PSEUDOCOMPACT VERSUS COUNTABLY COMPACT IN FIRST COUNTABLE SPACES

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Dedicated to the memory of Peter Nyikos

ABSTRACT. The primary objective of this work is to construct spaces that are "pseudocompact but not countably compact", abbreviated as P-NC, while endowing them with additional properties.

First, motivated by an old problem of van Douwen concerning first countable P-NC spaces with countable extent, we construct from CH a locally compact and locally countable first countable P-NC space with countable spread.

A space is deemed densely countably compact, denoted as DCC for brevity, if it possesses a dense, countably compact subspace. Moreover, a space qualifies as densely relatively countably compact, abbreviated as DRC, if it contains a dense subset D such that every infinite subset of D has an accumulation point in X.

A countably compact space is DCC, a DCC space is DRC, and a DRC space is evidently pseudocompact. The Tychonoff plank is a DCC space but is not countably compact. A Ψ -space belongs to the class of DRC spaces but is \neg DCC. Lastly, if $p \in \omega^*$ is not a P-point, then T(p), representing the type of p in ω^* , constitutes a pseudocompact subspace of ω^* that is \neg DRC.

When considering a topological property denoted as Q, we define a space X as "R-hereditarily Q" if every regular closed subspace of X also possesses property Q. The Tychonoff plank and the Ψ -space are not R-hereditary examples for separating the above-mentioned properties. However, the aforementioned space T(p) is an R-hereditary example, albeit not being first countable.

In this paper we want to find (first countable) examples which separates these properties R-hereditarily. We have obtained the following result.

- (1) There is a R-hereditarily "DCC, but not countably compact" space.
- (2) If CH holds, then there is a R-hereditarily "DRC, but ¬DCC" space.
- (3) If $\mathfrak{s}=\mathfrak{c}$, then there is a first countable, R-hereditarily "pseudocompact , but $\neg DRC$ " space.

In contrast to (2), it is unknown whether a first countable, R-here ditarily "DRC, but ¬DCC" space X can exist.

1. Introduction

The concept of *pseudo-compactness* was introduced by Hewitt in [7]. In [11] Mardesic and Papic proposed the notion of *feebly compact* spaces, and they established that a completely regular space is pseudocompact if and only if it is feebly compact.

A countably compact (abbreviated CC) Tychonoff space is necessarily pseudocompact. However, the reverse implication does not hold: both a Ψ -space and a Tychonoff plank serve as simple examples of *pseudocompact*, but not countably compact (abbreviated P-NC) spaces.

What weaker conditions lead to a space being pseudocompact?

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A space is pseudocompact if it has a *dense*, countably compact subspace, (in short, if the space is DCC). For example, $\omega_1 \times \omega$ is a dense, countably compact subspace of the Tychonoff plank. Answering affirmatively a question of Mardesic and Papic, in [12] Marjanovic showed that a Ψ -space is pseudo-compact space which is \neg DCC.

Let us say that a subspace D of a space X is relatively countably compact iff every infinite subset of D has a limit point in X. If a topological space contains a dense, relatively countably compact subset (it is DRC, in short), then X is clearly pseudocompact. For example, a Ψ -space is a DRC space because the isolated points form a dense, relatively countably compact subset.

In [2] Dorantes-Aldama and Shakhmatov introduced the following concept. A topological space X is called *selectively pseudocompact* (abbreviated SP) iff given any family $\{U_n : n \in \omega\}$ of non-empty open sets, it is possible to choose points $x_n \in U_n$ such that the set $\{x_n : n \in \omega\}$ has an accumulation point. Clearly every DRC space is SP, and all the SP spaces are feebly compact.

In [1, Section 2] Berner constructed a dense subspace of $\Sigma(2^{\omega_1})$, referred to as "Berner's Σ ", which is "SP, but $\neg DRC$ ".

In [1, Section 5] Berner introduced another example: a 0-dimensional, locally countable, first countable, "SP but $\neg DRC$ " space of cardinality \mathfrak{c}^+ , which will refer to as "Berner's monster".

Ginsburg and Sacks, [6], using a result of Frolik, proved that if $p \in \omega^*$ is not a P-point, then T(p), the type of p in ω^* , is a pseudocompact subspace of ω^* . In [10] Kunen constructed a weak P-point p which is not P-point in ZFC, and so the pseudocompact space T(p) mentioned above is an *anti-countably compact* space, i.e. no countable subset in it has a limit point. In [15] Shakhmatov constructed arbitrarily large pseudocompact, anti-countably compact spaces in ZFC.

The last two results addressed the following problem: To what extent can a pseudocompact space deviate from being countably compact?

Let us observe that some of the examples mentioned so far possess interesting additional properties. For a given topological property Q, a space X is defined to be R-hereditarily Q if every regular closed subspace of X also has property Q. For instance, every pseudocompact (DCC, DRC, SP) space is R-hereditarily pseudocompact (DCC, DRC, SP, respectively). The Ψ -space is first countable but not R-hereditarily " $\neg DCC$ ". Similarly, Berner's monster is first countable, but not R-hereditarily " $\neg SP$ " as it is locally compact.

On the other hand, the space T(p) and the space constructed by Shakhmatov are R-hereditarily "pseudocompact, but $\neg SP$," but neither of these spaces is first countable. Berner's Σ is R-hereditarily "SP, but $\neg DRC$ ", though it, too, is not first countable.

These observations raise the following question: Can we find examples that are both first countable and "R-hereditary", while being as far from being countably compact as possible, in other words, that contain as many closed discrete countable sets as possible? Can you find large "R-hereditary" examples, in particular, examples of sizes greater than 2^{ω} ?

Let us observe that if X is an "example", e.g. , X is "DRC, but \neg DCC", then the disjoint union of X and a compact space is also an example. Consequently, it is impossible to establish a cardinality bound for the sizes of spaces that are "DRC but \neg DCC".

However, the situation is entirely different when considering "R-hereditary" examples, as the disjoint union of an R-hereditary example and a compact space is not an R-hereditary example.

A first countable DCC space is countably compact, so we can not expect first countable examples separating CC and DCC. The Tychonoff plank is a DCC, but

 \neg CC space of size ω_1 , but it is not an R-hereditary example. However, we can construct arbitrarily large "R-hereditary" examples in ZFC.

Theorem 1.1. For each cardinal κ , there is an R-hereditarily "DCC but $\neg CC$ " space X with $|X| = \kappa^{\omega}$.

Proof. Our space X will be a dense subspace of the compact space ${}^{\omega}(\kappa^+ + 1)$, namely let

$$X = \{ f \in {}^{\omega}(\kappa^{+} + 1) : |\{n : f(n) = \kappa^{+}\}| < \omega \}.$$

The subspace $Y = {}^{\omega}(\kappa^+)$ of X is dense and countably compact. If ε is an elementary open set in X, i.e. $\operatorname{dom}(\varepsilon) \in [\omega]^{<\omega}$ and $\operatorname{ran}(\varepsilon)$ consists of open subsets of κ^+ , then define $\{f_n : n \in \omega\}$ as follows. For each $i \in \operatorname{dom}(\varepsilon)$ pick $\alpha_i \in \varepsilon(i)$ and let

$$f_n(i) = \begin{cases} \alpha_i & \text{if } i \in \text{dom}(\varepsilon), \\ \kappa^+ & \text{if } i \in n \setminus \text{dom}(\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_n : n \in \omega\} \subset X \cap [\varepsilon]$ is closed discrete in X because it converges to the function

$$\{\langle i, \alpha_i \rangle : i \in \mathrm{dom}(\varepsilon)\} \cup \{\langle n, \kappa^+ \rangle : n \in \omega \setminus \mathrm{dom}(\varepsilon)\} \in {}^{\omega}(\kappa^+ + 1) \setminus X.$$

So X is R-hereditarily $\neg CC$.

A Ψ -space is an example of a first countable space that is "DRC but \neg DCC", but it is not an R-hereditary example. In Section 2 we prove Theorem 2.1 which directly implies the following result:

Theorem 1.2. (1) If CH holds, then there is an R-hereditarily "DRC, but $\neg DCC$ " space X of size ω_1 . (2) It is consistent that CH holds, 2^{ω_1} is as large as you wish, and there is an R-hereditarily "DRC, but $\neg DCC$ " space X of size 2^{ω_1} .

We do not have even a consistent example of a first countable, R-here ditarily "DRC, but \neg DCC" space.

