Quotients

- 1D
- 2D
- Distance / Lines

We take old space, and "identify" points to make it same in new space.

Ex. $\mathbb{Z}/t_{12} = \{\bar{n} \mid n \in \mathbb{Z}\}$

a bunch of points

transl by 12

$\boxed{t_{12}(m) = m+12}$

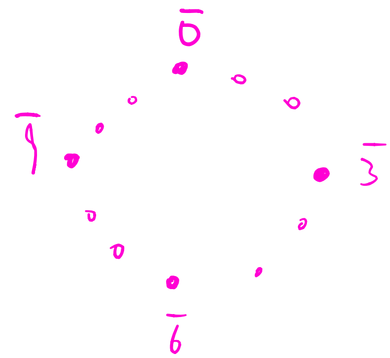
$\bar{n} = \{ \dots, n-36, n-24, n-12, n+12, n+24, n+36, \dots \}$

$= \{ \dots (t_{12})^{3}(m), (t_{12})^{-2}(m), \dots \}$

$\bar{6} + \bar{7} = \overline{13} = \bar{1}$

so, $\mathbb{Z}/t_{12} = \{\bar{0}, \bar{1}, \bar{2}, \dots \overline{11}\}$

↳ viusalize:



→ a clock

Ex2 $\mathbb{R}/_{t_1} = \{\bar{x} \mid \bar{x} \in \mathbb{R}\}$

$$\bar{x} = \{\dots, t_1^{-2}(x), t_1^{-1}(x), t_1^{0}(x), \dots\}$$

$$= \{\dots t_{-2}(x), t_{-1}(x)\ t_0(x) \dots\}$$

$$= \{\dots x-2, x-1, x \dots\}$$

Def: A fundamental domain of X/N is a subset of X and that no 2 points in X become identified. and X is as big as possible.

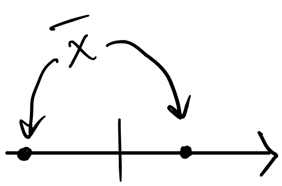
Ex $\mathbb{R}/F = \{\bar{x} \mid x \in \mathbb{R}\}$

↑

reflection in 0

$$\bar{x} = \{\dots \overset{x}{(F)^{-2}(x)}, \overset{-x}{(F)^{-1}(x)}, \overset{x}{F^{0}(x)}, \overset{-x}{(F)^{1}(x)}, \dots\}$$

$$\bar{x} = \{-x, x\}$$



$\mathbb{R}/\mathbb{Z}_2 \Rightarrow \mathbb{Z}_2$- action.

$F \cong [0, +\infty) = \bullet\!\longrightarrow$

Circle

$\mathbb{R}/\mathbb{Z} = \mathbb{R}/t_1 = S^1$

$\boxed{\mathbb{Z} \text{ act on } \mathbb{R} \text{ by } t_1}$

$= \{\mathbb{Z}x \mid x \in \mathbb{R}\}$

$\mathbb{Z}x = \{\dots, x-2, x-1, x, x+1, x+2, \dots\}$

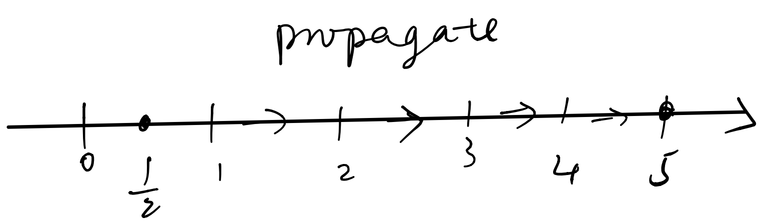
$= \mathbb{Z}$ orbit at x

Distance on $\mathbb{R}/\mathbb{Z}$:

$$d_{\mathbb{R}/\mathbb{Z}}(\mathbb{Z}_x, \mathbb{Z}_y) = \min\{d_{\mathbb{R}}(x', y') \mid x' \in \mathbb{Z}_x, y' \in \mathbb{Z}_y\}$$

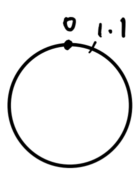
$$d_{\mathbb{R}/\mathbb{Z}}(\mathbb{Z}_{\frac{1}{2}}, \mathbb{Z}_5)$$

