A New IIR-Type Digital Fractional Order Dierentiator

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A new IIR-type digital fractional order differentiator

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Abstract

A new infinite impulse response (IIR)-type digital fractional order differentiator (DFOD) is proposed by using a new family of first-order digital differentiators expressed in the second-order IIR filter form. The integer first-order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule. The distinguishing point of the proposed DFOD lies in an additional tuning knob to compromise the high-frequency approximation accuracy.

Keywords: Digital fractional order differentiator (DFOD); Digital differentiators; Digital integrators; Simpson rule; Trapezoidal rule; Continued fraction expansion (CFE); IIR filter

1. Introduction

Fractional calculus is a 300-years-old topic. The theory of fractional-order derivative was developed mainly in the 19th century. Recent books [11,12,21,24] provide a good source of references on fractional calculus. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest [7,17,18,22,23]. For pioneering works, we cite [4,9,10,13].

In theory, the control systems can include both the fractional order dynamic system or plant to be controlled and the fractional-order controller (FOC). However, in control practice, it is more common to consider the fractional-order controller. This is due to the fact that the plant model may have already been obtained as an integer-order model in the classical sense. In most cases, our objective is to apply fractional-order control to enhance the system control performance. For example, as in the CRONE[1] control [14,17,18], *fractal*

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[1] CRONE is a French abbreviation for “*Contrôle Robuste d’Ordre Non Entier*” (which means non-integer order robust control).

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robustness is pursued. The desired frequency template leads to fractional transmittance [16,19] on which the CRONE controller synthesis is based. In the CRONE controller, the major ingredient is the fractional-order derivative s^r, where r is a real number and s is the Laplacian operator. Another example is the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller [20,22] which is actually an extension of the classical PID controller. In general form, the transfer function of $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ is given by $K_{\mathrm{p}}+T_{\mathrm{i}}s^{-\lambda}+T_{\mathrm{d}}s^{\mu}$, where λ and μ are positive real numbers; K_{p} is the proportional gain, T_{i} the integration constant and T_{d} the differentiation constant. Clearly, taking $\lambda=1$ and $\mu=1$, we obtain a classical PID controller. If $\lambda=0$ ($T_{\mathrm{i}}=0$) we obtain a PD^{μ} controller, etc. All these types of controllers are particular cases of the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller. It can be expected that the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of frequencies of practical interest [15,16]. Moreover, the fractional order can be a complex number as discussed in [15]. In this paper, we focus on the case where the fractional order is a real number.

The key step in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods [15], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator [8,27–29], continuous fractional expansion (CFE) of the Tustin operator [5,27–29], and numerical integration-based method [8,5]. However, as pointed out in [1–3], the Tustin operator-based discretization scheme exhibits large errors in high-frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [5] which applies the Al-Alaoui operator [1].

The above discretization methods for s^r lead naturally to the DFODs usually in IIR form. Recently, there are some methods to directly obtain the DFODs in finite impulse response (FIR) form [25,26]. However, using an FIR filter to approximate s^r may be less efficient due to the very high order of the FIR filter. In this paper, a new IIR (infinite impulse response)-type digital fractional order differentiator (DFOD) is proposed by using a new family of first-order digital differentiators expressed in the second-order IIR filter form. The integer first-order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule [2]. The distinguishing point of the proposed DFOD lies in an additional tuning knob to compromise the high-frequency approximation accuracy.

This paper is organized as follows: in Section 2, after a brief introduction of a new family of first-order digital differentiators expressed in the second-order IIR filter form by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule [2,3], the corresponding fractional-order digital differentiator via CFE truncation is presented. Section 3 presents some illustrative examples. Section 4 concludes this paper.

2. A new family of IIR-type fractional order digital differentiators

2.1. The concept of generating function

In general, the discretization of the fractional-order differentiator s^r (r is a real number) can be expressed by the so-called generating function $s=\omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [6]. For example, when a backward difference rule is used, i.e., $\omega(z^{-1})=(1-z^{-1})/T$ with T the sampling period, performing the PSE of $(1-z^{-1})^{\pm r}$ gives the discretization formula which is actually in FIR filter form [8,27]. In [5,29], the trapezoidal (Tustin) rule is used as a generating function

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}. \tag{1}$$

The DFOD is then obtained by using the CFE [29] or a new recursive expansion formula [5]. It is interesting to note that in [5], the so-called Al-Alaoui operator is used which is a mixed scheme of Euler and Tustin operators [1]. Correspondingly, the generating function for discretization is

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{8}{7T}\,\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{\pm r}. \tag{2}$$

Clearly, (2) is an infinite order of rational discrete-time transfer function. To approximate (2) with a finite-order rational transfer function, the CFE is an efficient way. In general, any function $G(z)$ can be represented by continued fractions in the form of

$$G(z) \simeq a_0(z) + \frac{b_1(z)}{a_1(z) + \frac{b_2(z)}{a_2(z)+\frac{b_3(z)}{a_3(z)+\dots}}}, \tag{3}$$

where the coefficients a_i and b_i are either rational functions of the variable z or constants. By truncation, an approximate rational function, $\hat{G}(z)$, can be obtained.

