CORE EQUALITY OF REAL SEQUENCES

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ABSTRACT. Given an ideal \mathcal{I} on ω and a bounded real sequence \boldsymbol{x} , we denote by $\operatorname{core}_{\boldsymbol{x}}(\mathcal{I})$ the smallest interval [a,b] such that $\{n \in \omega : x_n \notin [a-\varepsilon,b+\varepsilon]\} \in \mathcal{I}$ for all $\varepsilon > 0$ (which corresponds to the interval $[\liminf \boldsymbol{x}, \limsup \boldsymbol{x}]$ if \mathcal{I} is the ideal Fin of finite subsets of ω).

First, we characterize all the infinite real matrices A such that

$$core_{Ax}(\mathcal{J}) = core_{x}(\mathcal{I})$$

for all bounded sequences x, provided that \mathcal{J} is a countably generated ideal on ω and A maps bounded sequences into bounded sequences. Such characterization fails if both \mathcal{I} and \mathcal{J} are the ideal of asymptotic density zero sets. Next, we show that such equality is possible for distinct ideals \mathcal{I}, \mathcal{J} , answering an open question in [J. Math. Anal. Appl. **321** (2006), 515–523]. Lastly, we prove that, if $\mathcal{J} = \operatorname{Fin}$, the above equality holds for some matrix A if and only if $\mathcal{I} = \operatorname{Fin}$ or \mathcal{I} is an isomorphic copy of $\operatorname{Fin} \oplus \mathcal{P}(\omega)$ on ω .

1. Introduction

Let \mathcal{I} be an ideal on the natural numbers ω , that is, a family of subsets of ω closed under subsets and finite unions. Unless otherwise stated, it is assumed that \mathcal{I} contains the family Fin of finite sets and that $\omega \notin \mathcal{I}$. We denote by $\mathcal{I}^+ := \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^* := \{S \subseteq \omega : \omega \setminus S \in \mathcal{I}\}$ the family of \mathcal{I} -positive sets and the dual filter of \mathcal{I} , respectively. Ideals are regarded as subsets of the Cantor space $\{0,1\}^{\omega}$, hence it is possible to speak about F_{σ} -ideals, analytic ideals, meager ideals, etc. An important example of an ideal is the family \mathcal{Z} of sets $S \subseteq \omega$ with asymptotic density zero, that is, $|S \cap [0,n]| = o(n)$ as $n \to \infty$.

Given a sequence $\mathbf{x} = (x_n : n \in \omega)$ taking values in a topological space X, we define its \mathcal{I} -core by

$$\operatorname{core}_{\boldsymbol{x}}(\mathcal{I}) := \bigcap_{S \in \mathcal{I}^{\star}} \overline{\operatorname{co}} \left\{ x_n : n \in S \right\},$$

where co denotes the convex hull operator and \overline{co} its closure, see [14, 22]. In the cases where $\mathcal{I} = \text{Fin}$ and $\mathcal{I} = \mathcal{Z}$, the \mathcal{I} -core of \boldsymbol{x} is usually called "Knopp core" and "statistical core," respectively, see [12, 13, 18, 29]. Let also $\Gamma_{\boldsymbol{x}}(\mathcal{I})$ denote the set of \mathcal{I} -cluster points of \boldsymbol{x} , that is, the set of all $\eta \in X$ such that $\{n \in \omega : x_n \in U\} \in \mathcal{I}^+$ for all neighborhoods U of η . It has been shown in [22, Corollary 2.3] that, if \boldsymbol{x} is bounded real sequence, then $\operatorname{core}_{\boldsymbol{x}}(\mathcal{I}) = \operatorname{co}(\Gamma_{\boldsymbol{x}}(\mathcal{I}))$. In addition, since $\Gamma_{\boldsymbol{x}}(\mathcal{I})$ is a nonempty compact set, see e.g. [25, Lemma 3.1], it follows that

$$\operatorname{core}_{\boldsymbol{x}}(\mathcal{I}) = [\mathcal{I}-\liminf \boldsymbol{x}, \mathcal{I}-\limsup \boldsymbol{x}]$$
 (1)

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for all real bounded sequences x, where \mathcal{I} - $\liminf x := \min \Gamma_x(\mathcal{I})$ and \mathcal{I} - $\limsup x := \max \Gamma_x(\mathcal{I})$ (note that, if $\mathcal{I} = \text{Fin then } \mathcal{I}$ - $\lim \inf$ coincides with the ordinary $\lim \inf$, and analogously for

Given an infinite matrix $A = (a_{n,k} : n, k \in \omega)$, we denote by dom(A) its domain, that is, the set of real sequences $\mathbf{x} = (x_n)$ for which the A-transformed sequences $A\mathbf{x} := (A_n\mathbf{x} : n \in \omega)$ are well defined, which means that the series

$$A_n \boldsymbol{x} := \sum_{k \in \omega} a_{n,k} x_k$$

is convergent to a (finite) real number for each $n \in \omega$. We write $A \geq 0$ if $a_{n,k} \geq 0$ for all $n, k \in \omega$. Given sequence spaces $\mathcal{A}, \mathcal{B} \subseteq \mathbf{R}^{\omega}$, we denote by $(\mathcal{A}, \mathcal{B})$ the family of infinite matrices A such that $\mathcal{A} \subseteq \text{dom}(A)$ and $Ax \in \mathcal{B}$ for all $x \in \mathcal{A}$. For instance, it is well known that $A \in (\ell_{\infty}, \ell_{\infty})$ if and only if $A \in (c, c)$ if and only if $||A|| < \infty$, where ||A|| := $\sup_{n} \sum_{k} |a_{n,k}|$, see e.g. [5, Theorem 2.3.5]. Here, as usual, ℓ_{∞} and c stand for the vector space of bounded real sequences and convergent real sequences, respectively. ℓ_{∞} and all its subspaces are endowed with the topology induced by the supremum norm.

We denote by $c_b(\mathcal{I})$ the vector space of bounded real sequences x which, in addition, are \mathcal{I} -convergent to some $\eta \in \mathbf{R}$, namely, $\{n \in \omega : x_n \in U\} \in \mathcal{I}^*$ for all neighborhoods U of η (this will be shortened as \mathcal{I} - $\lim x = \eta$). Note that $c_b(\mathcal{I}) = \ell_\infty$ whenever \mathcal{I} is maximal (that is, \mathcal{I}^{\star} is a free ultrafilter on ω). Structural properties of bounded \mathcal{I} -convergent sequences, their \mathcal{I} -cluster points, and the relationship with A-summability have been extensively studied, see e.g. [2, 4, 10, 11, 15, 16, 21, 20, 26, 28] and references therein. Given ideals \mathcal{I}, \mathcal{J} on ω , we say that an infinite real matrix A is $(\mathcal{I}, \mathcal{J})$ -regular if it maps \mathcal{I} -convergent bounded sequences into \mathcal{J} -convergent bounded sequences preserving the corresponding ideal limits, namely,

$$A \in (c_b(\mathcal{I}), c_b(\mathcal{J}))$$
 and \mathcal{I} - $\lim x = \mathcal{J}$ - $\lim Ax$ for all $x \in c_b(\mathcal{I})$,

see e.g. [7, 19, 23]. Note that (Fin, Fin)-regular matrices are simply the classical regular matrices. Probably the most important regular matrix is the Cesàro matrix $C_1 = (a_{n,k})$ defined by $(a_{n,k}) = 1/n$ if $k \le n$ and $a_{n,k} = 0$ otherwise.

