Identification of the Heat Transfer Coefficient Using an Inverse Heat Conduction Model

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Abstract. Inverse problems of recovering heat transfer coefficient from integral measurements are considered. The heat transfer coefficient occurs in the transmission conditions of imperfect contact type or the Robin type boundary conditions. It is representable as a finite part of the Fourier series with time dependent coefficients. The additional measurements are integrals of a solution multiplied by some weights. Existence and uniqueness of solutions in Sobolev classes are proven and the conditions on the data are sharp. These conditions include smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in additional measurements. The proof relies on a priori bounds and the contraction mapping principle. The existence and uniqueness theorems are local in time.

Introduction

Under consideration is a parabolic equation of the form

$$Mu = u_t - Lu = f$$
, $Lu = \sum_{i,j=1}^n a_{ij}(t,x)u_{x_ix_j} - \sum_{i=1}^n a_i(t,x)u_{x_i} - a_0(t,x)u$, (0.1)

where $x \in G$ and $G \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of class C^2 (see the definitions in [1, Ch. 1]), $t \in (0,T)$. Let $Q = (0,T) \times G$, $S = (0,T) \times \Gamma$.

This equation is a vital tool in scientific and engineering applications to assess and forecast temperature changes over time. According to Animasaun I.L. et al. (2022) [2], it is commonly used to model heat conduction, diffusion, and numerous dynamic thermal processes. The problems of identification of the heat transfer coefficients arise in various problems of mathematical physics (see [3, 4, 5, 6]): diagnostics and identification of heat transfer in

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supersonic heterogeneous flows, modeling and description of heat transfer in heat-shielding materials and coatings, thermal protection design and control of heat transfer regimes, modeling of properties and thermal processes in reusable thermal protection of aerospace vehicles, composite materials, ecology, etc.

Two inverse problems are examined. In the former case, the heat transfer coefficient defined on the interface occurs in a transmission condition of imperfect contact type and in the latter in the Robin boundary condition. The statements of the problems are as follows. In the former case, the domain G is divided into two open sets G^+ and G^- , $\overline{G^-} \subset G$, $\overline{G^+} \cup \overline{G^-} = \overline{G}$, $G^+ \cap G^- = \emptyset$. Let $\Gamma_0 = \partial G^+ \cap \partial G^-$, $S_0 = \Gamma_0 \times (0,T)$. The equation (0.1) is supplemented with the initial and boundary conditions

$$B(t,x)u|_S = g, \quad u|_{t=0} = u_0(x),$$
 (0.2)

where $Bu = \frac{\partial u}{\partial N} + \beta u$ or Bu = u, $\frac{\partial u}{\partial N} = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j}(t,x)n_i$, with $\vec{n} = (n_1, n_2, \dots, n_n)$ the outward unit normal to S, and the transmission conditions

$$\frac{\partial u^{+}}{\partial N}(t,x) - \sigma(t,x)(u^{+}(t,x) - u^{-}(t,x)) = g^{+}(t,x), \quad (t,x) \in S_{0},$$
 (0.3)

$$\frac{\partial u^{-}}{\partial N}(t,x) = \frac{\partial u^{+}}{\partial N}(t,x), \quad (t,x) \in S_{0}, \tag{0.4}$$

where $\frac{\partial u^{\pm}}{\partial N}(t,x_0) = \lim_{x \in G^{\pm}, x \to x_0 \in \Gamma_0} \sum_{i,j=1}^n a_{ij} u_{x_i} \nu_j$ (ν is the unit outward normal to ∂G^-) and $u^{\pm}(t,x_0) = \lim_{x \in G^{\pm}, x \to x_0 \in \Gamma_0} u(t,x)$. The inverse problem is to determine a solution u to the problem (0.1)-(0.4) and the heat transfer coefficient $\sigma = \sum_{i=1}^m q_i(t) \Phi_i(t,x)$, where the functions q_i are unknowns and $\{\Phi_i(t,x)\}$ are some basis functions. It is naturally to assume that they depend only on x but for the sake of generality we take them depending on all variables.

In the latter case, the inverse problem is to determine a solution u to the problem (0.1)-(0.2) and the function $\beta = \sum_{i=1}^m q_i(t)\Phi_i(t,x)$, where the functions q_i are unknowns and $\{\Phi_i(t,x)\}$ are some basis functions.

