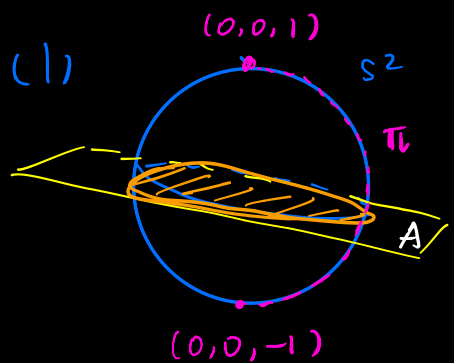
Lecture 17

(S^2, d_{S^2}), then discussed $R_{P,\theta} \in \mathrm{Iso}(S^2)$, here $R_{P,\theta} = R_{\vec{e},\theta}$.

Def: A set $L \subseteq S^2$ is said to be a line if $\exists$ P, Q s.t. L is equidistant set to P, Q.

i.e. $L = \{ R \in S^2 , d_{S^2}(R,P) = d_{S^2}(R,Q) \}$ ∎

Example (1)



Consider the set E

$\{z=0\} \cap \underbrace{\{x^2+y^2+z^2=1\}}_{S^2}$

$\subseteq S^2$

Q: Is E a line?

⚠ $d_{S^2}(P,Q) = \pi$ NOT 2

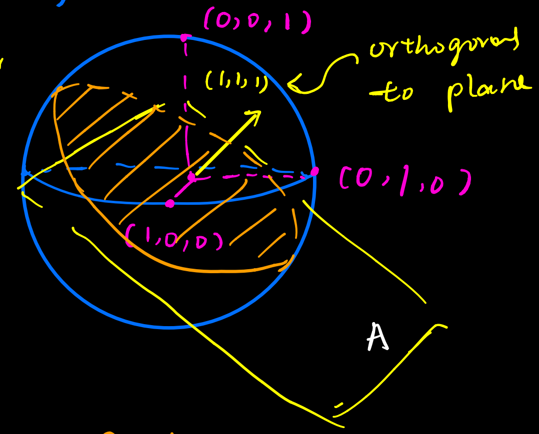
A: Yes. Choose $P=(0,0,1)$, $Q=(0,0,-1)$, then $\forall R \in E$,

we have to compute $d_{S^2}(R,P) = \frac{\pi}{2}$

$d_{S^2}(R,Q) = \frac{\pi}{2}$

(2) Let $\Lambda = \{\underbrace{x+y+z=0}_{\text{Normal vector } (1,1,1)}\} \cap S^2 \subseteq S^2$.

Is Λ a line?



We could choose

$P=(1,1,1)/\sqrt{3}$

$Q=-P$. Alternatively,

$\exists$ isometry bringing

Λ to E in Example 1. Since E is a line,

the Λ is a line.

• intersection part.

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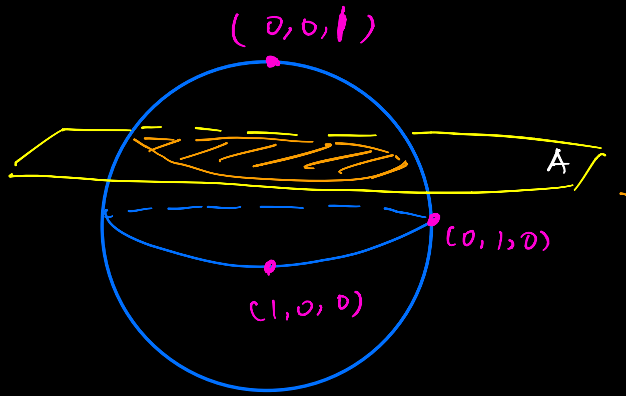
(3) (Non-Ex) Consider $\lambda = \{z = \frac{1}{2}\} \cap S^2 \subseteq S^2$

Q: is λ a line

No, it's not a line.

$\nexists$ $P, Q \in S^2$ s.t. λ is equid. to P, Q ∎



• intersection part or λ

Prop Let $L \subseteq S^2$ be a line. Then $\exists\, A \subseteq R^2$ a 2-plane through the origin s.t.

$$L = A \cap S^2$$

Pf: By def a line is the equidistant set to $P = (\alpha, \beta, \gamma)$, $Q = (\alpha', \beta', \gamma')$

i.e. $L = \{ (x, y, z) \in S^2 : (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = (x-\alpha')^2 + (y-\beta')^2 + (z-\gamma')^2 \}$

From simplifying, we will get 2-planes $A \subseteq R^3$ s.t. $A \cap S^2 = L$.

The equality $(*)$ reads as follows

$$-2x\alpha + \alpha^2 - 2y\beta + \beta^2 - 2z\gamma + \gamma^2 = -2x\alpha' + (\alpha')^2 - 2y\beta' + (\beta')^2 - 2z\gamma' + (\gamma')^2$$

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$\Longleftrightarrow$ $\boxed{(\alpha-\alpha')x+(\beta-\beta')y+(\gamma-\gamma')z=0}$

linearity gives 2-plane

~~through~~ the origin

our $A \subset \mathbb{R}^3$

§2. Reflections in S^2 along a line.

Def Let $E=\{z=0\} \cap S^2$ be the equator (which is a line)

Then, the reflection $r_E : S^2 \to S^2$ is the isometry given by

$$x \longrightarrow x$$

$$y \longrightarrow y$$

$$z \longrightarrow -z$$