# SUBGROUPS OF BOUNDED RANK IN HYPERBOLIC 3-MANIFOLD GROUPS

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ABSTRACT. We prove a finiteness theorem for subgroups of bounded rank in hyperbolic 3-manifold groups. As a consequence, we show that every bounded rank covering tower of closed hyperbolic 3-manifolds is a tower of finite covers associated to a fibration over a 1-orbifold.

### 1. Introduction

Suppose that M is an orientable 3-manifold and O is a 1-dimensional orbifold, both without boundary, so that O is homeomorphic to one of

(\$\dapprox\$) 
$$\mathbb{R}, \ \mathbb{R}/(x\mapsto -x), \ S^1, \ \text{or} \ S^1/(z\mapsto -z).$$

A fibration of M over O is a fiber orbibundle  $f: M \longrightarrow O$ , e.g. as defined in §3 of [4]. If  $p \in O$ , the preimage  $S = f^{-1}(p) \subset M$  is called a regular fiber or a singular fiber depending on whether p is a regular or singular point of O. Any singular fiber is a one-sided non-orientable surface embedded in M, while a regular fiber is a two-sided orientable surface. The preimage of any closed interval contained in the regular part of O is an embedded trivial interval bundle  $S \times [0,1] \hookrightarrow M$ , while the preimage of any closed interval containing a single singular point is a twisted interval bundle over the corresponding singular fiber. So, depending on the homeomorphism type of O as in  $(\diamond)$ , the manifold M is homeomorphic to either: a trivial interval bundle  $S \times \mathbb{R}$  over an orientable surface S, a twisted open interval bundle over a nonorientable surface, a mapping torus over an orientable surface, or a manifold obtained by gluing two copies of a twisted interval bundle over a non-orientable surface together along their boundaries.

A fibration of M over O gives a short exact sequence

$$(\heartsuit) 1 \longrightarrow \pi_1 S \longrightarrow \pi_1 M \longrightarrow Q \longrightarrow 1,$$

where S is a regular fiber, and the quotient Q is either trivial,  $\mathbb{Z}/2\mathbb{Z}$ , infinite cyclic or infinite dihedral, depending on O as in  $(\diamond)$ . We call the normal subgroup  $\pi_1 S \leq \pi_1 M$  the fiber subgroup; note that normality means that this subgroup is well-defined by the fibration, independent of basepoints.

Our main theorem is as follows.

**Theorem 1.1** (Finiteness for Bounded Rank Subgroups). Let M be an orientable, hyperbolizable 3-manifold with finitely generated fundamental group and no  $\mathbb{Z}^2$  subgroups in  $\pi_1 M$ . Given  $k \in \mathbb{N}$ , there is a finite set S of subgroups of  $\pi_1 M$ , such that for any subgroup  $H \leq \pi_1 M$  with rank at most k, we have either

(1)  $H = H_1 \star \cdots \star H_n \star F$ , where each  $H_i$  is conjugate in  $\pi_1 M$  to an element of our finite set of subgroups S, and  $F \leq G$  is free,

(2) the cover N corresponding to H is compact and fibers over a 1-orbifold, with fiber subgroup an element of S.

The proof uses technology developed by the author and Souto in [5]. Briefly, equip M with a convex-cocompact hyperbolic metric and let N be a cover of M such that  $\pi_1 N$  is freely indecomposable and has bounded rank. The main theorem of [5] says that the convex core of N decomposes as a union of product regions and building blocks. The product regions are homeomorphic to  $S \times I$ , in such a way that the level surfaces  $S \times t$  have bounded geometry, although the region itself can be very wide. The building blocks all have bounded diameter, and hence only finitely many topological types. We show that if N has a sufficiently wide product region, then it fibers as in (2), while otherwise, the convex core of N has bounded diameter, which implies  $\pi_1 N$  is one of finitely many subgroups  $H_i$ .

