1. $A \le_m \bar{A} \Rightarrow \bar{A} \le_m A$

A is TR, then $\bar{A}$ is TR, so A is Co-TR

$\because$ A is both TR and Co-TR, thus A is decidable.

2. Suppose T is decidable, then exists a TM A deciding T.

Construct M', s.t. $\langle M, w\rangle \in A_{TM}$, and prove $A_{TM} \le_M T$:

(1) for input x, if $x = 0^n 1^n$, then accept;

(2) otherwise, feed M with x, if M accepts then M' accepts.

when $\langle M, x\rangle \in A_{TM} \Rightarrow \langle M'\rangle \in T$

$\langle M, x\rangle \notin A_{TM} \Rightarrow \langle M'\rangle \notin T$

So $A_{TM} \le_M T$, and A_{TM} is undecidable. Proof.

3. (1) $\Leftarrow$: $A \le_m A_{TM} \Rightarrow A$ is TR

$\because$ A_{TM} is TR, so A is TR.

(2) $\Rightarrow$: A is TR $\Rightarrow$ $A \le_m A_{TM}$ then there is a TM M' recognizing A

So $x \in A \Rightarrow \langle M, x \rangle \in A_{TM}$, $A \le_m A_{TM}$

$\therefore$ A is TR $\Leftrightarrow$ $A \le_m A_{TM}$.

4. Suppose L is TR, then prove $L_{all} \le_m L$, i.e. $f(\langle M \rangle) = \langle M_1, M_2 \rangle$:

1. M_1 accepts x.
2. M_2 accepts x if M accepts x.

Thus, $\langle M \rangle \in L_{all} \Rightarrow \langle M_1, M_2 \rangle \in L$; $\langle M \rangle \notin L_{all} \Rightarrow \langle M_1, M_2 \rangle \notin L$

f can be computed.

However, L_{all} is not TR, proof.

5. (a) Suppose $OVERLAP_{DFA,TM}$ ($OL_{D,T}$) is decidable

then there is a TM A recognizing $OL_{D,T}$.

Construct a $\langle D, M' \rangle$ s.t. $\langle M, w \rangle \in A_{TM}$. $A_{TM} \leq_m OL_{D,T}$.

For input x: (1) D accepts x.

(2) if M accepts w, then M' accepts w. Otherwise, M' rejects w.

Thus, $\langle M, w \rangle \in A_{TM} \Rightarrow \langle D, M' \rangle \in OL_{DT}$

$\langle M, w \rangle \notin A_{TM} \Rightarrow \langle D, M' \rangle \notin OL_{D,T}$

And $f\langle M, w \rangle = \langle D, M' \rangle$ is computable.

However, $A_{TM} \leq_m OL_{D,T}$, A_{TM} is undecidable. Proof.

(b) Equals finding a TM. recognizing $OL_{D,T}$.

For input $\langle D, M \rangle$, use ordered enumerator to output every possible strings, making D and M recognizable.

(1) if a string can be accepted by both D and M, accept it.

(2) otherwise, wait for another string.

So M can recognize $OL_{D,T}$.

(c) $\overline{OL_{D,T}}$ is TR, construct a TM the same as above:

(1) if a string can be accepted by both D and M, reject it.

(2) if a string can be accepted by either D or M, accept it.

(3) otherwise, wait for another string.

6. (a) ✓ $f\langle M\rangle = \langle M\rangle \Rightarrow A \leq_m A$.

(b) ✓ $f\langle M_1\rangle = \langle M_2\rangle$, $h\langle M_2\rangle = \langle M_3\rangle \Rightarrow g\langle M_1\rangle = h(f\langle M_1\rangle) = \langle M_3\rangle$

(c) ✗ if A is CFG, and $B = \bar{A}$, then $A \leq_m A$ but $\bar{A} \not\leq_m A$.

(d) ✓ A is decidable, a^*b^* is regular.

(e) ✗ $A_{TM} \leq_m OL_{D,T}$, but $OL_{D,T} \not\leq_m A_{TM}$.

(f) X $L(A) = \{0\}$, $L(B) = \{1\}$,

In $A \le_m B \Rightarrow$ $f(0) = 1$, $f(1) = 0$

In $B \le_m A \Rightarrow$ $f(1) = 0$, $f(0) = 1$

But $A \neq B$