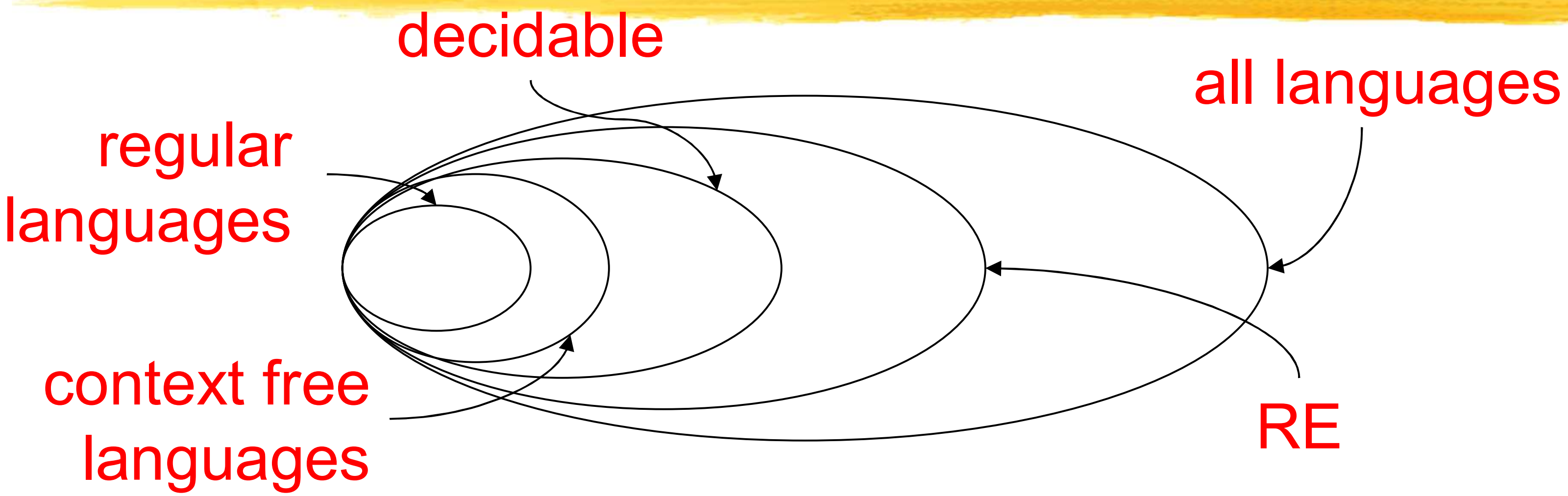
4. Decidability

- The diagonalization method
- The halting problem is undecidable
- co-HALT is unrecognizable

Undecidability



decidable $\subset$ RE $\subset$ all languages

our goal: prove these containments proper

Countable and Uncountable Sets

- the natural numbers $\mathbf{N} = \{1,2,3,\ldots\}$ are countable

- Definition: a set S is countable if it is finite, or it is infinite and there is a bijection

$$f: \mathbf{N} \to S$$

Example Countable Set

- The positive rational numbers Q = {m/n | m, n $\in$ **N** } are countable.
- Proof:

…

1/1	1/2	1/3	1/4	1/5	1/6	…
2/1	2/2	2/3	2/4	2/5	2/6	…
3/1	3/2	3/3	3/4	3/5	3/6	…
4/1	4/2	4/3	4/4	4/5	4/6	…
5/1	…					

Example Uncountable Set

Theorem: the real numbers **R** are NOT countable (they are "uncountable").

- How do you prove such a statement?
  - assume countable (so there exists bijection f)
  - derive contradiction (some element not mapped to by f)
  - technique is called diagonalization (Cantor)

Example Uncountable Set

- Proof:
  - suppose **R** is countable
  - list **R** according to the bijection f:

n	f(n)
1	3.14159…
2	5.55555…
3	0.12345…
4	0.50000…
…	

Example Uncountable Set

- Proof:
  - suppose **R** is countable
  - list **R** according to the bijection f:

n	f(n)
1	3.14159…
2	5.55555…
3	0.12345…
4	0.50000…
…	

set $x = 0.a_1a_2a_3a_4\ldots$

where digit $a_i \neq i^{th}$ digit after decimal point of f(i) (not 0, 9)

e.g. x = 0.2312…

x cannot be in the list!

