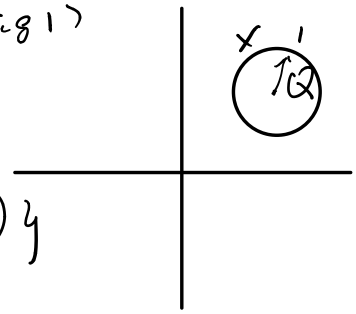
§1. Sets defined by distance to points

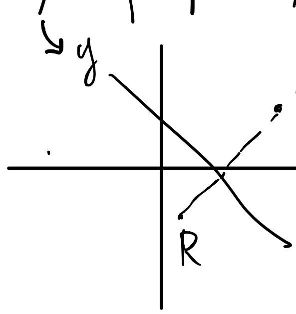
① Consider $X = \{ P \in \mathbb{R}^2 : d(P, Q) = 1 \} \subseteq \mathbb{R}^2$

(fig 1)



② Choose $Q, R \in \mathbb{R}^2$

$Y = \{ P \in \mathbb{R}^2, d(P, Q) = d(P, R) \}$



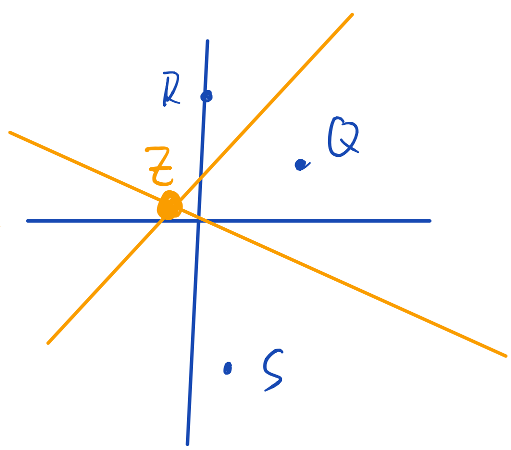
(fig 2)

③ Now, with 3 points $Q, R, S \in \mathbb{R}^2$ (non-colinear)

$Z := \{ P \in \mathbb{R}^2 : d(P, Q) = d(P, R) = d(P, S) \}$

Q: What is Z

Z is a unique pt.

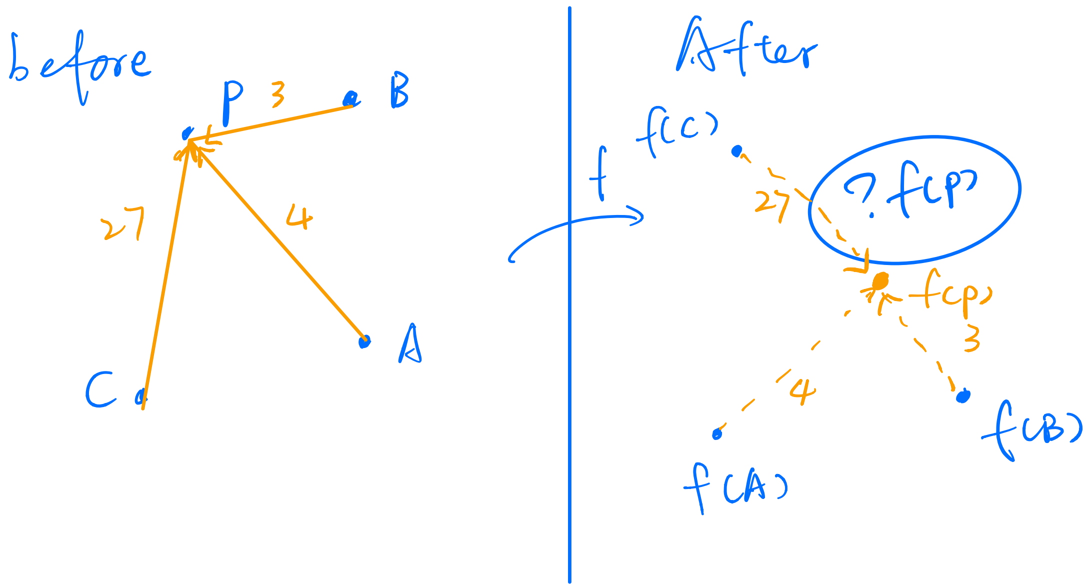


Proposition: Any isometry $f: \mathbb{R}^2 \to \mathbb{R}^2$ is uniquely determined by the images $f(A), f(B), f(C)$ of any 3 non-colinear pts $A, B, C \in \mathbb{R}^2$ ∎

$\longrightarrow$ A, B, C must be non-colinear because $f = \mathrm{Id}$, $f = r_L$ preserve the line $L = \langle A, B, C \rangle$ if A, B, C colinear.

Pf: First a pt P is uniquely determined by its distances to A, B, C. Need to know who $f(P) \in \mathbb{R}^2$ is

Now, $f(P)$ has distance to $f(A)$ exactly $d(P, A)$



MAIN THEOREM

set of isometries (group)

Any isometry $f \in Iso(\mathbb{R}^2)$ must be a product of 1, 2, 3 reflections ▨

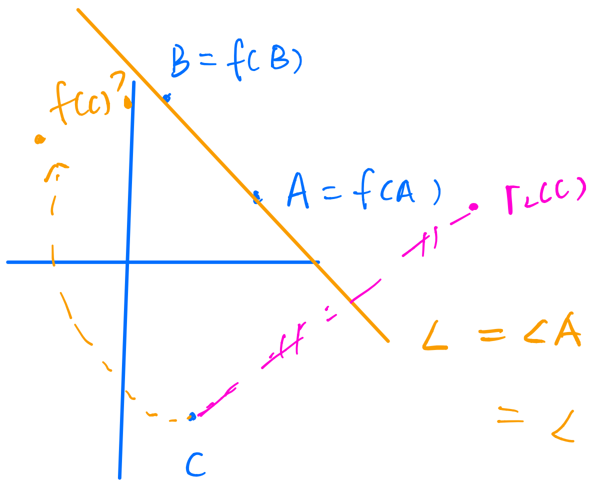
Pf Choose $A, B, C \in \mathbb{R}^2$ three non-colinear points. We study the cases for $f(A)$, $f(B)$ & $f(C)$

Case 0^{th} : $f(A) = A$, $f(B) = B$, $f(C) = C$.

Then $f = Id$. done.

Case 1st: $f(A) = A$, $f(B) = B$, $(f(C)?)$

Guess is that $f(C) = r_L(C)$



$L = \langle A, B \rangle$

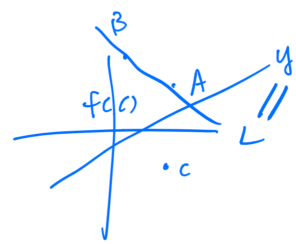
$= \langle f(A), f(B) \rangle$

By prop
f & r_L
coincide ABC
so $f = r_L$

Claim: $f(C) = r_L(C)$

Pf Consider the set $Y = \{ P \in \mathbb{R}^2,$

$d(P, C) = d(P, f(C)) \}$



Find $A \in Y$, because

$d(A, C) = d(f(A), f(C))$

$= d(A, f(C))$

Same argument says $B \in Y$

Hence Y = line spined A & B → so Y is actually L

Case 2^{nd}: $f(A) = A$, let $f(B) \neq B$ nor
$f(C) \neq C$

$\to$ f product of 2 refs

Case 3^{rd}: $f(A) \neq A$, $f(B) \neq B$, $f(C) \neq C$

$\to$ f product of 3 refs