$\mathbb{Z}_x = \mathbb{Z}_{\cancel{x}+5}$



<u>$\mathbb{Z}x$</u>

$$d_{\mathbb{R}/\mathbb{Z}} = \{\mathbb{Z}_0, \mathbb{Z}_{1.1}\} = 0.1$$



$$d_{\mathbb{R}/\mathbb{Z}} = \{\mathbb{Z}_0, \mathbb{Z}_{0.9}\} = 0.1$$

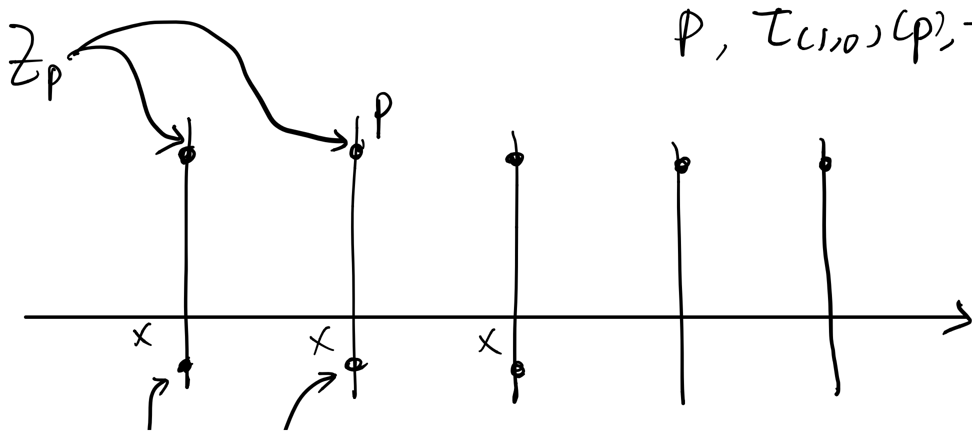
☆ $d_{\mathbb{R}/\mathbb{Z}}$ is continuous but the path that achieve the min might change drastically.

$d(0, 0.49) = 0.49$ $d(0, 0.52) = 0.48$

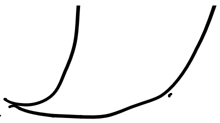
Cylinder: $C = \mathbb{R}^2/\mathbb{Z} \; \left(= \mathbb{R}^2/t_{(1,0)}\right)$

$$= \{ \mathbb{Z}p \mid p \in \mathbb{R}^2 \}$$

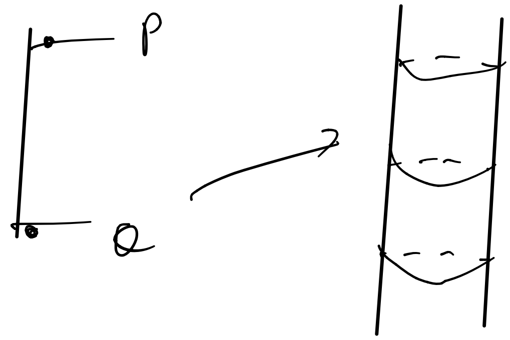
$\mathbb{Z}p = \{ \ldots \; t_{(-2,0)}(p), t_{(-1,0)}(p), p, t_{(1,0)}(p), t_{(2,0)}(p) \ldots \}$



