactly the minimizers in (3). Hence we conclude that $\sigma_i = \cos\theta_i$. Therefore, V_1 and V_2 are ξ-close if and only if the all principle angles between V_1 and V_2 lie in the interval $[0, \arccos(1-\xi)]$; the above argument also shows that we can find orthonormal bases for V_1 and V_2 so that corresponding vectors form exactly the principle angles.

A.3. Characterization of ξ-close Subspaces

It is naturally expected that the geometric interpretation should inspire more relationships among $P_1 = Q_1, P_2, Q_2, R_1, R_2$ and N_2. We would like to emphasize that P_1, P_2 and Q_1 are not confined to bases of eigenvector (eigenvectors are even not necessarily orthonormal). Meanwhile, since they are only used in the stability guarantee proof, we are granted the freedom to select any orthonormal bases. For simplicity, we will stick to the convention that $P_1 = Q_1$ (and thus $M_1 = N_1$). Further, in Lemma 8, such freedom is utilized to establish fundamental relationships between the bases in the above two decompositions. The results are concluded as follows.

Lemma 8 *Suppose $E_{\mathrm{u}}^\perp$ and E_{s} are ξ-close. Then we shall select P_2 and Q_2 such that*
(1) $\sigma_{\min}(P_2^\top Q_2) \geq 1 - \xi$, $\|P_1^\top Q_2\| \leq \sqrt{2\xi}$, $\|P_2 - Q_2\| \leq \sqrt{2\xi}$.
(2) $\|R_2\| \leq \frac{1}{1-\xi}$, $\|N_2\| \leq \frac{1}{1-\xi}\|A\|$.
(3) $\|P_1^\top - R_1\| \leq \frac{\sqrt{2\xi}}{1-\xi}$, $\|R_1\| \leq \frac{\sqrt{2\xi}}{1-\xi} + 1$.
(4) $\|\Delta\| \leq \frac{2-\xi}{1-\xi}\sqrt{2\xi}\|A\|$.

Proof (1) Following the above interpretation, take arbitrary orthonormal bases $\bar{P}_2$ and $\bar{Q}_2$ of $E_{\mathrm{u}}^\perp$ and E_{s}, respectively, and let $\bar{P}_2^\top \bar{Q}_2 = U \Sigma V^\top$ be the SVD, which translates to

$$(\bar{P}_2 U)^\top (\bar{Q}_2 V) = \Sigma =: \operatorname{diag}(\sigma_1, \cdots, \sigma_{n-k}).$$

Since U and V are orthonormal matrices, the columns of $\bar{P}_2 U$ and $\bar{Q}_2 V$ also form orthonormal bases of $E_{\mathrm{u}}^\perp$ and E_{s}, respectively. Then ξ-closeness basically says that there exist a basis $\{\alpha_1, \cdots, \alpha_{n-k}\}$ for $E_{\mathrm{u}}^\perp$, and a basis $\{\beta_1, \cdots, \beta_{n-k}\}$ for E_{s} (both are assumed to be orthonormal), such that

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij}\sigma_i = \begin{cases} \sigma_i \geq 1 - \xi & \text{for any } i = j \\ 0 & \text{for any } i \neq j \end{cases},$$

and we also have $\Pi_2 \beta_i = \sigma_i \alpha_i$ and $\Pi_1 \alpha_i = \sigma_i \beta_i$ (recall that Π_1, Π_2 are orthogonal projectors onto subspaces $E_{\mathrm{u}}, E_{\mathrm{u}}^\perp$, respectively). Therefore, without loss of generality, we shall always select

$P_2 = [\alpha_1 \ \cdots \ \alpha_{n-k}]$ and $Q_2 = [\beta_1 \ \cdots \ \beta_{n-k}]$, such that $P_2^\top Q_2 = \mathrm{diag}(\sigma_1, \cdots, \sigma_{n-k})$, and

$$\sigma_{\min}(P_2^\top Q_2) = \min_i |\sigma_i| \geq 1 - \xi.$$

Equivalently speaking, for any $\beta = Q_2\eta \in E_{\mathrm{s}}$, we have (note that $\|\eta\| = \|\beta\|$)

$$\|P_2^\top \beta\| = \|P_2^\top Q_2 \eta\| \geq \sigma_{\min}(P_2^\top Q_2)\|\eta\| \geq (1-\xi)\|\beta\|,$$

and consequently,

$$\|P_1^\top Q_2 \eta\| = \|P_1^\top \beta\| = \sqrt{\|\beta\|^2 - \|P_2^\top \beta\|^2} \leq \sqrt{2\xi}\|\beta\| = \sqrt{2\xi}\|\eta\|,$$

which further shows $\|P_1^\top Q_2\| \leq \sqrt{2\xi}$. To bound $\|P_2 - Q_2\|$, by definition we have

$$\begin{aligned}
\|P_2 - Q_2\| &= \max_{\|\eta\|=1} \|(P_2 - Q_2)\eta\| = \max_{\|\eta\|=1} \left\| \sum_i \eta_i(\alpha_i - \beta_i) \right\| \\
&= \max_{\|\eta\|=1} \sqrt{\sum_{i,j} \eta_i \eta_j (\alpha_i - \beta_i)^\top (\alpha_j - \beta_j)} \\
&= \max_{\|\eta\|=1} \sqrt{\sum_i 2(1-\mu_i)\eta_i^2} \\
&\leq \max_{\|\eta\|=1} \sqrt{2\xi \sum_i \eta_i^2} = \sqrt{2\xi}.
\end{aligned}$$

Here $\eta = [\eta_1, \cdots, \eta_{n-k}]$ is an arbitrary vector in $\mathbb{R}^{n-k}$.

(2) By definition, $I = QR = Q_1R_1 + Q_2R_2$. Also recall that $P_1 = Q_1$, so $P_1^\top Q_1 = I$ and $P_2^\top Q_1 = O$. Then by left-multiplying $P_2^\top$ to the equality, we have

$$P_2^\top = P_2^\top Q_1 R_1 + P_2^\top Q_2 R_2 = P_2^\top Q_2 R_2,$$

which further shows

$$\|R_2\| = \|(P_2^\top Q_2)^{-1} P_2^\top\| \leq \|(P_2^\top Q_2)^{-1}\| = \frac{1}{\sigma_{\min}(P_2^\top Q_2)} \leq \frac{1}{1-\xi}.$$

Therefore, since $N_2 = R_2 A Q_2$, we have

$$\|N_2\| = \|R_2 A Q_2\| \leq \|R_2\|\|A\|\|Q_2\| \leq \frac{1}{1-\xi}\|A\|.$$

(3) Similarly, by left-multiplying $P_1^\top$ to the equality, we have

$$P_1^\top = P_1^\top Q_1 R_1 + P_1^\top Q_2 R_2 = R_1 + P_1^\top Q_2 R_2,$$

which further shows

$$\|P_1^\top - R_1\| = \|P_1^\top Q_2 R_2\| \leq \|P_1^\top Q_2\|\|R_2\| \leq \frac{\sqrt{2\xi}}{1-\xi},$$

and therefore $\|R_1\| \leq \|P_1^\top - R_1\| + \|P_1^\top\| = 1 + \frac{\sqrt{2\xi}}{1-\xi}$.

(4) A combination of the above results gives

$$\begin{aligned}
\|\mathit{\Delta}\| &= \|P_1^\top A P_2\| = \|P_1^\top A P_2 - R_1 A Q_2\| \\
&\leq \|P_1^\top A(P_2 - Q_2)\| + \|(P_1^\top - R_1) A Q_2\| \\
&\leq \|P_1^\top\|\|A\|\|P_2 - Q_2\| + \|P_1^\top - R_1\|\|A\|\|Q_2\|
\end{aligned}$$

$$\leq \|A\|\sqrt{2\xi} + \frac{\sqrt{2\xi}}{1-\xi}\|A\| = \|\varDelta\| \leq \frac{2-\xi}{1-\xi}\sqrt{2\xi}\|A\|.$$

This completes the proof. ■

Appendix B. Solution to the Least Squares Problem in Stage 2

Lemma 9 gives the explicit form for the solution to the least squares problem (see Algorithm 1).

Lemma 9 *Given $D := [x_{t_0+1} \ \cdots \ x_{t_0+k}]$ and $\hat{P}_1\hat{P}_1^\top = \hat{\varPi}_1 = D(D^\top D)^{-1}D^\top$, the solution*

$$\hat{M}_1 = \arg\min_{M_1} \sum_{t=t_0+1}^{t_0+k} \|\hat{P}_1^\top x_{t+1} - M_1\hat{P}_1^\top x_t\|^2$$

is uniquely given by $\hat{M}_1 = \hat{P}_1^\top A\hat{P}_1$.

Proof Here we assume by default that the summation over t sums from t_0+1 to t_0+k. Since M_1 is a stationary point of $\mathcal{L}$, for any $\varDelta$ in the neighbourhood of O, we have

$$\begin{aligned}
0 \leq \mathcal{L}(M_1+\varDelta) - \mathcal{L}(M_1) &= \sum_t \|\hat{y}_{1,t+1} - M_1\hat{y}_{1,t} - \varDelta\hat{y}_{1,t}\|^2 - \sum_t \|\hat{y}_{1,t+1} - M_1\hat{y}_{1,t}\|^2 \\
&= \sum_t \langle \varDelta\hat{y}_{1,t}, \hat{y}_{1,t+1} - M_1\hat{y}_{1,t}\rangle + O(\|\varDelta\|^2) \\
&= \sum_t \operatorname{tr}\left(\hat{y}_{1,t}^\top \varDelta^\top(\hat{y}_{1,t+1} - A\hat{y}_{1,t})\right) + O(\|\varDelta\|^2) \\
&= \sum_t \operatorname{tr}\left(\varDelta^\top(\hat{y}_{1,t+1} - M_1\hat{y}_{1,t})\hat{y}_{1,t}^\top\right) + O(\|\varDelta\|^2) \\
&= \operatorname{tr}\left(\varDelta^\top \sum_t (\hat{y}_{1,t+1} - M_1\hat{y}_{1,t})\hat{y}_{1,t}^\top\right) + O(\|\varDelta\|^2).
\end{aligned}$$

Since it always holds for any $\varDelta$, we must have

$$\sum_t (\hat{y}_{1,t+1} - M_1\hat{y}_{1,t})\hat{y}_{1,t}^\top \ \Leftrightarrow \ M_1 \sum_t \hat{y}_{1,t}\hat{y}_{1,t}^\top = \sum_t \hat{y}_{1,t+1}\hat{y}_{1,t}^\top.$$

Plugging in $\hat{y}_{1,t} = \hat{P}_1^\top x_t$ and $\hat{y}_{1,t+1} = \hat{P}_1^\top Ax_t$, we further have

$$M_1\hat{P}_1^\top X\hat{P}_1 = M_1\sum_t \hat{P}_1^\top x_t x_t^\top \hat{P}_1 = \sum_t \hat{P}_1^\top Ax_t x_t^\top \hat{P}_1 = \hat{P}_1^\top AX\hat{P}_1,$$

where $X := \sum_t x_t x_t^\top = DD^\top$. Since the columns of $\hat{P}_1$ form an orthonormal basis of $\hat{E}_{\mathrm{u}}$, for any $x \in \hat{E}_{\mathrm{u}}$, $\hat{P}_1^\top x$ is the coordinate of x under that basis. The columns of D are linearly independent, so the columns of $\hat{P}_1^\top D$ are also linearly independent, which further yields

$$\operatorname{rank}(\hat{P}_1^\top X\hat{P}_1) = \operatorname{rank}\left((\hat{P}_1^\top D)(\hat{P}_1^\top D)^\top\right) = \operatorname{rank}(\hat{P}_1^\top D) = k.$$

