Lecture 2

§1. What function $f: \mathbb{R}^2 \to \mathbb{R}^2$ preserve distance?

By <u>def</u>: $f: \mathbb{R}^2 \to \mathbb{R}^2$ preserve distance d iff $d(P, Q) = d(f(P), f(Q))$

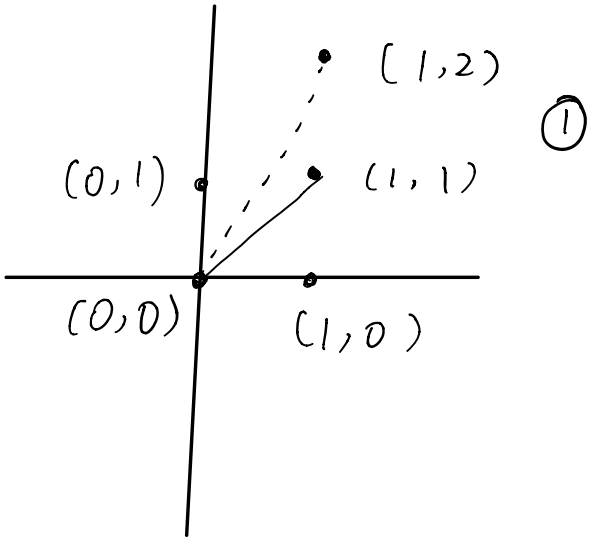
The 3 main classes of such f's

<u>Example</u>

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $(x, y) \to$ to	preserve d?
(1) $(x, y) \to (x, y^2)$	No, because distance is diff.....
(2) $(x, y) \to (x+1, y)$	Yes (just like moving plane/axis)
$(x, y) \to (x+19, y+23)$	
(3) $(x, y) \to (-x, -y)$	Yes (reflecting axis)
(4) $(x, y) \to (\cos x, \cot y)$	No
(5) $(x, y) \to (0, e^{x \log(y^2+1)})$	
$(0,0) \to (0,1)$	same pt → No.

(6) $(x, y) \to (\cos x, \sin y)$

$(0,0) \to (1,0)$
$(2\pi, 0) \to (1,0)$
> map to same point → No



preserve distance

① Translation ← $\alpha, \beta \in R$

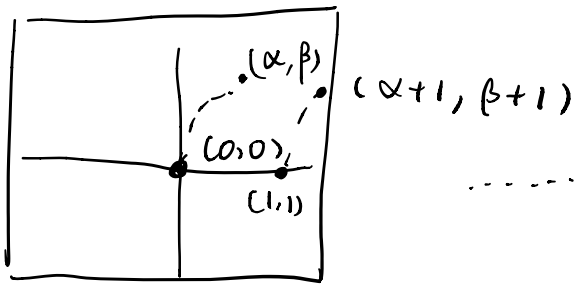
② Rot ← θ angle

③ Ref ← axis of reflection

✳ Translation : give $(\alpha, \beta) \in R^2$, then define :

$$t_{(\alpha,\beta)}(x, y) := (x+\alpha, y+\beta)$$

ex. of $R^2 \to R^2$



Lemma: $\forall (\alpha, \beta) \in \mathbb{R}^2$, $t_{(\alpha,\beta)}$ preserve d.

Pf: We need to check holds

eq1: $\sqrt{(x_1-x_2)^2+(y_1-y_2)}$

between (P, Q) $P(x_1, y_1)$ $Q(x_2, y_2)$

RHS:

$f(P) : (x_1+\alpha, y_1+\beta)$ $f(Q) = (x_2+\alpha, y_2+\beta)$

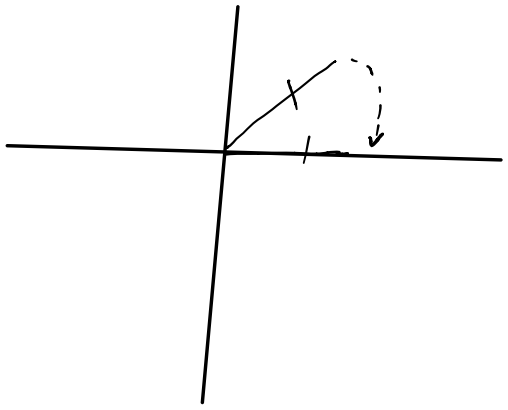
eq2: $\sqrt{((x_1+\alpha)-(x_2+\alpha))^2+((y_1+\beta)-(y_2+\beta))^2}$

then, eq1 = eq2.

✱ Rotation: given $\theta \in \mathbb{R}$, an angle.

$$\mathcal{R}_\theta(x, y) = (\cos\theta \cdot x - \sin\theta \cdot y, \sin\theta \cdot x + \cos\theta \cdot y)$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Lemma: $\forall\theta$, $\mathcal{R}_\theta\, \mathbb{R}^2 \to \mathbb{R}^2$, preserve d.

If need to check (#)

LHS: $d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

RHS: $d(\mathcal{R}_\theta P, \mathcal{R}_\theta Q) = \sqrt{\left[(\cos\theta \cdot x_1 - \sin\theta\, y_1) - [(\cos\theta\, x_2 - \sin\theta\, y_2)\right]^2 - (\sin\theta \cdot x_1 + \cos\theta \cdot y_1) - (\sin\theta\, x_2 + \cos\theta \cdot y_2)]^2}$