$\underline{Q1}$: if $P, Q \in H^2$, what is $d_{H^2}(P, Q)$?

$\longrightarrow$ We know $\overset{(\text{week } 1)}{d_{\mathbb{R}^2}}$, d_{S^2} (W5/W6)

$\underline{Q2}$: Given $\gamma \subseteq H^2$ a curve, what is the length of that curve?

$$\text{length}(\gamma) = \int_0^1 \| \gamma'(t) \| \, dt.$$

§1: The distance d_{H^2}

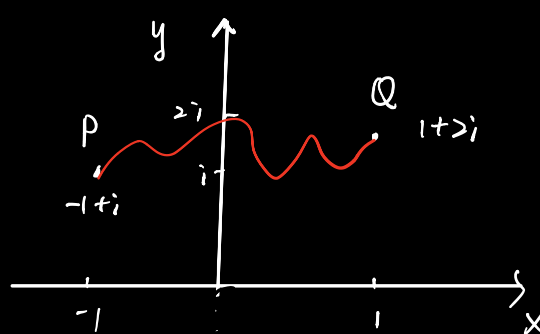
$$H^2 := \{ (x, y) \in \mathbb{R}^2 ; y > 0 \}$$

It is useful to write

$z = x + iy$.

$H^2 = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

$\uparrow$ complex numbers



With $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$
Real imaginary

Hyperbolic plane (by def^{n})

a.k.a upper half

Def The distance $d_{H^2}(P,Q)$ is defined by

$$d_{H^2}(P,Q) = \left(\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|} \right)$$

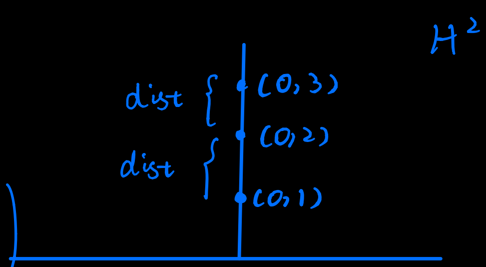
where $P=z$, $Q=w \in H^2$ $\qquad \bar{w} = x - iy$ if $w = x+iy$

Exercise: d_{H^2} is a distance, i.e. $d_{H^2} \geq 0$

and $d_{H^2}(P,Q) = 0$ iff $P=Q$

Also, $d_{H^2}(P,Q) = d_{H^2}(Q,P)$

Example



$P=(0,1)$ $\quad Q=(0,2)$

$$d_{H^2}(P,Q) = \ln\left(\frac{|i-(-2i)| + |i-2i|}{|i-(-2i)| - |i-2i|} \right)$$

$P = z = 0 + i1 = i$

$Q = w = 0 + 2i = 2i$

$$= \log\left(\frac{|3i| + |-i|}{|3i| - |-i|} \right)$$

$$= \log\left(\frac{4}{2}\right) = \boxed{\log 2}$$

V

Note ⚠ : $d_{H^2}((0,2),(0,3)) = \boxed{\log 3 - \log 2}$

$\neq d_{H^2}((0,1),(0,2))$

Preposition: If $P, Q \in H^2$ lie in the same vertical line, i.e. $\mathrm{Re}(P) = \mathrm{Re}(Q)$

then $d_{H^2}(P,Q) = \ln\left(\dfrac{\mathrm{Im}(Q)}{\mathrm{Im}(P)}\right)$

$= \ln(\mathrm{Im}(Q)) - \ln(\mathrm{Im}(P))$

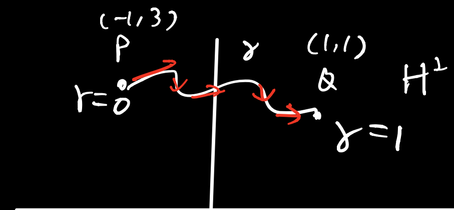
Remark This is NOT true if $\mathrm{Re}(P) \neq \mathrm{Re}(Q)$

§2. Length of the curves $\gamma \subseteq H^2$:

the distance d_{H_2} at infinitie Level read

as $ds = \dfrac{\sqrt{dx^2 + dy^2}}{y}$ $y > 0$

Let $\gamma : [0,1] \longrightarrow H^2$



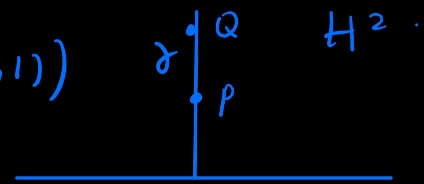
$$\gamma(t) = (x(t), y(t))$$

$$\text{length}(\gamma) := \int_0^1 \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{\underline{\underline{y(t)}}}\, dt$$

↳ $y(t)$ **NOT** $y'(t)$.

<u>Example</u> consider $P = (0,1)$ $Q = (0,2)$

$$\gamma = (0,1) + t\big((0,2) - (0,1)\big)$$
$$= (0, 1+t)$$



Straight vertical seg. from $P \to Q$

The tangent is $\gamma(t) = (0,1)$

$$|\gamma'(t)| = 1$$

↓

length(γ)

$$= \int_0^1 \frac{1}{1+t} = \log(1+t)\Big]_0^1$$
$$= \boxed{\log 2}$$

↙

a curve reach that dist is found.