Therefore, $\hat{P}_1^\top X\hat{P}_1$ is invertible, and M_1 is explicitly given by

$$M_1 = (\hat{P}_1^\top AX\hat{P}_1)(\hat{P}_1^\top X\hat{P}_1)^{-1}.$$

Note that $\hat{\varPi}_1 = \hat{P}_1\hat{P}_1^\top$ is the projector onto subspace $\operatorname{col}(D)$, we must have

$$\hat{P}_1\hat{P}_1^\top X = (\hat{\varPi}_1 D)D^\top = DD^\top = X,$$

which yields

$$M_1 = (\hat{P}_1^\top A(\hat{P}_1\hat{P}_1^\top X)\hat{P}_1)(\hat{P}_1^\top X\hat{P}_1)^{-1} = (\hat{P}_1^\top A\hat{P}_1)(\hat{P}_1^\top X\hat{P}_1)(\hat{P}_1^\top X\hat{P}_1)^{-1} = \hat{P}_1^\top A\hat{P}_1.$$

This completes the proof of Lemma 9. ■

Remarks *It might help understanding to note that, when $\hat{P}_1 = P_1$, for any $x_t, x_{t+1} \in E_\mathrm{u}$ we have*

$$P_1^\top A x_t = y_{t+1} = M_1 y_t = M_1 P_1^\top x_t,$$

which requires $P_1^\top A = M_1 P_1^\top$, or equivalently $M_1 = P_1^\top A P_1$ (recall $P_1^\top P_1 = I$).

Appendix C. Proof of Lemma 5

Lemma 5 is actually a direct corollary of the following lemma, for which we first need to define $\mathrm{gap}_i(A)$, the (*bipartite*) *spectral gap around λ_i with respect to* A, namely

$$\mathrm{gap}_i(A) := \begin{cases} \min_{\lambda_j \in \lambda(A_2)} |\lambda_i - \lambda_j| & \lambda_i \in \lambda(A_1) \\ \min_{\lambda_j \in \lambda(A_1)} |\lambda_i - \lambda_j| & \lambda_i \in \lambda(A_2) \end{cases},$$

where $\lambda(A)$ denotes the spectrum of A.

Lemma 10 *For 2-by-2 block matrices A and E in the form*

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}, \; E = \begin{bmatrix} O & E_{12} \\ E_{21} & O \end{bmatrix},$$

we have

$$|\lambda_i(A+E) - \lambda_i(A)| \le \frac{\kappa(A)\kappa(A+E)}{\mathrm{gap}_i(A)} \|E_{12}\|\|E_{21}\|.$$

Here $\kappa(A)$ is the condition number of the matrix consisting of A's eigenvectors as columns.

Proof The proof of the lemma can be found in existing literature like Nakatsukasa (2015). ■

Proof of Lemma 5 Lemma 10 basically guarantees that every eigenvalue of $A+E$ is within a distance of $O(\|E_{12}\|\|E_{21}\|)$ from some eigenvalue of A. Hence, by defining $\chi(A+E)$ as the maximum coefficient, namely

$$\chi(A+E) := \frac{\kappa(A)\kappa(A+E)}{\min_i\{\mathrm{gap}_i(A)\}},$$

we shall guarantee $|\rho(A+E) - \rho(A)| \le \chi(A+E)\|E_{12}\|\|E_{21}\|$. ■

Appendix D. Proof of Theorem 6 and Its Corollary

Without loss of generality, we shall write all matrices in the basis formed by unit eigenvectors $\{w_1, \cdots, w_n\}$ of A. Otherwise, let $W = [w_1 \; \cdots \; w_n]$, and perform change-of-coordinate by setting $\tilde{D} := W^{-1}DW$, $\tilde{\mathit{\Pi}}_1 := W^{-1}\mathit{\Pi}_1 W$, which further gives

$$\tilde{\hat{\mathit{\Pi}}}_1 = \tilde{D}(\tilde{D}^\top\tilde{D})^{-1}\tilde{D}^\top = (W^{-1}DW)(W^{-1}D^\top DW)^{-1}(W^{-1}D^\top W) = W^{-1}\hat{\mathit{\Pi}}_1 W.$$

Note that $\|W^{-1}\hat{\Pi}_1 W - W^{-1}\Pi_1 W\| \le \|W\|\|W^{-1}\|\|\hat{\Pi}_1 - \Pi_1\|$, where the upper bound is only magnified by a constant factor $\|W\|\|W^{-1}\|$ that is completely determined by A. Therefore, it is largely equivalent to consider $(\tilde{D}, \tilde{\Pi}_1, \tilde{\hat{\Pi}}_1)$ instead of $(D, \Pi_1, \hat{\Pi}_1)$.

Note that the matrix $D = [x_{t_0+1} \ \cdots \ x_{t_0+k}]$ can be written as

$$D = \begin{bmatrix} d_1 & \lambda_1 d_1 & \cdots & \lambda_1^{k-1} d_1 \\ d_2 & \lambda_2 d_2 & \cdots & \lambda_2^{k-1} d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_n & \lambda_n d_n & \cdots & \lambda_n^{k-1} d_n \end{bmatrix},$$

where $x_{t_0+1} =: [d_1, \cdots, d_n]^\top$. We first present a lemma that characterizes the explicit form of $\hat{\Pi}_1$, which is based on some well-known properties of Vandermonde matrices.

Lemma 11 *Given a Vandermonde matrix in variables $x_1, \cdots, x_n$ of order n*

$$V := V_n(x_1, \cdots, x_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix},$$

its determinant is given by

$$\det(V) = \sum_\pi (-1)^{\operatorname{sgn}(\pi)} x^0_{\pi(i_1)} x^1_{\pi(i_2)} \cdots x^{n-1}_{\pi(i_n)} = \prod_{j<\ell} (x_\ell - x_j), \tag{4}$$

and its (u, v)-cofactor is given by

$$\operatorname{cof}_{u,v}(V) = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{u-2} & \cdots & x_{v-1}^{u-2} & x_{v+1}^{u-2} & \cdots & x_n^{u-2} \\ x_1^{u} & \cdots & x_{v-1}^{u} & x_{v+1}^{u} & \cdots & x_n^{u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_{v-1}^{n-1} & x_{v+1}^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \sigma_{u,v} \prod_{j<\ell\neq v} (x_\ell - x_j), \tag{5}$$

where $\sigma_{u,v} := s_{n-u}(x_1, \cdots, x_{v-1}, x_{v+1}, \cdots, x_n)$, and $s_m(y_1, \cdots, y_n) := \sum_{i_1<\cdots<i_m} y_{i_1} \cdots y_{i_m}$.

Proof of Lemma 11 The proof of (4) can be found in any standard linear algebra textbook, and that of (5) can be found in Rawashdeh (2019). ■

Lemma 12 *The projector $\hat{\Pi}_1 = D(D^\top D)^{-1} D^\top$ has explicit form*

$$(\hat{\Pi}_1)_{uv} = \frac{\displaystyle\sum_{\substack{i_2<\cdots<i_k \\ \forall j: i_j \neq u,v}} \alpha_{u,i_2,\cdots,i_k} \alpha_{v,i_2,\cdots,i_k}}{\displaystyle\sum_{i_1<\cdots<i_k} \alpha^2_{i_1,\cdots,i_k}},$$

where the summand $\alpha_{i_1,\cdots,i_k}$ (with ordered subscript) is defined as

$$\alpha_{i_1,\cdots,i_k} := \prod_j d_{i_j} \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}).$$

Proof of Lemma 12 We start by deriving the explicit form of $(D^\top D)^{-1}$. Note that the determinant (which is also the denominator in the lemma) is given by

$$\begin{aligned}
\det(D^\top D) &= \sum_{i_1,\cdots,i_k} \begin{vmatrix} \lambda_{i_1}^0 d_{i_1}^2 & \lambda_{i_2}^1 d_{i_2}^2 & \cdots & \lambda_{i_k}^{k-1} d_{i_k}^2 \\ \lambda_{i_1}^1 d_{i_1}^2 & \lambda_{i_2}^2 d_{i_2}^2 & \cdots & \lambda_{i_k}^{k} d_{i_k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{k-1} d_{i_1}^2 & \lambda_{i_2}^k d_{i_2}^2 & \cdots & \lambda_{i_k}^{2k-2} d_{i_k}^2 \end{vmatrix} \\
&= \sum_{i_1,\cdots,i_k} d_{i_1}^2 \cdots d_{i_k}^2 \lambda_{i_1}^0 \lambda_{i_2}^1 \cdots \lambda_{i_k}^{k-1} \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \\
&= \sum_{i_1<\cdots<i_k} d_{i_1}^2 \cdots d_{i_k}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \sum_{\pi} (-1)^{\mathrm{sgn}(\pi)} \lambda_{\pi(j_1)}^0 \lambda_{\pi(j_2)}^1 \cdots \lambda_{\pi(j_k)}^{k-1} \\
&= \sum_{i_1<\cdots<i_k} d_{i_1}^2 \cdots d_{i_k}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j})^2 \\
&= \sum_{i_1<\cdots<i_k} \alpha_{i_1,\cdots,i_k}^2,
\end{aligned}$$

and the (u, v)-cofactor $\mathrm{cof}_{u,v}(D^\top D)$ is given by

$$\begin{aligned}
\mathrm{cof}_{u,v}(D^\top D) &= (-1)^{u+v} \sum_{i_1,\cdots,i_{k-1}} \begin{vmatrix} \lambda_{i_1}^0 d_{i_1}^2 & \cdots & \lambda_{i_{v-1}}^{v-2} d_{i_{v-1}}^2 & \lambda_{i_v}^{v} d_{i_v}^2 & \cdots & \lambda_{i_{k-1}}^{k-1} d_{i_{k-1}}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{u-2} d_{i_1}^2 & \cdots & \lambda_{i_{v-1}}^{u+v-4} d_{i_{v-1}}^2 & \lambda_{i_v}^{u+v-2} d_{i_v}^2 & \cdots & \lambda_{i_{k-1}}^{u+k-3} d_{i_{k-1}}^2 \\ \lambda_{i_1}^{u} d_{i_1}^2 & \cdots & \lambda_{i_{v-1}}^{u+v-2} d_{i_{v-1}}^2 & \lambda_{i_v}^{u+v} d_{i_v}^2 & \cdots & \lambda_{i_{k-1}}^{u+k-1} d_{i_{k-1}}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1}^{k-1} d_{i_1}^2 & \cdots & \lambda_{i_{u+v-2}}^{k+v-3} d_{i_{v-1}}^2 & \lambda_{i_v}^{k+v-1} d_{i_v}^2 & \cdots & \lambda_{i_{k-1}}^{2k-2} d_{i_{k-1}}^2 \end{vmatrix} \\
&= (-1)^{u+v} \sum_{i_1,\cdots,i_{k-1}} d_{i_1}^2 \cdots d_{i_{k-1}}^2 \lambda_{i_1}^0 \cdots \lambda_{i_{v-1}}^{v-2} \lambda_{i_v}^{v} \cdots \lambda_{i_{k-1}}^{k-1} s_{k-u} \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \\
&= (-1)^{u+v} \sum_{i_1<\cdots<i_{k-1}} s_{k-u} \cdot d_{i_1}^2 \cdots d_{i_{k-1}}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \cdot \\
&\qquad \sum_{\pi} (-1)^{\mathrm{sgn}(\pi)} \lambda_{\pi(i_1)}^0 \cdots \lambda_{\pi(i_{v-1})}^{v-2} \lambda_{\pi(i_v)}^{v} \cdots \lambda_{\pi(i_{k-1})}^{k-1} \\
&= (-1)^{u+v} \sum_{i_1<\cdots<i_{k-1}} s_{k-u} s_{k-v} \cdot d_{i_1}^2 \cdots d_{i_{k-1}}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j})^2,
\end{aligned}$$

where $s_{k-u}(\lambda_{i_1}, \cdots, \lambda_{i_{k-1}})$ is abbreviated to s_{k-u}.