In both problems, the additional integral measurements to determine the coefficients σ or β look as follows:

$$\int_{G} u(t,x)\varphi_{k}(x)dx = \psi_{k}(t), \ k = 1, 2, \dots, m.$$
 (0.5)

The transmission conditions (0.3), (0.4) agree with the conventional imperfect contact condition at the interface (see [5]). If $\sigma \to \infty$ then we come to the diffraction problem (see [1, Chapter 3, Sect. 13]) in which $u^+ = u^-$ and $\frac{\partial u^+}{\partial N} = \frac{\partial u^-}{\partial N}$ on S_0 .

At present, there are many publications on the numerical solution of the problems of the type (0.1)-(0.5) or (0.1), (0.2), (0.5) in the various statements. The most usable statement provides the pointwise additional measurements, in this case the condition (0.5) is replaced with the conditions $u(t, b_i) =$ $\psi_i(t)$ $(j=1,2,\ldots,m,\ b_i\in G)$. It is often the case when the coefficient σ depends only on time [6, 7, 8, 9] or space variables [10, 11, 12, 13] (see also the bibliography and the results in [14]-[17]). In almost all papers, the problem is reduced to some optimal control problem and the minimization of the corresponding quadratic functional (see [6, 14, 8, 15, 7, 10, 11]. Let us describe some of the already addressed results. In the case of a sole space or time variable, the heat transfer coefficient depending on the temperature is recovered numerically with the use of pointwise measurements in [6]. In [14] the authors determine the heat transfer coefficients that depend in a special manner on the additional parameters from a collection of values of a solution at given points. In [16, 10] the Monte-Carlo method is employed to restore the heat transfer coefficient depending on two space variables. The values of a solution on a part of the boundary serve as the overdetermination conditions. The simultaneous recovering of a coefficient in a parabolic equation and the heat transfer coefficient is realized in [7]. The pointwise overdetermination conditions are also used in [15], [17]. In [17] under consideration is a onedimensional inverse problem of simultaneous recovering the heat flow on one of the lateral boundaries and the thermal contact resistance at the interface. The article [11] implements the numerical determination of the heat transfer coefficient from measurements on the available part of the outer boundary of the domain.

Several existence results are known if the pointwise ovedetermination conditions are used instead of those in (0.5). If the measurement points lie on the boundary of the domain and the heat transfer coefficient occurring in the boundary condition is determined then the existence and uniqueness theorems can be found in [21], [22], [23]. The same results were obtained if the measurement points lie at the interface. The inverse problem of determination of the interface heat transfer coefficient under certain conditions is well-posed and the most general existence and uniqueness theorems can be found in [19, 20]. If the measurement points lie in G then the problem be-

comes ill-posed. The conditions (0.5) were used in [24] and [25] to determine the heat flux on the outer boundary and existence and uniqueness theorems are proven. It is often the case when the integrals in (0.5) are taken over the boundary of a domain [26]-[27] and the heat transfer coefficient depending on time or space variables is determined. In these articles, the problem is reduced to some control problem which is studied theoretically and some existence theory is presented. But these control problems are not equivalent to the initial ones.

As for the problem (0.1)-(0.5) of recovering the interface heat transfer coefficient σ and the problem (0.1), (0.2), (0.5) of recovering the coefficient β , there are no theoretical results on solvability or uniqueness of solutions to this problem in the literature except for our articles [28], [29]. In contrast to other articles, we look for the heat transfer coefficient in the form of a finite segment of the Fourier series and this statement allows to obtain an approximation to the heat transfer coefficient depending on all variables and the accuracy of determination depends on just a number of measurements. This article is actually a survey of the results obtained in the articles [28], [29]. The conditions on the data are described which allow to state that there are existence and uniqueness theorems in Sobolev classes for solutions to the above problems. These conditions include smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in additional measurements. The proof relies on a priori bounds and the contraction mapping principle. The existence and uniqueness theorems are local in time.

1 Preliminaries

The Lebesgue spaces $L_p(G; E)$ and the Sobolev spaces $W_p^s(G; E)$, $W_p^s(Q; E)$ of vector-valued functions taking the values in a Banach space E (see the definitions in [30], [31]) are used in the article. The Sobolev spaces are denoted by $W_p^s(G)$, $W_p^s(Q)$, etc., whenever $E = \mathbb{R}^n$. The inclusion $u = (u_1, u_2, \ldots, u_k) \in W_p^s(G)$ for a vector-function means that every of the component u_i of u belongs to $W_p^s(G)$. By a norm of a vector, we mean the sum of the norms of its coordinates. The Hölder spaces $C^{\alpha}(\overline{G})$, $C^{\alpha,\beta}(\overline{Q})$, $C^{\alpha,\beta}(\overline{S})$ are defined in [1] (see also [30]). Given an interval J = (0,T), put $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G))$ and $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma))$.

All coefficients of L are real as well as the corresponding function spaces.

