A MOEBIUS INVERSION FORMULA TO DISCARD TANGLED HYPERBOLIC SURFACES

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ABSTRACT. Recent literature on Weil–Petersson random hyperbolic surfaces has met a consistent obstacle: the necessity to condition the model, prohibiting certain rare geometric patterns (which we call *tangles*), such as short closed geodesics or embedded surfaces of short boundary length. The main result of this article is a Moebius inversion formula, allowing to integrate the indicator function of the set of tangle-free surfaces in a systematic, tractable way. It is inspired by a key step of Friedman's celebrated proof of Alon's conjecture. We further prove that our tangle-free hypothesis significantly reduces the number of local topological types of short geodesics, replacing the exponential proliferation observed on tangled surfaces by a polynomial growth.

1. Introduction

The study of random compact hyperbolic surfaces of large genus has drawn a lot of attention recently (see for instance [14, 11, 17, 13, 16] as a non-exhaustive list of recent papers). It has become apparent that there exists a set of "bad" hyperbolic surfaces, of small but non-zero probability, that one could benefit from discarding – in particular when studying the spectrum of the laplacian of random surfaces. These "bad" hyperbolic surfaces contain geometric patterns called *tangles*, a notion that was formalized and studied by the second author and Thomas in the context of hyperbolic geometry [15], and first arose in the literature about random graphs. Tangles are responsible for an exceptional proliferation of periodic geodesics. Our previous work [2] points to the fact that the existence of tangled surfaces is directly related to the existence of surfaces with small spectral gap. The necessity to remove tangled surfaces also appeared in [11] although the word *tangles* was not used there.

In fact, the word tangles was first coined by Friedman for large random regular graphs, in the proof of Alon's conjecture concerning the spectral gap [7]. The word was later used by Bordenave with a different meaning, but with the same purpose of discarding a set of small probability of "bad graphs" [4]. In this paper we use the notion of tangles introduced in [15] for hyperbolic surfaces, which is closer to Bordenave's. In the recent work of Huang and Yau on the fine spectral statistics of random regular graphs, it is again necessary to condition on a subset of the full set of regular graphs (the subset $\bar{\Omega}$ of [10, Definition 1.1]), characterized by the absence of too many cycles in balls of a certain radius.

Given two positive real numbers κ , \mathcal{L} , the definition of (κ, \mathcal{L}) -tangle is given in Definition 2.4: a (κ, \mathcal{L}) -tangle (or in short, a tangle), is either a simple smooth curve of length less than κ , or a hyperbolic pair-of-pants with all boundary lengths less than \mathcal{L} .

A hyperbolic surface is said to be (κ, \mathcal{L}) -tangle-free (or, in short, tangle-free) if it does not contain any (κ, \mathcal{L}) -tangle. We denote by $\mathcal{A}_{(g,n)}^{\kappa,\mathcal{L}}$ the set of (κ, \mathcal{L}) -tangle-free compact orientable hyperbolic surfaces of signature (g, n). It will be seen as a subset of the Teichmüller space or of the moduli space of compact hyperbolic surfaces.

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For fixed $\kappa, \mathcal{L} > 0$ and for any L > 0, Theorem 1.1 shows that the number of topologies of periodic geodesics of length $\leq L$ in a tangle-free hyperbolic surface is polynomial in L (instead of the expected exponential growth), provided the ratio $\frac{L}{C}$ stays bounded:

Theorem 1.1. Let $\operatorname{Loc}_{\chi}^{\kappa,\mathcal{L},L}$ be the number of local topological types of periodic geodesics of length $\leq L$, that can arise in a (κ,\mathcal{L}) -tangle-free surface of Euler characteristic $-\chi$. Then

$$\#\mathrm{Loc}_{\chi}^{\kappa,\mathcal{L},L} = \mathcal{O}_{\chi}\left(\frac{L}{\mathcal{L}}\left(1 + \frac{L}{\kappa}\right)^{18(1+\chi)\frac{L}{\mathcal{L}}}\right),\,$$

where $\mathcal{O}_{\chi}(\cdot)$ is the usual Landau "big-O" notation, with an implied constant depending on χ .

In other words, discarding tangle-free surfaces avoids the exponential proliferation of topological types of periodic geodesics. Precise definitions of the notion of local topological type and of the set $\text{Loc}_{\chi}^{\kappa,\mathcal{L},L}$ will be given in §3.1.

Lemma 10.4 in [2] treats the case where $\chi=1$; Theorem 1.1 is a generalization to arbitrary Euler characteristic. The theorem bears a similarity with a counting result obtained by Lipnowski and Wright [11, Theorem 2.1]. However, these authors counted closed geodesics in a given tangle-free hyperbolic surface, and their method of proof refers explicitly to the fixed hyperbolic metric. We wanted a statement about the number of topologies of periodic geodesics in all tangle-free hyperbolic surfaces.

In several models of random compact hyperbolic surfaces, and for the right choice of parameters κ , \mathcal{L} , the set $\mathbb{1}_{\mathcal{A}_g^{\kappa,L}}$ has probability close to 1. For instance, for the Weil-Petersson probability measure \mathbb{P}_g^{WP} on the moduli space of compact hyperbolic surfaces of genus g, it is known that for any small enough $\kappa > 0$, for $\mathcal{L} = \kappa \log g$,

$$1 - \mathbb{P}_g^{\text{WP}}\left(\mathcal{A}_g^{\kappa, \mathcal{L}}\right) \le \kappa^2 + \mathcal{O}\left(g^{\frac{3}{2}\kappa - 1}\right).$$

See [14, Theorem 4.2] and [15, Theorem 5]. An application of this was already used in [2]: we could condition on $\mathbb{1}_{\mathcal{A}_g^{\kappa,L}}$, losing only a fraction $\kappa^2 + \mathcal{O}\left(g^{\frac{3}{2}\kappa-1}\right)$ of the total moduli space, and avoid the exponential proliferation of topologies of geodesics of length comparable to $\log g$.

Conditioning a probability measure can be a dangerous operation, especially when we work with the Weil-Petersson measure on the moduli space of compact hyperbolic surfaces of genus g: this is a smooth measure, but the conditioning introduces an indicator function, with a discontinuity. Multiplying the Weil-Petersson measure by the indicator function $\mathbb{1}_{\mathcal{A}_g^{\kappa,L}}$ also ruins all possibilities to use the nice, algebraic integration formulas provided by Mirzakhani's works. The second result of this paper is Theorem 1.2, an identity apparented to the family of Moebius inversion formulas. For any (g,n), this identity allows to express the indicator function $\mathbb{1}_{\mathcal{A}_{(g,n)}^{\kappa,L}}$ on the moduli space as a sum of functions that can be explicitly lifted to the Teichmüller space, and hence that can be more easily integrated along the Weil-Petersson measure.

In §2.1.2 we define the *big moduli space* \mathbf{M} which is, roughly, the set of all compact hyperbolic surfaces, with all possible topologies (connected or not, with or without boundary – possibly with some degenerate components reduced to 1-dimensional compact manifolds). The set $\mathbf{D}_{\kappa,\mathcal{L}} \subset \mathbf{M}$ is the set of surfaces filled by tangles (see §2 for detailed definitions).

Theorem 1.2. There exists a (unique) function $\mu = \mu_{(\kappa,\mathcal{L})} : \mathbf{M} \longrightarrow \mathbb{R}$, such that

- the restriction of μ to each $\mathcal{M}^*_{(\mathbf{g},\mathbf{n})}$ is invariant under the action of the full diffeomorphism group (i.e. possibly permuting boundary components and connected components);
- For $Z \in \mathbf{M}$, $\mu(Z) = 0$ if $Z \notin \mathbf{D}_{\kappa,\mathcal{L}}$;
- For every $Z \in \mathbf{D}_{\kappa,\mathcal{L}}$,

$$1 = \sum_{\tau \in \mathcal{S}(Z)} \mu(\tau),$$

where the sum runs over all possible sub-surfaces of Z with geodesic boundary.

The full form of the theorem (Theorem 4.3) gives more precise information, in particular upper bounds, on the function μ . Like the original Moebius function in number theory, the function μ possesses a multiplicative property (Equation (4.4) or Remark 4.7). Theorem 1.2 has the following corollary, which is the main goal of the whole construction:

Corollary 1.3. For every g and n, for every compact hyperbolic surface X of signature (g, n),

(1.1)
$$1 - \mathbb{1}_{\mathcal{A}_{(g,n)}^{\kappa,L}}(X) = \sum_{\tau \in \mathcal{S}(X)} \mu(\tau).$$

One may wonder about the gain in such a formula: the left-hand side is the indicator function of the set of surfaces with tangles, a seemingly nice function taking only the values 0 and 1, whereas the right-hand side is a sum with an unbounded number of terms, and the function μ is not of constant sign. This is, of course, a typical feature of generalized Moebius inversion formulas. It turns out that the right-hand side of (1.1) can be easily integrated on the moduli space, because one can explicitly describe how to lift it to an integrable function on the Teichmüller space; on the opposite, it seems very difficult to lift the left-hand side, hence the potential interest of the formula.

We present the construction of the Moebius function in the special context of getting rid of tangles – with the purpose of applying it in the study of the spectral gap of random hyperbolic surfaces – but we believe that the construction is of wider interest, as it can straightforwardly be implemented to remove sets of surfaces containing other kinds of geometric patterns.

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2. Tangles in hyperbolic surfaces

As already mentioned, we are interested in tangles, that can be either 1-dimensional (short closed curves), or 2-dimensional (pairs of pants with short boundary). The Moebius function μ will be supported in the set of surfaces filled by tangles; these surfaces may be disconnected, and may have 1-dimensional connected components. We need an extended notion of moduli space which encompasses all these possibilities.

2.1. The big moduli space M. We denote \mathbb{N}_0 the set of nonnegative integers.

2.1.1. c-surfaces. We order the pairs $(g,n) \in \mathbb{N}_0^2$ by the order relation : $(g,n) \leq (g',n')$ if 2g' + n' - 2 > 2g + n - 2, or if 2g' + n' - 2 = 2g + n - 2 and $g' \ge g$. For $k \ge 1$, denote by $\mathcal{N}^{(k)}$ the set of 2k-tuples $(\mathbf{g}, \mathbf{n}) = ((g_1, n_1), \dots, (g_k, n_k)) \in (\mathbb{N}_0^2)^k$,

satisfying $2 - 2g_i - n_i \le 0$ and $(g_i, n_i) \le (g_{i+1}, n_{i+1})$.

Definition 2.1. A c-surface of signature $(\mathbf{g}, \mathbf{n}) \in \mathcal{N}^{(k)}$ is a non-empty topological space S with k connected components $S = \bigsqcup_{i=1}^k \tau_i$ labelled from 1 to k, where

- τ_i is a 1-dimensional oriented manifold diffeomorphic to a circle if $(g_i, n_i) = (0, 2)$;
- if $2-2g_i-n_i<0$, τ_i is a 2-dimensional orientable manifold with (or without) boundary, of signature (g_i, n_i) , with a numbering of boundary components by the integers $1, \ldots, n_i$.

We denote $\chi(\tau_i) = 2g_i - 2 + n_i$ and call it the absolute Euler characteristic of τ_i .

