CONNECTIONS BETWEEN K-STABILITY AND VOJTA'S CONJECTURE

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ABSTRACT. In this note, we use recent advances concerning the K-stability of Q-Fano varieties to provide settings for which Vojta's conjecture holds.

1. Introduction

In [Voj87], Vojta described a dictionary between value distribution theory and Diophantine geometry, which culminated in several conjectures that unify many aspsects of arithmetic geometry. Vojta's main conjecture [Voj87, Conjecture 3.4.3], which we recall in Conjecture 3.1, is perhaps the deepest conjecture in arithmetic geometry. Loosely speaking, this conjecture is a Diophantine approximation statement, which concerns the relationship between the heights of points on varieties with respect to three divisors on a projective variety X over number field F, namely a simple normal crossings divisor D, the canonical divisor K_X , and a big divisor A on the variety. We recall the relevant definitions in the statement of Vojta's conjecture in Section 2. This conjecture and its generalization [Voj87, Conjecture 5.2.6] are known to have a range of important consequences. For example, it generalizes Faltings' theorem concerning the Mordell conjecture, and it is also known to imply the Bombieri–Lang conjecture and the abc-conjecture [Voj87, Section 5.5].

1.1. **Main results.** The goal of this note is to use recent breakthroughs in the subject of K-stability of Q-Fano varieties to prove new instances of Vojta's conjecture for Q-Fano varieties. Using the theory of local Weil functions associated to b-divisors (Section 2.10), we can formulate a generalization of Vojta's conjecture (Conjecture 3.2) which incorporates b-divisors on *X*, and our first result gives examples of varieties and divisors which satisfy this conjecture. We refer the reader to Subsections 2.2 and 2.7 for our conventions concerning varieties and b-divisors.

Theorem A. Let F be a number field and let X be a Q-Fano F-variety such that $X_{\overline{F}}$ has canonical singularities and infinite automorphism group. Then there exists a finite extension F'/F and a b-divisor $\mathbb E$ on $X_{F'}$ such that inequalities predicted by the birational Vojta's conjecture are true in the following sense. Let $\mathbb D = \sum_{i=1}^q \mathbb D_i$ be a b-divisor on $X_{F'}$ such that the traces of each $\mathbb D_i$ are linearly equivalent to the trace of $\mathbb E$ on some fixed normal proper model Y of $X_{F'}$, and suppose the traces of $\mathbb D_1, \ldots, \mathbb D_q$ intersect properly on Y. Then, the birational Vojta's conjecture is true for $(X, \mathbb D)$ (Definition 3.4).

Our next results concern the classical version of Vojta's conjecture in the setting of a toric Fano variety X. For such a variety, work of Blum–Jonsson [BJ20] gives a combinatorial description of the δ -invariant in terms of the dot product of the barycenter of the polytope associated to $-K_X$ and

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the primitive generators of the one dimensional cones of Δ . Using their work, we show that for a toric Fano variety X there *always* exists a torus invariant divisor D for which Vojta's conjecture holds for (X, D). More precisely, we have the following.

Theorem B. Let F be a number field and let X be a toric Q-Fano projective F-variety. Then there exists a torus invariant divisor E on X that corresponds to a primitive generator of a one-dimensional cone of the fan associated to X such that inequalities predicted Vojta's conjecture are true in the following sense. Let $D = \sum_{i=1}^{q} D_i$ be a Cartier divisor on X such that each D_i is linearly equivalent to E on X, and suppose that D_1, \ldots, D_q are in general position on X. Then, Vojta's conjecture is true for (X, D) (Definition 3.4).

In Theorem A and Theorem B, we are only able to prove results for a single divisor (or a single b-divisor). The restriction to a single divisor is due to the fact that we only have an upper bound of 1 for the δ -invariant of the Fano varieties in these results. Our next result shows that the condition of K-semistability (Definition 4.9) will allow us to provide collections of torus invariant divisors on toric Fano varieties for which Vojta's conjecture is true.

