

STABLE MINIMAL HYPERSURFACES IN \mathbf{R}^5

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ABSTRACT. We show that a complete, two-sided stable minimal hypersurface in \mathbf{R}^5 is flat.

1. INTRODUCTION

A two-sided immersion $M^n \rightarrow \mathbf{R}^{n+1}$ is *minimal* if its mean curvature vector vanishes. A minimal immersion is *stable* if

$$\int_M |A_M|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2$$

for all $\varphi \in C_c^\infty(M)$, where A_M is the second fundamental form of the immersion. The stable Bernstein problem asks whether a complete two-sided stable minimal hypersurface in \mathbf{R}^{n+1} must be a flat affine hyperplane. We resolve here the stable Bernstein problem in \mathbf{R}^5 :

Theorem 1.1. *A complete, connected, two-sided stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$ is a flat affine hyperplane.*

The stable Bernstein problem was resolved in \mathbf{R}^3 by do Carmo–Peng, Fischer–Colbrie–Schoen, and Pogorelov [dCP79, FCS80, Pog81] and recently in \mathbf{R}^4 by the first- and second-named authors [CL24b] (subsequently, two alternative proofs in \mathbf{R}^4 were found in [CL23] and [CMR24]). After the first version of this paper appeared, Mazet was able to refine our methods to resolve the stable Bernstein problem in \mathbf{R}^6 in the affirmative [Maz24] (see Remark 1.7). The stable Bernstein problem remains open in \mathbf{R}^7 but holds assuming M has (extrinsic) Euclidean volume growth by work of Schoen–Simon and Simons [SS81, Sim67] (for embeddings) and Bellettini [Bel25] (for immersions). On the other hand, non-flat stable (area-minimizing) minimal immersions in \mathbf{R}^8 (and beyond) were found by Bombieri–de Giorgi–Giusti [BDGG69].

It is well known that the validity of the stable Bernstein property is equivalent to an *a priori* interior curvature bound for stable minimal hypersurfaces. We recall that a two-sided minimal immersion into a Riemannian manifold $M^n \rightarrow (X^{n+1}, g)$ is stable if

$$\int_M (|A_M|^2 + \text{Ric}_g(\nu, \nu)) \varphi^2 \leq \int_M |\nabla \varphi|^2$$

for all $\varphi \in C_c^\infty(M)$, where A_M is the second fundamental form of M and $\text{Ric}_g(\nu, \nu)$ is the Ricci curvature of the ambient metric g in the normal direction. As in [CLS22, Corollary 2.5], Theorem 1.1 implies interior curvature estimates for stable minimal immersions in 5-manifolds that only depend on a norm for the ambient sectional curvature.

Corollary 1.2. *Let (X^5, g) be a complete Riemannian manifold with bounded sectional curvature $|\text{sec}_g| \leq K$. Then any compact, two-sided stable minimal immersion $M^4 \rightarrow (X^5, g)$ satisfies*

$$|A_M|(q) \min\{1, d_M(q, \partial M)\} \leq C(K)$$

for all $q \in M$.

To prove Theorem 1.1, show that a complete, two-sided, stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has intrinsic Euclidean volume growth. In fact, as in [CL23], this can be localized in the spirit of Pogorelov's area bounds [Pog81] for stable minimal surfaces in \mathbf{R}^3 .

Theorem 1.3. *Let $F : M^4 \rightarrow \mathbf{R}^5$ be a simply connected, two-sided stable minimal immersion so that $F(x_0) = 0 \in \mathbf{R}^5$ for some $x_0 \in M$, ∂M is connected, and $F : M \rightarrow B_{\mathbf{R}^5}(0, 1)$ is proper. Then*

$$\mathcal{H}^4(M_{\rho_0}^*) \leq 8\pi^2,$$

where $M_{\rho_0}^*$ is the connected component of $F^{-1}(B_{\mathbf{R}^5}(0, \rho_0))$ containing x_0 and $\rho_0 = e^{-11\pi}$.

Similarly, we can give a geometric characterization of minimal hypersurfaces in \mathbf{R}^5 with finite Morse index, generalizing the well-known results of Gulliver, Fischer-Colbrie, and Osserman in \mathbf{R}^3 [Gul86, FC85, Oss64]. See [CL24b] for the corresponding result in \mathbf{R}^4 . Recall that a two-sided minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has *finite Morse index* if

$$\sup\{\dim V : V \subset C_c^\infty(M) \text{ is a subspace with } Q(f, f) < 0 \text{ for all } 0 \neq f \in V\} < \infty$$

where $Q(f, f) = \int_M |\nabla f|^2 - \int_M |A_M|^2 f^2$. We additionally recall [Sch83, Section 2] that an end E of a minimal immersion $M^4 \rightarrow \mathbf{R}^5$ is *regular at infinity* if it is contained in the graph of a function w on a hyperplane Π with asymptotics

$$w(x) = b + a|x|^{-2} + \sum_{j=1}^4 c_j x_j |x|^{-4} + O(|x|^{-4})$$

where a, b, c_1, \dots, c_4 are constants and x_1, \dots, x_4 are the Euclidean coordinates on Π .

Theorem 1.4. *A complete, two-sided minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has finite Morse index if and only if it has finite total curvature, i.e., $\int_M |A_M|^4 < \infty$, in which case M is properly immersed, has finitely many ends, each end of M is regular at infinity, and M has Euclidean volume growth, namely $|M \cap B_{\mathbf{R}^5}(0, \rho)| \leq C\rho^4$ for all $\rho > 0$; here $C = C(M)$.*

The results in this paper should be relevant to the study of stable/finite index minimal hypersurfaces in 5-manifolds in a manner similar to [CLS22].

1.1. Discussion of results and methods. Let $M^n \rightarrow \mathbf{R}^{n+1}$ be a complete, two-sided stable minimal immersion. The main difficulty in resolving the stable Bernstein problem is that the extrinsic and intrinsic geometry of $M^n \rightarrow \mathbf{R}^{n+1}$ could be *a priori* very complicated. For example, if g is the induced metric on M , the manifold (M, g) might have exponential volume growth. Furthermore, stability of the immersion does not directly imply any pointwise curvature condition on g , while minimality only implies that g has *non-positive* scalar curvature via the traced Gauss equation.

The strategy used in this article is motivated by the one developed by the first- and second-named authors for $M^3 \rightarrow \mathbf{R}^4$ in [CL23]. More precisely, we let g denote the induced metric on M and consider the conformally changed metric $\tilde{g} = r^{-2}g$, where r is the Euclidean distance function from the origin. This conformal change was first considered by Gulliver–Lawson in their study of isolated singularities in stable minimal hypersurfaces [GL86]. Note that if M is a flat hyperplane containing the origin, then (M, \tilde{g}) will be the standard round cylinder $\mathbf{R} \times \mathbf{S}^{n-1}$.

A key insight of Gulliver–Lawson is that \tilde{g} has uniformly positive scalar curvature in a weak spectral sense; namely, we have

$$(1.1) \quad -\tilde{\Delta} + \frac{1}{2} \left(\tilde{R} - \frac{n(n-2)}{2} \right) \geq 0,$$

where \tilde{R} is the scalar curvature and $\tilde{\Delta}$ is the (nonpositive) Laplace–Beltrami operator of \tilde{g} . It is now well known that a 3-manifold N^3 with uniformly positive scalar curvature has *macroscopic dimension one* in several ways:

- *Distance-sense*: If N is simply connected, then N has bounded Urysohn 1-width, see [Kat88] and [LM23].
- *Area-sense*: N admits equally spaced separating surfaces with uniformly bounded area; see [CL24a] and [CL23].
- *Volume-sense*: If N has nonnegative Ricci curvature, then N has linear volume growth, see [MW24] and [CLS23].

Since each of these results can be proved using μ -bubbles—a tool introduced by Gromov [Gro96] (see also [Gro18])—and since the weaker condition (1.1) suffices to carry out the μ -bubble argument, these observations hold for 3-manifolds satisfying (1.1). In particular, an appropriate version of the fact that (M, \tilde{g}) has *macroscopic dimension one* was used in [CL23] to deduce the stable Bernstein theorem for $M^3 \rightarrow \mathbf{R}^4$.

In higher dimensions, the positive scalar curvature property (1.1) of (M, \tilde{g}) appears to be too weak to deduce the stable Bernstein property. (In particular, $\mathbf{R}^{n-2} \times \mathbf{S}^2$ has uniformly positive scalar curvature.) The first main idea in the proof of Theorem 1.1 is to replace scalar curvature with a stronger curvature condition, namely *bi-Ricci curvature*. The bi-Ricci curvature of two orthonormal vectors $v, w \in T_p M$ is defined as

$$\text{BiRic}(v, w) = \text{Ric}(v, v) + \text{Ric}(w, w) - \text{R}(v, w, w, v)$$

where R is the curvature tensor. Alternatively, $\text{BiRic}(v, w)$ is the sum of sectional curvatures of 2-planes intersecting the plane spanned by v and w . We note that the bi-Ricci curvature of a 3-manifold is a multiple of the scalar curvature. Importantly, $\mathbf{R} \times \mathbf{S}^{n-1}$ has uniformly positive bi-Ricci curvature, while $\mathbf{R}^k \times \mathbf{S}^{n-k}$ does not for any $k > 1$.

