Bases for free Lie superalgebras

Michael Vaughan-Lee

January 2024

Abstract

We describe a basis for free Lie superalgebras which uses the theory of basic commutators. The only description of bases for free Lie superalgebras that I have found in the literature is in the book Infinite Dimensional Lie Superalgebras by Bahturin et al. [1]. Their bases make use of the theory of Shirshov bases in free Lie algebras, and I believe that there is a case for writing up an alternative approach using basic commutators. An additional reason for publishing this note is that I use the basis described here in a forthcoming paper where I prove that 5-Engel Lie algebras of characteristic p for p > 7 are nilpotent of class at most 11.

1 Introduction

Let L be a free Lie algebra over a commutative ring R with 1, and let L have free generating set A. Then L is a free R-module, and there are three standard ways described in the literature for obtaining a basis for L over R. The first two are due to Lyndon [2] and Shirshov [3], and the third uses Philip Hall's collection process. To obtain a Shirshov basis we proceed as follows. We let A^* be the set of all associative words $a_1a_2...a_n$ $(n \ge 1)$ where $a_1, a_2, ..., a_n \in A$ If $u = a_1a_2...a_n \in A^*$ and $v = b_1b_2...b_m \in A^*$ then we set

$$uv = a_1 a_2 \dots a_n b_1 b_2 \dots b_m,$$

turning A^* into a semigroup. We assume that there is a total order < on the set A, and if $u, v \in A^*$ we let u < v if u is lexicographically earlier than v. We say that $a_1 a_2 \ldots a_n$ is regular if

$$a_1 a_2 \dots a_n > a_{i+1} a_{i+2} \dots a_n a_1 a_2 \dots a_i$$

for all i = 1, 2, ..., n - 1. (So elements of A are regular.) We then define a map π from the set of regular words into L as follows. If $a \in A$ then we let $\pi(a) = a$. If w is a regular word of length greater than 1 then we write w = uv where v is chosen to have maximal length subject to v being regular and being a proper subword of w. It turns out that this

choice of v implies that u is also regular. Then we recursively define $\pi(uv) = [\pi(u), \pi(v)]$. For a proof that the image of π is a basis for L (the Shirshov basis) see Chapter 2 of [1]. The proof takes about 10 pages, and is extremely technical!

The Lyndon basis is defined similarly. A Lyndon word is an element $a_1 a_2 \dots a_n \in A^*$ such that

$$a_1 a_2 \dots a_n < a_{i+1} a_{i+2} \dots a_n a_1 a_2 \dots a_i$$

for all i = 1, 2, ..., n-1. (Elements of A are Lyndon words.) We define a map θ from the set of Lyndon words into L. If $a \in A$ then we let $\theta(a) = a$. And if w is a Lyndon word of length greater than 1 we write w = uv where v is chosen to have maximal length subject to v being a Lyndon word, and v being a proper subword of w. And as with the Shirshov basis we recursively set $\theta(uv) = [\theta(u), \theta(v)]$. The set $\{\theta(w) \mid w \text{ is a Lyndon word}\}$ is the Lyndon basis for L.

The third method of obtaining a basis for L as a free R-module is to show that the basic commutators (or basic Lie products) on the free generators A form an R-module basis for L.

The basic commutators of weight one are the elements $a \in A$, which we assume to be an ordered set.

The basic commutators of weight two are the elements [a, b] where $a, b \in A$ and a > b. These are ordered arbitrarily among themselves, and so that they follow basic commutators of weight one.

The basic commutators of weight three are the elements [a, b, c] where $a, b, c \in A$ and $a > b \le c$. (We use the left-normed convention so that [a, b, c] denotes [[a, b], c].) The basic commutators of weight three are ordered arbitrarily among themselves, and so that they follow the basic commutators of weight two.

In general, if k > 3 the basic commutators of weight k are the commutators [c, d] where for some m, n such that m + n = k,

- 1. c, d are basic commutators of weight m, n respectively,
- 2. c > d,
- 3. if, in the definition of basic commutators of weight m, c was defined to be [e, f] then $f \leq d$.