Note that symmetry of $D^\top D$ guarantees $\mathrm{cof}_{v,u}(D^\top D) = \mathrm{cof}_{u,v}(D^\top D)$, so we have

$$(D^\top D)^{-1}_{u,v} = \frac{\mathrm{cof}_{v,u}(D^\top D)}{\det(D^\top D)} = \frac{\mathrm{cof}_{u,v}(D^\top D)}{\det(D^\top D)}.$$

And eventually we shall derive that

$$\begin{aligned}
\hat{P}_{u,v} &= \sum_{p,q} D_{u,p} (D^\top D)^{-1}_{p,q} D^\top_{q,v} \\
&= \frac{1}{\det(D^\top D)} \sum_{p,q} D_{u,p} D_{v,q} \,\mathrm{cof}_{u,v}(D^\top D)
\end{aligned}$$

$$= \frac{1}{\det(D^\top D)} \sum_{p,q} \lambda_u^{p-1} d_u \lambda_v^{q-1} d_v \cdot (-1)^{p+q} \sum_{i_1<\cdots<i_{k-1}} s_{k-p} s_{k-q} \cdot d_{i_1}^2 \cdots d_{i_{k-1}}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j})^2$$

$$= \frac{1}{\det(D^\top D)} \sum_{i_1<\cdots<i_{k-1}} d_u d_v d_{i_1}^2 \cdots d_{i_{k-1}}^2 \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j})^2 \sum_{p=1}^{k} (-1)^p \lambda_u^{p-1} s_{k-p} \sum_{q=1}^{k} (-1)^q \lambda_v^{q-1} s_{k-q}$$

$$= \frac{1}{\det(D^\top D)} \sum_{i_1<\cdots<i_{k-1}} d_u d_{i_1} \cdots d_{i_{k-1}} \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \prod_{\ell} (\lambda_{i_\ell} - \lambda_u) \cdot$$

$$d_v d_{i_1} \cdots d_{i_{k-1}} \prod_{j<\ell} (\lambda_{i_\ell} - \lambda_{i_j}) \prod_{\ell} (\lambda_{i_\ell} - \lambda_v)$$

$$= \frac{1}{\det(D^\top D)} \sum_{\substack{i_2<\cdots<i_k \\ \forall j: i_j \neq u,v}} \alpha_{u,i_2,\cdots,i_k} \alpha_{v,i_2,\cdots,i_k},$$

which is in exact the same form as stated in the lemma. ■

Now we shall go back to the proof of the main result of this section.

Proof of Theorem 6 Recall that $d_i = \lambda_i^{t_0+1} x_{0,i}$. For the clarity of notations, let

$$\theta_{i_1,i_2,\cdots,i_k} := \frac{\alpha_{i_1,i_2,\cdots,i_k}}{\alpha_{1,2,\cdots,k}},$$

and it is evident that $|\theta_{i_1,i_2,\cdots,i_k}| = 1$ only if $(i_1, i_2, \cdots, i_k)$ is a permutation of $(1, 2, \cdots, k)$. For any other $(i_1, i_2, \cdots, i_k)$, by the definition in Lemma 12 we have

$$|\theta_{i_1,i_2,\cdots,i_k}| \le c_{i_1,i_2,\cdots,i_k} \cdot r^{\delta(i_1,i_2,\cdots,i_k)t_0} \le c \cdot r^{\delta(i_1,i_2,\cdots,i_k)t_0},$$

where $r = \max_i \{\frac{\lambda_{i+1}}{\lambda_i}\}$, $c := \max_{i_1,\cdots,i_k} \{c_{i_1,i_2,\cdots,i_k}\}$, and $\delta(i_1, i_2, \cdots, i_k) := \sum_j i_j - \frac{k(k+1)}{2} \in \mathbb{N}$. Therefore, $|\theta_{i_1,i_2,\cdots,i_k}|$ will be small when $(1, 2, \cdots, k)$ is "far away" from $(i_1, i_2, \cdots, i_k)$.

To get a tighter bound, we need to analyze the distribution of $\delta(\cdot)$ in the exponent. For any fixed $\delta = \delta(i_1, i_2, \cdots, i_k)$, there are $q(\delta + \frac{k(k+1)}{2}, k)$ different tuples, where $q(n, k)$ denotes the number of different methods to partition n into k distinct integer parts. Then we have

$$\sum_{i_1<\cdots<i_k} \theta_{i_1,\cdots,i_k}^2 - \theta_{1,\cdots,k}^2 = c \sum_{\delta=0}^{k(n-k)} q(\delta + \tfrac{k(k+1)}{2}, k) r^{2\delta t_0} \le c \cdot Q_k(r^{2t_0}) r^{-k(k+1)t_0},$$

where $Q_k(x) := \sum_n q(n, k) x^n$ is the generating function for $q(n, k)$ with fixed k, which is

$$Q_k(x) = x^{k(k+1)/2} \prod_{j=1}^{k} \frac{1}{1 - x^j},$$

Hence we conclude that

$$\sum_{i_1<\cdots<i_k} \theta_{i_1,\cdots,i_k}^2 - \theta_{1,\cdots,k}^2 \le c \left(\prod_{j=1}^{k} \frac{1}{1 - r^{2jt_0}} - 1 \right),$$

which monotone-increasingly converges to a constant $c\gamma(r, t_0) = \frac{c}{(r^{2t_0}; r^{2t_0})_\infty} - c$ as $k \to \infty$, where $(\cdot;\cdot)_\infty$ is the *q-Pochhammer symbol*. Note that

$$(x; x)_\infty = 1 - x + O(r^{4t_0}) \;\Rightarrow\; \gamma(r, t_0) = r^{2t_0} + O(r^{4t_0}),$$

we know that $\gamma(r,t_0) \le 2r^{2t_0}$ when r^{t_0} is sufficiently small. For the nominator, note that for each δ there are fewer entries with exponent δ in the nominator than in the denominator, so we also have

$$\left| \sum_{\substack{i_2<\cdots<i_k \\ \forall j: i_j \neq u,v}} \theta_{u,i_2,\cdots,i_k}\theta_{v,i_2,\cdots,i_k} \right| \le \begin{cases} c\gamma(r,t_0)+1 & u=v\le k \\ c\gamma(r,t_0) & \text{otherwise} \end{cases}.$$

Eventually, for any $\varepsilon > 0$, we shall select t_0 such that $c\gamma(r,t_0) < \frac{\varepsilon}{n^2}$, where the denominator is always bounded by

$$1 \le \sum_{i_1<\cdots<i_k} \theta^2_{i_1,\cdots,i_k} \le 1 + \frac{\varepsilon}{n^2}.$$

For the nominator, when $u = v \le k$, we have $\sum_{\substack{i_2<\cdots<i_k \\ \forall j: i_j \neq u}} \theta^2_{u,i_2,\cdots,i_k} \ge 1$, which shows

$$\left.\begin{aligned} (\hat{\Pi}_1)_{uv} \ge \left(1+\frac{\varepsilon}{n^2}\right)^{-1} \ge 1-\frac{\varepsilon}{n^2} \\ (\hat{\Pi}_1)_{uv} \le 1+\frac{\varepsilon}{n^2} \end{aligned}\right\} \;\Rightarrow\; \left|(\hat{\Pi}_1)_{uv} - (\Pi_1)_{uv}\right| \le \frac{\varepsilon}{n^2}.$$

Otherwise, the nominator cannot sum over a permutation of $(1,\cdots,k)$, which gives

$$\left|(\hat{\Pi}_1)_{uv} - (\Pi_1)_{uv}\right| = \left|(\hat{\Pi}_1)_{uv}\right| \le \frac{\varepsilon}{n^2}.$$

Therefore, the overall estimation error is bounded by

$$\|\hat{\Pi}_1 - \Pi_1\| \le \sum_{u,v} \left|(\hat{\Pi}_1)_{uv} - (\Pi_1)_{uv}\right| \le \varepsilon.$$

To achieve error threshold ε, it is required that $2cr^{2t_0} < \frac{\varepsilon}{n^2}$, or equivalently

$$t_0 = O\left(\frac{\log\frac{n}{\varepsilon}}{\log\frac{1}{r}}\right).$$

This completes the proof. ■

Proof of Corollary 7 We first construct a specific pair of orthonormal bases $(P_1^*, \hat{P}_1^*)$ that satisfy the corollary. To start with, take an arbitrary initial pair of orthonormal basis $(P_1^\circ, \hat{P}_1^\circ)$, and consider the SVD $(P_1^\circ)^\top \hat{P}_1^\circ = U\varSigma V^\top$, which is equivalent to $(P_1^\circ U)^\top(\hat{P}_1^\circ V) = \varSigma$. Note that the columns of $P_1^\circ U = [w_1 \ \cdots w_k]$ and $\hat{P}_1^\circ V = [\hat{w}_1 \ \cdots \hat{w}_k]$ form orthonormal bases of $\mathrm{col}(\Pi_1)$ and $\mathrm{col}(\hat{\Pi}_1)$, respectively; furthermore, these bases project onto each other accordingly by subscripts, namely

$$\Pi_1\hat{w}_i = \sigma_i w_i, \; \hat{\Pi}_1 w_i = \sigma_i \hat{w}_i.$$

Now we set $P_1^* := P_1^\circ U$ and $\hat{P}_1^* := \hat{P}_1^\circ V$. Note that

$$|1-\sigma_i| = \|(\hat{\Pi}_1 - \Pi_1)\hat{w}_i\| < \varepsilon,$$

which shows, by using property of projection matrix Π_1,

$$\|w_i - \hat{w}_i\| = \sqrt{\|w_i - \Pi_1\hat{w}_i\|^2 + \|\Pi_1\hat{w}_i - \hat{w}_i\|^2} = \sqrt{|1-\sigma_i|^2 + \|(\hat{\Pi}_1 - \Pi_1)\hat{w}_i\|^2} < \sqrt{2}\varepsilon,$$

and thus

$$\|P_1^* - \hat{P}_1^*\| = \max_{\|z\|=1} \|(P_1^* - \hat{P}_1^*)z\| = \max_{\|z\|=1} \left\| \sum_i z_i(w_i - \hat{w}_i) \right\| \le \sqrt{k}\cdot\sqrt{2}\varepsilon.$$

To bridge the gap of generalizing to an arbitrary $\hat{P}_1$, we only have to note that there exists an orthonormal matrix T that maps the basis $\hat{P}_1^*$ to $\hat{P}_1 = \hat{P}_1^* T$. Now take $P_1 = P_1^* T$, and we have

$$\|\hat{P}_1 - P_1\| = \|(\hat{P}_1^* - P_1^*)T\| = \|\hat{P}_1^* - P_1^*\| < \sqrt{2k}\varepsilon.$$

As for the estimation error bound for M_1, we can directly write

$$\begin{aligned} \|P_1^\top A P_1 - \hat{P}_1^\top A \hat{P}_1\| &\le \|P_1^\top A P_1 - P_1^\top A \hat{P}_1\| + \|P_1^\top A \hat{P}_1 - \hat{P}_1^\top A \hat{P}_1\| \\ &\le \|A\|\|P_1 - \hat{P}_1\| + \|A\|\|P_1 - \hat{P}_1\| \\ &< 2\|A\|\delta, \end{aligned}$$

This completes the proof of the corollary. ■

Recall that we are allowed to take any orthonormal basis P_1 for E_u. Hence we shall always assume by default that P_1 in the proofs are selected as shown in the proof above.