Non-RE Languages

<u>Theorem</u>: there exist languages that are not Recursively Enumerable.

Proof outline:

- the set of all TMs is countable
- the set of all languages is uncountable
- the function L: {TMs} $\rightarrow${languages} cannot be onto

Non-RE Languages

- Lemma: the set of all TMs is countable.
- Proof:
  - each TM M can be described by a finite-length string <M>
  - can enumerate these strings, and give the natural bijection with **N**

Non-RE Languages

- Lemma: the set of all languages is uncountable
- Proof:
  - fix an enumeration of all strings s_1, s_2, s_3, …
  (for example, lexicographic order)
  - a language L is described by its characteristic vector χ_L whose i^{th} element is 0 if s_i is not in L and 1 if s_i is in L

Non-RE Languages

- suppose the set of all languages is countable
- list characteristic vectors of all languages according to the bijection f:

n	f(n)
1	0101010…
2	1010011…
3	1110001…
4	0100011…
…	

Non-RE Languages

- suppose the set of all languages is countable
- list characteristic vectors of all languages according to the bijection f:

n	f(n)
1	0101010…
2	1010011…
3	1110001…
4	0100011…
…	

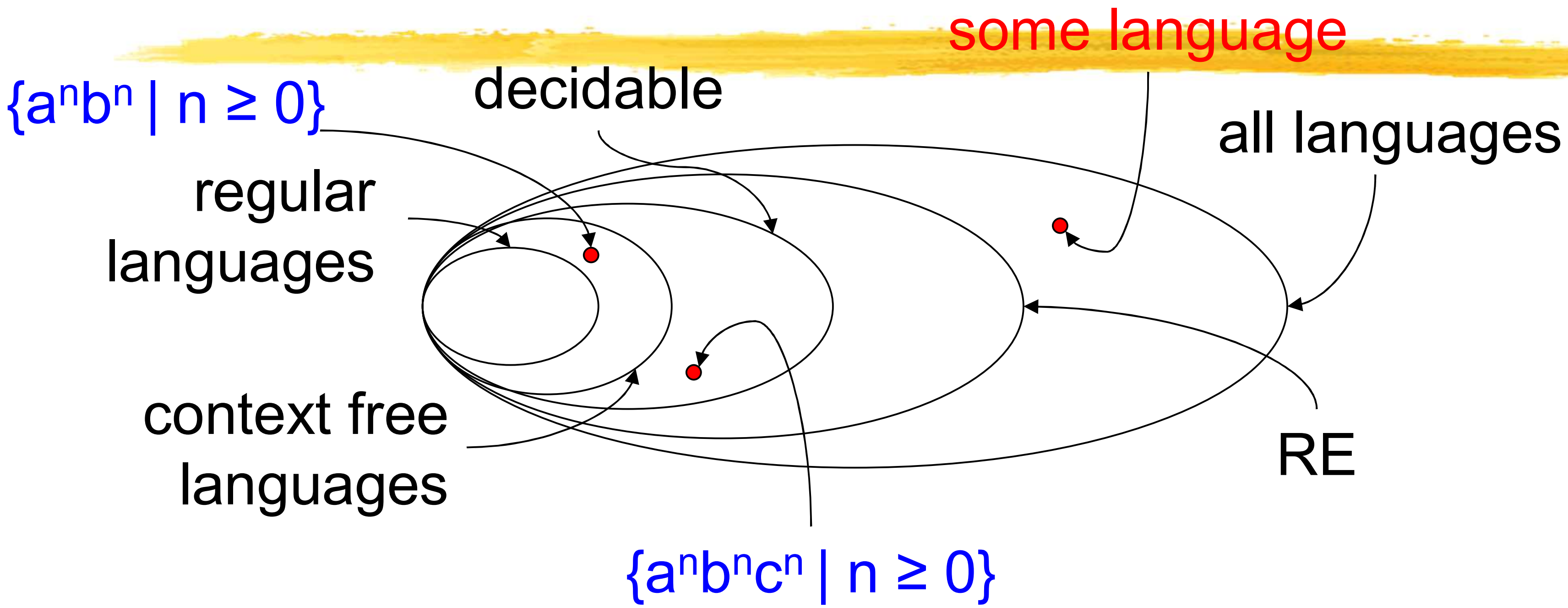
set x = 1101…

where i^{th} digit ≠ i^{th} digit of f(i)

x cannot be in the list!

therefore, the language with characteristic vector x is not in the list

So far…



- This language might be an esoteric, artificially constructed one. Do we care?
- We will show a natural undecidable L next.

