S^2 2-sphere, ds^2 distance, $P \in S^2$ points, $L \subseteq S^2$ lines

↳ isometries → $\boxed{\text{Rotation}}$ — 2 fixed pt. — $P \in S^2$, θ angle. $(\varphi, R_{t\theta}, \varphi)$

↳ isometries → $\boxed{\text{reflections}}$ — inf fixed pt — $L \subseteq S^2$ a line $\Longleftrightarrow$ 2-plane $A \subseteq \mathbb{R}^3$ with $(0,0,0) \in A$

$\bar{F}_E$

$\bar{F}_L = \varphi^{-1} \circ r_E \circ \varphi$

(conjugation)

§1 Types of Isometries S^2

Consider $a: \mathbb{R}^3 \to \mathbb{R}^3$ which sends $a(x,y,z) = (-x,-y,-z)$

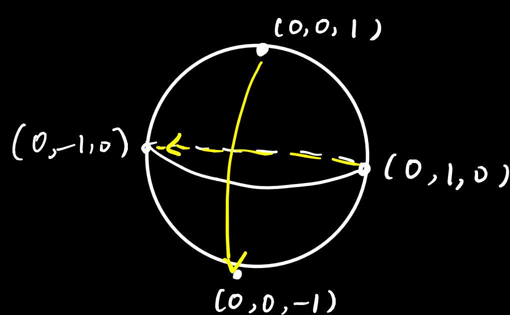
Now, a is isometry of $\mathbb{R}^3$ hence yields $a: S^2 \to S^2$

Q: Does a $S^2 \to S^2$ has fixed pts?

A: if $\underbrace{(-x,-y,-z)}_{a(p)} = (x,y,z)$

then $(x,y,z) = (0,0,0) \notin S^2$

So NO Fixed pts.



This map $a: S^2 \to S^2$ called the antipodal map, is an isometry of S^2 with no fixed pts.

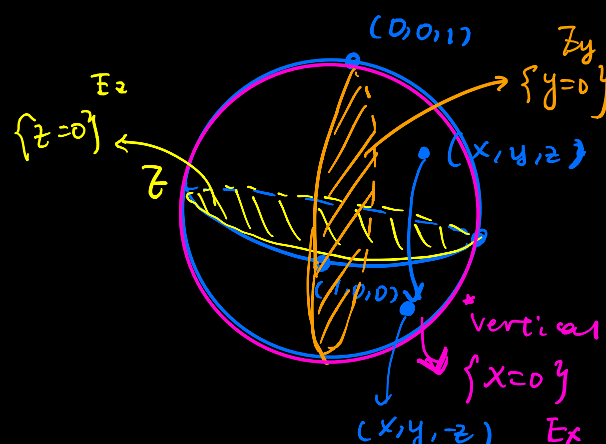
Thm (Classif. of $Iso(S^2)$) Every isometry $f: S^2 \to S^2$ is a compo. of 1, 2 or 3 refs.

Ex: In particular $a: S^2 \to S^2$ is a compo. of reflections.

We have $\bar{r}_{E_z}(x,y,z) = (x,y,-z)$

$\bar{r}_{E_x}(x,y,z) = (-x,y,z)$

$\bar{r}_{E_y}(x,y,z) = (x,-y,z)$



Hence

$$\bar{r}_{E_x}\bar{r}_{E_y}\bar{r}_{E_z}(x,y,z) = (-x,-y,-z)$$

$$\text{i.e.} = a(x,y,z)$$

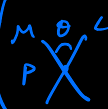
§2. Group of Rotations in $Iso(S^2)$

Thm A rotation $R_{p,\theta} \in Iso(S^2)$ is a product of 2 refs.

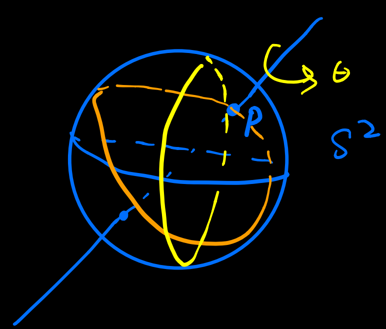
Conversely, compo. of 2 refs is a rotation.

Cor: The compo. of $R_{P,\theta}, R_{Q,\theta}$ is a rotation $R_{R,\varphi}$.

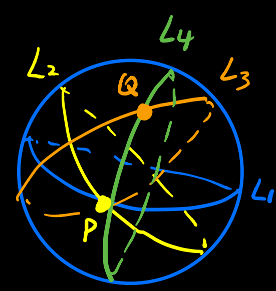
Pf_Cor: Let $R_{P,\theta} = \overline{r}_{L_2}\overline{r}_{L_1}$, $R_{Q,\phi} = \overline{r}_{L_4}\overline{r}_{L_3}$. We need to find lines $L_5, L_6 \subseteq S^2$ s.t. $R_{Q,\phi} \circ R_{P,\theta} = \overline{r}_4\overline{r}_3\overline{r}_2\overline{r}_1 = \overline{r}_6\overline{r}_5$

Ex Following our $\mathbb{R}^2$ results, ( $R_{P,\theta} = \overline{r}_M\overline{r}_L$)

We observe that $R_{P,\theta} \in \mathrm{Iso}(S^2)$ be expressed as $\overline{r}_L\overline{r}_{L'}$ with $P \in L \cap L'$ with min. angle between $(\measuredangle)(L, L') = \theta/2$



Pf_cnt'd



Now, choose $L \subseteq S^2$ the line with $P, Q \in L$ and find $\underset{L_6}{N}, \underset{L_5}{M} \subseteq S^2$ lines s.t. $\overline{r}_{L_2}\overline{r}_{L_1} = \overline{r}_L\overline{r}_M$

$\overline{r}_{L_4}\overline{r}_{L_3} = \overline{r}_N\overline{r}_L$ Then, $\overline{r}_N\underbrace{\overline{r}_L\overline{r}_L}_{id}\overline{r}_M = \overline{r}_N\overline{r}_M$