The basic commutators of weight k are then ordered arbitrarily among themselves, and so that they follow the basic commutators of weight k-1.

A proof that the basic commutators form an R-module basis for L can be found in my book [4].

2 Lie superalgebras

A Lie superalgebra is a \mathbb{Z}_2 -graded algebra $L = L_0 \oplus L_1$ with a bilinear product [,] such that

$$[L_0, L_0], [L_1, L_1] \le L_0,$$

 $[L_0, L_1], [L_1, L_0] \le L_1.$

Elements in L_0 are said to be even elements, and elements in L_1 are said to be odd. If a is even then we set |a| = 0, and if a is odd then we set |a| = 1. Odd elements and even elements are said to be homogeneous. Finally, the product [,] must satisfy the following relations for all homogeneous elements a, b, c.

$$\begin{split} [b,a] &= -(-1)^{|a|.|b|}[a,b].\\ (-1)^{|a|.|c|}[a,[b,c]] &+ (-1)^{|b|.|a|}[b,[c,a]] + (-1)^{|c|.|b|}[c,[a,b]] = 0. \end{split}$$

It is helpful to note that these relations imply that if a, b, c are homogeneous elements then

$$[a, [b, c]] = [a, b, c] - (-1)^{|b| \cdot |c|} [a, c, b].$$

We also add in the requirement that [a, a] = 0 for even elements, and the requirement that [a, a, a] = 0 for odd elements. (These extra requirements are redundant if 2 and 3 are invertible.)

Now let $L = L_0 \oplus L_1$ be a free Lie superalgebra over a ring R with 1, freely generated by an ordered set (A, <). We assume that the elements of A are homogeneous, some even and some odd. Let $S = \{\pi(w) \mid w \text{ is a regular word in } A^*\}$, and let (C, <) be a (complete) set of basic commutators on A. Note that the definitions of S and C are independent of the \mathbb{Z}_2 -grading on L. Note also that the elements of S and C are homogeneous elements of S.

Theorem 1 (Bahturin et al. [1]) The free Lie superalgebra L is a free R-module with basis $S \cup \{[w, w] \mid w \in S \text{ is odd}\}.$

Theorem 2 The free Lie superalgebra L is a free R-module with basis

$$C \cup \{[w,w] \,|\, w \in \mathit{C} \,\, is \,\, odd\}.$$

3 Proof of Theorem 2

Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a commutative ring R with 1, and suppose that L is generated by a set A whose elements are all homogeneous. (We are not assuming here that L is free.) Let (C, <) be a complete set of basic commutators on the generators in A. As a preliminary step to proving Theorem 2 we show that $C \cup \{[c, c] \mid c \in C \text{ is odd}\}$ spans L as an R-module.

Lemma 3 Let a_1, a_2, \ldots, a_n be homogeneous elements in L and let c be a Lie product of a_1, a_2, \ldots, a_n in some order with some bracketing. Then c is a \mathbb{Z} -linear combination of left-normed Lie products $[a_1, a_{2\sigma}, a_{3\sigma}, \ldots, a_{n\sigma}]$ where σ is a permutation of $\{2, 3, \ldots, n\}$.

Proof. The proof is by induction on n, the case n=1 being trivial, and the case n=2 following from the fact that $[a_2, a_1] = \pm [a_1, a_2]$. So suppose that n > 2, and suppose that our claim holds true for smaller values of n. Let a_1, a_2, \ldots, a_n be homogeneous elements in L and let c be a Lie product of a_1, a_2, \ldots, a_n in some order with some bracketing. Let c = [u, v]. Since $[u, v] = \pm [v, u]$ it is sufficient to consider the case when u involves a_1 . We next show that it is sufficient to consider the case when v has weight one as a Lie product in (some of) a_2, a_3, \ldots, a_n . If v has weight greater than 1 we write v = [x, y] and then

$$[u, v] = [u, [x, y]] = [[u, x], y] \pm [[u, y], x].$$

So [u, v] is a \mathbb{Z} -linear combination of two Lie products [u', v'] where u' = [u, x] or [u, y], and where the weight of v' is smaller than the weight of v. Repeating this argument as necessary we see that c = [u, v] is a \mathbb{Z} -linear combination of Lie products [u', v'] where u' involves a_1 and where v' has weight one. By induction u' is a \mathbb{Z} -linear combination of left-normed Lie products with first entry a_1 , and this completes the proof of the lemma. \square

