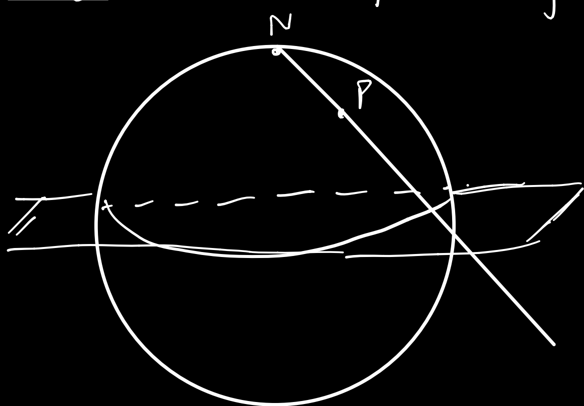
Stereographic Projection

$S^2 \to \mathbb{R}^2 \qquad \mathbb{R}^2 \to S^2$

<u>Def</u> $\pi : S^2 - \{(0,0,1)\} \to \mathbb{R}^2$



let $p = (x, y, z) \in S^2 - N$.
in $\mathbb{R}^3$.

Draw a line from $N \to P$

$\pi(p)$ is the unique pt where l cross the x-y plane.

π is a bijection : $\pi^{-1} : \mathbb{R}^2 \longrightarrow S^2 - N$

Take $(u, v) \in \mathbb{R}^2$, draw a line from (u, v) to N. Then $\pi^{-1}(u,v)$ = place where l parallel S^2.

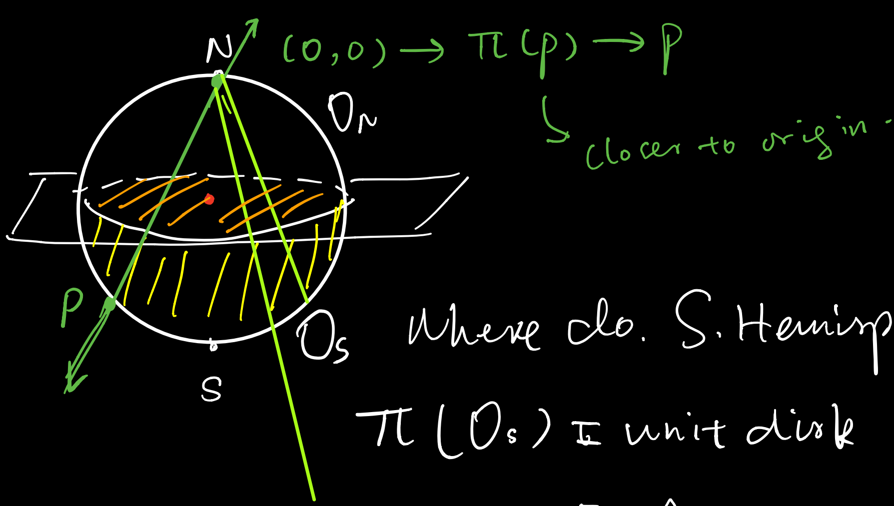
$$S^2 - N \longleftrightarrow \mathbb{R}^2$$
$$(x, y, z) \longrightarrow \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

$$\left(\frac{2u,\ 2v,\ u^2+v^2-1}{u^2+v^2+1} \right) \longleftarrow (u,v)$$

might mix not exactly $O \to O$, line $\to$ line.

- Where do circles/lines in S^2 go in $\mathbb{R}^2$?
- -- — — — ——— in $\mathbb{R}^2$ ... S^2?

Note: the equator $\{x^2+y^2=1\}$ in S^2 is fixed by π.



Where do. S. Hemisphere go?

$\pi(O_s)$ = unit disk

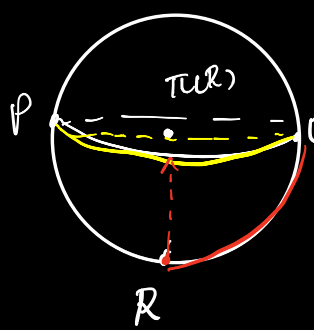
$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$

Area$(O_s) = 2\pi$, but Area$(\pi(O_s)) = \pi$.

$\to$ π does not preserve area

$\times$ preserve distance.