The c-surface S can be decomposed as $S = (c, \sigma)$, where $c = (\tau_1, \ldots, \tau_j) = (c_1, \ldots, c_j)$ is a collection of 1-dimensional compact manifolds, and $\sigma = (\tau_{j+1}, \ldots, \tau_{j+m}) = (\sigma_1, \ldots, \sigma_m)$ is a collection of 2-dimensional orientable compact manifolds with boundary (k = j + m) is the total number of connected components). We will call c the 1-dimensional part of S and σ the 2-dimensional part of S. We say that S is purely 1d (resp. purely 2d) if m = 0 (resp. j = 0). We shall denote by |S| = j the number of 1d components, and $\chi(S)$ the total absolute Euler characteristic, that is $\chi(S) = \sum_{i=1}^m \chi(\sigma_i)$ where $\chi(\sigma_i) = 2g_{i+j} + n_{i+j} - 2$.

Definition 2.2. Let S be a c-surface and E a subset of S. We say that E fills S if the connected components of $S \setminus E$ are either contractible, or annular regions around a boundary component of the 2d-part of S. In particular, E must contain all the 1d components of S.

2.1.2. Extended notion of moduli space. For 2-2g-n<0, we define $\mathcal{T}_{g,n}^*=\{(\mathbf{x},Y),\mathbf{x}\in(\mathbb{R}_{>0}^*)^n,Y\in\mathcal{T}_{g,n}(\mathbf{x})\}$ — where $\mathcal{T}_{g,n}(\mathbf{x})$ is the standard Teichmüller space of hyperbolic surfaces of genus g, with n labelled boundary components of lengths $\mathbf{x}=(x_1,\ldots,x_n)$. This notation was already used in our previous papers [2, 1].

We denote by $\mathcal{M}_{g,n}^*$ the corresponding moduli spaces, obtained by quotienting the Teichmüller space by the action of the Mapping Class Group. Recall that the latter is (modulo isotopy) the group of orientation preserving homeomorphisms that fix the boundary, so that an element of $\mathcal{M}_{g,n}^*$ still has a labelled boundary. If $Z=(\mathbf{x},Y)\in\mathcal{M}_{g,n}^*$ with 2-2g-n<0, denote by $\ell(\partial Z)=\sum_{i=1}^n x_i$ the total boundary length of Z, and $\ell^{\max}(\partial Z)=\max_{i=1}^n x_i$ the length of the longest boundary component of Z.

We also use this notation for g=0 and n=2, corresponding to 2-2g-n=0: $\mathcal{T}_{0,2}^*=\mathcal{M}_{0,2}^*$ is the space of Riemannian metrics on an oriented circle, modulo orientation preserving isometry. If $Z \in \mathcal{M}_{0,2}^*$, we denote by $\ell(Z)$ the length of Z. The map $Z \mapsto \ell(Z)$ allows to identify $\mathcal{M}_{0,2}^*$ with $\mathbb{R}_{>0}$.

For $(\mathbf{g}, \mathbf{n}) = ((g_1, n_1), \dots, (g_k, n_k)) \in \mathcal{N}^{(k)}$, we define

$$\mathcal{M}^*_{(\mathbf{g},\mathbf{n})} = \prod_{i=1}^k \mathcal{M}^*_{g_i,n_i}.$$

This can be interpreted as the moduli space of Riemannian metrics on a c-surface with k labelled connected components, of respective signatures $(g_1, n_1), \ldots, (g_k, n_k)$ (with a hyperbolic metric on the 2d components).

Definition 2.3. We define the big moduli space,

$$\mathbf{M} = \bigsqcup_{k=1}^{+\infty} \bigsqcup_{(\mathbf{g},\mathbf{n}) \in \mathcal{N}^{(k)}} \mathcal{M}^*_{(\mathbf{g},\mathbf{n})}.$$

2.2. **Definition of tangles.** We introduce a notion of tangles and derived tangles, associated with two parameters $\kappa, \mathcal{L} > 0$ with $\kappa < \mathcal{L}$.

Definition 2.4. We call $Z \in \mathbf{M}$ a (κ, \mathcal{L}) -tangle (or in short, a tangle) if

- either $Z \in \mathcal{M}_{0,2}^*$ and $\ell(Z) \le \kappa$;
- or $Z \in \mathcal{M}_{g,n}^*$ with 2 2g n = -1, and $\ell^{\max}(\partial Z) \leq \mathcal{L}$.

The following definition is directly inspired by the work of Friedman [7].

Definition 2.5. We call $Z \in \mathbf{M}$ a (κ, \mathcal{L}) -derived tangle (or in short, a derived tangle) if it is filled by (κ, \mathcal{L}) -tangles. In other words, $Z \in \prod_{i=1}^k \mathcal{M}_{g_i,n_i}^*$ has k connected components, some of them circles and some of them hyperbolic manifolds with geodesic boundary; and there is a finite or infinite sequence of submanifolds $(Z_n)_{n\geq 1}$ which are either periodic geodesics, or surfaces of Euler characteristic -1, which are tangles in Z, and whose union fill Z.

We denote by $\mathbf{D}_{\kappa,\mathcal{L}} \subset \mathbf{M}$ the set of all (κ,\mathcal{L}) -derived tangles.

The following proposition is obvious:

Proposition 2.6. If X is a compact oriented hyperbolic surface, then X contains a maximal derived tangle, defined as the surface with geodesic boundary filled by all the tangles contained in X. We denote it by $\tau(X)$.

Remark 2.7. We caution that $\tau(X)$ may not be a *sub-c-surface* of X in the sense of our forthcoming Definition 4.1, because several of the boundary components of $\tau(X)$ may be freely homotopic in X (and hence they are equal, since we take the geodesic representative). This is a harmless remark, linked to the fact that in our Definition 2.2 of *filled surface* we do not allow essential cylinders in $S \setminus E$.

Proposition 2.8. Let
$$(\mathbf{g}, \mathbf{n}) = (\underbrace{(0, 2), \dots, (0, 2)}_{j}, (g_1, n_1), \dots, (g_m, n_m)) \in \mathcal{N}^{(j+m)}$$
, with $\chi_i = 2g_i - 2 + n_i > 0$ for $i = 1, \dots, m$. If $Z = (c, \sigma) \in \mathcal{M}^*_{(\mathbf{g}, \mathbf{n})}$ is a (κ, \mathcal{L}) -derived tangle, where $c = (c_1, \dots, c_j)$ is the 1d-part of Z and $\sigma = (\sigma_1, \dots, \sigma_m)$ is its 2d-part, then

 $\ell(c_i) < \kappa$

for
$$i = 1, ..., j$$
, and
$$\ell(\partial \sigma_i) \leq 3\mathcal{L}\chi(\sigma_i).$$

Furthermore, σ_i admits a maximal multi-curve $(\gamma_1, \ldots, \gamma_{3g_i-3+2n_i})$, cutting σ_i into surfaces of Euler characteristic -1, such that $\ell(\gamma_t) \leq 3\mathcal{L}\chi(\sigma_i)$ for all $t = 1, \ldots, 3g_i - 3 + 2n_i$.

Proof. The statement for c_1,\ldots,c_j is obvious: the components c_i are 1-dimensional manifolds and thus cannot contain a surface, so they must be 1-dimensional tangles, that is to say, curves of length $\leq \kappa$. To prove the rest of the proposition, it suffices to treat the case $j=0,\ m=1$, that is to say, the case where Z is a connected hyperbolic surface. The surface Z is filled by a sequence of submanifolds $(Z_n)_{n\geq 1}$ which are either 1-dimensional tangles, or tangles of Euler characteristic -1. Let $S_n\subset Z$ be the surface with geodesic boundary filled by Z_1,\ldots,Z_n . The sequence (S_n) is increasing for inclusion; because Z is 2-dimensional and connected, we can reorder the sequence $(Z_n)_{n\geq 1}$ to ensure that the S_n are 2-dimensional and connected for n large enough (it suffices that Z_{n+1} intersects S_n). Let χ_n be the absolute Euler characteristic of S_n . The sequence χ_n is a non-decreasing sequence of integers, bounded by $\chi(Z)$: thus, there is a finite set $A\subset \mathbb{N}$, or cardinality $\#A\leq \chi(Z)$, such that $\chi_n<\chi_{n+1}$ for $n\in A$, and $\chi_n=\chi_{n+1}$ for $n\notin A$. This is equivalent to saying that $S_n=S_{n+1}$ iff $n\notin A$.

Since $\ell(\partial Z_n) \leq 3\mathcal{L}$ if Z_n is 2-dimensional, and $\ell(Z_n) \leq \kappa$ if Z_n is 1-dimensional, we always have $\ell(\partial S_{n+1}) \leq \ell(\partial S_n) + \max(2\kappa, 3\mathcal{L}) = \ell(\partial S_n) + 3\mathcal{L}$. As a consequence $\ell(\partial S_n) \leq 3\#A\mathcal{L}$ for all n, which proves (2.1). For the last statement, see Lemma 6.5 and Remark 6.6.

Notation. We denote as $\mathcal{A}_{(g,n)}^{\kappa,\mathcal{L}}$ the set of hyperbolic surfaces of signature (g,n) that do not contain (κ,\mathcal{L}) -tangles, and call them tangle-free surfaces.

3. Counting topological types of geodesics in tangle-free surfaces

3.1. Local topological types. Recall from [2] the notion of filling type and of local topological type of closed curve. We omit some of the notations from [2] which are not used in the present paper; the reader can consult the full article for additional details.

Notation 3.1. To any pair of integers (g_S, n_S) such that $2g_S - 2 + n_S \ge 0$, we shall associate a *fixed* smooth oriented surface S of signature (g_S, n_S) . We denote by $\chi(S) = 2g_S - 2 + n_S$ the absolute value of the Euler characteristic of S.

Definition 3.2. A local loop is a pair (S, γ) , where S is a filling type and γ is a primitive loop filling S. Two local loops (S, γ) and (S', γ') are said to be locally equivalent if S = S' (i.e. $g_S = g_{S'}$ and $n_S = n_{S'}$), and there exists a positive homeomorphism $\psi : S \to S$, possibly

permuting the boundary components of S, such that $\psi \circ \gamma$ is freely homotopic to γ' . This defines an equivalence relation \sim on local loops. Equivalence classes for this relation are denoted as $[S, \gamma]_{loc}$ and called *local (topological) types* of loops.

Let us now define a notion of local topological type for loops on a compact hyperbolic surface $S_{g,n}$ of signature (g,n). If $\tilde{\gamma}$ is a loop in a surface $S_{g,n}$, we denote by $S_{g,n}(\tilde{\gamma}) \subset S_{g,n}$ the surface filled by $\tilde{\gamma}$ in $S_{g,n}$, as defined for instance in [2, §4.3, Definition 4.1]: we take a regular neighbourhood \mathcal{R} of $\tilde{\gamma}$, to which we add the contractible components $S_{g,n} \setminus \mathcal{R}$ if there are any.

Definition 3.3. Let $[S, \gamma]_{loc}$ be a local topological type. A loop $\tilde{\gamma}$ in a hyperbolic surface $S_{g,n}$ is said to belong to the local topological type $[S, \gamma]_{loc}$ if there exists a positive homeomorphism $\phi: S_{g,n}(\tilde{\gamma}) \to S$ such that the loops $\phi \circ \tilde{\gamma}$ and γ are freely homotopic in S.

Two loops γ_1 , γ_2 in two hyperbolic surfaces S_1, S_2 are said to be *locally equivalent* if they belong to the same local topological type.

3.2. The counting result.

Definition 3.4. For $0 < \kappa < \mathcal{L}$, L > 0 and S a filling type, we denote by $\operatorname{Loc}_{S}^{\kappa,\mathcal{L},L}$ the set of local topological types $[S,\gamma]_{\operatorname{loc}}$ such that there exists a (κ,\mathcal{L}) -tangle-free surface Z and a homeomorphism $\phi: S \longrightarrow Z$, with $\ell_Z(\phi(\gamma)) \leq L$.

