Lecture 12

Theorem (W. Killing – H. Hopf) Every complete connected Euclidean surface S' must be of the form $S \cong \mathbb{R}^2/\Gamma$.

§1. Definition in the abstract: "Euclidean Surface"

Surface: The data needed is (S, ds) with S a set (of points) and a function $ds: S \times S \longrightarrow \mathbb{R}$ (think distance)

s.t. ① ds symmetric

② $ds \geq 0$ and 0 only for $ds(p,p)$

③ Triangle ineq.

$$ds(P,Q) \leq ds(P,R) + ds(R,Q)$$

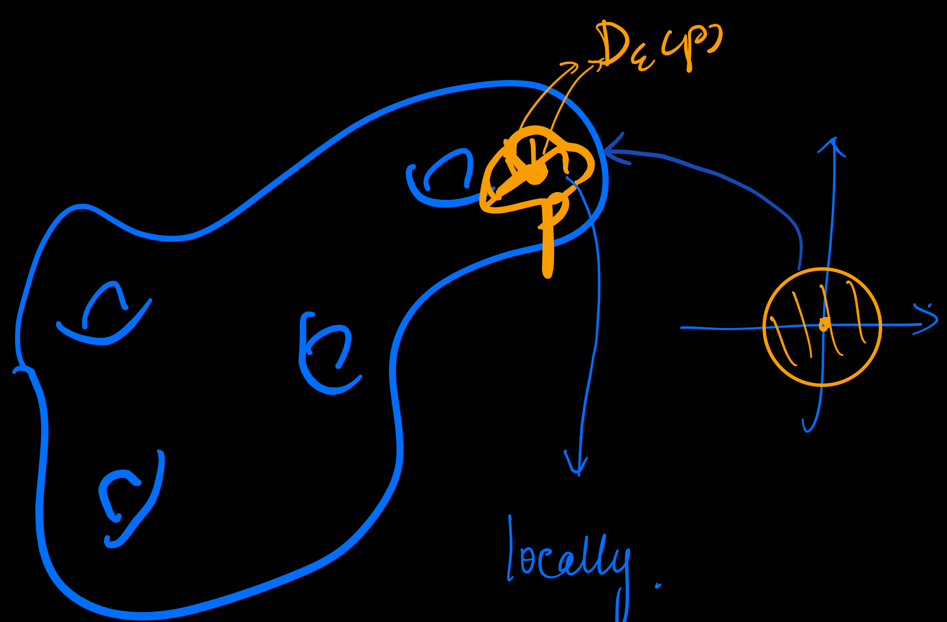
$\underline{\text{Def}^n}$ (Euclidean Surface) A set (S, d_S) is said to be an Eu. Surface if for any $P \in S$ we have $\exists \varepsilon > 0$ s.t.

(1) $D_\varepsilon(p) := \{ d_S(P,Q) < \varepsilon \} \subseteq S$ is in bijection with a disk $D_\varepsilon^{\mathbb{R}^2}(0,0) \subseteq \mathbb{R}^2$, s.t. bijection preserve distances.

Euclidean.

1 to 1 & onto

$\underline{\text{Ex.}}$

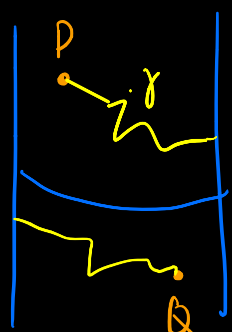


Locally, cylinder = torus

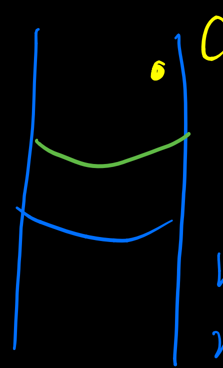
Globally: cylinder $\neq$ torus

Def^n A Eu. Surface (S, d_S) is <u>connected</u> if $\forall\, P, Q \in S\ \exists\, \gamma \subseteq S$, a polygonal path from P to Q.

Ex.