A characterization of $(\mathcal{I}, \mathcal{J})$ -regular matrices has been recently proved by the author and Jeff Connor in [7, Theorem 1.2 and Theorem 1.3], see also [19, Corollary 2.11].

Theorem 1.1. Let A be an infinite real matrix and fix ideals \mathcal{I} , \mathcal{J} on ω . Suppose also that $A \geq 0$ or $\mathcal{I} = \text{Fin or } \mathcal{J}$ is countably generated.

Then A is $(\mathcal{I}, \mathcal{J})$ -regular if and only if:

- (T1) $||A|| < \infty$;
- (T2) \mathcal{J} - $\lim_{n} \sum_{k} a_{n,k} = 1;$ (T3) \mathcal{J} - $\lim_{n} \sum_{k \in E} |a_{n,k}| = 0$ for all $E \in \mathcal{I}$.

In the statement above, recall that an ideal \mathcal{J} on ω is countably generated if there exists a sequence $(Q_k : k \in \omega)$ of subsets of ω such that $S \in \mathcal{I}$ if and only if $S \subseteq \bigcup_{k \in F} Q_k$ for some finite $F \in \text{Fin.}$ Examples of countably generated ideals include Fin and the isomorphic copies on ω of Fin $\times \{\emptyset\} := \{S \subseteq \omega^2 : \exists n \in \omega, S \subseteq [0, n] \times \omega\}$ and Fin $\oplus \mathcal{P}(\omega) := \{S \subseteq \omega^2 : \exists n \in \omega, S \subseteq [0, n] \times \omega\}$ $\{0,1\} \times \omega : |S \cap (\{0\} \times \omega)| < \infty\}$, cf. [19, Remark 2.16]. Hereafter, an ideal \mathcal{I} on a countably

infinite set W is said to be an isomorphic copy of an ideal \mathcal{J} on ω if there exists a bijection $f: W \to \omega$ such that $S \in \mathcal{J}$ if and only if $f^{-1}[S] \in \mathcal{I}$ for each $S \subseteq \omega$.)

The above result extends the classical Silverman–Toeplitz characterization, which corresponds to the case $\mathcal{I} = \mathcal{J} = \text{Fin.}$ Lastly, it is worth mentioning that Theorem 1.1 does not hold for arbitrarily ideals: indeed, there exists a $(\mathcal{Z}, \mathcal{Z})$ -regular which does not satisfy condition (T3), see [7, Theorem 1.4].

2. Main results

Given ideals \mathcal{I}, \mathcal{J} on ω , we study the core equality problem

$$\operatorname{core}_{Ax}(\mathcal{J}) = \operatorname{core}_{x}(\mathcal{I}) \quad \text{for all sequences } x \in \ell_{\infty}.$$
 (2)

More explicitly, we obtain necessary and sufficient conditions on the entries of A to satisfy equality (2) and, then, we study the existence of such matrices A.

The first result in this direction has been obtained by Allen [1], which provides a characterization of the matrices A which satisfy equality (2) in the case $\mathcal{I} = \mathcal{J} = \text{Fin}$:

Theorem 2.1. Let A be an infinite real matrix and suppose that $\mathcal{I} = \mathcal{J} = \text{Fin}$. Then equality (2) holds if and only if:

- (A1) A is regular;
- (A2) $\lim_{n \to \infty} |a_{n,k}| = 1$;
- (A3) for each infinite $E \subseteq \omega$, there exists a strictly increasing sequence $(n_i : i \in \omega)$ in ω such that $\lim_i \sum_{k \in E} a_{n_i,k} = 1$.

Note that, taking into account (A2), condition (A3) can be rewritten equivalently as $\limsup_n \sum_{k \in E} |a_{n,k}| = 1$ for all infinite $E \subseteq \omega$. In addition, condition (A2) implies that A maps bounded sequences into bounded sequences, i.e., $A \in (\ell_{\infty}, \ell_{\infty})$.

A partial extension of the result above has been obtained by Connor, Fridy, and Orhan in the case where all the entries of A are nonnegative, see [6, Theorem 2.1].

Theorem 2.2. Let $A \in (\ell_{\infty}, \ell_{\infty})$ be an infinite real matrix, let \mathcal{I} , \mathcal{J} be ideals on ω , and suppose that $A \geq 0$.

Then equality (2) holds if and only if:

- (C1) A is $(\mathcal{I}, \mathcal{J})$ -regular;
- (C2) \mathcal{J} - $\lim \sup_{n} \sum_{k \in E} a_{n,k} = 1 \text{ for all } E \in \mathcal{I}^+.$

Our first main result removes the hypotheses that the entries of A are nonnegative and, on the other hand, it requires that \mathcal{J} is countably generated. Hence it provides a generalization of Theorem 2.1.

Theorem 2.3. Let $A \in (\ell_{\infty}, \ell_{\infty})$ be an infinite real matrix, let \mathcal{I} , \mathcal{J} be ideals on ω , and suppose that \mathcal{J} is countably generated.

Then equality (2) holds if and only if:

- (L1) A is $(\mathcal{I}, \mathcal{J})$ -regular;
- (L2) \mathcal{J} - $\limsup_{n} \sum_{k \in E} |a_{n,k}| = 1 \text{ for all } E \in \mathcal{I}^+.$

The proof of Theorem 2.3 recovers also Theorem 2.2, see Remark 3.2 below. In addition, the above characterization does *not* hold without any constraint on the ideals \mathcal{I}, \mathcal{J} . Indeed, it fails for $\mathcal{I} = \mathcal{J} = \mathcal{Z}$, see Remark 3.3.

At this point, another result by Connor, Fridy, and Orhan proves that there are no regular matrices A satisfying equality (2) if $\mathcal{I} = \mathcal{Z}$ and $\mathcal{J} = \text{Fin}$, see [6, Theorem 2.4]. Accordingly, the authors left as open question whether there exist distinct ideals \mathcal{I}, \mathcal{J} on ω for which equality (2) holds for some matrix $A \in (\ell_{\infty}, \ell_{\infty})$. Next, we show that the answer is affirmative. To this aim, recall that an ideal \mathcal{I} is $Rudin-Keisler\ below$ an ideal \mathcal{J} , shortened as $\mathcal{I} \leq_{RK} \mathcal{J}$, if there exists a map $h: \omega \to \omega$ such that $S \in \mathcal{I}$ if and only if $h^{-1}[S] \in \mathcal{J}$.

Theorem 2.4. Let \mathcal{I} , \mathcal{J} be ideals on ω such that $\mathcal{I} \leq_{RK} \mathcal{J}$. Then there exists an infinite real matrix $A \in (\ell_{\infty}, \ell_{\infty})$ which satisfies equality (2).