To simplify the exposition, we suppose below that p > n+2. Denote $(u,v) = \int_G u(x)v(x)dx$. Introduce the notations $Q^{\tau} = (0,\tau) \times G$, $S^{\tau} = (0,\tau) \times \Gamma$, $S_0^{\tau} = (0,\tau) \times \Gamma_0$, $Q^{\pm} = (0,T) \times G^{\pm}$, $Q_{\pm}^{\tau} = (0,\tau) \times G^{\pm}$. Let $B_{\delta}(b)$ be a ball centered at b of radius δ . The symbol $\rho(X,M)$ stands for the distance between the sets $X, M \subset \mathbb{R}^n$.

2 Identification of the interface heat transfer coefficient

Describe the conditions on the data ensuring solvability of the problem. The operator L is assumed to be elliptic, i. e., there exists a constant $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \ge \delta_0|\xi|^2 \ \forall \xi \in \mathbb{R}^n, \ \forall (t,x) \in Q.$$
 (2.1)

The conditions on the coefficients are as follows:

$$a_i \in L_p(Q) \ (i \ge 0), \ a_{ij} \in C(Q^{\pm}) \ (i, j = 1, \dots, n);$$
 (2.2)

the functions $a_{ij}|_{Q^{\pm}}$ admits extensions to continuous functions of class $C(\overline{Q^{\pm}})$ and

$$a_{ij}^{\pm}|_{S_0} \in W_p^{s_0,2s_0}(S_0), \ a_{ij}|_{Q^{\pm}} \in C([0,T];W_p^1(G^{\pm})), \ a_{ij}|_S \in W_p^{s_0,2s_0}(S), \ (2.3)$$

where i, j = 1, ..., n, $a_{ij}^{\pm}(t, x_0) = \lim_{x \in G^{\pm}, x \to x_0 \in \Gamma_0} a_{ij}(t, x)$, the last inclusion in (2.3) is fulfilled provided that $Bu \neq u$ in (0.2);

$$a_{ij}, a_k \in L_{\infty}(G; W_p^{s_0}(0, T)) \ (k = 0, 1, \dots, n, \ i, j = 1, \dots, n).$$
 (2.4)

The main conditions on the data are of the form

$$f \in L_p(Q), \quad u_0(x) \in W_p^{2-2/p}(G^{\pm}), \quad g \in W_p^{k_0, 2k_0}(S), \quad g^+ \in W_p^{s_0, 2s_0}(S_0), \quad (2.5)$$

where $k_0 = 1 - 1/2p$ in the case of Bu = u and $k_0 = 1/2 - 1/2p$ otherwise;

$$\beta \in W_p^{s_0, 2s_0}(S), \ g(0, x)|_{\Gamma} = B(0, x)u_0|_{\Gamma}, \ \frac{\partial u_0^+}{\partial N} = \frac{\partial u_0^-}{\partial N}, \ x \in \Gamma_0;$$
 (2.6)

$$\varphi_k|_{G^{\pm}} \in W^1_{\infty}(G^{\pm}), \ \Phi_k \in W^{s_0,2s_0}_p(S_0), \ \psi_k \in W^{s_0+1}_p(0,T),$$

$$(f,\varphi_k) \in W^{s_0}_p(0,T), \ k = 1, 2, \dots, m. \quad (2.7)$$

Assume that a pair (u, \vec{q}) , $\vec{q} = (q_1, q_2, \dots, q_m)$ is a solution to the problem (0.1)-(0.5). Multiply (0.1) by φ_i and integrate over G. Integrating by parts and using the transmission conditions, we infer

$$\psi_{i}'(t) + \sum_{j=1}^{m} q_{j}(t) \int_{\Gamma_{0}} \Phi_{j}(t, x) (u^{+}(t, x) - u^{-}(t, x)) (\varphi_{i}^{+}(x) - \varphi_{i}^{-}(x)) d\Gamma_{0} - \int_{\Gamma_{0}} \frac{\partial u}{\partial N} \varphi_{i}(x) d\Gamma + \int_{\Gamma_{0}} g^{+}(\varphi_{i}^{+}(x) - \varphi_{i}^{-}(x)) d\Gamma + a(u, \varphi_{i}) = \int_{G} f(t, x) \varphi_{i}(x) dx.$$

$$a(u, \varphi_{i})(t) = \sum_{k,l=1}^{n} \int_{G} a_{kl} u_{x_{l}} \varphi_{ix_{k}}(x) dG + \int_{G} (\sum_{k=1}^{n} a_{k} u_{x_{k}} + a_{0} u) \varphi_{i}(x) dG, \quad (2.8)$$