The case of Theorem 1.1 (1) where H has infinite index (or equivalently, the fact that there are only finitely many conjugacy classes of one-ended infinite index subgroups  $H \leq \pi_1 M$  of bounded rank) can be deduced from earlier arguments in Kapovich-Weidmann [13], specifically their Theorem 7.5. Very recently, Weidmann-Weller [20] have extended Kapovich-Weidmann's work to apply to M that have rank 2 cusps. At heart, all these arguments are similar, in that Kapovich-Weidmann and Weidmann-Weller use an algebraic version of the 'carrier graphs' that are the essential tool in part of Biringer-Souto [5]. However, we do not know how to prove Theorem 1.1 in full without relying on the rest of [5]. Also, the proof of Theorem 1.1 via the machinery of [5] is quick and geometrically transparent.

As an application, we can characterize infinite chains of bounded rank subgroups of closed hyperbolic 3-manifold groups. If M is a closed 3-manifold that fibers over a 1-orbifold, then we have the short exact sequence  $(\heartsuit)$ , with  $Q = \mathbb{Z}$  or  $D_{\infty}$ , and any chain of finite index subgroups in Q pulls back to a chain of finite index subgroups of G with uniformly bounded rank. We show that this is the only way to produce infinite chains of bounded rank.

Corollary 1.2. Suppose that M is a closed, orientable hyperbolic 3-manifold, and

$$\pi_1 M \geq H_1 > H_2 > \cdots$$

is a chain of finite index subgroups such that  $\sup_i \operatorname{rank} H_i < \infty$ . Then after passing to a subsequence,  $H_1$  is fibered over  $S^1$  or  $S^1/(z \mapsto -z)$  with associated SES

$$1 \longrightarrow \pi_1 S \longrightarrow H_1 \longrightarrow Q \longrightarrow 1$$
,

and  $(H_i)$  is a chain of finite index subgroups of  $H_1$  that is the preimage of a chain of finite index subgroups of Q.

*Proof.* Say  $M, H_i$  are as above. Since the subgroups  $H_i$  are finite index in M, are 1-ended, and hence do not split nontrivially as free products. As the index of  $H_i$  increases with i, none of these groups are conjugate to each other, so Theorem 1.1 says that there are only finitely many  $H_i$  that do not fiber over a 1-orbifold with fiber subgroup an element of S. Remove these finitely many  $H_i$ , and pass to a subsequence so that for each i, the fiber subgroup is some fixed  $K \leq \pi_1 M$ .

For context, let M be a closed 3-manifold. Lackenby [15] defined the rank gradient of a chain  $\pi_1 M \geq H_1 \geq H_2 \geq \cdots$  of finite index subgroups to be the limit

$$RG(H_i) := \lim_{i \to \infty} \frac{\operatorname{rank}(H_i) - 1}{[\pi_1 M : H_i]}.$$

Multiple authors have studied chains where  $RG(H_i) = 0$ , i.e. where the rank grows sublinearly in the index. For instance, DeBlois–Frield–Vidussi [9] have shown that if  $\phi: \pi_1 M \longrightarrow \mathbb{Z}$  is a homomorphism, the chain  $H_i = \phi^{-1}(i\mathbb{Z})$  has zero rank gradient if and only if  $\phi$  is the surjection in the short exact sequence ( $\heartsuit$ ) associated to a fibration of M over the circle. In general, if M is hyperbolic then every known example of a chain of subgroups of  $\pi_1 M$  with zero rank gradient is constructed by modifying a chain coming from a fibration over a 1-orbifold. It is unknown whether there are qualitatively different examples. For instance, let  $(H_i)$  be a chain of subgroups of  $\pi_1 M$ , let  $N_i$  be the associated tower of covers of M, and let  $h(N_i)$  be their Cheeger constants. Is it true that  $RG(H_i) = 0$  implies  $h(N_i) \to 0$ ? It is easy to see that the Cheeger constant goes to zero in a chain coming from a fibration. A positive answer to this question would also disprove Gaboriau's Fixed Price Conjecture in measurable dynamics, via work of Abert-Nikolov [1].