Now let c be a Lie product of weight n in elements a_1, a_2, \ldots, a_n from the generating set A, and suppose that $a_1 \leq a_i$ for $i = 2, 3, \ldots, n$. By Lemma 3 we can express c as a \mathbb{Z} -linear combination of left-normed Lie products $[c_1, c_2, \ldots, c_n]$ where $c_1 = a_1$ and c_2, c_3, \ldots, c_n is a permutation of a_2, a_3, \ldots, a_n .

Suppose first that $c_1 < c_2$. Then $[c_2, c_1]$ is a basic commutator in C, and

$$[c_1, c_2, \ldots, c_n] = \pm [[c_2, c_1], c_3, \ldots, c_n].$$

If we set $d_2 = [c_2, c_1]$, $d_i = c_i$ for i = 3, ..., n, then we see that

$$[c_1, c_2, \dots, c_n] = \pm [d_2, d_3, \dots, d_n]$$

and that $[d_2, d_3, \ldots, d_n]$ is a Lie product of basic commutators d_2, d_3, \ldots, d_n which have the property that if $2 \le i \le n$ and if $d_i = [e, f]$ in its definition as a basic commutator, then $f \le d_j$ for $j = 2, 3, \ldots, n$.

Next suppose that $c_1 = c_2$. If c_1 is even then $[c_1, c_2] = 0$, so we may assume that c_1 is odd. If n = 2 then $[c_1, c_2, \ldots, c_n] = [c_1, c_1]$ where c_1 is an odd basic commutator. If $n \ge 3$ then we may suppose that $c_3 > c_1$ since $[c_1, c_1, c_1] = 0$. But then

$$[c_1, c_2, c_3] = [[c_1, c_1], c_3] = -[c_3, [c_1, c_1]] = -2[c_3, c_1, c_1],$$

and $[c_3, c_1, c_1]$ is a basic commutator in C. In this case, if we set $d_3 = [c_3, c_1, c_1]$, $d_i = c_i$ for i = 4, ..., n, then we see that

$$[c_1, c_2, \dots, c_n] = -2[d_3, d_4, \dots, d_n]$$

and that $[d_3, d_4, \ldots, d_n]$ is a Lie product of basic commutators d_3, d_4, \ldots, d_n which have the property that if $3 \le i \le n$ and if $d_i = [e, f]$ in its definition as a basic commutator, then $f \le d_j$ for $j = 3, 4, \ldots, n$.

So we see that if c is a Lie product of weight n in elements a_1, a_2, \ldots, a_n from the generating set A, then c is a \mathbb{Z} -linear combination of elements of the following three types:

- 1. a basic commutator in C,
- 2. [c,c] where c is an odd basic commutator in C,
- 3. $[c_1, c_2, \ldots, c_k]$ (1 < k < n) where c_1, c_2, \ldots, c_k are basic commutators in C with the property that if $1 \le i \le k$ and if c_i is defined to be [e, f] then $f \le c_j$ for $j = 1, 2, \ldots, k$.

Consider a Lie product of type 3. By Lemma 3 we can write it as a \mathbb{Z} -linear combination of Lie products $[d_1, d_2, \ldots, d_k]$ where d_1, d_2, \ldots, d_k is a permutation of c_1, c_2, \ldots, c_k and where $d_1 \leq d_i$ for $i = 1, 2, \ldots, k$. Let $[d_1, d_2, \ldots, d_k]$ be one of the Lie products in this linear combination.

