## ON THE COMPANION OF SPACES HAVING DENSE, RELATIVELY COUNTABLE COMPACT SUBSPACES

ISTVÁN JUHÁSZ, LAJOS SOUKUP, AND ZOLTÁN SZENTMIKLÓSSY

ABSTRACT. A topological space is said to be DRC (DRS) iff it possesses a dense, relatively countably compact (or relatively sequentially compact, respectively) subspace.

The concept of selectively pseudocompact game  $\operatorname{Sp}(X)$  and the selectively sequentially pseudocompact game  $\operatorname{Sp}(X)$  were introduced by Dorantes-Aldama and Shakhmatov. They explored the relationship between the existence of a winning strategy and a stationary winning strategy for player P in these games. In particular, they observed that there exists a stationary winning strategy in the game  $\operatorname{Sp}(X)$  ( $\operatorname{Sp}(X)$ ) for Player P iff X is DRC (or DRS, respectively).

In this paper we introduce natural weakening of the properties DRC and DRS: a space X is  $DRC_{\omega}$  ( $DRS_{\omega}$ ) iff there is a sequence  $\langle D_n : n \in \omega \rangle$  of dense subsets of X such that every sequence  $\langle d_n : n \in \omega \rangle$  with  $d_n \in D_n$  has an accumulation point (or contains a convergent subsequence, respectively).

These properties are also equivalent to the existence of some limited knowledge winning strategy on the corresponding games Sp(X) and Ssp(X).

Clearly, DRS implies DRC and  $DRS_{\omega}$ , DRC or  $DRS_{\omega}$  imply  $DRC_{\omega}$ . The main part of this paper is devoted to prove that apart from these trivial implications, consistently there are no other implications between these properties.

## 1. Introduction

The notion of pseudocompactness was introduced by Hewitt in [5]. The concept of relatively countably compact subspaces were explored by Marjanovic in [11] to show that a  $\Psi$ -space is pseudocompact. Berner [1] constructed pseudocompact spaces with and without having dense, relatively countably compact subspaces.

To simplify the formulation of our results, a topological space is said to be DRC (DRS) iff it possesses a dense, relatively countably compact (or relatively sequentially compact, respectively) subspace.

The concept of selective pseudocompact and selectively sequentially pseudocompact spaces were introduced in [2]. In a subsequent work, [3], the same authors introduced two related topological games and explored the connection between these classes of spaces and the existence of certain type of winning strategies in the defined games.

To start with, we recall some definitions from [2] and [3]. Given a topological space X and a sequence  $\vec{a} \in {}^{\omega}X$  write

$$acc(\vec{a}) = \{x \in X : \{n \in \omega : \vec{a}(n) \in U\} \text{ is infinite for each open } U \ni x\}.$$

A subspace  $Y \subset X$  is relatively countably compact iff  $acc(\vec{y}) \neq \emptyset$  for every  $\vec{y} \in {}^{\omega}Y$ . A subspace  $Z \subset X$  is relatively sequentially compact iff every sequence  $\vec{z} \in {}^{\omega}Z$  contains a subsequence converging to some point in X.

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A space X is selectively pseudocompact ([2, Def 2.2.]) if for every  $\vec{U} \in {}^{\omega}(\tau_X^+)$  there is  $\vec{x} \in {}^{\omega}X$  with  $\vec{x}(n) \in \vec{U}(n)$  such that  $acc(\vec{x}) \neq \emptyset$ .

A space X is selectively sequentially pseudocompact ([2, Def 2.3]) iff for every  $\vec{U} \in {}^{\omega}(\tau_X^+)$  there is  $\vec{x} \in {}^{\omega}X$  with  $\vec{x}(n) \in \vec{U}(n)$  such that  $\vec{x}$  contains a converging subsequence.

The following games were introduced in [3, Definition 5.1]. Given a space X define the games Sp(X) and Ssp(X) between players O and P as follows. The games are played in  $\omega$  turns. In both game in the  $n^{th}$  turn O picks a non-empty open set  $U_n$ , and then P picks a point  $x_n \in U_n$ .

In the game Sp(X) Player P wins iff  $acc(\langle x_n : n \in \omega \rangle) \neq \emptyset$ .

In the game Ssp(X) player P wins iff the sequence  $\langle x_n : n \in \omega \rangle$  contains a convergent subsequence.

In [3, Theorem 4.5] the authors observed that the statements (a) and (b) below are equivalent: (a) the space X is contains a dense, relatively countably compact subset, (b) P has a stationary winning strategy in the game Sp(X). Let us recall that strategy of P is stationary iff the moves of P depends on only from the last move of the opponent.

They also showed that (c) and (d) below are similarly equivalent: (c) the space X contains a dense, relatively sequentially compact subset, (b) P has a stationary winning strategy in the game Spp(X).

In [3] the authors refrain from introducing specific names for spaces having property (a) or property (c). To simplify the formulation of our results we introduce the following names and notations. We say that a space X is DRC ("densely relatively countably compact") iff it contains a dense, relatively countably compact subset, and Y is DRS ("densely relatively sequentially compact") iff it contains a dense, relatively sequentially compact subset.