It is worth to recall that, if \mathcal{J} is a P-ideal (that is, for all increasing sequences (J_n) in \mathcal{J} there exists $J \in \mathcal{J}$ such that $J_n \setminus J \in \text{Fin}$ for all $n \in \omega$) and $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$, then it is possible to choose a witnessing function h which is finite-to-one, see [9, Proposition 1.3.1]. Rudin–Keisler ordering and the latter stronger variant (known as Rudin–Blass ordering) on the maximal ideals are extensively studied in the literature, cf. [9, Section 1.3] and references therein. Examples of (distinct) ideals \mathcal{I}, \mathcal{J} on ω such that $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ are abundant. For instance, it is known that $\text{Fin} \leq_{\text{RK}} \mathcal{J}$ for all meager ideals \mathcal{J} , see [9, Corollary 3.10.2]. In addition, $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ for all Erdős–Ulam ideals \mathcal{I}, \mathcal{J} (where an ideal \mathcal{H} is said to be Erdős–Ulam if there exists a nonnegative real sequence (h_n) such that $\sum_n h_n = \infty$, $h_n = o(\sum_{k \leq n} h_k)$ as $n \to \infty$, and $S \in \mathcal{H}$ if and only if $\sum_{k \in S, k \leq n} h_k = o(\sum_{k \leq n} h_k)$ as $n \to \infty$), see [9, Lemma 1.13.10], cf. also [17, Corollary 1]. Other examples can be found within the class of summable ideals, see [9, Section 1.12].

Our last result extends the latter [6, Theorem 2.4] by finding all ideals \mathcal{I} for which equality (2) holds with $\mathcal{J} = \text{Fin}$ and some matrix A:

Theorem 2.5. There exists an infinite real matrix $A \in (\ell_{\infty}, \ell_{\infty})$ satisfying equality (2) with $\mathcal{J} = \text{Fin } if \ and \ only \ if \ \mathcal{I} = \text{Fin } or \ \mathcal{I} \ is \ an \ isomorphic \ copy \ of \ \text{Fin} \oplus \mathcal{P}(\omega) \ on \ \omega$.

(Equivalently, the latter condition means $\mathcal{I} = \{S \subseteq \omega : S \cap T \in \text{Fin}\}$ for some infinite $T \subseteq \omega$.) It is worth noting that Theorem 2.5 is also related to the question posed by Mazur in *The Scottish Book* whether the notion of statistical convergence (i.e., \mathcal{Z} -convergence) of bounded sequences is equivalent to some matrix summability method, see [10] and references therein. A positive answer has been given by Khan and Orhan in [16, Theorem 2.2].

Based on the previous observations, we leave as an open question to check whether there exists a matrix $A \in (\ell_{\infty}, \ell_{\infty})$ satisfying equality (2) with $\mathcal{I} = \text{Fin}$ if and only if \mathcal{J} is meager. In the same direction, it would be interesting to know if the condition $\mathcal{I} \leq_{RK} \mathcal{J}$ is also necessary in the statament of Theorem 2.4.

3. Proofs

We start with an auxiliary lemma and, then, we proceed to the proofs of our results.

Lemma 3.1. Let x and y be two relatively compact sequences taking values in a locally convex topological vector space X. Let \mathcal{I} be an ideal on ω . Then

$$core_{\boldsymbol{x}}(\mathcal{I}) = core_{\boldsymbol{y}}(\mathcal{I})$$

whenever \mathcal{I} - $\lim(\boldsymbol{x} - \boldsymbol{y}) = 0$.

Proof. Since X is, in particular, a topological group, it follows by [25, Lemma 3.5] and the hypothesis \mathcal{I} - $\lim(x-y)=0$ that $\Gamma_x(\mathcal{I})=\Gamma_y(\mathcal{I})$. Now, let K_x be the closure of the image $\{x_n:n\in\omega\}$, which is a compact subset of the locally convex space X. Of course, this implies that $\{x_n:n\in\omega\}\subseteq K_x+U$ for every open neighborhood U of 0 (which means that x is " \mathcal{I}^* -asymptotically K_x -controlled," using the terminology in [14, Definition 3.2]). It follows by [14, Theorem 3.4] that $\operatorname{core}_x(\mathcal{I})=\overline{\operatorname{co}}\,\Gamma_x(\mathcal{I})$; cf. also [22, Theorem 2.2] in the case of first countable locally convex spaces. With an analog reasoning on the sequence y, we conclude that $\operatorname{core}_x(\mathcal{I})=\overline{\operatorname{co}}\,\Gamma_x(\mathcal{I})=\overline{\operatorname{co}}\,\Gamma_y(\mathcal{I})=\operatorname{core}_y(\mathcal{I})$.

Proof of Theorem 2.3. ONLY IF PART. Pick a sequence $\mathbf{x} \in c_b(\mathcal{I})$ and define $\eta := \mathcal{I}\text{-}\lim \mathbf{x}$, so that $\Gamma_{\mathbf{x}}(\mathcal{I}) = \{\eta\}$. It follows that $A\mathbf{x}$ is well-defined bounded sequence and, thanks to equality (2), that $\operatorname{core}_{A\mathbf{x}}(\mathcal{I}) = \operatorname{core}_{\mathbf{x}}(\mathcal{I}) = \{\eta\}$. We conclude by [22, Proposition 4.2] that $\mathcal{I}\text{-}\lim A\mathbf{x} = \eta$, therefore A is $(\mathcal{I}, \mathcal{I})$ -regular.

At this point, pick a set $E \in \mathcal{I}^+$. Recall that the hypothesis $A \in (\ell_{\infty}, \ell_{\infty})$ is equivalent to $\sup_{n \in \omega} \sum_k |a_{n,k}| < \infty$, see e.g. [5, Theorem 2.3.5]. In addition, it follows by Theorem 1.1 that \mathcal{J} - $\lim_n |a_{n,k}| = 0$ for all $k \in \omega$. This implies that conditions (K1)–(K3) in the statement of [19, Corollary 4.3] are satisfied (in the case where d = m = 1, so that $a_{n,k}(i,j)$ is simply $a_{n,k}$ considering that both i and j can take only one value). Hence, thanks to [19, Corollary 4.3], there exists a $\{-1,0,1\}$ -valued sequence \boldsymbol{x} supported on E such that

$$\mathcal{J}\text{-}\limsup_{n\to\infty} \sum_{k\in E} |a_{n,k}| = \mathcal{J}\text{-}\limsup_{n\to\infty} |A_n \boldsymbol{x}|.$$
 (3)

(Again, equation (3) is a rewriting of the claim in [19, Corollary 4.3] taking into account that both variables i and j can take only value.) Define $F := \{n \in E : x_n = 1\}$ and $G := \{n \in E : x_n = -1\}$, so that $\{F, G\}$ is a partition of E and $\mathbf{x} := \mathbf{1}_F - \mathbf{1}_G$ (hereafter, $\mathbf{1}_S$ stands for the characteristic function of E). Note that, since E is an ideal, then at least one between E and E is an E-positive set. For convenience, let E is an ideal, then defined by E is an ideal positive set. For convenience, let E is an ideal partition defined by E is continuous, it follows by [24, Proposition 3.2] that

$$\Gamma_{h(A\boldsymbol{x})}(\mathcal{J}) = h \left[\Gamma_{A\boldsymbol{x}}(\mathcal{J}) \right],$$

where $h(A\boldsymbol{x}) := (h(A_n\boldsymbol{x}) : n \in \omega)$. Since \boldsymbol{x} is a bounded real sequence, $A \in (\ell_{\infty}, \ell_{\infty})$, and h is continuous, then $h(A\boldsymbol{x})$ is a relatively compact sequence, so that the above sets are nonempty, see e.g. [25, Lemma 3.1(vi)]. In particular, it follows that max $\Gamma_{h(A\boldsymbol{x})}(\mathcal{J}) = \max h [\Gamma_{A\boldsymbol{x}}(\mathcal{J})]$. Taking into account equality (2) (so that \mathcal{J} -lim sup and \mathcal{J} -lim inf are preserved) and that

the sequence $x = 1_F - 1_G$ has at least a \mathcal{J} -cluster point in $\{-1,1\}$, we conclude that

$$\begin{split} \mathcal{J}\text{-}\lim\sup_{n\to\infty} \sum_{k\in E} |a_{n,k}| &= \max \ \Gamma_{h(A\boldsymbol{x})}(\mathcal{J}) = \max \ h\left[\Gamma_{A\boldsymbol{x}}(\mathcal{J})\right] \\ &= \max\{|\mathcal{J}\text{-}\limsup A\boldsymbol{x}|, |\mathcal{J}\text{-}\liminf A\boldsymbol{x}|\} \\ &= \max\{|\mathcal{J}\text{-}\limsup \boldsymbol{x}|, |\mathcal{J}\text{-}\liminf \boldsymbol{x}|\} = \max \ h\left[\Gamma_{\boldsymbol{x}}(\mathcal{J})\right] = 1. \end{split}$$

Therefore both conditions (L1) and (L2) hold.

