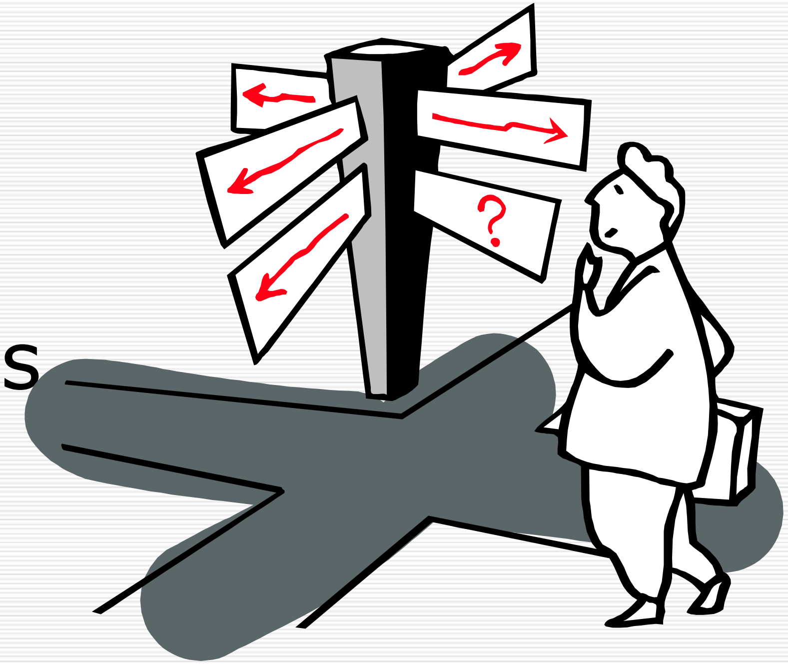
1.3 Properties of Regular Languages

- Pumping Lemma
- Closure properties
- Decision properties
- Minimization of DFAs



Limits on the Power of FA

- Is every language describable by a sufficiently complex regular expression?
- If someone asks you to design a FA for a language that seems hard, how do you know when to give up?
- An FA is limited on finite memory, i.e. finite set of states

Non-Regular Languages

- Is this language regular?
  - L = {w | w has equal numbers of ‘01’ and ‘10’}

    = 0(0+1)*0+1(0+1)*1
  - L = {w | w has equal numbers of ‘0’ and ‘1’}
- How to prove that there is no Finite Automaton recognizing a given language?
- Every regular language satisfies pumping lemma --- necessary condition

Pumping Lemma

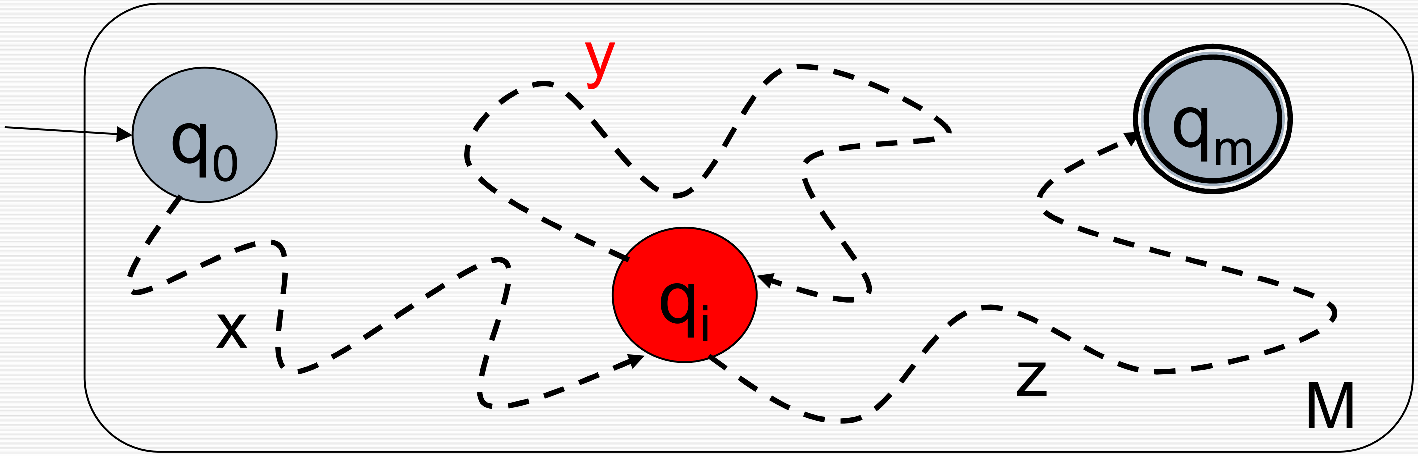
- If L is a regular language, then there exists a constant n (pumping length) such that every string w in L with $|w| >= n$, can be written as $w = xyz$, where:
  - $|y| > 0$
  - $|xy| <= n$
  - For all $i >= 0$, xy^iz is also in L