The Halting Problem

- Definition of the "Halting Problem":

  HALT = { <M, x> | TM M halts on input x }

- HALT is recursively enumerable.
  - proof? We can easily construct a Universal TM that recognizes the language HALT

- Is HALT decidable?

The Halting Problem

Theorem: HALT is not decidable (undecidable).

Proof:

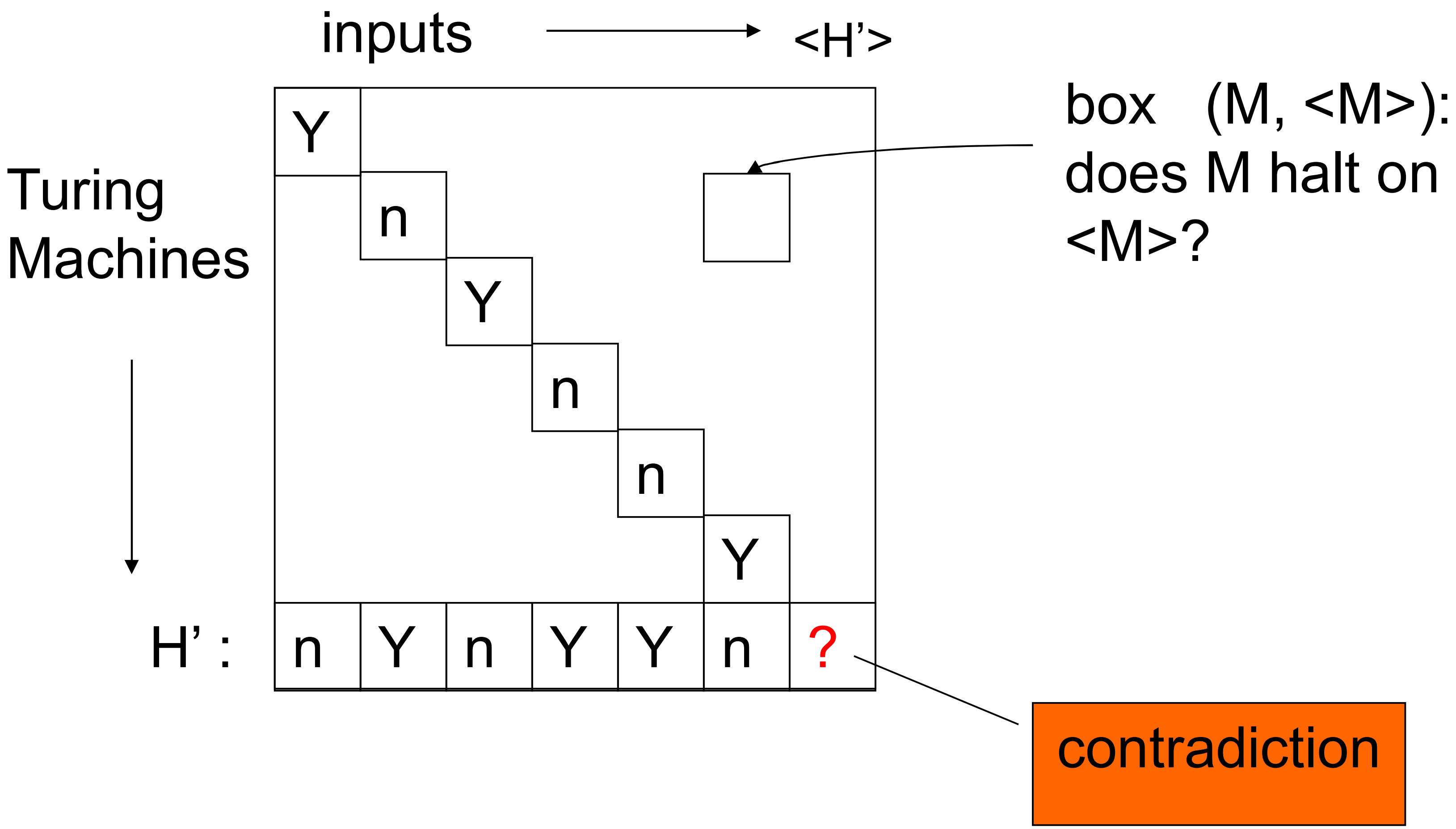
- Suppose TM H decides HALT
- Define new TM H': on input <M>
  - if H accepts <M, <M>>, then loop
  - if H rejects <M, <M>>, then halt