IF PART. Conversely, let $A = (a_{n,k} : n, k \in \omega) \in (\ell_{\infty}, \ell_{\infty})$ be a $(\mathcal{I}, \mathcal{J})$ -regular matrix which satisfies condition (L2). Then we get

$$1 = \mathcal{J}\text{-}\lim_{n \to \infty} \sum_{k \in \omega} a_{n,k} \leq \mathcal{J}\text{-}\liminf_{n \to \infty} \sum_{k \in \omega} |a_{n,k}| \leq \mathcal{J}\text{-}\limsup_{n \to \infty} \sum_{k \in \omega} |a_{n,k}| = 1,$$

so that \mathcal{J} - $\lim_n \sum_k |a_{n,k}| = \mathcal{J}$ - $\lim_n \sum_k a_{n,k} = 1$. Decomposing each $a_{n,k}$ into its positive and negative part as $a_{n,k}^+ - a_{n,k}^-$ for all $n, k \in \omega$, it follows that

$$\mathcal{J}\text{-}\lim_{n\to\infty}\sum_{k\in\omega}a_{n,k}^-=0\quad\text{ and }\quad \mathcal{J}\text{-}\lim_{n\to\infty}\sum_{k\in\omega}a_{n,k}^+=1. \tag{4}$$

At this point, pick a sequence $\boldsymbol{x} \in \ell_{\infty}$, define $\eta := \mathcal{I}$ - $\limsup \boldsymbol{x}$. Considering that $\operatorname{core}_{\boldsymbol{x}+\kappa \mathbf{1}_{\omega}}(\mathcal{I}) = \operatorname{core}_{\boldsymbol{x}}(\mathcal{I}) + \{\kappa\}$ and also, by the $(\mathcal{I}, \mathcal{J})$ -regularity of A, that $\operatorname{core}_{A(\boldsymbol{x}+\kappa \mathbf{1}_{\omega})}(\mathcal{J}) = \operatorname{core}_{A\boldsymbol{x}}(\mathcal{J}) + \{\kappa\}$ for all $\kappa \in \mathbf{R}$, we can suppose without loss of generality that $\eta > 0$. Fix an arbitrary $\varepsilon > 0$ and define

$$\delta := \min \left\{ \frac{\varepsilon}{2 + \eta + 4\|x\|}, 1 \right\} \quad \text{and} \quad E := \left\{ k \in \omega : x_k \ge \eta - \delta \right\}.$$

Note that $\delta > 0$ and E is an \mathcal{I} -positive set since η is an \mathcal{I} -cluster point of \boldsymbol{x} . It follows by (4) and condition (L2) that \mathcal{J} - $\limsup_{n} \sum_{k \in E} a_{n,k}^+ = 1$. Thus, define

$$S := \left\{ n \in \omega : 1 - \delta \le \sum_{k \in E} a_{n,k}^+ \le \sum_{k \in \omega} |a_{n,k}| \le 1 + \delta \right\}.$$
 (5)

Observe the first inequality in the definition of S holds on a \mathcal{J} -positive set, the second one for all n, and the latter one on \mathcal{J}^* . Therefore $S \in \mathcal{J}^+$. For each $n \in S$, it also follows that $|\sum_{k \in \omega} a_{n,k}^+ - 1| \leq \delta$, so that

$$\left| \sum_{k \in \omega} a_{n,k}^{-} \right| \le 2\delta \quad \text{and} \quad \left| \sum_{k \in E^{c}} a_{n,k}^{+} \right| \le 2\delta.$$
 (6)

Putting all together, we obtain that, for all $n \in S$,

$$A_{n}x = \sum_{k \in E} a_{n,k}^{+} x_{k} + \sum_{k \in E^{c}} a_{n,k}^{+} x_{k} - \sum_{k \in \omega} a_{n,k}^{-} x_{k}$$

$$\geq (1 - \delta)(\eta - \delta) - 2\delta ||x|| - 2\delta ||x||$$

$$\geq \eta - \delta(1 + \eta + 4||x||) \geq \eta - \varepsilon.$$

At the same time, define $E' := \{k \in \omega : x_k \leq \eta + \delta\}$, which belongs to \mathcal{I}^* and note, similarly as above, that \mathcal{J} - $\lim_n \sum_{k \in E'} |a_{n,k}| = \mathcal{J}$ - $\lim_n \sum_{k \in E'} a_{n,k}^+ = 1$

and \mathcal{J} - $\lim_n \sum_{k \in \omega} a_{n,k}^- = 0$. Let S' be the set defined as in (5) replacing E with E', and note that $S' \in \mathcal{J}^*$. Similarly, estimates (6) hold for all $n \in S'$ replacing E with E'. Putting again all together, we obtain that, for all $n \in S'$,

$$A_{n}x = \sum_{k \in E} a_{n,k}^{+} x_{k} + \sum_{k \in E^{c}} a_{n,k}^{+} x_{k} - \sum_{k \in \omega} a_{n,k}^{-} x_{k}$$

$$\leq (1 + \delta)(\eta + \delta) + 2\delta ||x|| + 2\delta ||x||$$

$$\leq \eta + \delta(2 + \eta + 4||x||) \leq \eta + \varepsilon.$$

Since ε is arbitrary, we conclude that \mathcal{J} -lim sup $A\boldsymbol{x}=\eta$. Therefore A preserves the ideal superior limits for all bounded sequences \boldsymbol{x} . Replacing \boldsymbol{x} with $-\boldsymbol{x}$, A preserves also the corresponding ideal inferior limits. It follows by identity (1) that equality (2) holds, concluding the proof.

Remark 3.2. It is clear from the proof above that the IF PART holds without any additional hypothesis on \mathcal{J} . Moreover, the fact the \mathcal{J} is countably generated has been used only once in the proof of the ONLY IF PART, precisely in the existence of a $\{-1,0,1\}$ -valued sequence \boldsymbol{x} supported on a given $E \in \mathcal{I}^+$ and satisfying equality (3). The latter is trivial if $A \geq 0$ by choosing $\boldsymbol{x} = \mathbf{1}_E$. In this sense, we recover also Theorem 2.2.