Remark 1.5. The bi-Ricci curvature was introduced by Shen–Ye [SY96, SY97] motivated by the relationship between bi-Ricci curvature and stable minimal hypersurfaces. More recently, Brendle–Hirsch–Johne studied bi-Ricci curvature as part of a general notion of curvatures that interpolate between Ricci and scalar curvature [BHJ24].

In Theorem 3.1, we prove that for $M^4 \rightarrow \mathbf{R}^5$ a complete, two-sided stable minimal immersion, the Gulliver–Lawson conformal metric (M, \tilde{g}) has uniformly positive bi-Ricci curvature in the weak spectral sense; namely, we have

$$(1.2) \quad -\tilde{\Delta} + (\tilde{\lambda}_{\text{BiRic}} - 1) \geq 0,$$

where $\tilde{\lambda}_{\text{BiRic}}(x)$ is the smallest bi-Ricci curvature of \tilde{g} at x . This computation follows the general strategy introduced by Gulliver–Lawson [GL86], but is considerably more involved. At this point, we must leverage the improved positivity (1.2) to prove “one-dimensionality” of (M^4, \tilde{g}) , and then use this to conclude the stable Bernstein theorem.

In [Xu23], Xu showed that if N^n has uniformly positive bi-Ricci curvature and $n \leq 5$, then N admits μ -bubbles with uniformly positive Ricci curvature in the weak spectral sense. We

generalize his arguments (for $n = 4$) to *spectral* uniformly positive bi-Ricci curvature. This construction produces an exhaustion by (warped) μ -bubbles that satisfy

$$(1.3) \quad -\Delta^\Sigma + \frac{3}{4} \left(\lambda_{\text{Ric}}^\Sigma - \frac{1}{2} \right) \geq 0,$$

where $\lambda_{\text{Ric}}^\Sigma(x)$ is the smallest eigenvalue of the Ricci curvature of Σ at x . The spectral condition (1.3) can be thought of as a weak form of uniform positivity of the Ricci curvature of the μ -bubble Σ . Indeed, (1.3) implies that Σ has uniformly bounded diameter by a result of Shen–Ye [SY97] (see Theorem 5.1). In particular, this implies that (M, \tilde{g}) has bounded Urysohn 1-width.

The fact that (M, \tilde{g}) has bounded Urysohn 1-width does confirm that a complete, two-sided stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has controlled geometry in a certain sense, but does not seem sufficient to conclude that M is a flat hyperplane. Indeed, in the proof of the stable Bernstein theorem for $M^3 \rightarrow \mathbf{R}^4$ in [CL23], the key property of the μ -bubbles is that they have bounded area (in addition to bounded diameter). In this case, when the μ -bubbles are two-dimensional, the area bound follows directly from a spectral condition like (1.3) thanks to the Gauss–Bonnet theorem.

In higher dimensions (as considered in this paper) the Bishop–Gromov volume comparison gives uniform volume upper bounds from Ricci curvature lower bounds. We consider a weighted version of Bray’s proof [Bra97] of the Bishop theorem via the isoperimetric profile and use this to deduce a sharp volume comparison theorem for 3-dimensional manifolds satisfying spectrally positive Ricci curvature (see Theorem 5.1).

Remark 1.6. A different spectral version of the Bishop–Gromov theorem was obtained by Carron for manifolds with a Euclidean Sobolev inequality and a (strong) form of non-negative Ricci curvature in a spectral sense [Car19]. Note that this result does not seem applicable to our setting since the μ -bubbles are compact and thus cannot admit a Euclidean Sobolev inequality.

Granted these two ingredients, i.e., spectral positivity of bi-Ricci curvature for the Gulliver–Lawson conformal metric and volume bounds for the 3-dimensional μ -bubbles, we can follow the arguments of [CL23] to deduce the stable Bernstein theorem for $n = 4$.

Remark 1.7. There have been several exciting developments in this area that occurred after this paper was first posted. Antonelli–Xu have generalized the spectral Bishop–Gromov result to hold in all dimensions [AX24a]. Mazet subsequently combined their spectral Bishop–Gromov result with a delicate refinement of the strategy used here to resolve the stable Bernstein problem in \mathbf{R}^6 [Maz24]. (See also [AX24b, Tam24].)

1.2. Organization. We review the notation and conventions in Section 2. Then, we compute the spectral curvature properties of the Gulliver–Lawson conformal metric in Section 3. We discuss μ -bubble existence and stability in Section 4 and then use the stability inequality to prove geometric estimates for the μ -bubbles in Section 5. Finally, in Section 6 we prove the results stated in the introduction.

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2. NOTATION AND CONVENTIONS

Let (M^n, g) be a Riemannian manifold. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $T_p M$. We write $R(\cdot, \cdot, \cdot, \cdot)$ for the curvature operator with the convention that $R(e_i, e_j, e_j, e_i)$ is the sectional curvature of the 2-plane spanned by e_i, e_j . At $p \in M$, we define

- *The Ricci curvature tensor:*

$$\text{Ric}(e_1, e_1) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1).$$

- *The minimum Ricci curvature scalar:*

$$\lambda_{\text{Ric}} = \inf_{|v|=1} \text{Ric}(v, v).$$

- *The bi-Ricci curvature:*

$$\text{BiRic}(e_1, e_2) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1) + \sum_{j=3}^n R(e_2, e_j, e_j, e_2).$$

- *The minimum bi-Ricci curvature scalar:*

$$\lambda_{\text{BiRic}} = \inf_{\{v, w\} \text{ orthonormal}} \text{BiRic}(v, w).$$

Let $M^n \rightarrow (X^{n+1}, g_X)$ be a two-sided codimension one smooth immersion. Let ν be a smooth unit normal vector field along M . We define

- *The second fundamental form:*

$$A(X, Y) = -\langle \nabla_X^N Y, \nu \rangle.$$

- *The mean curvature:*

$$H = \sum_{i=1}^n A(e_i, e_i).$$

In particular, the unit sphere in \mathbf{R}^{n+1} has positive mean curvature with respect to the outward unit normal. Given our conventions, for a smooth family of immersions $\{F_t : M^n \rightarrow N^{n+1}\}_t$, we have

$$\frac{d}{dt} \mathcal{H}_{F_t^* g_X}^n(M) = \int_M H_{F_t} \left\langle \nu_{F_t}, \frac{dF_t}{dt} \right\rangle d\mathcal{H}_{F_t^* g}^n,$$

where $F_t^* g_X$ is the pullback metric under F_t , H_{F_t} is the mean curvature of the immersion F_t , ν_{F_t} is the unit normal vector field to F_t , and \mathcal{H}_h^n is the n -dimensional Hausdorff measure with respect to the metric h .

3. CONFORMAL CHANGE FOR STABLE MINIMAL HYPERSURFACES

We study the geometry of stable minimal hypersurfaces in Euclidean space under a conformal change introduced by Gulliver–Lawson [GL86] to study singularities of stable minimal hypersurfaces.

Let $F : M^n \rightarrow \mathbf{R}^{n+1}$ be a complete two-sided stable minimal immersion. Let g be the pullback metric on M . Let r denote the Euclidean distance from 0 in \mathbf{R}^{n+1} . Consider the conformal metric $\tilde{g} = r^{-2}g \equiv e^{2\phi}g$ on $N = M \setminus F^{-1}(\{0\})$, as in [GL86]. Note that (N, \tilde{g}) is complete. Henceforth, we use tildes to denote quantities with respect to \tilde{g} ; and otherwise

we use the metric g . For instance, we let $d\mu$ and $d\tilde{\mu}$ denote the volume measures on N with respect to g and \tilde{g} respectively.

We prove here the following result.

Theorem 3.1. *Suppose $n = 4$. Then there is a smooth function V on (N, \tilde{g}) so that*

$$V \geq 1 - \tilde{\lambda}_{\text{BiRic}}$$

and

$$\int_N |\tilde{\nabla} \psi|_{\tilde{g}}^2 d\tilde{\mu} \geq \int_N V \psi^2 d\tilde{\mu}$$

for any $\psi \in C_c^\infty(N, \tilde{g})$.

We note that Theorem 3.1 shows that (N, \tilde{g}) satisfies a weak form of the condition $\widetilde{\text{BiRic}} \geq 1$.

3.1. Standard calculations for Euclidean minimal hypersurfaces. We begin with some standard calculations about the Euclidean distance function r on minimal hypersurfaces.

