Lecture 4

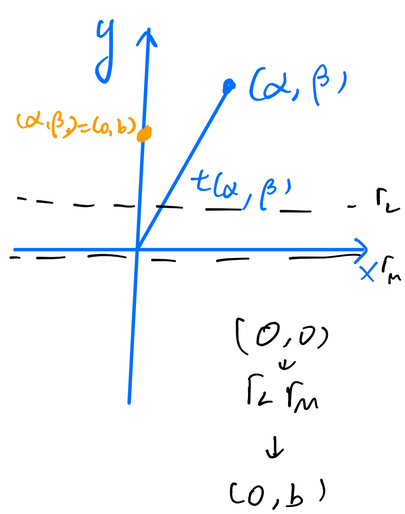
Thm: 1a) Any translation $t_{(\alpha,\beta)}$ is the composition of 2 reflections "product"

r_m, r_L, $M, L \subseteq \mathbb{R}^2$

1b) Any rotation $R_{\theta,P}$ is the composition of 2 reflections

2) Any product/compos. of 2 reflect. $r_m \cdot r_L$ is either a rotation or translation

Pf of (1a): We have $t_{(\alpha,\beta)}$. We want $M, L \subseteq \mathbb{R}^2$ lines s.t. we obtain $t_{(\alpha,\beta)} = r_M r_L$. By conjugation, we assume



that $(\alpha, \beta) = (0, b)$

Choose $M = \{y = \frac{b}{2}\}$, $L = \{y = 0\}$ We want

$t_{(0,b)} = \bar{r}_{L_{\frac{b}{2}}} \bar{r}$.

$$\bar{r}_{L_{\frac{b}{2}}} \bar{r}(x, y) = \underbrace{\bar{r}_{L_{\frac{b}{2}}}}_{t_{(0,\frac{b}{2})} \bar{r}\, t_{(0,-\frac{b}{2})}} (x, -y)$$

$$= t_{(0,\frac{b}{2})} \bar{r} (x, -y - \frac{b}{2})$$

$$= t_{(0,\frac{b}{2})} (x, y + \frac{b}{2})$$

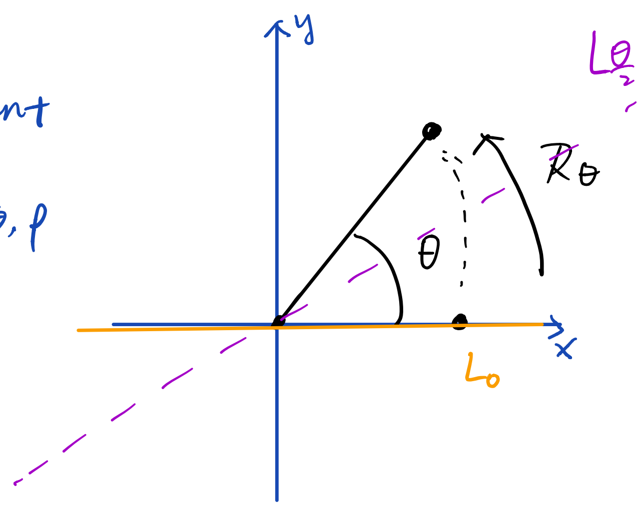
$$= (x, y+b) = t_{(0,b)}(x, y)$$

<u>Pf of (1b)</u>

We have $R_{\theta,p}$. we want $M, L \subseteq \mathbb{R}^2$, s.t. $\bar{r}_M \bar{r}_L = R_{\theta,p}$

By conjugating, we assume $P = (0,0)$

We consider $L_0 = \{y = 0\}$ and $L_{\frac{\theta}{2}} = \{ y = \frac{\theta}{2} . r\sin\frac{\theta}{2} \}$



$$\overline{\Gamma_{L_{\frac{\theta}{2}}}} \cdot \overline{\Gamma_{L_0}}(x,y) = \overline{\Gamma_{L_{\frac{\theta}{2}}}}(x,-y)$$

$$= R_{\frac{\theta}{2}}\overline{\Gamma_0}R_{-\frac{\theta}{2}}(x,-y)$$

$$= R_{\frac{\theta}{2}}\overline{\Gamma_0}\left(\cos\frac{\theta}{2}x - \sin(-\frac{\theta}{2})(-y),\ -\sin\frac{\theta}{2}x + \cos(-\frac{\theta}{2})y\right)$$

$$= R_{\frac{\theta}{2}}\overline{\Gamma_0}\left(\cos\frac{\theta}{2}x - \sin(\frac{\theta}{2})y,\ -\sin\frac{\theta}{2}x - \cos(\frac{\theta}{2})y\right)$$

$$= R_{\frac{\theta}{2}}\left(\cos\frac{\theta}{2}x - \sin\frac{\theta}{2}y,\ \sin\frac{\theta}{2}x + \cos\frac{\theta}{2}y\right)$$

$$= \left[\cos\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}x - \sin\frac{\theta}{2}y\right) - \sin\frac{\theta}{2}\cdot\left(\sin\frac{\theta}{2}x + \cos\frac{\theta}{2}y\right),\ \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}x - \sin\frac{\theta}{2}y\right) + \cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}x + \cos\frac{\theta}{2}y\right)\right]$$

$$= \left(\cos\theta\cdot x - \sin\theta\cdot y,\ \sin\theta\cdot x + \cos\theta\, y\right)$$

$$= R_\theta(x,y)$$

Note : $\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos(2\frac{\theta}{2})$

$$2\cos(\frac{\theta}{2})\sin\frac{\theta}{2} = \sin\theta$$