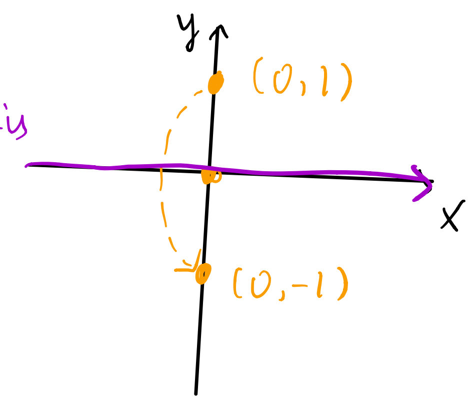
Lecture 3

§1: Reflection → isometric.

Def: The reflection $\bar{r}: \mathbb{R}^2 \to \mathbb{R}^2$ along x-axis is

$$\bar{r}(x,y) := (x,-y)$$

pts on x-axis are fixed.



Illus.

Lemma $\bar{r}$ is an isometry.

Pf: $d(P,Q)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

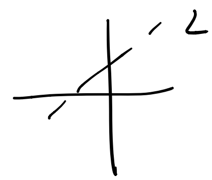
$$d(\bar{r}(P),\bar{r}(Q))=\sqrt{(x_1-x_2)^2+\underbrace{(-y_1-(-y_2))^2}_{(y_2-y_1)^2}}$$

preserve!

§2. How to obtain more interesting isometry?

Ex1 Given $P \in \mathbb{R}^2$, how do we get rotation on $R_\theta P$ of angle θ centered at p

Ex2 How to reflect any line L?



§3. Composition of Isometries.

Given $f, g \in \mathrm{Isom}(\mathbb{R}^2) := \{f : \mathbb{R}^2 \to \mathbb{R}^2 \mid d(P,Q) = d(f(P), f(Q))\}, \forall P, Q \in \mathbb{R}^2$

set of isometries

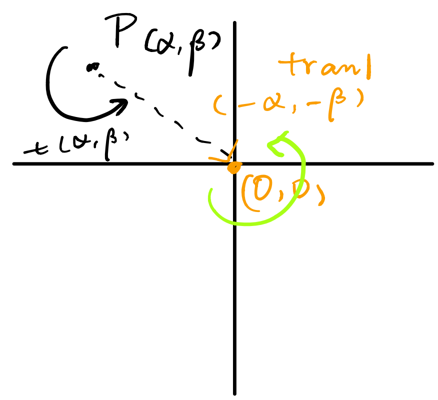
We can consider

$$f \circ g : \mathbb{R}^2 \underset{\text{isom}}{\xrightarrow{g}} \mathbb{R}^2 \underset{\text{isom}}{\xrightarrow{f}} \mathbb{R}^2 \Big\} \text{ is an isometry}$$

assume
$$\left.\begin{aligned} d(P,Q) &= d(f(P), f(Q)) \\ d(P,Q) &= d(gP, gQ) \end{aligned}\right\} \ldots d(fg(P), fg(Q))$$

§3.1 . Rot. along any center

Given $P \in \mathbb{R}^2$, the rotation of angle θ defined at P.

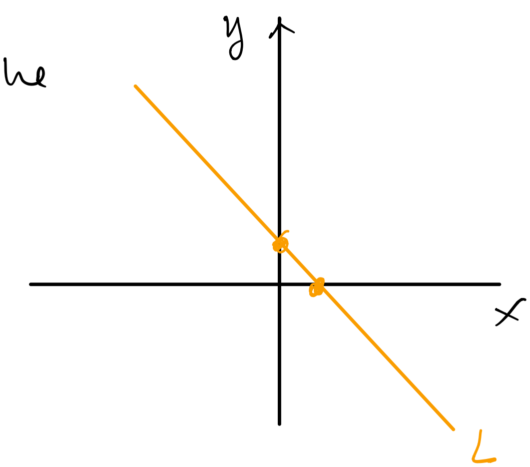


$$\boxed{R_{\theta, P} := t_{(\alpha,\beta)} \circ R_\theta \circ t_{(-\alpha,-\beta)}}$$

where $P(\alpha, \beta)$

§3.2 Ref. along arbitany line L

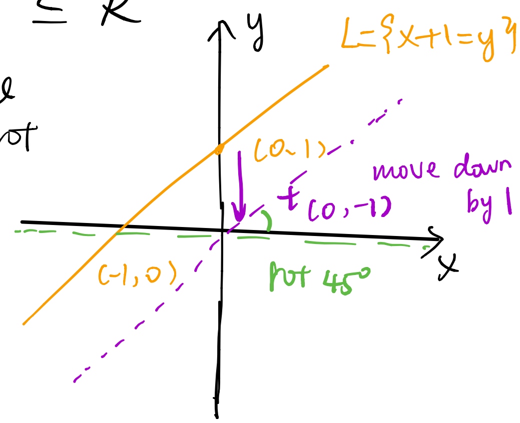
Given $L \subseteq \mathbb{R}^2$ a line, the reflection F_L along is:



$F_L = f^{-1} F f$, where f

is any isometry bring L to X-axis

Ex. $L = \{x+1=y\} \leq R^2$



By def, choosing

-clockwise
-positive rot

$f = R_{\frac{\pi}{4}} \circ t_{(0,-1)}$

$$\boxed{\overline{\Gamma}_L := t_{(0,1)} \circ R_{\frac{\pi}{4}} \circ \Gamma \circ R_{\frac{\pi}{4}} \circ t_{(0,-1)}}$$

note $AB = B^{-1}A^{-1}$

the reverse f^{-1} of $R_{\frac{\pi}{4}} \circ t_{(0,-1)}$

§4. The set of isometric generated by trans, rot, refl?

$\{\text{Rot } R_{\theta,\phi}\} \overset{=?}{\leq} \langle \text{Rot } R_{\theta,p} \rangle :=$ all compo. of rot $R_{\theta,p}$

generated by rot

$\{\text{tranl } t_{(\alpha,\beta)}\} \overset{=?}{\subseteq} \langle \text{tranl}, t_{\alpha,\beta} \rangle$ = inclusion

$\{\text{ref } \overline{\Gamma}_L\} \overset{=?}{\subseteq} \langle \text{ref } \overline{\Gamma}_L \rangle$

$t_{(\delta,\gamma)} \circ t_{(\alpha,\beta)} = t_{(\delta+\alpha,\gamma+\beta)}$

§4.1 The inclusion $\{Refl\} \subseteq \langle Refl \rangle$

Ex. We choose two lines L = x-axis

M = y-axis

<u>ALGEBRA</u> $r_M r_L(x,y) = r_M(x,-y) = (-x,-y)$

Q: is this refl?

$\exists N$ st $\bar{r}_N = \bar{r}_M \bar{r}_L$?