Proof of Pumping Lemma

- L is regular, then
  - Let M be a FA that recognizes L.
  - Set n = number of states of M.
  - Consider $w \in L$, say $w = a_1 a_2 \ldots a_m$ $(m >= n)$. Let $q_i = \delta^*(q_0, a_1 \ldots a_i)$, q_0 is the start state.
  - Since there are only n different states, two of $q_0 q_1 \ldots q_m$ ($m >= n$) must be the same (**pigeon hole principle**); say $q_i = q_j$ where $0 <= i < j <= n$ (among the first $n+1$ states).

Proof of Pumping Lemma (cont'd)

- Let $x=a_1\ldots a_i$; $y=a_{i+1}\ldots a_j$; $z=a_{j+1}\ldots a_m$.
- Then by repeating the loop from q_i to q_i with label $a_{i+1}\ldots a_j$ zero or more times, we can show that xy^iz is accepted by A.



Use Pumping Lemma

- Use pumping lemma to prove that L is not regular
  - assume L is regular;
  - then there exists a pumping length *n*;
    - We may not know what *n* is, but we can work the rest of the “game" with *n* as a parameter.
  - select a string $w \in$ L such that $|w|>=n$;
  - Applying the PL, we know *w* can be broken into *xyz*, satisfying the PL properties
  - We derive a contradiction by picking $i>=0$ such that xy^iz is not in L (whatever *x*, *y*, *z* are)

Example 1.38:

- $B = \{0^n1^n \mid n>=0\}$
- Assume B is regular, let p be the pumping length
- Choose $w = 0^p1^p$ in B ($|w| > p$)
- Applying PL, $w = xyz$, where $|y|>0$, such that xy^iz in B for all $i >= 0$
- Three possible cases for y:
  - $y = 0^k$, $(k>0)$, then $xyyz = 0^{p+k}1^p$, not in B
  - $y = 1^k$, $(k>0)$, then $xyyz = 0^p1^{k+p}$, not in B
  - $y = 0^k1^l$, $(k+l>0)$, then $xyyz = 0^p1^l0^k1^p$, not in B
- Contradiction, B is not regular

Example 1.39:

- $C = \{w \mid w$ has equal number of 0s and 1s$\}$
- Assume C is regular, let p be the pumping length
- Choose $w = 0^p1^p$ in C ($|w| > p$)
  - We did not choose $w = (01)^p$, why?
- Applying PL, $w = xyz$, where $|y|>0$, $|xy|<=p$, such that xy^iz in C for all $i >= 0$
- Since $|xy|<=p$, then $y = 0^k$, $(k>0)$
- Then $xyyz = 0^{p+k}1^p$, not in C
- Contradiction, C is not regular

Example 1.40:

- $F = \{ww \mid w \in \{0, 1\}^*\}$
- Assume F is regular, let p be the pumping length
- Choose $w = 0^p 1 0^p 1$ in F ($|w| > p$)
  - We did not choose $w = 0^p 0^p$
- Applying PL, $w = xyz$, where $|y|>0$, $|xy|<=p$, such that xy^iz in F for all $i >= 0$
- Since $|xy|<=p$, then $y = 0^k$, $(k>0)$
- Then $xyyz = 0^{p+k} 1 0^p 1$, not in F
- Contradiction, F is not regular

Example 1.41:

- $D = \{0^{n^2} \mid n >= 0\}$
- Assume D is regular, let p be the pumping length
- Choose $w = 0^{p^2}$ in D ($|w| > p$)
- Applying PL, $w = xyz$, where $|y|>0$, $|xy|<=p$, such that xy^iz in D for all $i >= 0$
- $|xyz| = p^2$, $0 < |y| <= p$, then $p^2 < |xyyz| <= p^2+p < (p+1)^2$, so xyyz is not in D
- Contradiction, D is not regular

Example 1.42:

- $E = \{0^i 1^j \mid i > j\}$
- Assume E is regular, let p be the pumping length
- Choose $w = 0^{p+1}1^p$ in E ($|w| > p$)
- Applying PL, $w = xyz$, where $|y|>0$, $|xy|<=p$, such that xy^iz in D for all $i >= 0$
- Since $|xy|<=p$, then $y=0^k$, ($k>0$)
- Then $xyyz = 0^{p+1+k}1^p$, in E…
- However $xy^0z = xz = 0^{p+1-k}1^p$, not in E
- Contradiction, E is not regular