where $\varphi_k^{\pm}(x_0) = \lim_{x \to x_0, x \in G^{\pm}} \varphi_k(x)$. Define the function $\varphi_i^0(x) = \varphi_i^+(x) - \varphi_i^-(x)$ ($x \in \Gamma_0$). We would like to have that the system (2.8) is uniquely solvable relatively the vector-function \vec{q} , i. e., $|\det B(t)| \ge \delta_0 > 0 \ \forall t \in [0, T]$, where B(t) is the matrix with entries $\int_{\Gamma} \Phi_j(t, x) (u^+(t, x) - u^-(t, x)) (\varphi_i^+(x) - \varphi_i^-(x)) d\Gamma$. Let $B_0 = B(0)$. Taking t = 0, we obtain the condition

$$|\det B_0| \neq 0, \ b_{ij} = \int_{\Gamma} \Phi_j(0, x) (u_0^+(x) - u_0^-(x)) (\varphi_i^+(x) - \varphi_i^-(x)) d\Gamma.$$
 (2.9)

Let t = 0 in (2.8). We arrive at the system

$$\psi_{i}'(0) + \sum_{j=1}^{m} q_{j}(0) \int_{\Gamma_{0}} \Phi_{j}(0, x) (u_{0}^{+}(x) - u_{0}^{-}(x)) \varphi_{i}^{0}(x) d\Gamma_{0} - \int_{\Gamma_{0}} \frac{\partial u_{0}}{\partial N} \varphi_{i}(x) d\Gamma + \int_{\Gamma_{0}} g^{+}(0, x) \varphi_{i}^{0}(x) d\Gamma + a(u_{0}, \varphi_{i}) = (f(0, x), \varphi_{i}), \quad (2.10)$$

where i = 1, 2, ..., m. Under the condition (2.9), there exists a unique solution $(q_1(0), ..., q_m(0))$ to the system (2.10). Thus, we have determined the function $\sigma(0, x) = \sum_{i=1}^{m} q_i(0)\Phi_i(0, x)$. Taking t = 0 at (0.3), (0.5) and using the initial conditions (0.2), we come to the necessary consistency conditions

$$\frac{\partial u_0^+}{\partial N} - \sigma(0, x)(u_0^+ - u_0^-)\big|_{\Gamma} = g^+(0, x), \ \int_G u_0(x)\varphi_k(x) \, dx = \psi_k(0), \quad (2.11)$$

where k = 1, ..., m. The main result of this section is the following theorem.

Theorem 2.1 Let the conditions (2.1)-(2.7), (2.9), (2.11) hold. Then, on some segment $[0, \tau_0]$ ($\tau_0 \leq T$), there exists a unique solution (u, \vec{q}) ($\vec{q} = (q_1, \ldots, q_m)$) to the problem (0.1)-(0.5) such that $u|_{Q^{\pm}} \in W_p^{1,2}(Q_{\pm}^{\tau_0}), \vec{q} \in W_p^{s_0}(0, \tau_0)$.

Proof. Outline the proof (see [28]). Let a pair $u \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$, $\vec{q} \in W_p^{s_0}(0,T)$ be a solution to the problem (0.1)-(0.5). As before, we can find constants $q_i(0)$. Let $\sum_{i=1}^m q_i(0)\Phi_i(t,x) = \sigma_0(t,x)$ and denote by $v \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$ a solution to the auxiliary transmission problem

$$Mu = f(t, x), (t, x) \in Q, Bu|_{S} = g, u|_{t=0} = u_0,$$
 (2.12)

$$B^{+}u = \frac{\partial u^{+}}{\partial N} - \sigma_0(u^{+} - u^{-}) = g^{+}, \quad \frac{\partial u^{+}}{\partial N} = \frac{\partial u^{-}}{\partial N}, \quad (t, x) \in S_0$$
 (2.13)

whose solvability is established with the use of Theorem 1 in [19]. Make the change of variables u = v + w. Inserting this function u in (0.1) and involving the equation (2.12), we obtain that the function $w \in W_p^{1,2}(Q^+) \cap W_p^{1,2}(Q^-)$ is a solution to the problem

$$w_t - Lw = 0$$
, $Bw|_{\Gamma} = 0$, $\frac{\partial w^+}{\partial N} = \frac{\partial w^-}{\partial N}$, $w|_{t=0} = 0$,
 $\frac{\partial w^+}{\partial N} - \sigma_0(w^+ - w^-) = (\sigma - \sigma_0)(v^+ + w^+ - v^- - w^-)$. (2.14)