Proposition 3.2.

$$\text{Hess}^M r = r^{-1}g - r^{-1}dr \otimes dr - r^{-1}\langle \vec{x}, \nu \rangle A.$$

Proof. We compute

$$\partial_i r = \frac{x_i}{r}, \quad \partial_i \partial_j r = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$

Thus

$$\text{Hess}^{\mathbf{R}^{n+1}} r = r^{-1}g_{\text{Euc}} - r^{-1}dr \otimes dr.$$

Hence, for X and Y tangent to M , we have

$$\begin{aligned} \text{Hess}^M r(X, Y) &= \langle \nabla_X^M \nabla^M r, Y \rangle \\ &= \langle \nabla_X^{\mathbf{R}^{n+1}} \nabla^{\mathbf{R}^{n+1}} r, Y \rangle - \langle \nabla_X^{\mathbf{R}^{n+1}} (\nabla^{\mathbf{R}^{n+1}} r)^\perp, Y \rangle \\ &= \text{Hess}^{\mathbf{R}^{n+1}} r(X, Y) + \left\langle \frac{\vec{x}^\perp}{r}, (\nabla_X^{\mathbf{R}^{n+1}} Y)^\perp \right\rangle \\ &= r^{-1}g(X, Y) - r^{-1}(dr \otimes dr)(X, Y) - r^{-1}\langle \vec{x}, \nu \rangle A(X, Y), \end{aligned}$$

which concludes the proposition. \square

We write $\phi = -\log r$ so that $\tilde{g} = e^{2\phi}g$.

Proposition 3.3.

$$\text{Hess}^M(\log r) = r^{-2}g - 2r^{-2}dr \otimes dr - r^{-2}\langle \vec{x}, \nu \rangle A.$$

Proof. For any $f > 0$, we compute

$$\text{Hess}^M(\log f)(X, Y) = \left\langle \nabla_X^M \left(\frac{\nabla^M f}{f} \right), Y \right\rangle = f^{-1} \text{Hess}^M f(X, Y) - f^{-2}(df \otimes df)(X, Y).$$

We conclude by Proposition 3.2. \square

We define the tensor

$$T = \text{Hess}^M \phi - d\phi \otimes d\phi + \frac{1}{2}|d\phi|^2 g.$$

This quantity appears in the formula for curvature in the conformal metric \tilde{g} (see [Lee18, Theorem 7.30]). We now use Proposition 3.3 to compute T .

Proposition 3.4.

$$T = -r^{-2}g + r^{-2}dr \otimes dr + \frac{1}{2}r^{-2}|dr|^2 g + r^{-2}\langle \vec{x}, \nu \rangle A.$$

3.2. Curvature in the conformal metric. We now compute relevant curvature quantities related to the conformal metric \tilde{g} .

First, we set up a convenient orthonormal basis for $T_p M$ with respect to the metrics g and \tilde{g} . Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $T_p M$ with respect to g . Then $\{\tilde{e}_i = r e_i\}_{i=1}^n$ is an orthonormal basis for $T_p M$ with respect to \tilde{g} .

We are now equipped to compute the sectional curvatures of (N, \tilde{g}) . We will write $R_{ijji} = R(e_i, e_j, e_j, e_i)$, $A_{ij} = A(e_i, e_j)$, and $T_{ij} = T(e_i, e_j)$. We also denote $R_{ijji} = \tilde{R}(\tilde{e}_i, \tilde{e}_j, \tilde{e}_j, \tilde{e}_i)$.

Proposition 3.5.

$$r^2 R_{ijji} = \tilde{R}_{ijji} - 2 + |dr|^2 + (dr(e_i))^2 + (dr(e_j))^2 + \langle \vec{x}, \nu \rangle (A_{ii} + A_{jj}).$$

Proof. Using the formula for the Riemann curvature tensor under a conformal change [Lee18, Theorem 7.30] and the tensor T as defined in Proposition 3.4, we compute

$$\begin{aligned} \tilde{R}_{ijji} &= \tilde{R}(\tilde{e}_i, \tilde{e}_j, \tilde{e}_j, \tilde{e}_i) = r^4 \tilde{R}(e_i, e_j, e_j, e_i) \\ &= r^2 (R_{ijji} - T_{ii} - T_{jj}) \\ &= r^2 R_{ijji} + 2 - |dr|^2 - (dr(e_i))^2 - (dr(e_j))^2 - \langle \vec{x}, \nu \rangle (A_{ii} + A_{jj}), \end{aligned}$$

which concludes the proposition. \square

Taking the appropriate combinations of sectional curvatures, we use Proposition 3.5 to compute the bi-Ricci curvatures of (N, \tilde{g}) .

Proposition 3.6.

$$\begin{aligned} r^2 \text{BiRic}(e_1, e_2) &= \widetilde{\text{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n - 6) + (2n - 1)|dr|^2 \\ &\quad + (n - 3)(dr(e_1)^2 + dr(e_2)^2) \\ &\quad + (n - 3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}). \end{aligned}$$

Proof. Using Proposition 3.5, we compute

$$\begin{aligned} r^2 \text{BiRic}(e_1, e_2) &= \sum_{i=2}^n r^2 R_{1i i 1} + \sum_{j=3}^n r^2 R_{2j j 2} \\ &= \widetilde{\text{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n - 6) + (2n - 3)|dr|^2 \\ &\quad + 2|dr|^2 + (n - 3)(dr(e_1)^2 + dr(e_2)^2) \\ &\quad + 2\langle \vec{x}, \nu \rangle \text{Tr}(A) + (n - 3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}) \\ &= \widetilde{\text{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n - 6) + (2n - 1)|dr|^2 \\ &\quad + (n - 3)(dr(e_1)^2 + dr(e_2)^2) \end{aligned}$$

$$+ (n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}).$$

In the last inequality we used that the trace of A is zero for a minimal hypersurface. \square

To exploit the stability inequality, we use the Gauss equation to express BiRic in terms of the second fundamental form of M .

Proposition 3.7.

$$\text{BiRic}(e_1, e_2) = - \sum_{i=1}^n A_{1i}^2 - \sum_{j=2}^n A_{2j}^2 - A_{11}A_{22}.$$

Proof. Using the Gauss equation and $\text{Tr } A = 0$, we compute

$$\begin{aligned} \text{BiRic}(e_1, e_2) &= \sum_{i=2}^n R_{1ii1} + \sum_{j=3}^n R_{2jj2} \\ &= \sum_{i=2}^n (A_{11}A_{ii} - A_{1i}^2) + \sum_{j=3}^n (A_{22}A_{jj} - A_{2j}^2) \\ &= - \sum_{i=1}^n A_{1i}^2 - \sum_{j=2}^n A_{2j}^2 - A_{11}A_{22}. \end{aligned}$$

This completes the proof. \square

Choose the basis vectors e_1 and e_2 so that $\widetilde{\text{BiRic}}(\tilde{e}_1, \tilde{e}_2) = \tilde{\lambda}_{\text{BiRic}}$. We can now bound $|A|^2$ in terms of $\tilde{\lambda}_{\text{BiRic}}$.

Proposition 3.8. *For $n \geq 3$,*

$$r^2|A|^2 \geq \frac{2}{n-2} \left((3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\text{BiRic}} \right)$$

Proof. Combining Propositions 3.6 and 3.7, we estimate

$$\begin{aligned} (3.1) \quad r^2 \left(\sum_{i=1}^n A_{1i}^2 + \sum_{j=2}^n A_{2j}^2 + A_{11}A_{22} \right) &+ (n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}) \\ &= (4n-6) - (2n-1)|dr|^2 - (n-3)(dr(e_1)^2 + dr(e_2)^2) - \tilde{\lambda}_{\text{BiRic}}. \end{aligned}$$

Using $\langle \vec{x}, \nu \rangle = r dr(\nu)$ and Young's inequality, we have

$$|(n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22})| \leq (n-3)dr(\nu)^2 + \frac{n-3}{4}r^2(A_{11} + A_{22})^2.$$

Combined with (3.1) and the fact that $dr(e_1)^2 + dr(e_2)^2 + dr(\nu)^2 \leq 1$, we find

$$r^2 \left(\sum_{i=1}^n A_{1i}^2 + \sum_{j=2}^n A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11} + A_{22})^2 \right) \geq (3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\text{BiRic}}.$$

Using the fact that $\text{Tr } A = 0$, we now compute

$$\begin{aligned} A_{11}^2 + A_{22}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11} + A_{22})^2 &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}(A_{11} + A_{22})^2 \\ &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}\sigma(A_{11} + A_{22})^2 + \frac{n-1}{4}(1-\sigma)(A_{33} + \dots + A_{nn})^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{2} + \frac{n-1}{2}\sigma \right) (A_{11}^2 + A_{22}^2) + \frac{(n-1)(n-2)}{4}(1-\sigma)(A_{33}^2 + \dots + A_{nn}^2) \\
&= \frac{n-2}{2}(A_{11}^2 + \dots + A_{nn}^2),
\end{aligned}$$

where we took $\sigma = \frac{n-3}{n-1}$ in the last line. Hence, for $n \geq 3$, we have

$$\begin{aligned}
\frac{n-2}{2}r^2|A|^2 &\geq r^2 \left(\frac{n-2}{2} \sum_{i=1}^n A_{ii}^2 + \sum_{i=2}^n A_{1i}^2 + \sum_{j=3}^n A_{2j}^2 \right) \\
&\geq r^2 \left(\sum_{i=1}^n A_{1i}^2 + \sum_{j=2}^n A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11} + A_{22})^2 \right) \\
&\geq (3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\text{BiRic}},
\end{aligned}$$

as asserted. \square

3.3. Stability inequality in the conformal metric.

Proposition 3.9. *We have $\tilde{\Delta}_M(\log r) = n - n|dr|^2$.*

Proof. By the formula for the Laplace–Beltrami operator under a conformal transformation and Proposition 3.3, we have

$$\begin{aligned}
\tilde{\Delta}_M(\log r) &= r^2(\Delta_M(\log r) - (n-2)r^{-2}|dr|^2) \\
&= n - 2|dr|^2 - (n-2)|dr|^2 = n - n|dr|^2,
\end{aligned}$$

as desired. \square

We can now rewrite the stability inequality in the metric \tilde{g} .

