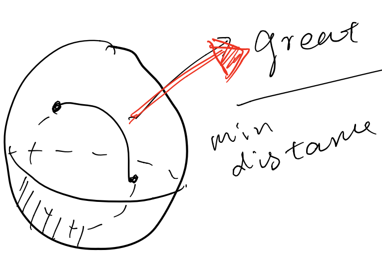
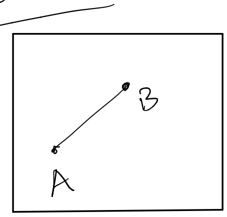
Lecture 1 : Intro

Roger Casals (MSB 3214)

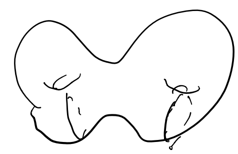
§1. What are surfaces?

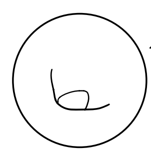


2-Sphere



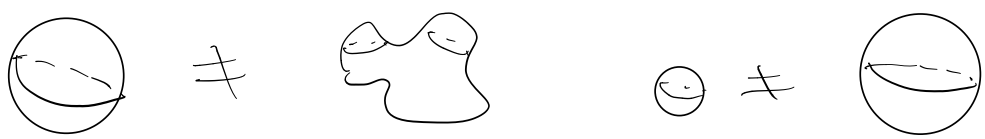
Euclidean plane





genus 1
"torus"

§2. What is geometry?



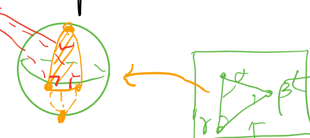
Shape matters

$\rightarrow \boxed{\text{Distance}}$

- Which motions preserve distance? Rot.
- What min. distance? (见上)
- Properties of regions?

Σ angles > π

90°



$\alpha+\beta+\gamma=\pi$

lines in spheres will be great circles

side: lines

§3. Euclidean Plane

Def: The Euclidean plane is

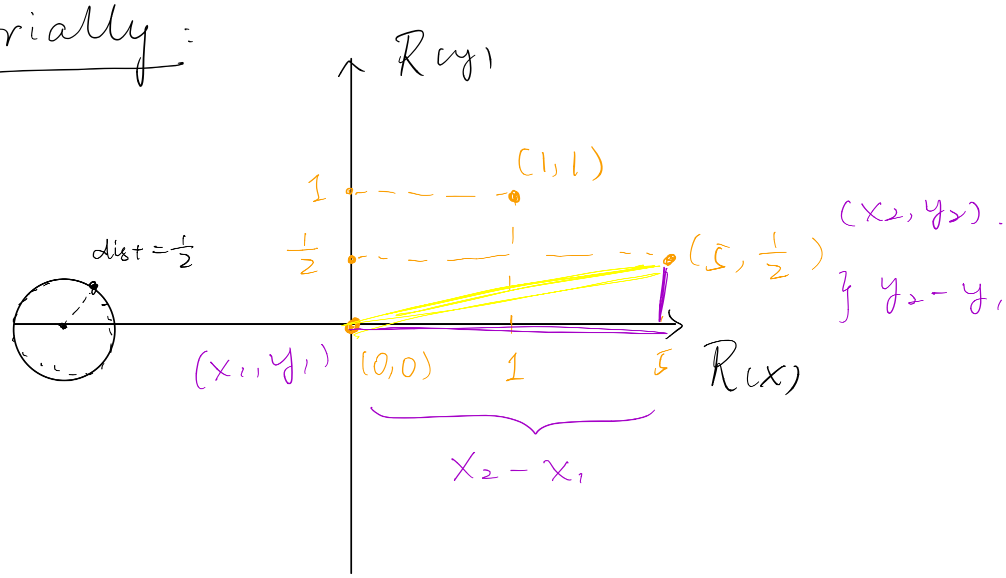
main obj.

$$\mathbb{R}^2 := \{(x, y) \text{ s.t. } x \in \mathbb{R}, y \in \mathbb{R}\}$$

A point in $\mathbb{R}^2$ is an element (x, y), $x, y \in \mathbb{R}$ e.g. $(0,0)$, $(1,2)$, $(\sqrt{\pi}, e)$

*: line: a set of real numbers.

Pictorially:



The geometry in $\mathbb{R}^2$ will be given:

the distance.

$$\operatorname{dist}\big((x_1, y_1), (x_2, y_2)\big) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ are two points in plane

<u>Properties</u> of the distance $dist : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

(a) $dist \geqslant 0$, and if $dist(P, Q) = 0$ then $P = Q$

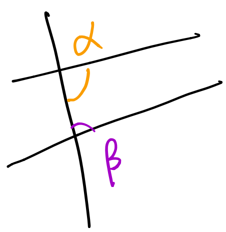
(b) $dist(P, Q) = dist(Q, P)$

(c) $dist(P, R) + dist(R, Q) \geqslant dist(P, Q)$

§ Disc

1. Parallel Postulate

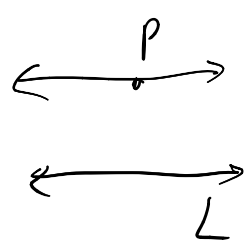
if $\alpha + \beta < 180°$, the two lines meet on the same side



$\Updownarrow$

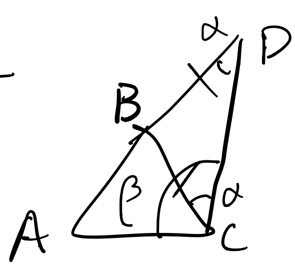
Playfair's Axiom

Given a line L_1, and a point P not in L_1, $\exists$ line through P, parallel to L