The condition (0.5) is rewritten as follows:

$$\int_{G} w\varphi_k(x) dx = \psi_k - \int_{G} v(t, x)\varphi_k(x) dx = \tilde{\psi}_k, \ k = 1, 2, \dots, m.$$
 (2.15)

In view of (2.7) and (2.11), $\tilde{\psi}_k(0) = 0$ and $\tilde{\psi}_k(t) \in W_p^1(0,T)$. Multiply the equation in (2.14) by $\varphi_k(x)$ and integrate over G. Integrating by parts yields

$$\tilde{\psi}_{i}'(t) + \sum_{j=1}^{m} \tilde{q}_{j}(t) \int_{\Gamma_{0}} \Phi_{j}(t,x) (w^{+}(t,x) - w^{-}(t,x) + v^{+}(t,x) - v^{-}(t,x)) \varphi_{i}^{0}(x) d\Gamma_{0} - v^{-}(t,x) + v^{-}(t,x) +$$

$$\int_{\Gamma} \frac{\partial w}{\partial N} \varphi_i(x) d\Gamma + \int_{\Gamma_0} \sigma_0(w^+(t,x) - w^-(t,x)) \varphi_i^0(x) d\Gamma + a(w,\varphi_i) = 0, \quad (2.16)$$

where i = 1, ..., m and $\tilde{q}_i = q_i - q_i(0)$. The equality (2.16) is rewritten as

$$\sum_{j=1}^{m} \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x) (v^+ - v^-) \varphi_i^0(x) d\Gamma_0 = -a(w,\varphi_i) + \int_{\Gamma} \frac{\partial w}{\partial N} \varphi_i(x) d\Gamma - \tilde{\psi}_i'(t) + \int_{\Gamma_0} \sigma_0(w^+ - w^-) \varphi_i^0(x) d\Gamma - \sum_{i=1}^{m} \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t,x) (w^+ - w^-) \varphi_i^0(x) d\Gamma_0.$$

and, thereby, we have the operator equation

$$B(t)\vec{q} = \vec{F}, \quad F_k = -a(w, \varphi_i) - \tilde{\psi}_i'(t) - \int_{\Gamma_0} \sigma_0(w^+ - w^-)\varphi_i^0(x) d\Gamma + \int_{\Gamma} \frac{\partial w}{\partial N} \varphi_i(x) d\Gamma + \sum_{j=1}^m \tilde{q}_j(t) \int_{\Gamma_0} \Phi_j(t, x) (w^+ - w^-)\varphi_i^0(x) d\Gamma_0,$$

where $\vec{F} = (F_1, \dots, F_m)^T$, $\vec{q} = (\tilde{q}_1, \dots, \tilde{q}_m)^T$ and B(t) is the matrix with entries $b_{ij} = \int_{\Gamma_0} \Phi_j(t, x) (v^+(t, x) - v^-(t, x)) \varphi_i^0(x) d\Gamma_0$. Moreover, $B(0) = B_0$ and the matrix B_0 is nondegenerate. The embedding theorems imply that $v \in C(\overline{Q})$, $\Phi_i \in C(\overline{S})$ (even more $v \in C^{1-(n+2)/2p,2-(n+2)/p}(\overline{Q})$) and thereby there exist parameters τ_0 and $\delta_1 > 0$ such that

$$|\det B(t)| \ge \delta_1 \ \forall t \in [0, \tau_0].$$

For $\tau \leq \tau_0$, we have that

$$\vec{q} = B^{-1}\vec{F} = R(\vec{q}) = \vec{g}_0 + R_0(\vec{q}),$$
 (2.17)

where $\vec{g}_0 = B^{-1}\vec{\Psi}$ and the kth coordinate Ψ_k of the vector $\vec{\Psi}$ is of the form $\Psi_k(t) = -\tilde{\psi}_k'(t)$. This equation is used to determine \vec{q} . It is not difficult to demonstrate that $\int_G v_t(t,x)\varphi_k(x) dx \in W_p^{s_0}(0,T)$ and, thus, $\tilde{\psi}_k(t) \in W_p^{1+s_0}(0,T)$, $\tilde{\psi}_k(0) = \tilde{\psi}_k'(0) = 0$. Next, using the conventional estimates for solutions to parabolic problems, we can show that the operator R is a contraction in the ball $B_{R_0} = \{\vec{q} \in \tilde{W}_p^{s_0}(0,\tau) : \|\vec{q}\|_{\tilde{W}_p^{s_0}(0,\tau)} \leq R_0\}$ and takes this ball into itself provided that the parameter τ is sufficiently small, where $R_0 = 2\|\vec{g}_0\|_{\tilde{W}_p^{s_0}(0,T)}$. The contraction mapping principle implies that the equation (3.16) is solvable locally in time (see the complete proof in [28]). Thus, the equation (3.16) is solvable and we have determined the vector \vec{q} . A solution w in this case is a solution to the problem (2.14). Validate the