Proposition 3.10. *For any $\psi \in C_c^\infty(N, \tilde{g})$ we have*

$$\int_N |\tilde{\nabla} \psi|_{\tilde{g}}^2 d\tilde{\mu} \geq \int_N \left(r^2|A|^2 - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} \right) |dr|^2 \right) \psi^2 d\tilde{\mu}.$$

Proof. Using

$$d\tilde{\mu} = r^{-n}d\mu \quad \text{and} \quad |\tilde{\nabla} f|_{\tilde{g}}^2 = r^2|\nabla f|^2,$$

the stability inequality for M can be written as

$$\int_N r^{n-2} |\tilde{\nabla} f|_{\tilde{g}}^2 d\tilde{\mu} \geq \int_N r^{n-2} (r^2|A|^2) f^2 d\tilde{\mu}$$

for any $f \in C_c^{0,1}(N, \tilde{g})$. We take $f = r^{\frac{2-n}{2}}\psi$ for $\psi \in C_c^{0,1}(N, \tilde{g})$. Then

$$\tilde{\nabla} f = r^{\frac{2-n}{2}} \tilde{\nabla} \psi - \frac{n-2}{2} r^{-\frac{n}{2}} \psi \tilde{\nabla} r.$$

In particular,

$$|\tilde{\nabla} f|_{\tilde{g}}^2 = a + b + c$$

where

$$a := r^{2-n} |\tilde{\nabla} \psi|_{\tilde{g}}^2, \quad b := \frac{(n-2)^2}{4} r^{-n} \psi^2 |\tilde{\nabla} r|_{\tilde{g}}^2, \quad c := -(n-2) r^{1-n} \psi \langle \tilde{\nabla} \psi, \tilde{\nabla} r \rangle_{\tilde{g}}.$$

We have

$$\int_N r^{n-2} a \, d\tilde{\mu} = \int_N |\tilde{\nabla} \psi|_{\tilde{g}}^2 d\tilde{\mu}.$$

Since $r^{-2} |\tilde{\nabla} r|_{\tilde{g}}^2 = |dr|^2$, we have

$$\int_N r^{n-2} b \, d\tilde{\mu} = \int_N \frac{(n-2)^2}{4} |dr|^2 \psi^2 d\tilde{\mu}.$$

Finally, we use integration by parts and Proposition 3.9 to compute

$$\begin{aligned} \int_N r^{n-2} c \, d\tilde{\mu} &= - \int_N \frac{n-2}{2} \langle \tilde{\nabla}(\psi^2), \tilde{\nabla}(\log r) \rangle_{\tilde{g}} d\tilde{\mu} \\ &= \int_N \frac{n-2}{2} \tilde{\Delta}(\log r) \psi^2 d\tilde{\mu} \\ &= \int_N \left(\frac{n(n-2)}{2} - \frac{n(n-2)}{2} |dr|^2 \right) \psi^2 d\tilde{\mu}. \end{aligned}$$

The assertion follows from the above expressions. \square

We now rephrase our estimate for $r^2 |A|^2$ from Proposition 3.8 to suit this form of the stability inequality.

Proposition 3.11. *For $3 \leq n \leq 5$,*

$$r^2 |A|^2 - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} \right) |dr|^2 \geq \frac{2}{n-2} \left(\frac{(2-n)(n^2-4n-4)}{8} - \tilde{\lambda}_{\text{BiRic}} \right).$$

Proof. By Proposition 3.8, we have

$$\begin{aligned} r^2 |A|^2 - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} \right) |dr|^2 \\ \geq \frac{6(n-1)}{n-2} - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} - \frac{2(2n-1)}{n-2} \right) |dr|^2 - \frac{2}{n-2} \tilde{\lambda}_{\text{BiRic}}. \end{aligned}$$

Note that the coefficient of $|dr|^2$ on the right-hand side is negative for $3 \leq n \leq 5$, so we can use $|dr|^2 \leq 1$ to conclude that the left hand side is greater than or equal to

$$\frac{2}{n-2} \left(\frac{(2-n)(n^2-4n-4)}{8} - \tilde{\lambda}_{\text{BiRic}} \right),$$

as desired. \square

Theorem 3.1 follows by plugging $n = 4$ into Proposition 3.11 and applying the stability inequality as formulated in Proposition 3.10.

4. μ -BUBBLES IN SPECTRAL POSITIVE BI-RICCI CURVATURE

We generalize the μ -bubble construction in uniformly positive bi-Ricci curvature of [Xu23] to manifolds with spectral uniformly positive bi-Ricci curvature. Our main tool is the notion of *warped μ -bubbles* (see [CL24a]), extending the standard μ -bubbles used in [Xu23].

Suppose that (N^n, g_N) is a smooth complete noncompact Riemannian manifold that admits a smooth function V so that

$$V \geq 1 - \lambda_{\text{BiRic}}(g_N)$$

and

$$\int_N |\nabla \psi|^2 \geq \int_N V \psi^2$$

for all $\psi \in C_c^\infty(N, g_N)$. Recall from [FCS80, Theorem 1] that there is a positive function u on N satisfying

$$(4.1) \quad -\Delta_N u = Vu \geq (1 - \lambda_{\text{BiRic}}(g_N))u.$$

We prove the following theorem in dimension $n = 4$.

Theorem 4.1. *Let $X \subset N^4$ be a closed subset with smooth boundary $\partial X = \partial_+ X \sqcup \partial_- X$ for some nonempty smooth hypersurfaces $\partial_\pm X$. Suppose $d_N(\partial_+ X, \partial_- X) \geq 10\pi$. Then there is a connected, relatively open subset $\Omega \subset X$ with smooth boundary $\partial \Omega = \partial_- X \sqcup \Sigma$ so that*

- $\partial_- X \subset \Omega$,
- $\Sigma \subset X \setminus \partial X$ is a closed submanifold
- $\Omega \subset B_{10\pi}(\partial_- X)$,
- there is a smooth function $W \in C^\infty(\Sigma)$ so that

$$W \geq \frac{3}{4} \left(\frac{1}{2} - \lambda_{\text{Ric}}(\Sigma) \right)$$

and

$$\int_\Sigma |\nabla \psi|^2 \geq \int_\Sigma W \psi^2$$

for all $\psi \in C^\infty(\Sigma)$.

The strategy to prove Theorem 4.1 is to minimize a certain warped prescribed mean curvature functional.

Let w be a smooth positive function on N . Let $\Omega \subset N$ be an open set with smooth boundary (or more generally a set of finite perimeter). Let h be a smooth function defined in a neighborhood of $\partial \Omega$. We study minimizers of the warped prescribed mean curvature functional

$$\mathcal{A}(\Omega) = \int_{\partial \Omega} w d\mathcal{H}^{n-1} - \int_\Omega h w d\mathcal{H}^n.$$

Ultimately, we will take $w = u$ to be our warping function, but we leave w general for most of the calculations. Note that this functional may be viewed as a μ -bubble functional on a warped manifold, see [CL24a, Remark 11].

4.1. First variation formula. The first variation formula for \mathcal{A} can be computed as.

Proposition 4.2. *Let $\{\Omega_t\}_{|t| < \varepsilon}$ be a smooth family of sets of finite perimeter with $\Omega_0 = \Omega$ and variation vector field V_t . Then*

$$\frac{d}{dt} \mathcal{A}(\Omega_t) = \int_{\partial \Omega_t} \langle \nabla^N w, V_t^\perp \rangle + w H_t \langle \nu_t, V_t \rangle - w h \langle \nu_t, V_t \rangle d\mathcal{H}^{n-1}.$$

Hence, critical points for \mathcal{A} satisfy

$$H = h - w^{-1} \langle \nabla^N w, \nu \rangle.$$

4.2. Second variation formula. We now prove the following second variation formula. Although the theorem statement below only holds in dimension $n = 4$, although we carry out the computations for general n until plugging in $n = 4$ at the very end of the proof.

Theorem 4.3. *Let $\Omega \subset N^4$ be an open set with smooth boundary that's stationary and stable for \mathcal{A} with weight function $w = u$. Let $\Sigma = \partial\Omega$. Let γ denote the pullback metric on Σ . Then there is a smooth function W on Σ so that*

$$W \geq \frac{3}{4} \left(\frac{1}{2} - \lambda_{\text{Ric}}(\gamma) \right)$$

and

$$\int_{\Sigma} |\nabla^{\Sigma} \psi|^2 \geq \int_{\Sigma} W \psi^2 + \frac{3}{8} \int_{\Sigma} (1 + h^2 - 2|\nabla^N h|) \psi^2$$

for any $\psi \in C_c^{\infty}(\Sigma)$.