2. <u>Thm</u> Triangle Inequality $AC < AB + BC$

<u>Pf.</u>



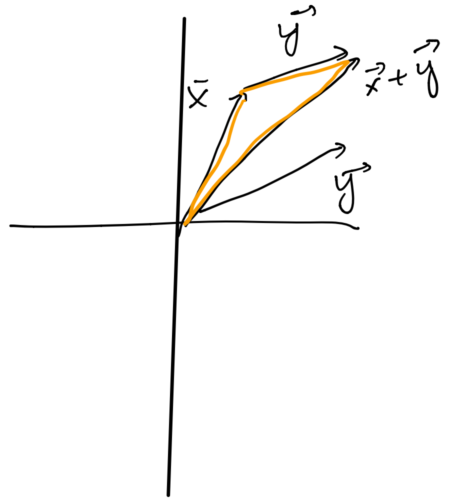
$\triangle BDC$ is isosceles

$BD = BC$

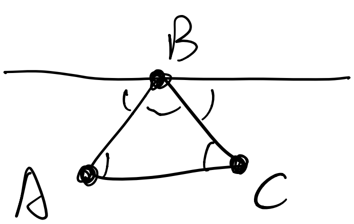
$$AD > AC$$
$$AB + BD > AC$$
$$BD = BC$$

3. $\vec{x}, \vec{y} \in R^2$

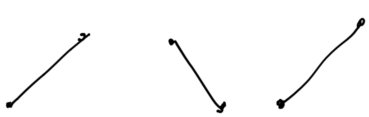
$$|\vec{x} + \vec{y}| \le |\vec{x}| + |\vec{y}|$$



4. <u>Thm</u> △ 内角和 180°



5. isometry



distance all same.
no matter how change.

6. $f \quad R^2 \to R^2$

$(x, y) \to (x, y^2)$ NOT ISOMETRY

if $|(0,2)| = |(0,4)| = 4$

ori. $|(0,2)| = 2$

7. $g(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an ISOMETRY

not linear map (translation)

8. Linear map is a func. $A: R^2 \to R^2$

s.t. ① $A(\vec{x} + \vec{y}) = A(\vec{x}) + A(\vec{y})$

② $A(\lambda \vec{x}) = \lambda A(\vec{x})$

$\forall \vec{x}, \vec{y} \in R^2 \quad \lambda \in R$

$A(\vec{0}) = A(0 \cdot \vec{x}) = 0 A(\vec{x}) = \vec{0}$

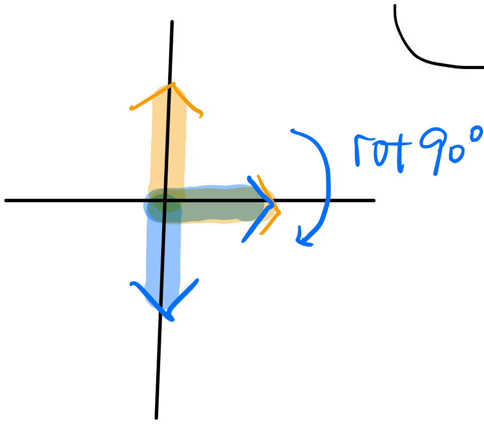
9. Every linear map "is" a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10. Prop $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is an isometry

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

$$= d\left(\begin{bmatrix} y_1 \\ -x_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ -x_2 \end{bmatrix} \right)$$

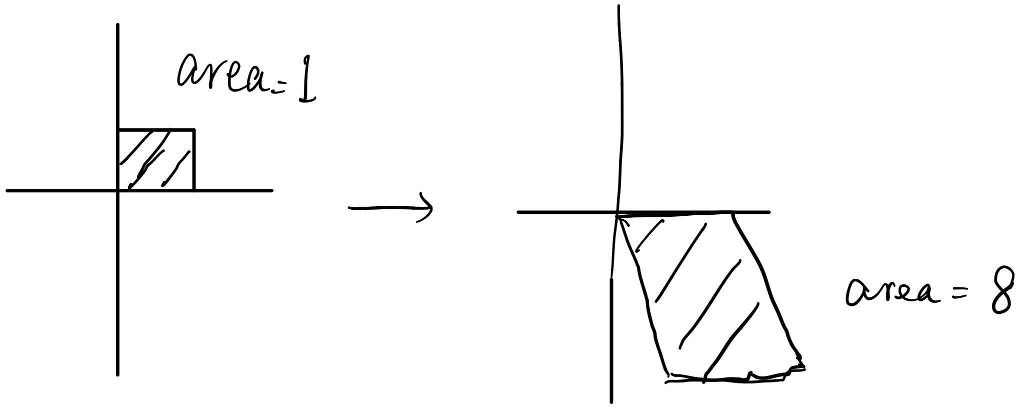


$$\to \sqrt{(y_1 - y_2)^2 + (-x_1 - (-x_2))^2}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= d\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

11. $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$ $\det A = -8$



12. <u>Thm</u> All linear isometry have det $+1$ or -1

The converse is FALSE

e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$ $\quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\det A = 1$

$\downarrow \qquad \downarrow$

len $= 1$ $\quad$ len $= \sqrt{2}$

13. Thm Linear maps with nontrivial

① kernel are not isometry

$\ker A \neq \{\vec{0}\}$

Pf. $\ker A \neq \{\vec{0}\}$ Take $\vec{x} \in \mathbb{R}^2$, $\vec{x} \in \ker A$

$\vec{x} \neq \vec{0}$

then $|\vec{x}| \neq 0$ But $|A(\vec{x})| = |\vec{0}| = 0$

So A is not isometry.

课后问题:

1. 什么是 kernel?
2. 温习线性代数.