conditions (2.15). Multiply the equation in (2.14) by φ_k and integrate the result over G. Integrating by parts yields

$$\int_{G} w_{t} \varphi_{k} dx + \sum_{j=1}^{m} \tilde{q}_{j}(t) \int_{\Gamma_{0}} \Phi_{j}(t, x) (w^{+}(t, x) - w^{-}(t, x) + v^{+}(t, x) - v^{-}(t, x)) \varphi_{i}^{0}(x) d\Gamma_{0}$$
$$- \int_{\Gamma} \frac{\partial w}{\partial N} \varphi_{i}(x) d\Gamma + \int_{\Gamma_{0}} \sigma_{0}(w^{+} - w^{-}) \varphi_{i}^{0}(x) d\Gamma + a(w, \varphi_{i}) = 0.$$

Subtracting this equality from (2.16), we infer

$$\int_{G} w_t \varphi_k \, dx = \tilde{\psi}'_k, \quad k = 1, \dots, m.$$

Integrating this equality with respect to t, we establish (2.15). The uniqueness of solutions follows from the estimates obtained in the proof and standard arguments.

Remark 2.2 Generally speaking, the interface Γ_0 as well as the outer boundary Γ can consist of several connectedness components. In particular, we can have several heat transfer coefficients occurring in different transmission conditions. The claim of the theorem remains valid under the same conditions.

3 Identification of the heat transfer coefficients in the Robin boundary condition

The problem (0.1), (0.2), (0.5) of recovering the coefficient β in the Robin boundary condition is considered. Proceed with the conditions on the data of the problem. They are quite similar to those in the previous section. The conditions on the coefficients are as follows:

$$a_{ij} \in C([0,T]; W_p^1(G)), \ a_{ij}|_S \in W_p^{s_0,2s_0}(S) \ (s_0 = 1/2 - 1/2p),$$
 (3.1)

$$a_{ij}, a_k \in L_{\infty}(G; W_p^{s_0}(0, T)) \ (k = 0, 1, \dots, n, i, j = 1, \dots, n, p > n+2).$$
 (3.2)

The main conditions on the data of the problem have the form

$$f \in L_p(Q), \ u_0(x) \in W_p^{2-2/p}(G), \ g \in W_p^{s_0,2s_0}(S).$$
 (3.3)

Write out the additional conditions

$$\varphi_k \in W^1_{\infty}(G), \ \Phi_k \in W^{s_0,2s_0}_p(S), \ \psi_k \in W^{s_0+1}_p(0,T),
(f,\varphi_k) \in W^{s_0}_p(0,T), \ k = 1, 2, \dots, m.$$
 (3.4)

Assume that a pair $u \in W_p^{1,2}(Q)$, $\vec{q} = (q_1, q_2, \dots, q_m)$ is a solution to the problem (0.1), (0.2), (0.5). Multiply (0.1) by φ_i and integrate over G. Integrating by parts, we infer

$$\psi_i'(t) + \sum_{j=1}^m q_j(t) \int_{\Gamma} \Phi_j(t, x) u(t, x) \varphi_i(x) d\Gamma + \sum_{k,l=1}^n \int_{G} a_{kl} u_{x_l} \varphi_{x_k i}(x) dG$$
$$- \int_{\Gamma} g(t, x) \varphi_i(x) d\Gamma + \int_{G} (\sum_{k=1}^n a_k u_{x_k} + a_0 u) \varphi_i(x) dG = (f, \varphi_i). \quad (3.5)$$

It is naturally to assume that this system is uniquely solvable relatively the vector-function \vec{q} . Thus, it is desirable to have that $|\det B(t)| \ge \delta_0 > 0 \ \forall t \in [0,T]$, where B(t) is the matrix with entries $\int_{\Gamma} \varphi_i(x) \Phi_j(t,x) u(t,x) d\Gamma$. At t=0 we must have

$$|\det B_0| \neq 0 \ B_0 = B(0), \ b_{ij} = \int_{\Gamma} \varphi_i(x) \Phi_j(0, x) u_0(x) d\Gamma.$$
 (3.6)