Berner's Σ is R-hereditarily "SP but $\neg DRC$ ", but its character is ω_1 . On the other hand, Berner's monster is a first countable, SP but $\neg DRC$ space, but it is not an R-hereditarily example, as it is locally compact. In Section 3 we will prove the following result (see Theorems 3.1 and 3.3).

Theorem 1.3. If $\mathfrak{s} = \mathfrak{c}$, then there is a first countable, R-hereditarily "SP but $\neg DRC$ " space of size \mathfrak{c} .

The space T(p) and the example of Shakmatov are anti-countably compact, so they are R-hereditarily "pseudocompact, but $\neg SP$ ". A first countable pseudocompact space is selectively pseudocompact, so we can not expect first countable examples separating these properties.

Figure 1 provides a summary of our findings. The symbol f indicates the non-existence of corresponding spaces, while $\sqrt{}$ denotes the presence of examples with stronger properties in certain cells. Examples are presented with slanted line background when they represent consistent constructions. Question mark indicates the absence of an example.

The actual starting point of our investigation was a problem posed by van Douwen. As we remarked, both a Ψ -space and a Tychonoff plank serve as simple examples of *pseudocompact*, but not countably compact. Notably, a Ψ -space is first countable but has uncountable extent, while the Tychonoff plank has countable extent, but fails to be first countable. So it is a natural question is whether there are P-NC spaces with small extent and countable character?

	$\neg CC \land DCC$	¬DCC ∧ DRC	$\neg \mathbf{DRC} \wedge \mathbf{SP}$	$\neg \mathbf{SP} \ \land \ \mathbf{P}$
_	Tychonoff plank	√	√	√
R-hereditary	Thm 1.1	Thm 1.2	Berner's Σ	T(p), Shakmatov
\mathbf{M}_1	£	Ψ-space	Berner's monster	£
R-hereditary, M ₁	£	??	Thm 1.3	£

FIGURE 1. Examples separating classes of pseudocompact spaces

Eric Van Douwen and Peter Nyikos constructed two distinct examples of such spaces, assuming $\mathfrak{b}=\omega_1$ (as discussed in [4, Notes to Section 13], where Nyikos provided an example) and assuming $\mathfrak{b}=\mathfrak{c}$ (see [4, Ex. 13.3]), respectively. In [4, Question 12.5 and 12.6], van Douwen posed two related questions: the first concerns the minimum cardinality of a first countable P-NC space, known to lie between \mathfrak{b} and \mathfrak{a} .

The second question asks whether it is possible to create a first countable P-NC space with countable extent in ZFC.

While we could not resolve the first question, we made some progress on the second. In Section 4 we prove the following statement (which follows immediately from Theorem 4.2):

Theorem 1.4. If CH holds, then there is a first countable, pseudocompact, but not countably compact space with $s(X) = \omega$.

It is important to emphasize that ZFC can not guarantee the existence of such a space, as its existence would imply the existence of an S-space (see Proposition 4.1).

Notions and notations. Given a space X and a set $A \subset X$ write

$$acc(A, X) = \{ p \in X : p \text{ is an accumulation point of } A \text{ in } X \},$$

and let

$$CD(X) = \{A \in [X]^{\omega} : acc(A, X) = \emptyset\} = \{A \in [X]^{\omega} : A \text{ is closed discrete}\}.$$

Definition 1.5. Let X be a topological space and $Y \subset X$. We say that Y is relatively countably compact in X, and we write $Y \subset {}^{RC} X$ iff every infinite subset of Y has an accumulation point in X. (In [1], Berner referred to this as "conditionally compact".)

We write $Y \subset^{DRC} X$ if Y is both dense and relatively countably compact in X. We say that $Y \subset X$ is anti-countably compact (AC, in short) in X iff $[Y]^{\omega} \subset CD(X)$.

2. DRC but
$$\neg DCC$$
 spaces.

In this section we will construct consistent examples of R-hereditarily "DRC but ¬DCC" spaces.

Theorem 2.1. (1) If CH holds, then there is a crowded 0-dimensional T_2 space X such that

- (a) X has a partition $S \cup Y$, where S is countable and dense, and $|\overline{A}| = |X|$ for each $A \in [S]^{\omega}$,
- (b) every $B \in [Y]^{\omega}$ is closed and discrete in X,
- (c) every countably compact subset of X is scattered.

(2) It is consistent that CH holds, 2^{ω_1} is as large as you wish, and there is a 0-dimensional T_2 space X with $|X| = 2^{\omega_1}$ such that (a)-(c) above hold for X.

Proof of Theorem 1.2 from Theorem 2.1. X is DRC as (a) implies that S is relatively countably compact in X. Moreover, since X is crowded, (c) implies that a dense subset of a non-empty regular closed subset H of X can not be countably compact.

Before proving Theorem 2.1 we need some preparation.

Definition 2.2. (1) A triple $\mathfrak{X} = \langle \mathcal{X}, \mathcal{B}, \mathcal{F} \rangle$ is a *nice triple* iff

- (a) $\mathcal{X} = \langle X, \tau \rangle$ is a crowded, 0-dimensional space,
- (b) $X = C \cup \mathbb{Q}$ for some set C of ordinals,
- (c) $\mathcal{B} = \{B_i : i \in I\}$ is a clopen base of \mathcal{X} , where I is a set of ordinals with |I| = |X|,
- (d) the set \mathbb{Q} is dense in \mathcal{X} ,
- (e) $\mathcal{F} \subset X \times [\mathbb{Q}]^{\omega}$ and $|\mathcal{F}| \leq |X|$,
- (f) if $\langle a, A \rangle \in \mathcal{F}$, then $a \in \text{acc}(A, \mathcal{X})$.

We say that \mathfrak{X} is *countable* iff X is countable.

Observe that we did not assume that the topology τ is T_2 .

If \mathfrak{X}_{ℓ} is a nice triple, we will use the notation \mathcal{X}_{ℓ} , X_{ℓ} , τ_{ℓ} , C_{ℓ} , \mathcal{B}_{ℓ} , I_{ℓ} , $B_{\ell}(i)$ for $i \in I_{\ell}$, and \mathcal{F}_{ℓ} .

- (2) If \mathfrak{X}_0 and \mathfrak{X}_0 are nice triples, then we say that \mathfrak{X}_1 is an extension of \mathfrak{X}_0 , and we write $\mathfrak{X}_1 \ll \mathfrak{X}_0$, iff
 - (i) $C_0 \subset C_1$ and $I_0 \subset I_1$,
- (ii) $B_0(i) = B_1(i) \cap X_0$ for each $i \in I_0$,
- (iii) if $B_0(i) \subset B_0(i')$ then $B_1(i) \subset B_1(i')$ for each $i, i' \in I_0$,
- (iv) if $B_0(i) \cap B_0(i') = \emptyset$ then $B_1(i) \cap B_1(i') = \emptyset$ for each $i, i' \in I_0$,
- (v) $\mathcal{F}_0 \subset \mathcal{F}_1$.

Lemma 2.3. Assume that $\langle L, \triangleleft \rangle$ is a directed poset, and $\{\mathfrak{X}_i : i \in L\}$ is a family of countable nice triples such that $i \triangleleft j$ implies that $\mathfrak{X}_j \ll \mathfrak{X}_i$.

Then there is a unique nice triple \mathfrak{X}_* denoted by $\lim_{\zeta \in L} \mathfrak{X}_{\zeta}$, such that

- (e1) $\mathfrak{X}_* \ll \mathfrak{X}_{\zeta}$ for each $\zeta \in L$,
- (e2) $X_* = \bigcup_{\zeta \in L} X_{\zeta}$.
- (e3) $I_* = \bigcup_{\zeta \in L} I_{\zeta}$.
- (e4) $\mathcal{F}_* = \bigcup_{\zeta \in L} \mathcal{F}_{\zeta}$.

If $|L| \leq \omega$, then $\lim_{\zeta \in L} \mathcal{X}_{\zeta}$ is countable.

Proof. Write $C_* = \bigcup_{\zeta \in L} C_{\zeta}$, $X_* = C_* \cup \mathbb{Q}$, $I_* = \bigcup_{\zeta \in L} I_{\zeta}$, $\mathcal{F}_* = \bigcup_{\zeta \in L} \mathcal{F}_{\zeta}$, for $i \in I_*$ let

$$B_*(i) = \bigcup \{B_{\xi}(i) : i \in I_{\xi}\},\$$

and $\mathcal{B}_* = \{B_*(i) : i \in I_*\}$. Then \mathcal{B}_* is a base of a 0-dimensional topology τ_* on X_* . Write $\mathcal{X}_* = \langle X_*, \tau_* \rangle$. Then $\mathfrak{X}_* = \langle \mathcal{X}_*, \mathcal{B}_*, \mathcal{F}_* \rangle$ is a nice triple which meets the requirements, and it is clearly unique.