Theorem C. Let F be a number field, let X be a toric Q-Fano projective F-variety, let $\{D_1, \ldots, D_q\}$ be a collection of torus invariant divisors on X that correspond to primitive generators of one-dimensional cones of the fan associated to X and lie in general position, and let $D = \sum_{i=1}^q D_i$. If $X_{\overline{F}}$ is K-semistable, then Vojta's conjecture is true for (X,D) (Definition 3.4).

- 1.2. **Ideas and relation to other works.** We now explain how results in K-stability help, and along the way we recall earlier works in a similar direction. In the Fano setting, Vojta's conjecture is more accessible due in part to recent work of Ru–Vojta [RV20], where they showed it suffices to study the asymptotic volume constant (Example 4.3), or equivalently in the K-stability language, the volume of a filtration, as studied in [BHJ17, BJ20, BJ21] in much greater generality. More precisely, the first instance of using results of K-stability to understand the asymptotic volume constants appeared in the work of Grieve [Gri20], where the valuative criterion for K-(un)stability (i.e., the δ -invariant being less than 1) provides a prime divisor E over X for which the asymptotic volume constant of $-K_X$ along E is $\geqslant 1$. This is exactly what Ru–Vojta needed. Our main observation in this note is that the condition $\delta \leqslant 1$ is in fact well-understood in K-stability theory, see e.g. [BX19, LXZ22] (see also [HR22] for a different upper bound, in a setting where X is not necessarily Fano), and it holds for any Fano variety with infinite automorphism group.
- 1.3. **Organization.** In Section 2, we establish notation and recall background on algebraic geometry and on the theory of heights. We state Vojta's conjecture in Section 3, and we also describe a slight generalization of his conjecture which incorporates local b-Weil functions. In Section 4, we review the notion of a volume of filtration on the section ring of a line bundle and relate this to the asymptotic volume constant of a big line bundle along an effective Cartier divisor. We also describe results of Ru–Vojta and Grieve illustrating the importance of this volume in the arithmetic setting. Additionally, we discuss background on K-stability and recall the works that allow us to compute the asymptotic volume constant of the anti-canonical divisor of a Q-Fano variety along a b-Cartier divisor. Finally, in Section 5, we prove our results.

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2. Preliminaries

In this section, we establish conventions we use throughout the work and recall some definitions and concepts from algebraic geometry and height theory.

- 2.1. **Fields.** We will use F to denote a number field, F'/F to denote a finite extension of F, and let M_F denote the set of places of F. We also let K denote an arbitrary field of characteristic zero. As usual, the notation \overline{F} (resp. \overline{K}) will refer to an algebraic closure of F (resp. K).
- 2.2. **Varieties and divisors.** In this work, a K-variety is a geometrically integral separated scheme of finite type over Spec(K). For any extension K'/K, we will denote the base change of a K-variety X to Spec(K') by $X_{K'}$.

Definition 2.3. A Q-Fano K-variety X is a geometrically normal, geometrically irreducible projective K-variety such that $X_{\overline{K}}$ has at most log-terminal singularities and such that the anti-canonical divisor $-K_X$ is an ample Q-Cartier divisor.

We will primarily consider integral divisors on normal projective K-varieties. We now record the definition of a collection of divisors being in general position.

Definition 2.4. Let D_1, \ldots, D_q be effective Cartier divisors on a K-variety X of dimension n.

- (1) We say that D_1, \ldots, D_q intersect properly if for any subset $I \subset \{1, \ldots, q\}$ and any $x \in \bigcap_{i \in I} supp(D_i)$, the sequence $(\varphi_i)_{i \in I}$ is a regular sequence in the local ring $\mathscr{O}_{X,x}$ where φ_i are the local defining equations of D_i for $1 \leqslant i \leqslant q$.
- (2) We say that D_1, \ldots, D_q lie in *general position* if for any $I \subseteq \{1, \ldots, q\}$, we have that

$$dim\left(\bigcap_{i\in I} supp(D_i)\right) = n - \#I$$

if $\#I\leqslant \mathfrak{n}$ and $\bigcap_{i\in I} supp(D_i)=\emptyset$ if $\#I>\mathfrak{n}.$

By [RV20, Remark 2.2], these conditions are equivalent when X is Cohen–Macaulay.