Taking t = 0 in (3.5), we arrive at the system

$$\psi_{i}'(0) + \sum_{j=1}^{m} q_{j}(0) \int_{\Gamma} \Phi_{j}(0, x) u_{0}(x) \varphi_{i}(x) d\Gamma + \sum_{k,l=1}^{n} \int_{G} a_{kl}(t, 0) u_{0x_{l}} \varphi_{x_{k}i}(x) dG$$
$$- \int_{\Gamma} g(0, x) \varphi_{i}(x) d\Gamma + \int_{G} (\sum_{k=1}^{n} a_{k} u_{0x_{k}} + a_{0} u_{0}) \varphi_{i}(x) dG = (f(0, x), \varphi_{i}), \quad (3.7)$$

where i = 1, 2, ..., m. Under the condition (3.6), there exists a unique solution $(q_1(0), ..., q_m(0))$ to this system. Thus, we have determined the function $\beta(0, x) = \sum_{i=1}^m q_i(0)\Phi_i(0, x)$. The consistency conditions at t = 0 provide the equalities

$$\frac{\partial u_0}{\partial N} + \beta(0, x)u_0\big|_{\Gamma} = g(0, x) \ (x \in \Gamma), \ \int_G u_0(x)\varphi_k(x) \ dx = \psi_k(0), \ k = 1, \dots, m;$$
(3.8)

The main result of this section is the following theorem.

Theorem 3.1 Let the conditions (2.1), (3.1)-(3.4), (3.6), (3.8) hold. Then on some segment $[0, \tau_0]$ ($\tau_0 \leq T$) there exists a unique solution (u, \vec{q}) ($\vec{q} = (q_1, \ldots, q_m)$) to the problem (0.1), (0.2), (0.5) such that $u \in W_p^{1,2}(Q^{\tau_0})$, $\vec{q} \in W_p^{s_0}(0, \tau_0)$.

Let a pair $u \in W_p^{1,2}(Q)$, $\vec{q} \in W_p^{s_0}(0,T)$ is a solution to the problem (0.1)-(0.5). In view of (3.6), we can find constants $q_i(0)$ from the system (3.7). Let $\sum_{i=1}^m q_i(0)\Phi_i(t,x) = \beta_0(t,x)$ and denote by $v \in W_p^{1,2}(Q)$ a solution to the problem

$$v_t - Lv = f$$
, $\frac{\partial v}{\partial N} + \beta_0(t, x)v|_{\Gamma} = g(t, x)$, $v|_{t=0} = u_0(x)$. (3.9)

Note that $\vec{q} \in W_p^{s_0}(0,T)$ and $\Phi_j \in W_p^{s_0,2s_0}(S)$ then $q_i(t)\Phi_i(t,x) \in W_p^{s_0,2s_0}(S)$, and $g \in W_p^{s_0,2s_0}(S)$ as well. Make the change of variables u = v + w. The function $w \in W_p^{1,2}(Q)$ is a solution to the problem

$$w_t - Lw = 0$$
, $\frac{\partial w}{\partial N} + \beta_0(t, x)w\big|_{\Gamma} = (\beta_0 - \beta)(v + w)$, $\omega|_{t=0} = 0$. (3.10)

The condition (0.5) is rewritten as follows:

$$\int_{G} w\varphi_k(x) dx = \psi_k - \int_{G} v(t, x)\varphi_k(x) dx = \tilde{\psi}_k, \ k = 1, 2, \dots, m.$$
 (3.11)

In view of (3.8), $\tilde{\psi}_k(0) = 0$ and at least $\tilde{\psi}_k(t) \in W_p^1(0,T)$. It is easy to demonstrate that $\tilde{\psi}_k(t) \in W_p^{1+s_0}(0,T)$ and, thus, $\int_G v_t(t,x)\varphi_k(x)\,dx \in W_p^{s_0}(0,T)$. Multiply the equation in (3.10) by $\varphi_k(x)$ and integrate over G. We obtain that $(w_t, \varphi_k) = (Lw, \varphi_k)$. Integrating by parts, we infer

$$\tilde{\psi}'_k(t) + a(w, \varphi_k) + \int_{\Gamma} \beta_0 w \varphi_k \, d\Gamma + \sum_{i=1}^m \tilde{q}_i(t) \int_{\Gamma} (v + w) \Phi_i \varphi_k \, d\Gamma = 0, \quad (3.12)$$

where i, k = 1, ..., m and $a(w, \varphi_k) = \int_G \sum_{i,j=1}^n a_{ij} \omega_{x_j} \varphi_{kx_i} + (\sum_{i=1}^n a_i \omega_{x_i} + a_0 \omega) \varphi_k dx$. The last equality is rewritten as