If $d_1 < d_2$ then $[d_2, d_1]$ is a basic commutator in C and

$$[d_1, d_2, \dots, d_k] = \pm [[d_2, d_1], d_3, \dots, d_k].$$

Setting $e_2 = [d_2, d_1]$, $e_i = d_i$ for i = 3, 4, ..., k, we see that $[[d_2, d_1], d_3, ..., d_k]$ is a left-normed Lie product of length k - 1 in basic commutators $e_2, e_3, ..., e_k$ with the property that if $2 \le i \le k$ and if e_i is defined to be [e, f] then $f \le e_j$ for j = 2, 3, ..., k.

If $d_1 = d_2$ then (as above) we see that either $[d_1, d_2, \ldots, d_k] = 0$, or if k = 2 then $[d_1, d_2, \ldots, d_k] = [c, c]$ where c is an odd basic commutator in C, or if k > 2 then $d_3 > d_1$, $[d_3, d_1, d_1]$ is a basic commutator in C, and

$$[d_1, d_2, \dots, d_k] = -2[[d_3, d_1, d_1], d_4, \dots, d_k].$$

If we set $e_3 = [d_3, d_1, d_1]$, $e_i = d_i$ for i = 4, ..., k, then $[[d_3, d_1, d_1], d_4, ..., d_k]$ is a left-normed Lie product of length k-2 in basic commutators $e_3, e_4, ..., e_k$ with the property that if $3 \le i \le k$ and if e_i is defined to be [e, f] then $f \le e_j$ for j = 3, ..., k.

Continuing in this way by reverse induction on k we see that any Lie product of elements from the generating set A is a \mathbb{Z} -linear combination of elements of type 1 or type 2 above. This proves the following lemma.

Lemma 4 Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a commutative ring R with 1, and suppose that L is generated by a set A whose elements are all homogeneous. Let (C, <) be a complete set of basic commutators on the generators in A. Then

$$C \cup \{[c,c] \mid c \in C \text{ is odd}\}$$

spans L as an R-module.

To complete the proof of Theorem 2 we need to show that in the case when L is freely generated by A then $C \cup \{[c,c] \mid c \in C \text{ is odd}\}$ is linearly independent over R. It is sufficient to prove that this is the case when L is a free Lie superalgebra over the integers \mathbb{Z} , as this will imply that $L \otimes_{\mathbb{Z}} R$ is a free Lie superalgebra over R. We let $A = A_0 \oplus A_1$ be the free associative superalgebra over \mathbb{Z} , freely generated by a_1, a_2, \ldots, a_r , where these free generators are homogeneous (some even and some odd). So if a and b are two elements in A and either both lie in A_0 or both lie in A_1 then $ab \in A_0$, and if a and b are two elements in A with one element in A_0 and the other in A_1 the $ab \in A_1$. We define a bilinear bracket product [,] on A by setting

$$[a,b] = ab - (-1)^{|a|.|b|}ba$$

for homogeneous elements $a, b \in A$. It is straightforward to see that this turns A into a Lie superalgebra. We show that if L is the Lie sub-superalgebra of A generated by a_1, a_2, \ldots, a_r then L is free, and has a \mathbb{Z} -module basis $C \cup \{[c, c] \mid c \in C \text{ is odd}\}$ where C is a complete set of basic commutators on the generators.

Let A_n be the \mathbb{Z} -module spanned by all products $a_{i_1}a_{i_2}\ldots a_{i_n}$ of length n in the generators a_1, a_2, \ldots, a_r . Clearly A_n is a free \mathbb{Z} -module of rank r^n , with these products forming a basis. We define a basic product in A to be an element of the form $c_1c_2\ldots c_m$ $(m \geq 1)$ where c_1, c_2, \ldots, c_m are basic commutators in C, and where $c_1 \leq c_2 \leq \ldots \leq c_m$. And we define the weight of a basic product $c_1c_2\ldots c_m$ to be $\sum_{i=1}^m \operatorname{wt}(c_i)$.