Later, we will need the notion of a local Weil function associated to b-divisor. To simplify this description, we will recall the notion of the presentation of a Cartier divisor from [BG06, Section 2.2.1].

Definition 2.5. Let X be a normal projective K-variety, let D be a Cartier divisor on X with associated line bundle $\mathscr{O}_X(D)$ and rational section s_D . It is well-known that there exist base point free line bundles \mathscr{L} and \mathscr{M} on X such that $\mathscr{O}_X(D) \cong \mathscr{L} \otimes \mathscr{M}^{-1}$. For generating sections s_0, \ldots, s_n of \mathscr{L} and t_0, \ldots, t_m of \mathscr{M} , we call the data

$$\mathcal{D} = (s_D; \mathcal{L}, s; \mathcal{M}, t)$$

a presentation of D where $\underline{s} = \{s_0, \dots, s_n\}$ and $\underline{t} = \{t_0, \dots, t_m\}$

- 2.6. **Toric geometry.** We will make use of the standard notation for toric varieties from [Ful93]. In this work, a *toric variety* is a normal K-variety X that is equipped with a faithful action of a split algebraic torus $G^n_{m,K}$ over K which has a dense orbit in X. Note that by [CLS11, Theorem 9.2.9] and faithfully flat descent, a toric variety is Cohen–Macaulay. Also, by the theory of toric varieties (see e.g., [CLS11, Ful93]), one can associate to a projective toric variety X a complete fan Δ , and we will primarily denote a toric variety by X_{Δ} where Δ is the associated fan.
- 2.7. b-divisors. We recall the notions of models of varieties and b-Cartier divisors. The notion of b-divisor is originally due to Shokurov; see [Cor07, Definition 1.7.4 and Section 2.3] for details. The letter "b" refers to birational.

Let X be a proper K-variety. Recall that a *model* of X is a proper birational morphism $Y \to X$ defined over K where Y is a proper K-variety, and we consider the category of models of X. A b-*Cartier divisor* (resp. Q-b-*Cartier divisor*) on X is an equivalence class $\mathbb D$ of pairs (Y, D) where Y is a model of X and D is a Cartier (resp. Q-Cartier) divisor on Y, and the equivalence relation is given by saying that $(Y_1, D_1) \sim (Y_2, D_2)$ if Y_1 dominates Y_2 via $\phi: Y_1 \to Y_2$ and $D_1 = \phi^*D_2$. We represent a Q-b-Cartier divisor $\mathbb D$ as an equivalence class of pairs (Y, D) as above. In this case, we will say that D is the *trace* of $\mathbb D$ on Y. We say that a Q-b-Cartier divisor $\mathbb D$ on X is *effective* if it is represented by a pair (Y, D) where D is an effective Q-divisor on Y.

2.8. **Local Weil functions and height functions.** Next, we will recall the notion of local Weil functions. Let \mathcal{O}_F denote the ring of integers of F, recall that M_F is the set of places of F, and let F_{ν} denote the completion of F with respect to the place $\nu \in M_F$.

Let X be a projective F-variety. Using [Voj87], one can associate to every Cartier divisor D on X and every place $v \in M_F$, a local Weil function $\lambda_{D,v} \colon X(F) \setminus \operatorname{supp}(D) \to \mathbb{R}$ where $\operatorname{supp}(D)$ is the support of D. For an effective D, $\lambda_{D,v}$ measures the v-adic distance of a point to D. Using local Weil functions, we can define the height function of a divisor.

Definition 2.9. For a divisor D, we define the height function associated to D to be

$$h_D(x) = \sum_{\nu \in M_F} \lambda_{D,\nu}(x)$$

for all $x \in X(F) \setminus \text{supp}(D)$. Up to a bounded constant function, h_D is independent of the choices of local Weil functions.