Proof. We take a variation $\{\Omega_t\}$ where $\Omega_0 = \Omega$ is a critical point for \mathcal{A} . We can choose our variation so that $D_t V_t = 0$ by taking the normal exponential flow of $V_0 = \phi\nu$.

We compute

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{A}(\Omega_t) &= \int_{\Sigma} \phi^2 \text{Hess}^N w(\nu, \nu) - \phi \langle \nabla^{\Sigma} w, \nabla^{\Sigma} \phi \rangle + \phi^2 \langle \nabla^N w, \nu \rangle H \\ &\quad - \int_{\Sigma} w(\phi \Delta^{\Sigma} \phi + (|A_{\Sigma}|^2 + \text{Ric}_{g_N}(\nu, \nu)) \phi^2) \\ &\quad - \int_{\Sigma} \phi^2 \langle \nabla^N w, \nu \rangle h + \phi^2 w \langle \nabla^N h, \nu \rangle. \end{aligned}$$

We use integration by parts on the $-w\phi\Delta^{\Sigma}\phi$ term and the formula.

$$\text{Hess}^N f(\nu, \nu) = \Delta^N f - \Delta^{\Sigma} f - \langle \nabla^N f, \nu \rangle H,$$

This gives

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \mathcal{A}(\Omega_t) &= \int_{\Sigma} \phi^2 (\Delta^N w - \Delta^{\Sigma} w) \\ &\quad + \int_{\Sigma} w(|\nabla^{\Sigma} \phi|^2 - (|A_{\Sigma}|^2 + \text{Ric}_{g_N}(\nu, \nu)) \phi^2) \\ &\quad - \int_{\Sigma} \phi^2 \langle \nabla^N w, \nu \rangle h - \int_{\Sigma} \phi^2 w \langle \nabla^N h, \nu \rangle. \end{aligned}$$

Since Ω is a stable critical point of \mathcal{A} , we thus have

$$\begin{aligned} (4.2) \quad \int_{\Sigma} w |\nabla^{\Sigma} \phi|^2 - \phi^2 \Delta^{\Sigma} w &\geq \int_{\Sigma} (-\Delta^N w + (|A_{\Sigma}|^2 + \text{Ric}_{g_N}(\nu, \nu)) w) \phi^2 \\ &\quad + \int_{\Sigma} (h \langle \nabla^N w, \nu \rangle + w \langle \nabla^N h, \nu \rangle) \phi^2. \end{aligned}$$

Take $\phi = w^{-1/2} \psi$. We compute

$$\nabla^{\Sigma} \phi = w^{-1/2} \nabla^{\Sigma} \psi - \frac{1}{2} w^{-3/2} \psi \nabla^{\Sigma} w.$$

Write

$$w |\nabla^{\Sigma} \phi|^2 = a + b + c$$

where

$$a := |\nabla^\Sigma \psi|^2, \quad b := -w^{-1} \psi \langle \nabla^\Sigma w, \nabla^\Sigma \psi \rangle, \quad \text{and} \quad c := \frac{1}{4} w^{-2} \psi^2 |\nabla^\Sigma w|^2.$$

We have

$$\begin{aligned} \int_\Sigma a &= \int_\Sigma |\nabla^\Sigma \psi|^2, \\ \int_\Sigma c &= \frac{1}{4} \int_\Sigma |\nabla^\Sigma \log w|^2 \psi^2, \end{aligned}$$

and

$$\begin{aligned} \int_\Sigma b - \phi^2 \Delta^\Sigma w &= \int_\Sigma w^{-1} \psi \langle \nabla^\Sigma w, \nabla^\Sigma \psi \rangle - \int_\Sigma \psi^2 |\nabla^\Sigma \log w|^2 \\ &\leq \left(\frac{\varepsilon}{2} - 1 \right) \int_\Sigma \psi^2 |\nabla^\Sigma \log w|^2 + \frac{1}{2\varepsilon} \int_\Sigma |\nabla^\Sigma \psi|^2 \end{aligned}$$

for all $\varepsilon > 0$. Taking $\varepsilon = 3/2$, we have

$$\int_\Sigma w |\nabla^\Sigma \phi|^2 - \phi^2 \Delta^\Sigma w \leq \frac{4}{3} \int_\Sigma |\nabla^\Sigma \psi|^2.$$

Combined with (4.2) we have

$$\begin{aligned} (4.3) \quad \frac{4}{3} \int_\Sigma |\nabla^\Sigma \psi|^2 &\geq \int_\Sigma \left(-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \text{Ric}_{g_N}(\nu, \nu) - \frac{1}{2} H^2 - \frac{1}{2} \right) \psi^2 \\ &\quad + \int_\Sigma \left(\frac{1}{2} + \frac{1}{2} H^2 + h \langle \nabla^N \log w, \nu \rangle + \langle \nabla^N h, \nu \rangle \right) \psi^2. \end{aligned}$$

Since Σ is a critical point for \mathcal{A} , we have

$$H^2 = h^2 + \langle \nabla^N \log w, \nu \rangle^2 - 2h \langle \nabla^N \log w, \nu \rangle \geq h^2 - 2h \langle \nabla^N \log w, \nu \rangle.$$

Hence, we have

$$\begin{aligned} (4.4) \quad \frac{4}{3} \int_\Sigma |\nabla^\Sigma \psi|^2 &\geq \int_\Sigma \left(-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \text{Ric}_{g_N}(\nu, \nu) - \frac{1}{2} H^2 - \frac{1}{2} \right) \psi^2 \\ &\quad + \frac{1}{2} \int_\Sigma (1 + h^2 - 2|\nabla^N h|) \psi^2. \end{aligned}$$

It remains to find a good lower bound for the first integrand on the right-hand side of (4.4). Using the Gauss equation, we compute

$$\begin{aligned} \text{Ric}_\gamma(e_1, e_1) &= \sum_{i=1}^{n-1} \text{R}_\gamma(e_1, e_i, e_i, e_1) \\ &= \sum_{i=1}^{n-1} (\text{R}_{g_N}(e_1, e_i, e_i, e_1) + A_{11} A_{ii} - A_{1i}^2) \\ &= \text{BiRic}_{g_N}(e_1, \nu) - \text{Ric}_{g_N}(\nu, \nu) + A_{11} \sum_{i=2}^{n-1} A_{ii} - \sum_{i=2}^{n-1} A_{1i}^2. \end{aligned}$$

Moreover, using $\text{Tr}(A) = H$, we have

$$\begin{aligned}
A_{11} \sum_{i=2}^{n-1} A_{ii} &= -A_{11}^2 + A_{11}H = -A_{11}^2 - H \sum_{i=2}^{n-1} A_{ii} + H^2 \\
&\geq -A_{11}^2 - \frac{1}{2\varepsilon} \left(\sum_{i=2}^{n-1} A_{ii} \right)^2 + \left(1 - \frac{\varepsilon}{2}\right) H^2 \\
&\geq -A_{11}^2 - \frac{n-2}{2\varepsilon} \sum_{i=2}^{n-1} A_{ii}^2 + \left(1 - \frac{\varepsilon}{2}\right) H^2 \\
&= -\sum_{i=1}^{n-1} A_{ii}^2 + \frac{6-n}{4} H^2,
\end{aligned}$$

where we took $\varepsilon = \frac{n-2}{2}$ in the last line. Choosing e_1 so that $\lambda_{\text{Ric}}(\gamma) = \text{Ric}_\gamma(e_1, e_1)$, we have

$$|A_\Sigma|^2 + \text{Ric}_{g_N}(\nu, \nu) \geq \lambda_{\text{BiRic}}(g_N) - \lambda_{\text{Ric}}(\gamma) + \frac{6-n}{4} H^2.$$

Taking $n = 4$ and $w = u$, we have

$$-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \text{Ric}_{g_N}(\nu, \nu) - \frac{1}{2} H^2 - \frac{1}{2} \geq \frac{1}{2} - \lambda_{\text{Ric}}(\gamma),$$

which completes the proof. \square

Proof of Theorem 4.1. Equipped with the second variation formula from Theorem 4.3, the proof now follows by taking h to be the standard μ -bubble prescribing function, chosen precisely so that $1 + h^2 - 2|\nabla^N h| \geq 0$, and then minimizing the functional \mathcal{A} (see [CL23, Lemma 24] for the choice of h and see [CL24a, Proposition 12] or [Zhu21, Proposition 2.1] for the existence theory).

