We defined $S^2 \in \mathbb{R}^3$ as a set, and endowed it w/ distance d_{S^2}

(i) What are isometries of (S^2, d_{S^2})?

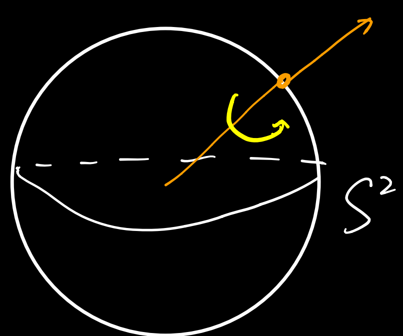
$\boxed{\text{Rot}}$ $\boxed{\text{Ref}}$

§1. Rotations in S^2

For a rot we need: $P \in S^2$, $\theta \in S^1$

or equivalently $\vec{\ell}$ axis in $\mathbb{R}^3$ and $\theta \in S^1$

through origin & oriented.



Note: $R_{\ell,\theta}$ or $R_{P,\theta}$ is a map $\mathbb{R}^3 \to \mathbb{R}^3$ which restrict to a map $S^2 \to S^2$

<u>Def</u> : The Rot $R_{z,\theta}$ along z-axis $(p=(0,0,1))$ is given by $(x,y,z) \longmapsto (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y, z)$

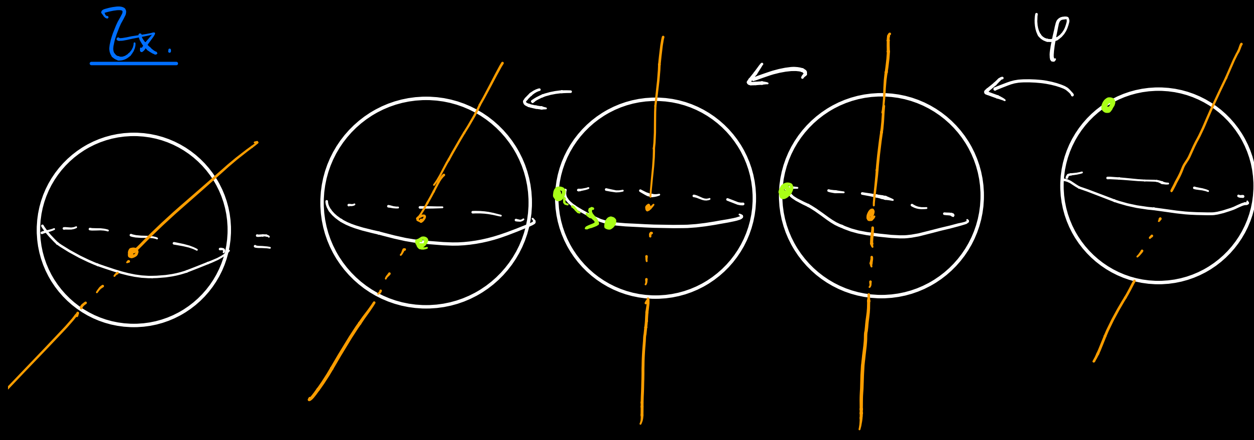
$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Rot of angle in } \mathbb{R}^3 \text{ spinned by}}$$

<u>Remark</u> : This restricted to map $R_{p,\theta}$ $S^2 \to S^2$ because it preserves the distance $d_{\mathbb{R}^3}$, in particular it is an isometry for d_{S^2}.

Now, given any axis $\vec{l} \subset \mathbb{R}^3$, the rot $R_{\vec{l},\theta}$ is defined as $R_{\vec{l},\theta} := \varphi^{-1} \circ R_{z,\theta} \circ \varphi$.

Where $\varphi : S^2 \to S^2$ is an isometry s.t. $\varphi(\vec{l}) = z$-axis

Ex.



Prop: The rotation along x-axis, noted $R_{x,\theta}$ is given by $(x, y, z) \longmapsto$

$$(x, \cos\theta y - \sin\theta z, \sin\theta y + \cos\theta z)$$

Proof: Choose $\varphi: S^2 \to S^2$ s.t. $\varphi(x\text{-axis}) = z\text{-axis}$

Let's find $\hat{\varphi}: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\varphi(x\text{-axis}) = z\text{-axis}$ and $\hat{\varphi}$ isometry.

$$\hat{\varphi} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$ is an isometry that works.

(Linear Algebra Interlude) $A: \mathbb{R}^3 \to \mathbb{R}^3$,

A preserve distances in $\mathbb{R}^3$ $\Leftrightarrow$ $\langle Av, Aw\rangle = \langle u, w\rangle$

A preserve the inner product

$\Leftrightarrow$ $\langle v, A^t A w\rangle = \langle v, w\rangle$ for all $u, w \in \mathbb{R}^3$

$\Leftrightarrow$ $\boxed{A^t A = id}$ (Equivalently $AA^t = id$)

(Pf-cont) we want the composition $(\varphi^{-1} \circ R_{z,\theta} \circ \varphi)|_{S^2}$

$$(x, y, z) \xrightarrow{\varphi} (z, -y, x) \xrightarrow{R_{z,\theta}} (\cos\theta\, z + \sin\theta\, y,\ \sin\theta\, z - \cos\theta\, y,\ x)$$

$$\xrightarrow{\varphi^{-1}} (x,\ \cos\theta\, y - \sin\theta\, z,\ \cos\theta\, z + \sin\theta\, y)$$

Remark: for $R_{y,\theta}$, a choice of $\hat{\varphi}$ is

$$\hat{\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

§2. Reflections in S^2

The data needed for reflections $F: S^2 \to S^2$ is a line $L \subseteq S^2$. But wait what is a line?

<u>Def</u> $L \subseteq S^2$ is a line iff $\exists P, Q \in S^2$ s.t.

$$L = \{ R \in S^2 ,\ d_{S^2}(R, P) = d_{S^2}(R, Q) \}$$

→ 详见 Lecture 17