2.10. **Birational local Weil functions.** To conclude our preliminaries section, we describe how one can define the local Weil function of a b-Cartier divisor. We will follow the treatment from [RV20]. The reader may also consult [Gri20, Birational Weil functions] for an equivalent characterization of these functions.

Definition 2.11. Let X be a projective F-variety, and let $v \in M_F$.

(1) A local b-Weil function on X (resp. a local Q-b-Weil function on X) is an equivalence class of pairs (U, λ) where U is a non-empty Zariski open subset of X and $\lambda \colon U(F_{\nu}) \to \mathbb{R}$ is a function such that there exists a model $\phi \colon Y \to X$ and a Cartier divisor (resp. Q-Cartier divisor) D on Y such that $\lambda \circ \phi$ extends to a Weil function for D (resp. such that $n\lambda \circ \phi$

- extends to a Weil function for nD for some non-zero integer n for which nD is a Cartier divisor). Pairs (U,λ) and (U',λ') are *equivalent* if $\lambda = \lambda'$ on $(U \cap U')(F_{\nu})$.
- (2) Let λ be a local b-Weil function on X, and let $\mathbb D$ be a b-Cartier divisor on X. We say that λ is a *local* b-Weil function for $\mathbb D$ if $\mathbb D$ is represented by a pair (Y,D) such that if $\phi\colon Y\to X$ is the structural morphism of Y, then $\lambda\circ \phi$ extends to a Weil function for $\mathbb D$ on Y.

To conclude, we recall some properties of these local b-Weil functions.

Proposition 2.12 ([RV20, Proposition 4.6]). *Let X be a projective F-variety.*

- (1) Let λ be a local b-Weil function on X. Then there is a unique b-Cartier divisor $\mathbb D$ such that λ is a local b-Weil function for $\mathbb D$.
- (2) Let \mathbb{D} be a b-Cartier divisor on X. Then there is a local b-Weil function λ for \mathbb{D} .

Using Proposition 2.12, we say that $\lambda_{\mathbb{D},\nu}(\cdot)$ is the local b-Weil function for a b-Cartier divisor \mathbb{D} . In different settings, we will consider $\lambda_{\mathbb{D},\nu}$ as a function defined on a dense Zariski open subset of X or on a dense Zariski open subset of Y where (Y, \mathbb{D}) is a presentation of \mathbb{D} . It will be clear from context what we are considering as the domain of $\lambda_{\mathbb{D},\nu}$.

To conclude this section, we describe an important example of a local b-Weil function associated a b-divisor $\mathbb D$ on X defined over a finite extension extension F'/F.

Example 2.13. Let X be a projective F-variety, and let \mathbb{D} be a b-divisor on X defined over a finite extension F'/F. Suppose that \mathbb{D} is represented by the pair (Y, D) where $Y \to X_{F'}$ is a model of $X_{F'}$ and D is a Cartier divisor on Y, so D is the trace of \mathbb{D} on Y. Let $\mathcal{D} = (s_D; \mathcal{L}, \underline{s}; \mathcal{M}, \underline{t})$ be a presentation for the divisor $\mathcal{O}_Y(D)$ for Y a model of X defined over F'. For any place $w \in M_{F'}$ such that $w \mid v$, the local b-Weil function associated to \mathbb{D} is defined for $x \in Y \setminus \text{supp}(D)$ as

$$\lambda_{\mathbb{D},\nu}(x) = \max_{k} \min_{\ell} \log \left| \frac{s_k}{t_\ell s_D}(x) \right|_{w} = \max_{k} \min_{\ell} \log \left| N_{F_w'/F_{\nu}} \left(\frac{s_k}{t_\ell s_D}(x) \right) \right|_{\nu}^{1/[F_w':F_{\nu}]}$$

where $N_{F'_w/F_v}$ is the norm map from $F'_w \to F_v$ and $|\cdot|_v$ is the standard absolute value associated to v. Note that when F' = F and $\mathbb D$ is a Cartier divisor on X, we recover the construction of a local Weil function associated to D.