If χ a positive integer, we let

$$\operatorname{Loc}_{\chi}^{\kappa,\mathcal{L},L} = \bigcup_{S,\chi(S) \le \chi} \operatorname{Loc}_{S}^{\kappa,\mathcal{L},L}$$

where the union is over the finite set of filling types S with $\chi(S) \leq \chi$.

Obviously, $\operatorname{Loc}_{\chi}^{\kappa,\mathcal{L},L}$ is decreasing as a function of \mathcal{L} and of κ , and increasing as a function of L. The following statement claims that the cardinality of $\operatorname{Loc}_{g}^{\kappa,\mathcal{L},L}$ is at most polynomial in L if $\kappa > 0$ is fixed and $\frac{L}{L}$ is bounded:

Theorem 3.5. For a given filling type S,

$$\#\operatorname{Loc}_{S}^{\kappa,\mathcal{L},L} = \mathcal{O}_{S}\left(\frac{L}{\mathcal{L}}\left(3\left(16\frac{L^{2}}{\kappa^{2}}+1\right)\right)^{3\frac{L}{\mathcal{L}}(6g_{S}-6+2n_{S})} \times \left(\frac{L}{\kappa}+1\right)^{3\frac{L}{\mathcal{L}}}\right)$$

and as a consequence, for a given χ ,

$$\#Loc_{\chi}^{\kappa,\mathcal{L},L} = \mathcal{O}_{\chi} \left(\frac{L}{\mathcal{L}} \left(3 \left(16 \frac{L^2}{\kappa^2} + 1 \right) \right)^{9\chi \frac{L}{\mathcal{L}}} \times \left(\frac{L}{\kappa} + 1 \right)^{3\frac{L}{\mathcal{L}}} \right).$$

This directly implies Theorem 1.1. When applying this result to random hyperbolic surfaces, the case $\mathcal{L} = \kappa \log g$ and $L = A \log g$ is especially relevant; for this choice of parameters, we have the following corollary:

Corollary 3.6. Let $\chi \geq 1$, $0 < \kappa < 1$, $\mathcal{L} = \kappa \log g$, $A \geq 1$. Then

$$\#\mathrm{Loc}_{\chi}^{\kappa,\mathcal{L},A\log g}=\mathcal{O}_{\kappa,A,\chi}\left((\log g)^{\beta_{\kappa,A,\chi}}\right)$$

for a $\beta_{\kappa,A,\chi} > 0$ depending only on κ , A and χ .

For the sake of readability we postpone the proof of Theorem 3.5 to §6.

4. Moebius inversion formula

4.1. Sub-c-surfaces and their geodesic representatives.

Definition 4.1. If S is a c-surface, we call sub-c-surface of S a subset $S' \subset S$ which (for the induced topology and differential structure) is a c-surface, and such that

- the 1d components of S' are homotopically non-trivial in S;
- the boundary curves of the 2d components of S' are homotopically non-trivial in S, and are paiwise non-homotopic.

In other words, we ask that the boundary $\partial S'$ be a multicurve in S.

Assume that S has signature $(\mathbf{g}, \mathbf{n}) \in \mathcal{N}^{(k)}$ and S' has signature $(\mathbf{g}', \mathbf{n}') \in \mathcal{N}^{(k')}$. Let $S' = (c', \sigma')$ be the decomposition of S' into its 1d and its 2d parts, where $c' = (c'_1, \ldots, c'_{j'})$ and $\sigma' = (\sigma'_1, \ldots, \sigma'_{m'})$. If S is endowed with a metric $Z \in \mathcal{T}^*_{(\mathbf{g}, \mathbf{n})}$, then the collection (c', σ') is isotopic to a geodesic sub-c-surface (c'_Z, σ'_Z) . By this, we mean that σ'_Z has geodesic boundary components, and that $c'_{Z,i}$ is an oriented periodic geodesic. We call (c'_Z, σ'_Z) the geodesic representative of (c', σ') in Z.

Then (c'_Z, σ'_Z) (endowed with the Riemannian metric induced by Z) is a representative of an element of the moduli space $\mathcal{M}^*_{(\mathbf{g}',\mathbf{n}')}$.

We denote by S(Z) the set of geodesic sub-c-surfaces in the hyperbolic surface Z. If $\tau \in S(Z)$, we denote by $\bar{\tau}$ its equivalence class in M (since the moduli space is define as a space of Riemannian manifolds modulo isometry).

Remark 4.2. By definition, the boundary components of σ_i and those of σ'_j are numbered, but even if $\sigma'_j \subset \sigma_i$ we do not impose any condition relating the respective numberings of the boundary components. Similarly, if $c'_j = c_i$ (as sets), we do not impose that the orientations of c'_j and c_i coincide.

4.2. The "Moebius function". Let κ, \mathcal{L} be two real numbers with $\kappa < \mathcal{L}$. Assume as before (mostly for simplicity) that $\kappa < \mathcal{L}$, and that $\kappa < 2 \operatorname{argsh}(1)$, so that all closed geodesics of length $\leq \kappa$ in a hyperbolic manifold are simple and pairwise disjoint [5, Theorem 4.1.6].

Theorem 4.3. There exists a (unique) function $\mu = \mu_{(\kappa,\mathcal{L})} : \mathbf{M} \longrightarrow \mathbb{R}$, such that

- the restriction of μ to each $\mathcal{M}^*_{(\mathbf{g},\mathbf{n})}$ is invariant under the action of the full diffeomorphism group (i.e. possibly permuting boundary components and connected components);
- For $Z \in \mathbf{M}$, $\mu(Z) = 0$ if $Z \notin \mathbf{D}_{\kappa,\mathcal{L}}$;
- For every $Z \in \mathbf{D}_{\kappa,\mathcal{L}}$,

$$(4.1) 1 = \sum_{\tau \in \mathcal{S}(Z)} \mu(\tau),$$

where $\mu(\tau)$ is defined as $\mu(\bar{\tau})$, where $\bar{\tau} \in \mathbf{M}$ is the equivalence class of τ .

In addition, the function μ we construct satisfies the following:

• If $(\mathbf{g}, \mathbf{n}) = (\underbrace{(0, 2), \dots, (0, 2)}_{j})$, and $Z = (c_1, \dots, c_j) \in \mathcal{M}^*_{(\mathbf{g}, \mathbf{n})}$, then

(4.2)
$$\mu(Z) = \frac{(-1)^{j+1}}{2^j i!}$$

if $\ell(c_i) < \kappa$ for all i, and $\mu(Z) = 0$ otherwise;

• If $(\mathbf{g}, \mathbf{n}) = ((g_1, n_1), \dots, (g_m, n_m))$ with $\chi_i = 2g_i - 2 + n_i > 0$ for $i = 1, \dots, m$, if $Z = (\sigma_1, \dots, \sigma_m) \in \mathcal{M}^*_{(\mathbf{g}, \mathbf{n})}$, then

$$(4.3) |\mu(Z)| \le F(\chi(Z)) e^{\mathcal{L}G(\chi(Z))}$$

where $\chi(Z) = \sum_{i=1}^{m} \chi_i$ is the absolute Euler characteristic of Z, where $(F(n))_{n\geq 0}$, $(G(n))_{n\geq 0}$ are two increasing sequences defined respectively by (4.13) and by G(n) = G(n-1) + 9n(n-1) and G(1) = 0;

• If $Z = (c, \sigma)$ is the decomposition of Z into 1d and 2d parts, then

(4.4)
$$\mu(Z) = -\mu(c)\mu(\sigma).$$

Remark 4.4. Denoting \emptyset the empty subsurface and deciding that $\mu(\emptyset) = -1$, one can conveniently rewrite (4.1) as

$$(4.5) 0 = \sum_{\tau \in \mathcal{S}_0(Z)} \mu(\tau)$$

where $S_0(Z)$ is the set of all geodesic subsurfaces of Z, including the empty one. This convention is also compatible with the multiplicative property (4.4).

Proof. The existence and uniqueness of μ is proven recursively. We first prove the claim in the case where Z is purely 1d, with an induction on |Z|, the number of 1d components. We then prove that if μ exists, then it must satisfy (4.4): this is proven by induction on the pair $(\chi(Z), |Z|)$ with lexicographic order. We are left with proving the result when Z is purely 2d: this is proven by induction on $\chi(Z)$.

Introduce

$$N(Z) := \#\{\tau \in \mathcal{S}(Z), \tau \text{ diffeomorphic to } Z\}.$$

In the special case where $Z=(c_1,\ldots,c_j)$ and all the c_i are 1-dimensional, then $N(Z)=N_j:=2^jj!$, corresponding to permutations and orientation changes of the components c_i . For more general Z,N(Z) may be explicitly described in terms of permutations of components having the same topology, and re-labelling of boundary components, but we won't need a precise formula. In general, if $Z=(c,\sigma)$ where $c=(c_1,\ldots,c_j)$ is the one-dimensional part of Z and $\sigma=(\sigma_1,\ldots,\sigma_m)$ is the two-dimensional part, then $N(Z)=N(j)N(\sigma)$, simply because a 1-dimensional manifold cannot contain a 2-dimensional one.

For $\chi(Z)=0$, i.e. $Z=(c_1,\ldots,c_j)$ is purely 1d, the recursion to prove (4.1) is on j=|Z|. If $Z=(c_1)$, a single circle of length $\leq \kappa$, (4.1) reads $1=N_1\mu(c_1)=2\mu(c_1)$, so that the formula holds with $\mu(c_1)=1/2$. Suppose that (4.2) if known for all values j< n, and denote $\mu_j:=\frac{(-1)^{j+1}}{2^jj!}$. If $Z=(c_1,\ldots,c_n)$, then (4.1) gives the recursion formula

(4.6)
$$1 = N_n \mu(Z) + \sum_{1 \le j \le n} \frac{n!}{(n-j)!} 2^j \mu_j.$$

The term $\frac{n!}{(n-j)!}$ is the number of ordered subsets of cardinality j in a set of n elements (here the components of Z); the factor 2^j corresponds to the number of possible orientations of a multicurve with j components. The identity (4.6) leads to (4.2) at rank n, thanks to the standard identity

$$\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} (-1)^{j} = 0.$$

We now prove that if μ exists, then it must satisfy (4.4). We prove (4.4) by induction on the pair $(\chi(Z), |Z|)$ ordered by lexicographic order. It is obviously true for $(\chi(Z), |Z|) = (0, 0)$ with the convention of Remark 4.4. Let $Z = (c, \sigma)$ be the decomposition of Z into 1d and 2d part. Assume $|Z| \ge 1$, otherwise the claim us again obvious. Denote j := |Z|. We can write the

necessary condition (4.1) as

(4.7)
$$1 = \mu(Z)N(Z) + \sum_{\tau \in \mathcal{S}(\sigma)} \mu(\tau) + \sum_{l=1}^{j} \sum_{l' \geq 0} \sum_{\substack{s \in \mathcal{S}(c), \tau \in \mathcal{S}_{0}(\sigma) \\ |s| = l, |\tau| = l', \chi(\tau) < \chi(\sigma)}} \frac{(l+l')!}{l'!l!} \mu(s,\tau) + \sum_{l=1}^{j-1} \sum_{\substack{s \in \mathcal{S}(c), \tau \in \mathcal{S}_{0}(\sigma) \\ |s| = l, \chi(\tau) = \chi(\sigma)}} \mu(s,\tau).$$

The term $\mu(Z)N(Z)$ corresponds to all sub-c-surfaces of Z that are diffeomorphic to Z. The other terms corresponds to strict subsurfaces of Z.

