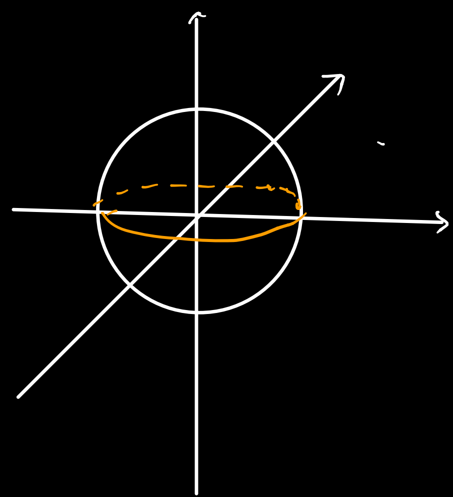
Recall: the 2-sphere is

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$$



with $d_{S^2}(P, Q) = 2\arcsin\left(\frac{d_{\mathbb{R}^3}(P,Q)}{2}\right)$

where $d_{\mathbb{R}^3}$ is Eu. distance.

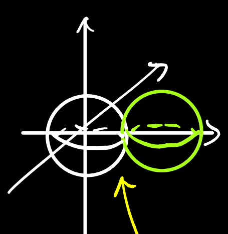
Relationships of points in d_{S^2} $\Longleftrightarrow$ in $\mathbb{R}^3$

* What are the isometries of S^2?

⚠ No translation

$t_{(\alpha,\beta,\gamma)}\ \mathbb{R}^3 \to \mathbb{R}^3$

$S^2 \to \mathbb{R}^3$



after transl, some points may not on sphere.

$$t_p\left(\frac{p}{\|p\|}\right) = \frac{p}{\|p\|} + p = \underbrace{\frac{p}{\|p\|}}_{\text{Norm}} \underbrace{(1 + \|p\|)}_{\text{Norm}+1} \in S^2$$

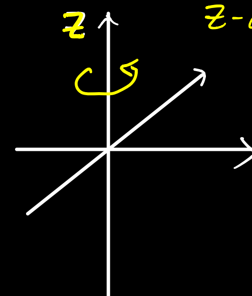
Rotation (through a point)

Reflections (in a line, i.e. a great circle)

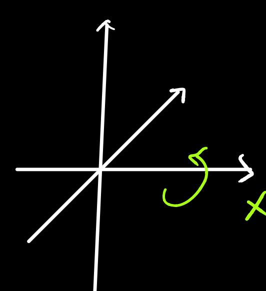
ROT

$$R_{\underset{(0,0,1)}{z},\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

around Z-axis

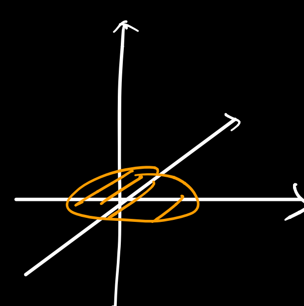


$$R_{\underset{(1,0,0)}{x},\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



REF

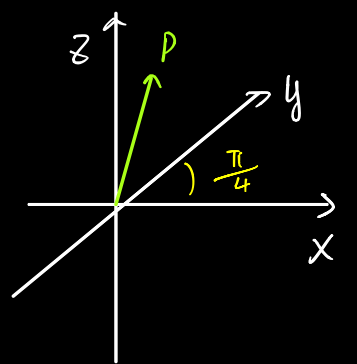
$$F_E = (x, y, z) = (x, y, -z)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



around a plane

Q: How to perform $R_{p,\theta}$, where

$p = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$?



Conjugation

$$\text{Thing at } p = \underbrace{\begin{pmatrix}\text{Transform} \\ \text{back} \\ \text{to } p\end{pmatrix}}_{T^{-1}} \circ \underbrace{\begin{pmatrix}\text{Standard} \\ \text{operation}\end{pmatrix}}_{S = R_{z,\theta}} \circ \underbrace{\begin{pmatrix}\text{Transform } p \\ \text{to the standard} \\ \text{place}\end{pmatrix}}_{T}$$

Rotations that bring p to $(0,0,1)$

$T = R_{x,\frac{\pi}{4}}$

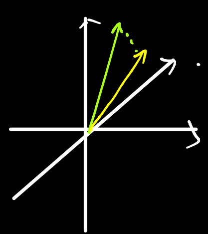
$$R_{p,\theta} = R_{x,-\frac{\pi}{4}} \circ R_{z,\theta} \circ R_{x,\frac{\pi}{4}}$$

Q: How to perform reflection $\bar{r}_M$, where M is the great circle in S^2 cut by the plane $\{y+z=0\}$?

$S = \bar{r}_E$ $\quad$ T must bring M to E

$\iff \langle 0, 1, 1 \rangle$ to z-axis

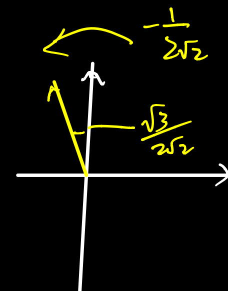
Q: $R_{Q,\theta}$ $\quad Q = \left(-\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in S^2$



① Rotate to yz-plane w/ R_z

② Rotate up to the z-axis w/ R_x

① $\theta = \arcsin\left(\frac{1/2\sqrt{2}}{\sqrt{3}/2\sqrt{2}}\right) = \frac{\pi}{6}$



② rotate $Q' = R_{z,\frac{\pi}{2}}(Q)$ to $(0,0,1)$

$= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P$

$$T = R_{x,\frac{\pi}{4}} \circ R_{z,-\frac{\pi}{4}}$$

$$R_{Q,\theta} = T^{-1} \circ R_{z,\theta} \circ T$$

$$T = \left(\begin{array}{l}\text{Rotate to}\\ \text{z-axis (0,0,1)}\\ \text{w/ } R_x\end{array}\right) \circ \left(\begin{array}{l}\text{Rotate to}\\ \text{Yz plane (so}\\ \text{sit above y-axis)}\\ \text{w/ } R_z\end{array}\right)$$