Lecture 11

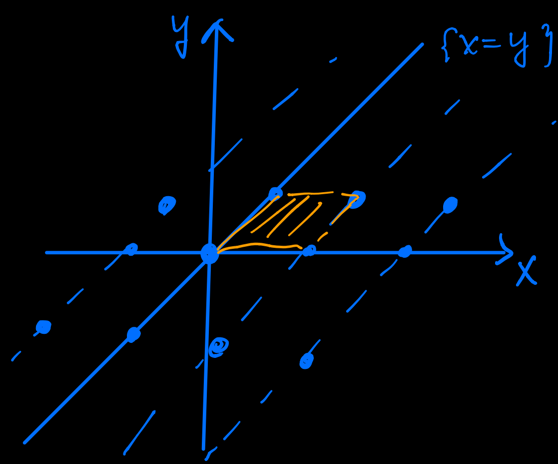
* **Thm** Let $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^2)$ be a subgroup.

Then,

LHS $\begin{bmatrix} \Gamma \text{ discontinuous} \\ \text{and fixed pt free} \end{bmatrix} \Leftrightarrow$ RHS $\begin{bmatrix} \forall p \in \mathbb{R}^2, \exists \varepsilon > 0 \text{ s.t.} \\ |D(p) \cap \Gamma Q| \leq 1 \\ \text{for all } Q \in \mathbb{R}^2 \end{bmatrix}$

Example

① $\Gamma = \langle t_{(1,0)}, t_{(1,1)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$



It will be the case that $\mathbb{R}^2/\Gamma$ is equivalent to 2-torus

$\exists$ bijection isometry between

$\mathbb{R}^2/\Gamma \cong \mathbb{R}^2/\langle t_{(1,0)}, t_{(0,1)} \rangle$

* **Thm** Let $\Gamma \leq Iso(\mathbb{R}^2)$ be discontinuous and fixed point free. Then Γ is generated by 1 or 2 elements.

Example $\Gamma = \langle \underline{t_{(1,0)}, t_{(0,1)}}, t_{(3,4)} \rangle$

$= \langle t_{(1,0)}, t_{(0,1)} \rangle$

$t_{(3,4)} = t_{(1,0)}^{3} \cdot t_{(0,1)}^{4}$

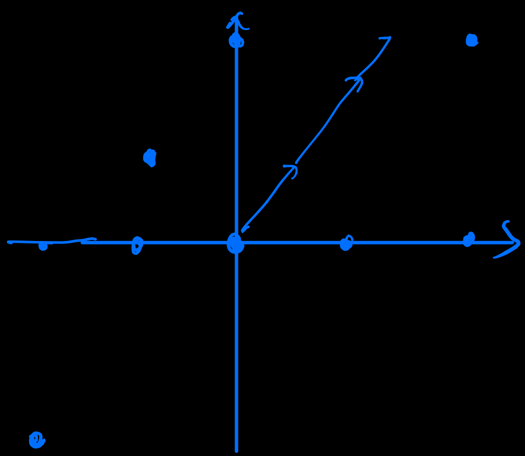
Proof: By our classification of elements in $Iso(\mathbb{R}^2)$, Γ must only contains translations and glide reflections

Note: $\otimes$ rotation and ref NOT Fixed pt free

For simlicity, we will do just tranlations:

Assume Γ is gen. by translations, we want $\Gamma = \langle t_1 \rangle$ or $\Gamma = \langle t_1, t_2 \rangle$ [Hypo + Goal]

Let $P \in \mathbb{R}^2$ be a point. Then choose $t_1 \in \Gamma$ such that $\text{dist}(p, t_1(p))$ is smaller (or equal) than $\text{dist}(p, gp)$



Now, either $\langle t_1 \rangle = \Gamma$ or not

if not, we need to find at least one generator

Assertion: if $\Gamma = \langle t_1, t_2 \rangle$ with t_1 as above. Then t_2 is in a different direction from t_1

Proof: if t is in direction of t_1, choose $m \in \mathbb{Z}$ s.t. $t(p)$ is closest to $t_1^m(p)$. Then, $t^{-1} \circ t_1^m(p)$ is closer to p than $t_1(p)$ ∎

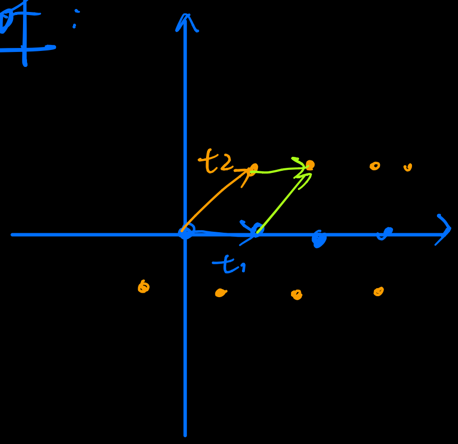
Contradiction.

Choose t_2 s.t. $t_2(p)$ minimize distance

(note: $t_2 \in \Gamma$ not in $t_2 \notin \langle t_1 \rangle$, good by assertion)

<u>Assertion II</u>: if $\Gamma \neq \langle t_1, t_2 \rangle$ then contradiction.

<u>Proof</u>:



if $t_3 \subseteq \Gamma$ and $t_3 \notin \langle t_1, t_2 \rangle$

then $t_3(p)$

Choose $n, m \in \mathbb{Z}$ s.t.

$t_1^n t_1^m$ is closest to $t_3(p)$

$\Rightarrow$ contradiction with being shorter than the longest side. ∎