Lemma 5 There are r^n basic products of weight n, and they span A_n .

Proof. We use the Hall collection process to express the generators $a_{i_1}a_{i_2} \dots a_{i_n}$ of A_n as linear combinations of basic products. Consider a generator $a_{i_1}a_{i_2} \dots a_{i_n}$. First we collect entries a_1 in the product towards the left. We look for a subword $a_{i_j}a_{i_{j+1}}$ where $i_j > 1$ and $i_{j+1} = 1$. If there are no such subwords then all entries a_1 are collected to the left, and

$$a_{i_1}a_{i_2}\dots a_{i_n} = a_1^m a_{j_1}a_{j_2}\dots a_{j_{n-m}}$$

for some $m \geq 0$, and some $j_1, j_2, \ldots, j_{n-m} > 1$. If there is a subword $a_{i_j}a_{i_{j+1}}$ of this form then $a_{i_j}a_{i_{j+1}} = [a_{i_j}, a_1] + \varepsilon a_1 a_{i_j}$ where $\varepsilon = \pm 1$, and $[a_{i_j}, a_1] \in C$. We then replace $a_{i_1}a_{i_2}\ldots a_{i_n}$ by

$$a_{i_1}a_{i_2}\ldots a_{i_{j-1}}[a_{i_j},a_1]a_{i_{j+2}}\ldots a_{i_n}+\varepsilon a_{i_1}a_{i_2}\ldots a_{i_{j-1}}a_1a_{i_j}a_{i_{j+2}}\ldots a_{i_n}.$$

We continue collecting a_1 to the left in each of the products

$$a_{i_1}a_{i_2}\dots a_{i_{j-1}}[a_{i_j},a_1]a_{i_{j+2}}\dots a_{i_n}, \ a_{i_1}a_{i_2}\dots a_{i_{j-1}}a_1a_{i_j}a_{i_{j+2}}\dots a_{i_n}$$

in turn. If $i_{i+2} = 1$ then

$$[a_{i_j}, a_1]a_{i_{j+2}} = [a_{i_j}, a_1]a_1 = [a_{i_j}, a_1, a_1] + \eta a_1[a_{i_j}, a_1]$$

where $\eta = \pm 1$, and we substitute $[a_{i_j}, a_1, a_1] + \eta a_1[a_{i_j}, a_1]$ for $[a_{i_j}, a_1]a_{i_{j+2}}$ in the product

$$a_{i_1}a_{i_2}\ldots a_{i_{j-1}}[a_{i_j},a_1]a_{i_{j+2}}\ldots a_{i_n}.$$

If $i_{j+2} > 1$ then we search $a_{i_1} a_{i_2} \dots a_{i_{j-1}} [a_{i_j}, a_1] a_{i_{j+2}} \dots a_{i_n}$ for a subword $a_i a_1$ with i > 1, and if we find one then we substitute $[a_i, a_1] \pm a_1 a_i$ for $a_i a_1$, as before. We apply the same procedure to

$$a_{i_1}a_{i_2}\ldots a_{i_{j-1}}a_1a_{i_j}a_{i_{j+2}}\ldots a_{i_n}.$$

We introduce the notation [b, a] (r = 0, 1, 2, ...) for repeated commutators, setting [b, a] = b, [b, a] = [b, a], [b, a] = [b, a, a], and so on. At any stage in the process of collecting entries a_1 to the left we have an expression for $a_{i_1}a_{i_2}...a_{i_n}$ as a linear combination of products of the form

$$a_1^{e_1}[a_{j_1,m_1} a_1] a_1^{e_2}[a_{j_2,m_2} a_1] a_1^{e_3} \dots [a_{j_k,m_k} a_1] a_1^{e_{k+1}}$$

$$(1)$$

where (for some k) $e_1, e_2, \ldots, e_{k+1}, m_1, m_2, \cdots, m_k \ge 0$, and where $j_1, j_2, \ldots, j_k > 1$. (Note that the commutators $[a_j, m, a_1]$ are basic commutators.) If $e_2 = \ldots = e_{k+1} = 0$ then all entries a_1 in (1) are collected to the left. If $e_{s+1} > 0$ then