3. Vojta's conjecture

In this section, we recall Vojta's conjecture [Voj87] and state a generalization of this conjecture which incorporates b-divisors. First, we start with Vojta's conjecture, which is one of the deepest conjectures in Diophantine geometry.

Conjecture 3.1 (Vojta's Conjecture [Voj87, Conjecture 3.4.3]). Let F be a number field, let X be a smooth proper F-variety with canonical divisor K_X , let D be a normal crossings divisor on X defined over F, let S be a finite set of places of k, and let A be a big divisor class on X. For any $\varepsilon > 0$, there exists a Zariski-closed subset $Z = Z(F, X, D, \varepsilon, S, A)$ of X such that

$$\sum_{\nu \in S} \lambda_{D,\nu}(x) + h_{K_X}(x) \leqslant \varepsilon h_A(x) + O(1)$$
(3.1.1)

for all points $x \in X(F) \setminus Z(F)$.

To conclude this section, we present a "birational" version of Vojta's conjecture. By "birational", we mean that the proximity function with respect to $\mathbb D$ and $\mathbb S$ from Conjecture 3.1 is replaced by the the proximity function with respect to $\mathbb D$ and $\mathbb S$ where $\mathbb D$ is some b-Cartier divisor over $\mathbb X$. Since local Weil functions are local b-Weil functions, this "birational" version of Vojta's conjecture is a slight generalization of Conjecture 3.1.

Conjecture 3.2 (Birational Vojta's Conjecture). Let F be a number field, let X be a smooth proper F-variety with canonical divisor K_X , let $\mathbb D$ be an effective Cartier b-divisor on X defined over a finite extension F'/F such that if (Y,D) represents $\mathbb D$, then D is a simple normal crossings divisor on Y, let S be a finite set of places of F, and let A be a big divisor class on X. For any $\varepsilon > 0$, there exists a Zariski-closed subset $Z = Z(F,X,\mathbb D,\varepsilon,S,A)$ of X such that

$$\sum_{v \in S} \lambda_{\mathbb{D}, v}(x) + h_{K_X}(x) \leqslant \varepsilon h_A(x) + O(1)$$
(3.2.1)

for all points $x \in X(F) \setminus Z(F)$ where $\lambda_{\mathbb{D},v}$ is the local b-Weil function defined in Example 2.13.

Remark 3.3. We note that our "birational" Vojta's conjecture does not concern the behavior of Conjecture 3.1 under birational modifications of X as the height functions h_{K_X} and h_A are defined for divisors K_X and A on X and not on some model of X. For a discussion on this topic, we refer the reader to [Voj87, Example 3.5.4].

In order to simplify certain statements, we will introduce a definition which capture when the inequality in Conjecture 3.1 and Conjecture 3.2 hold.

Definition 3.4. Let F be a number field, let X be a normal projective F-variety, and let D (resp. \mathbb{D}) be an effective Cartier divisor (resp. an effective Cartier b-divisor) on X. If the inequality (3.1.1) (resp. (3.2.1)) holds for F, X, D (resp. F, X, D) and any choices of ε , S, A outside of a Zariski-closed subset Z(F, X, D, ε , S, A) (resp. Z(F, X, D, ε , S, A)), then we say that *Vojta's conjecture* (resp. *the birational Vojta's conjecture*) *holds for* (X, D) (resp. (X, D)).

4. Volume of a filtration

In this section, we begin by recalling the notion of the volume of a filtration, see for instance [BHJ17, BJ20, BJ21] for more details. Special attention will be paid to filtrations arising from an effective Cartier divisor or a prime divisor over X. Then we state what is needed and known about the volume of a filtration in these two special cases. Unless otherwise stated, X is a normal projective \overline{K} -variety, and \mathcal{L} is a big line bundle on X. Also, we will freely use results concerning the volume of a line bundle from [Laz04, Section 2.2.C], and when dealing with volumes of divisors, we will need to work with \overline{K} -divisors as discussed in [Laz04, Section 1.3].