We finish this section with simple but frequently-used bounds on $\|\hat{P}_1^\top P_1\|$ and $\|\hat{P}_1^\top P_2\|$. These factors represent an additional error introduced by using the inaccurate projector $\hat{P}_1$.

Proposition 13 *Under the premises of Corollary 7, $\|I_k - \hat{P}_1^\top P_1\| < \delta$, $\|\hat{P}_1^\top P_2\| < \delta$.*

Proof Note that $P_1^\top P_1 = I_k$ and $P_1^\top P_2 = O$, it is evident that

$$\|I_k - \hat{P}_1^\top P_1\| = \|(P_1 - \hat{P}_1)^\top P_1\| < \delta,$$
$$\|\hat{P}_1^\top P_2\| = \|(\hat{P}_1 - P_1)^\top P_2\| = \|\hat{P}_1 - P_1\| < \delta.$$

This finishes the proof. ■

Appendix E. Proof of Theorem 4

We start by showing the estimation bound for B_1, which is straight-forward since $\Delta = O$. Note that the upper bound of the norm of our controller K_1 appears as a natural corollary of it.

Proposition 14 *Under the premises of Theorem 4, $\|\hat{B}_1 - B_1\| < 4\|A\|\sqrt{k}\delta$.*

Proof Note that the column vector b_i has estimation error bound

$$\begin{aligned} \|b_i - \hat{b}_i\| &= \frac{1}{\|x_{t_i}\|} \left\| \left(P_1^\top x_{t_i+1} - M_1 P_1^\top x_{t_i}\right) - \left(\hat{P}_1^\top x_{t_i+1} - \hat{M}_1 \hat{P}_1^\top x_{t_i}\right) \right\| \\ &\le \frac{1}{\|x_{t_i}\|} \left(\|(P_1^\top - \hat{P}_1^\top) A x_{t_i}\| + \|(M_1 P_1^\top - \hat{M}_1 \hat{P}_1^\top) x_{t_i}\| \right) \\ &\le \|P_1^\top - \hat{P}_1^\top\|\|A\| + \|M_1 P_1^\top - M_1 \hat{P}_1^\top\| + \|M_1 \hat{P}_1^\top - \hat{M}_1 \hat{P}_1^\top\| \\ &< \|A\|\delta + \|M_1\|\|P_1^\top - \hat{P}_1^\top\| + \|M_1 - \hat{M}_1\| \\ &< \|A\|\delta + \|A\|\delta + 2\|A\|\delta = 4\|A\|\delta, \end{aligned}$$

where we repeatedly apply Corollary 7 and the fact that $\|M_1\| \le \|A\|$. Then, to bound the error of the whole matrix, we simply apply the definition

$$\|\hat{B}_1 - B_1\| = \max_{\|u\|=1} \|(\hat{B}_1 - B_1)u\| \le \max_{\|u\|=1} \sum_{i=1}^{k} |u_i|\|\hat{b}_i - b_i\| < 4\|A\|\sqrt{k}\delta.$$

This completes the proof. ■

Corollary 15 *Under the premises of Theorem 4,* $\|K_1\| < \frac{2\|A\|}{c\|B\|}$.

Proof By Proposition 14, it is evident that

$$\sigma_{\min}(\hat{B}_1) \geq \sigma_{\min}(B_1) - \|\hat{B}_1 - B_1\| > (c - 4\|A\|\sqrt{k}\delta)\|B\| > \frac{c}{2}\|B\|,$$

where the last inequality requires

$$\delta < \frac{c}{8\|A\|\sqrt{k}}. \tag{6}$$

Recall that $K_1 = \hat{B}_1^{-1}\hat{M}_1$, and note that $\|\hat{B}_1^{-1}\| \leq \frac{1}{\sigma_{\min}(\hat{B}_1)}$, so we have

$$\|K_1\| = \|\hat{B}_1^{-1}\hat{M}_1\| \leq \frac{\|\hat{P}_1^\top A\hat{P}_1\|}{\sigma_{\min}(\hat{B}_1)} < \frac{2\|A\|}{c\|B\|}.$$

This completes the proof. ■

Recall that to apply Lemma 5, we need a bound on the spectral radii of diagonal blocks. The top-left block has already been eliminated to approximately O by the design of K_1, but the bottom-right block needs some extra work — although M_2 is known to be stable, the inaccurate projection introduces an extra error that perturbs the spectrum. To bound the perturbed spectral radius, we will apply the following bound known as Elsner's Theorem, the statement of which requires the following definitions: the *spectral variation* of B with respect to A is defined to be

$$\mathrm{sv}_A(B) := \max_i \min_j |\lambda_i(B) - \lambda_j(A)|,$$

and the *Hausdorff distance* between A and B is defined to be

$$\mathrm{hd}(A, B) := \max\{\mathrm{sv}_A(B), \mathrm{sv}_B(A)\}.$$

Geometrically, every eigenvalue of A lies within a disk of radius $\mathrm{hd}(A, B)$ from some eigenvalue of B, and vice versa. Hence it is evident that $|\rho(A) - \rho(B)| \leq \mathrm{hd}(A, B)$.

Lemma 16 (Elsner's Theorem) *For any n-by-n matrices A and B,*

$$|\rho(A) - \rho(B)| \leq \mathrm{hd}(A, B) \leq (\|A\| + \|B\|)^{1-1/n}\|A - B\|^{1/n}.$$

Proof The proof is well-known and can be found in, e.g., Elsner (1985). ■

Now we are ready to prove the main theorem for any symmetric dynamical matrix A.

Proof of Theorem 4 With $\tau = 1$, the controlled dynamics under estimated controller $\hat{K}_1$ becomes

$$\hat{L}_1 = \begin{bmatrix} M_1 + P_1^\top BK_1\hat{P}_1^\top P_1 & P_1^\top BK_1\hat{P}_1^\top P_2 \\ P_2^\top BK_1\hat{P}_1^\top P_1 & M_2 + P_2^\top BK_1\hat{P}_1^\top P_2 \end{bmatrix}.$$

We first guarantee that the diagonal blocks are stable. For the top-left block,

$$\begin{aligned} \|M_1 + P_1^\top BK_1\| &= \|M_1 - B_1\hat{B}_1^{-1}\hat{M}_1\hat{P}_1^\top P_1\| \\ &\leq \|M_1 - \hat{M}_1\| + \|\hat{M}_1 - B_1\hat{B}_1^{-1}\hat{M}_1\| + \|B_1\hat{B}_1^{-1}\hat{M}_1(I_k - \hat{P}_1^\top P_1)\| \\ &\leq \|M_1 - \hat{M}_1\| + \|\hat{B}_1 - B_1\|\|K_1\| + \|B\|\|K_1\|\|I_k - \hat{P}_1^\top P_1\| \\ &< 2\|A\|\delta + \frac{8\|A\|^2\sqrt{k}}{c\|B\|}\delta + \frac{2\|A\|}{c}\delta \end{aligned} \tag{7}$$

$$= \frac{2\big(4\sqrt{k}\|A\| + (c+1)\|B\|\big)\|A\|}{c\|B\|}\delta,$$

where in (7) we apply Corollary 7, Corollary 15, and Proposition 13. Meanwhile, for the bottom-right block, note that the norm of the error term is bounded by

$$\|P_2^\top BK_1\hat{P}_1^\top P_2\| \leq \|B\|\|\hat{B}_1^{-1}\|\|\hat{M}_1\|\|\hat{P}_1^\top P_2\| \leq \frac{2\|A\|}{c}\delta.$$

Hence, by Lemma 16, the spectral radius of the bottom-right block is bounded by

$$\begin{aligned}\rho(M_2 + P_2^\top BK_1\hat{P}_1^\top P_2) &\leq \rho(M_2) + (2\|M_2\| + \tfrac{2}{c}\|A\|\delta)^{1-1/(n-k)}(\tfrac{2}{c}\|A\|\delta)^{1/(n-k)} \\ &< |\lambda_{k+1}| + 3\|M_2\|\left(\tfrac{2\|A\|}{3c\|M_2\|}\delta\right)^{1/(n-k)} \\ &< 1,\end{aligned}$$

where we require

$$\delta < \min\left\{\frac{2\|A\|}{c\|M_2\|}, \frac{3c\|M_2\|}{2\|A\|}\left(\frac{1-|\lambda_{k+1}|}{3\|M_2\|}\right)^{n-k}\right\}. \tag{8}$$

To apply the lemma, it only suffices to bound the spectral norms of off-diagonal blocks. Note that the top-right block is bounded by

$$\|P_1^\top BK_1\hat{P}_1^\top P_2\| \leq \|B\|\|K_1\|\|\hat{P}_1^\top P_2\| < \frac{2\|A\|}{c}\delta,$$

and the bottom-left block is bounded by

$$\|P_2^\top BK_1\hat{P}_1^\top P_1\| \leq \|B\|\|K_1\| \leq \frac{2\|A\|}{c}.$$

Now, by Lemma 5, we can guarantee that

$$\rho(\hat{L}_1) \leq \max\left\{\frac{2\big(4\sqrt{k}\|A\| + 2(c+1)\|B\|\big)\|A\|}{c\|B\|}\delta, |\lambda_{k+1}| + \|B\|\|K_1\|\delta\right\} + \frac{4\|A\|^2\chi(A,E)}{c^2}\delta < 1,$$

where we require

$$\delta < \min\left\{\frac{1}{\frac{2\left(4\sqrt{k}\|A\|+2(c+1)\|B\|\right)\|A\|}{c\|B\|} + \frac{4\|A\|^2\chi(A,E)}{c^2}}, \frac{1-|\lambda_{k+1}|}{\frac{2\|A\|}{c} + \frac{4\|A\|^2\chi(A,E)}{c^2}}\right\}. \tag{9}$$

So far, it is still left to recollect all the constraints we need on δ (see (6), (8) and (9)), and check whether they are in the expected order. This completes the proof of Theorem 4. ■

Appendix F. Proof of the Main Theorem

Technically, we would like to bound the spectral radius of the matrix

$$\hat{L}_\tau = \begin{bmatrix} M_1^\tau + P_1^\top A^{\tau-1}BK_1\hat{P}_1^\top P_1 & \Delta_\tau + P_1^\top A^{\tau-1}BK_1\hat{P}_1^\top P_2 \\ P_2^\top A^{\tau-1}BK_1\hat{P}_1^\top P_1 & M_2^\tau + P_2^\top A^{\tau-1}BK_1\hat{P}_1^\top P_2. \end{bmatrix}$$

using Lemma 5. The proof is split into two major building blocks: on the one hand, we introduce the well-known Gelfand's Formula to bound matrices appearing with exponents; on the other hand, we establish the estimation bound of B_τ (parallel to Lemma 14) and proceed to bound $\|K_1\|$, for which we rely on the instability results shown in Section F.2. Finally, a combination of these building blocks naturally establishes the main theorem.