$$[a_{j_s,m_s} a_1]a_1^{e_{s+1}} = [a_{j_s,m_s} a_1]a_1a_1^{e_{s+1}-1} = [a_{j_s,m_s+1} a_1]a_1^{e_{s+1}-1} + \varepsilon a_1[a_{j_s,m_s} a_1]a_1^{e_{s+1}-1}$$

for some $\varepsilon = \pm 1$, and we substitute

$$[a_{j_s,m_s+1}a_1]a_1^{e_{s+1}-1} + \varepsilon a_1[a_{j_s,m_s}a_1]a_1^{e_{s+1}-1}$$

for $[a_{j_s,m_s} a_1] a_1^{e_{s+1}}$ in (1). Continuing in this way we eventually obtain an expression for $a_{i_1} a_{i_2} \dots a_{i_n}$ as a linear combination of products

$$a_1^{e_1}[a_{j_1,m_1} a_1][a_{j_2,m_2} a_1] \dots [a_{j_k,m_k} a_1]$$
(2)

where all entries a_1 are collected to the left. Now the spanning elements $a_{i_1}a_{i_2}\dots a_{i_n}$ for A_n can all be expressed in the form

$$a_1^{e_1} a_{j_1} a_1^{m_1} a_{j_2} a_1^{m_2} \dots a_{j_k} a_1^{m_k} \tag{3}$$

where $e_1, m_1, \ldots, m_k \geq 0, j_1, j_2, \ldots, j_k > 1$ and there is a natural 1-1 correspondence between the set of all possible products of weight n of the form (2) and the set of all possible products of weight n of the form (3). So there are r^n possible products of weight n of the form (2), where all entries a_1 have been collected to the left.

We illustrate this by applying the Hall collection to a product baba where a and b are even generators of A and b > a.

$$baba = [b, a]ba + abba.$$

And

$$[b, a]ba = [b, a][b, a] + [b, a]ab = [b, a][b, a] + [b, a, a]b + a[b, a]b,$$

 $abba = ab[b, a] + abab = ab[b, a] + a[b, a]b + aabb.$

So

$$baba = aabb + ab[b, a] + 2a[b, a]b + [b, a][b, a] + [b, a, a]b.$$

Note that there are six possible products in a and b which have weight two in each of a and b:

And there are six possible products of this weight in which all entries a have been collected to the left:

$$aabb, \ a[b,a]b, \ ab[b,a], \ [b,a,a]b, \ [b,a][b,a], \ b[b,a,a].$$

Once we have an expression for $a_{i_1}a_{i_2}\ldots a_{i_n}$ as a linear combination of products where all the entries a_1 have been collected to the left, we collect entries a_2 in these products to the left, and then entries a_3 , and so on. There are only finitely many basic commutators in C which have weight at most n. Let these be b_1, b_2, \ldots, b_m with $b_1 < b_2 < \ldots < b_m$. Suppose we have reached the point where all entries b_1, b_2, \ldots, b_k have been collected to the left, but no b_i with i > k has been collected to the left. Then we have an expression for $a_{i_1}a_{i_2}\ldots a_{i_n}$ as a linear combination of products of the form

$$b_1^{e_1}b_2^{e_2}\dots b_k^{e_k}b_{k+1}^{e_{k+1}}c_1b_{k+1}^{r_1}c_2b_{k+1}^{r_2}\dots c_tb_{k+1}^{r_t}$$

$$\tag{4}$$

where $e_1, e_2, \ldots, e_{k+1}, r_1, r_2, \ldots, r_t \geq 0$, and where c_1, c_2, \ldots, c_t are basic commutators in C with $c_i > b_{k+1}$ for $i = 1, 2, \ldots, t$, and where if c_i is defined to be [e, f] then $f \leq b_k$. We assume by induction that there are r^n possible products of the form (4) in A_n . We then collect entries b_{k+1} to the left, introducing basic commutators $[c_{i,r}, b_{k+1}]$ (r > 0). When all entries b_{k+1} have been collected to the left then we will have an expression for $a_{i_1}a_{i_2}\ldots a_{i_n}$ as a linear combination of products of the form