We write  $P \uparrow_1 Sp(X)$  iff P has a stationary winning strategy in the game Sp(X). We define  $P \uparrow_1 Spp(X)$  similarly.

Next, we introduce natural weakenings of the DRC and DRS properties which are still stronger than the selectively pseudocompactness and selectively sequentially pseudocompactness, respectively.

## **Definition 1.1.** Let X be a topological space.

- (i) X is  $DRC_{\omega}$  iff there is a sequence  $\langle D_n : n \in \omega \rangle$  of dense subsets of X such that every sequence  $\langle d_n : n \in \omega \rangle$  with  $d_n \in D_n$  has an accumulation point.
- (ii) X is  $DRS_{\omega}$  iff there is a sequence  $\langle d_n : n \in \omega \rangle$  of dense subsets of X such that every sequence  $\langle d_n : n \in \omega \rangle$  with  $d_n \in D_n$  contains a convergent subsequence.

As it turns out, these properties have characterizations using the games Sp and Spp. To formulate our observation we need to introduce the following types of strategies of games which use only restricted information.

Write  $P \uparrow_{1,n} Sp(X)$  iff P has a winning strategy in the game such Sp(X) that the  $n^{th}$  move of P depends on only n and  $U_n$ . We define  $P \uparrow_{1,n} Ssp(X)$  similarly.

**Proposition 1.2.** A topological space X is  $DRC_{\omega}$  iff  $P \uparrow_{1,n} Sp(X)$ . A topological space Y is  $DRC_{\omega}$  iff  $P \uparrow_{1,n} Sp(Y)$ .

*Proof.* Assume that X is  $DRC_{\omega}$  witnessed by the sequence  $\langle D_n : n \in \omega \rangle$  of dense subsets of X. Let P play the following strategy: in the  $n^{th}$ -turn if O pick  $U_n$ , then let P choose an arbitrary  $d_n \in D_n \cap U_n$ . Then the sequence  $\langle d_n : n < \omega \rangle$  has an accumulation point so P wins. Moreover, the move of P depends on only n and  $U_n$ .

Assume know that  $\sigma: \omega \times \tau_X^+ \to X$  is a winning strategy of P. For  $n \in \omega$ , let

$$D_n = {\sigma(n, U) : U \in \tau^+(X)}.$$

Then  $D_n$  is dense because  $\sigma(n, U) \in U$ .

Moreover, if  $\langle d_n : n \in \omega \rangle$  is a sequence with  $d_n \in D_n$ , then we can pick  $U_n \in \tau_X^+$  with  $d_n = \sigma(n, U_n)$ . If O plays  $U_n$  in the  $n^{th}$  turn, then P produces the sequence  $\langle d_n : n < \omega \rangle$ . Since P wins,  $\langle d_n : n < \omega \rangle$  has accumulation point. So the sequence  $\langle D_n : n < \omega \rangle$  witnesses that X is  $DRC_{\omega}$ .

The second equivalence can be proved similarly.

Figure 1 summarizes the equivalences and the straightforward implications between these properties.

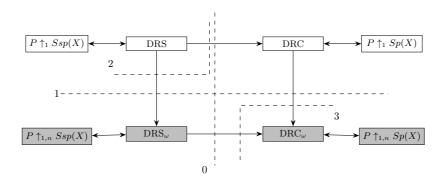


FIGURE 1. DRC and its companion

The aim of this paper is to show that these implications are not reversible. Since there are countably compact spaces without having any convergent sequences (for example,  $\omega^*$ ), DRC does not imply DRS $_{\omega}$ .

The main result of this paper is Theorem 1.3 below, which implies that the other implications are also not reversible.

Given a topological property Q, we say that a space X is hereditary Q iff every non-empty regular closed subset of X has property Q. For example, a pseudocompact space is "hereditary pseudocompact", although a closed subspace of a pseudocompact space is not necessarily pseudocompact.

**Theorem 1.3** (CH). There is a crowded, 0-dimensional Hausdorff space X of cardinality  $\omega_1$ , and X has three dense subspaces  $Z_1$ ,  $Z_2$  and  $Z_3$  such that

- (1)  $Z_1$  is hereditary  $(DRS_{\omega}, but not DRC)$ ,
- (2)  $Z_2$  is hereditary (DRS $_{\omega}$  and DRC, but not DRS),
- (3)  $Z_3$  is hereditary  $(DRC_{\omega}, but neither DRC, nor <math>DRS_{\omega})$ .

Proof of Theorem 1.3. Let K, L, M be disjoint countable sets, let  $\{K_n\}_{n \in \omega}$  and  $\{L_n\}_{n \in \omega}$  be partitions of K and L, respectively, into infinite pieces. Write

$$\mathcal{K} = \{ K' \subset K : \forall n \mid K' \cap K_n \mid < \omega \},$$
  

$$\mathcal{L} = \{ L' \subset L : \forall n \mid L' \cap L_n \mid < \omega \},$$
  

$$\mathcal{M} = [M]^{\omega}.$$

The underlying set of our topological space will be

$$K \cup L \cup M \cup \omega_1$$
.