Note: iso preserve area



$d_{S^2}(P,Q)=\pi$

$d_{\mathbb{R}^2}(\pi(P),\pi(Q)) = 2 = d_{\mathbb{R}^3}(P,Q)$

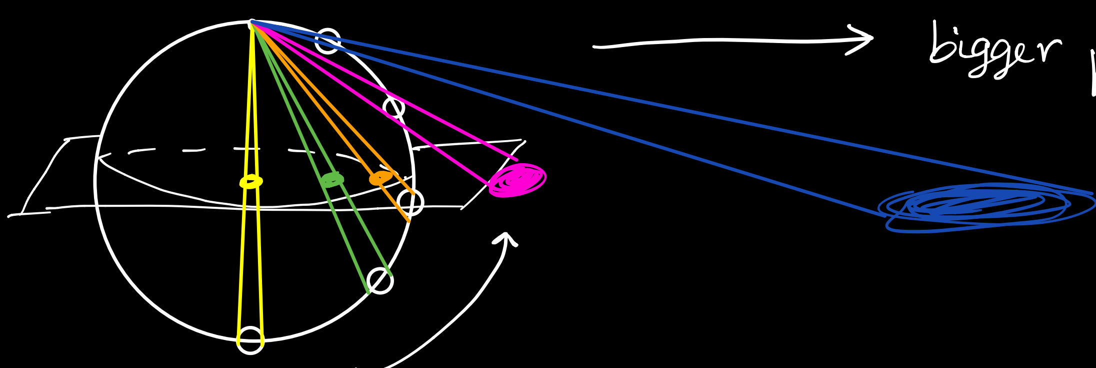
$d_{S^2}(R,Q)=\frac{\pi}{2}$

$d_{\mathbb{R}^2}(\pi(R),\pi(Q)) = 1$

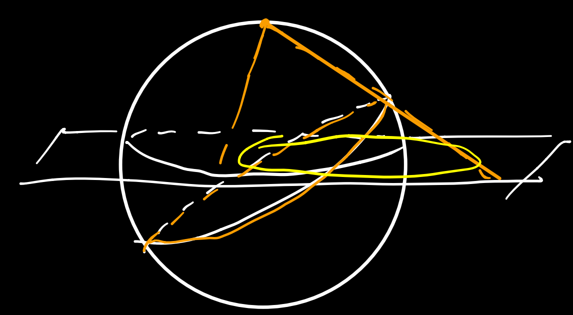
$d_{\mathbb{R}^2}(R,Q)=\sqrt{2}$.

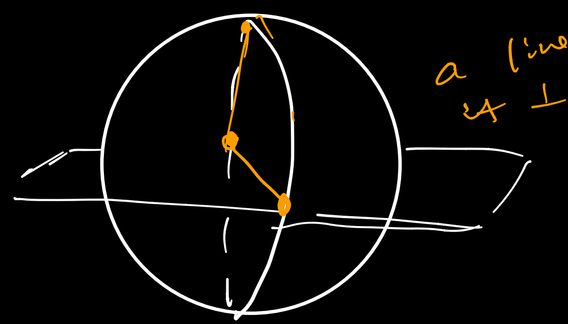
$\pi(O_N) = \{(x,y)\in \mathbb{R}^2 \mid x^2+y^2>1\}$.

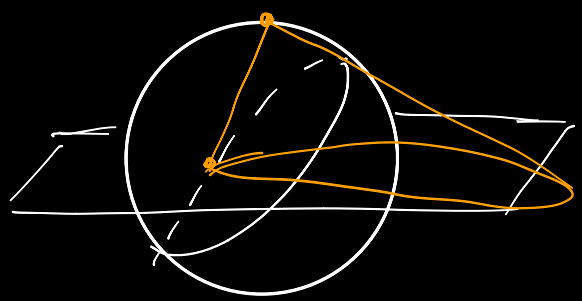
Take small disk starting at S, moving up.



$\longrightarrow$ bigger projection to inf of plane lines in $S^2 \longrightarrow \mathbb{R}^2$





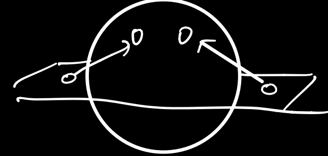


- lines that X intersect North Pole $\rightarrow$ circles in $\mathbb{R}^2$
- line that ✓ intersect N $\rightarrow$ lines in $\mathbb{R}^2$

What about $\mathbb{R}^2 \rightarrow S^2$?

Do circles in $\mathbb{R}^2$ goes to lines (big circles) in S^2

NO



Small O disjointed, under π^{-1}

their π^{-1} are

not lines

$$\begin{Bmatrix} \text{circles} \\ \text{lines} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{circles} \\ \text{lines} \end{Bmatrix}.$$

S^2 $\qquad\qquad$ $\mathbb{R}^2$

or say

$$\{\text{circles}\} \longleftrightarrow \{\text{generalized circles}\}$$

$$\text{generalized circles} = \text{a line or a circle}.$$