$$\sum_{i=1}^{m} \tilde{q}_{i}(t) \int_{\Gamma} \Phi_{i} \varphi_{k} v(t, x) d\Gamma = -\sum_{i=1}^{m} \tilde{q}_{i}(t) \int_{\Gamma} \Phi_{i} \varphi_{k} w d\Gamma - \tilde{\psi}'_{k}(t) - a(\omega, \varphi_{k}) - \int_{\Gamma} \beta_{0} w \varphi_{k} d\Gamma \quad (3.13)$$

and thereby

$$B(t)\vec{q} = \vec{F}, \ \vec{F} = (F_1, \dots, F_m)^T, \ \vec{q} = (\tilde{q}_1, \dots, \tilde{q}_m)^T,$$
 (3.14)

where $F_k = -\sum_{i=1}^m \tilde{q}_i(t) \int_{\Gamma} \Phi_i \varphi_k w \, d\Gamma - \tilde{\psi}_k'(t) - a(\omega, \varphi_k) - \int_{\Gamma} \beta_0 w \varphi_k \, d\Gamma$ and B(t) is the matrix with entries $b_{ij} = \int_{\Gamma} \Phi_j \varphi_i v(t, x) \, d\Gamma$. Note that $B(0) = B_0$ and the matrix B_0 is nondegenerate. The embedding theorems imply that $v \in C(\overline{Q})$, $\Phi_i \in C(\overline{S})$ (even more $v \in C^{1-(n+2)/2p,2-(n+2)/p}(\overline{Q})$) and thereby there exist parameters τ_0 and $\delta_1 > 0$ such that

$$|\det B(t)| \ge \delta_1 \quad \forall t \in [0, \tau_0]. \tag{3.15}$$

The function w in (3.14) is a solution to the problem (3.10). For $\tau \leq \tau_0$, we have that

$$\vec{q} = B^{-1}\vec{F} = R(\vec{q}) = \vec{g}_0 + R_0(\vec{q}),$$
 (3.16)

where $\vec{g}_0 = B^{-1}\vec{\Psi}$ and the k-th coordinate Ψ_k of the vector $\vec{\Psi}$ is of the form $\Psi_k(t) = -\tilde{\psi}_k'(t)$. We use this equation to determine \vec{q} . Next, using the known estimates for solutions to parabolic problems, we demonstrate that the operator R is a contraction in the ball $B_{R_0} = \{\vec{q} \in \tilde{W}_p^{s_0}(0,\tau) : \|\vec{q}\|_{\tilde{W}_p^{s_0}(0,\tau)} \le R_0\}$ and takes it into itself if the parameter τ is sufficiently small, where $R_0 = 2\|\vec{g}_0\|_{\tilde{W}_p^{s_0}(0,T)}$. The contraction mapping principle implies the solvability of the equation (3.16). We have determined the vector-function \vec{q} on some time segment. A solution w in this case is a solution to the problem (3.10). Show that the conditions (3.11) hold for a solution to the problem (3.10). Multiply the equation in (3.10) by φ_k and integrate the result over G. Integrating by parts, we infer

$$\int_{\Gamma} w_t \varphi_k \, dx + = -a(w, \varphi_k) - \int_{\Gamma} \beta_0 w \varphi_k \, d\Gamma - \sum_{i=1}^m \tilde{q}_i(t) \int_{\Gamma} (v+w) \Phi_i \varphi_k \, d\Gamma, \quad (3.17)$$

Subtracting this equality from (3.12), we conclude that

$$\int_{G} w_t \varphi_k \, dx = \tilde{\psi}'_k, \quad k = 1, \dots, m,$$

Integrating this equality with respect to t, we establish the equality (3.11). The uniqueness of solutions follows from the estimates obtained in the proof.

4 Discussion

We consider inverse problems of recovering the heat transfer coefficient from integral measurements. These problems arise in some practical applications, but there are no theoretical results concerning the existence and uniqueness questions. The results can be used in developing new numerical algorithms and provide new conditions of existence and uniqueness of solutions to these problems. We consider a model case, but it is clear what changes should be made in the general case for validating similar results. The main conditions on the data are conventional. The proof relies on a priori bounds and the contraction mapping principle. We think that the results will allow to establish some global existence and uniqueness results based on the maximum principle and additional conditions on the data. Similar results are valid in the Hölder spaces. But we think that from the viewpoint of applications it is better to deal with the Sobolev spaces.

5 Conclusions

The existence and uniqueness theorems in inverse problems of recovering the heat transfer coefficient from the integral measurements are proven locally in time. The heat transfer coefficient occurs in the transmission conditions of imperfect contact type. It is sought in the form of a finite segment of the Fourier series with coefficients depending on time. The proof relies on a priori bounds and fixed point theorem. The conditions on the data ensuring existence and uniqueness of solutions in Sobolev classes are sharp. They are smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in additional measurements.

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