**Proposition 1.4** (CH). There exists a 0-dimensional  $T_2$  topological space  $X = \langle K \cup L \cup M \cup \omega_1, \tau \rangle$  such that

(a) 
$$\forall U \in \tau^+$$

$$|U \cap \omega_1| = \omega_1 \wedge |U \cap M| = \omega \wedge \forall^{\infty} n \ |U \cap K_n| = |U \cap L_n| = \omega,$$

- (b) every  $A \in \mathcal{K}$  contains a convergent subsequence.
- (c) If  $A \in [X]^{\omega}$  is convergent, then  $A \setminus K$  is finite and  $A \cap K \in \mathcal{K}$ ,
- (d) if  $B \in \mathcal{K} \cup \mathcal{L}$ , then  $acc(B) = acc(B) \cap \omega_1 \neq \emptyset$ ,
- (e) if  $B \in \mathcal{M}$ , then  $acc(B) \cap \omega_1 \neq \emptyset$ ,
- (f)  $acc(K_n) = acc(L_n) = acc(\Gamma) = \emptyset$  for each  $n < \omega$  and  $\Gamma \in [\omega_1]^{\omega}$

We show that the space X from Proposition 1.4 by taking

$$Z_1 = K \cup \omega_1,$$
  

$$Z_2 = (K \cup M) \cup \omega_1,$$
  

$$Z_3 = L \cup \omega_1$$

satisfies the requirements of Theorem 1.3. First, X is 0-dimensional,  $T_2$  and  $|X| = \omega_1$ .

(1) Assume that  $Y \subset Z_1$  is relatively countably compact. Then, by (f),  $Y \cap \omega_1$  is finite, and  $Y \cap K_n$  is also finite for each n. Thus,  $Y \cap K \in \mathcal{K}$ . Hence,  $acc(Y \cap K) \subset \omega_1$ , and so  $Y \cap K$  is nowhere dense by (a). Since  $Y \cap \omega_1$  is finite, it follows that Y is nowhere dense, and so no regular closed subset of  $Z_1$  is DRC.

Next we show that  $Z_1$  is  $DRS_{\omega}$ . Fix first a partition  $\{I_k : k \in \omega\}$  of  $\omega$  into infinite pieces, and write  $D_k = \bigcup \{K_n : n \in I_k\}$ . Then, by (a), every  $D_k$  is dense.

If  $d_k \in D_k$ , then  $\{d_k : k \in \omega\} \in \mathcal{K}$ , so it contains a convergent subsequence by (b). Thus, the sequence  $\{D_k : k \in \omega\}$  witnesses that  $Z_1$  is  $DRS_{\omega}$ .

(2) Assume that  $Y \subset Z_2$  is relatively sequentially compact. Then,  $Y \setminus K$  is finite, and  $Y \cap K_n$  is finite for each n by (c). Thus,  $Y \cap K \in \mathcal{K}$ . Hence  $acc(Y \cap K) \subset \omega_1$  by (d). Thus  $Y \cap K$  is nowhere dense by (a). Since  $Y \setminus K$  is finite, it follows that Y is nowhere dense, and so no regular closed subset of  $Z_2$  is DRS.

The subspace  $Z_2$  is DRC, because M is relatively countably compact by (d) and M is dense by (a).

Since  $Z_1$  is a dense subspace of  $Z_2$  because K is dense in X by (a), and  $Z_1$  is  $DRS_{\omega}$ , it follows that  $Z_2$  is also  $DRS_{\omega}$ .

(3) By (c),  $Z_3$  does not contain convergent sequences, so no subspace of  $Z_2$  is  $\mathrm{DRS}_{\omega}$ . Assume that  $Y \subset Z_3$  is relatively countably compact. Then, by (f),  $Y \cap \omega_1$  is finite, and  $Y \cap L_n$  is also finite for each n. Thus,  $Y \cap L \in \mathcal{L}$ . Hence  $acc(Y \cap L) \subset \omega_1$ , and so  $Y \cap L$  is nowhere dense by (d). Since  $Y \cap \omega_1$  is finite, it follows that Y is nowhere dense, and so no regular closed subset of  $Z_3$  is DRC.

Next we show that  $Z_3$  is  $DRC_{\omega}$ . Fix first a partition  $\{I_k : k \in \omega\}$  of  $\omega$  into infinite pieces, and write  $D_k = \bigcup \{K_n : n \in I_k\}$ . Then, by (a), every  $D_k$  is dense.

If  $d_k \in D_k$ , then  $\{d_k : k \in \omega\} \in \mathcal{L}$ , so it has an accumulation point (d). Thus, the sequence  $\{D_k : k \in \omega\}$  witnesses that  $Z_3$  is  $DRC_{\omega}$ .

So we have verified that it is really enough to prove Proposition 1.4 which we will do in the next section after some preparation.  $\Box$ 

## 2. Main construction

In this sections let the sets K, L, M, the partitions  $\{K_n : n \in \omega\}$  and  $\{L_n : n \in \omega\}$ , and the families  $\mathcal{K}$ ,  $\mathcal{L}$  and  $\mathcal{M}$  be fixed as in the proof of Theorem 1.3.

**Definition 2.1.** A pair  $\mathbb{D} = \langle X, \mathcal{B} \rangle$  is a description iff  $\mathcal{B} = \{B_i : i \in \nu\}$  is a clopen subbase of a 0-dimensional  $T_2$  topology  $\tau$  on X.