Remark 3.3. On the other hand, if $\mathcal{I} = \mathcal{J} = \mathcal{Z}$, then the analogue of Theorem 2.3 does *not* hold. Indeed, thanks to the proof of [7, Theorem 1.4] there exists a matrix $A \in (\ell_{\infty}, c_0(\mathcal{Z}) \cap \ell_{\infty})$ and an infinite set $I \in \mathcal{I}$ such that

$$\mathcal{Z}\text{-}\limsup_{n\to\infty}\sum_{k\in I}|a_{n,k}|=1.$$

(Here, $c_0(\mathcal{Z})$ stands for the vector space of sequences which are \mathcal{Z} -convergent to 0.) At this point, define $B := A + \mathrm{Id}$, where Id stands for the infinite identity matrix. On one hand, for each $\mathbf{x} \in \ell_{\infty}$ we have $B\mathbf{x} = A\mathbf{x} + \mathbf{x}$ and $A\mathbf{x} \in c_0(\mathcal{Z}) \cap \ell_{\infty}$, hence by Lemma 3.1 we get $\mathrm{core}_{B\mathbf{x}}(\mathcal{Z}) = \mathrm{core}_{\mathbf{x}}(\mathcal{Z})$. Thus equality (2) holds for the matrix B. On the other hand,

$$\mathcal{Z}\text{-}\limsup_{n\to\infty}\sum_{k\in\omega}|b_{n,k}|=1+\mathcal{Z}\text{-}\limsup_{n\to\infty}\sum_{k\in\omega}|a_{n,k}|\geq 1+\mathcal{Z}\text{-}\limsup_{n\to\infty}\sum_{k\in I}|a_{n,k}|=2.$$

This shows that B does not satisfy condition (L2).

Proof of Theorem 2.4. Since $\mathcal{I} \leq_{RK} \mathcal{J}$, there exists a map $h : \omega \to \omega$ such that $S \in \mathcal{I}$ if and only if $h^{-1}[S] \in \mathcal{J}$. Now, let $A = (a_{n,k} : n, k \in \omega)$ be the matrix defined by

$$a_{n,k} = \begin{cases} 1 & \text{if } k = h(n), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A \in (\ell_{\infty}, \ell_{\infty})$ since every row contains a single 1 (however, A is not necessarily regular if the witnessing map h cannot be chosen finite-to-one). Fix $\mathbf{x} \in \ell_{\infty}$, let $U \subseteq \mathbf{R}$ be a nonempty open set, and define $S := \{n \in \omega : x_n \in U\}$. Observe also that $A_n \mathbf{x} = \sum_k a_{n,k} x_k = x_{h(n)}$ for all $n \in \omega$. It follows that $S \in \mathcal{I}$ if and only if

$$h^{-1}[S] = \{ n \in \omega : x_{h(n)} \in U \} = \{ n \in \omega : A_n \mathbf{x} \in U \} \in \mathcal{J}.$$

This implies that $\Gamma_{\boldsymbol{x}}(\mathcal{I}) = \Gamma_{A\boldsymbol{x}}(\mathcal{I})$, so that by [22, Corollary 2.3] we get

$$\operatorname{core}_{\boldsymbol{x}}(\mathcal{I}) = \operatorname{co}(\Gamma_{\boldsymbol{x}}(\mathcal{I})) = \operatorname{co}(\Gamma_{A\boldsymbol{x}}(\mathcal{J})) = \operatorname{core}_{A\boldsymbol{x}}(\mathcal{J}).$$

Therefore equality (2) holds.

For our last proof, we need to recall that an ideal \mathcal{I} on ω is said to be:

(i) a P-ideal if for all increasing sequences (I_n) in \mathcal{I} there exists $I \in \mathcal{I}$ such that $I_n \setminus I \in \mathcal{I}$ for all $n \in \omega$;

- (ii) a P^+ -ideal if for all decreasing sequences (I_n) in \mathcal{I}^+ there exists $I \in \mathcal{I}^+$ such that $I \setminus I_n \in \text{Fin for all } n \in \omega$;
- (iii) tall if every infinite set $S \subseteq \omega$ contains an infinite subset $I \subseteq S$ such that $I \in \mathcal{I}$;
- (iv) nowhere tall if, for every $S \in \mathcal{I}^+$, the ideal $\mathcal{I} \upharpoonright S := \mathcal{I} \cap \mathcal{P}(S)$ is not tall.

Proof of Theorem 2.5. If PART. Suppose that $\mathcal{I} = \{S \subseteq \omega : S \cap T \in \text{Fin}\}$ for some infinite $T \subseteq \omega$ (i.e., $\mathcal{I} = \text{Fin}$ if T is cofinite or \mathcal{I} is an isomorphic copy of $\text{Fin} \oplus \mathcal{P}(\omega)$ on ω otherwise). Then $\mathcal{I} \leq_{\text{RK}}$ Fin. In fact, if $(t_n : n \in \omega)$ denotes the increasing enumeration of elements of T, one can choose the witnessing map $h : \omega \to \omega$ defined by $h(n) = t_n$ for all $n \in \omega$. The claim follows by Theorem 2.4.

ONLY IF PART. Let \mathcal{I} be an ideal on ω for which there exists a matrix $A \in (\ell_{\infty}, \ell_{\infty})$ which satisfies equality (2) with $\mathcal{I} = \text{Fin.}$ We divide the remaining proof in several claims.

CLAIM 1. \mathcal{I} is an analytic ideal.

Proof. Observe that, for each $E \subseteq \omega$, we have $E \in \mathcal{I}$ if and only if \mathcal{I} - $\limsup \mathbf{1}_E \leq 0$. It follows by equality (2) that

$$\mathcal{I} = \{ E \subseteq \omega : \limsup A \mathbf{1}_E \le 0 \} = \bigcap_{p \in \omega} \bigcup_{q \in \omega} \bigcap_{n \ge q} G_{p,n},$$

where $G_{p,n} := \{ E \subseteq \omega : \sum_{k \in E} a_{n,k} < 2^{-p} \}$ for all $n, p \in \omega$. Hence, it is sufficient to show that each $G_{n,p}$ is open. For, fix $n, p \in \omega$. If $G_{n,p} = \emptyset$, then it is open. Otherwise fix $E \in G_{n,p}$. Since $\sum_{k} |a_{n,k}| \le ||A|| < \infty$, there exists $k_0 \in \omega$ such that

$$\sum_{k>k_0} |a_{n,k}| < \frac{1}{2} \left(2^{-p} - \sum_{k \in E} a_{n,k} \right).$$

Now, let $F \subseteq \omega$ be a set such that $E \cap [0, k_0] = F \cap [0, k_0]$. It follows that

$$\sum_{k \in F} a_{n,k} \le \sum_{k \in F \cap [0,k_0]} a_{n,k} + \sum_{k \in F \setminus [0,k_0]} |a_{n,k}|$$

$$\le \sum_{k \in E \cap [0,k_0]} a_{n,k} + \sum_{k > k_0} |a_{n,k}|$$

$$\le \sum_{k \in E} a_{n,k} + 2 \sum_{k > k_0} |a_{n,k}| < 2^{-p}.$$

This shows that $F \in G_{n,p}$. Hence E is an interior point, so that $G_{n,p}$ is open. Therefore \mathcal{I} is a $G_{\delta\sigma\delta}$ -ideal.

CLAIM 2. \mathcal{I} is a P^+ -ideal.