The sum $\sum_{\tau \in S(\sigma)}$ corresponds to sub-c-surfaces τ of Z which are entirely contained in the 2-dimensional part σ of Z. The two other sums corresponds to all strict sub-c-surfaces of Z sharing at least one 1-dimensional component with Z. Such a subsurface can be decomposed into s, consisting of exactly l components of c, and τ a subsurface of σ (τ itself may have 1d components, their number it denoted by l').

If $\chi(\tau) < \chi(\sigma)$ then l may take any value between 1 and j. If $\chi(\tau) = \chi(\sigma)$ then $\tau = \sigma$ as a manifold, and τ and σ just differ in the numbering of the boundary components. In this case, τ does not contain any 1-dimensional component. In other words, in the last sum, $|\tau|$ has to be 0. The last sum has to be stopped at j-1: for |s|=j we would have (s,τ) diffeomorphic to Z, contradicting the fact that (s,τ) was asked to be a strict sub-c-surfaces of Z.

The combinatorial term $\frac{(l+l')!}{l'!l!}$ it the number of partitions of a set of cardinality l+l' into two sets of cardinality l and l'. It takes into account the fact that the 1-dimensional components of τ may be intertwined, in their numbering, with the components of s.

Let us call σ' is the 2-dimensional component of τ . Because either $\chi(\sigma') < \chi(\sigma)$ or $l+l' \leq j-1$, we can use the induction assumption and write $\mu(s,\tau) = -\mu_{l+l'}\mu(\sigma')$. We now use the following property of the sequence (μ_n) :

$$-\frac{(l+l')!}{\nu!l!}\mu_{l+l'} = \mu_l \mu_{l'}.$$

Again by the induction assumption, $\mu_{l'}\mu(\sigma') = -\mu(\tau)$, so

$$\frac{(l+l')!}{l'!l!}\mu(s,\tau) = -\mu_l\mu(\tau) = -\mu(s)\mu(\tau).$$

Equation (4.7) becomes

$$\begin{split} 1 &= \mu(Z)N(Z) + \sum_{\tau \in \mathcal{S}(\sigma)} \mu(\tau) - \sum_{l=1}^{j} \sum_{\substack{s \in \mathcal{S}(c), \tau \in \mathcal{S}_{0}(\sigma) \\ |s| = l, \chi(\tau) < \chi(\sigma)}} \mu(s)\mu(\tau) \\ &- \sum_{l=1}^{j-1} \sum_{\substack{s \in \mathcal{S}(c), \tau \in \mathcal{S}_{0}(\sigma) \\ |s| = l, \chi(\tau) = \chi(\sigma)}} \mu(s)\mu(\tau) \\ &= \mu(Z)N(Z) + 1 - \Big(\sum_{l=1}^{j} \sum_{\substack{s \in \mathcal{S}(c) \\ |s| = l}} \mu(s)\Big) \Big(\sum_{\substack{\tau \in \mathcal{S}_{0}(\sigma) \\ \chi(\tau) < \chi(\sigma)}} \mu(\tau)\Big) \\ &- \Big(\sum_{l=1}^{j-1} \sum_{\substack{s \in \mathcal{S}(c) \\ |s| = l}} \mu(s)\Big) \Big(\sum_{\substack{\tau \in \mathcal{S}_{0}(\sigma) \\ \chi(\tau) = \chi(\sigma)}} \mu(\tau)\Big) \\ &= \mu(Z)N(Z) + 1 - 1(-N(\sigma)\mu(\sigma)) - (1 - N_{j}\mu_{j})N(\sigma)\mu(\sigma), \end{split}$$

yielding

$$\mu(Z)N(Z) = -N(\sigma)\mu(\sigma)N_i\mu_i$$

which gives $\mu(Z) = -\mu(\sigma)\mu_i$ because we already know that $N(Z) = N_i N(\sigma)$.

Finally, suppose that $Z = (\sigma_1, \ldots, \sigma_m) \in \mathcal{M}^*_{(\mathbf{g}, \mathbf{n})}$ is purely 2d. Then we can proceed by induction on $\chi(Z)$ to show the existence and uniqueness of μ , because the only sub-c-surfaces of Z that have the same Euler characteristic as Z, are equal to Z, up to permutation of (diffeomorphic) connected components and relabelling of boundary components. In this case, (4.1) reads

$$1 = \mu(Z)N(Z) + \sum_{\substack{\tau \in \mathcal{S}(Z) \\ \chi(\tau) < \chi(Z)}} \mu(\tau)$$

Because N(Z) is a positive integer, this defines $\mu(Z)$ uniquely, by recursion on $\chi(Z)$. There remains to prove (4.3), again by induction. First note that

(4.8)
$$|\mu(Z)| \le 1 + \sum_{\substack{\tau \in \mathcal{S}(Z) \\ \chi(\tau) < \chi(Z)}} |\mu(\tau)|.$$

Assume first that m=1 and σ_1 has Euler characteristic -1. Then the only possible derived tangles in Z of Euler characteristic 0 are the unions of simple oriented geodesics Z of lengths $\leq \kappa$. There are 3 non-oriented simple geodesics if Z is a three-holed sphere; if Z is a once-punctured torus, there can be at most two non-oriented simple geodesics of length $\leq \kappa$ because we chose $\kappa < 2 \operatorname{argsh}(1)$. In any case, (4.8) implies that

$$(4.9) |\mu(Z)| \le 1 + 3 \times 2|\mu_1| + 6 \times 2^2|\mu_2| + 6 \times 2^3|\mu_3| = 8.$$

Assume that $\chi(Z) = n$ with n > 1, and that (4.3) holds for c-surfaces of absolute Euler characteristic < n. We then have

$$|\mu(Z)| \leq 1 + \sum_{\substack{\tau \in \mathcal{S}(Z) \\ \chi(\tau) < \chi(Z)}} |\mu(\tau)| \leq 1 + \sum_{\substack{1 \leq j < n \\ \chi(\tau) = j}} \sum_{\tau \in \mathcal{S}(Z)} F(j) e^{\mathcal{L}G(j)} \mathbb{1}_{\tau \in \mathbf{D}_{\kappa, \mathcal{L}}}$$

$$\leq 1 + F(n-1) e^{\mathcal{L}G(n-1)} \sum_{1 \leq j < n} \#\{\tau \in \mathcal{S}(Z), \chi(\tau) = j, \ell(\partial \tau) \leq 3\mathcal{L}j\}$$

where we have relied on (2.1) in the last line.

To conclude, we need a rough estimate on $\#\{\tau \in \mathcal{S}(Z), \chi(\tau) = j, \ell(\partial \tau) \leq 3\mathcal{L}j\}$. A sub-c-surface $\tau \in \mathcal{S}(Z)$ is fully characterized by the data of the multicurve $\partial \tau = (\gamma_1, \ldots, \gamma_N)$ in Z, and of a partition of $\{1, \ldots, N\} = \sqcup \Lambda_l$ into non-empty intervals, such that i, j belong to the same Λ_l if and only if γ_i and γ_j belong to the same connected component of τ . Remark that $N \leq 3n$.

We shall use the following estimate:

Lemma 4.5. [2, Lemma 2.1], adapted from [5, Theorem 4.1.6 and Lemma 6.6.4] Let X be a hyperbolic surface, compact or bordered. For any L > 0,

$$\#\{\gamma \in \mathcal{P}(X) \mid \ell_X(\gamma) \le L\} \le 205 \,\chi(X) \,e^L.$$

where $\chi(X)$ is the absolute Euler characteristic of X. As a consequence, if F is a function supported in [-L, L],

(4.11)
$$\sum_{\gamma \in \mathcal{P}(X)} |F(\ell(\gamma))| \le 560 \, \chi(X) L \|F(\ell)e^{\ell}\|_{\infty}.$$

This lemma implies that the number of multi-curves with $N \leq 3n$ components of length less than $3\mathcal{L}(n-1)$ is less that

$$3n(205ne^{3\mathcal{L}(n-1)})^{3n}.$$

This gives the very rough upper bound,

(4.12)
$$1 + \sum_{1 \le j < n} \#\{\tau \in \mathcal{S}(Z), \chi(\tau) = j, \ell(\partial \tau) \le 3\mathcal{L}j\} \le 3n^2 T(n) (205ne^{3\mathcal{L}(n-1)})^{3n}$$

where T(n) is an upper bound of the number of partitions of a set of cardinality $\leq 3n$. From (4.3), we can infer that

$$|\mu(Z)| \le F(n)e^{\mathcal{L}G(n)}$$

if we define F by the recursion relation

(4.13)
$$F(n) = F(n-1)3n^2T(n)(205n)^{3n}.$$

and G by G(n) = G(n-1) + 9n(n-1). The initial bound (4.9) gives the initial values F(1) = 8 and G(1) = 0.

Remark 4.6. The method leading to (4.12) also shows that

$$(4.14) 1 + \sum_{1 \le j < M} \#\{\tau \in \mathcal{S}(Z), \chi(\tau) = j, |\tau| = 0, \ell(\partial \tau) \le 3\mathcal{L}j\} \le 3M^2 T(M) (205ne^{3\mathcal{L}(M-1)})^{3M}$$

where n is still the Euler characteristic of Z.

Remark 4.7. Let $(\mathbf{g}, \mathbf{n}) \in \mathcal{N}^{(k)}$. There exists a uniquely defined partition $(\Lambda_l)_{l=1,\ldots,M}$ of $\{1,\ldots,k\}$ into sub-intervals, and a strictly increasing sequence $(G_l,N_l)_{l=1,\ldots,M}$, such that $(g_i,n_i)=(G_l,N_l)$ for all $i \in \Lambda_l$. If $Z=(\sigma_1,\ldots,\sigma_k) \in \mathcal{M}^*_{(\mathbf{g},\mathbf{n})}$, let $Z_l=((\sigma_i)_{i\in\Lambda_l})$ be the collection of components of signature (G_l,N_l) . The argument yielding (4.4) also shows a more general multiplicative property (that will not be used in this work):

$$\mu(Z) = (-1)^{m-1} \prod_{l=1}^{m} \mu(Z_l).$$

5. Comparison with other inclusion-exclusion formulas

5.1. Comparison with tangles in Friedman's work. In his seminal work about random regular graphs, J. Friedman uses a formula similar to (4.1), calling it *generalized Moebius inversion*: see [7, Proposition 9.5].

In Friedman's work on random graphs, the construction of the Moebius function is very short, and there is absolutely no need of any explicit description of μ , whereas for random surfaces, we needed the explicit formula (4.2) in the 1d case (or at least the strong decay with respect to j), and the rough upper bound (4.3) in the 2d case. There are several reasons for these complications.

For surfaces, we have to deal with the possibility of closed geodesics of length $\leq \kappa$, for κ arbitrarily small, as one of the reasons for proliferation of long periodic geodesics. As a consequence, some of the tangles to be removed are circles. Our derived tangles may thus have an arbitrary number of connected components, while keeping zero Euler characteristic. On the opposite, on a graph, a closed geodesic path always has length ≥ 2 , so there is no need to remove short closed geodesics. Tangles in Friedman's work have non-zero Euler characteristic: bounding the Euler characteristic of a derived tangle by M automatically implies bounding its number of connected components (interestingly, this remark was done by J. Friedman just after Lemma 9.6 in [7]).