We say that  $\langle X, \mathcal{B} \rangle$  is *countable* iff  $X \cup \nu$  is countable.

If  $\mathbb{D}^{\alpha}$  is a description, we assume that  $\mathbb{D}^{\alpha} = \langle X^{\alpha}, \mathcal{B}^{\alpha} \rangle$ ,  $\mathcal{B}^{\alpha} = \{B_{i}^{\alpha} : i < \nu^{\alpha}\}$ , and  $\mathcal{B}^{\alpha}$  is the base of the topology  $\tau^{\alpha}$ .

We write

$$\langle X^0, \mathcal{B}^0 \rangle \preceq \langle X^1, \mathcal{B}^1 \rangle$$

iff

- (i)  $X^0 \subset X^1$  and  $\nu^0 \subset \nu^1$ ,

- (ii)  $B_i^1 \cap X^0 = B_i^0$  for each  $i \in \nu^0$ , (iii)  $B_i^0 \subset B_j^0$  iff  $B_i^1 \subset B_j^1$  for each  $i, j \in \nu_0$ , (iv)  $B_i^0 \cap B_j^0 = \emptyset$  iff  $B_i^1 \cap B_j^1 = \emptyset$  for each  $i, j \in \nu_0$ .

If  $\beta$  is a limit ordinal and  $\vec{\mathbb{D}} = \langle \mathbb{D}^{\alpha} : \alpha < \beta \rangle$  is a  $\preceq$ -increasing sequence of descriptions, then we take

- (a)  $X^{\beta} = \bigcup_{\alpha < \beta} X^{\alpha}$  and  $\nu^{\beta} = \bigcup_{\alpha < \beta} \nu^{\alpha}$ ,
- (b)  $B_i^{\beta} = \bigcup \{B_i^{\alpha} : i \in \nu^{\alpha}\} \text{ for } i \in \nu^{\beta},$

and write

$$\lim \vec{\mathbb{D}} = \left\langle X^{\beta}, \{B_i^{\beta} : i \in \nu^{\beta}\} \right\rangle.$$

The statement of the following lemma is straightforward.

**Lemma 2.2.**  $\lim \vec{\mathbb{D}}$  is a description and  $\mathbb{D}^{\alpha} \leq \lim \vec{\mathbb{D}}$  for each  $\alpha < \beta$ .

**Definition 2.3.** A triple  $\mathbb{A} = \langle X, \mathcal{B}, \mathcal{F} \rangle$  is an approximation iff

- (i)  $X = K \cup L \cup M \cup \gamma$  for some  $\gamma < \omega_1$ ,
- (ii)  $\langle X, \mathcal{B} \rangle$  is a countable description,  $\mathcal{B} = \{B_i : i < \nu\}$  for some  $\nu < \omega_1$ ,
- (iii)  $\mathcal{F} = \{F_y : y \in \gamma\}, \text{ and } F_y \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{M},$
- (iv)  $F_y \to_{\tau} y$  for each  $y \in \gamma$ .
- (v) For each  $B \in \mathcal{B}$ ,  $|B \cap M| = \omega$  and  $\forall^{\infty} n \in \omega \ |B \cap K_n| = |B \cap L_n| = \omega$ .

$$desc(\mathbb{A}) = \langle X, \mathcal{B} \rangle$$
.

If  $\mathbb{A}^{\nu}$  is an approximation, we assume that  $\mathbb{A}^{\nu} = \langle X^{\nu}, \mathcal{B}^{\nu}, \mathcal{F}^{\nu} \rangle$ , and  $\mathcal{F}^{\nu} = \{ F_{\nu}^{\nu} : \mathcal{F}^{\nu} \}$ 

Assume that  $\mathbb{A}^k = \langle X^k, \mathcal{B}^k, \mathcal{F}^k \rangle$  are approximations for k < 2. Write  $\mathbb{A}^0 \leq \mathbb{A}^1$ 

- $\begin{array}{ll} (1) \; \operatorname{desc}(\mathbb{A}^0) \preceq \operatorname{desc}(\mathbb{A}^1), \\ (2) \; \text{if} \; F_y^0 \in \mathcal{K}, \, \text{then} \; F_y^1 = F_y^0, \\ (3) \; \text{if} \; F_y^0 \in \mathcal{L} \cup \mathcal{M}, \, \text{then} \; F_y^1 \subset^* F_y^0. \end{array}$

**Lemma 2.4.** Assume that  $\mathbb{A} = \langle \mathbb{A}^{\alpha} : \alpha < \beta \rangle$  is a  $\preceq$ -increasing sequence of approximations. Write  $\langle X^{\beta}, B^{\beta} \rangle = \lim \langle desc(\mathbb{A}^{\alpha}) : \alpha < \beta \rangle$ .