4.1. General definitions and arithmetic importance. Denote by $R = \bigoplus_{\mathfrak{m}} R_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}} H^0(X, \mathfrak{m}\mathscr{L})$ the section ring of \mathscr{L} . By a *filtration* \mathscr{F} on R we mean a decreasing, multiplicative filtration on R satisfyting $\mathscr{F}^0R = R$. To such a filtration and $t \in \mathbb{R}_{\geqslant 0}$, set

$$\operatorname{vol}(\mathscr{F}^{(t)}R) \coloneqq \lim_{m \to \infty} \frac{\dim \mathscr{F}^{mt}R_m}{m^n/n!}.$$

The filtration \mathscr{F} is linearly bounded if $\sup\{t: vol(\mathscr{F}^{(t)}R)>0\}<\infty.$

Definition 4.2. To each linearly bounded filtration, one can define the *volume* of \mathscr{F} to be

$$\beta(\mathcal{L}, \mathcal{F}) := \frac{1}{\operatorname{vol}(\mathcal{L})} \int_0^\infty \operatorname{vol}(\mathcal{F}^{(t)} R) dt.$$

The following two examples show the two kinds of filtration that we will consider.

Example 4.3 (Asymptotic volume constants). Let D be an an effective Cartier divisor on X. As first appeared in the works [Aut09] and [MR15], the *asymptotic volume constant of* $\mathscr L$ *along* D, which is denoted by $\beta(\mathscr L,D)$ in [RV20], is the volume of the filtration

$$\mathscr{F}^{\mathfrak{m}} H^{0}(X, \mathscr{L}^{N}) := H^{0}(X, \mathscr{L}^{N} - \mathfrak{m}D).$$

It is not hard to see that $\beta(\mathscr{L},D) = \lim\inf_{N\to\infty} \frac{\sum_{m\geqslant 1} h^0(X,\mathscr{L}^N(-mD))}{Nh^0(X,\mathscr{L}^N)}$, which is the definition in [RV20]. See [BJ20, Corollary 2.12] for a proof.

Example 4.4 (S-invariant in K-stability). Let $E \subset Y \to X$ be a prime divisor over X where Y is a normal model of X. Then the associated divisorial valuation ord_E defines a filtration via

$$\mathscr{F}_F^{\lambda}R_m := \{s \in R_m : ord_E(s) \geqslant \lambda\}.$$

We write $\beta(\mathcal{L}, E) := \beta(\mathcal{L}, \mathcal{F}_E)$. In the case when X is Q-Fano, we put $\beta(-K_X, E) := \frac{1}{r}\beta(\mathcal{L}, E)$, where r is such that $\mathcal{L} = -rK_X$ is Cartier. It is also worth noting that β is pullback invariant under birational morphisms.

For our purposes, it is important to know when $\beta \ge 1$, as suggested by the following result.

Theorem 4.5 ([RV20, Corollary 1.13] & [Gri20, Theorem 4.1]). Let F be a number field, and let S be a finite set of places of F containing all the Archimedean places. Let X be a Q-Fano F-variety, and let D_1, \ldots, D_q be reduced irreducible Cartier divisors over X which are defined over a finite extension F'/F. Let $\mathbb{D}_1, \ldots, \mathbb{D}_q$ be the b-Cartier divisor determined by D_1, \ldots, D_q , respectively, let $\varphi \colon Y \to X_{F'}$ be a normal proper model such that D_1, \ldots, D_q is the trace of $\mathbb{D}_1, \ldots, \mathbb{D}_q$, and let $\mathbb{D} = \mathbb{D}_1 + \cdots + \mathbb{D}_q$ and $D = D_1 + \cdots + D_q$. If D_1, \ldots, D_q intersect properly (Definition 2.4) and $\beta(-K_{X_{F'}}, D) \geqslant 1$, then the birational version of Vojta's conjecture (Conjecture 3.2) is true for (X, \mathbb{D}) i.e., for any $\epsilon > 0$, there exists a Zariski-closed subset $Z = Z(F, X, \mathbb{D}, \epsilon, S, A)$ of X such that

$$\sum_{\nu \in S} \lambda_{\mathbb{D},\nu}(x) + h_{K_X}(x) \leqslant \varepsilon h_A(x) + O(1)$$

for all points $x \in X(F) \setminus Z(F)$ where $\lambda_{\mathbb{D},\nu}$ is the local b-Weil function defined in Example 2.13.

