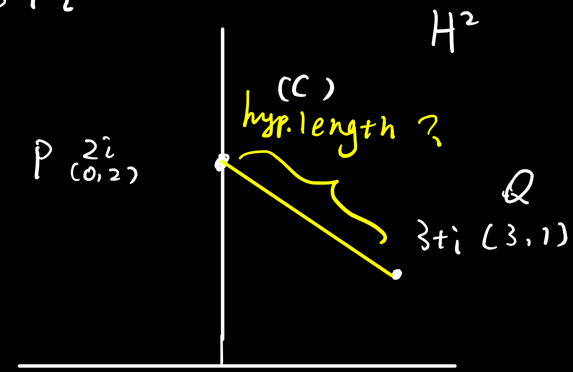
Lecture

Exercise 1: $P = 2i \quad Q = 3+i$

(a) $d_{H^2}(P, Q)$

(b) What is the hyberbolic line containing P & Q



Basic facts : $P = z \quad Q = w$, the

(i) $$d_{H^2}(P, Q) = \ln\left(\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\right)$$

(ii) There are 2 types of hyper. lines

→ Euclidean vertical line

→ Euclidean Semicircles centered at x-axis

$\gamma(t) = (x(t), y(t))$

(iii) $$\text{Length}(\gamma) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|_{\mathbb{R}^2}}{\gamma_2(t)}\, dt = \int_{t_0}^{t_1} \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

↳ γ parametrized from $[t_0, t_1]$

Solution:

(a) We have $z = 2i$, $w = 3+i$, $\overline{w} = 3-i$

So,

$$d_{H^2} = \ln\left(\frac{|3i-3| + |-3+i|}{|3i-3| - |-3+i|}\right)$$

$$= \ln\left(\frac{\sqrt{9+9} + \sqrt{9+1}}{\sqrt{9+9} - \sqrt{9+1}}\right)$$

$$= \ln\left(\frac{\sqrt{2}(3+\sqrt{5})}{\sqrt{2}(3-\sqrt{5})}\right) = \ln\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)$$

$$\approx 1.9$$

distance $d_{H^2}(P,Q)$ $=$ min length (γ) between P & Q

(b) The equation for the center of the semicircle is

$$|2i - c| = |3 + i - c|$$

$c \in \mathbb{R}$

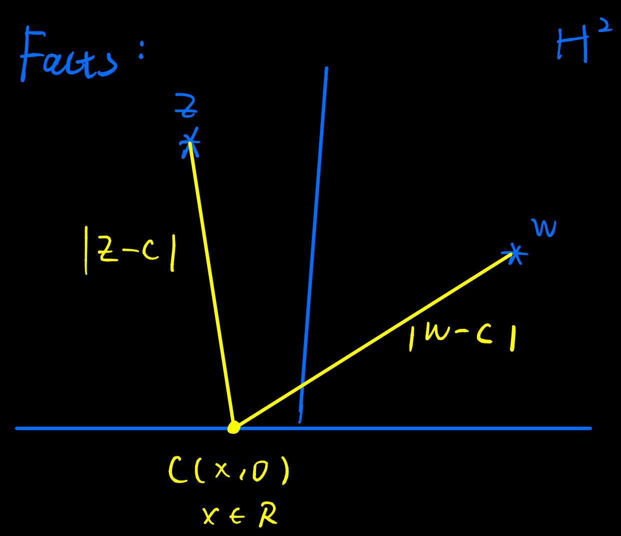
$$\Longleftrightarrow$$

$$|2i - c|^2 = |(3-c) + i|^2$$

$$\Longleftrightarrow$$

$$c^2 + 4 = (3-c)^2 + 1$$

Facts:



