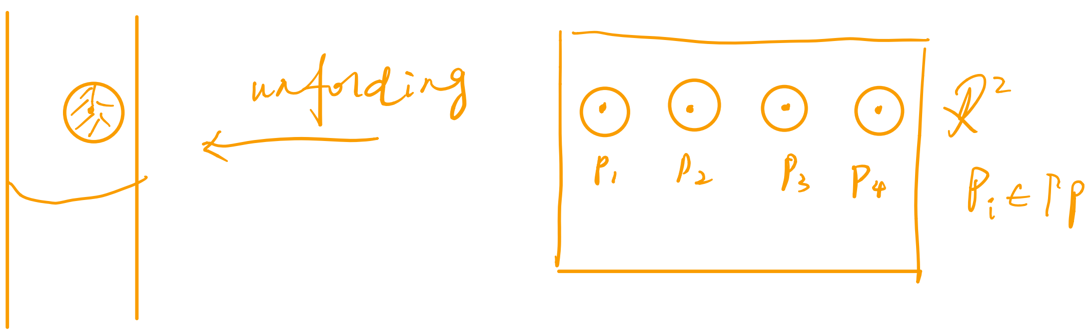
$\hookrightarrow \Gamma$

Lecture 9

§1. Distances in $\mathbb{R}^2/\Gamma$:

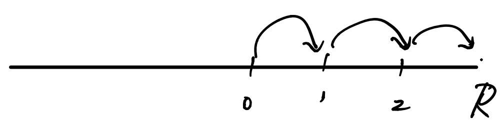
$$d(\Gamma P, \Gamma Q) = \min\{d(P', Q'),\ P' \in \Gamma_P,\ Q' \in \Gamma_Q\}$$

Locally, at a point $\Gamma P \in \mathbb{R}^2$, consider the disk of radius $\varepsilon \ll 1$

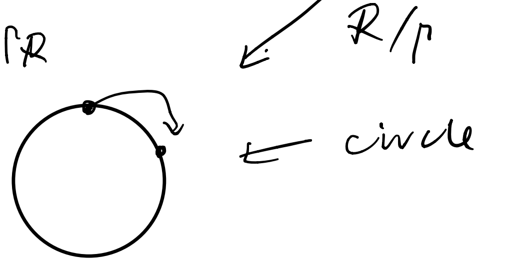


Q: What can go wrong?

(1st) Limit points:



Consider $\Gamma = \langle t, \rangle$



but also t_α, $\alpha \in \Gamma \mathbb{R}$

(δ_2)

$t^\alpha = t_{n\alpha}$

$n\alpha \in \mathbb{Z}$?

II $\exists\ \alpha$

In precise terms, 2 issues may arise:

(1st) Limit points:

distance become ill-defined and geometry in $\mathbb{R}^2/\Gamma$ will be locally different.

dist = 0

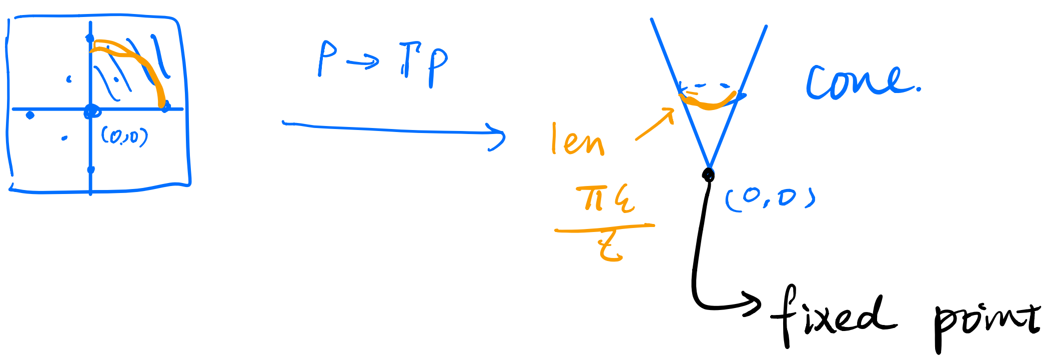
$\Downarrow$

same pt.

(2nd) Fixed point: if $\exists\ p$ s.t. $g(p) = p$ $\forall g \in \Gamma$, then we have issues

Example 2.: Why fixed points are trouble?

Consider $\mathbb{R}^2$, $\Gamma \in \langle R_{\pi/2} \rangle \leq Iso(\mathbb{R}^2)$



Thm Let $\Gamma \leq \mathcal{I}_{so}(\mathbb{R}^2)$ act on $\mathbb{R}^2$, Then,

$$\left.\begin{array}{l} \text{1. No Limited points} \\ \text{2. Fixed point free} \end{array}\right\} \Longleftrightarrow \begin{array}{l} \forall p \in \mathbb{R}^2, \\ \exists \text{ disk } D_p \in \mathbb{R}^2 \\ \text{s.t. } D_p \cap \Gamma(p) = \{p\} \end{array}$$

(In fact, $|D_p \cap \Gamma(a)| \leq 1$)

Corollary: if Γ has no fixed point nor limited point, then $\mathbb{R}^2/\Gamma$ is locally isometric to $\mathbb{R}^2$

(i.e. $\forall p \in \mathbb{R}^2/\Gamma$, $\exists$ disk D_p, s.t. the geometry in D_p is the geometry as that of a disk in $\mathbb{R}^2$)

Proof: ($\Leftarrow$) By contradiction, Assume $g \in \Gamma$ has a fixed point

(1) if Limited points existed, then any disk D_p will have elements of the limited point orbit

(2) For reflections or rotations, the fixed point cannot have a neighborhood w. disjoint pa image.

$(\Rightarrow)$ consider $p \in \mathbb{R}^2$. Now $g(p) \neq p$,

$\underbrace{\forall g}_{\text{no fixed pt.}} \in \Gamma$, $g \neq id$

$p \xrightarrow{\delta} g(p)$

Because no limit point exist. $\exists\, \delta > 0$ s.t.

$\mathrm{dist}(p, g(p)) > \delta$, $\forall g \in \Gamma$

Then $D_p := D_{\delta/3}(p)$

disk $\rightarrow$ radius $\delta/3$