For completeness, we construct the function h . Let φ_0 be a smoothing of the function $d(\partial_- X, \cdot)$ so that $|\nabla \varphi_0| \leq 2$ and $\varphi_0|_{\partial_- X} \equiv 0$. Let $\varepsilon \in (0, 1/2)$ so that ε and $4\pi + 2\varepsilon$ are regular values of φ_0 . Define

$$\varphi = \frac{\varphi_0 - \varepsilon}{4 + \frac{\varepsilon}{\pi}} - \frac{\pi}{2}.$$

Then $|\nabla \varphi| \leq \frac{1}{2}$, and the set $\Omega_1 = \{-\pi/2 < \varphi < \pi/2\}$ has smooth boundary and satisfies $\Omega_1 \subset B_{10\pi}(\partial_- X)$. On Ω_1 , we define

$$h = -\tan(\varphi).$$

Since

$$\nabla h = -(1 + \tan^2(\varphi)) \nabla \varphi = -(1 + h^2) \nabla \varphi,$$

it holds that

$$2|\nabla h| \leq 1 + h^2.$$

Hence, h has the desired property; the rest of the proof follows as in [CL23, Lemma 24]. \square

5. GEOMETRIC ESTIMATES FOR μ -BUBBLES

We prove that the μ -bubbles constructed in the previous section have uniformly bounded diameter and volume.

Theorem 5.1. *Suppose that (Σ^3, γ) is a connected closed 3-manifold that admits a smooth function W and a constant $\alpha \in (0, 2]$ so that*

$$W \geq \alpha^{-1}(2 - \lambda_{\text{Ric}}(\gamma))$$

and

$$\int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} W \psi^2$$

for all $\psi \in C^\infty(\Sigma)$. Then

$$\text{diam}(\Sigma, \gamma) \leq \pi$$

and

$$\text{Vol}(\Sigma, \gamma) \leq 2\pi^2.$$

Note that both inequalities are sharp for the round 3-sphere (see also Remark 5.7 explaining the rigidity statement for the volume estimate).

Proof of diameter bound. A smooth positive first eigenfunction θ of the operator $-\Delta - W$ satisfies

$$-\Delta \theta \geq \alpha^{-1}(2 - \lambda_{\text{Ric}}(\gamma))\theta.$$

Hence,

$$\text{Ric}_\gamma^{(\theta, \alpha)} \equiv \text{Ric}_\gamma - \alpha(\theta^{-1} \Delta \theta) \gamma \geq \text{Ric}_\gamma - \lambda_{\text{Ric}} \gamma + 2\gamma \geq 2\gamma,$$

where we adapt the notation of [SY97]. Since $\alpha \leq \frac{4}{3-1} = 2$, we have

$$\text{diam}(\Sigma, \gamma) \leq \pi,$$

by [SY97, Corollary 1]. This completes the proof of the diameter bound. \square

It remains to prove the volume bound for Σ . The strategy is to exploit the concavity properties of a weighted isoperimetric profile. These arguments extend the strategy of Bray's proof of the Bishop volume comparison from [Bra97] (we follow the exposition in [Bre12]).

5.1. Weighted isoperimetric profile. For an open set $\Omega \subset \Sigma$ with smooth boundary, we define a weighted area and volume functional by

$$a(\Omega) = \int_{\partial\Omega} \theta^\alpha d\mathcal{H}_\gamma^2 \text{ and } v(\Omega) = \int_{\Omega} \theta^\alpha d\mathcal{H}_\gamma^3,$$

where θ be the unique first eigenfunction of $-\Delta - W$ with $\min \theta = 1$. Namely, θ satisfies

$$(5.1) \quad -\Delta \theta \geq \alpha^{-1}(2 - \lambda_{\text{Ric}}(\gamma))\theta.$$

The *weighted isoperimetric profile* is the function $\mathcal{I} : (0, v(\Sigma)) \rightarrow \mathbf{R}$ given by

$$\mathcal{I}(v) = \inf\{a(\Omega) \mid v(\Omega) = v\}.$$

By compactness of Caccioppoli sets and [Mor03, §3.10], there's $\Omega \subset \Sigma$ achieving $\mathcal{I}(v)$ for all $v \in (0, v(\Sigma))$.

Proposition 5.2. *\mathcal{I} is continuous.*

Proof. By the compactness theory for Caccioppoli sets and the lower semi-continuity of mass, we have

$$\liminf_{v \rightarrow v_0} \mathcal{I}(v) \geq \mathcal{I}(v_0).$$

By the existence of a continuous upper barrier function for \mathcal{I} at v_0 for any $v_0 \in (0, v(\Sigma))$ (see §5.4 below), we also have

$$\limsup_{v \rightarrow v_0} \mathcal{I}(v) \leq \mathcal{I}(v_0),$$

so \mathcal{I} is continuous. \square

5.2. First variation. We compute the first variation of the functionals a and v . Let $\{\Omega_t\}_{|t|<\varepsilon}$ be a smooth family of open sets with smooth boundary whose variation vector field along $\partial\Omega = \partial\Omega_0$ is $f\nu$, where ν is the unit normal field to $\partial\Omega$ pointing out of Ω . The following computation is standard.

Proposition 5.3. *We have*

$$\frac{d}{dt}\Big|_{t=0} a(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f\theta^\alpha \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} f\theta^\alpha.$$

5.3. Second variation. We compute the second variation of the functionals a and v . We consider the same setup as the previous subsection.

Proposition 5.4. *We have*

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) &= \int_{\partial\Omega} |\nabla^{\partial\Omega} f|^2 \theta^\alpha - (\text{Ric}_\Sigma(\nu, \nu) + |A_{\partial\Omega}|^2) f^2 \theta^\alpha + \alpha(\Delta^\Sigma \theta - \Delta^{\partial\Omega} \theta) f^2 \theta^{\alpha-1} \\ &\quad + \int_{\partial\Omega} \alpha(\alpha-1) \langle \nabla^\Sigma \theta, \nu \rangle^2 f^2 \theta^{\alpha-2} + H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f^2 \theta^\alpha, \end{aligned}$$

and

$$\frac{d^2}{dt^2}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f^2 \theta^\alpha.$$

Proof. By Proposition 5.3, we compute

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) &= \int_{\partial\Omega} (-\Delta^{\partial\Omega} f - (\text{Ric}_\Sigma(\nu, \nu) + |A_{\partial\Omega}|^2) f) f\theta^\alpha \\ &\quad + \int_{\partial\Omega} \alpha \text{Hess}^\Sigma \theta(\nu, \nu) f^2 \theta^{\alpha-1} - \alpha \langle \nabla^{\partial\Omega} \theta, \nabla^{\partial\Omega} f \rangle f\theta^{\alpha-1} \\ &\quad + \int_{\partial\Omega} \alpha H \langle \nabla^\Sigma \theta, \nu \rangle f^2 \theta^{\alpha-1} + \alpha(\alpha-1) \langle \nabla^\Sigma \theta, \nu \rangle^2 f^2 \theta^{\alpha-2} \\ &\quad + \int_{\partial\Omega} H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f^2 \theta^\alpha. \end{aligned}$$

Using the formula

$$\text{Hess}^\Sigma \phi(\nu, \nu) = \Delta^\Sigma \phi - \Delta^{\partial\Omega} \phi - H \langle \nabla^\Sigma \phi, \nu \rangle$$

and applying integration by parts to the $-f\theta^\alpha \Delta^{\partial\Omega} f$ term, we deduce

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) &= \int_{\partial\Omega} |\nabla^{\partial\Omega} f|^2 \theta^\alpha - (\text{Ric}_\Sigma(\nu, \nu) + |A_{\partial\Omega}|^2) f^2 \theta^\alpha + \alpha(\Delta^\Sigma \theta - \Delta^{\partial\Omega} \theta) f^2 \theta^{\alpha-1} \\ &\quad + \int_{\partial\Omega} \alpha(\alpha-1) \langle \nabla^\Sigma \theta, \nu \rangle^2 f^2 \theta^{\alpha-2} + H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f^2 \theta^\alpha. \end{aligned}$$

Similarly, by Proposition 5.3, we compute

$$\frac{d^2}{dt^2}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f^2\theta^\alpha.$$

This completes the proof. \square

5.4. Differential inequality in the barrier sense. Fix $v_0 \in (0, v(\Sigma))$.

Let Ω be a weighted isoperimetric surface for the problem $\mathcal{I}(v_0)$. Let $\{\Omega_t\}_{|t|<\varepsilon}$ be a smooth family of open sets with smooth boundary with $\Omega_0 = \Omega$ whose variation vector field at $t = 0$ is $\theta^{-\alpha}\nu$, where ν is the outward pointing unit normal vector field along $\partial\Omega$.

We note that $v(t) := v(\Omega_t)$ is a smooth function. By Proposition 5.3, we have

$$v'(0) = \frac{d}{dt}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} 1 > 0.$$

By the inverse function theorem, there is some small $\sigma > 0$ and a smooth function

$$t : (v_0 - \sigma, v_0 + \sigma) \rightarrow \mathbf{R}$$

that is the inverse of $v(t)$.

Let $u : (v_0 - \sigma, v_0 + \sigma) \rightarrow \mathbf{R}$ be defined by $u(v) = a(t(v))$. Note that $u(v_0) = a(0) = \mathcal{I}(v_0)$. Moreover, since $v(\Omega_{t(v)}) = v$, we have $u(v) \geq \mathcal{I}(v)$ for all $v \in (v_0 - \sigma, v_0 + \sigma)$.