The basic idea of this paper is to define a new generating function by applying a new family of first-order digital differentiators expressed in the second-order IIR filter form [2] via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

2.2. Integer order IIR-type digital integrator

It was pointed out in [1,3] that the magnitude of the frequency response of the ideal integrator $1/s$ lies between that of the Simpson and trapezoidal digital integrators. It is reasonable to "interpolate" the Simpson and trapezoidal digital integrators to compromise the high-frequency accuracy in frequency response. This leads to the following hybrid digital integrator:

$$H(z) = aH_S(z) + (1-a)H_T(z), \quad a \in [0,1], \tag{4}$$

where a is actually a weighting factor or tuning knob. $H_S(z)$ and $H_T(z)$ are the z-transfer functions of the Simpson's and the trapezoidal integrators given, respectively, as follows:

$$H_S(z) = \frac{T(z^2+4z+1)}{3(z^2-1)} \tag{5}$$

and

$$H_T(z) = \frac{T(z+1)}{2(z-1)}. \tag{6}$$

The overall weighted digital integrator with the tuning parameter a is hence given by

$$H(z) = \frac{T(3-a)\{z^2+[2(3+a)/(3-a)]z+1\}}{6(z^2-1)} = \frac{T(3-a)(z+r_1)(z+r_2)}{6(z^2-1)}, \tag{7}$$

where

$$r_1 = \frac{3+a+2\sqrt{3a}}{3-a}, \quad r_2 = \frac{3+a-2\sqrt{3a}}{3-a}.$$

It is interesting to note the fact that $r_1 = 1/r_2$ and $r_1 = r_2 = 1$ only when $a = 0$ (trapezoidal). For $a \neq 0$, $H(z)$ must have one non-minimum phase (NMP) zero.

2.3. IIR-type fractional order digital differentiator

Firstly, we can obtain a family of new integer-order digital differentiators from the digital integrators introduced in Section 2.2. Direct inversion of $H(z)$ will give an unstable filter since $H(z)$ has an NMP zero r_1. By reflecting the NMP r_1 to $1/r_1$, i.e. r_2, we have

$$\tilde{H}(z) = K\,\frac{T(3-a)(z+r_2)^2}{6(z^2-1)}.$$

To determine K, let the final value of the impulse responses of $H(z)$ and $\tilde{H}(z)$ be the same, i.e., $\lim_{z\to 1}(z-1)H(z) = \lim_{z\to 1}(z-1)\tilde{H}(z)$, which gives $K = r_1$. Therefore, a new family of the digital differentiators can be given by

$$\omega(z) = \frac{1}{\tilde{H}(z)} = \frac{6(z^2-1)}{r_1 T(3-a)(z+r_2)^2} = \frac{6r_2(z^2-1)}{T(3-a)(z+r_2)^2}. \tag{8}$$

We can regard $\omega(z)$ in (8) as the generating function introduced in Section 2.1. Finally, we can obtain the expression for the DFOD as

$$G(z^{-1}) = (\omega(z^{-1}))^r = k_0\left(\frac{1-z^{-2}}{(1+bz^{-1})^2}\right)^r, \tag{9}$$

where, without loss of generality, $r\in[0,1]$, $k_0 = (6r_2/T(3-a))^r$ and $b = r_2$. Note that if r is a real number greater than one, i.e., $r = [r] + r'$, the integer part will result in a usual integer-order differentiator $s^{[r]}$ which can be discretized using existing methods including the scheme used above.

It is well known that, compared to the PSE method, the CFE is a method for evaluation of functions with faster convergence and larger domain of convergence in the complex plane. Using CFE, an approximation for a irrational transfer function $G(z^{-1})$ can be expressed in the form of (3). Similar to (2), here, the irrational transfer function $G(z^{-1})$ in (9) can be expressed by an infinite order of rational discrete-time transfer function by CFE method as shown in (3).