Remark 4.6. Note that when $D_1, ..., D_q$ are reduced irreducible Cartier divisor on X and F = F', then Theorem 4.5 is equivalent to [RV20, Corollary 1.13].

4.7. **K-stability input.** We now summarize recent results in K-stability which give $\beta \ge 1$. From now on, we assume that X is Q-Fano variety (Definition 2.3).

The volume of a prime divisor over X in Example 4.4 is related to K-stability of Fano varieties via the following valuative criterion.

Definition 4.8. Let X be a Q-Fano \overline{K} -variety. The δ -invariant of X is defined to be

$$\delta(X) = \delta(X, -K_X) := \inf_{E} \frac{A_X(E)}{\beta(-K_X, E)},$$

where the infimum runs through all prime divisors over X, and $A_X(E) = 1 + ord_E(K_{Y/X})$.

Definition 4.9 ([Fuj19, Li17, BJ20, LXZ22]). A Q-Fano \overline{K} -variety X is K-semistable (resp. K-stable) if $\delta(X) \ge 1$ (resp. $\delta(X) > 1$), and K-unstable otherwise.

Theorem 4.10. Let X be a \mathbb{Q} -Fano \overline{K} -variety with infinite automorphism group. Then $\delta(X) \leq 1$, and there is a prime divisor E over X computing $\delta(X)$ i.e.,

$$\delta(X) = \frac{A_X(E)}{\beta(-K_X, E)}.$$

Proof. The result should be well-known to experts in K-stability. We will recall the proof for the readers who are not familiar with the K-stability literature. If $\delta(X) > 1$, then by [BX19, Corollary 1.3] the automorphism group of X has to be finite, contradicting our assumption. The equality in the latter claim is a consequence of [LXZ22, Theorem 1.2] (see also [XZ22] for a different proof). \Box

Corollary 4.11. Let X be a \mathbb{Q} -Fano \overline{K} -variety with infinite automorphism group and canonical singularities. Then there is some divisorial valuation $v = \operatorname{ord}_{E} with \beta(-K_{X}, E) \geqslant 1$.

Proof. By Theorem 4.10, we may choose a prime divisor E over X with $A_X(E) \le \beta(-K_X, E)$. Since X has canonical singularities, we have that $A_X(E) \ge 1$, and hence $\beta(-K_X, E) \ge 1$.

4.12. **The toric case.** To conclude, we recall work of Blum–Jonsson [BJ20] which provides a combinatorial description of the δ -invariant in the setting where X is a toric Q-Fano F-variety. Recall our conventions for toric varieties (cf. Subsection 2.6). Fix a projective toric F-variety $X = X_{\Delta}$ associated to the fan Δ and suppose that K_X is Q-Cartier. Let v_1, \ldots, v_d be primitive generators of one-dimensional cones of Δ , and let D_1, \ldots, D_d be the corresponding toric invariant divisors. Then $-K_X = D_1 + \cdots + D_d$. The polytope associated to $-K_X$ is given by

$$P_{-K_X} = \{u \in M_\mathbb{R} : \langle u, \nu_i \rangle \geqslant -1, i = 1, \dots, d\}.$$

Under this setup, we recall the following results from [BJ20].

Proposition 4.13 ([BJ20, Corollary 7.7 & 7.16]). With notation as above,

$$\delta(X) = \min_{i=1,\dots,d} \frac{1}{\beta(-K_X, D_i)} = \min_{i=1,\dots,d} \frac{1}{\langle \bar{u}, \nu_i \rangle + 1},$$

where $\overline{\mathbf{u}}$ is the barycenter of P_{-K_X} .