Proof. Let us suppose for the sake of contradiction that \mathcal{I} is not a P^+ -ideal, hence it is possible to fix a strictly decreasing sequence $(I_n:n\in\omega)$ in \mathcal{I}^+ such that for all sequences $(F_n:n\in\omega)$ of finite sets with $F_n\subseteq I_n\setminus I_{n+1}$ we have $\bigcup_n F_n\in\mathcal{I}$. Since $I_0\in\mathcal{I}^+$ it follows by equality (2) that $\limsup A\mathbf{1}_{I_0}=1$. Hence there exists $n_0\in\omega$ such that $A_{n_0}\mathbf{1}_{I_0}>1-2^{-0}$. Set $p_0:=0$ and pick an integer $q_0>p_0$ such that $\sum_{k\in F_0}a_{n_0,k}\mathbf{1}_{I_0}(k)>1-2^{-0}$, where $F_0:=\omega\cap[p_0,q_0]$. Recall also that A is $(\mathcal{I},\operatorname{Fin})$ -regular since it satisfies (2), hence it is regular. In particular, by Theorem 1.1, we have $\lim_k a_{n,k}=0$ and $\sum_k |a_{n,k}|<\infty$ for all $n,k\in\omega$. At this point, suppose that n_{i-1} and $F_{i-1}:=\omega\cap[p_{i-1},q_{i-1}]$ have been defined for some positive integer i. Then, proceed recursively as follows:

(i) Pick an integer $p_i > q_{i-1}$ with the property that

$$\sum_{k \ge p_i} |a_{n_j,k}| < 2^{-i}$$

for all $j \in \omega \cap [0, i-1]$.

(ii) Let $n_i > n_{i-1}$ be an integer such that

$$A_{n_i} \mathbf{1}_{I_i} > 1 - 2^{-i}$$
 and $\sum_{k < p_i} |a_{n,k}| < 2^{-i}$

for all integers $n \geq n_i$.

(iii) Let $q_i > p_i$ be an integer such that

$$\sum_{k \in F_i} a_{n_i,k} \mathbf{1}_{I_i}(k) > 1 - 2^{1-i},$$

where $F_i := \omega \cap [p_i, q_i]$. (Note that this is possible because $\sum_{k \geq p_i} a_{n_i,k} \mathbf{1}_{I_i}(k)$ is at least $A_{n_i} \mathbf{1}_{I_i} - \sum_{k < p_i} |a_{n_i,k}| > 1 - 2^{1-i}$.)

To conclude, define $F := \bigcup_i F_i \cap I_i$. By the standing hypothesis, we have $F \in \mathcal{I}$, hence by equality (2) we get $\limsup A\mathbf{1}_F = \mathcal{I}$ - $\limsup \mathbf{1}_F = 0$. On the other hand, it follows by the construction above that, for all $i \geq 1$,

$$\limsup_{n \to \infty} A_n \mathbf{1}_F \ge \limsup_{i \to \infty} \sum_{k \in F} a_{n_i,k}$$

$$\ge \limsup_{i \to \infty} \left(\sum_{k \in F_i \cap I_i} a_{n_i,k} - \sum_{k < p_i} |a_{n_i,k}| - \sum_{k \ge p_{i+1}} |a_{n_i,k}| \right)$$

$$\ge \limsup_{i \to \infty} (1 - 2^{1-i} - 2^{-i} - 2^{-1-i}) = 1.$$

This contradiction proves that \mathcal{I} is a P^+ -ideal.

CLAIM 3. \mathcal{I} is a P-ideal.

Proof. Let us suppose for the sake of contradiction that \mathcal{I} is not a P-ideal, hence it is possible to fix an increasing sequence $(I_n : n \in \omega)$ in \mathcal{I} such that, for all sequences $(F_n : n \in \omega)$ with

 $F_n \subseteq D_n := I_{n+1} \setminus I_n$ for each n, we have $\bigcup_n (D_n \setminus F_n) \in \mathcal{I}^+$. Without loss of generality, we can assume that $I_0 = \emptyset$. Define

$$S := \{ n \in \omega : D_n \notin \operatorname{Fin} \}.$$

It is easy to see that, if S is finite, then $(I_n : n \in \omega)$ cannot be a sequence witnessing that \mathcal{I} is not a P-ideal: indeed, in such case, $I := I_0$ if $S = \emptyset$ or $I := I_{1+\max S}$ if $S \neq \emptyset$ satisfies $I_n \setminus I \in F$ in for all $n \in \omega$. Hence S has to be infinite, which implies that, passing to a suitable subsequence, we can assume without loss of generality that D_n is infinite for all $n \in \omega$.

Now, note that, since each I_n belongs to \mathcal{I} , then $\lim A\mathbf{1}_{I_n} = 0$ for all $n \in \omega$ by equality (2). Let $(k_n : n \in \omega)$ be a strictly increasing sequence in ω such that

$$\sum_{k>k_n} |a_{n,k}| < 2^{-n} \quad \text{for all } n \in \omega.$$
 (7)

It follows by the IF PART in the proof of Theorem 2.3 with $\mathcal{J} = \text{Fin that } \lim_n \sum_k a_{n,k}^- = 0$. In particular, there exists a strictly increasing sequence $(h_m : m \in \omega)$ such that

$$\sum_{k \in \omega} a_{n,k}^- < 2^{-m} \quad \text{for all } n \ge h_m. \tag{8}$$

Let also $(t_m : m \in \omega)$ be a strictly increasing sequence in ω such that

$$t_m \ge h_m \text{ and } \left| \sum_{i \le m} A_n \mathbf{1}_{D_i} \right| < 2^{-m} \quad \text{for all } m \in \omega \text{ and } n \ge t_m.$$
 (9)

To conclude, define $F_n := \omega \cap [0, k_{t_n}]$ for all $n \in \omega$ and set $D_\infty := \bigcup_n (D_n \setminus F_n)$. On the one hand, it follows by the standing hypothesis that $D_\infty \in \mathcal{I}^+$, hence by equality (2) we have $\limsup A\mathbf{1}_{D_\infty} = 1$. On the other hand, pick $m \in \omega$ and fix $n \in [t_m, t_{m+1})$. It follows that

$$|A_n \mathbf{1}_{D_{\infty}}| = \left| \sum_{i \in \omega} A_n \mathbf{1}_{D_i \setminus F_i} \right|$$

$$\leq \sum_{k > k_n} |a_{n,k}| + \left| \sum_{i \in \omega} \sum_{k \le k_n} a_{n,k} \mathbf{1}_{D_i \setminus F_i}(k) \right|$$

$$\leq 2^{-n} + \left| \sum_{k \le k_n} \sum_{i \in \omega} a_{n,k} \mathbf{1}_{D_i \setminus F_i}(k) \right|,$$

where at the last inequality we used (7). At this point, notice that, if $k \leq k_n$ and i > m then $\mathbf{1}_{D_i \setminus F_i}(k) = 0$ since $\min(D_i \setminus F_i) > \max(F_i) = k_{t_i} \geq k_{t_{m+1}} > k_n$. Taking into account that

 $n \geq t_m \geq m$, inequality (8), and that $t_m \geq h_m$, we obtain

$$|A_n \mathbf{1}_{D_{\infty}}| \leq 2^{-m} + \left| \sum_{k \leq k_n} \sum_{i \leq m} a_{n,k} \mathbf{1}_{D_i \setminus F_i}(k) \right|$$

$$\leq 2^{-m} + \left| \sum_{k \leq k_n} \sum_{i \leq m} a_{n,k} \mathbf{1}_{D_i}(k) \right| + \sum_{k \in \omega} a_{n,k}^-$$

$$\leq 2^{1-m} + \left| \sum_{k \leq k_n} \sum_{i \leq m} a_{n,k} \mathbf{1}_{D_i}(k) \right|.$$

Lastly, using also inequality (9), we get

$$|A_{n}\mathbf{1}_{D_{\infty}}| \leq 2^{1-m} + \left| \sum_{k \in \omega} \sum_{i \leq m} a_{n,k} \mathbf{1}_{D_{i}}(k) \right| + \left| \sum_{k > k_{n}} \sum_{i \leq m} a_{n,k} \mathbf{1}_{D_{i}}(k) \right|$$

$$\leq 2^{1-m} + \left| \sum_{i \leq m} A_{n}\mathbf{1}_{D_{i}} \right| + \sum_{k > k_{n}} |a_{n,k}|$$

$$\leq 2^{1-m} + 2^{-m} + 2^{-n} \leq 4^{1-m}.$$

This proves that $\lim A\mathbf{1}_{D_{\infty}}=0$, which gives the desired contradiction.