Let primes denote derivatives with respect to v and dots denote derivatives with respect to t .

Proposition 5.5. *The function u satisfies*

$$u''(v_0) \leq -\left(2 + \frac{1}{2}u'(v_0)^2\right)u(v_0)^{-1}.$$

Proof. We have

$$t'(v) = \frac{1}{\dot{v}(t(v))} \text{ and } t''(v) = -\frac{\ddot{v}(t(v))}{\dot{v}(t(v))^3}.$$

By Proposition 5.3 and 5.4, we have

$$t'(v_0) = \left(\int_{\partial\Omega} 1\right)^{-1} \text{ and } t''(v_0) = -\left(\int_{\partial\Omega} 1\right)^{-3} \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle)\theta^{-\alpha}.$$

We note that

$$\frac{d}{dv}a(t(v)) = \dot{a}(t(v))t'(v)$$

and

$$\frac{d^2}{dv^2}a(t(v)) = \ddot{a}(t(v))t'(v)^2 + \dot{a}(t(v))t''(v).$$

We compute

$$u'(v_0) = H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle,$$

where we use the fact that isoperimetric surfaces satisfy the equation $H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle = \lambda$ for some $\lambda \in \mathbf{R}$.

By (5.1) and Propositions 5.3 and 5.4 with $f = \theta^{-\alpha}$, we have

$$u''(v_0) \left(\int_{\partial\Omega} 1\right)^2 = (\ddot{a}(0)(t'(v_0))^2 + \dot{a}(0)t''(v_0)) \left(\int_{\partial\Omega} 1\right)^2$$

$$\begin{aligned}
&= \int_{\partial\Omega} |\nabla^{\partial\Omega}(\theta^{-\alpha})|^2 \theta^\alpha - (\text{Ric}_\Sigma(\nu, \nu) + |A_{\partial\Omega}|^2) \theta^{-\alpha} \\
&\quad + \int_{\partial\Omega} \alpha(\Delta^\Sigma \theta - \Delta^{\partial\Omega} \theta) \theta^{-\alpha-1} + \alpha(\alpha-1) \langle \nabla^\Sigma \theta, \nu \rangle^2 \theta^{-\alpha-2} \\
&\quad + \int_{\partial\Omega} H(H + \alpha \theta^{-1} \langle \nabla^\Sigma \theta, \nu \rangle) \theta^{-\alpha} - (H + \alpha \theta^{-1} \langle \nabla^\Sigma \theta, \nu \rangle)^2 \theta^{-\alpha} \\
&= \int_{\partial\Omega} (\alpha \theta^{-1} \Delta^\Sigma \theta - \text{Ric}_\Sigma(\nu, \nu)) \theta^{-\alpha} - \alpha |\nabla^{\partial\Omega} \theta|^2 \theta^{-\alpha-2} - |A_{\partial\Omega}|^2 \theta^{-\alpha} \\
&\quad + \int_{\partial\Omega} -\alpha H \langle \nabla^\Sigma \theta, \nu \rangle \theta^{-\alpha-1} - \alpha \langle \nabla^\Sigma \theta, \nu \rangle^2 \theta^{-\alpha-2} \\
&\leq \int_{\partial\Omega} -2\theta^{-\alpha} - \frac{1}{2} H^2 \theta^{-\alpha} - \alpha H \langle \nabla^\Sigma \theta, \nu \rangle \theta^{-\alpha-1} - \alpha \langle \nabla^\Sigma \theta, \nu \rangle^2 \theta^{-\alpha-2} \\
&= \int_{\partial\Omega} -\left(2 + \frac{1}{2} (H + \alpha \theta^{-1} \langle \nabla^\Sigma \theta, \nu \rangle)^2\right) \theta^{-\alpha} + \frac{1}{2} \alpha(\alpha-2) \langle \nabla^\Sigma \theta, \nu \rangle^2 \theta^{-\alpha-2} \\
&\leq -\left(2 + \frac{1}{2} u'(v_0)^2\right) \int_{\partial\Omega} \theta^{-\alpha},
\end{aligned}$$

where we used $0 < \alpha \leq 2$. In particular, $u''(v_0) \leq 0$.

By Hölder's inequality, we have

$$\left(\int_{\partial\Omega} 1\right)^2 \leq \int_{\partial\Omega} \theta^\alpha \int_{\partial\Omega} \theta^{-\alpha} = u(v_0) \int_{\partial\Omega} \theta^{-\alpha}.$$

Hence, u satisfies

$$u''(v_0) \leq -\left(2 + \frac{1}{2} u'(v_0)^2\right) u(v_0)^{-1},$$

which concludes the proof. \square

We consider a power of \mathcal{I} and u to simplify the corresponding differential inequality. We let $\mathcal{F}(v) = \mathcal{I}(v)^{3/2}$. By Proposition 5.5, we have the following result.

Proposition 5.6. *For any $v_0 \in (0, V)$, there is a smooth function $U : (v_0 - \sigma, v_0 + \sigma) \rightarrow \mathbf{R}$ satisfying*

- $U(v_0) = \mathcal{F}(v_0)$,
- $U(v) \geq \mathcal{F}(v)$ for all $v \in (v_0 - \sigma, v_0 + \sigma)$,
- $U''(v_0) \leq -3U(v_0)^{-1/3}$.

Proof. We take $U(v) = u(v)^{3/2}$ as in Proposition 5.5, and conclude that

$$U'(v) = \frac{3}{2} u^{1/2}(v) u'(v)$$

and

$$\begin{aligned}
U''(v_0) &= \frac{3}{4} u^{-1/2}(v_0) u'(v_0)^2 + \frac{3}{2} u(v_0)^{1/2} u''(v_0) \\
&\leq \frac{3}{4} u^{-1/2}(v_0) u'(v_0)^2 - 3u(v_0)^{-1/2} - \frac{3}{4} u(v_0)^{-1/2} (u'(v_0))^2 = -3U(v_0)^{-1/3},
\end{aligned}$$

as desired. \square

5.5. Integrating the differential inequality. We study solutions to the ODE

$$(5.2) \quad f''(v) = -3f(v)^{-1/3}.$$

By Propositions 5.2 and 5.6 combined with the fact that $-3f^{-1/3}$ is increasing in f , it follows from a standard ODE comparison that no solution to (5.2) can touch $\mathcal{F}(v)$ from below unless they are equal.

Observe that $g : [0, 1) \rightarrow [0, \frac{\pi}{4})$ given by

$$g(x) = \frac{1}{3} \int_0^x \frac{dt}{\sqrt{1-t^{2/3}}}$$

is a diffeomorphism. As such, the map $v \mapsto g^{-1}(\frac{\pi}{4} - v)$ extends to an even smooth map $\tilde{f} : [-\frac{\pi}{4}, \frac{\pi}{4}] \rightarrow \mathbf{R}$ so that $\tilde{f}(0) = 1$ and $\tilde{f}''(v) = -3\tilde{f}(v)^{-1/3}$. For $z > 0$ define $f_z : (-\frac{\pi}{4}z, \frac{\pi}{4}z) \rightarrow \mathbf{R}$ by

$$v \mapsto z^{\frac{3}{2}} \tilde{f}(z^{-1}v).$$

Note that $f_z(v)$ solves (5.2) and has $f'_z(0) = 0$ and $f_z(0) = z^{3/2}$. We set $\beta(z) := \frac{\pi}{4}z$ and observe that $f_z(\pm\beta(z)) = 0$. We extend f_z to all of \mathbf{R} by zero.

Assume for the sake of contradiction that $v(\Sigma) = \int_{\Sigma} \theta^\alpha > 2\pi^2$.

Claim. *There is a $\delta > 0$ so that for $z = 4\pi + \delta$, we have*

$$(5.3) \quad \mathcal{F}(v) \geq f_z(v - \beta(z))$$

for all $v \in (0, 2\pi^2)$.

Proof. Indeed, let $\delta > 0$ and $\varepsilon > 0$ be sufficiently small so that $\frac{\pi}{2}z + \varepsilon z < v(\Sigma)$ for $z \in (0, 4\pi + \delta)$, which is possible since $v(\Sigma) > 2\pi^2 = 2\beta(4\pi)$. Consider the graph of

$$g_z(v) = f_z(v - \beta(z) - \varepsilon z)$$

for $v \in [\varepsilon z, 2\beta(z) + \varepsilon z]$. Note that

$$g_z(\varepsilon z) = g_z(2\beta(z) + \varepsilon z) = 0 < \min\{\mathcal{F}(\varepsilon z), \mathcal{F}(2\beta(z) + \varepsilon z)\}.$$

Moreover, g_z converges uniformly to zero as $z \rightarrow 0$. Hence, if $g_{z^*}(v^*) > \mathcal{F}(v^*)$ for some v^* and z^* , then there must be some $z \in (0, z^*]$ so that g_z touches \mathcal{F} from below, which contradicts Proposition 5.6. Therefore, we have $\mathcal{F} \geq g_z$ for every $z \in (0, 4\pi + \delta)$. We send z to $4\pi + \delta$ and then ε to 0. This proves the assertion. \square

We now study the asymptotic behavior of \mathcal{F} and $f_{4\pi+\delta}(v - \beta(4\pi + \delta))$ as $v \rightarrow 0$.