Lemma 2.4. If \mathfrak{X}_0 is a countable nice triple, then there is a countable extension \mathfrak{X}_1 of \mathfrak{X}_0 such that $X_1 = X_0$, \mathcal{X}_1 is T_2 , and C_0 is a closed discrete subspace in \mathcal{X}_1 .

Proof. We can assume that $\langle x, \mathbb{Q} \rangle \in \mathcal{F}_0$ for each $x \in X_0$ because \mathbb{Q} is dense in \mathcal{X}_0 . Consider the family

$$\mathcal{M} = \{B_0(i) \cap F : i \in I_0, \langle \gamma, F \rangle \in \mathcal{F}_0, \gamma \in B_0(i)\}.$$

Since $\mathcal{M} \subset [\mathbb{Q}]^{\omega}$ and $|\mathcal{M}| \leq \omega$, we can choose a family $\mathcal{S} = \{S_n : n < \omega\} \subset [\mathbb{Q}]^{\omega}$ such that

$$\forall \varepsilon \in Fn(\omega, 2) \ \forall M \in \mathcal{M} \ |M \cap S[\varepsilon]| = \omega,$$

where $S[\emptyset] = \mathbb{Q}$, and $S[\varepsilon] = \bigcap_{\varepsilon(n)=1} S_n \cap \bigcap_{\varepsilon(n)=0} (\mathbb{Q} \setminus S_n)$ for $\varepsilon \neq \emptyset$.

Fix an enumeration $\{\{x_n, y_n\} : n < \omega\}$ of $[X_0]^2$, and let

$$T_n = S_n \cup \{x_n\} \setminus \{y_n\}.$$

Consider the family

$$\mathcal{B}' = \{ B_0(i) \cap T[\varepsilon] : i \in I_0, \varepsilon \in Fn(\omega, 2) \},\$$

where $T[\emptyset] = X_0$, and $T[\varepsilon] = \bigcap_{\varepsilon(n)=1} T_n \cap \bigcap_{\varepsilon(n)=0} (X_0 \setminus T_n)$ for $\varepsilon \neq \emptyset$.

Then \mathcal{B}' is a neighborhood base of a 0-dimensional topology τ_1 on $\mathbb{Q} \cup C_0$. The topology is T_2 because $\{x_n, y_n\} \in [X_0]^2$ are separated by $T[\{\langle n, 1 \rangle\}] = T_n \ni x_n$ and $T[\{\langle n, 0 \rangle\}] = X_0 \setminus T_n \ni y_n$.

The subset C_0 is closed discrete, because $x_n \in T[\{\langle n, 1 \rangle\}]$ and $(C_0 \setminus \{x_n\}) \subset T[\{\langle n, 0 \rangle\}]$.

Moreover, $a \in \operatorname{acc}(A, \tau_1)$ for each $\langle a, A \rangle \in \mathcal{F}_0$. Indeed, if $\langle a, F \rangle \in \mathcal{F}_0$, and $a \in B_0(i) \cap T[\varepsilon]$ then $F \cap B_0(i)$ is infinite as $a \in B_0(i)$. Since $F \cap B_0(i) \in \mathcal{M}$, it follows that $F \cap B_0(i) \cap T[\varepsilon]$ is also infinite. Since $\langle x, \mathbb{Q} \rangle \in \mathcal{F}_0$ for each $x \in X_0$, τ_1 is crowded and \mathbb{Q} is dense in it.

Fix an enumeration
$$\{B_1(i): i \in I_1\}$$
 of \mathcal{B}' such that $B_1(i) = B_0(i)$ for $i \in I_0$.
Then $\mathfrak{X}_1 = \langle \langle X_0, \tau_1 \rangle, \mathcal{B}_1, \mathcal{F}_0 \rangle$ meets the requirements.

Lemma 2.5. If \mathfrak{X}_0 is a nice countable triple, and $A \in [\mathbb{Q}]^{\omega}$, then there is a countable extension \mathfrak{X}_1 of \mathfrak{X}_0 such that $X_1 = X_0$ and A contains an infinite closed discrete subset B in \mathcal{X}_1 .

Proof of Lemma 2.5. By Lemma 2.4, we can assume that \mathcal{X}_0 is T_2 . We can also assume that A is not closed discrete in \mathcal{X}_0 . Thus, A should contain convergent sequences. So we can assume that A converges to some γ in \mathcal{X}_0 .

Let $\{B'(\ell) : \ell < \omega\}$ be an enumeration of \mathcal{B}_0 , and let $\{F_n : n < \omega\}$ be an ω -abundant enumeration of $\{F : \langle \gamma, F \rangle \in \mathcal{F}\}$.

By induction on n, choose $U_n \in \mathcal{B}_0$ and $d_n \in A$ such that

- (i) $U_n \subset \bigcap \{B'(\ell) : \ell < n, \gamma \in B'(\ell)\} \setminus \bigcup \{B'(\ell) : \ell < n, \gamma \notin B'(\ell)\},\$
- (ii) $U_n \cap F_n \neq \emptyset$,
- (iii) $\gamma \notin U_n$, $\{d_m : m < n\} \cap U_n = \emptyset$,
- (iv) $d_n \in A \setminus \{d_m : m < n\} \setminus \bigcup \{U_m : m \le n\}.$

Let

$$V = \{\gamma\} \cup \bigcup_{n \in \omega} U_n,$$

and write

$$\mathcal{B}_1 = \mathcal{B}_0 \cup \{V \cap B : \gamma \in B \in \mathcal{B}_0\}.$$

Then \mathcal{B}_1 is the neighborhood base of a 0-dimensional topology τ_1 on X_0 such that $B = A \setminus V$ is an infinite, closed discrete set in τ_1 .

By (ii), $\gamma \in acc(F, \mathcal{X}_1)$ for each $\langle \gamma, F \rangle \in \mathcal{F}_0$.

Fix an enumeration $\{B_1(i): i \in I_1\}$ of \mathcal{B}_1 such that $B_1(i) = B_0(i)$ for $i \in I_0$. Then $\mathfrak{X}_1 = \langle \langle X_0, \tau_1 \rangle, \mathcal{B}_1, \mathcal{F}_0 \rangle$ meets the requirements.

Lemma 2.6. If \mathfrak{X}_0 is a nice countable triple, $A \in [\mathbb{Q}]^{\omega}$ is closed discrete in \mathcal{X}_0 , and $z \notin C_0$ is an ordinal, then there is a countable extension \mathfrak{X}_1 of \mathfrak{X}_0 such that $C_1 = C_0 \cup \{z\}$ and $\langle z, A \rangle \in \mathcal{F}_1$.

Proof of Lemma 2.6. We can assume that $\langle x, \mathbb{Q} \rangle \in \mathcal{F}_0$ for each $a \in X_0$.

Let $\{B_i : i < \omega\}$ be an enumeration of the base \mathcal{B}_0 .

By induction choose a decreasing sequence $\{A_n : n < \omega\}$ of infinite subsets of A such that

$$A_n \subset B_n \text{ or } A_n \cap B_n = \emptyset$$

for $n < \omega$. Pick pairwise distinct $a_n \in A_n$ for $n \in \omega$, then choose pairwise disjoint clopen neighborhoods U_n of a_n such that $U_n \subset B_i$ iff $a_n \in B_i$ and $U_n \cap B_i = \emptyset$ iff $a_n \notin B_i$ for each $i \le n$.

Then, for each $i < \omega$,

$$\forall^{\infty} n(U_n \subset B_i) \ \lor \ \forall^{\infty} n(U_n \cap B_i) = \emptyset.$$

Let $C_1 = C_0 \cup \{z\}$, and $I_1 = I_0 \cup \{\zeta_n : n < \omega\}$, where $\zeta_n \notin I_0$. For $\zeta \in I_0$ let

$$B_1(\zeta) = \begin{cases} B_0(\zeta) & \text{if } \forall^{\infty} n(U_n \cap B_0(\zeta) = \emptyset), \\ B_0(\zeta) \cup \{z\} & \text{if } \forall^{\infty} n(U_n \subset B_0(\zeta)). \end{cases}$$

Moreover, for $n < \omega$ let

$$B_1(\zeta_n) = \{z\} \cup \bigcup_{m \ge n} U_m.$$

Let τ_1 be the topology generated by $\mathcal{B}_1 = \{B_1(j) : j \in I_1\}$ as a base. To show that every $B_1(i)$ is closed, assume that $z \notin B_1(j)$. Then there is $m \in \omega$ such that $U_n \cap B_i(j) = \emptyset$ for each $n \geq m$. Thus $B_1(j) \cap B_1(\zeta_m) = \emptyset$.