F.1. Gelfand's Formula

In this section, we will show norm bounds for factors that contain matrix exponents. It is natural to apply the well-known Gelfand's formula as stated below.

Lemma 17 (Gelfand's formula) *For any square matrix X, we have*

$$\rho(X) = \lim_{t\to\infty} \|X^t\|^{1/t}. \tag{10}$$

In other words, for any $\varepsilon > 0$, there exists a constant $\zeta_\varepsilon(X)$ such that

$$\sigma_{\max}(X^t) = \|X^t\| \le \zeta_\varepsilon(X)(\rho(X)+\varepsilon)^t. \tag{11}$$

Further, if X is invertible, let $\lambda_{\min}(X)$ denote the eigenvalue of X with minimum modulus, then

$$\sigma_{\min}(X^t) \ge \frac{1}{\zeta_\varepsilon(X^{-1})}\left(\frac{|\lambda_{\min}(X)|}{1+\varepsilon|\lambda_{\min}(X)|}\right)^t. \tag{12}$$

Proof The proof of (10) can be easily found in existing literature (e.g., Horn and Johnson (2013), Corollary 5.6.14), and (11) follows by the definition of limits. For (12), note that

$$\sigma_{\min}(X^t) = \frac{1}{\sigma_{\max}((X^{-1})^t)} \ge \frac{1}{\zeta_\varepsilon(X^{-1})(\rho(X^{-1})+\varepsilon)^t} = \frac{1}{\zeta_\varepsilon(X^{-1})}\left(\frac{|\lambda_{\min}(X)|}{1+\varepsilon|\lambda_{\min}(X)|}\right)^t,$$

where we apply $\sigma_{\min}(X^t) = \sigma_{\max}((X^{-1})^t)^{-1}$ and $\rho(X^{-1}) = |\lambda_{\min}(X)|^{-1}$. ■

It is evident that $\rho(A) = \rho(M_1) = \rho(N_1) = |\lambda_1|$, $\lambda_{\min}(M_1) = \lambda_{\min}(N_1) = |\lambda_k|$ and $\rho(M_2) = \rho(N_2) = |\lambda_{k+1}|$ (for $\rho(M_2)$, note that the union of spectra of M_1 and M_2 is equal to the spectrum of A). Therefore, we can use Gelfand's formula to bound the relevant factors appearing in $\hat{L}_\tau$.

Proposition 18 *Under the premises of Theorem 3, the following results hold for any $t \in \mathbb{N}$:*
(1) $\|B_t\| \le \zeta_{\varepsilon_1}(A)(\varepsilon_1+|\lambda_1|)^{t-1}\|B\|$;
(2) $\|P_2^\top A^t\| \le \zeta_{\varepsilon_2}(M_2)(\varepsilon_2+|\lambda_{k+1}|)^t$;
(3) $\|\Delta_t\| \le C_\Delta(\varepsilon_1+|\lambda_1|)^t$, *where* $C_\Delta = \zeta_{\varepsilon_1}(M_1)\zeta_{\varepsilon_2}(M_2)\frac{(2-\xi)\sqrt{2\xi}\|A\|}{1-\xi}\frac{2|\lambda_{k+1}|}{|\lambda_1|+\varepsilon_1-|\lambda_{k+1}|-\varepsilon_2}$.

Proof (1) This is a direct corollary of Gelfand's Formula, since

$$\|B_t\| = \|P_1^\top A^{t-1}B\| \le \|A^{t-1}\|\|B\| \le \zeta_{\varepsilon_1}(A)(\varepsilon_1+|\lambda_1|)^{t-1}\|B\|.$$

(2) It only suffices to recall $\rho(M_2) = |\lambda_{k+1}|$, and note that

$$P_2^\top A^t = P_2^\top P M^t P^{-1} = [O\ I_{n-k}]M^t P^\top = M_2^t P_2^\top.$$

Hence by Gelfand's formula we have $\|P_2^\top A^t\| = \|M_2^t\| \le \zeta_{\varepsilon_2}(M_2)(\varepsilon_2+|\lambda_{k+1}|)^t$.

(3) This is a direct corollary of Lemma 8(4) and Gelfand's formula, since

$$\begin{aligned}\|\Delta_t\| &= \left\|\sum_i M_1^i \Delta M_2^{t-1-i}\right\| \le \|\Delta\|\sum_i \|M_1^i\|\|M_2^{t-1-i}\| \\ &\le \zeta_{\varepsilon_1}(M_1)\zeta_{\varepsilon_2}(M_2)\frac{(2-\xi)\sqrt{2\xi}\|A\|}{1-\xi}\sum_i(\varepsilon_1+|\lambda_1|)^i(\varepsilon_2+|\lambda_{k+1}|)^{t-1-i} \\ &= C_\Delta(\varepsilon_1+|\lambda_1|)^t.\end{aligned}$$

This finishes the proof of the proposition. ■

Proposition 19 *Under the premises of Theorem 3,*

$$\|\hat{M}_1^\tau - M_1^\tau\| < 2\tau\|A\|\zeta_{\varepsilon_1}(A)^2(\varepsilon_1 + |\lambda_1|)^{\tau-1}\delta.$$

Proof Recall that Corollary 7 gives $\|M_1 - \hat{M}_1\| < 2\|A\|\delta$. Meanwhile, by Gelfand's Formula,

$$\|M_1^t\| = \|P^\top A^t P\| \le \|A^t\| \le \zeta_{\varepsilon_1}(A)(\varepsilon_1 + |\lambda_1|)^t,$$
$$\|\hat{M}_1^t\| = \|\hat{P}^\top A^t \hat{P}\| \le \|A^t\| \le \zeta_{\varepsilon_1}(A)(\varepsilon_1 + |\lambda_1|)^t.$$

Then we have the following bound by telescoping

$$\begin{aligned}
\|M_1^\tau - \hat{M}_1^\tau\| &= \left\| \sum_{i=1}^{\tau} \left(M_1^i \hat{M}_1^{\tau-i} - M_1^{i-1}\hat{M}_1^{\tau-i+1} \right) \right\| \\
&\le \sum_{i=1}^{\tau} \|M_1^{i-1}\| \|\hat{M}_1^{\tau-i}\| \|M_1 - \hat{M}_1\| \\
&< \tau \cdot \zeta_{\varepsilon_1}(A)^2(\varepsilon_1 + |\lambda_1|)^{\tau-1} \cdot 2\|A\|\delta \\
&= 2\tau\|A\|\zeta_{\varepsilon_1}(A)^2(\varepsilon_1 + |\lambda_1|)^{\tau-1}\delta.
\end{aligned}$$

This finishes the proof. ■

Corollary 20 *Under the premises of Theorem 3,*

$$\|\hat{M}_1^\tau\| < \big(\zeta_{\varepsilon_1}(M_1)(\varepsilon_1 + |\lambda_1|) + 2\|A\|\zeta_{\varepsilon_1}(A)\big)(\varepsilon_1 + |\lambda_1|)^{\tau-1}.$$

Proof A combination of Gelfand's Formula and Proposition 19 yields

$$\begin{aligned}
\|\hat{M}_1^\tau\| &\le \|M_1^\tau\| + \|\hat{M}_1^\tau - M_1^\tau\| \\
&\le \zeta_{\varepsilon_1}(M_1)(\varepsilon_1 + |\lambda_1|)^\tau + 2\tau\|A\|\zeta_{\varepsilon_1}(A)^2(\varepsilon_1 + |\lambda_1|)^{\tau-1}\delta \\
&< \big(\zeta_{\varepsilon_1}(M_1)(\varepsilon_1 + |\lambda_1|) + 2\tau\|A\|\zeta_{\varepsilon_1}(A)\delta\big)(\varepsilon_1 + |\lambda_1|)^{\tau-1},
\end{aligned}$$

where the last inequality requires $\delta < \frac{1}{\tau}$. This completes the proof. ■

F.2. Instability of the Unstable Component

We have been referring to E_s (and approximately, $E_\mathrm{u}^\perp$) as "stable", and E_u as "unstable". This leads us to think that the unstable component will constitute a exploding proportion of the state as the system evolves with zero control input. However, in some cases it might happen that the proportion of unstable component does not increase within the first few time steps, although eventually it will explode. This motivates us to formally characterize such instability of the unstable component.

In this section, we aim to establish a fundamental property of A^ω (for large enough ω, of course) that it "almost surely" increases the norm of the state. By "almost surely" we mean that the initial state should have non-negligible unstable component, which happens with probability $1 - \varepsilon$ when we uniformly sample the initial state from the surface of unit hyper-sphere in $\mathbb{R}^n$.

Throughout this section, we use γ to denote the ratio of the unstable component over the stable component within some state x (i.e., $\frac{\|R_1 x\|}{\|R_2 x\|}$). Note that

$$x = \mathit{\Pi}_\mathrm{u} x + \mathit{\Pi}_\mathrm{s} x = Q_1 R_1 x + Q_2 R_2 x,$$

where Q_1, Q_2 are orthonormal. Hence

$$\|R_1x\| - \|R_2x\| \le \|x\| \le \|R_1x\| + \|R_2x\|.$$

As a consequence, when $\frac{\|R_1x\|}{\|R_2x\|} > \gamma > 1$, we also know that

$$\frac{\|R_1x\|}{\|x\|} \ge \frac{\|R_1x\|}{\|R_1x\| + \|R_2x\|} > \frac{\gamma}{\gamma+1}, \quad \frac{\|R_2x\|}{\|x\|} \le \frac{\|R_2x\|}{\|R_1x\| - \|R_2x\|} < \frac{1}{\gamma-1}.$$

The following results are presented to fit in the framework of an inductive proof. We first establish the inductive step, where Proposition 21 shows that the unstable component eventually becomes dominant with a non-negligible initial γ, and Proposition 23 shows that the unstable component will still constitute a non-negligible part after a control input of mild magnitude is injected. Meanwhile, Proposition 24 shows that the initial unstable component is non-negligible with large probability.

Proposition 21 *Given a dynamical matrix A and a constant $\gamma > 0$, for any state x such that $\frac{\|R_1x\|}{\|R_2x\|} > \gamma$, for any $\omega \in \mathbb{N}$, we have*

$$\frac{\|R_1A^\omega x\|}{\|R_2A^\omega x\|} > \gamma_\omega := C_\gamma \left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)}\right)^\omega,$$

where $C_\gamma := \frac{1}{(1+\frac{1}{\gamma})\zeta_{\varepsilon_3}(N_1^{-1})\zeta_{\varepsilon_2}(N_2)\|R_2\|}$ is a constant related to γ. Specifically, for any $\gamma_+ > 0$, there exists a constant $\omega_0(\gamma,\gamma_+) = O(\log\frac{\gamma_+}{\gamma})$, such that for any $\omega > \omega_0(\gamma,\gamma_+)$, $\frac{\|R_1x\|}{\|R_2x\|} > \gamma_+$.