- 1.1. **Organization.**  $\S 2$  contains background necessary for the proof. In particular, in  $\S 2.1$  we review some hyperbolic geometry and prove a lemma that allows us to recognize when an embedded incompressible surface represents a fiber in a fibration of M over a 1-orbifold, while in  $\S 2.2$  we review the main theorem of Biringer-Souto [5]. The proof of Theorem 1.1 is presented in  $\S 3$ .
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#### 2. Background

2.1. Hyperbolic geometry. A (complete) hyperbolic 3-manifold is a quotient  $M = \Gamma \backslash \mathbb{H}^3$ , where  $\Gamma$  acts freely and properly discontinuously by isometries. A topological 3-manifold is called hyperbolizable if it is homeomorphic to a hyperbolic 3-manifold as above. If M is hyperbolic, the convex core of M is the quotient

$$CC(M) := \Gamma \backslash CH(\Lambda(\Gamma)),$$

where  $\Lambda(\Gamma) \subset \partial \mathbb{H}^3$  is the *limit set* of  $\Gamma$ , and  $CH(\cdot)$  denotes the hyperbolic convex hull. See e.g. [16, 3] for details. Since CC(M) may be less than 3-dimensional, it is sometimes more convenient to work with its closed 1-neighborhood  $CC_1(M)$ .

If M is an orientable hyperbolic 3-manifold with finitely generated fundamental group, the Tameness Theorem of Agol [2] and Calegari-Gabai [7] says that M is homeomorphic to the interior of a compact 3-manifold with boundary. Equivalently, the ends of M are all 'tame', i.e. they all have neighborhoods homeomorphic to  $S \times (0, \infty)$ , where S is a closed orientable surface. Assuming for simplicity that M has no cusps, work of Thurston, Bonahon and Canary [6, 8] implies that each end  $\mathcal{E}$  of M is either convex cocompact, meaning that it has a neighborhood that lies outside CC(M), or degenerate, meaning that it has a neighborhood  $U \cong S \times (0, \infty)$  that contains a sequence of bounded area surfaces  $f_i: S \longrightarrow U$  in the homotopy class of a level surface, where  $f_i$  exits the end as  $i \to \infty$ . See e.g. [16] for more details and precise definitions. As an example, if M fibers over the circle with fibers homeomorphic to a surface S, then the obvious infinite cyclic cover N of M is homeomorphic to  $S \times \mathbb{R}$  and both ends of N are degenerate: one can take the required bounded area surfaces to be all the lifts of a fixed surface  $S \longrightarrow M$  in the homotopy class of the fiber.

Suppose that M is an orientable hyperbolic 3-manifold and  $S \hookrightarrow M$  is an immersed  $\pi_1$ -injective closed surface. The cover  $M_S$  of M corresponding to  $\pi_1 S$  is homeomorphic to  $S \times \mathbb{R}$ ; this follows from the Tameness Theorem and some standard 3-manifold topology arguments, c.f. [12]. We call  $M_S$  doubly degenerate if  $M_S$  has two degenerate ends; in this case we also call S 'doubly degenerate'.

**Lemma 2.1.** Suppose that M is an orientable hyperbolic 3-manifold and  $S \hookrightarrow M$  is a doubly degenerate, <u>embedded</u>  $\pi_1$ -injective closed orientable surface. Then S is a regular fiber in a fibration of M over a 1-orbifold.

*Proof.* Let  $M_S$  be the cover of M corresponding to S, so  $M_S$  is a doubly degenerate hyperbolic 3-manifold homeomorphic to  $S \times \mathbb{R}$ . Thurston's covering theorem [18, Theorem 9.2.2] says that either  $\pi: M_S \longrightarrow M$  is finite-to-one, or  $\pi$  factors as

$$(1) M_S \longrightarrow N \stackrel{\rho}{\longrightarrow} M,$$

where the first map is a cyclic covering map onto a closed hyperbolic 3-manifold fibering over the circle and the second map is a finite cover.