- (1) For each  $\alpha < \beta$  and  $y \in \gamma_{\beta}$ ,
- (a) if  $F_y^{\alpha} \in \mathcal{K}$ , then  $F_y \to_{\tau_{\beta}} y$ . (b) if  $F_y^{\alpha} \in \mathcal{L} \cup \mathcal{M}$ , then  $y \in acc_{\tau_{\beta}}(F_y^{\alpha})$ .
- (2) If  $\beta$  is countable, there is a countable approximation  $\mathbb{A}^{\beta} = \langle X^{\beta}, \mathcal{B}^{\beta}, \mathcal{F}^{\beta} \rangle$  such that  $\mathbb{A}^{\alpha} \prec \mathbb{A}^{\beta}$  for each  $\alpha < \beta$  (recall that  $\langle X^{\beta}, B^{\beta} \rangle = \lim(\langle \langle X^{\alpha}, B^{\alpha} \rangle : \alpha < \beta \rangle)$ ).

Proof of the Lemma. (1) is straightforward. To prove (2) we should define  $F_y^{\beta}$  for  $y \in \gamma^{\beta}$ . If  $F_y^{\alpha} \subset K$  for some  $\alpha < \beta$ , then let  $F_y^{\beta} = F_y^{\alpha}$ .

If  $F_y^{\alpha} \subset L \cup M$ , then  $\langle F_y^{\alpha} : y \in \gamma_{\alpha}, \alpha < \beta \rangle$  is a countable, mod-finite decreasing sequence of infinite sets. So we can choose  $F_y^\beta$  such that  $F_\beta^y \subset^* F_y^\alpha$  for each  $\alpha < \beta$ 

Then 
$$\mathbb{A}^{\beta} = \langle X^{\beta}, \mathcal{B}^{\beta}, \{F_{y}^{\beta} : y \in \gamma^{\beta}\} \rangle$$
 is a suitable approximation.

**Definition 2.5.** If  $\mathbb{A} = \langle X, \mathcal{B}, \mathcal{F} \rangle$  is an approximation and  $n \in \omega$ , let

$$\mathcal{D}(\mathbb{A}, n) = \{ B \cap K_j, B \cap L_j, B \cap M : B \in \mathcal{B}, n \le j < \omega \}.$$

**Lemma 2.6.** If  $\mathbb{A}^0 = \langle X^0, \mathcal{B}^0, \mathcal{F}^0 \rangle$  is an approximation and  $\langle H_0, H_1 \rangle$  is a partition of  $X^0$  such that for some  $n \in \omega$ ,

- (1)  $|H_i \cap D| = \omega$  for each i < 2 and  $D \in \mathcal{D}(\mathbb{A}^0, n) \cap [X^0]^{\omega}$ ,
- (2)  $F_y^0 \subset^* H_i$  for each i < 2 and  $y \in H_i \cap \gamma$ ,

then there is an approximation  $\mathbb{A}^1 \geq \mathbb{A}^0$  such that  $H_0, H_1 \in \mathcal{B}^1$ .

*Proof.* Let  $\mathcal{B}_1 = \{B_{\zeta}^1 : \zeta < \gamma^1\}$  be an enumeration of  $\mathcal{B}^0 \cup \{B \cap H_i : B \in \mathcal{B}^0, i < \gamma^1\}$  $\{H_0, H_1\}$  such that  $B_{\zeta}^1 = B_{\zeta}^0$  for  $\zeta < \gamma^0$ . Then  $\mathcal{B}^1$  is a base of a 0-dimensional  $T_2$  topology on X and  $H_0, H_1 \in \mathcal{B}^1$ . Then  $\mathbb{A}^1 = \langle X^0, \mathcal{B}^1, \mathcal{F}^0 \rangle$  is an approximation which meets the requirements.  $\square$ 

**Lemma 2.7.** Assume that  $\mathbb{A}^0 = \langle X^0, \mathcal{B}^0, \mathcal{F}^0 \rangle$  is an approximation,  $n \in \omega$   $k \in K$ ,  $\ell \in L, m \in M, y \in \gamma^0, K' \in \mathcal{K}, L' \in \mathcal{L} \text{ such that } F_y \cap (K' \cup L') \text{ is finite. Then there is an approximation } \mathbb{A}^1 \geq \mathbb{A}^0 \text{ such that } X^1 = X^0, \mathcal{F}^1 = \mathcal{F}^0, \text{ and}$ 

$$\{k,\ell,m,y\} \cap acc(K' \cup K_{\leq n} \cup L' \cup L_{\leq n} \cup \gamma^0,\tau^1) = \emptyset.$$

*Proof.* Choose  $K'' \in \mathcal{K}$  and  $L'' \in \mathcal{L}$  such that  $K'' \supset K'$ ,  $L'' \supset L'$  and  $F_z^0 \subset^* K'' \cup L''$ 

provided  $F_z^0 \subset K \cup L$  for each  $z \in \gamma^0$ . Choose  $M'' \in \mathcal{M}$  such that  $F_z^0 \subset M$  implies  $F_z^0 \subset^* M''$  for each  $z \in \gamma^0$ , and  $|B \setminus M''| = \omega$  for each  $B \in \mathcal{B}$ .

Write  $N'' = K'' \cup L'' \cup M'' \cup K_{\leq n} \cup L_{\leq n}$ .