Proof Recall that $R_1A^\omega = N_1^\omega R_1$ and $R_2A^\omega = N_2^\omega R_2$. By Gelfand's Formula we have

$$\begin{aligned}\frac{\|R_1A^\omega x\|}{\|R_2A^\omega x\|} &= \frac{\|N_1^\omega R_1x\|}{\|N_2^\omega R_2x\|} \ge \frac{\sigma_{\min}(N_1^\omega)\|R_1x\|}{\|N_2^\omega\|\|R_2\|\|x\|} > \frac{\sigma_{\min}(N_1^\omega)}{(1+\frac{1}{\gamma})\|N_2^\omega\|\|R_2\|} \\ &\ge \frac{\big(|\lambda_k|/(1+\varepsilon_3|\lambda_k|)\big)^\omega}{(1+\frac{1}{\gamma})\zeta_{\varepsilon_3}(N_1^{-1})\zeta_{\varepsilon_2}(N_2)(|\lambda_{k+1}|+\varepsilon_2)^\omega\|R_2\|} \\ &= \frac{1}{(1+\frac{1}{\gamma})\zeta_{\varepsilon_3}(N_1^{-1})\zeta_{\varepsilon_2}(N_2)\|R_2\|}\left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)}\right)^\omega.\end{aligned}$$

Therefore, we shall take

$$\omega_0(\gamma,\gamma_+) = \frac{\log\gamma_+/C_\gamma}{\log(|\lambda_k|)/\big((1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)\big)} = O\left(\log\frac{\gamma_+}{\gamma}\right),$$

and the proof is completed. ■

Corollary 22 *Under the premises of Proposition 21, for any $\omega > \omega_0(\gamma,\gamma_+)$,*

$$\frac{\|P_1^\top A^\omega x\|}{\|A^\omega x\|} > 1 - \frac{2}{\gamma_\omega - 1}, \quad \frac{\|P_2^\top A^\omega x\|}{\|A^\omega x\|} < \frac{1}{\gamma_\omega - 1}.$$

Proof Note that we have decomposition $x = \Pi_{\mathrm{u}}x + \Pi_1\Pi_{\mathrm{s}}x + \Pi_2\Pi_{\mathrm{s}}x$, where $\|\Pi_{\mathrm{u}}x\| = \|R_1x\|$ and $\|\Pi_{\mathrm{s}}x\| = \|R_2x\|$. Hence, for any $\omega > \omega_0(\gamma,\gamma_+)$, we can show that

$$\frac{\|P_1^\top A^\omega x\|}{\|A^\omega x\|} = \frac{\|\Pi_{\mathrm{u}}A^\omega x + \Pi_1\Pi_{\mathrm{s}}A^\omega x\|}{\|A^\omega x\|}$$

$$
\begin{aligned}
&\geq \frac{\|\Pi_{\mathrm{u}} A^\omega x\| - \|\Pi_1 \Pi_{\mathrm{s}} A^\omega x\|}{\|A^\omega x\|} \\
&\geq \frac{\|R_1 A^\omega x\| - \|R_2 A^\omega x\|}{\|A^\omega x\|} \\
&> \frac{\gamma_\omega}{\gamma_\omega + 1} - \frac{1}{\gamma_\omega - 1} > 1 - \frac{2}{\gamma_\omega - 1},
\end{aligned}
$$

and similarly,

$$
\frac{\|P_2^\top A^\omega x\|}{\|A^\omega x\|} = \frac{\|\Pi_2 \Pi_{\mathrm{s}} A^\omega x\|}{\|A^\omega x\|} \leq \frac{\|\Pi_{\mathrm{s}} A^\omega x\|}{\|A^\omega x\|} < \frac{1}{\gamma_\omega - 1}.
$$

The proof is completed. ■

Proposition 23 *Given dynamical matrices A, B and constants $\gamma > 0, \gamma_+ > 1$, for any state x such that $\frac{\|R_1 x\|}{\|R_2 x\|} > \gamma_+$, suppose we feed a control input $\|u\| \leq \alpha \|x\|$ and observe the next state $x' = Ax + Bu$, where α satisfies*

$$
\alpha < \frac{\frac{\gamma_+}{\gamma_+ + 1} \sigma_{\min}(M_1) - \frac{\gamma}{\gamma_+ - 1} \frac{1}{1-\xi} \|A\|}{(1 + \frac{\sqrt{2\xi}}{1-\xi} + \frac{\gamma}{1-\xi}) \|B\|}. \tag{13}
$$

Then we can guarantee that $\frac{\|R_1 x'\|}{\|R_2 x'\|} > \gamma$.

Proof The proposition can be shown by direct calculation. Let $z = Rx = [z_1^\top, z_2^\top]^\top$. Recall that

$$
Rx' = z' = \begin{bmatrix} N_1 z_1 + R_1 B u \\ N_2 z_2 + R_2 B u \end{bmatrix},
$$

and note that $\frac{\|z_1\|}{\|x\|} > \frac{\gamma_+}{\gamma_+ + 1}$, $\frac{\|z_2\|}{\|x\|} < \frac{1}{\gamma_+ - 1}$ under the assumptions, so we have

$$
\begin{aligned}
\frac{\|R_1 x'\|}{\|R_2 x'\|} &= \frac{\|N_1 z_1 + R_1 B u\|}{\|N_2 z_2 + R_2 B u\|} \geq \frac{\|N_1 z_1\| - \|R_1 B u\|}{\|N_2 z_2\| + \|R_2 B u\|} \\
&\geq \frac{\sigma_{\min}(N_1)\|z_1\| - \|R_1 B\| \|u\|}{\|N_2\| \|z_2\| + \|R_2 B\| \|u\|} \\
&\geq \frac{\sigma_{\min}(N_1) \frac{\gamma_+}{\gamma_+ + 1} \|x\| - \alpha \|R_1\| \|B\| \|x\|}{\|N_2\| \frac{1}{\gamma_+ - 1} \|x\| + \alpha \|R_2\| \|B\| \|x\|} \\
&\geq \frac{\sigma_{\min}(M_1) \frac{\gamma_+}{\gamma_+ + 1} \|x\| - \alpha (1 + \frac{\sqrt{2\xi}}{1-\xi}) \|B\| \|x\|}{\frac{1}{1-\xi} \|A\| \frac{1}{\gamma_+ - 1} \|x\| + \alpha \frac{1}{1-\xi} \|B\| \|x\|} \\
&> \gamma,
\end{aligned}
$$

where we apply Lemma 8 and the fact that $N_1 = M_1$. ■

Proposition 24 *Suppose a state x is sampled uniformly randomly from the unit hyper-sphere surface $\mathbb{B}_n \subset \mathbb{R}^n$, then for any constant $\gamma < \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2/(\sigma_{\min}(R_1) k) + 1}}\right\}$, we have*

$$
\Pr_{x \sim \mathcal{U}(\mathbb{B}_n)} \left[\frac{\|R_1 x\|}{\|R_2 x\|} > \gamma \right] > 1 - \theta(\gamma),
$$

where $\theta(\gamma) = \frac{8\sqrt{2}}{\mathrm{B}(\frac{1}{2}, \frac{n-1}{2}) \sqrt{\sigma_{\min}(R_1)}} \gamma = O(\gamma)$ is a constant bounded linearly by γ.

Proof Note that

$$\|R_1 x\| > \frac{\gamma}{1-\gamma}\|x\| \;\Rightarrow\; \|R_2 x\| < \|x\| + \|R_1 x\| < \frac{1}{1-\gamma}\|x\| \;\Rightarrow\; \frac{\|R_1 x\|}{\|R_2 x\|} > \gamma.$$

so we only have to show that $\Pr_{x\sim\mathcal{U}(\mathbb{B}_n)}\left[\|R_1 x\| \le \frac{\gamma}{1-\gamma}\right] < \theta(\gamma)$. Now let $R_1^\top R_1 = S^\top D S$ be the eigen-decomposition of $R_1^\top R_1$, where S is selected to be orthonormal such that

$$D = \operatorname{diag}(d_1, \cdots, d_k, 0, \cdots, 0).$$

Note that the vector $y = Sx =: [y_1, \cdots, y_n]$ also obeys a uniform distribution over $\mathbb{B}_n$, so we have

$$\begin{aligned}
\Pr\left[\|R_1 x\| \le \tfrac{\gamma}{1-\gamma}\right] &= \Pr\left[x^\top R_1^\top R_1 x \le (\tfrac{\gamma}{1-\gamma})^2\right] = \Pr\left[y^\top D y \le (\tfrac{\gamma}{1-\gamma})^2\right] \\
&\le \Pr\left[d_i y_i^2 \le \tfrac{1}{k}(\tfrac{\gamma}{1-\gamma})^2,\ \forall i = 1, \ldots, k\right] \\
&\le \sum_{i=1}^{k} \Pr\left[y_i^2 \le \tfrac{1}{d_i k}(\tfrac{\gamma}{1-\gamma})^2\right].
\end{aligned}$$

It suffices to bound the probability $\Pr_{y\sim\mathcal{U}(B)}\left[y_i^2 \le \eta\right]$. Note that y can be obtained by first sampling a Gaussian random vector $z \sim \mathcal{N}(0, I_n)$, and then normalize it to get $y = \frac{z}{\|z\|}$. Hence

$$\Pr_{y\sim\mathcal{U}(\mathbb{B}_n)}\left[y_i^2 \le \eta\right] = \Pr_{z\sim\mathcal{N}(0,I_n)}\left[z_i^2 \le \eta\|z\|^2\right] = \Pr_{z\sim\mathcal{N}(0,I_n)}\left[\frac{z_i^2}{\sum_{j\neq i} z_j^2} \le \frac{\eta}{1-\eta}\right],$$

where $w := \frac{z_i^2}{\sum_{j\neq i} z_j^2}$ is known to obey an F-distribution $w \sim \mathcal{F}(1, n-1)$. The c.d.f. of w is known to be $I_{w/(w+n-1)}(\frac{1}{2}, \frac{n-1}{2})$, where I denotes the *regularized incomplete Beta function*. Note that

$$I_{w/(w+n-1)}\left(\frac{1}{2}, \frac{n-1}{2}\right) = \frac{2w^{1/2}}{(n-1)^{1/2}\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})} - \frac{n w^{3/2}}{3(n-1)^{3/2}\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})} + O(n^{5/2}),$$

it can be shown that $I_{w/(w+n-1)}\left(\frac{1}{2}, \frac{n-1}{2}\right) < \frac{4\sqrt{w}}{\sqrt{n-1}\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})}$. Hence

$$\Pr_{y\sim\mathcal{U}(\mathbb{B}_n)}\left[y_i^2 \le \eta\right] = \Pr_{z\sim\mathcal{N}(0,I_n)}\left[\frac{z_i^2}{\sum_{j\neq i} z_j^2} \le \frac{\eta}{1-\eta}\right] < \frac{4\sqrt{\frac{\eta}{1-\eta}}}{\sqrt{n-1}\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})},$$

which further gives

$$\Pr\left[\|R_1 x\| \le \tfrac{\gamma}{1-\gamma}\right] < \sum_{i=1}^{k} \frac{4\sqrt{\frac{2}{d_i k}(\frac{\gamma}{1-\gamma})^2}}{\sqrt{n-1}\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})} < \frac{8\sqrt{2}}{\mathrm{B}(\frac{1}{2}, \frac{n-1}{2})\sqrt{\sigma_{\min}(R_1)}}\gamma = O(\gamma)$$

where we require $\gamma < \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2/(\sigma_{\min}(R_1)k)+1}}\right\}$. ∎

Combining the previous three propositions, we have shown in an inductive way that the algorithm guarantees $\frac{\|P_2^\top x_{t_i}\|}{\|x_{t_i}\|}$ is constantly upper bounded at each time step t_i ($i = 1, \cdots, k$), which is critical to the estimation error bound of B_τ. This is concluded as the following lemma.