Suppose first that  $\pi: M_S \longrightarrow M$  is finite-to-one. Then  $\pi_1 M$  has a finite index surface subgroup, so it is finitely generated and does not split as a free product. Hence M is tame and any standard compact core  $C \subset M$  has incompressible boundary. Here, a 'standard compact core' C is a compact submanifold such that  $M \setminus int(C)$  is homeomorphic to  $\partial C \times [0, \infty)$ . Each component  $T \subset \pi^{-1}(\partial C)$  is then an incompressible, embedded closed surface in  $M_S \cong S \times \mathbb{R}$ , and therefore is a 'level surface', isotopic to  $S \times \{t\}$ . The restriction  $\pi|_T$  is an embedding: if not, it would nontrivially cover a component of  $\partial C$ , but since  $\pi|_T$  is homotopic in M to the embedded surface S, work of Freedman-Hass-Scott (c.f. Lemma 3.1 in [5]) implies that  $\pi|_T$  is homotopic to an embedding within an arbitrarily small neighborhood of its image, which is impossible since components of  $\partial C$  are two-sided, and hence their regular neighborhoods are products. Since  $\pi^{-1}(C)$  is connected and is bounded by level surfaces, it must be a trivial interval bundle bounded by  $\pi^{-1}(\partial C)$ , which has two components. Since each of these components embeds under  $\pi$ , the covering map  $\pi:\pi^{-1}(C)\longrightarrow C$  is either a homeomorphism or is 2-1, depending on whether the two components of  $\pi^{-1}(\partial C)$  have distinct  $\pi$ -images or not. In the first case, C is a trivial interval bundle, and hence M is an open trivial interval bundle with Sa fiber, while in the second case Proposition 4.1 in [19] implies that M is an open twisted interval bundle with S a regular fiber. So, we're done.

Now suppose that  $\pi$  factors as in (1). Every component of  $\rho^{-1}(S) \subset N$  is an incompressible embedded surface that is homotopic to a finite cover of the fiber of N, so every component is actually in the homotopy class of the fiber. Hence  $\rho^{-1}(S)$  cuts N into a collection of trivial interval bundles  $S \times [0,1]$ , and each component of  $\rho^{-1}(S)$  projects homeomorphically into M. As in the previous paragraph, this implies that S cuts M into pieces that are either trivial interval bundles  $S \times [0,1]$  or twisted interval bundles with S a regular fiber. So, M fibers.

2.2. Thick manifolds with bounded rank. In this subsection we review some material from [5]. Let M be an orientable hyperbolic 3-manifold. A product region in M is the image  $U \subset M$  of a proper embedding

$$\Sigma_q \times I \longrightarrow M, \quad I = [0, 1], [0, \infty), \text{ or } (-\infty, \infty),$$

such that for some regular neighborhood  $\mathcal{N}(U) \supset U$ , we have:

- (1) every point  $p \in U$  is in the image of a NAT simplicial ruled surface  $\Sigma_g \longrightarrow \mathcal{N}(U)$  that is a homotopy equivalence,
- (2) each component  $S \subset \partial U$  lies in the 1-neighborhood of another such NAT simplicial ruled surface.

Here, a simplicial ruled surface (SRS) is a map from a triangulated surface  $\Sigma$ , where edges map to geodesics and where the image of each triangle is foliated by geodesics. Equipping  $\Sigma_g$  with the pullback metric, we say that the SRS is NAT (or 'not accidentally thin') if there is no simple closed curves on  $\Sigma_g$  with length less than  $\epsilon$  that is nullhomotopic in M. The reader can see [5, §5] for precise definitions, but the main point is that NAT SRSs  $\Sigma_g \longrightarrow M$  in  $\epsilon$ -thick<sup>1</sup> hyperbolic 3-manifolds M satisfy a 'bounded diameter lemma', i.e. the diameter of  $\Sigma_g$  in the pullback metric is bounded above by a constant depending only on  $g, \epsilon$ . See §5.5 of [5]. For the purposes of this paper, it's sufficient to think of a product region as just a submanifold homeomorphic to a surface cross an interval, where the geometry is bounded in the surface direction. The phrasing with SRSs is just a useful way to formulate this without specifying a priori what 'bounded' means.

In [5, Theorem 13.1], the authors prove the following geometric decomposition theorem for convex cores of  $\epsilon$ -thick hyperbolic 3-manifolds with bounded rank.

**Theorem 2.2.** Fix  $k \in \mathbb{N}$  and some sufficiently small  $\epsilon > 0$ . Then there are constants n = n(k), g = g(k),  $B = B(k, \epsilon)$  as follows.

Suppose M is a complete, orientable hyperbolic 3-manifold with

$$\operatorname{rank}(\pi_1(M)) \le k, \quad \operatorname{inj}(M) \ge \epsilon,$$

and assume that  $\pi_1 M$  is freely indecomposable. Then CC(M) contains a collection  $\mathcal{U}$  of at most n product regions, each with genus at most g, such that every component of  $CC_1(M) \setminus (\bigcup_{u \in \mathcal{U}} int(U))$  has diameter at most B.