Since  $D \setminus N''$  is infinite for each  $D \in \mathcal{D}(\mathbb{A}^0, n)$ , there is a partition  $\langle G_0, G_1 \rangle$  of  $(K \cup L \cup M) \setminus N''$  such that  $G_i \cap D$  is infinite for each infinite  $D \in \mathcal{D}(\mathbb{A}^0, n)$ .

$$H_0 = G_0 \cup F_y \cup \{y, k, \ell.m\},$$
  
 $H_1 = G_1 \cup (N'' \setminus (F_y \cup \{k, \ell, m\}) \cup (\gamma^0 \setminus \{y\}).$ 

Clearly,  $H_1 = X^0 \setminus H_0$ . Then  $\langle H_0, H_1 \rangle$  is a partition of  $X^0$  and we can apply Lemma 2.6 to obtain  $\mathbb{A}^1$ .

**Lemma 2.8.** Assume that  $\mathbb{A}^0$  is a countable approximation, and  $K' \in \mathcal{K}$ . Then there is a countable approximation  $\mathbb{A}^1 \geq \mathbb{A}^0$  such that  $F_y^1 \cap K$  is infinite for some

Proof of Lemma 2.8. We can assume that  $K' \cap F_y^0$  is finite for each  $y \in \gamma^0$ .

Next, applying Lemma 2.7  $\omega$  times, we can get  $\mathbb{A}^2 \geq \mathbb{A}^0$  such that,  $X^2 = X^1$ ,  $\mathcal{F}^2 = \mathcal{F}^1$ , K' is closed discrete in  $\tau^2$  and the  $K_n$ -s and  $L_n$ -s are also closed discrete (in each step we guarantee that a point  $x \in X^0$  is not an accumulation point of  $K' \cup K_n \cup L_n$ ).

Then, by induction on  $n \in \omega$ , we can pick  $k_n \in K'$  and  $x_n \in U_n \in \mathcal{B}^0$  such that  $\{B_n: n < \omega\}$  is a locally finite family of disjoint open sets such that

$$\forall B \in \mathcal{B}^2 \ (\forall^{\infty} n \in \omega \ U_n \subset B \ \lor \forall^{\infty} n \in \omega \ U_n \cap B = \emptyset \ ).$$

Let  $\gamma^1 = \gamma^2 + 1$ ,  $\nu^1 = \nu^2 + \omega$ , and let

$$B^1_{\nu^2+n} = \{\nu^2\} \cup \bigcup \{U_j : j \ge n\}.$$

For  $\zeta < \nu^2$  let

$$B_{\zeta}^{1} = \begin{cases} B_{\zeta}^{2} \cup \{\gamma^{0}\} & \text{if } \forall^{\infty} n \in \omega \ U_{n} \subset B_{\zeta}^{2}, \\ \\ B_{\zeta}^{2} & \text{if } \forall^{\infty} n \in \omega \ U_{n} \cap B_{\zeta}^{2} = \emptyset. \end{cases}$$

Let  $F_{\gamma^0}^1 = \{x_n : n < \omega\}$ . Then  $\mathbb{A}^1 = \langle X^1, \mathcal{B}^1, \mathcal{F}^1 \rangle$  meets the requirements.

**Lemma 2.9.** Assume that  $\mathbb{A}^0$  is a countable approximation, and  $L' \in \mathcal{L}$ . Then there is a countable approximation  $\mathbb{A}^1 \geq \mathbb{A}^0$  such that  $F_y^1 \subset^* L'$  for some  $y \in \gamma^1$ .

Proof of Lemma 2.9. Imitating the proof of Lemma 2.8 we obtain  $\mathbb{A}^2 \geq \mathbb{A}^0$  such that  $F_y^2 \cap L'$  is infinite for some  $y \in \gamma^2$ . Define  $\mathbb{A}^1 = \langle X^2, \mathcal{B}^2, \mathcal{F}^1 \rangle$  such that

$$F_y^1 = \left\{ \begin{array}{ll} F_y^2 \cap L' & \text{if } F_y^2 \cap L' \text{ is infinite,} \\ \\ F_y^2 & \text{if } F_y^2 \cap L' \text{ is finite.} \end{array} \right.$$

Then  $\mathbb{A}^1$  satisfies the requirements.

**Lemma 2.10.** Assume that  $\mathbb{A}^0$  is a countable approximation, and  $M' \in \mathcal{M}$ . Then there is a countable approximation  $\mathbb{A}^1 \geq \mathbb{A}^0$  such that M' contains an infinite closed discrete subset.

Proof of Lemma 2.11. We can assume that  $M' \to_{\tau^0} x$  for some  $x \in X$ . Pick  $K' \in \mathcal{K}$ such that  $F_y^0 \subset^* K'$  for each  $y \in \gamma^0$  with  $F_y^0 \subset K$ . Let

$$\mathcal{D}=\mathcal{D}(\mathbb{A}^0,0)\cup(\{B\cap F_z^0:z\in\gamma^0,B\in\mathcal{B}^0\}\cap[L\cup M]^\omega)\cup\{M'\cap B:B\in\mathcal{B}^0\}.$$

Let  $\langle G_0, G_1 \rangle$  be a partition of  $(K \cup L \cup M) \setminus K'$  such that  $G_i \cap D$  is infinite for each infinite  $D \in \mathcal{D}$ .