In addition, in [7, §9], Friedman uses in a crucial way an order relation on graphs, such that if two graphs G, G' satisfy $G \leq G'$ then the counting function of geodesics on G' is smaller than the counting function of geodesics on G. He can then reduce the discussion to the set of *minimal tangles* for this order relation, which is finite for a given Euler characteristic. See also [8]. We don't know if an analogue of this order relation can be implemented for surfaces. In the graph

case, the upshot is that it is obvious that the Moebius function is bounded on the (finite) set of minimal derived tangles of a given Euler characteristic; in the surface case, some work was needed to obtain the bound (4.3).

5.2. Comparison with inclusion-exclusion in [11] and in [2]. In [11], the only tangles to be removed are the short periodic geodesics: the standard inclusion-exclusion principle is used in Lemma 3.4 of [11], to write that the κ -thin set $\mathcal{M}_q^{<\kappa}$ satisfies

(5.1)
$$\mathbb{1}_{\mathcal{M}_g^{\leq \kappa}}(X) = \sum_{j=1}^{+\infty} \mu_j \sum_{(c_1, \dots, c_j) \subset X} \mathbb{1}_{[0, \kappa)}(\ell(c_1)) \dots \mathbb{1}_{[0, \kappa)}(\ell(c_j)),$$

where the sum runs over all multicurves $(c_1, \ldots, c_j) \subset X$. This in exact correspondence with our formula (4.2) for the "Moebius function" (the additional factor $2^j j!$ in this paper just comes from a different convention on how to count multicurves).

In our previous work [2] we needed to remove 2d-tangles. We used the standard inclusion-exclusion formula in [2, §10.2]. In other words, we decomposed $1 - \mathbbm{1}_{\mathcal{A}_{(g,n)}^{\kappa,L}}$ in a formula similar to (5.1), where the components (c_1, \ldots, c_j) may either be short curves or small pairs-of-pants. Sorting the tangled surfaces according to the number of tangles j turned out not to be so useful, and we had to do extra work in [2, §10.3.1] to show that the occurrence of a derived tangle of Euler characteristic ≤ -2 falls into a remainder term $\mathcal{O}(\frac{1}{g^2})$. This required enumerating all possible ways that two tangles can intersect one another. If the tangles are just short curves, this is easy because short curves of length $\kappa < \operatorname{argsh}(1)$ are always disjoint in any hyperbolic surfaces. There is no equivalent property for 2-dimensional tangles.

In this paper, the event "existence of tangles in X", instead of being sorted according to the *number* of tangles, is directly sorted according to the *surface filled* by tangles. This shunts the step of enumerating all possible ways that j tangles can intersect one another, which seems daunting once tangles are allowed to be pairs-of-pants and not just short curves, and for $j \geq 3$.

6. Counting topological types

6.1. Counting simple curves in surfaces. Our counting argument relies mostly on the following estimate, based on Thurston's description of the space of simple curves, as exposed in [6]. We denote by $i(\gamma_1, \gamma_2)$ the geometric intersection of two loops γ_1, γ_2 in minimal position In all the forthcoming discussion, S is a fixed compact surface of signature (g_S, n_S) .

Proposition 6.1. Let S be a compact surface of signature (g_S, n_S) . Let $\lambda = (\lambda_1, \ldots, \lambda_{3g_S-3+n_S})$ be a multicurve cutting S into surfaces of Euler characteristic -1. Let N, I, J be three positive integers. Call $\mathcal{C}_{\lambda}(N, I, J)$ the set of N-tuples of simple curves $(\gamma_1, \ldots, \gamma_N)$ in minimal position, such that

$$i(\gamma_j, \lambda_k) \leq I$$

for all $j = 1, ..., N, k = 1, ..., 3g_S - 3 + n_S$, and such that

$$i(\gamma_i, \gamma_k) \leq J$$

for all $j \neq k \in \{1, ..., N\}$. Consider on $\mathcal{C}_{\lambda}(N, I, J)$ the equivalence relation:

$$(6.1) \qquad (\gamma_1, \dots, \gamma_N) \ \mathcal{R} \ (\gamma_1', \dots, \gamma_N')$$

iff there exists a homeomorphism ϕ such that $\phi(\gamma_j)$ is freely homotopic to γ'_j . Call $\overline{\mathcal{C}}_{\lambda}(N,I,J)$ the quotient set. Then

(6.2)
$$\#\overline{\mathcal{C}}_{\lambda}(N,I,J) \le 2^{3g_S - 3 + n_S} \left((3(I+1))^{(3g_S - 3 + n_S)} (J+1)^{3g_S - 3 + n_S} \right)^N.$$

Proof. We shall even prove the upper bound (6.2) for equivalence classes of a stronger equivalence relation, where we ask ϕ to be in the group generated by Dehn twists around curves in λ . We follow standard arguments borrowed from Thurston's description of the space of simple curves. We rely on the reference [6, Exposé 4].

Let
$$(\gamma_1, \ldots, \gamma_N) \in \mathcal{C}_{\lambda}(N, I, J)$$
. Let

$$m^j(k) := i(\gamma_j, \lambda_k).$$

If B is a subset of $\{1,\ldots,3g_S-3+n_S\}$, call $\mathcal{C}_{\lambda,B}(N,I,J)$ the set of elements of $\mathcal{C}_{\lambda}(N,I,J)$ such that $m^j(k)=0$ for all $j=1,\ldots,N$ and for all $k\in B$. Let $\overline{\mathcal{C}}_{\lambda,B}(N,I,J)$ be the set of corresponding equivalence classes. We shall show that

$$\#\overline{\mathcal{C}}_{\lambda,B}(N,I,J) \le \left((3(I+1))^{(3g_S-3+n_S-\#B)} (J+1)^{3g_S-3+n_S-\#B} \right)^N$$

for all B, and the upper bound (6.2) is obtained by summing over all $B \subset \{1, \ldots, 3g_S - 3 + n_S\}$. So let us fix $B \subset \{1, \ldots, 3g_S - 3 + n_S\}$ and describe a "normal form" for a representative of the equivalence class of any $(\gamma_1, \ldots, \gamma_N) \in \mathcal{C}_{\lambda, B}(N, I, J)$.

Let $Q_1, \ldots, Q_{2g_S-2+n_S}$ be the connected components of $S \setminus \lambda$. Fix $j \in \{1, \ldots, N\}$. It is classical that the data of the family $(m^j(k))_{k \notin B}$ determines exactly one topology for each restriction $\gamma_j \cap Q_m$, where $m = 1, \ldots, 2g_S-2+n_S$. To exploit this fact, we follow quite closely the discussion of [6, Exposé 4, §III] but introduce some small variants, suited to our counting problem.

Let P^2 be a fixed pair of pants with boundary components numbered B_1, B_2, B_3 , For each triple of non-negative integers (M_1, M_2, M_3) such that $M_1 + M_2 + M_3$ is even, there exists a model of multiple arc in P^2 having M_1, M_2, M_3 intersections with each of the three boundary components. The list of models is given in [6, Exposé 4, §III, Figure 3].

We denote by $\lambda_k \times [-1, +1]$ some tubular neighbourhoods of each λ_k , chosen to be pairwise disjoint in S. The complement of their union is made of disjoint pairs-of-pants $R_1, \ldots, R_{2g_S-2+n_S}$, with $R_m \subset Q_m$. Let us fix once and for all a homeomorphism ϕ_m between each R_m and the reference pair of pants P^2 , thus obtaining a collection of models of multiple arcs in R_m .

The data of the family $(m^j(k))_{k\notin B}$ determines a model of multiple arc in each R_m , such that $\gamma_j \cap R_m$ is isotopic to the model.

There exists a curve γ'_j isotopic to γ_j , that coincides with the model in each R_m (furthermore, by [6, Exposé 4, Lemme 5], any two curves γ'_j , γ''_j satisfying this requirement are isotopic in each cylinder $\lambda_k \times [-1, +1]$, for an isotopy fixing the boundary $\lambda_k \times \{-1, +1\}$). The "normal form" γ'_j is fully determined by the data of $(m^j(k))_{k \notin B}$ and of the isotopy class of $\gamma'_j|_{\lambda_k \times [-1, +1]}$ for all $k \notin B$. Let s^j_k , t^j_k be two distinct transversals to this cylinder, joining $\lambda_k \times \{-1\}$ and $\lambda_k \times \{+1\}$, sharing the same endpoints and intersecting only at those endpoints. The isotopy class of $\gamma_j|_{\lambda_k \times [-1, +1]}$ is fully parametrized by s^j_k and t^j_k , defined as the geometric intersection of $\gamma_j|_{\lambda_k \times [-1, +1]}$ respectively with s^j_k and t^j_k . Besides, we have either $m^j(k) = s^j_k + t^j_k$, or $s^j_k = m^j(k) + t^j_k$ or $t^j_k = m^j(k) + s^j_k$, see [6, Exposé 4, §III, Figure 5].

The following remark is crucial: applying a Dehn twist to the transversal s_k^j while keeping the parameters s_k^j constant amounts to applying the same Dehn twist on γ_j . Since we are only counting equivalence classes, it is enough to show that there exists a choice of transversal s_k^j for which we can bound all the s_k^j by the integer J.

Given $k \notin B$, we choose (for instance) the smallest $j_o \in [1, ..., N]$ such that γ_{j_o} intersects λ_k , and we choose s_k^j to be a branch of $\gamma'_{j_o}|_{\lambda_k \times [-1,+1]}$. Our assumptions on the intersection numbers both imply that $0 \le s_k^j \le J$ (giving at most J+1 possibilities for each $k \notin B$, and for $j=1,\ldots,N$), that $0 \le m^j(k) \le I$ for $k \notin B$, and for $j=1,\ldots,N$, giving I+1 possibilities for $m^j(k)$. This upper bound to be multiplied by three for each $k \notin B$, and for each $j=1,\ldots,N$, corresponding to the three situations $m^j(k) = s_k^j + t_k^j$, or $s_k^j = m^j(k) + t_k^j$ or $t_k^j = m^j(k) + s_k^j$.

Counting all possibilities gives the desired upper bound.

6.2. Sequence of volutes associated with a non-simple loop. Let γ be a non-simple (oriented) loop in minimal position, contained in a surface S. The following construction is inspired by a discussion from Luo–Tan [12, §1.2], and relies on the fact that the complement of γ does not contain any bigon.

Let x_0 be a point on γ , assumed not to be a self-intersection point of γ , and choose a parametrization $\gamma: \mathbb{R} \longrightarrow S$ such that $\gamma(0) = x_0$. Consider the path $A_t^- = \gamma([-t,0])$ for t > 0. Let $t_- > 0$ be the smallest positive number such that A_{t_-} is not a simple arc (say, $\gamma(-t_-) = \gamma(t')$ with $t' \in (-t_-,0]$). Next, let $t_+ > 0$ be the smallest positive time so that $\gamma(t_+) \in \gamma([-t_-,t_+))$ (say, $\gamma(t_+) = \gamma(t'')$ with $t'' \in [-t_-,t_+)$). The graph $G(x_0) = \gamma([-t_-,t_+])$ has Euler characteristic -1; call $\Sigma(x_0)$ a regular neighbourhood, it is either a three holed sphere or a once-holed torus. The graph $G(x_0)$ may be decomposed into two rooted simple oriented loops $g_1(x_0) = \gamma_{[-t_-,t']}, g_2(x_0) = \gamma_{[t'',t_+]}$ (with respective roots $\gamma(-t_-)$ and $\gamma(t_+)$), and a simple path $\gamma(t_0) = \gamma([t',t''])$ joining the root of $\gamma(t_0)$ to the root of $\gamma(t_0)$.