Finally, put $\mathcal{F}_1 = \mathcal{F}_0 \cup \{\langle z, A \rangle\}.$

Then
$$\mathfrak{X}_1 = \langle \langle X_1, \tau_1 \rangle, \mathcal{B}_1, \mathcal{F}_1 \rangle$$
 satisfies the requirements.

Proof of Theorem 2.1.(1). Let $\langle K_0, K_1 \rangle$ be a partition of ω_1 into uncountable pieces, and let $\{A_{\xi} : \xi \in K_1\}$ be an ω_1 -abundant enumeration of the family $[\mathbb{Q}]^{\omega}$.

We define a \ll -decreasing sequence $\langle \mathfrak{X}_{\zeta} : \zeta \leq \omega_1 \rangle$ of nice triples such that

- (i) $C_{\zeta} \in \omega_1 + 1$, and $|X_{\zeta}| = |\zeta| + \omega$,
- (ii) $X_0 = \mathbb{Q}$ and τ_0 is the usual topology on \mathbb{Q} ,
- (iii) if ζ is a limit ordinal, let $\mathfrak{X}_{\zeta} = \lim_{\xi \in \zeta} \mathfrak{X}_{\xi}$ (see Lemma 2.3).
- (iv) Assume that $\zeta = \xi + 1$, and $\xi \in K_0$.

Apply Lemma 2.4 for \mathfrak{X}_{ξ} to obtain a countable nice triple \mathfrak{X}_{ζ} such that \mathcal{X}_{ζ} is T_2 and the countable subset C_{ξ} closed discrete in \mathcal{X}_{ζ} .

(v) Assume $\zeta = \xi + 1$, and $\xi \in K_1$.

First, apply Lemma 2.5 for the nice triple \mathfrak{X}_{ξ} and A_{ξ} to find a countable extension \mathfrak{X}'_{ξ} of \mathfrak{X}_{ξ} such that in \mathfrak{X}'_{ξ} the set A_{ξ} contains an infinite closed discrete set B_{ξ} .

Then, applying Lemma 2.6 for \mathfrak{X}'_{ξ} and B, we can obtain a countable extension \mathfrak{X}_{ζ} of \mathfrak{X}'_{ξ} such that $\langle a, B \rangle \in \mathcal{F}_{\zeta}$. We can assume that $C_{\zeta} = C_{\xi} \cup \{a\} \in \omega_1$.

Finally, \mathcal{X}_{ω_1} satisfies the requirements. It is T_2 because \mathcal{X}_{ζ} is T_2 for cofinally many ζ and $\mathfrak{X}_{\omega_1} \ll \mathfrak{X}_{\zeta}$. It is DRC because \mathbb{Q} is a dense, relatively countably compact subset.

We also have $\Delta(\mathcal{X}_{\omega_1}) = \omega_1$. Indeed, if $B_i \in \mathcal{B}_{\omega_1}$, then let $A = B_i \cap \mathbb{Q}$. Then $J = \{\xi : A_{\xi} = A\}$ is uncountable, and for each $\xi \in J$ we added a new accumulation point to A. But these points are in B_i .

To prove (c) assume, for contradiction, that $Z \subset X_{\omega_1}$ is a countably compact set that is not scattered. Then, there exists an open set U such that $T = Z \setminus U$ is crowded. Since T is countably compact, we must have $|T| \geq \omega_1$. Hence, $T \cap \omega_1$ is infinite, which is a contradiction because every infinite countable subset of ω_1 is closed discrete.

Proof of Theorem 2.1.(2). Assume that GCH holds in the ground model, and let $\kappa > \omega_1$ be an arbitrarily large regular cardinal.

Consider the poset $\mathcal{P} = \langle P, \ll \rangle$, where

$$P = \{\mathfrak{X}_* : \mathfrak{X}_* \text{ is a nice triple}, C_* \cup I_* \in [\kappa]^{\leq \omega} \}.$$

If D and E are sets of ordinals with tp(D) = tp(E), denote $\rho_{D,E}$ the unique \in -preserving bijection between D and E.

Definition 2.7. Two conditions \mathfrak{X}_0 and \mathfrak{X}_1 are twins iff

- (1) $tp(C_0) = tp(C_1)$ and $tp(I_0) = tp(I_1)$,
- (2) $\rho_{C_0,C_1} \upharpoonright C_0 \cap C_1 = id$, and $\rho_{I_0,I_1} \upharpoonright I_0 \cap I_1 = id$,
- (3) for each $i \in I_0$,

$$B_1(\rho_{I_0,I_1}(i)) = (B_0(i) \cap \mathbb{Q}) \cup \rho''_{C_0,C_1}(B_0(i) \cap \kappa),$$

(4) $\mathcal{F}_1 = \{ \langle \rho(a), A \rangle : \langle a, A \rangle \in \mathcal{F}_0 \}.$

Lemma 2.8. If \mathfrak{X}_0 and \mathfrak{X}'_1 are twins, then they are compatible in P.

Proof. Let $C_2 = C_1 \cup C_2$, $X_2 = \mathbb{Q} \cup C_2$, $I_2 = I_0 \cup I_1$, and for $i \in I_2$ let

$$B_2(i) = \begin{cases} B_0(i) \cup B_1(\rho_{I_0,I_1}(i)) & \text{if } i \in I_0 \setminus I_1, \\ B_1(i) \cup B_0(\rho_{I_0,I_1}^{-1}(i)) & \text{if } i \in I_1 \setminus I_0, \\ B_0(i) \cup B_1(i) & \text{if } i \in I_0 \cap I_1. \end{cases}$$

Then $\{B_2(i): i \in I_2\}$ is a base of a 0-dimensional (but typically not Hausdorff) topology τ_2 on $X_0 \cup X_1$. Moreover, \mathcal{X}_0 and \mathcal{X}_1 are subspaces of \mathcal{X}_2 .

Finally, the triple $\langle \langle X_2, \tau_2 \rangle, \mathcal{B}_2, \mathcal{F}_0 \cup \mathcal{F}_1 \rangle \in P$ is a common extension of \mathfrak{X}_0 and \mathfrak{X}_1 .

The previous lemma clearly implies the following statement:

Lemma 2.9. P satisfies ω_2 -c.c.

Since \mathcal{P} is σ -closed by Lemma 2.3, forcing with P preserves cardinals, and 2^{ω_1} in the generic extension will be $((|P|)^{\omega_1})^V = \kappa$.

Let $\mathcal{G} \subset P$ be a generic filter. By Lemma 2.3, we can consider the nice triple $\mathbb{X}_* = \lim \mathcal{G}$. By trivial density arguments, we obtain that $X_* = \mathbb{Q} \cup \kappa$, $I_* = \kappa$ and X_* is T_2 by Lemma 2.4.

So we obtain a 0-dimensional T_2 space \mathcal{X}_* in $V[\mathcal{G}]$. We show that \mathcal{X}_* satisfies the requirements.

Lemma 2.10. $|\operatorname{acc}(A, \tau_*)| = \kappa \text{ for each } A \in [\mathbb{Q}]^{\omega}.$

Proof. Since \mathcal{P} is σ -complete, A is in the ground model. Fix $\delta < \kappa$. By applying Lemma 2.6, we obtain that

$$E_{A,\delta} = \{\mathfrak{X}_0 \in P : \langle \gamma, A \rangle \in \mathcal{F}_0 \text{ for some } \delta < \gamma < \kappa \}$$

is dense in P. Thus, there is \mathfrak{X}) $\in \mathcal{G} \cap E_{A,\delta}$. Hence, $\operatorname{acc}(A, \tau_*) \setminus \delta \neq \emptyset$. Thus, $|\operatorname{acc}(A, \tau_*)| = \kappa$.

Lemma 2.11. Every $A \in [\kappa]^{\omega}$ is closed discrete in \mathcal{X}_* .