The statement above is slightly different from that given in [5, Theorem 13.1]. First, the 'freely indecomposable' assumption in the statement above implies that there are no essential simple closed curves on  $\partial CC(M)$  that are compressible in M; this is a stronger version of a related hypothesis in the statement in [5]. Also, in [5] the authors work with the convex core and its interior rather than the 1-neighborhood and the convex core itself, under a standing assumption that the convex core is 3-dimensional, but the version above follows formally from theirs.

## 3. The proof of Theorem 1.1

Fix an orientable hyperbolic 3-manifold M with finitely generated fundamental group and no rank two cusps. Here's what we will actually prove.

**Theorem 3.1.** Given  $k \in \mathbb{N}$ , there is some g = g(k) such that there are only finitely many isomorphism types of subgroups  $H \leq \pi_1 M$  such that

- (a) H is freely indecomposable and rank(H) < k,
- (b) it's not the case that M is compact,  $H \leq \pi_1 M$  has finite index, and the associated cover N of M fibers over a compact 1-orbifold with regular fiber of genus at most g.

Let's show how to derive Theorem 1.1 from Theorem 3.1.

<sup>&</sup>lt;sup>1</sup>The *injectivity radius* of a hyperbolic 3-manifold M, written  $\operatorname{inj}(M)$ , is the half the length of the shortest homotopically essential loop in M, and M is called  $\epsilon$ -thick if  $\operatorname{inj}(M) \geq \epsilon$ .

Proof of Theorem 1.1. Let  $S^1$  be a minimal set of subgroups of G representing all the conjugacy classes of one-ended subgroups  $H \leq G$  with rank at most k that satisfy (b) above. When M is noncompact, we set  $S = S^1$ . Otherwise, we set S to be the union of  $S^1$  with all doubly degenerate closed surface subgroups of G that have genus at most g.

Since G is a hyperbolic group, for any fixed finitely presented one-ended hyperbolic group H, there are a finite number of conjugacy classes of subgroups of G isomorphic to H, see Delzant [10]. Any one-ended group is freely indecomposable, so by this and Theorem 3.1, the set  $\mathcal{S}^1$  is finite. Delzant's theorem also implies that up to conjugacy, there are only finitely many doubly degenerate closed surface subgroups  $K \leq G$  with genus at most g. However, if M is compact, Thurston's covering theorem [18, Theorem 9.2.2] implies that any such  $K \leq G$  is a normal subgroup of a finite index subgroup  $H_K \leq G$ . Since there only finitely many conjugacy classes of such K, we can take the indices  $[G:H_K]$  to be bounded, and hence finitely many  $H_K$  suffice, implying that there are only finitely many  $K \leq G$ , even without identifying conjugates. So,  $\mathcal{S}$  is always finite.

Now suppose  $H \leq G$  has rank at most k. Since G is torsion free, it follows from Grushko's Theorem and Stallings' Theorem, c.f. [17], that H can be written as

$$H = H_1 \star \cdots \star H_n \star F$$
,

where the  $H_i$  are one-ended, F is a free group, and all these free factors have rank at most k. If all the  $H_i$  satisfy (b) above, they are conjugate into  $\mathcal{S}$  and we're done. Otherwise, some  $H_i$  has finite index, so the free product decomposition must be trivial, i.e.  $H = H_i$ , and the cover  $N \longrightarrow M$  corresponding to H fibers over a compact 1-orbifold with regular fiber a surface of genus at most g, and the fiber subgroup of H lies in  $\mathcal{S}$  by construction.

The rest of the section is devoted to the proof of Theorem 3.1. Since the conclusion of the theorem is topological, we may assume that M is convex co-compact, say by Thurston's Haken hyperbolization theorem [14] in the noncompact case.

Claim 3.2. There is some L = L(g, M) as follows. Suppose that  $N \longrightarrow M$  is a locally isometric covering map and  $U \subset N$  is a product region with genus at most g. Then either width $(U) \leq L$ , or there is a fibration of N over a 1-dimensional orbifold where U is a collar neighborhood of a regular fiber.