Corollary 4.14 ([BJ20, Corollary 7.17, 7.19]). If X is a toric Q-Fano variety, then $\delta(-K_X) \leq 1$. The equality is achieved of and only if $X_{\overline{F}}$ is K-semistable or equivalently if the barycenter of the polytope associated to $-K_X$ is equal to the origin, in which case $\beta(-K_X, D_i) = 1$ for all i.

Proof. We apply Proposition 4.13 with $b_i = 1$ for all i. Then there is some i such that $\langle \overline{u}, \nu_i \rangle \geqslant 0$. Indeed, if $\overline{u} \neq 0$, then there must be some i for which $\langle \overline{u}, \nu_i \rangle > 0$, since otherwise all ν_i would lie in some half-space, contradicting the fan being complete; if $\overline{u} = 0$, then $\beta(-K_X, D_i) = 1$ for all i. \square

Remark 4.15. Note that Δ being complete implies that we cannot have $\beta(-K_X, D_i) > 1$ for all i.

5. Proof of Main Theorems and Examples

In this section, we prove our main theorems and provide examples of varieties satisfying the conditions of our main theorems.

<i>Proof of Theorem A.</i> This follows from Corollary 4.11 and Theorem 4.5.	
Proof of Theorem B. This follows from Corollary 4.14 and Theorem 4.5.	
<i>Proof of Theorem C.</i> This follows from Corollary 4.14 and Theorem 4.5.	

To conclude, we give examples of or provide references to literature describing varieties satisfying the assumptions of these results.

- 5.1. Q-Fano varieties with canonical singularities and infinite automorhpism group. These varieties satisfy the condition of Theorem A. In dimension 1, the situation is not very interesting as the only Q-Fano curve is \mathbb{P}^1 which has automorhpism group PGL_2 , which is infinite. In dimension 2, the work [CP21] gives a full classification of del Pezzo surfaces with Du Val singularities (or equivalently canonical singularities) that have an infinite automorphism group. In dimension 3, the works [KPS18, PCtS19] provide a classification of smooth Q-Fano threefolds which have infinite automorhpism group.
- 5.2. **Toric** Q-**Fano varieties**. These varieties satisfy the condition of Theorem B. The dimension 1 situation is the same as above. In dimension 2, the smooth toric Q-Fano varieties are the well-known smooth toric del Pezzo surfaces: \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, the rational normal scroll \mathbb{F}_1 , and \mathbb{P}^2 blown-up at two and three points. In dimension 3, Batryev [Bat81] and Wantanabe–Wantanabe [WW82] have completely classified the smooth toric Q-Fano 3-folds, and also in dimension 4, Batryev [Bat99] classified smooth toric Q-Fano 4-folds. A classification in higher dimensions is unknown, but we refer the reader to [Sat00] for a discussion on this problem as well as a method for characterizing toric Q-Fano varieties.
- 5.3. **K-semistable toric Q-Fano varieties.** These varieties satisfy the conditions of Theorem C. Again, the dimension 1 setting is the same as above. In [ACC⁺23], the authors studied the Calabi problem, which aims to classify Fano 3-folds which are K-polystable. In doing so, the authors gives a complete classification of smooth toric **Q-Fano** surfaces and 3-folds that are K-semistable.

Proposition 5.4 ([ACC⁺23, Section 2]). Let X be a smooth toric \mathbb{Q} -Fano projective F-surface. $X_{\overline{F}}$ is K-semistable if and only if $X_{\overline{F}}$ is isomorphic to \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, or the blow-up of \mathbb{P}^2 at three points.

Theorem 5.5 ([ACC⁺23, Table 3.1]). Let X be a smooth toric Q-Fano projective F-threefold. $X_{\overline{F}}$ is K-semistable if and only if $X_{\overline{F}}$ is isomorphic to \mathbb{P}^3 , $\mathbb{P}^1 \times \mathbb{P}^2$, the blow-up of \mathbb{P}^3 are two disjoint lines, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^1 \times \mathbb{S}$ where S is a smooth toric del Pezzo surface with $K_S^2 = 6$.

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