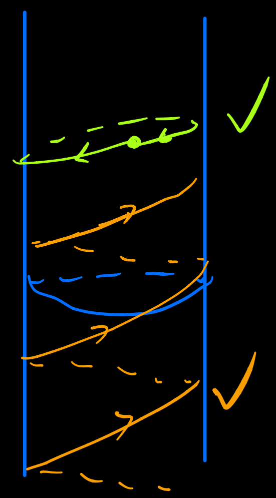
Non-Ex



C is an Eu. Surface but it is not connected

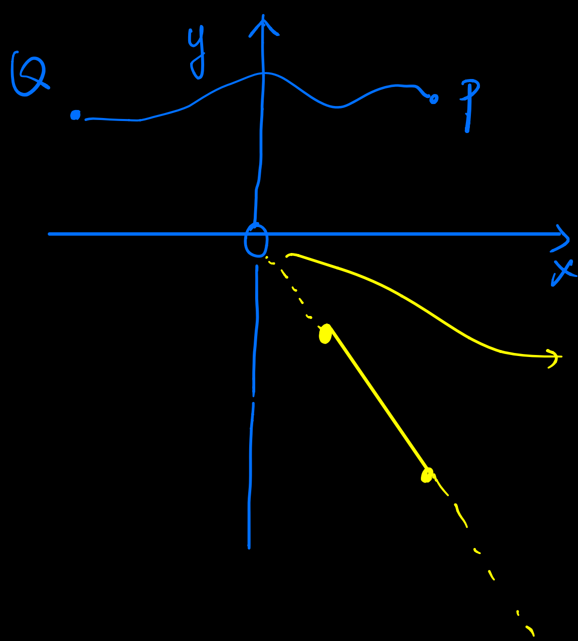
Def^n An Eu. Surface (S, d_S) is complete if every line segment can be extended indifinitely. ▨

Example the cylinder $\mathbb{R}^2/\langle t_{1,0}\rangle$



除去 $(0,0)$

Non-Ex. $\mathbb{R}^2 \setminus \{0\}$ is Eu. Surface
is connected. but
not complete.



extend forever here.

Back to main theorem

(1) Create a map $\mathbb{R}^2 \xrightarrow{P} S$

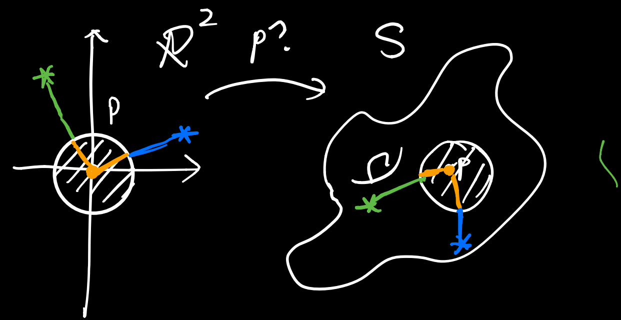
(2) Find Γ s.t. $p\ \mathbb{R}^2 \to S$ same as

$\mathbb{R}^2 \to \mathbb{R}^2/\Gamma$

§2. Proof of Killing-Hopf

STEP 1

The map P is defined by



$P(Q) := \{$ intersect $\vec{Q}$ with $D_\varepsilon(0)$, then

$Q \in \mathbb{R}^2$

Conside the image of this segment in $D_\varepsilon^S(p)$ and shoot from there for distance $\|Q\|\}$

STEP 2 : Where is Γ? $\Gamma :=$ "symmetries of map p"

Rigoriously, $\Gamma := \{ g \in \mathrm{Iso}(\mathbb{R}^2) \text{ s.t. } p \circ g = p \}$

(1) Γ is a group

(2) The map $p : \mathbb{R}^2 \xrightarrow{p} S$ same as $\mathbb{R}^2 \xrightarrow[\text{map}]{\text{orbit}} \mathbb{R}^2/\Gamma$

↳ if $P, Q \in \mathbb{R}^2$ and $p(P) = p(Q)$

then $\exists\, g \in \Gamma$ s.t. $g(P) = Q$.