Let  $H_0 = G_0 \cup K' \cup \gamma^0$  and  $H_1 = X^0 \setminus H_0$ . Let  $\mathcal{B}_1 = \{B_{\zeta}^1 : \zeta < \gamma^1\}$  be an enumeration of  $\mathcal{B}^0 \cup \{B \cap H_i : B \in \mathcal{B}^0, i < \gamma^1\}$  $2\} \cup \{H_0, H_1\}$  such that  $B_{\zeta}^1 = B_{\zeta}^0$  for  $\zeta < \gamma^0$ . Then  $\mathcal{B}^1$  is a base of a 0-dimensional  $T_2$  topology on X and  $H_0, H_1 \in \mathcal{B}^1$ .

Then  $\mathbb{A}^1 = \langle X^0, \mathcal{B}^1, \mathcal{F}^0 \rangle$  is an approximation which meets the requirements. because  $M' \cap H_1$  is closed discrete.

**Lemma 2.11.** Assume that  $\mathbb{A}^0$  is a countable approximation, and  $M' \in \mathcal{M}$ . Then there is a countable approximation  $\mathbb{A}^1 \geq \mathbb{A}^0$  such that  $F_y^1 \subset^* M'$  for some  $y \in \gamma^1$ .

Proof of Lemma 2.11. By Lemma 2.10 we can assume that M' is closed discrete. Then, by induction on  $n \in \omega$ , we can pick  $k_n \in M'$  and  $x_n \in B_n \in \mathcal{B}^0$  such that  $\{B_n: n<\omega\}$  is a locally finite family of disjoint open sets such that

$$\forall B \in \mathcal{B}^2 \ (\forall^{\infty} n \in \omega \ U_n \subset B \ \lor \forall^{\infty} n \in \omega \ U_n \cap B = \emptyset \ ).$$

Let  $\gamma^1 = \gamma^2 + 1$ ,  $\nu^1 = \nu^2 + \omega$ , and let

$$B_{\nu^2+n}^1 = \{\nu^2\} \cup \bigcup \{U_j : j \ge n\}.$$

For  $\zeta < \nu^2$  let

$$B_{\zeta}^{1} = \begin{cases} B_{\zeta}^{2} \cup \{\gamma^{0}\} & \text{if } \forall^{\infty} n \in \omega \ U_{n} \subset B_{\zeta}^{2}, \\ \\ B_{\zeta}^{2} & \text{if } \forall^{\infty} n \in \omega \ U_{n} \cap B_{\zeta}^{2} = \emptyset. \end{cases}$$

Let  $F_{\gamma^0}^2 = \{x_n : n < \omega\}$ . Then  $\mathbb{A}^2 = \langle X^2, \mathcal{B}^2, \mathcal{F}^2 \rangle \leq \mathbb{A}^1$  and there is  $y \in \gamma^2$  such that  $M' \cap F_y^2$  is infinite.

Define  $\mathbb{A}^{1'} = \langle X^2, \mathcal{B}^2, \mathcal{F}^1 \rangle$  such that

$$F_y^1 = \left\{ \begin{array}{ll} F_y^2 \cap L' & \text{if } F_y^2 \cap L' \text{ is infinite} \\ \\ F_y^2 & \text{if } F_y^2 \cap L' \text{ is finite.} \end{array} \right.$$

Then  $\mathbb{A}^1$  satisfies the requirements.

**Lemma 2.12.** Assume that  $\mathbb{A}^0$  is a countable approximation,  $y \in \gamma^0$  and  $L' \in \mathcal{L}$ . Then there is a countable approximation  $\mathbb{A}^1 \leq \mathbb{A}^0$  such that  $' \not\to_{\tau^1} y$ .

Proof of Lemma 2.12. Let  $K' \in \mathcal{K}$  such that  $F_z^0 \subset^* K'$  for each  $z \in \gamma^0$  with  $F_z^0 \subset K$ .

Let

$$\mathcal{D} = \mathcal{D}(\mathbb{A}^0, n) \cup (\{B \cap F_z^0 : z \in \gamma^0, B \in \mathcal{B}^0\} \cap [L \cup M]^\omega) \cup \{L' \cap B : B \in \mathcal{B}^0\}.$$

Let  $\langle G_0, G_1 \rangle$  be a partition of  $(K \cup L \cup M) \setminus K_0$  such that  $H_i \cap D$  is infinite for each infinite  $D \in \mathcal{D}$ .

Let  $H_0 = G_0 \cup K' \cup Y$  and  $H_1 = X^0 \setminus H_0$ .

Let  $\mathcal{B}_1 = \{B_{\zeta}^1 : \zeta < \gamma^1\}$  be an enumeration of  $\mathcal{B}^0 \cup \{B \cap H_i : B \in \mathcal{B}^0, i < \gamma^1\}$  $\{H_0, H_1\}$  such that  $B_{\zeta}^1 = B_{\zeta}^0$  for  $\zeta < \gamma^0$ . Then  $\mathcal{B}^1$  is a base of a 0-dimensional  $T_2$  topology on X and  $H_0, H_1 \in \mathcal{B}^1$ . Then  $\mathbb{A}^1 = \langle X^0, \mathcal{B}^1, \mathcal{F}^0 \rangle$  is an approximation which meets the requirements.