Thanks to Claims 1, 2, and 3, \mathcal{I} is an analytic P-ideal which is also a P^+ -ideal. Although it will not be used in the following results, it follows by [3, Theorem 2.5] that \mathcal{I} is a necessarily F_{σ} P-ideal (we omit details).

CLAIM 4. \mathcal{I} is not tall.

Proof. Let us suppose for the sake of contradiction that \mathcal{I} is tall. Define the infinite matrix $A^+ := (a_{n,k}^+ : n, k \in \omega)$. Since $\lim_n \sum_k a_{n,k}^- = 0$ (cf. the proof of Claim 3), it follows that A^+ is a nonnegative $(\mathcal{I}, \operatorname{Fin})$ -regular matrix; in particular, it is a nonnegative regular matrix. At this point, define the map $\mu^* : \mathcal{P}(\omega) \to \mathbf{R}$ by

$$\mu^*(S) := \limsup A^+ \mathbf{1}_S \quad \text{ for all } S \subseteq \omega.$$

Note also that $\lim(A\boldsymbol{x}-A^+\boldsymbol{x})=0$ for all $\boldsymbol{x}\in\ell_{\infty}$, hence by Lemma 3.1 and equality (2)

$$\operatorname{core}_{A^+\boldsymbol{x}}(\operatorname{Fin}) = \operatorname{core}_{\boldsymbol{x}}(\mathcal{I})$$
 for all sequences $\boldsymbol{x} \in \ell_{\infty}$.

Thus $\mathcal{I} = \{S \subseteq \omega : \mathcal{I}\text{-}\lim \mathbf{1}_S = 0\} = \{S \subseteq \omega : \lim A^+\mathbf{1}_S = 0\} = \{S \subseteq \omega : \mu^*(S) = 0\}.$ Since \mathcal{I} is not tall, it follows by [8, Proposition 7.2] that $\lim_n \sup_k a_{n,k}^+ = 0$. In addition, recalling that A^+ is a nonnegative matrix, we have also that $\lim_n a_{n,k}^+ = 0$ and $\sum_k a_{n,k}^+ < \infty$ for all $n, k \in \omega$ by Theorem 1.1. It follows by [8, Theorem 6.2] that the function μ^* has the weak Darboux property, i.e., for each $S \subseteq \omega$ and $y \in [0, \mu^*(S)]$ there exists $X \subseteq S$ such that $\mu^*(X) = y$, cf. [27, Section 2]. This implies that there exists a decreasing sequence $(I_m : m \in \omega)$ of subsets of ω such that

$$\mu^{\star}(I_m) = 2^{-m}$$
 for all $m \in \omega$.

At this point, let $I \subseteq \omega$ be a set such that $J_m := I \setminus I_m \in \text{Fin for all } m \in \omega$. Since μ^* is monotone and subadditive, we obtain

$$\mu^*(I) \le \mu^*(I_m) + \mu^*(J_m) = 2^{-m} + \limsup_{n \to \infty} \sum_{k \in J_m} a_{n,k} = 2^{-m}$$
 for all $m \in \omega$.

Hence $\mu^*(I) = 0$, i.e., $I \in \mathcal{I}$. This proves that \mathcal{I} is not a P^+ -ideal, which contradicts Claim 2. Therefore \mathcal{I} cannot be tall.

CLAIM 5. \mathcal{I} is a nowhere tall ideal.

Proof. Fix a set $S \in \mathcal{I}^+$ and consider the ideal $\tilde{\mathcal{I}} := \mathcal{I} \upharpoonright S$. Since \mathcal{I} is analytic by Claim 1 and $\mathcal{P}(S)$ is closed, then $\tilde{\mathcal{I}}$ is analytic as well. Moreover, since \mathcal{I} is both a P-ideal and P^+ -ideals by Claims 2 and 3, respectively, it is immediate that the same properties hold for $\tilde{\mathcal{I}}$. Let $\tilde{A} = (\tilde{a}_{n,k} : n, k \in \omega)$ be the matrix defined by $\tilde{a}_{n,k} := a_{n,k}$ if $k \in S$ and $\tilde{a}_{n,k} := 0$ otherwise. Now, note that, by equality (2),

$$\tilde{\mathcal{I}} = \{ X \subseteq S : \mathcal{I}\text{-}\lim \mathbf{1}_X = 0 \} = \{ X \subseteq S : \lim A \mathbf{1}_X = 0 \}$$
$$= \{ X \subseteq S : \lim \tilde{A} \mathbf{1}_X = 0 \} = \{ X \subseteq S : \tilde{\mu}^*(X) = 0 \},$$

where $\tilde{\mu}^*(X) := \limsup_n \tilde{A}^+ \mathbf{1}_X$ for each $X \subseteq S$. Lastly, observe that \tilde{A}^+ has the following properties: $\limsup_n \sum_k \tilde{a}_{n,k}^+ = 1$, $\lim_n \tilde{a}_{n,k}^+ = 0$, and $\sum_k \tilde{a}_{n,k}^+ < \infty$ for all $n, k \in \omega$. In particular, even if \tilde{A}^+ is not necessarily regular, it satisfies the hypotheses of [8, Theorem 6.2 and Proposition 7.2]. Hence, we proceed verbatim as in Claim 4 and we obtain that $\tilde{\mathcal{I}}$ is not tall. Since S is arbitrary, we conclude that \mathcal{I} is a nowhere tall ideal.

Thanks to Claims 1, 3, and 5 we know that \mathcal{I} is an analytic P-ideal which is also nowhere tall (notice that such properties are invariant under isomorphisms). Then, it is known that \mathcal{I} is necessarily Fin or (an isomorphic copy on ω of) Fin $\oplus \mathcal{P}(\omega)$ or $\{\emptyset\} \times$ Fin, see e.g. [9, Corollary 1.2.11] or [11, Theorem 2.26]. Finally, \mathcal{I} has to be also a P^+ -ideal by Claim 2. Hence, it is immediate to check that \mathcal{I} cannot be a copy of $\{\emptyset\} \times$ Fin. This concludes the proof.

