§1.

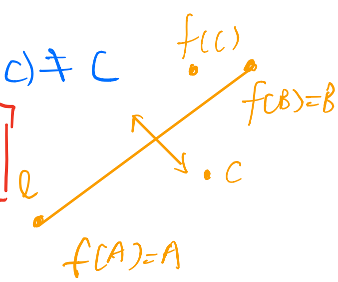
Thm 1 Every isometry of $\mathbb{R}^2$ is a product of 1, 2 or 3 reflections

Proof: Consider A, B, C non colinear.

Then the cases are:

⓪ if $f(A)=A$, $f(B)=B$, $f(C)=C$, the $f = Id$

① $f(A)=A$, $f(B)=B$, $f(C)\neq C$



we argue that $\boxed{\bar{r}_L(C) = f(C)}$

Indeed, $l_{C,f(C)}$ satisfies that $\bar{r}_{l_{C,f(C)}}(C) = f(C)$

$\bar{r}_L(A) = A$

$\bar{r}_L(B) = B$

If we conclude $l_{C,f(C)} = L$, the $\bar{r}_L(C) = f(C)$

$\downarrow$

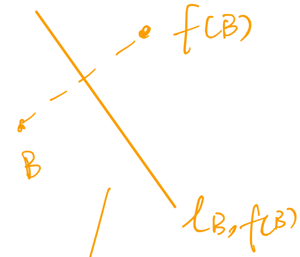
L contains A & B

$$\begin{array}{ccc} d(A,C) & = & d(A,C) \\ \| & & \| \\ d(A,f(C)) & = & d(f(A),f(C)) \end{array} \leftarrow \text{isometry} \Bigg\} \boxed{A \in l_{C,f(C)}} \Rightarrow \boxed{\bar{r}_L = f}$$

$\uparrow$ $A = f(A)$

② $f(A)=A,\quad f(B)\neq B,\quad f(C)\neq C$

consider $\ell_{B,f(B)}$, then $\overline{\Gamma}_{\ell_{B,f(B)}}(B) = f(B)$



Do we have $\overline{\Gamma}_{\ell_{B,f(B)}}(A) = f(A)$

Since $d(A,B) = d(A, f(B))$,

because $f(A) = A$.

$$d(A,B) = d(A, f(B)) = d(f(A), f(B))$$

$A \in \ell_{B,f(B)}$, and then

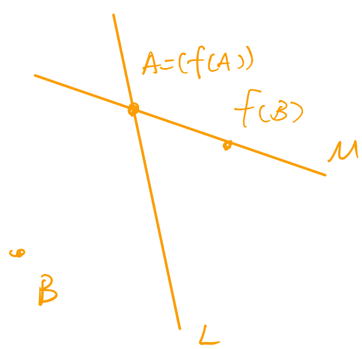
$\overline{\Gamma}_{B,f(B)}(A) = A = f(A)$

<u>Remark</u>: If $\overline{\Gamma}_{\ell_{B,f(B)}}(C) = C$, then $f = \overline{\Gamma}_{\ell_{B,f(B)}}$ and we are done.

We know $\overline{\Gamma}_{\ell_{B,f(B)}}(A) = f(A)$, $\overline{\Gamma}_{\ell_{B,f(B)}}(B) = f(B)$

In general, $f(C) \neq \overline{\Gamma}_{\ell_{B,f(B)}}(C)$

$\boxed{\text{claim } f = \overline{\Gamma}_M \overline{\Gamma}_L}$



↳ sufficient to check

$A = f(A) == \overline{\Gamma}_M\overline{\Gamma}_L(A) = A$ ✓

$f(B) \in M \leftarrow f(B) = f(B) = \overline{\Gamma}_M\overline{\Gamma}_L(B) = f(B)$ ✓

$f(C) = \overline{\Gamma}_M\overline{\Gamma}_L(C)$

Finally, verify that $f(C) = \overline{r}_M \overline{r}_L(C)$

It suffices to show that $M = \ell_{\overline{r}_L(C), f(C)}$

$$\downarrow$$

$$\overline{r}_M(\overline{r}_L(C)) = f(C)$$

We know $A = f(A), f(B) \in M$. * Need to check $A, f(B) \in \ell_{\overline{r}_L(C), f(C)}$

Check $A \in \ell_{\overline{r}_L(C), f(C)}$: $d(A, \overline{r}_L(C)) =$

$$d(A, f(C)) = d(A, C)$$

$$\parallel$$

$$d(\overline{r}_L(A), \overline{r}_L(C))$$

$A, f(A)$

③ Last case: $f(A) \neq A$

$f(B) \neq B$

$f(C) \neq C$

<u>HINT</u>: Start by considering $A, f(A) \in L$. Look at $\overline{r}_L$. In the end,

$$f = \overline{r}_N \overline{r}_M \overline{r}_L, \text{ where } \begin{cases} A, f(A) \in L \\ B, f(B) \in M \\ C, f(C) \in N \end{cases}$$

§ 2. GLIDE REFLECTIONS

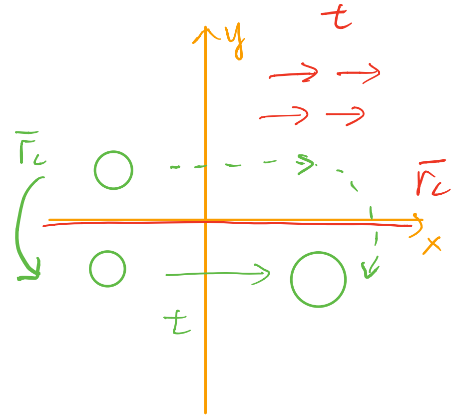
Thm II Every isometry of $\mathbb{R}^2$ is a translation,

2 refs rotation, or a glide reflection → 1 or 3 refs

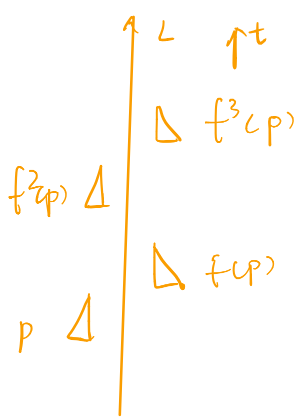
Def of glide ref.

A glide reflection is a composition of a reflection $\bar{r}_L$ and a translation along the line L.

Ex.1 $t_{(1,0)} \circ \bar{r}$



Ex.2 The world's most famous glide ref.



Note, known by Thm I that a glide reflection is a product of refs.

How many? $\boxed{3}$

Remark: a ref. is a glide ref

Prop: The product of 3 reflections is a glide reflection.