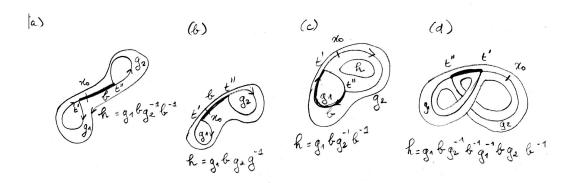


FIGURE 1. The four pictures for the volute $G(x_0)$.

In Figure 1, we assumed for symmetry reasons that at time t' the curve γ crosses itself from the right. The picture shows the four possible situations, depending on whether $t' \leq t''$ or t'' > t', and whether at time t'' the curve γ crosses itself from the right or from the left. In cases (a), (b), (c), $\Sigma(x_0)$ is a pair-of-pants, its boundary is made of two disjoint simple curves $\gamma_1(x_0), \gamma_2(x_0)$ homotopic respectively to $g_1(x_0), g_2(x_0)$, and of a third simple curve $h(x_0)$, homotopic to

(6.3)
$$h'(x_0) = g_1(x_0) \cdot b(x_0) \cdot g_2(x_0)^{\pm 1} \cdot b(x_0)^{-1},$$

where • means the concatenation of paths. In case (d), $\Sigma(x_0)$ is a once-holed torus : the boundary of $\Sigma(x_0)$ is made of only one curve $h(x_0)$ homotopic to

$$g_1(x_0) \cdot b(x_0) \cdot g_2(x_0)^{-1} \cdot b(x_0)^{-1} g_1(x_0)^{-1} \cdot b(x_0) \cdot g_2(x_0) \cdot b(x_0)^{-1}$$
.

The surface $\Sigma(x_0)$ is filled by two simple curves $\gamma_1(x_0), \gamma_2(x_0)$ intersecting once, homotopic respectively to $g_1(x_0), g_2(x_0)$. To fix ideas we can decide to put their intersection point at $\gamma(t')$.

Definition 6.2. We call $G(x_0)$ the *volute* associated with x_0 .

We now take $\epsilon > 0$ very small, so that $\gamma([t'', t_+ + \epsilon])$ still intersects exactly once, at $\gamma(t_+)$. Take as new origin $x_1 := \gamma(t_+ + \epsilon)$ and repeat the same construction with x_0 replaced by x_1 . We obtain a volute $G(x_1)$ formed of $g_1(x_1), g_2(x_1), b(x_1)$, and a surface $\Sigma(x_1)$ of Euler characteristic -1 containing three simple curves $(\gamma_1(x_1), \gamma_2(x_1), h(x_1))$. If $\Sigma(x_1)$ is a pair-of-pants then $(\gamma_1(x_1), \gamma_2(x_1), h(x_1))$ are disjoint simple curves bordering $\Sigma(x_1)$. If $\Sigma(x_1)$ is a once-holed torus then $h(x_1)$ is homotopic to the boundary, and $\gamma_1(x_1), \gamma_2(x_1)$ are simple curves

intersecting once and filling $\Sigma(x_1)$. Remark that the rooted oriented loops $g_2(x_0)$ and $g_1(x_1)$ coincide, so we can impose $\gamma_2(x_0) = \gamma_1(x_1)$.

Repeating this construction, we obtain a sequence of volutes $(G(x_n))$, with $g_2(x_n) = g_1(x_{n+1})$, and a surface $\Sigma(x_n)$ of Euler characteristic -1 containing three simple curves $(\gamma_1(x_n), \gamma_2(x_n), h(x_n))$, with $\gamma_2(x_n) = \gamma_1(x_{n+1})$. Since γ is a loop, there is N such that $G(x_N) = G(x_0)$ and we stop the construction.

Keep in mind that $h(x_n)$ is homotopic to $h'(x_n)$, formed like in (6.3) by a concatenation of $g_1(x_n)$, $g_2(x_n)$ and copies of $b(x_n)$. As a consequence of the construction, for any Riemannian metric on S, we have

(6.4)
$$\ell(G(x_n)) \ge \max\left(\ell(g_1(x_n)), \ell(g_2(x_n)), \frac{\ell(h'(x_n))}{2}\right).$$

We will also use the fact that

(6.5)
$$\ell(\gamma) \le \sum_{n=0}^{N-1} \ell(G(x_n)) \le 3\ell(\gamma).$$

We now denote

- $g_n = \gamma_2(x_n) = \gamma_1(x_{n+1}) \ (n \in \mathbb{Z}/N\mathbb{Z});$
- $h_n = h(x_n)$, $\Sigma_n = \Sigma(x_n)$. If Σ_n form a pair-of-pants, (g_{n-1}, g_n, h_n) are its boundary components. If Σ_n is a once-holed torus, h_n is its boundary and g_{n-1}, g_n are two simple curves, disjoint from h_n , intersecting once and filling Σ_n .

We take the collection of curves $((g_{n-1}, g_n, h_n))_{n \in \mathbb{Z}/N\mathbb{Z}}$ in minimal position. Remark that if both Σ_n and Σ_{n+1} are once-holed tori, the curves g_{n-1}, g_n and g_{n+1} are not arbitrary: for instance, g_{n+1} must intersect g_{n-1} . However, since we just need to find upper bounds on the number of topological types, it will not be necessary to determine all conditions fulfilled by the curves g_n .

The segments I_n and Condition (V). If Σ_{n+1} is a pair-of-pant, we consider a segment I_{n+1} going from from g_n to g_{n+1} inside Σ_{n+1} . If Σ_{n+1} is a once-holed torus, we define the segment I_{n+1} as reduced to a point, the intersection point between g_n and g_{n+1} .

If Σ_n and Σ_{n+1} are pairs-of-pants, they must satisfy the following condition, referred below as Condition (V): the first exit point of I_{n+1} out of Σ_n cannot be through the boundary g_n .

We write $I_{n+1} = [O_{n+1}, T_{n+1}]$ where $T_n, O_{n+1} \in g_n$ for all n. The point O_{n+1} will serve as base-point of g_n .

Note that the prescriptions above only define I_{n+1} is modulo homotopy with gliding endpoints along g_n, g_{n+1} , so we can impose further conditions. Assume first that both Σ_n and Σ_{n+1} are pairs-of-pants. If I_{n+1} intersects h_n , by gliding O_{n+1} along g_n , we can impose that $O_{n+1} \neq T_n$ and that $[O_{n+1}, T_{n+1}]$ is in minimal position with $[O_n, T_n]$. If I_{n+1} intersects g_{n-1} we impose that $O_{n+1} = T_n$ and (O_{n+1}, T_{n+1}) contains (T_n, O_n) . Note that we can have $T_{n+1} = O_n$ if $g_{n+1} = g_{n-1}$ and $h_{n+1} = h_n$ (which implies, in particular, that $\Sigma_{n+1} = \Sigma_n$).

If Σ_n is a pair-of-pants and Σ_{n+1} a once-holed torus, we impose that I_n is in minimal position with g_{n+1} . If Σ_n is a once-holed torus and Σ_{n+1} a pair-of-pants, we impose that g_n is in minimal position with I_{n+1} .

We let c_n be the smallest subpath of g_n , oriented in the direction of g_n , and going from T_n to O_{n+1} (in case $T_n \neq O_{n+1}$, this is a strict subsegment of g_n ; in case $T_n = O_{n+1}$, it is a trivial subpath). Call

$$(6.6) p_n = g_{n-1} \cdot [O_n, T_n] \cdot c_n,$$

a continuous path going from O_{n-1} to O_n . Then p_{n+1} is compatible with p_n , in the sense that the origin of p_{n+1} is the terminus of p_n .

Remark 6.3. A finite sequence of continuous paths $p_j : [0, t_j] \longrightarrow S$ (indexed by j = 1, ..., N) is said to be compatible if $p_j(t_j) = p_{j+1}(0)$. We define the homotopy relation between two sequences of N compatible paths $(p_1, ..., p_N), (q_1, ..., q_N)$ as the usual homotopy relation between

 p_j and q_j , with the additional condition that the deformation must preserve the compatibility condition.

Coming back to (6.6), there exists integers $k_n \geq 0$, $n \in \mathbb{Z}/N\mathbb{Z}$ such that the loop γ is freely homotopic to

$$\bullet_{n=0}^{N-1}(p_n\bullet g_n^{k_n}).$$

If both Σ_n and Σ_{n+1} are once-holed tori then $k_n = 0$. If Σ_n or Σ_{n+1} is a pair-of-pants then any loop homotopic to γ contains at least $k_n - 1$ disjoint copies of a closed curve homotopic to g_n .

6.3. A surjective map. Below we describe a surjective map, from a set \mathcal{P} onto the set of free homotopy classes of non-simple loops in S. We later identify a subset $\mathcal{P}^{\kappa,\mathcal{L},L} \subset \mathcal{P}$ that is mapped onto a set of representatives of $\text{Loc}_S^{\kappa,\mathcal{L},L}$. Our upper bound on $\#\text{Loc}_S^{\kappa,\mathcal{L},L}$ will be obtained by evaluating the cardinality of $\mathcal{P}^{\kappa,\mathcal{L},L}$, modulo homeomorphisms and homotopy.

For any integer $N \geq 1$, denote by \mathcal{P}_n the set of sequences $(P_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$, where $P_n = (g_{n-1}, g_n, h_n)$ is a collection of three simple curves in S (with oriented g_{n-1}, g_n), either disjoint and bordering a pair-of-pants Σ_n , or such that g_{n-1}, g_n intersect once and fill a once-holed torus with boundary h_n . If Σ_n and Σ_{n+1} are pairs-of-pants, we further impose that they satisfy Condition (V).

Define

$$\mathcal{P} = \bigsqcup_{N=1}^{+\infty} \mathcal{P}_N \times \mathbb{Z}^{\mathbb{Z}/N\mathbb{Z}}.$$

Suppose we have an integer $N \geq 1$, and an element $(P_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$ of \mathcal{P}_N . Write $(\partial P_n)_{n \in \mathbb{Z}/N\mathbb{Z}} := (g_{n-1}, g_n, h_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$. Let us consider a sequence of segments $I_n = [O_n, T_n]$ satisfying the prescriptions of the previous section. The collection of compatible paths $(p_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$ is completely determined by the data of the sequence $(\partial P_n)_{n \in \mathbb{Z}/N\mathbb{Z}} = (g_{n-1}, g_n, h_n)_{n \in \mathbb{Z}/N\mathbb{Z}}, I_n = [O_n, T_n]$ and the choice of an orientation of each g_n . The data of a family of positive integers $(k_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$ then defines a loop γ via (6.7). If we just give $(\partial P_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$ and $(k_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$, there is some freedom in the choice of $I_n = [O_n, T_n]$, but different choices will give loops γ in the same homotopy class.

What we have just described is a map

$$\Phi: \mathcal{P} \longrightarrow \pi_1^*(S)$$

onto the set of free homotopy classes of loops on S, which can then be composed by the projection $\pi: \pi_1^*(S) \longrightarrow \text{Top}(S)$, the set of local topological types of loops in S.

If γ is a non-simple loop, and if $P = (P_1, \dots, P_N, k_1, \dots, k_N) \in \mathcal{P}_N^{S'} \times \mathbb{N}^{\mathbb{Z}/N\mathbb{Z}}$ is the sequence associated to γ via the construction of §6.2, then $\Phi(P)$ is homotopic to γ . This shows that the map Φ is surjective. This is even true if we impose the restriction $k_n = 0$ if Σ_n and Σ_{n+1} are once-holed tori.