The Halting Problem

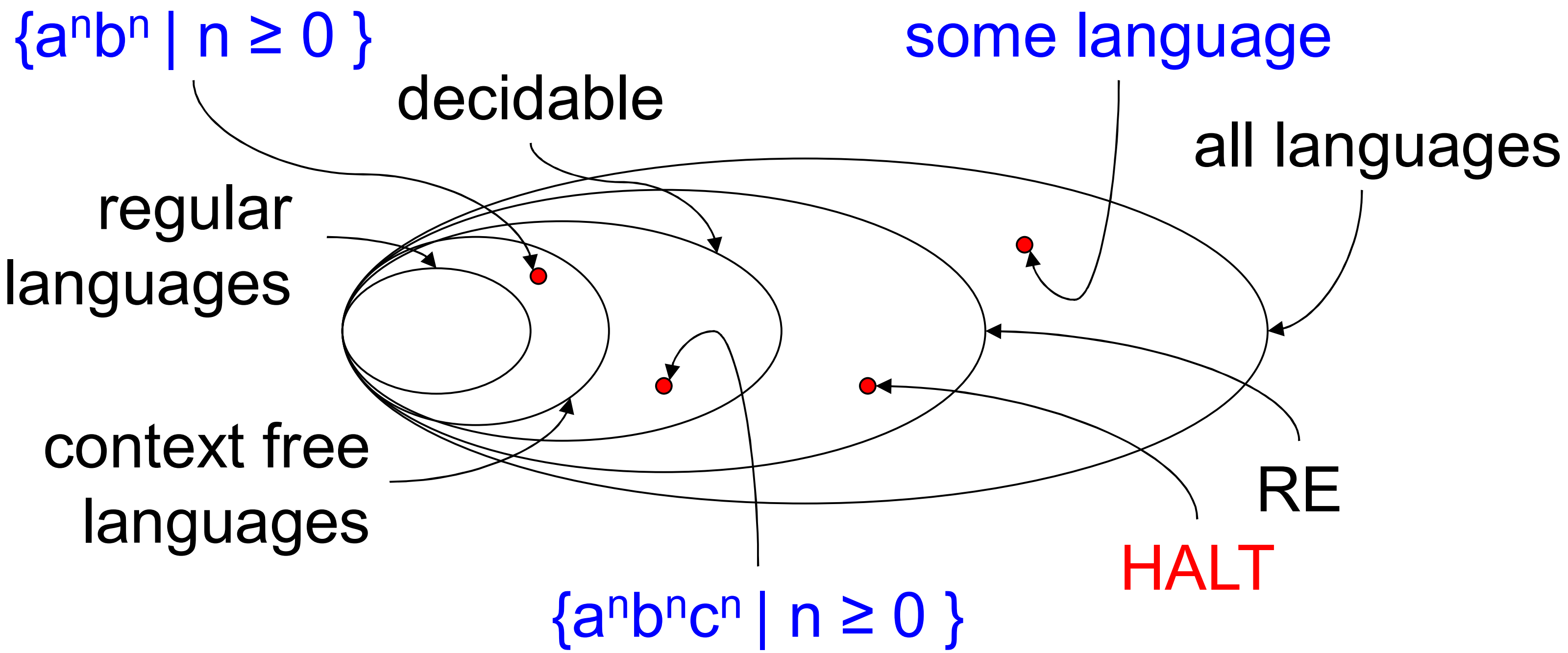
Proof:

- define new TM H’: on input <M>
  - if H accepts <M, <M>>, then loop
  - if H rejects <M, <M>>, then halt
- consider H’ on input <H’>:
  - if it halts, then H rejects <H’, <H’>>, which implies it cannot halt
  - if it loops, then H accepts <H’, <H’>>, which implies it must halt
- contradiction. Thus neither H nor H’ can exist

The Halting Problem



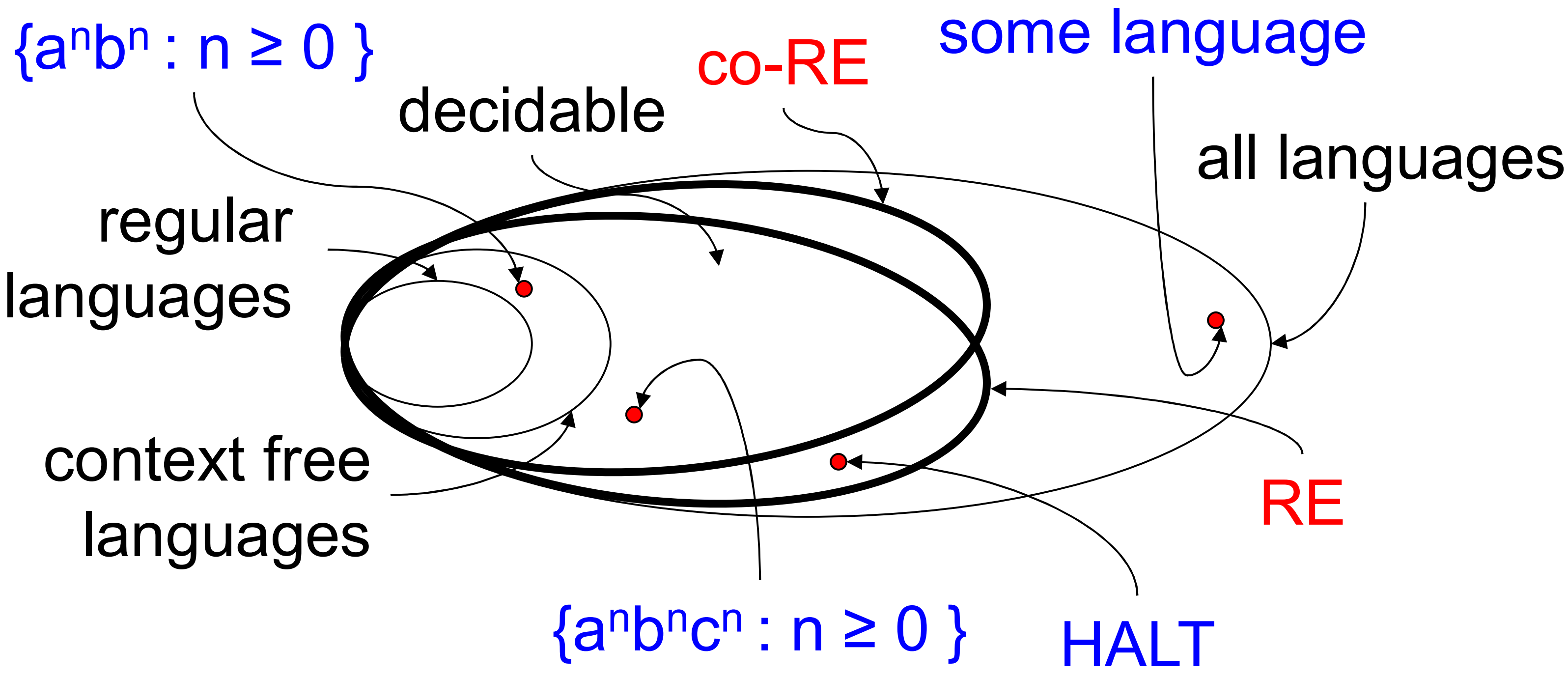
So far…



- Can we exhibit a natural language that is non-RE?

RE and co-RE

- The complement of a RE language is called a co-RE language



RE and co-RE

Theorem: a language L is decidable if and only if L is RE and L is co-RE.

Proof:

($\Rightarrow$) we already know decidable implies RE

- if L is decidable, then complement of L is decidable by flipping accept/reject.
- so L is in co-RE.

RE and co-RE

Theorem: a language L is decidable if and only if L is RE and L is co-RE.

