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Homework week 2.

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a) Gaussian distribution is normalized.

Gaussian distribution function is

$$N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Normalization is given by

$$\int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = 1$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx = 1$$

assume mean equals to 0, so we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx = 1$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx = \sqrt{2\pi\sigma^2}$$

Let

$$I = \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\sigma^2}x^2\right) dx$$

Take square of both sides on the above function

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\sigma^2}x^2\right) \exp\left(\frac{-1}{2\sigma^2}y^2\right) dx\,dy$$

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then we will change from (x,y) coordinate to (r,θ) coordinate, assume that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Based on the fact that

$$\int_0^{\infty} \exp\left(\frac{-r^2}{2\sigma^2}\right) r\,dr$$
$$= -\sigma^2 \exp\left(-\frac{r^2}{2\sigma^2}\right)\Big|_0^{+\infty} = -\sigma^2(0-1) = \sigma^2$$

therefore:

$$I^2 = \int_0^{2\pi} \sigma^2 d\theta = 2\pi\sigma^2, \Rightarrow I = \sqrt{2\pi}\sigma$$

prove normalize:

$$\int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu^2)\right\} dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2} y^2\right\} dy$$

Since $y = x - \mu$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2} y^2\right) dy$$
$$= 1.$$

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b) expectation of Gaussian distribution is μ
probability density function is

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Expected value formula

$$E(x) = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

so

$$E(x) = \frac{1}{\sigma\sqrt{2\pi}} = \int_{-\infty}^{\infty} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma + \mu) \exp(-t^2)\, dt$$

$$= \frac{1}{\pi} \left(\sqrt{2}\sigma \int_{-\infty}^{\infty} t \exp(-t^2)\, dt + \right.$$

$$\left. \mu \int_{-\infty}^{\infty} \exp(-t^2)\, dt\right)$$

$$= \frac{1}{\pi} \left(\sqrt{2}\sigma \left(-\frac{1}{2}\exp(-t)\right)\Big|_{-\infty}^{\infty} + \mu\sqrt{\pi}\right)$$

$$= \frac{\mu\sqrt{\pi}}{\sqrt{\pi}}$$

$$= \mu$$

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and cov matrix Σ given by

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

$$\rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$$

Σ is symmetric so Σ_{aa} and Σ_{bb} are sym.
while $\Sigma_{ab} = \Sigma_{ba}^T$

we're looking for conditional dist $p(x_a | x_b)$

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) = -\frac{1}{2}(x-\mu)^T A (x-\mu)$$

$$= -\frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{ab}(x_b-\mu_b)$$

$$-\frac{1}{2}(x_b-\mu_b)^T A_{ba}(x_a-\mu_a) - \frac{1}{2}(x_b-\mu_b)^T A_{bb}(x_b-\mu_b)$$

$$= -\frac{1}{2} x_a^T A_{aa}^{-1} x_a + x_a^T (A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) + \text{const}$$

It's quadratic form of x_a hence cond dist $p(x_a|x_b)$ will be Gaussian, because this dist is characterized by its mean and its variance. Compare with Gauss dist.

$$\Delta^2 = -\frac{1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + \text{const}$$

$$\Sigma_{a|b} = A_{aa}^{-1}$$

$$\mu_{a|b} = \Sigma_{a|b}(A_{aa}\mu_a - A_{ab}(x_b-\mu_b))$$

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$$= \mu_a - A_{aa}^{-1} A_{ab} (x_b - \mu_b)$$

using Schur

$$\rightarrow A_{aa} = (\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba})^{-1}$$

$$A_{ab} = -(\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba})^{-1} \Sigma_{ab} \Sigma_{bb}^{-1}$$

As a result

$$\mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (x_b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

$$\rightarrow p(x_a | x_b) = N(x_{a|b} | \mu_{a|b}, \Sigma_{a|b})$$

b)

The marginal dist

$$p(x_a) = \int p(x_a, x_b) \, dx_b$$

intergrate out x_b by looking the quadratic form relate to x_b

$$-\frac{1}{2} x_b^T A_{bb} x_b + x_b^T m$$

$$= -\frac{1}{2} (x_b - A_{bb}^{-1} m)^T A_{bb} (x_b - A_{bb}^{-1} m)$$

$$+ \frac{1}{2} m^T A_{bb}^{-1} m$$

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with $m = \Lambda_{bb}\mu_b - \Lambda_{ba}(x_a - \mu_a)$, intergrate over unnormalized Gaussian.

$$\int \exp\left(-\frac{1}{2}(x_b - \Lambda_{bb}^{-1}m)^T \Lambda_{bb}(x_b - \Lambda_{bb}^{-1}m)\right)dx_b$$

remaining term

$$-\frac{1}{2}x_a^T(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})x_a + x_a^T(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})^{-1} \times \mu_a + \text{const}$$

Similarly

$$E(x_a) = \mu_a$$
$$\text{cov}(x_a) = \Sigma_{aa}$$
$$\rightarrow p(x_a) = N(x_a \mid \mu_a, \Sigma_{aa})$$