It it clear that an isotopy of the collection $(P_n)_{n\in\mathbb{Z}/N\mathbb{Z}}$ will result in an isotopy of the collection $(p_n)_{n\in\mathbb{Z}/N\mathbb{Z}}$, and thus that the map Φ passes to the quotient of \mathcal{P} by the *isotopy* equivalence relation. However, it is much easier to count *homotopy* classes of families $(P_n)_{n\in\mathbb{Z}/N\mathbb{Z}}$. By Graaf and Schrijver [9], the isotopy equivalence relation is the same as the homotopy equivalence relation, modulo finite sequences of third Reidemeister moves. As a consequence, it remains true that the homotopy class of the collection $(P_n)_{n\in\mathbb{Z}/N\mathbb{Z}}$ determines the homotopy class of the collection $(p_n)_{n\in\mathbb{Z}/N\mathbb{Z}}$.

Hence, let us define $\tilde{\mathcal{P}}$ to be the quotient of \mathcal{P} by the equivalence relation

(6.8)
$$(P_n, k_n)_{n \in \mathbb{Z}/N\mathbb{Z}} \sim (P'_n, k'_n)_{n \in \mathbb{Z}/N'\mathbb{Z}} \iff \left[N = N', \exists j \in \mathbb{Z}/N\mathbb{Z}, \forall n, k_n = k'_{n+j} \right]$$

and there exists a homeomorphism $\phi: S \longrightarrow S$, $\phi((P_n)_{n \in \mathbb{Z}/N\mathbb{Z}})$ homotopic to $(P'_{n+j})_{n \in \mathbb{Z}/N\mathbb{Z}}$.

We have shown that the map

$$\tilde{\Phi}: \tilde{\mathcal{P}} \longrightarrow \operatorname{Top}(S)$$

is well defined and surjective.

Notation 6.4. Later on, we may denote $\Phi^S, \mathcal{P}^S, \tilde{\Phi}^S, \tilde{\mathcal{P}}^S$ etc. if we wish to highlight the fact that the construction takes place in S.

6.4. Filling and pairs-of-pants decomposition. Before moving to the proof of Theorem 3.5, we clarify a fact that will help us count topological types.

Lemma 6.5. Let S be a filling type of signature (g_S, n_S) with $2g_S - 2 + n_S > 0$. Let c = (c_1,\ldots,c_N) be a finite or infinite collection of non-contractible loops in minimal position, which fills S. Then there exists a decomposition of $S = Q_1 \cup Q_2 \cup \ldots Q_{2q_S-2+n_S}$ into surfaces of Euler characteristics -1 with disjoint interiors, and with the following property: if Z is a bordered hyperbolic surface, if $\phi: S \longrightarrow Z$ is a homeomorphism such that $\sum_{i=1}^{N} \ell_Z(\phi(c_i)) \leq L$, then

- $\ell(\partial Z) < 2L$;
- $\ell_Z(\partial(\phi(Q_m))) \le 2L;$ $\ell_Z(\partial(\phi(S_m))) \le 2L \text{ if } S_m := Q_1 \cup \ldots \cup Q_m.$

Later on, we shall write $Q(c) = (Q_1(c), \dots, Q_{2g_S-2+n_S}(c))$ if we want to refer explicitly to the decomposition into pairs-of-pants associated to c via Lemma 6.5. The multi-curve cutting S into $Q_1 \cup Q_2 \cup \ldots Q_{2g_S-2+n_S}$ will be denoted by $\lambda(c) = (\lambda_1(c), \ldots, \lambda_{3g_S-3+n_S}(c))$.

Remark 6.6. (Variants of Lemma 6.5) If instead of assuming that $\sum_{i=1}^{N} \ell_Z(\phi(c_i)) \leq L$ we just assume that $\ell_Z(\phi(c_i)) \leq L$ for all i, the proof gives $\ell_Z(\partial(\phi(Q_m))) \leq 2(m+1)L$ and $\ell_Z(\partial(\phi(S_m))) \le 2(m+1)L.$

If the c_i , instead of being closed curves, are either closed curves with $\ell_Z(\phi(c_i)) \leq L$ or pairs of pants with $\ell_Z^{\max}(\partial \phi(c_i)) \leq \mathcal{L}$, then an adaptation of the lemma gives $\ell_Z(\partial (\phi(Q_m))) \leq \mathcal{L}$ $(m+1)\max(2L,3\mathcal{L})$ and $\ell_Z(\partial(\phi(S_m))) \leq (m+1)\max(2L,3\mathcal{L})$.

Proof. The first item is proven in [2, Lemma 4.4].

To simplify the discussion, we endow S with an auxiliary Riemannian metric Y and assume that the c_i are geodesics. We construct S_m and Q_m inductively, they are defined to have geodesic boundary for the auxiliary metric Y.

In our construction, S_m will be filled by a union $p_1 \cup \ldots \cup p_m \cup p_{m+1}$, where each p_j is a closed subsegment of c, joining two intersection points of c. The p_j only intersect transversally; the origin of p_{m+1} lies on $p_1 \cup \ldots \cup p_m$, and its endpoint lies either in $p_1 \cup \ldots \cup p_m$, or in p_{m+1} (the segment p_{m+1} deprived of its boundary points).

 Q_1 is constructed as follows: if one of the loops c_i is non-simple, we take p_1 to be a volute of c_i and $p_2 = \emptyset$. Then $Q_1 = S_1$ is the surface filled by p_1 . If all the loops c_i are simple, then there must exists i < j such that $c_i \cap c_j \neq \emptyset$. We take $p_1 = c_i$ and p_2 a sub-arc arc of c_j beginning and ending on c_i . Then $Q_1 = S_1$ is the surface filled by $p_1 \cup p_2$.

Suppose we have constructed $p_1, \ldots, p_m, Q_1, \ldots, Q_{m-1}$, with $S_{m-1} = Q_1 \cup \ldots \cup Q_{m-1}$ filled by $p_1 \cup \ldots \cup p_m$.

If $S_{m-1} = S$, there is nothing to be done. Otherwise, there is a (geodesic) subarc p of c leaving from ∂S_{m-1} and returning to it, contained in $S \setminus S_{m-1}$.

The surface filled by $p_1 \cup \ldots \cup p_m \cup p$ is strictly larger than S_{m-1} . If p is simple, it intersects either one or two boundary components of ∂S_{m-1} . Then $p \cap (S \setminus S_{m-1})$, together with those boundary components, fill a surface Q_{m+1} , of Euler characteristics -1. The full geodesic containing p must enter and exit S_{m-1} . Hence, it must intersect the already constructed $p_1 \cup \ldots \cup p_m$ (otherwise, $p_1 \cup \ldots \cup p_m$ would not fill S_{m-1}). Thus, there is a geodesic arc p_{m+1} containing p, ending and beginning at $p_1 \cup \ldots \cup p_m$, such that $p_{m+1} \setminus p$ is fully contained in S_{m-1} .

If p is not simple, then parametrize p by $p:[0,T] \longrightarrow S \setminus S_{m-1}$, such that $p(0) \in \partial S_{m-1}$. Take the smallest $t \leq T$ such that $p(t) \in p([0,t))$. We define Q_{m+1} to be the surface filled by p([0,T]) and the component of ∂S_{m-1} containing p(0). We extend p to p_{m+1} as previously.

Now, let Z be a bordered hyperbolic surface, and let $\phi: S \longrightarrow Z$ is a homeomorphism. If $\gamma = (\gamma_1, \ldots, \gamma_N)$ are the geodesic representatives of $(\phi(c_1), \ldots, \phi(c_N))$ for the metric Z, then $(\gamma_1, \ldots, \gamma_N)$ are obtained from $(\phi(c_1), \ldots, \phi(c_N))$ by an isotopy followed by a finite sequence of third Reidemeister moves (Graaf and Schrijver [9]). This operation maps each subarc $\phi(p_j)$ to a subarc q_j of γ , such that $q_1 \cup \ldots \cup q_m$ fills a surface isotopic to $\phi(S_{m-1}) = \phi(Q_1) \cup \ldots \cup \phi(Q_{m-1})$. By construction, for $i \neq j$, $q_i \cap q_j$ consists of isolated points, so $\ell(q_1 \cup \ldots \cup q_m) = \sum_{j=1}^m \ell(q_j) \leq \ell(\gamma)$.

The assumption $\ell_Z(c) \leq L$ means that $\ell(\gamma) \leq L$, so that $\ell(q_1 \cup \ldots \cup q_m) \leq \ell(\gamma) \leq L$. By [2, Lemma 4.4], we can say that $\ell_Z(\partial(\phi(S_{m-1}))) \leq 2L$. Even more precisely, $\ell_Z(\partial(\phi(S_{m-1}))) \leq 2\ell(q_1 \cup \ldots \cup q_m)$ and $\ell(\partial Q_{m+1}) \leq \ell_Z(\partial(\phi(S_m))) + 2\ell(q_{m+1}) \leq 2\ell(q_1 \cup \ldots \cup q_{m+1}) \leq 2L$. This ends the proof.

6.5. **Proof of Theorem 3.5.** We now describe a subset of $\tilde{\mathcal{P}}$ which projects onto $\operatorname{Loc}_{S}^{\kappa,\mathcal{L},L}$, and evaluate its cardinality. Because of the structure of the equivalence relation (6.8), we can write $\tilde{\mathcal{P}} = \bigsqcup_{N=1}^{+\infty} \tilde{\mathcal{P}}_{N}$, where $\tilde{\mathcal{P}}_{N}$ is the set of equivalence classes restricted to $\mathcal{P}_{N} \times \mathbb{Z}^{\mathbb{Z}/N\mathbb{Z}}$.

Our construction starts with the following three observations.

Lemma 6.7. If S is endowed with a (κ, \mathcal{L}) tangle-free hyperbolic metric Z, and if γ is a loop in S such that $\ell_Z(\gamma) \leq L$, then

$$i(\gamma, \gamma) \le \frac{4}{\kappa^2} L^2.$$

See for instance [6, Exposé 4, Lemme 2]; the constant C given in this proof is smaller than $\frac{4}{\kappa^2}$, where κ is a lower bound on the injectivity radius.

Lemma 6.8. Supposed S is endowed with a (κ, \mathcal{L}) tangle-free hyperbolic metric Z, and γ is a loop filling S such that $\ell_Z(\gamma) \leq L$. Then $\lambda = (\lambda_1(\gamma), \ldots, \lambda_{3g_S-3+n_S}(\gamma))$ be the multicurve cutting S into surfaces of Euler characteristics -1, defined by Lemma 6.5. Then

$$i(\gamma, \lambda_j) \le \frac{4}{\kappa^2} 2L^2,$$

for all $j = 1, ..., 3g_S - 3 + n_S$. More generally,

$$i(c, \lambda_j) \le \frac{4}{\kappa^2} 2L\ell_Z(c),$$

for any multicurve c.

This follows again from [6, Exposé 4, Lemme 2], and the lower bound κ on the injectivity radius.