Proof of Proposition 1.4. Let

$$Task = \{ \langle 1, Y \rangle : Y \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{M} \} \cup \{ \langle 2, Z \rangle : Z \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{M} \cup \omega_1 \cup \{ L_n, K_n : n < \omega \} \} \cup \{ \langle 3, T \rangle : T \in \mathcal{L} \cup \mathcal{M} \}.$$

and fix an  $\omega_1$ -abundant enumeration  $\{t_{\zeta}: \zeta < \omega_1\}$  of Task.

We will define a  $\leq$ -increasing sequence  $\langle \mathbb{A}^{\alpha} = \langle X^{\alpha}, \mathcal{B}^{\alpha}, \mathcal{F}^{\alpha} \rangle : \alpha < \omega_1 \rangle$  of countable approximations by transfinite recursion.

Let  $\mathbb{A}^0$  be a countable approximation such that  $\gamma^0 = 0$ .

If  $\zeta$  is a limit ordinal, apply Lemma 2.4(2) to obtain  $\mathbb{A}^{\zeta}$  from  $\langle \mathbb{A}^{\eta} : \eta < \zeta \rangle$ .

Assume that  $\zeta = \eta + 1$  and we have  $\mathbb{A}^{\eta}$ .

Case 1.  $t_{\eta} = \langle 1, Y \rangle$ .

If  $Y \in \mathcal{K}$ , then apply Lemma 2.8 to find  $\mathbb{A}^{\zeta}$  such that  $Y \cap F_{\eta}^{\zeta}$  is infinite for some

If  $Y \in \mathcal{L} \cup \mathcal{M}$ , then apply Lemma 2.9 or Lemma 2.12to find  $\mathbb{A}^{\zeta}$  such that  $F_{\eta}^{\zeta} \subset F$ for some  $y < \gamma^{\zeta}$ .

Case 2.  $t_n = \langle 2, Z \rangle$ .

If  $Z \in \mathcal{K} \cup \mathcal{L}$ , then apply Lemma 2.7  $\omega$  times to obtain  $\mathbb{A}^{\zeta}$  such that  $acc(Z, \tau^{\zeta}) \subset$ 

If  $Z \in \omega_1 \cup \{K_n, L_n : n < \omega\}$ , then apply Lemma 2.7  $\omega$  times to obtain  $\mathbb{A}^{\zeta}$  such that  $acc(Z, \tau_{\zeta}) = \emptyset$ .

Case 3.  $t_{\eta} = \langle 3, T \rangle$ . Apply Lemma 2.10 or Lemma 2.12 to obtain  $\mathbb{A}^{\zeta}$  such that T does not converge in  $\tau^{\zeta}$ .

Finally let  $\langle X, \mathcal{B} \rangle = \lim \langle \langle X^{\zeta}, \mathcal{B}^{\zeta} \rangle : \zeta < \omega_1 \rangle$ . Then the space  $\langle X^{\omega_1}, \tau^{\omega_1} \rangle$  satisfies the requirements.

References

- [1] A. J. Berner. Spaces with dense conditionally compact subsets. Proc. Amer. Math. Soc., 81(1):137–142, 1981.
- [2] A. Dorantes-Aldama and D. Shakhmatov. Selective sequential pseudocompactness. Topology Appl., 222:53-69, 2017.
- [3] A. Dorantes-Aldama and D. Shakhmatov. Reprint of: Compactness properties defined by open-point games Topology Appl., 281:107416, 21, 2020.
- [4] R. Engelking. General topology, volume 6 of Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.
- [5] E. Hewitt. Rings of real-valued continuous functions. I. Trans. Amer. Math. Soc., 64:45–99,
- [6] K. Kunen. On the compactification of the integers. Notices Amer. Math. Soc, 17(299), 1970. abstract 70T-G7.

- [7] K. Kunen. Some points in  $\beta N.$  Math. Proc. Cambridge Philos. Soc., 80(3):385–398, 1976.
- [8] K. Kunen. Weak P-points in N\*. In Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), volume 23 of Colloq. Math. Soc. János Bolyai, pages 741–749. North-Holland, Amsterdam-New York, 1980.
- [9] K. Kunen and J. E. Vaughan, editors. Handbook of set-theoretic topology. North-Holland Publishing Co., Amsterdam, 1984.
- [10] S. Mardešić and P. Papić. Sur les espaces dont toute transformation réelle continue est bornée. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II, 10:225–232, 1955.
- [11] M. M. Marjanović. A pseudocompact space having no dense countably compact subspace. Glasnik Mat. Ser. III, 6(26):149–151, 1971.
- [12] J. A. Martínez-Cadena and R. G. Wilson. Maximal densely countably compact topologies. Acta Math. Hungar., 151(2):259–270, 2017.
- [13] M. E. Rudin. Partial orders on the types in  $\beta N$ . Trans. Amer. Math. Soc., 155:353–362, 1971.
- [14] D. B. Shakhmatov. A pseudocompact Tychonoff space all countable subsets of which are closed and  $C^*$ -embedded. Topology Appl., 22(2):139–144, 1986.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUN-REN Email address: juhasz@renyi.hu

Alfréd Rényi Institute of Mathematics, Hun-Ren

 $Email\ address: {\tt soukup@renyi.hu}$ 

Eötvös University of Budapest Email address: szentmiklossyz@gmail.com