Remark 3.4. Pick an infinite set $T \subseteq \omega$ which is not cofinite and define $\mathcal{J} := \{S \subseteq \omega : S \cap T \in \text{Fin}\}$ (hence, \mathcal{J} is an isomorphic copy on ω of $\text{Fin} \oplus \mathcal{P}(\omega)$). Pick also an ideal \mathcal{I} on ω such that $\mathcal{I} \leq_{RK} \mathcal{J}$. Thanks to Theorem 2.4, there exists an infinite matrix $A \in (\ell_{\infty}, \ell_{\infty})$ such that equality (2) holds. Now, as it has been observed in the proof of the IF PART of Theorem 2.5, we have also $\mathcal{J} \leq_{RK} \text{Fin.}$ Hence, with the same argument, there exists a matrix $B \in (\ell_{\infty}, \ell_{\infty})$ such that $\text{core}_{Bx}(\text{Fin}) = \text{core}_{x}(\mathcal{J})$ for all sequences $x \in \ell_{\infty}$. In addition, by the proof of Theorem 2.4, it is possible to assume that each row of B contains a single 1. Set C := BA (observe that each entry of C is well defined) and note that, if $x \in \ell_{\infty}$ then Cx = B(Ax) is bounded, hence $C \in (\ell_{\infty}, \ell_{\infty})$. It follows also that

$$\forall x \in \ell_{\infty}, \quad \operatorname{core}_{Cx}(\operatorname{Fin}) = \operatorname{core}_{Ax}(\mathcal{J}) = \operatorname{core}_{x}(\mathcal{I}).$$

This does not contradict the claim of Theorem 2.5. In fact, we claim that, if $\mathcal{I} \leq_{RK} \mathcal{J}$ then either $\mathcal{I} = \text{Fin or } \mathcal{I}$ is an isomorphic copy of $\text{Fin} \oplus \mathcal{P}(\omega)$.

For, let $h: \omega \to \omega$ be a map which witnesses $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$. Observe that $W := h[T] \in \mathcal{I}^+$ since $T \in \mathcal{J}^+$, hence W is an infinite set. Considering that $h^{-1}[F] \in \mathrm{Fin}$ if and only if $F \in \mathrm{Fin}$ for each $F \subseteq \omega$, we obtain that

$$\forall S \in \omega, \quad S \in \mathcal{I} \iff h^{-1}[S] \in \mathcal{J} \iff h^{-1}[S] \cap T \in \operatorname{Fin} \\ \iff h^{-1}[S \cap W] \in \operatorname{Fin} \iff S \cap W \in \operatorname{Fin}.$$

To sum up, W is an infinite set and $\mathcal{I} = \{S \subseteq \omega : S \cap W \in \text{Fin}\}$. Therefore either $\mathcal{I} = \text{Fin}$ (in the case where W is cofinite) or \mathcal{I} is isomorphic to $\text{Fin} \oplus \mathcal{P}(\omega)$ (in the opposite case).

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References

- 1. H. S. Allen, *T-transformations which leave the core of every bounded sequence invariant*, J. London Math. Soc. **19** (1944), 42–46.
- A. Aveni and P. Leonetti, Most numbers are not normal, Math. Proc. Cambridge Philos. Soc. 175 (2023), no. 1, 1–11.
- 3. M. Balcerzak and P. Leonetti, On the relationship between ideal cluster points and ideal limit points, Topology Appl. **252** (2019), 178–190.
- 4. A. Bartoszewicz, S. Głąb, and A. Wachowicz, *Remarks on ideal boundedness, convergence and variation of sequences*, J. Math. Anal. Appl. **375** (2011), no. 2, 431–435.
- 5. J. Boos, Classical and modern methods in summability, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Assisted by Peter Cass, Oxford Science Publications.
- 6. J. Connor, J. A. Fridy, and C. Orhan, Core equality results for sequences, J. Math. Anal. Appl. 321 (2006), no. 2, 515–523.
- 7. J. Connor and P. Leonetti, A characterization of $(\mathcal{I}, \mathcal{J})$ -regular matrices, J. Math. Anal. Appl. **504** (2021), no. 1, Paper No. 125374, 10.
- 8. L. Drewnowski and P. J. Paúl, *The Nikodým property for ideals of sets defined by matrix summability methods*, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.) **94** (2000), no. 4, 485–503, Perspectives in mathematical analysis (Spanish).
- 9. I. Farah, Analytic quotients: theory of liftings for quotients over analytic ideals on the integers, Mem. Amer. Math. Soc. 148 (2000), no. 702, xvi+177.
- R. Filipów and J. Tryba, Ideal convergence versus matrix summability, Studia Math. 245 (2019), no. 2, 101–127.
- 11. R. Filipów and J. Tryba, Representation of ideal convergence as a union and intersection of matrix summability methods, J. Math. Anal. Appl. 484 (2020), no. 2, 123760, 21.
- 12. J. A. Fridy and C. Orhan, Statistical core theorems, J. Math. Anal. Appl. 208 (1997), no. 2, 520-527.
- J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior, Proc. Amer. Math. Soc. 125
 (1997), no. 12, 3625–3631.
- 14. V. Kadets and D. Seliutin, On relation between the ideal core and ideal cluster points, J. Math. Anal. Appl. 492 (2020), no. 1, 124430, 7.
- 15. T. Kania, A letter concerning Leonetti's paper 'Continuous projections onto ideal convergent sequences', Results Math. 74 (2019), no. 1, Paper No. 12, 4.
- M. K. Khan and C. Orhan, Matrix characterization of A-statistical convergence, J. Math. Anal. Appl. 335 (2007), no. 1, 406–417.
- 17. A. Kwela, Erdos-Ulam ideals vs. simple density ideals, J. Math. Anal. Appl. 462 (2018), no. 1, 114-130.

- 18. G. Laush and S. Park, Knopp's core theorem and subsequences of a bounded sequence, Proc. Amer. Math. Soc. 13 (1962), 971–974.
- 19. P. Leonetti, *Regular matrices of unbounded linear operators*, Proc. Roy. Soc. Edinburgh Sect. A, to appear (doi:10.1017/prm.2024.1).
- 20. P. Leonetti, Rough families, cluster points, and cores, J. Convex Anal. 32 (2025), no. 4, 1083–1090.
- 21. P. Leonetti, Continuous projections onto ideal convergent sequences, Results Math. 73 (2018), no. 3, Paper No. 114, 5.
- 22. P. Leonetti, Characterizations of the ideal core, J. Math. Anal. Appl. 477 (2019), no. 2, 1063–1071.
- 23. P. Leonetti, *Tauberian theorems for ordinary convergence*, J. Math. Anal. Appl. **519** (2023), no. 2, Paper No. 126798, 10.
- 24. P. Leonetti and M. Caprio, Turnpike in infinite dimension, Canad. Math. Bull. 65 (2022), no. 2, 416–430.
- 25. P. Leonetti and F. Maccheroni, *Characterizations of ideal cluster points*, Analysis (Berlin) **39** (2019), no. 1, 19–26.
- P. Leonetti and C. Orhan, On some locally convex FK spaces, Topology Appl. 322 (2022), Paper No. 108327, 9.
- 27. P. Leonetti and S. Tringali, Upper and lower densities have the strong Darboux property, J. Number Theory 174 (2017), 445–455.
- 28. M. A. Rincon-Villamizar and C. Uzcategui Aylwin, Banach spaces of *I-convergent sequences*, J. Math. Anal. Appl. **536** (2024), no. 2, Paper No. 128271, 19.
- 29. A. A. Ščerbakov, Cores of sequences of complex numbers and their regular transformations, Mat. Zametki **22** (1977), no. 6, 815–823.

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