The CFE expansion can be automated by using a symbolic computation tool such as the MATLAB Symbolic Toolbox. To illustrate, let us denote $x = z^{-1}$. Referring to (9), the task is to perform the following expansion:

$$\mathrm{CFE}\left(\frac{1-x^2}{(1+bx)^2}\right)^r$$

to the desired order n. The explicit presentation for the above expansion, though tedious, is possible. Here we resort to the symbolic computation. The following MATLAB script will generate the above CFE with `p1` and `q1` containing, respectively, the numerator and denominator polynomials in `x` or z^{-1} with their coefficients being functions of `b` and `r`.

```
clear all;close all;syms x r b;maple('with(numtheory)');
aas = ((1 - x * x)/(1 + b * x)^2)^r; n = 3; n2 = 2 * n;
maple(['cfe := cfrac(' char(aas) ',x,n2);']);
pq=maple('P_over_Q:=nthconver','cfe',n2);
p0=maple('P:=nthnumer','cfe',n2);
q0=maple('Q:=nthdenom','cfe',n2);
p=(p0(5:length(p0))); q=(q0(5:length(q0)));
p1=collect(sym(p),x); q1=collect(sym(q),x);
```

3. Illustrative examples

Here we present some results for $r=0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. Let $T=0.001$ s. We have the following:

$$G_{(3,0.00)}(z^{-1})=\frac{357.8-178.9z^{-1}-178.9z^{-2}+44.72z^{-3}}{8+4z^{-1}-4z^{-2}-z^{-3}},$$

$$G_{(3,0.25)}(z^{-1})=\frac{392.9-78.04z^{-1}-349.8z^{-2}+88.97z^{-3}}{11.32+4z^{-1}-5.66z^{-2}-z^{-3}},$$

$$G_{(3,0.50)}(z^{-1})=\frac{1501-503.6z^{-1}-1289z^{-2}+446.5z^{-3}}{47.26+4z^{-1}-23.63z^{-2}-z^{-3}},$$

$$G_{(3,0.75)}(z^{-1})=\frac{968.1-442z^{-1}-820.8z^{-2}+363z^{-3}}{32.47-4z^{-1}-16.24z^{-2}+z^{-3}},$$

$$G_{(3,1.00)}(z^{-1})=\frac{353.1-208z^{-1}-297.4z^{-2}+164.7z^{-3}}{12.46-4z^{-1}-6.228z^{-2}+z^{-3}}, \tag{10}$$

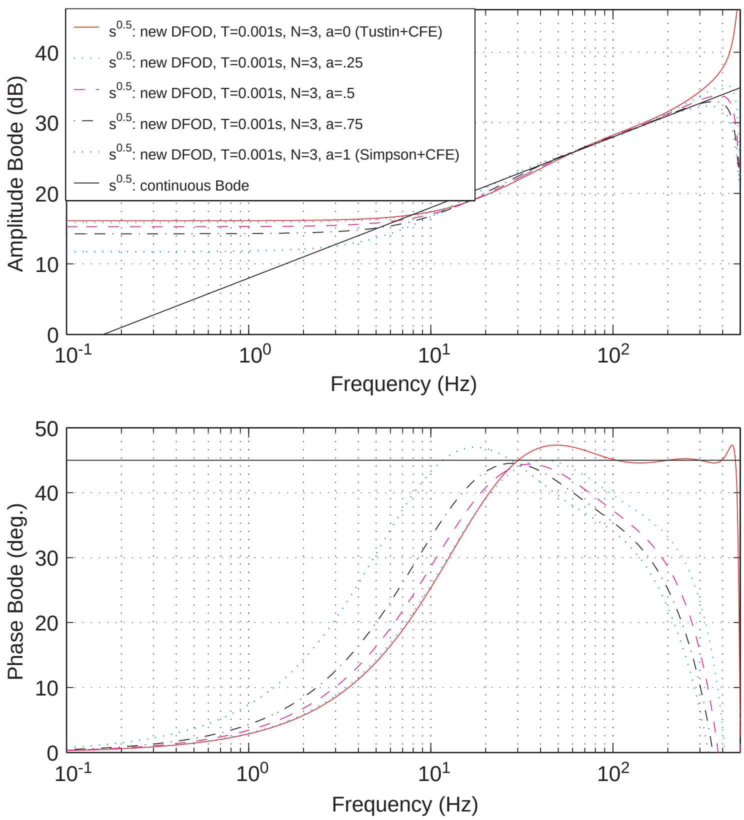

Fig. 1. Bode plot comparison for $r=0.5$, $n=3$ and $a=0,0.25,0.5,0.75,1$.