Lemma 25 *Under the premises of Theorem 3, for any constant* $\gamma < \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2/(\sigma_{\min}(R_1)k)+1}}\right\}$ *and* $\gamma < t_0$*, the algorithm guarantees*

$$\frac{\|P_2^\top x_{t_i}\|}{\|x_{t_i}\|} < \frac{1}{\gamma_\omega - 1}, \ \forall i = 1, \cdots, k$$

with probability $1-\theta(\gamma)$ *over the initialization of* x_0 *on the unit hyper-sphere surface* $\mathbb{B}_n$*, where*

$$\gamma_\omega := C_\gamma \left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)} \right)^\omega .$$

Proof We proceed by showing that $\frac{\|R_1 x_{t_i}\|}{\|R_2 x_{t_i}\|} > \gamma_\omega$ for $i = 1, \cdots, k$ in an inductive way.

For the base case, Proposition 24 guarantees that x_0 satisfies $\frac{\|R_1 x_0\|}{\|R_2 x_0\|} > \gamma$ with probability $1-\theta(\gamma)$, and Proposition 21 further guarantees $\frac{\|R_1 x_{t_1}\|}{\|R_2 x_{t_1}\|} > \gamma_\omega$. Here we require $t_0 > \omega$.

For the inductive step, suppose we have shown $\frac{\|R_1 x_{t_i}\|}{\|R_2 x_{t_i}\|} > \gamma_\omega$. Since $\|u_{t_i}\| = \alpha\|x_{t_i}\|$, by Proposition 23 we have $\frac{\|R_1 x_{t_i+1}\|}{\|R_2 x_{t_i+1}\|} > \gamma$, and again Proposition 21 guarantees $\frac{\|R_1 x_{t_{i+1}}\|}{\|R_2 x_{t_{i+1}}\|} > \gamma_\omega$.

Now it only suffices to apply Corollary 22 to complete the proof. ∎

F.3. Estimation Error of B_τ

Proposition 26 *Under the premises of Theorem 3,*

$$\|\hat{B}_\tau - B_\tau\| < C_B(|\lambda_1|+\varepsilon_1)^{\tau-1}\delta,$$

where $C_B := \frac{2\sqrt{k}\zeta_{\varepsilon_1}(A)^2\left((2\tau+2)\|A\|+\|B\|\right)}{\alpha}$.

Proof This is parallel to Lemma 14. Note that we have to subtract an additional term (induced by non-zero $\mathit{\Delta}_\tau$ in M^τ) to calculate the actual b_i, so we have

$$\begin{aligned}
\|b_i - \hat{b}_i\| &= \frac{1}{\alpha\|x_{t_i}\|}\left\|\left(P_1^\top x_{t_i+\tau} - M_1^\tau P_1^\top x_{t_i} - \mathit{\Delta}_\tau P_2^\top x_{t_i}\right) - \left(\hat{P}_1^\top x_{t_i+\tau} - \hat{M}_1^\tau \hat{P}_1^\top x_{t_i}\right)\right\| \\
&\le \frac{1}{\alpha\|x_{t_i}\|}\left(\|(P_1-\hat{P}_1)^\top(A^\tau x_{t_i} + B_\tau u_{t_i})\| + \|M_1^\tau P_1^\top x_{t_i} - \hat{M}_1^\tau \hat{P}_1^\top x_{t_i}\| + \|\mathit{\Delta}_\tau P_2^\top x_{t_i}\|\right) \\
&< \frac{1}{\alpha}\left(\zeta_{\varepsilon_1}(A)^2(|\lambda_1|+\varepsilon_1)^{\tau-1}\big((2\tau+2)\|A\|+\|B\|\big)\delta + \delta\right).
\end{aligned}$$

Here the first term is bounded by

$$\begin{aligned}
\|(P_1-\hat{P}_1)^\top(A^\tau x_{t_i} + B_\tau u_{t_i})\| &\le \|P_1 - \hat{P}_1\|(\|A^\tau\| + \|A^{\tau-1}B\|)\|x_{t_i}\| \\
&< \|x_{t_i}\|\zeta_{\varepsilon_1}(A)(|\lambda_1|+\varepsilon_1)^{\tau-1}(\|A\|+\|B\|)\delta,
\end{aligned}$$

where in the last inequality we apply Corollary 7; the second term is bounded by

$$\begin{aligned}
\|M_1^\tau P_1^\top x_{t_i} - \hat{M}_1^\tau \hat{P}_1^\top x_{t_i}\| &\le (\|M_1^\tau(P_1^\top - \hat{P}_1^\top)\| + \|(M_1^\tau - \hat{M}_1^\tau)\hat{P}_1^\top\|)\|x_{t_i}\| \\
&< \big(\zeta_{\varepsilon_1}(A)(|\lambda_1|+\varepsilon_1)^{\tau-1}\|A\|\delta \\
&\quad + 2\tau\|A\|\zeta_{\varepsilon_1}(A)^2(|\lambda_1|+\varepsilon_1)^{\tau-1}\delta\big)\|x_{t_i}\| &(14)\\
&\le \|x_{t_i}\|\zeta_{\varepsilon_1}(A)^2(|\lambda_1|+\varepsilon_1)^{\tau-1}(2\tau+1)\|A\|\delta, &(15)
\end{aligned}$$

where in (14) we apply Proposition 19, and in (15) we apply a simple fact that $\varepsilon_{\varepsilon_1}(A) \ge 1$; the third term is bounded by

$$\frac{\|\mathit{\Delta}_\tau\|\|P_2^\top x_{t_i}\|}{\|x_{t_i}\|} \le \frac{C_\Delta(\varepsilon_1+|\lambda_1|)^\tau}{\left[C_\gamma\left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)}\right)^\omega - 1\right]} \tag{16}$$

$$< \frac{2C_\Delta(\varepsilon_1 + |\lambda_1|)^\tau}{C_\gamma \left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)}\right)^\omega} \tag{17}$$

$$< \delta, \tag{18}$$

where in (16) we apply Lemma 25, while in (17) and (18) we require

$$\omega > \max\left\{\frac{\log 2/C_\gamma}{\log\left(|\lambda_k|/(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)\right)}, \frac{\log(2C_\Delta)/(C_\gamma\delta) + \tau\log(\varepsilon_1 + |\lambda_1|)}{\log\left(|\lambda_k|/(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)\right)}\right\}. \tag{19}$$

Finally, to bound the error of the whole matrix, we simply apply the definition

$$\begin{aligned}
\|\hat{B}_\tau - B_\tau\| &= \max_{\|u\|=1}\|(\hat{B}_\tau - B_\tau)u\| \le \max_{\|u\|=1}\sum_{i=1}^k |u_i|\|\hat{b}_i - b_i\| \\
&< \frac{\sqrt{k}}{\alpha}\left(\zeta_{\varepsilon_1}(A)^2(|\lambda_1|+\varepsilon_1)^{\tau-1}\big((2\tau+2)\|A\| + \|B\|\big) + 1\right)\delta \\
&< \frac{2\sqrt{k}\zeta_{\varepsilon_1}(A)^2\big((2\tau+2)\|A\| + \|B\|\big)}{\alpha}(|\lambda_1|+\varepsilon_1)^{\tau-1}\delta.
\end{aligned}$$

This completes the proof. ■

Corollary 27 *Under the premises of Theorem 3,*

$$\sigma_{\min}(\hat{B}_\tau) > \frac{c\|B\|}{4\zeta_{\varepsilon_3}(N_1^{-1})}\left(\frac{|\lambda_k|}{1+\varepsilon_3|\lambda_k|}\right)^{\tau-1}.$$

Proof We apply the $E_\mathrm{u} \oplus E_\mathrm{s}$-decomposition. Note that

$$B_\tau = P_1^\top A^{\tau-1}B = P_1^\top(Q_1N_1^{\tau-1}R_1 + Q_2N_2^{\tau-1}R_2)B = N_1^{\tau-1}R_1B + P_1^\top Q_2N_2^{\tau-1}R_2B,$$

so by Gelfand's Formula and Lemma 8 we have

$$\begin{aligned}
\sigma_{\min}(B_\tau) &= \sigma_{\min}(N_1^{\tau-1}R_1B + P_1^\top Q_2N_2^{\tau-1}R_2B) \\
&\ge \sigma_{\min}(N_1^{\tau-1})\sigma_{\min}(R_1B) - \|P_1^\top Q_2\|\|N_2^{\tau-1}\|\|R_2\|\|B\| \\
&\ge \frac{c\|B\|}{\zeta_{\varepsilon_3}(N_1^{-1})}\left(\frac{|\lambda_k|}{1+\varepsilon_3|\lambda_k|}\right)^{\tau-1} - \frac{\sqrt{2\xi}\zeta_{\varepsilon_2}(N_2)\|B\|}{1-\xi}(\varepsilon_2 + |\lambda_{k+1}|)^{\tau-1} \\
&> \frac{c\|B\|}{2\zeta_{\varepsilon_3}(N_1^{-1})}\left(\frac{|\lambda_k|}{1+\varepsilon_3|\lambda_k|}\right)^{\tau-1}
\end{aligned}$$

where the last inequality requires

$$\frac{\sqrt{2\xi}\zeta_{\varepsilon_2}(N_2)\zeta_{\varepsilon_3}(N_1^{-1})}{c(1-\xi)}\left(\frac{(\varepsilon_2 + |\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1} < \frac{1}{2},$$

or equivalently,

$$\tau > \frac{\log\frac{c(1-\xi)}{2\sqrt{2\xi}\zeta_{\varepsilon_2}(N_2)\zeta_{\varepsilon_3}(N_1^{-1})}}{\log\frac{(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}} + 1. \tag{20}$$

Therefore, using Proposition 26, $\sigma_{\min}(\hat{B}_\tau)$ is lower bounded by

$$\sigma_{\min}(\hat{B}_\tau) \ge \sigma_{\min}(B_\tau) - \|\hat{B}_\tau - B_\tau\|$$

$$
\begin{aligned}
&> \frac{c\|B\|}{2\zeta_{\varepsilon_3}(N_1^{-1})} \left(\frac{|\lambda_k|}{1+\varepsilon_3|\lambda_k|} \right)^{\tau-1} - \frac{2\sqrt{k}\zeta_{\varepsilon_1}(A)^2\left((2\tau+2)\|A\|+\|B\|\right)}{\alpha}(|\lambda_1|+\varepsilon_1)^{\tau-1}\delta \\
&> \frac{c\|B\|}{4\zeta_{\varepsilon_3}(N_1^{-1})} \left(\frac{|\lambda_k|}{1+\varepsilon_3|\lambda_k|} \right)^{\tau-1},
\end{aligned}
$$

where the last inequality requires

$$
\delta < \frac{\alpha c\|B\|}{8\sqrt{k}\zeta_{\varepsilon_1}(A)^2\zeta_{\varepsilon_3}(N_1^{-1})\left((2\tau+2)\|A\|+\|B\|\right)} \left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_1|+\varepsilon_1)} \right)^{\tau-1}. \tag{21}
$$

This completes the proof. ■

Finally, using the above bounds, we can easily upper bound the norm of our controller K_1.

