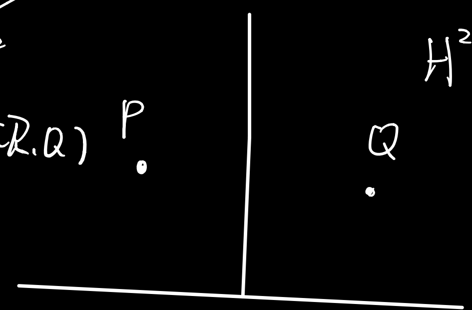
Lecture 24

Main Question What are lines in (H^2, d_{H^2})?

$R \in H^2$

s.t. $d(R,P) = d(R,Q)$

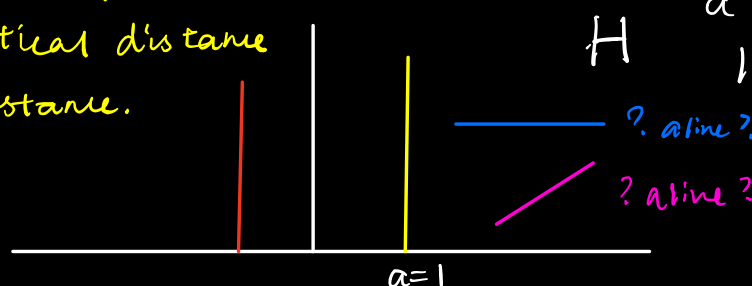


$$d_{H^2}(z,w) = \underbrace{\log\left(\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\right)}_{\text{distance.}}$$

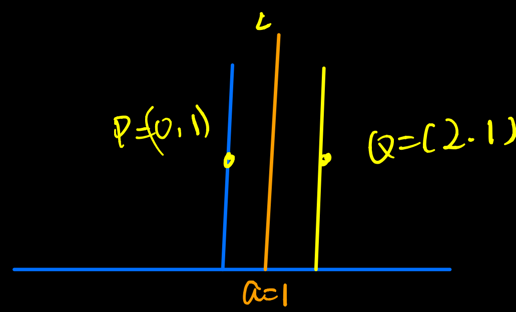
§ 1. First type of line

Thm I. Let $L \subseteq H^2$ be of the form

$L = \underbrace{\{x = a\}}_{\text{vertical distance in } d_{\mathbb{R}^2} \text{ distance.}}$ for $a \in \mathbb{R}$, then L is a hyperbolic line.



Example:



Choose $P=(0,1)=i$

$Q=(2,1)=2+i$ $\qquad \bar{Q}=2-i$

Since $\tanh\left(\dfrac{d_{H^2}(P,Q)}{2}\right)=\dfrac{|z-w|}{|z-\bar{w}|}$, then

for $R=a+ib\in H^2$

$$d_{H^2}(\underset{z}{R},\underset{w}{P}) = d_{H^2}(\underset{z}{R},\underset{w}{Q})$$

iff

$$\underbrace{\frac{|a+(b-1)i|}{|a+(b+1)i|}}_{\left(\tanh\left(\frac{d_{H^2}(R,P)}{2}\right)\right)} = \underbrace{\frac{|(a-2)+(b-1)i|}{|(a-2)+(b+1)i|}}_{\left(\tanh\left(\frac{d_{H^2}(R,Q)}{2}\right)\right)}$$

$$\Updownarrow$$

$$\frac{a^2+(b-1)^2}{a^2+(b+1)^2}=\frac{(a-2)^2+(b-1)^2}{(a-2)^2+(b+1)^2}$$

$$(a-2)^2(b-1)^2 + a^2(b+1)^2 = (a-2)^2(b+1)^2 + a^2(b-1)^2$$

$$\Updownarrow$$

$$(a-2)^2((b+1)^2-(b-1)^2) + a^2((b-1)^2-(b+1)^2)=0$$

$$\Updownarrow$$

$$4b(a-2)^2 - 4b\,a^2 = 0$$

since $b>0$

$$(a-2)^2 - a^2 = 0$$

$$-4a+4 = 0 \qquad \boxed{a=1}$$

$L = \{(x,y) \in H^2 \mid x=a, a \in R\}$

Pf of thm: The map $t_\alpha : H^2 \xrightarrow{\alpha \in R} H^2$,

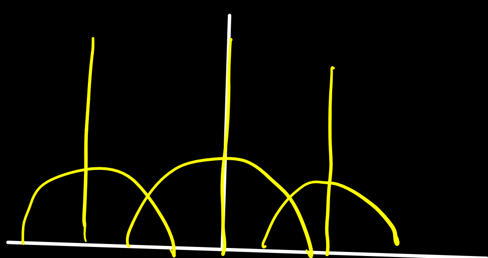
$$z \longmapsto z+\alpha$$

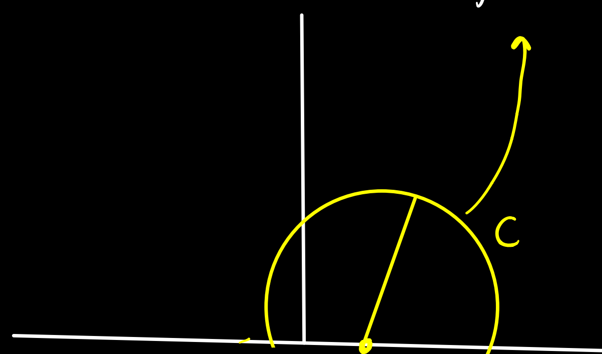
$(x,y) \qquad (x+\alpha, y)$ preserves distance.

hence, it preserves lines. In particular, in set $t_\alpha(L)$ is a line, and thus any Euclidean vertical lines is a hyperbolic line.

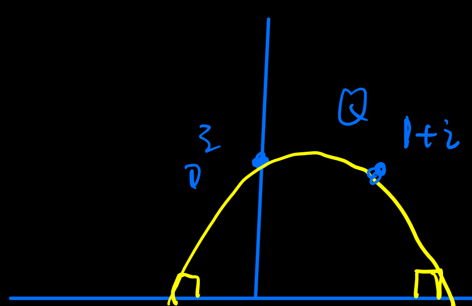
§2. Second type of line.

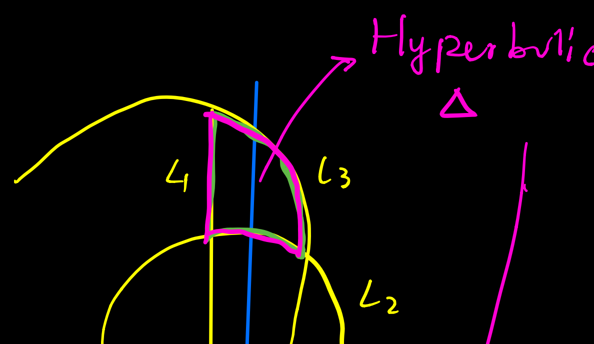
Thm II: Let $C \subseteq H^2$ be a Euclidean semicircle centered at x axis. Then C is a hyperbolic line.





Example 1. Let $P = i$, $Q = 1+i$





Thm For $T_{\alpha,\beta,\gamma}$ hyperbolic

$$\boxed{\alpha+\beta+\gamma = \pi - \text{Area}(T_{\alpha,\beta,\gamma})}$$