Lemma 6.9. Suppose S is endowed with a (κ, \mathcal{L}) tangle-free hyperbolic metric Z, and γ is a loop on S such that $\ell_Z(\gamma) \leq L$. Let $P = (P_n, k_n)_{n \in \mathbb{Z}/N\mathbb{Z}} \in \mathcal{P}_N \times \mathbb{N}^{\mathbb{Z}/N\mathbb{Z}}$ be the sequence associated to γ via the construction of §6.2. Then

$$N \le 3\frac{L}{\mathcal{L}}$$

and each k_n satisfies

$$0 \le k_n \le 1 + \frac{L}{\kappa}.$$

Proof. Write $\partial P_n = (g_{n-1}, g_n, h_n)$. The tangle-free hypothesis implies that, for each $j = 1, \ldots, N$, $\max(\ell_Z(g_n), \ell_Z(g_{n-1}), \ell_Z(h_n)) \geq \mathcal{L}$. As a consequence of (6.4), the volutes $G(x_0), \ldots, G(x_{N-1})$ have lengths at least \mathcal{L} , and by (6.5),

$$\ell_Z(\gamma) \ge N \frac{\mathcal{L}}{3},$$

which is the announced result.

To obtain the upper bound on k_n , we recall that γ contains at least $k_n - 1$ copies of closed curves homotopic to g_n . By the minimizing property of closed geodesics, this implies

$$\ell_Z(\gamma) \ge (k_n - 1)\ell_Z(g_n)$$

By the tangle-free condition, $\ell_Z(g_n) \geq \kappa$, which gives the desired bound.

We now use Lemmas 6.7 and 6.8 to describe a finite set $\tilde{\mathcal{P}}_{N}^{\kappa,\mathcal{L},L} \subset \tilde{\mathcal{P}}_{N}$, such that the map

(6.9)
$$\tilde{\Phi}: \tilde{\mathcal{P}}^{\kappa,\mathcal{L},L} := \bigsqcup_{N=1}^{3\frac{L}{\mathcal{L}}} \tilde{\mathcal{P}}_{N}^{\kappa,\mathcal{L},L} \times \left[0, 1 + \frac{L}{\kappa}\right]^{\mathbb{Z}/N\mathbb{Z}} \longrightarrow \operatorname{Loc}_{S}^{\kappa,\mathcal{L},L}$$

is surjective. Theorem 3.5 will follow from an estimate of $\#\tilde{\mathcal{P}}_{N}^{\kappa,\mathcal{L},L}$.

Let $\bar{\Lambda}$ be the set of decompositions of S into surfaces of Euler characteristic -1. Let us fix a finite family $\Lambda_S \subset \bar{\Lambda}$, containing exactly one representative of each orbit under the group of homeomorphisms on Λ . The cardinality of Λ_S is a function of (g_S, n_S) , that we do not need to make explicit. For a given decomposition $\lambda = (\lambda_1, \dots, \lambda_{3g_S-3+n_S}) \in \Lambda$, we denote by $\tilde{\mathcal{P}}_N^{\kappa, \mathcal{L}, \mathcal{L}, \lambda}$ the subset of $\tilde{\mathcal{P}}_N$, formed of elements $(\bar{P}_1,\ldots,\bar{P}_N)$ satisfying : $i(\partial P_j,\partial P_k) \leq \frac{4}{\kappa^2}(2L)^2$ for all $j \neq k$ and $i(\partial P_j, \lambda_k) \leq \frac{4}{\kappa^2} (2L)^2$ for all j, k. Next, define $\tilde{\mathcal{P}}_N^{\kappa, \mathcal{L}, L} = \bigcup_{\lambda \in \Lambda_S} \tilde{\mathcal{P}}_N^{\kappa, \mathcal{L}, L, \lambda}$. If S is endowed with a tangle-free metric Z such that $\ell(\gamma) \leq L$, by (6.4), $\ell_Z(\partial P_j) \leq 2L$.

Lemmas 6.7, 6.8 and 6.9 imply that (6.9) is surjective.

Proposition 6.1 with $I = J = \frac{4}{\kappa^2}(2L)^2 = \frac{16}{\kappa^2}L^2$ implies that

$$\#\tilde{\mathcal{P}}_{N}^{\kappa,\mathcal{L},L} \le 2^{3g_S-3+n_S} \#\Lambda_S \left(3(\frac{16}{\kappa^2}L^2+1)\right)^{2(3g_S-3+n_S)})^N$$

Finally, summing over all elements of $\left[0, 1 + \frac{L}{\kappa}\right]^{\mathbb{Z}/N\mathbb{Z}}$ and over all possible $N \leq 3\frac{L}{\mathcal{L}}$,

$$\# \text{Loc}_{S}^{\kappa,\mathcal{L},L} \leq \# \tilde{\mathcal{P}}^{\kappa,\mathcal{L},L} \\
\leq 3 \frac{L}{\mathcal{L}} 2^{3g_{S}-3+n_{S}} \# \Lambda_{S} \left(3 \left(\frac{16}{\kappa^{2}} L^{2} + 1 \right) \right)^{2(3g_{S}-3+n_{S})} \right)^{3 \frac{L}{\mathcal{L}}} \times \left(1 + \frac{L}{\kappa} \right)^{3 \frac{L}{\mathcal{L}}},$$

which proves Theorem 3.5.

APPENDIX A. MORE COUNTING!

Here we include an additional counting result whose proof uses similar arguments as for Theorem 3.5, and which is needed in our paper [3]. The objects that we need to count are now pairs formed by a loop γ and a tangle τ .

A.1. Local topological types of lc-surfaces. The topological object to consider is now a pair formed by (γ, τ) , where γ is a loop and τ is a c-surface. Such a pair will be called an lc-surface. We shall define topological types of such objects, following the ideas of [2, §4.2].

Notation A.1. To any $k \geq 0$ and any $(\mathbf{g}, \mathbf{n}) \in \mathcal{N}^{(k)}$ we shall associate a fixed smooth oriented c-surface S of signature (\mathbf{g}, \mathbf{n}) . The data of (\mathbf{g}, \mathbf{n}) , or equivalently of the c-surface S, is called a c-filling type.

Definition A.2. A local lc-surface is a triplet (S, γ, τ) , where S is a c-filling type, γ is a loop in S, τ is a sub-c-surface of S, and the pair (γ, τ) fills S. Two local lc-surfaces (S, γ, τ) and (S', γ', τ') are said to be *locally equivalent* if S = S' (i.e. $g_S = g_{S'}$ and $n_S = n_{S'}$), and there exists a positive homeomorphism $\psi: S \to S$, possibly permuting the boundary components of S and the connected components of S, such that $\psi \circ \gamma$ is freely homotopic to γ' , and $\psi(\tau) = \tau'$. The latter equality means that for $\tau = (\tau_1, \dots, \tau_k), \tau' = (\tau'_1, \dots, \tau'_{k'})$, we have $k = k', \psi(\tau_j) = \tau'_j$, ψ respects the orientation of τ_k if τ_k is 1d, or the numbering of the boundary components of τ_k

if τ_k is 2d. This defines an equivalence relation \sim on local lc-surfaces. Equivalence classes for this relation are denoted as $[S, \gamma, \tau]_{loc}$ and called *local (topological) types* of lc-surfaces.

Definition A.3. Denote by $\operatorname{Tan}_{S,\chi,M}^{\kappa,\mathcal{L},L}$ the set of local topological types $[S,\gamma,\tau]_{\operatorname{loc}}$ of lc-surfaces of filling type S, such that $[S,\gamma]_{\operatorname{loc}}=T\in\operatorname{Loc}_{\chi}^{\kappa,\mathcal{L},L},\ \chi(\tau)< M,$ and : there exists a hyperbolic surface Z, a homeomorphism $\phi:S\longrightarrow Z$, with $\ell_Z(\phi(\gamma))\leq L$, and such that the geodesic representative of $\phi(\tau)$ is a (κ,\mathcal{L}) -derived tangle in Z.

We shall prove:

Theorem A.4. Assume that |S| = 0, that is to say, S is purely 2-dimensional. Then

$$\#\mathrm{Tan}_{S,\chi,M}^{\kappa,\mathcal{L},L} \leq \mathcal{O}_{\chi,S} \left(\frac{L}{\mathcal{L}} \left(3(\frac{16}{\kappa^2}L^2 + 1) \right)^{9\chi \frac{L}{\mathcal{L}}} \times \left(1 + \frac{L}{\kappa} \right)^{3\frac{L}{\mathcal{L}}} \left(2Le^{3M\mathcal{L}} \right)^{3\chi(S)} \right).$$

As a corollary,

Corollary A.5. If $L = A \log g$, $\mathcal{L} = \kappa \log g$, and if $\chi(S) < \chi'$,

$$\#\operatorname{Tan}_{S,\chi,M}^{\kappa,\mathcal{L},L} \leq \mathcal{O}_{\chi',\kappa,A}\left(g^{10M\chi'\kappa}\right).$$

Note that we are no longer invoking the absence of tangles in Definition A.3 and Theorem A.4. As a consequence, we cannot use lower bounds on the injectivity radius to find upper bounds on the intersection numbers. The upper bound is still polynomial in L, but has an exponential factor in \mathcal{L} .

Instead of the tangle-free assumption, we use the following fact in hyperbolic trigonometry to restrict the intersection numbers :

Lemma A.6. There exists a universal constant C such that, for any $\mathcal{L} > 0$, for any hyperbolic surface Z of Euler characteristic -1 having $\ell^{\max}(\partial Z) \leq \mathcal{L}$, any segment c going from the boundary ∂Z to itself, and not homotopic to a portion of ∂Z , has length

$$\ell(c) \ge Ce^{-\mathcal{L}}.$$

Corollary A.7. Let Y be a (κ, \mathcal{L}) -derived tangle of signature (g, n), with 0 < 2g - 2 + n < M. Consider a maximal multi-curve $\lambda = (\lambda_1, \ldots, \gamma_{3g-3+2n})$, cutting Y into surfaces of Euler characteristic -1, such that $\ell(\lambda_t) \leq 3\mathcal{L}M$ for all $t = 1, \ldots, 3g - 3 + 2n$ (the existence of which was proven in Proposition 2.8).

If γ is a geodesic in Y, of length $\ell(\gamma) \leq L$, then $(i(\gamma,\lambda)-1)Ce^{-3\mathcal{L}M} \leq L$. In other words,

$$i(\gamma, \lambda) \le 1 + \frac{1}{C}e^{3\mathcal{L}M}L.$$

Proof. If $i(\gamma, \lambda) = n$, we can find n-1 disjoint segments on γ , joining components of λ , of length $\geq Ce^{-3\mathcal{L}M}$ by Lemma A.6.

The proof of Theorem A.4 is based on the following facts: with the notation of Definition A.3, let in addition S' be the surface filled by γ . Denote as previously $\lambda = (\lambda_1, \ldots, \lambda_{3g_{S'}-3+n_{S'}}) = (\lambda_1(\gamma), \ldots, \lambda_{3g_{S'}-3+n_{S'}}(\gamma))$ be the multicurve associated to γ via Lemma 6.5. Denote also $\lambda_{3g_{S'}-3+n_{S'}+1}, \ldots, \lambda_{3g_{S'}-3+2n_{S'}}$ the boundary components of S'. Let $P = (P_1, \ldots, P_N, k_1, \ldots, k_N) \in \mathcal{P}_N^{S'} \times \mathbb{N}^{\mathbb{Z}/N\mathbb{Z}}$ be the sequence associated to γ via the construction of §6.2. Then, in addition to the restrictions on P that we found before, Corollary A.6 tells us that we must have

$$i(\lambda_j, \partial \tau) \le \frac{1}{C} 2Le^{3M\mathcal{L}}$$

for all $j = 1, ..., 3g_{S'} - 3 + 2n_{S'}$ and

$$i(P_j, \partial \tau) \le \frac{1}{C} 2Le^{3M\mathcal{L}}$$

for all j = 1, ..., N. The arguments are then similar to those for Theorem 3.5 and we skip details.

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