$\Rightarrow |z-c|^2 = |w-c|^2$

eq for c, find c for $\mathbb{R}$

$\Longleftrightarrow$

$c^2+4 = 9-6c+c^2+1$

$\Longrightarrow \boxed{c=1}$

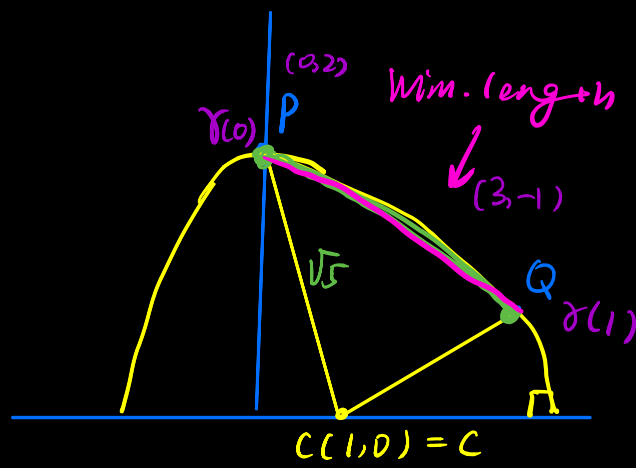


figure for this example.

(C) Consider the Euclidean segment between P, Q, What is the hyperbolic length?

(shown on purple color)

Parameter via $\gamma(t) = (0,2) + t(3,-1)$

$$= (\underbrace{3t}_{x(t)}, \underbrace{2-t}_{y(t)})$$

Now, $\gamma'(t) = (3,-1)$

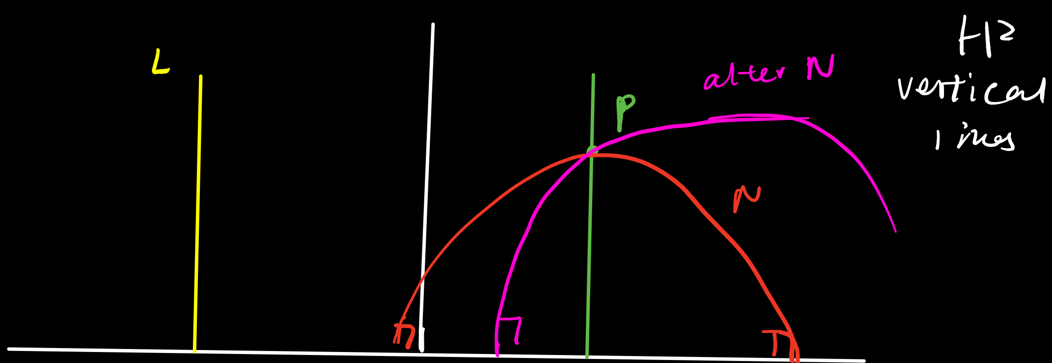
So $\sqrt{(x'(t))^2 + y'(t))^2} = \sqrt{10}$

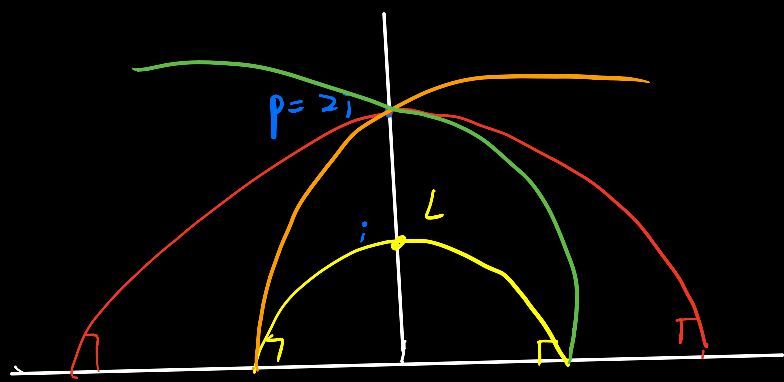
$$\text{Length}(\gamma) = \int_0^1 \frac{\sqrt{10}}{2-t}\,dt = \sqrt{10}\cdot\Big[\ln|2-t|\Big]_0^1$$

$$= \boxed{\sqrt{10}\cdot\ln(2)}$$

$$\approx 2.2$$

<u>Exercise 2.</u>: Find a line $\ell \subseteq H^2$ and a point P in H^2 s.t. $\exists\, M, N \subseteq H^2$ hyperbolic lines parallel and containing P i.e. $P \in M$, $P \in N$ (find M, N implicitly)





No vertical lines