Proposition 28 *Under the premises of Theorem 3,*

$$
\|K_1\| < C_K \left(\frac{(\varepsilon_1+|\lambda_1|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|} \right)^{\tau-1},
$$

where $C_K := \frac{4\zeta_{\varepsilon_3}(N_1^{-1})\left(\zeta_{\varepsilon_1}(M_1)(\varepsilon_1+|\lambda_1|)+2\|A\|\zeta_{\varepsilon_1}(A)\right)}{c\|B\|}$

Proof Recall that the controller is constructed as $K_1 = \hat{B}_\tau^{-1}\hat{M}_1^\tau\hat{P}_1^\top$, so we have

$$
\|K_1\| \le \|\hat{B}_\tau^{-1}\|\|\hat{M}_1^\tau\| = \frac{\|\hat{M}_1^\tau\|}{\sigma_{\min}(\hat{B}_\tau)},
$$

and the bound is merely a combination of Corollary 20 and Corollary 27 whenever $\delta < \frac{1}{\tau}$. ■

F.4. Proof of Theorem 3

Now we are ready to combine the above building blocks and present the complete proof of Theorem 3. Note that, with all the bounds established above, the proof structure parallels that of Theorem 4, the special case with a symmetric dynamical matrix A.

Proof of Theorem 3 The proof is again based on Lemma 5. We first guarantee that the diagonal blocks are stable. For the top-left block,

$$
\begin{aligned}
\|M_1^\tau + P_1^\top A^{\tau-1}BK_1\| &= \|M_1^\tau - B_\tau\hat{B}_\tau^{-1}\hat{M}_1^\tau\hat{P}_1^\top P_1\| \\
&\le \|M_1^\tau - \hat{M}_1^\tau\| + \|(B_\tau - \hat{B}_\tau)\hat{B}_\tau^{-1}\hat{M}_1^\tau\| + \|B_\tau\hat{B}_\tau^{-1}\hat{M}_1^\tau(I-\hat{P}_1^\top P_1)\| \\
&\le \|M_1^\tau - \hat{M}_1^\tau\| + \|B_\tau - \hat{B}_\tau\|\|K_1\| + \|B_\tau\|\|K_1\|\|I-\hat{P}_1^\top P_1\| \\
&\le 2\tau\|A\|\zeta_{\varepsilon_1}(A)^2(\varepsilon_1+|\lambda_1|)^{\tau-1}\delta \\
&\quad + C_B C_K \left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|} \right)^{\tau-1}\delta \qquad (22) \\
&\quad + \zeta_{\varepsilon_1}(A)\|B\|C_K \left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|} \right)^{\tau-1}\delta \\
&< (C_B C_K + \zeta_{\varepsilon_1}(A)\|B\|C_K + 1) \left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|} \right)^{\tau-1}\delta \qquad (23)
\end{aligned}
$$

$$< \frac{1}{2}, \tag{24}$$

where in (22) we apply Propositions 19, 26, 28, and 13; in (23) we require

$$\frac{1}{\tau}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1} > 2\|A\|\zeta_{\varepsilon_1}(A)^2; \tag{25}$$

and in (24) we require

$$\delta < \frac{1}{2(C_BC_K+\zeta_{\varepsilon_1}(A)\|B\|C_K+1)}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-(\tau-1)}. \tag{26}$$

For the bottom-right block, it is straight-forward to see that

$$\begin{aligned}
\|M_2^\tau + P_2^\top A^{\tau-1}BK_1\hat{P}_1^\top P_2\| &\le \|M_2^\tau\| + \|P_2^\top A^{\tau-1}\|\|B\|\|K_1\|\|\hat{P}_1^\top P_2\| \\
&\le \zeta_{\varepsilon_2}(M_2)(\varepsilon_2+|\lambda_{k+1}|)^\tau \\
&\quad + \zeta_{\varepsilon_2}(M_2)\|B\|C_K\left(\frac{(\varepsilon_1+|\lambda_1|)(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1}\delta \\
&< 1
\end{aligned}$$

where the last inequality requires

$$\tau > \frac{\log 1/(4\zeta_{\varepsilon_2}(M_2))}{\log(\varepsilon_2+|\lambda_{k+1}|)}, \tag{27}$$

$$\delta < \frac{1}{4\zeta_{\varepsilon_2}(M_2)\|B\|C_K}\left(\frac{(\varepsilon_1+|\lambda_1|)(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-(\tau-1)}. \tag{28}$$

Now it only suffices to bound the spectral norms of off-diagonal blocks. Note that, by applying Proposition 28 and Proposition 18, the top-right block is bounded as

$$\begin{aligned}
\|\Delta_\tau + P_1^\top A^{\tau-1}BK_1\hat{P}_1^\top P_2\| &\le \|\Delta_\tau\| + \|B_\tau\|\|K_1\|\|\hat{P}_1^\top P_2\| \\
&< C_\Delta(\varepsilon_1+|\lambda_1|)^\tau \\
&\quad + \zeta_{\varepsilon_1}(A)\|B\|C_K\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1}\delta \\
&< (C_\Delta+1)(\varepsilon_1+|\lambda_1|)^\tau
\end{aligned}$$

where the last inequality requires

$$\delta < \frac{(\varepsilon_1+|\lambda_1|)^2}{\zeta_{\varepsilon_1}(A)\|B\|C_K}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-\tau}; \tag{29}$$

and the bottom-left block is bounded as

$$\begin{aligned}
\|P_2^\top A^{\tau-1}BK_1\hat{P}_1^\top P_1\| &\le \|P_2^\top A^{\tau-1}\|\|B\|\|K_1\| \\
&< \zeta_{\varepsilon_2}(M_2)\|B\|C_K\left(\frac{(\varepsilon_1+|\lambda_1|)(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1}.
\end{aligned}$$

Now, by Lemma 5, we can guarantee that

$$\rho(\hat{L}_\tau) \le \frac{1}{2} + \chi(\hat{L}_\tau)\frac{(C_\Delta+1)\zeta_{\varepsilon_2}(M_2)\|B\|C_K}{\varepsilon_1+|\lambda_1|}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{\tau-1} < 1,$$

which requires

$$\tau > \frac{\log \frac{2(\varepsilon_1+|\lambda_1|)}{\chi(\hat{L}_\tau)(C_\Delta+1)\zeta_{\varepsilon_2}(M_2)\|B\|C_K}}{\log \frac{(\varepsilon_1+|\lambda_1|)^2(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}}. \tag{30}$$

Note that the above constraint makes sense only if $|\lambda_1|^2|\lambda_{k+1}| < 1$.

So far, it is still left to recollect all the constraints we need on the parameters $\tau, \alpha, \delta, \gamma$ and ω. To start with, all constraints on τ (see (20), (25), (27) and (30)) can be summarized as

$$\begin{aligned}\tau > \max\Bigg\{ & \frac{\log \frac{c(1-\xi)}{2\sqrt{2\xi}\zeta_{\varepsilon_2}(N_2)\zeta_{\varepsilon_3}(N_1^{-1})}}{\log \frac{(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}} + 1, \frac{\log 1/(4\zeta_{\varepsilon_2}(M_2))}{\log(\varepsilon_2 + |\lambda_{k+1}|)}, \frac{\log \frac{2(\varepsilon_1+|\lambda_1|)}{\chi(\hat{L}_\tau)(C_\Delta+1)\zeta_{\varepsilon_2}(M_2)\|B\|C_K}}{\log \frac{(\varepsilon_1+|\lambda_1|)^2(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}}, \\ & \phi_{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)/|\lambda_k|}\left(\frac{2\|A\|\zeta_{\varepsilon_1}(A)^2(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)\Bigg\} \\ = {} & O(1),\end{aligned}$$

where $\phi_a(x)$ denotes the inverse function of $\frac{a^x}{x}$ on the interval $[\frac{1}{\log a}, +\infty)$ (where it is monotone increasing). Meanwhile, we shall select any $\gamma < \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2/(\sigma_{\min}(R_1)k)+1}}\right\}$ such that

$$\gamma = O(k^{-\ell}),$$

select ω such that (see (19), and note that $C_\gamma = O(\gamma) = O(k^{-\ell})$)

$$\omega > \max\left\{\frac{\log 2/C_\gamma}{\log\left(|\lambda_k|/(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)\right)}, \frac{\log(2C_\Delta)/(C_\gamma\delta) + \tau\log(\varepsilon_1+|\lambda_1|)}{\log\left(|\lambda_k|/(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)\right)}\right\} = O(\ell\log k),$$

and select α such that (see (13), and note that $\gamma_\omega = \Omega(1)$)

$$\alpha < \frac{\frac{\gamma_\omega}{\gamma_\omega+1}\sigma_{\min}(M_1) - \frac{\gamma}{\gamma_\omega-1}\frac{1}{1-\xi}\|A\|}{(1+\frac{\sqrt{2\xi}}{1-\xi}+\frac{\gamma}{1-\xi})\|B\|} = O(1).$$

Finally, constraints on δ (see (21), (26), (28) and (29)) can be summarized as

$$\begin{aligned}\delta < \min\Bigg\{ & \frac{\alpha c\|B\|}{8\sqrt{k}\zeta_{\varepsilon_1}(A)^2\zeta_{\varepsilon_3}(N_1^{-1})\left((2\tau+2)\|A\|+\|B\|\right)}\left(\frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_1|+\varepsilon_1)}\right)^{\tau-1}, \\ & \frac{1}{2(C_BC_K+\zeta_{\varepsilon_1}(A)\|B\|C_K+1)}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-(\tau-1)}, \\ & \frac{1}{4\zeta_{\varepsilon_2}(M_2)\|B\|C_K}\left(\frac{(\varepsilon_1+|\lambda_1|)(\varepsilon_2+|\lambda_{k+1}|)(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-(\tau-1)}, \\ & \frac{(\varepsilon_1+|\lambda_1|)^2}{\zeta_{\varepsilon_1}(A)\|B\|C_K}\left(\frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|}\right)^{-\tau}\Bigg\} \\ = {} & \min\{O(|\lambda_k|^\tau), O(|\lambda_1|^{-2\tau})\} = O(1).\end{aligned}$$

Note that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are taken to be small enough, so that

$$|\lambda_{k+1}| + \varepsilon_2 < 1, \quad \frac{(\varepsilon_1+|\lambda_1|)^2(1+\varepsilon_3|\lambda_k|)}{|\lambda_k|} < 1, \quad \frac{|\lambda_k|}{(1+\varepsilon_3|\lambda_k|)(|\lambda_{k+1}|+\varepsilon_2)} > 1.$$

Also, the probability of sampling an admissible x_0 is $1 - \theta(\gamma) = 1 - O(k^{-\ell})$. ∎