Proof. Since \mathcal{P} is σ -complete, A is in the ground model. Fix $a \in \kappa \setminus A$. By Lemma 2.4, the set

$$D_{a,A} = \{\mathfrak{X}_0 \in P : a \notin \operatorname{acc}(A, \tau_0)\}$$

is dense in P. Thus, there is $\mathfrak{X}_0 \in \mathcal{G} \cap D_{a,A}$. Hence, $a \notin \operatorname{acc}(A, \tau_*)$.

We can prove (c) in the same manner as in proof of part (1). Assume, for contradiction, that $Z \subset X_*$ is a countably compact set that is not scattered. Then there exists an open set $U \in \tau_*$ such that $T = Z \setminus U$ is crowded. Since T is countably compact, it follows that $|T| \geq \omega_1$. However, this implies that $T \cap \kappa$ is infinite, which is a contradiction because every infinite countable subset of κ is closed and discrete.

(a) holds by Lemma 2.10, and (b) holds by Lemma 2.11.

This completes the proof of Theorem 2.1(2).

3. PSEUDOCOMPACT SPACES WITHOUT DENSE, RELATIVELY COUNTABLY COMPACT SUBSPACES

Berner's Σ is R-hereditarily "SP, but $\neg DRC$ ", but it is not first countable. Berner's monster is first countable, but not R-hereditarily " $\neg DRC$ ".

In this section, we construct a first countably, R-hereditarily "SP, but ¬DRC" space which contains as many countable discrete subsets as possible. A pseudocompact, first countable space cannot be anti-countably compact, as it must contain convergent sequences. As the next best alternative, in Corollary 3.2, we construct spaces where every uncountable subset contains an infinite closed discrete subset.

Theorem 3.1. If $\mathfrak{s} = \mathfrak{c}$, then there is an SP, crowded, first countable 0-dimensional T_2 space X with $\Delta(X) = \mathfrak{c}$ which is left separated in type \mathfrak{c} .

Assuming CH, we can get a bit more.

Corollary 3.2. If CH holds, then there is a selectively pseudocompact, crowded, first countable 0-dimensional T_2 space X with $\Delta(X) = \omega_1$ such that relatively countable compact subset is countable, and every countable set is nowhere dense.

To obtain Theorem 1.3 and Corollary 3.2 from Theorem 3.1, we need to formulate some results which excludes the existence of certain relatively countably compact subspaces in certain left separated spaces.

Theorem 3.3. (1) A left separated, crowded regular space Y is not DRC. (2) A first countable 0-dimensional T_2 space which is left separated in type ω_1 is not DRC.

Observe that in (2) we do not assume that the space is crowded.

Proof of Theorem 1.3 from Theorem 3.1 and Theorem 3.3.(1). Consider the space X we obtain from Theorem 3.1. If H is a regular closed subset of X, then H is crowed, so it is not DRC by Theorem 3.3.(1).

Proof of Corollary 3.2 from Theorem 3.1 and Theorem 3.3.(2). Consider the space X we obtain from Theorem 3.1. Then X is left-separated in type $\mathfrak{c} = \omega_1$. Let Y be an uncountable subset of X. Then Y is also left-separated in type ω_1 , and so it is not DRC by Theorem 3.3.(2).

Proof of Theorem 3.3(1). Let $\{y_{\alpha} : \alpha < \kappa\}$ be a left-separating enumeration of Y and let $D \subset Y$ be dense.

By recursion on $n \in \omega$ pick $y_{\alpha_n} \in D$ and $U_n, V_n \in \tau_Y$ as follows.

Let $y_{\alpha_0} \in D$ be arbitrary.

If y_{α_n} is given, let U_n be a left-separating neighborhood of y_{α_n} . Since Y is regular, we can choose $V_n \in \tau_Y^+$ such that $\overline{V_n} \subset U_n$.

Since Y is crowded, we can pick $y_{\alpha_{n+1}} \in D \cap (V_n \setminus \{y_{\alpha_n}\})$.

We claim that $\{y_{\alpha_n} : n < \omega\} \in [D]^{\omega}$ is closed discrete in Y. Indeed, $\alpha_n < \alpha_{n+1}$ by the construction. Let $\alpha = \sup\{\alpha_n : n < \omega\}$. Then

$$\{y_{\alpha_n}: n < \omega\}' \subset \bigcap_{n < \omega} \overline{\{y_{\alpha_m}: m \ge n\}} \subset \overline{\{y_{\zeta}: \zeta < \alpha\}} \cap \bigcap_{n < \omega} \overline{V_n} \subset \{y_{\zeta}: \zeta < \alpha\} \cap \bigcap_{n < \omega} \{y_{\zeta}: \zeta < \alpha\} \cap \{y_{\zeta}: \alpha \le \zeta\} = \{y_{\zeta}: \zeta < \alpha\} \cap \{y_{\zeta}: \alpha \le \zeta\} = \emptyset.$$

Proof of Theorem 3.3(2). We can assume that $Y = \omega_1$. Let $D \subset Y$ be dense. Let $\{B(\alpha, i) : \alpha < \omega_1, i < \omega\}$ be a clopen base of Y such that $B(\alpha, i) \supset B(\alpha, i + 1)$ and $B(\alpha, 0) \cap \alpha = \emptyset$.

By induction on n pick $\alpha_n \in D$, $\beta_n \in Y$ and $k_n, i_n \in \omega$ such that

- $(1) \ \alpha_{n-1} < \alpha_n,$
- $(2) \{\beta_n : n < \omega\} = \bigcup \{\alpha_n : n < \omega\},\$
- (3) $Y_n = Y \setminus \bigcup \{B(\beta_m, i_m) : m \leq n\}$ is uncountable,
- (4) $\alpha_n \in Y_n \cap D$.

Assume that we have $\alpha_m, \beta_m, k_m, i_m$ for m < n.

Using a bookkeeping function choose β_n such that (2) will hold. Since Y_{n-1} is uncountable, we can choose i_n such that $Y_n = Y_{n-1} \setminus B(\beta_n, i_n)$ is still uncountable.

Since D is dense, Y_n is uncountable clopen, and Y is left-separated, we can pick $\alpha_n \in D \cap (Y_n \setminus \max(\alpha_{n-1} + 1, \beta_n + 1).$

Let $\delta = \bigcup \{\alpha_n : n < \omega\} = \{\beta_n : n < \omega\}$. Then $\overline{\{\alpha_n : n < \omega\}} \subset \delta$ because Y is left-separated, and β_m is not an accumulation point of $\{\alpha_n : n < \omega\}$ because

$$B(\beta_m, i_m) \cap \{\alpha_n : n < \omega\} \subset \{\alpha_k : k \le m\}.$$

Thus, $\{\alpha_n : n < \omega\} \subset D$ is closed discrete in Y. So D is not relatively countably compact.

Before proving Theorem 3.1, we need to prove some lemmas.

Definition 3.4. (1) A triple $\mathfrak{X} = \langle \mathcal{X}, \mathcal{B}, \prec_X \rangle$ is a good triple iff

- (t1) $\mathcal{X} = \langle X, \tau \rangle$ is a left-separated, crowded, first countable, 0-dimensional T_2 -space,
- (t2) $\mathcal{B} = \langle B(x,i) : x \in X, i \in \omega \rangle$ is a family of clopen sets,
- (t3) $\{B(x,i): i \in \omega\}$ is a neighborhood base at x in X consisting of clopen subsets such that $B(x,i) \supset B(x,i+1)$ for each $i < \omega$.
- (t4) \prec_X is a left separating well-ordering of X,

If \mathfrak{X}_{ℓ} is good triple, write $\mathfrak{X}_{\ell} = \langle \mathcal{X}_{\ell}, \mathcal{B}_{\ell}, \prec_{\ell} \rangle$, $\mathcal{X}_{\ell} = \langle X_{\ell}, \tau_{\ell} \rangle$, moreover let $\mathcal{B}_{\ell} = \langle B_{\ell}(x, i) : x \in X_{\ell}, i < \omega \rangle$.