In the statement of the claim, if U is a compact product region, width (U) is defined to be the infimal length of a path in U between the two boundary components of U. If U is noncompact, we set width  $(U) := \infty$ .

*Proof.* It suffices to prove the claim for product regions of fixed genus g. Hoping for a contradiction, let  $\rho_i: N_i \longrightarrow M$  be a sequence of locally isometric covers containing product regions  $U_i \subset N_i$  with genus g, where width $(U_i) \to \infty$ , and where no  $U_i$  is a collar neighborhood of a regular fiber in a fibration of  $N_i$ .

Pick base points  $p_i \in U_i$  such that  $d(p_i, \partial U_i) \to \infty$ . By Lemma 6.20<sup>2</sup> in [5], we can assume after passing to a subsequence that the sequence  $(N_i, p_i)$  converges geometrically to a pointed doubly degenerate hyperbolic 3-manifold  $(N_{\infty}, p_{\infty})$ , where

<sup>&</sup>lt;sup>2</sup>In [5], results are often stated using 'width relative to the  $\epsilon$ -thin part' instead of width as defined here. However, in the current setting we are only looking at covers of a fixed convex cocompact M, so if  $\epsilon$  is chosen smaller than the injectivity radius of M then width and relative width coincide.

 $N_{\infty} \cong \Sigma_g \times \mathbb{R}$ . Moreover, if we choose level surfaces  $S_i \subset U_i$  at bounded distance from  $p_i$  (see Lemma 6.5 of [5]), then if  $(\phi_i)$  is a sequence of almost isometric maps witnessing the geometric convergence (see Definition 9.1 of [5]), for large i we have that the image of  $\phi_i$  contains  $S_i$  and  $\phi_i^{-1}(S_i)$  is a level surface in  $N_{\infty}$ .

By Arzela-Ascoli, after passing to a subsequence we can assume that the maps  $\rho_i \circ \phi_i$  converge to a locally isometric covering map  $\rho_\infty : N_\infty \longrightarrow M$ . It follows that for large i, the groups  $(\rho_i)_*(\pi_1 S_i)$  and  $(\rho_\infty)_*(\pi_1 N_\infty)$  are conjugate in  $\pi_1 M$ . So,  $S_i$  is a doubly degenerate incompressible embedded surface in  $N_i$ , and hence is a regular fiber in some fibration over a 1-orbifold by Lemma 2.1, a contradiction.  $\square$ 

So, let  $\epsilon > 0$  be smaller than the injectivity radius of M, and small enough so that Theorem 2.2 holds. Let g = g(k) be as in Theorem 2.2. Let N be a cover of M with rank $(N) \leq k$ , with  $\pi_1 N$  freely indecomposable, and assume that it's not the case that N is compact and fibers over a 1-orbifold with regular fiber a genus at most g surface. We will show that N is homotopy equivalent to a simplicial complex with at most V = V(k, M) simplices; it will follow that  $\pi_1 N$  has one of finitely many isomorphism types.

First, it could be that N is noncompact, but fibers over a 1-orbifold with a regular fiber of genus at most g. In this case, N is homotopy equivalent to a (possibly non-orientable) surface with bounded complexity, so we're done. We may then assume that N does not fiber over a 1-orbifold with regular fiber of genus at most g, in which case Claim 3.2 says that all genus at most g product regions in N have width at most some L = L(k, M). Since the width of a thick product region bounds its diameter [5, Fact 6.4], by Theorem 2.2 we have that  $CC_1(N)$  has diameter at most some constant D = D(k, M). Pick a maximal  $\epsilon/2$ -separated set  $S \subset CC_1(N)$ . The number of such points is at most linear is vol  $CC_1(N)$ , which is bounded above by a function of D. Let  $\mathcal N$  be the nerve complex of the set of  $\epsilon$ -balls around points in S. Then  $\mathcal N$  has bounded degree, so the number of simplices in  $\mathcal N$  is bounded by some V = V(k, M). By the nerve lemma (c.f. Corollary 4G.3 in [11]),  $\mathcal N$  is homotopy equivalent to the union of all the associated  $\epsilon$ -balls. This union deformation retracts onto  $CC_1(N)$ , via the closest point retraction, and N does as well, so N and  $\mathcal N$  are homotopy equivalent as desired.

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