$$b_1^{e_1}b_2^{e_2}\dots b_k^{e_k}b_{k+1}^{e_{k+1}}[c_{1,r_1}b_{k+1}][c_{2,r_2}b_{k+1}]\dots [c_{t,r_t}b_{k+1}].$$
 (5)

There is a 1-1 correspondence between the set of all possible products of weight n of the form (4) and the set of all possible products of weight n of the form (5). So we may assume that there are r^n possible products of weight n of the form (5). Continuing in this way we eventually obtain an expression for $a_{i_1}a_{i_2}\ldots a_{i_n}$ as a linear combination of products of weight n in which all entries b_1, b_2, \ldots, b_m have been collected to the left. These must all be basic products, and by induction the number of possible basic products of weight n is r^n . \square

Corollary 6 The set $C \cup \{[c,c] | c \in C \text{ is odd}\}$ is linearly independent.

Proof. Since there are r^n basic products of weight n, and since they span A_n , they must form a \mathbb{Z} -module basis for A_n and must be linearly independent. If $c \in C$ has weight n then c is a basic product of weight n. And if $c \in C$ is odd and [c, c] has weight n then $[c, c] = 2c^2$ where c^2 is a basic product of weight n. So

$$\{c \in C \mid \text{wt}(c) = n\} \cup \{[c, c] \mid c \text{ is odd, wt}([c, c]) = n\}$$

is linearly independent. Since

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_n \oplus \ldots$$

this implies that $C \cup \{[c, c] | c \in C \text{ is odd}\}$ is linearly independent.

Now let M be a free Lie superalgebra over \mathbb{Z} , freely generated by x_1, x_2, \ldots, x_r , and let $A = A_0 \oplus A_1$ be the free associative superalgebra over \mathbb{Z} , freely generated by a_1, a_2, \ldots, a_r , where $|a_i| = |x_i|$ for $i = 1, 2, \ldots, r$. As above we define a bracket product [,] on A by setting

$$[a,b] = ab - (-1)^{|a|.|b|}ba$$

for homogeneous elements $a, b \in A$. And as above, we let L be the Lie sub-superalgebra of A generated by a_1, a_2, \ldots, a_r . Let D be a complete set of basic commutators on x_1, x_2, \ldots, x_r , where we assume that $x_1 < x_2 < \ldots < x_r$. By Lemma 4, M is spanned by

$$D \cup \{[d,d] \mid d \in D \text{ is odd}\}.$$

There is a homomorphism from M onto L mapping x_i to a_i for i = 1, 2, ..., r, and this homomorphism maps D onto a complete set of basic commutators on $a_1, a_2, ..., a_r$. The image of $D \cup \{[d, d] \mid d \in D \text{ is odd}\}$ under this homomorphism is linearly independent, and so $D \cup \{[d, d] \mid d \in D \text{ is odd}\}$ is linearly independent. This proves Theorem 2 for free Lie superalgebras of finite rank. But if Theorem 2 holds for free Lie superalgebras of finite rank then it certainly holds true for free Lie superalgebras of arbitrary rank.

References

- [1] Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky, and M.V. Zaicev, *Infinite dimensional Lie superalgebras*, De Gruyter Expositions in Mathematics, 7, De Gruyter, Berlin, New York, 1992.
- [2] R.C. Lyndon, On Burnside's problem, Trans. Amer. Math. Soc. 77 (1954), 202–215.
- [3] A.I. Shirshov, On the bases of free Lie algebras, Algebra i Logika ${f 1}$ (1962), 14–19.
- [4] M.R. Vaughan-Lee, *The restricted Burnside problem*, second ed., Oxford University Press, 1993.