- (2) Given good triples $\mathfrak{X}_{\ell} = \langle \mathcal{X}_{\ell}, \mathcal{B}_{\ell}, \prec_{\ell} \rangle$ for $\ell \in 2$, we say that \mathfrak{X}_{1} is an extension of \mathfrak{X}_{0} , and we write $\mathfrak{X}_{1} \ll \mathfrak{X}_{0}$, iff
- (e1) $X_0 \subset X_1$,
- (e2) $B_0(x,i) = B_1(x,i) \cap X_0$ for each $x \in X_0$ and $i \in \omega$,
- (e3) if $B_0(x,i) \subset B_0(x',i')$ and $x' \notin B_0(x,i)$ then $B_1(x,i) \subset B_1(x',i')$ for each $x, x' \in X_0$ and $i, i' < \omega$,
- (e4) if $B_0(x,i) \cap B_0(x',i') = \emptyset$ then $B_1(x,i) \cap B_1(x',i') = \emptyset$ for each $x, x' \in X_0$ and $i, i' < \omega$,
- (e5) $\prec_0 \subset \prec_1$ and X_0 is an initial segment in $\langle X_1, \prec_1 \rangle$.

Key Lemma 3.5. Assume that

(a) \mathcal{X}_0 is a good triple,

- (b) $|X_0| < \mathfrak{s}$,
- (c) the family $\{B_0(\zeta, j(\zeta)) : \zeta \in K\}$ is locally finite in X for some $K \in [X]^{\omega}$ and $j : K \to \omega$,

Then there is a good triple \mathfrak{X}_1 such that

- (1) $\mathfrak{X}_1 \ll \mathfrak{X}_0$,
- (2) the family $\{B_1(\zeta, j(\zeta)) : \zeta \in K\}$ is not locally finite in \mathcal{X}_1 .
- (3) $|X_1| = |X_0|$.

Proof of the Key Lemma 3.5. For $\zeta \in K$ pick $\eta_{\zeta} \in B_0(\zeta, j(\zeta)) \setminus \{\zeta\}$. Let $K_* = \{\eta_{\zeta} : \zeta \in K\}$. Since $|X_0| < \mathfrak{s}$, the family $\{B_0(x,i) \cap K_* : x \in X_0, i \in \omega\}$ can not be a splitting family on $[K_*]^{\omega}$. So, there is a set $L \in [K]^{\omega}$ such that writing $L_* = \{\eta_{\zeta} : \zeta \in K_*\}$ for each $\langle x, i \rangle \in X \times \omega$ we have

$$L_* \subset^* B_0(x,i) \vee L_* \subset^* X_0 \setminus B_0(x,i).$$

The underlying set of the extension \mathfrak{X}_1 will be

$$X_1 = X_0 \cup \{p\} \cup (X_0 \times \mathbb{Q}),$$

where p is a new element.

For $q \in \mathbb{Q}$ let $\{I(q,i) : i \in \omega\}$ be a clopen neighborhood base of q in \mathbb{Q} . Fix an enumeration $\{\zeta_n : n < \omega\}$ of L.

Define $B_1(y, i)$ for $y \in X_1$ and $i < \omega$ as follows.

Case 1. $y = \langle x, q \rangle \in X_0 \times \mathbb{Q}$.

Let

$$B_1(y,i) = \{x\} \times I(q,i).$$

Case 2. y = p.

Let

$$B_1(p,i) = \{p\} \cup \bigcup_{n \ge i} (\{\eta_{\zeta_n}\} \times \mathbb{Q}).$$

Case 3. $y \in X$.

Let

$$B'(y,i) = B_0(y,i) \cup (B_0(y,i) \setminus \{y\}) \times \mathbb{Q},$$

and

$$B_1(y,i) = \begin{cases} B'(y,i) & \text{if } L_* \subset^* X_0 \setminus B_0(y,i), \\ B'(y,i) \cup \{p\} & \text{if } L_* \subset^* B_0(y,i). \end{cases}$$

Finally, let \prec_Q be a well-ordering of \mathbb{Q} in type ω , and define \prec_1 as follows.

- (a) $\prec_0 \subset \prec_1$,
- (b) $\forall x \in X_0 \ x \prec_1 p$,
- (c) $\forall y \in X_0 \times \mathbb{Q} \ p \leq_1 y$,
- (d) $\prec_1 \upharpoonright X_0 \times \mathbb{Q}$ is the lexicographical product of \prec_0 and \prec_Q .

In that way we defined \mathfrak{X}_1 . We should check first that \mathfrak{X}_1 is a good triple extending \mathfrak{X}_0 .

Claim 3.5.1. (e1), (e2), (e4) and (e5) hold for \mathfrak{X}_0 and \mathfrak{X}_1 .

Trivial from definition.

Claim 3.5.2. If $B_0(x,i) \subset B_0(x',i')$ and $p \in B_1(x,i)$, then $p \in B_1(x',i')$.

Indeed, if $L_* \subset^* B_0(x,i)$, then $L_* \subset^* B_0(x',i')$.

Claim 3.5.3. (e3) holds for \mathfrak{X}_0 and \mathfrak{X}_1 .

Proof. Indeed, if $B_0(x,i) \subset B_0(x',i')$ and $x' \notin B_0(x,i)$ then

$$B_1(x,i) \cap (X_0 \times \mathbb{Q}) = (B_0(x,i) \setminus \{x\}) \times \mathbb{Q} \subset (B_0(x',i') \setminus \{x'\}) \times \mathbb{Q} = B_1(x',i') \cap (X_0 \times \mathbb{Q}),$$

and $p \in B_1(x, i)$ implies $p \in B_1(x', i)$ by Claim 3.5.2.

Claim 3.5.4. $\{B_1(y,n): n \in \omega\}: y \in X_1\}$ is a neighborhood system of a topology τ_1 on X_1 .

Proof of the Claim. By [5, Proposition 1.2.3], we should check that

(BP1) $y \in B_1(y, n)$ for each $y \in Y$ and $n < \omega$,

(BP2) if $z \in B_1(y, n)$ then $B_1(z, m) \subset B_1(y, n)$ for some $m < \omega$,

(BP3) for each $x \in X_1$ and for each $n, m < \omega$ there is $k < \omega$ such that $B_1(x, k) \subset B_1(x, n) \cap B_1(x, m)$.

Conditions (BP1) and (BP3) are trivial.

To check (BP2), assume that $z \in B_1(y,n)$, $z \neq y$. If $y = \langle x,q \rangle \in X_0 \times \mathbb{Q}$, then $z = \langle x,r \rangle$ for some $r \in I(q,n)$. Thus, there is m with $I(r,m) \subset I(q,n)$ and so $B_1(z,m) \subset B_1(y,n)$.

If y = p, then $z = \langle x, r \rangle$, where $x = \eta_{\zeta_k}$ for some $k \geq n$, and $r \in \mathbb{Q}$, and so $B_1(z, m) \subset \{\eta_{\zeta_k}\} \times \mathbb{Q} \subset B_1(p, n)$ for each $m \in \omega$.

Finally, consider the case $y \in X_0$. If $z \in X_0$, then pick m such that $B_0(z, m) \subset B_0(y, n) \setminus \{y\}$. Then $B_1(z, m) \subset B_1(y, n)$ by Claim 3.5.3.

If $z = \langle x', q \rangle \in X_0 \times \mathbb{Q}$, then $x' \neq x$ by the definition of $B_1(x, n)$. Thus, $B_1(z, m) \subset \{x'\} \times \mathbb{Q} \subset B_1(x, n)$ for each $m \in \omega$.

Now, assume that z = p. Then there is $m \in \omega$ such that $\eta_{\zeta_k} \in B_0(x, n)$ for each $k \geq m$. Hence $B_1(p, m) \subset B_1(x, n)$.

Claim 3.5.5. τ_1 is T_2 .

Proof. Fix $\{y, z\} \in [X_1]^2$.

Assume first that $y \in X_0$ and z = p. Since the family $\{B_0(\zeta, j(\zeta)) : \zeta \in K\}$ is locally finite in \mathcal{X}_0 , there are $i, m \in \omega$ such that $B_0(\zeta_n, g(\zeta_n)) \cap B_0(y, i) = \emptyset$ for each $n \geq m$. In particular, $\eta_{\zeta_n} \notin B_0(y, i)$ for $n \geq m$, and so $B_1(p, m) \cap B_1(y, i) = \emptyset$.

If $y \in X_0$ and $z = \langle x, q \rangle \in (X_0 \setminus \{y\}) \times \mathbb{Q}$ then pick i such that $x \notin B_0(y, i)$. Then $B_1(y, i) \cap B_1(z, j) \subset B_1(y, i) \cap (\{x\} \times \mathbb{Q}) = \emptyset$ for each $j \in \omega$.