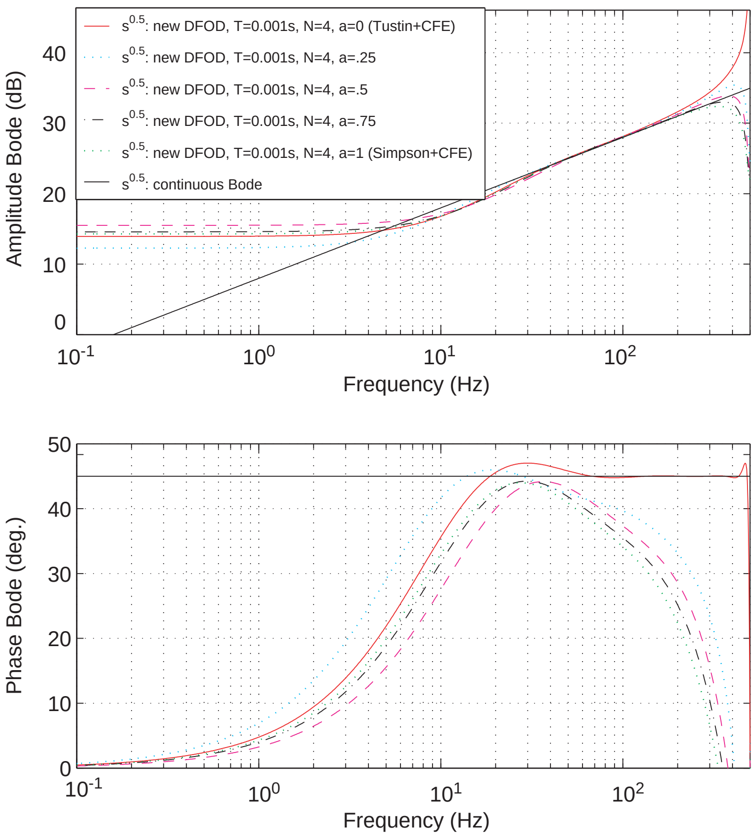

Fig. 2. Bode plot comparison for $r = 0.5$, $n = 4$ and $a = 0, 0.25, 0.5, 0.75, 1$.

$$G_{(4,0.00)}(z^{-1}) = \frac{715.5 - 357.8z^{-1} - 536.7z^{-2} + 178.9z^{-3} + 44.72z^{-4}}{16 + 8z^{-1} - 12z^{-2} - 4z^{-3} + z^{-4}},$$

$$G_{(4,0.25)}(z^{-1}) = \frac{555.3 - 392.9z^{-1} - 477.2z^{-2} + 349.8z^{-3} - 19.56z^{-4}}{16 - 2.489z^{-1} - 12z^{-2} + 1.245z^{-3} + z^{-4}},$$

$$G_{(4,0.50)}(z^{-1}) = \frac{508.1 - 1501z^{-1} - 4.478z^{-2} + 1289z^{-3} - 382.9z^{-4}}{16 - 40.54z^{-1} - 12z^{-2} + 20.27z^{-3} + z^{-4}},$$

$$G_{(4,0.75)}(z^{-1}) = \frac{477 + 968.1z^{-1} - 919z^{-2} - 820.8z^{-3} + 422.7z^{-4}}{16 + 37.8z^{-1} - 12z^{-2} - 18.9z^{-3} + z^{-4}},$$

$$G_{(4,1.00)}(z^{-1}) = \frac{453.6 + 353.1z^{-1} - 661.7z^{-2} - 297.4z^{-3} + 221.5z^{-4}}{16 + 16.74z^{-1} - 12z^{-2} - 8.371z^{-3} + z^{-4}}. \qquad (11)$$

The Bode plot comparisons for the above two groups of approximate fractional-order digital differentiators are summarized in Figs. 1 and 2, respectively. We can clearly observe the improvement in high-frequency magnitude response. If the trapezoidal scheme is used, the high-frequency magnitude response is far from the ideal one. The role of the tuning knob a is obviously useful in some applications. MATLAB code for this new DFOD is available upon request.

4. Conclusions

We have presented a new IIR (infinite impulse response)-type digital fractional order differentiator (DFOD) with a tuning knob to compromise the high-frequency approximation accuracy. The basic idea is the use of a new family of first-order digital differentiators expressed in the second-order IIR filter form [2] via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

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