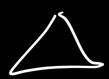
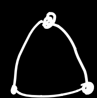
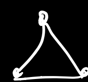
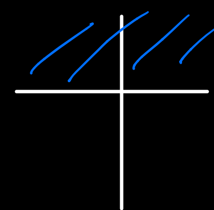
- Hyperbolic plane H^2

E^2 (R^2) $\qquad S^2 \qquad H^2 = R^2_+ = \{(x,y) \in R^2, y > 0\}$

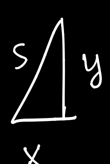
   

$= 180°$ $\qquad > 180° \qquad < 180°$

Culuture $\quad 0 \qquad + \qquad -$

$$ds = \frac{dx^2 + dy^2}{y}$$

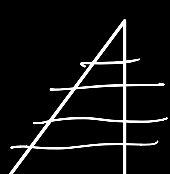
In, E^2 Δ satisfy $s^2 = x^2 + y^2$



(true everywhere)

$ds^2 = dx^2 + dy^2$

little



$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{in } H_2$$

in E^2, what is the length of curve?

$\gamma : [a, b] \to R^2 \quad \gamma(t) = (x(t), y(t))$

$$l_E(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_\gamma \sqrt{dx^2+dy^2} = \int_\gamma ds_{E^2}$$

$$dx = x'(t)\,dt$$

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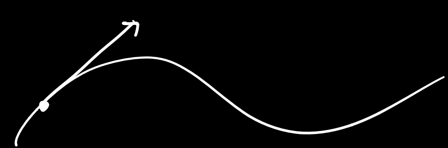
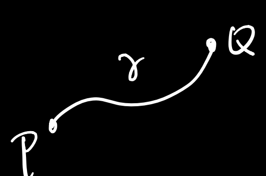
in H^2, $\gamma : [a,b] \to H^2$



$$\gamma'(t) = x'(t) + y'(t)$$

<u>Def</u> The distance function on H^2 is

$$d_{H^2}(P,Q) = \underbrace{\inf^{\to \text{ min is fine}}}_{\substack{\text{all path } \gamma \\ \text{from } P \text{ to } Q.}} \ell_{H^2}(\gamma)$$



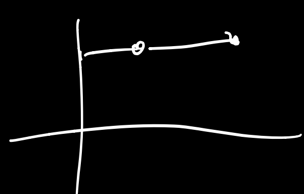
(inf is basically min)

Def Line is $L_{P,Q} = \{ R \in H^2 ,\ \text{dist}(R,P) = \text{dist}(R,Q) \}$

- length is preserved under hori. shifts.
- Vertical shifts are NOT!

Q: What is the length of hori. and vertical shift segments?

Hori



$\gamma : [0, a] \to H^2,$

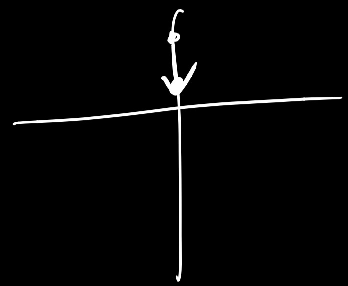
$\gamma(t) = (t, b)$

$\gamma'(t) = (1, 0)$

$l_{\mathbb{E}^2} = a$

$$\ell_{H^2}(\gamma) = \int_b^a \frac{\sqrt{1^2+0^2}}{b}\, dt = \frac{a}{b}$$

Vertical

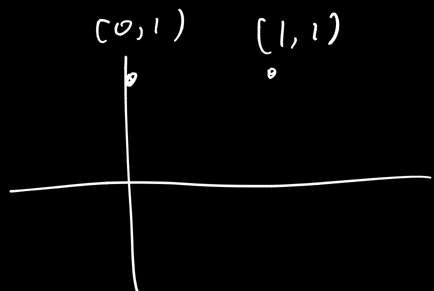


$\gamma\,[b, b+a] \to H^2$

$\gamma(t) = (0, t)$

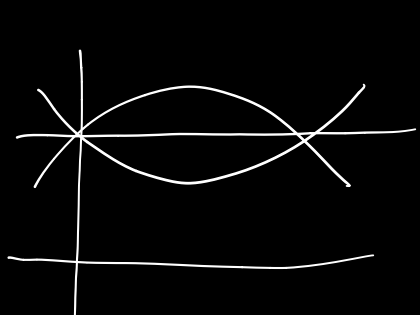
$\gamma'(t) = (0, 1)$

$$l_{H^2}(\gamma) = \int_b^{b+a} \frac{\sqrt{0^2+1^2}}{t} = \log\left(\frac{b+a}{b}\right)$$



$\mathrm{dist}\big((0,1),(1,1)\big) \leq 1$

Q: How low can it go?



Parameterization

$y = kx(1-x)$