If $y \in X_0$ and $z = \langle y, q \rangle \in \{y\} \times \mathbb{Q}$ then $B_1(y, i) \cap B_1(z, k) \subset B_1(y, i) \cap (\{y\} \times \mathbb{Q}) = \emptyset$ for each $i, k \in \omega$.

The remaining cases are trivial.

Claim 3.5.6. Every $B_1(y,i)$ is closed, so τ_1 is zero-dimensional.

Proof. Fix $z \in X_1$ with $z \notin B_1(y, i)$.

If $\{y, z\} \in [X_0]^2$, then $z \notin B_0(y, i)$, so we can pick k such that $B_0(z, k) \cap B_0(y, i) = \emptyset$. Then $B_1(z, k) \cap B_1(y, i) = \emptyset$ by (e4).

Since $X_0 \times \mathbb{Q}$ is an open subspace in \mathcal{X}_1 and the subspace topology on $X_0 \times \mathbb{Q}$ is the product topology of the discrete topology on X_0 and the topology of \mathbb{Q} , it follows that if $\{y,z\} \in [X_0 \times \mathbb{Q}]^2$ then there is k such that $B_1(z,k) \cap B_1(y,i) = \emptyset$.

Consider next the case when $y \in X_0$ and $z = \langle x, q \rangle \in X_0 \times \mathbb{Q}$. Then $z \notin B_0(y, i)$ implies $B_1(y, i) \cap (\{x\} \times \mathbb{Q}) = \emptyset$ and so $B_1(y, i) \cap B_1(z, k) = \emptyset$ for each $k \in \omega$.

Assume next that $y \in X_0$ and z = p. Then $z \notin B_1(y,i)$ implies that there is $m \in \omega$ such that $\eta_n \notin B_0(y,i)$ for each $n \ge m$. Thus, $B_1(y,i) \cap B_1(p,m) = \emptyset$.

Finally, assume that y = p.

Consider first that case $z \in X_0$. Since the family $\{B_0(\zeta, j(\zeta)) : \zeta \in K\}$ is locally finite, there is m such that $K \cap B_0(z, m)$ contains at most one element, namely z. Then, $B_1(p, i) \cap B_1(z, m) = \emptyset$.

Now, assume that $z = \langle x, q \rangle \in X \times \mathbb{Q}$. Then $x \notin \{\eta_{\zeta_{\ell}} : i \leq \ell < \omega\}$, so $B_1(p,i) \cap B_1(z,j) = \emptyset$ for each $j \in \omega$.

Claim 3.5.7. \prec_1 is a left-separating well ordering of X_1 .

Proof. Trivial.
$$\Box$$

Putting together these observations we obtain that

$$\mathfrak{X}_1 = \langle \mathcal{X}_1, \{B_1(y,i) : y \in X_1, i \in \omega\}, \prec_1 \rangle$$

is a good triple and $\mathfrak{X}_1 \ll \mathfrak{X}_0$. Moreover, p is an accumulation point of the family $\{B_1(\zeta, j(\zeta)) : \zeta \in K\}$.

Lemma 3.6. Assume that $\langle I, A \rangle$ is a directed poset, and $\{\mathfrak{X}_i : i \in I\}$ is a family of good triples such that i A j implies that $\mathfrak{X}_j \ll \mathfrak{X}_i$. Then there is a good triple $\langle \mathcal{X}_*, \mathcal{B}_*, \mathcal{A}_* \rangle$ denoted by $\lim_{i \in I} \mathfrak{X}_i$, such that

- (a) $\lim_{i \in I} \mathfrak{X}_i \ll \mathfrak{X}_i$ for each $i \in I$,
- (b) $X_* = \bigcup_{i \in I} X_i$.

Proof. Write $X_* = \bigcup_{i \in I} X_i$, and for $x \in X_*$ and for $n \in \omega$ let

$$B_*(x,n) = \bigcup \{B_j(x,n) : x \in X_j\},\$$

and put

$$\prec_* = \bigcup_{i \in I} \prec_i$$
.

Then, \mathcal{B}_* is a base of a 0-dimensional T_2 topology τ_* on X_* . Thus, writing $\mathcal{X}_* = \langle X_*, \tau_* \rangle$ the triple $\mathcal{Z}_* = \langle \mathcal{X}_*, \{B_*(x,n) : x \in X_*, n \in \omega\}, \prec_* \rangle$ satisfies the requirements.

Proof of Theorem 3.1. Let $\{\langle K_{\alpha}, j_{\alpha} \rangle : \alpha < \mathfrak{c} \}$ be a \mathfrak{c} -abundant enumeration of the family

$$\{\langle K, j \rangle : K \in [\mathfrak{c}]^{\omega}, j : K \to \omega\}.$$

We define a \ll -decreasing sequence $\langle \mathfrak{X}_{\zeta} : \zeta \leq \mathfrak{c} \rangle$ of good triples such that

- (i) X_{ζ} is an ordinal, $|X_{\zeta}| = |\zeta| + \omega$, and \prec_{ζ} is the natural ordering of ordinals,
- (ii) X_0 is a crowded 0-dimensional, first countable T_2 topology on ω ,
- (iii) if ζ is a limit ordinal, let $\mathfrak{X}_{\zeta} = \lim_{\xi \in \zeta} \mathfrak{X}_{\xi}$ (see Lemma 3.6).
- (iv) If $\zeta = \xi + 1$, do the following.
 - (a) Consider K_{ξ} and j_{ξ} .
 - (b) If $K_{\xi} \notin [X_{\xi}]^{\omega}$ or $\{B_{\xi}(k, j_{\xi}(k)) : k \in K_{\xi}\}$ is not a locally finite family of open sets in \mathcal{X}_{ξ} , then we do nothing, i.e. let $\mathfrak{X}_{\zeta} = \mathfrak{X}_{\xi}$.
 - (c) If $K_{\xi} \in [X_{\xi}]^{\omega}$ and $\{B_{\xi}(k, j_{\xi}(k)) : k \in K_{\xi}\}$ is a locally finite family of open sets in \mathcal{X}_{ξ} , apply Lemma 3.5 for \mathfrak{X}_{ξ} and $\{B_{\xi}(k, j_{\xi}(k)) : k \in K_{\xi}\}$ to obtain \mathfrak{X}_{ζ} . Hence, $\{B_{\xi+1}(k, j_{\xi}(k)) : k \in K_{\xi}\}$ is not locally finite. We can assume that $X_{\zeta} \in \mathfrak{c}$ is an ordinal, and \prec_{ζ} is the natural ordering on that ordinal.

Finally, $X_{\mathfrak{c}}$ satisfies the requirements. To show that $\mathcal{X}_{\mathfrak{c}}$ is SP, let $\{B_{\mathfrak{c}}(k,j(k)): k \in K\}$ be a family of basic open sets. There is $\xi < \mathfrak{c}$ such that $K_{\xi} = K \in [X_{\xi}]^{\omega}$ and $j_{\xi} = j$. Then, by the construction, $\{B_{\xi+1}(k,j(k)): k \in K\}$ is not locally finite, it has an accumulation point p. Since $\chi(p, \mathcal{X}_{\xi+1}) = \omega$, we can pick $x_k \in B_{\xi}(k,j(k))$ for $k \in K$ such that $p \in \operatorname{acc}(\{x_k : k \in K\}, \tau_{\xi+1})$. Since $\mathfrak{X}_{\mathfrak{c}} \ll \mathfrak{X}_{\xi+1}$, we have $\tau_{\xi+1} = \{U \cap X_{\xi+1} : U \in \tau_{\mathfrak{c}}\}$. Hence, $p \in \operatorname{acc}(\{x_k : k \in K\}, \tau_{\mathfrak{c}})$.

4. A PSEUDOCOMPACT, BUT NOT COUNTABLY COMPACT SPACE WITH COUNTABLE SPREAD

First, we make the following observation: the proposition below implies that ZFC alone is insufficient to construct a space as required in Theorem 1.4.

Proposition 4.1. If there is a pseudocompact, but not countably compact, regular space X with $s(X) = \omega$, then there is an S-space.

Proof. A Lindelöf pseudocompact space is compact, so X can not be Lindelöf, and so it contains a right-separated subspace $Y \in [X]^{\omega_1}$. Since $s(Y) \leq s(X) = \omega$, it follows that $z(Y) = \omega$ as well. Thus, Y is an S-space.

Theorem 4.2. If CH holds, then there is a DRC, but $\neg DCC$, locally countable, locally compact, first countable, 0-dimensional T_2 space X with cardinality ω_1 and $s(X) = \omega$.