Proof:

($\Leftarrow$) we have TM M that recognizes L, and TM M' recognizes complement of L.

- on input x, simulate M, M' in parallel
- if M accepts, accept; if M' accepts, reject.

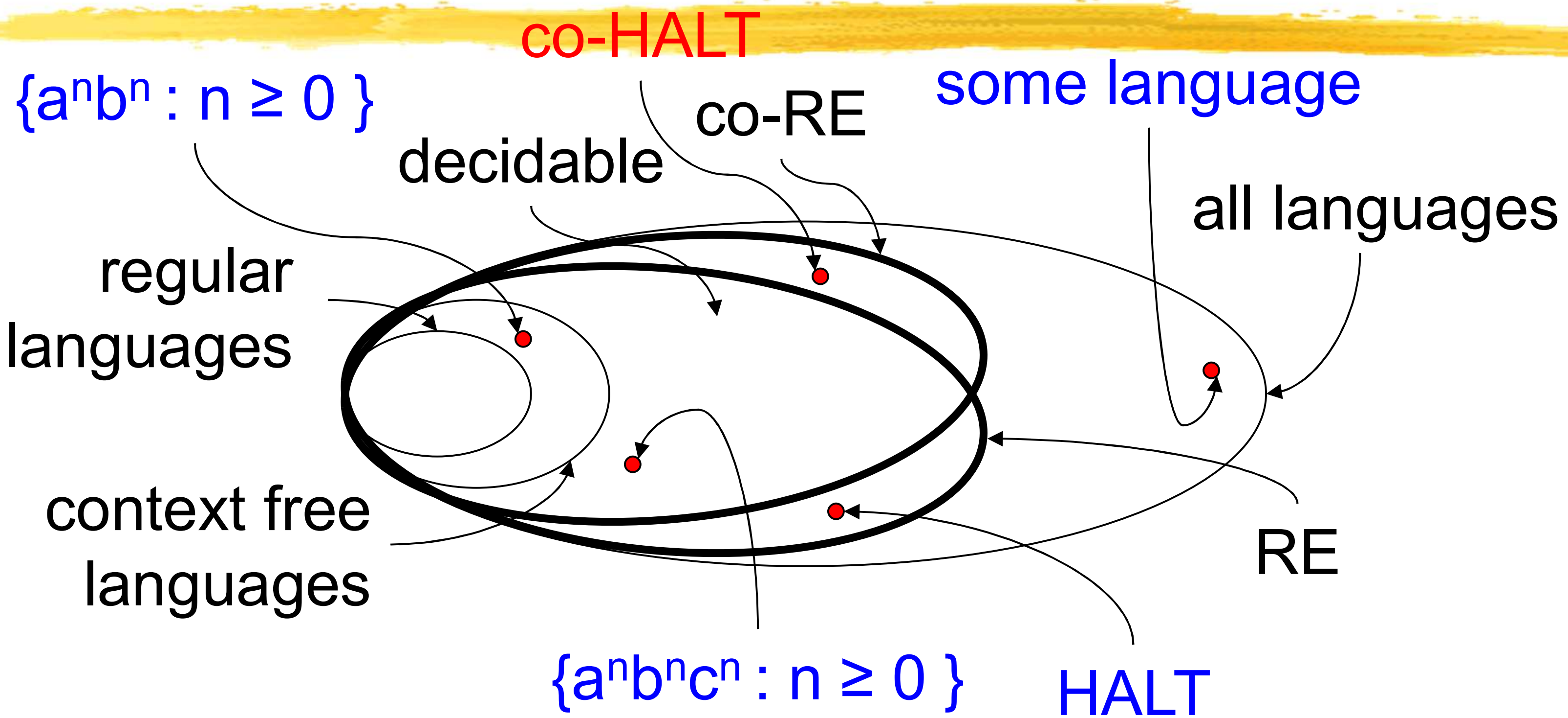
A natural non-RE Language

Theorem: the complement of HALT is not recursively enumerable.

Proof:

- we know that HALT is RE
- suppose complement of HALT is RE
- then HALT is co-RE
- implies HALT is decidable. Contradiction.

Summary



Punch line: some problems have no algorithms, HALT in particular.