Z_Q

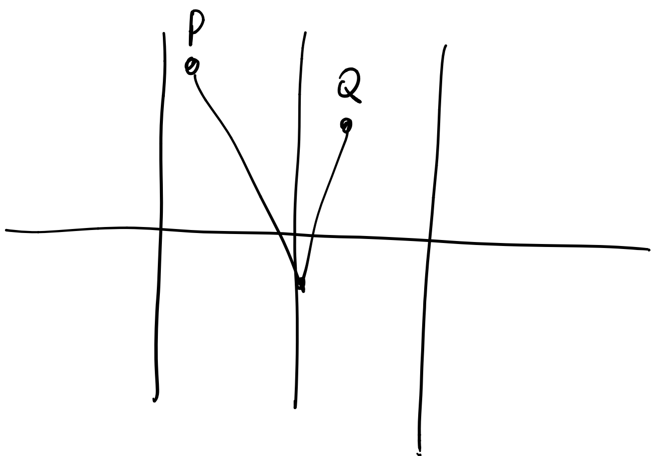


$FQ =$



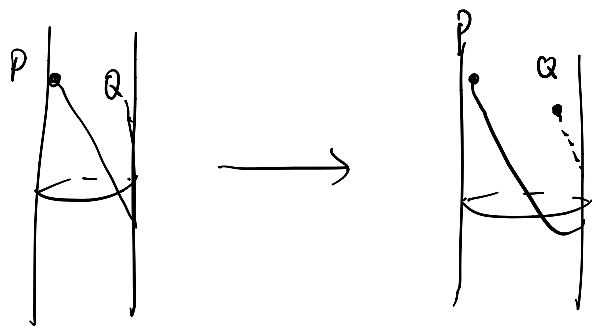
$\subseteq \mathbb{R}^3$

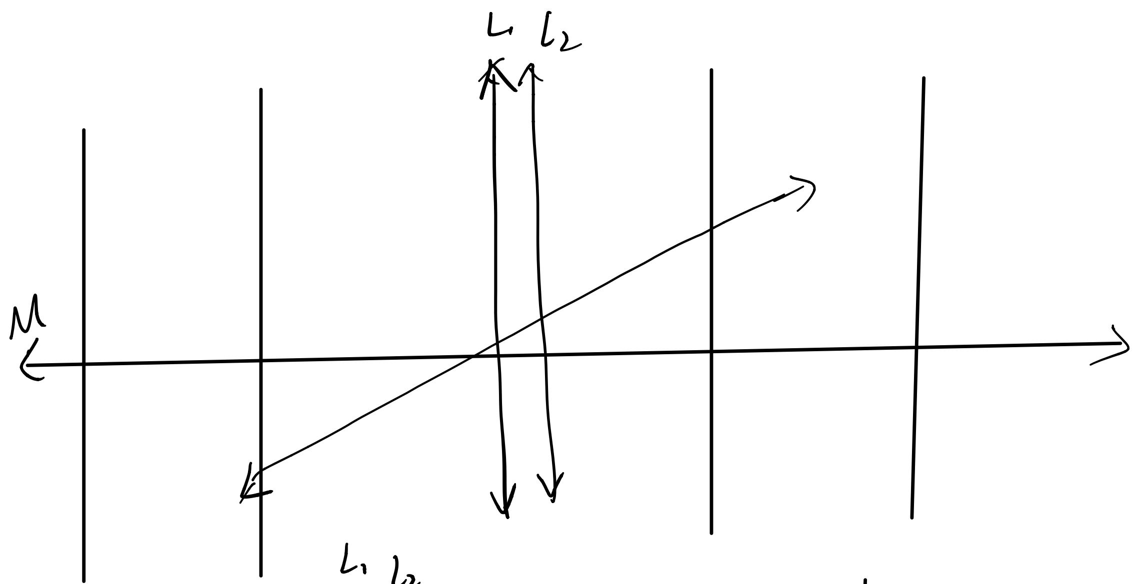
$$d_C(Z_P, Z_Q) = \{ d_{\mathbb{R}^2}(P', Q') \mid Q' \in Z_Q \}$$

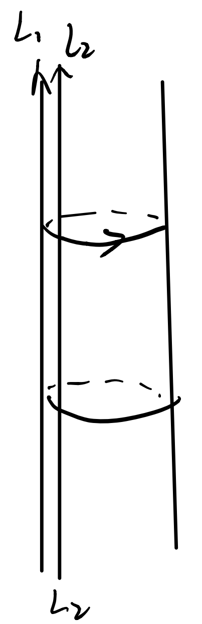


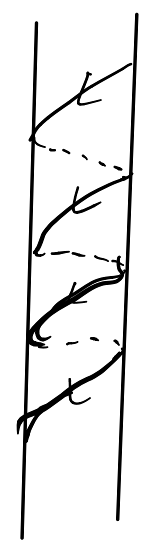
$\Downarrow$

shortest path.









helix.

Exercise

Ex 2.2.1 on book.