We do not know if we can find an R-hereditary example for the problem we addressed in the previous theorem.

Before proving Theorem 4.2 we need some preparation. The first statement is well-known:

Lemma 4.3. If Y is a countable, regular space, $D \subset Y$ is closed discrete, then there is a neighborhood assignment $W: D \to \tau_Y$ such that the family $\{W(d): d \in D\}$ is closed discrete.

The Euclidean topology on $\mathbb R$ is denoted by ε . The next lemma is the key of our proof.

Lemma 4.4. Assume that

- (a) $Y = \langle Y, \tau_Y \rangle$ is a countable, locally compact T_2 space,
- (b) $Y \cap \mathbb{R}$ is closed in Y and $p \in \mathbb{R} \setminus Y$,
- (c) the topology $\tau_Y \upharpoonright Y \cap \mathbb{R}$ refines the Euclidean topology on $Y \cap \mathbb{R}$,
- (d) $E \in [Y \setminus \mathbb{R}]^{\omega}$ is closed discrete in Y,
- (e) $\mathcal{D} \subset [Y \cap \mathbb{R}]^{\omega}$, \mathcal{D} is countable.

Then there is a space $Z = \langle Z, \tau_Z \rangle$ such that

- (a') Z is locally compact T_2 ,
- (b') $Z = Y \cup \{p\}$ and $\tau_Y = \tau_Z \cap \mathcal{P}(Y)$,
- (c') the topology $\tau_Z \upharpoonright Z \cap \mathbb{R}$ refines the Euclidean topology on $Z \cap \mathbb{R}$,
- $(d') p \in \overline{E}^Z$,
- (e') for each $D \in \mathcal{D}$ if $p \in \overline{D}^{\varepsilon}$, then $p \in \overline{D}^{Z}$.

Proof of Lemma 4.4. Write $S = Y \cap \mathbb{R}$ and $A = Y \setminus \mathbb{R}$. Fix an enumeration $E = \{e_n : n \in \omega\}$.

Case 1. $p \notin \overline{S}^{\varepsilon}$.

By Lemma 4.3, there is a neighborhood assignment $W: E \to \tau_Y$ such that the family $\{W(e): e \in E\}$ is closed discrete. For each $k \in \omega$ write

$$V_k = \{p\} \cup \{W(e_n) : n \ge k\},\$$

and define the topology of τ_Z as follows:

- (a) $\langle Y, \tau_Y \rangle$ is an open subspace of $\langle Z, \tau_Z \rangle$,
- (b) $\{V_k : k \in \omega\}$ is a neighborhood base of p in $\langle Z, \tau_Z \rangle$.

Then $\langle Z, \tau_Z \rangle$ clearly satisfies the requirements.

Case 2. $p \in \overline{S}^{\varepsilon}$.

Choose a sequence $P = \{p_n : n < \omega\} \subset S$ such that

$$\lim_{\varepsilon} \left\{ p_n : n < \omega \right\} = p,$$

and for each $D \in \mathcal{D}$, if $p \in \overline{D}^{\varepsilon}$, then $D \cap \{p_n : n < \omega\}$ is infinite.

Let U_n be a compact open neighborhood of p_n in τ_Y for $n \in \omega$ such that the family $\{U_n \cap \mathbb{R} : n < \omega\}$ converges to p in the Euclidean topology.

Since S is closed in Y, we have that $P \cup E$ is closed discrete in Y. Thus, by Lemma 4.3, there is a neighborhood assignment $W: P \cup E \to \tau_Y$ such that the family $\{W(x): x \in P \cup E\}$ is closed discrete. We can assume that $W(p_n) \subset U_n$.

For $k \in \omega$ write

$$V_k = \{p\} \cup \bigcup_{n \ge k} (W(p_n) \cup W(e_n)).$$

Define the topology of τ_Z as follows:

- (a) $\langle Y, \tau_Y \rangle$ is an open subspace of $\langle Z, \tau_Z \rangle$,
- (b) $\{V_k : k \in \omega\}$ is a neighborhood base of p in $\langle Z, \tau_Z \rangle$.

This construction clearly works.

Proof of Theorem 4.2. Let $\{D_{\zeta}: \zeta < \omega_1\} = [\mathbb{R}]^{\omega}$, $\{p_{\xi}: \xi < \omega_1\} = \mathbb{R}$, and $\{E_{\xi}: \xi < \omega_1\} = [\omega \times \omega]^{\omega}$.

We will define a sequence $\langle \langle X_{\alpha}, \tau_{\alpha} \rangle : \alpha \leq \omega_1 \rangle$ of countable, locally compact T_2 spaces such that

- (a) $X_{\alpha} = (\omega \times (\omega + 1)) \cup \{p_{\zeta} : \zeta < \alpha\},$
- (b) $\tau_{\beta} \cap \mathcal{P}(X_{\alpha}) = \tau_{\alpha} \text{ for } \alpha < \beta$,
- (c) $\omega \times \omega$ is dense in τ_{α} ,
- (d) E_{α} has an accumulation point in $\tau_{\alpha+1}$,
- (e) if $\zeta < \alpha$ and $D_{\zeta} \subset X_{\alpha}$ and $p_{\alpha} \in \overline{D_{\zeta}^{\varepsilon}}$, then $p_{\alpha} \in \overline{D_{\zeta}^{\tau_{\alpha+1}}}$.

We have $X_0 = (\omega \times (\omega + 1))$, and let topology τ_0 on $(\omega \times (\omega + 1))$ be the product topology.

In limit step, take the direct limit.

To get $X_{\alpha+1}$ from X_{α} apply Lemma 4.4 for $Y=X_{\alpha}, \mathcal{D}=\{D_{\xi}: \xi<\alpha\}, E=E_{\beta(\alpha)}$ and $p=p_{\alpha}$, where

$$\beta(\alpha) = \min\{\beta : E_{\beta} \text{ is closed discrete in } \tau_{\alpha}\}.$$

The space $\mathcal{X} = \langle X_{\omega_1}, \tau_{\omega_1} \rangle$ is clearly locally countable, locally compact, 0-dimensional T_2 with cardinality ω_1 . The subspace $\omega \times \omega$ is dense and relatively countably compact because every E_{α} has accumulation point, so X is DRC.

If $D \subset X_{\omega_1}$ is dense, then D should contain the isolated point: $\omega \times \omega \subset D$. Since $\{n\} \times \omega$ converges to $\langle n, \omega \rangle$, if D is countable compact, then $E = \{\langle n, \omega \rangle : n < \omega\} \subset D$. But E is closed discrete in \mathcal{X} , so D can not be countably compact. Thus, \mathcal{X} is $\neg DCC$.

Finally, if $D \in [\mathbb{R}]^{\omega_1}$, then D has a countable ε -dense subset D_{ζ} . Pick $\alpha > \zeta$ such that $p_{\alpha} \in D \cap \overline{D_{\zeta}}^{\varepsilon}$. Then $p_{\alpha} \in \overline{D_{\zeta}}^{\tau_{\omega_1}}$, so D is not discrete. Hence, $s(\mathcal{X}) = \omega$. \square

5. Problems.

By [4, 12.5], if X is a regular, feebly compact, first countable space with $|X| < \mathfrak{b}$, then X is countably compact.

Problem 5.1. In the statement above, is it necessary to assume that X is T_3 ? What about T_2 spaces?

Problem 5.2. Is there a regular, feebly compact, but not countably compact, first countable space with $|X| = \mathfrak{b}$ in ZFC?

Theorem 4.1 shows that the following question arises naturally.

Problem 5.3. Does the existence of an S-space imply the existence of a first countable P-NC space with countable spread?

In Theorem 1.2 we obtain only consistency.

Problem 5.4. Is there, in ZFC, a 0-dimensional R-hereditarily "DRC but $\neg DCC$ " T_2 space ?

Concerning the next problem, we have a consistency result without assuming the first countability (see Theorem 1.2).

Problem 5.5. Is it consistent that there exists a first countable, 0-dimensional T_2 , R-hereditarily ""DRC but $\neg DCC$ " space ?

We know that there exist arbitrarily large R-hereditarily "DCC but \neg CC" (or "pseudocompact, but \neg SP") spaces.

Problem 5.6. Are there arbitrarily large R-hereditarily "DRC but $\neg DCC$ " (or "SP but $\neg DRC$ ") spaces?

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