Since $f_{4\pi+\delta}(-\beta(4\pi + \delta)) = 0$, we have $f'_{4\pi+\delta}(-\beta(4\pi + \delta)) = 3\sqrt{4\pi + \delta}$. Hence, we have

$$(5.4) \quad f_{4\pi+\delta}(v - \beta(4\pi + \delta)) = 3\sqrt{4\pi + \delta}v + o(v)$$

as $v \rightarrow 0$.

On the other hand, we compute an upper bound for \mathcal{F} as $v \rightarrow 0$ by studying small geodesic balls. Take x_0 so that $\theta(x_0) = \min \theta = 1$. A straightforward computation gives

$$v(B_r(x_0)) = \frac{4}{3}\pi r^3 + o(r^3)$$

and

$$a(B_r(x_0)) = 4\pi r^2 + o(r^2)$$

as $r \rightarrow 0$. Solving the first equation for r and plugging into the second equation, we deduce

$$\mathcal{I}(v) \leq (36\pi)^{1/3}v^{2/3} + o(v^{2/3}),$$

so

$$(5.5) \quad \mathcal{F}(v) \leq 6\sqrt{\pi}v + o(v).$$

However, (5.3), (5.4), and (5.5) imply $3\sqrt{4\pi + \delta} \leq 6\sqrt{\pi}$, which yields a contradiction. Therefore, since we normalized so that $\min \theta = 1$, we have

$$(5.6) \quad \text{Vol}(\Sigma, \gamma) \leq \int_{\Sigma} \theta^{\alpha} = v(\Sigma) \leq 2\pi^2,$$

which concludes the proof of Theorem 5.1.

Remark 5.7. We remark that the rigidity case of Theorem 5.1 follows easily. We proved above that $v(\Sigma) \leq 2\pi^2$. As such, if $\text{Vol}(\Sigma, \gamma) = 2\pi^2$ then (5.6) and the normalization of θ gives that $\theta \equiv 1$ on Σ . Thus, (5.1) gives $\lambda_{\text{Ric}}(\gamma) \geq 2$. As such, the rigidity case in the classical Bishop–Gromov theorem implies that (Σ, γ) is isometric to the round 3-sphere.

6. STABLE BERNSTEIN THEOREM

We use the estimates of the previous sections to prove the stable Bernstein theorem in \mathbf{R}^5 as well as the various consequences indicated in the introduction.

Let $F : M^4 \rightarrow \mathbf{R}^5$ be a complete, simply connected, two-sided stable minimal hypersurface, and let g denote the pullback metric on M . On $N = M \setminus F^{-1}(0)$, let $\tilde{g} = r^{-2}g$, where r is the Euclidean distance function from 0.

Combining Theorem 3.1, Theorem 4.1, and Theorem 5.1 (with the appropriate rescaling of the μ -bubble metric), we have the following tool.

Lemma 6.1. *Let $X \subset N$ be a closed subset with boundary $\partial X = \partial_+ X \sqcup \partial_- X$. Suppose $d_{\tilde{g}}(\partial_+ X, \partial_- X) \geq 10\pi$. Then there is a connected, relatively open subset $\Omega \subset X$ with smooth boundary $\partial\Omega = \partial_- X \sqcup \Sigma$ so that*

- $\partial_- X \subset \Omega$,
- $\Sigma \subset X \setminus \partial X$ is a closed hypersurface,
- $\Omega \subset B_{10\pi}(\partial_- X)$, and
- any connected component Σ_0 of Σ has intrinsic diameter at most 2π and volume at most $16\pi^2$.

We first prove *a priori* that any complete, simply connected, two-sided stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has Euclidean volume growth. The proof follows the same strategy as [CL23].

Theorem 6.2. *Any complete, simply connected, two-sided stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$ satisfies*

$$\mathcal{H}^4(B_{\rho}(x_0) \subset M) \leq 8\pi^2 e^{44\pi} \rho^4$$

for all $\rho > 0$ and $x_0 \in M$.

Proof. Without loss of generality we can assume that $0 \in F(M)$ and that $F(x_0) = 0$. By Lemma 6.1, there is a relatively open subset $\tilde{\Omega} \subset N \setminus B_{\rho}(x_0)$ such that

- $\partial B_{\rho}(x_0) \subset \tilde{\Omega}$,
- $\tilde{\Omega} \subset \tilde{B}_{10\pi}(\partial B_{\rho}(x_0))$, and
- any connected component of $\partial\tilde{\Omega} \setminus \partial B_{\rho}(x_0)$ is smooth with \tilde{g} -volume at most $16\pi^2$.

Converting back to the metric g (see [CL23, Lemma 6.2]), we have

$$\tilde{\Omega} \subset \{x \in M : d_M(x, \partial B_\rho(x_0)) < \rho e^{10\pi}\} \subset B_{\rho e^{11\pi}}(x_0).$$

Hence, any connected component of $\partial\tilde{\Omega} \setminus \partial B_\rho(x_0)$ has g -volume at most $16\pi^2 e^{33\pi} \rho^3$. Since M is simply connected and has one end (by [CSZ97]), there is a precompact open set Ω containing $B_\rho(x_0) \cup \tilde{\Omega}$ that has exactly one boundary component, which is one of the components of $\partial\tilde{\Omega} \setminus \partial B_\rho(x_0)$. By the isoperimetric inequality for minimal hypersurfaces in \mathbf{R}^{n+1} [MS73] (we use the sharp version due to Brendle [Bre21] to improve the resulting constant), we have

$$\mathcal{H}^4(B_\rho(x_0) \subset M) \leq c_{\text{iso}}(16\pi^2)^{4/3} e^{44\pi} \rho^4,$$

where $c_{\text{iso}} = (128\pi^2)^{-1/3}$ in this dimension. This concludes the proof. \square

Proof of Theorem 1.1. Suppose that $M^4 \rightarrow \mathbf{R}^5$ is a complete, two-sided stable minimal immersion. Since stability passes to the universal cover, we can assume that M is simply connected. Theorem 6.2 implies that M has (intrinsic) Euclidean volume growth. Thus, [SSY75] implies that M is flat. \square

Proof of Corollary 1.2. Arguing exactly as in [CLS22, Corollary 2.5] if the asserted curvature estimates failed then there would exist a non-flat complete, two-sided stable minimal immersion $M^4 \rightarrow \mathbf{R}^5$. This contradicts Theorem 1.1. \square

Proof of Theorem 1.3. By Lemma 6.1, there is a relatively open set $\tilde{\Omega} \subset M \setminus F^{-1}(\{0\})$ so that

- $\partial M \subset \tilde{\Omega}$,
- $\tilde{\Omega} \subset \tilde{B}_{10\pi}(\partial M)$, and
- any connected component of $\partial\tilde{\Omega} \setminus \partial M$ is smooth with \tilde{g} -volume at most $16\pi^2$.

Since M is simply connected and ∂M is connected, we can find an open subset $M' \Subset \text{Int}(M)$ so that $\partial M'$ is connected and consists of one of the components of $\partial\tilde{\Omega} \setminus \partial M$. Converting between distances in g and distances in \tilde{g} by [CL23, Lemma 25], we have $M_{\rho_0}^* \subset M'$, where $\rho_0 = e^{-11\pi}$ and $M_{\rho_0}^*$ is the connected component of $F^{-1}(B_{\mathbf{R}^5}(0, \rho_0))$ that contains x_0 . By the same calculation as in the proof of Theorem 6.2, we have

$$\mathcal{H}^4(M_{\rho_0}^*) \leq 8\pi^2.$$

This finishes the proof. \square

Proof of Theorem 1.4. We follow [CL24b, §6]. By [Tys89, §3], if a complete, two-sided, minimal immersion $M^4 \rightarrow \mathbf{R}^5$ has $\int_M |A_M|^4 < \infty$ then M has finite index. As such, it suffices to consider a complete, two-sided, minimal immersion $M^4 \rightarrow \mathbf{R}^5$ with finite Morse index. Fix $x_0 \in M$ and consider (M, \tilde{g}) the Gulliver–Lawson conformal metric. Arguing as in [CL24b, Lemma 21], there's $k < \infty$ so that for any $\rho > 0$ sufficiently large so that $M \setminus B_\rho(x_0)$ (where $B_\rho(x_0)$ is the g -metric ball) is stable and if $B_\rho(x_0) \subset \Omega$ is bounded with smooth boundary and $M \setminus \Omega$ has only unbounded components, then $\partial\Omega$ has k components.

The proof of Theorem 6.2 directly carries over to prove that

$$\mathcal{H}^4(B_\rho(x_0) \subset M) \leq k 8\pi^2 e^{44\pi} \rho^4$$

for all $\rho \gg 0$ sufficiently large. Using the L^4 -estimates from [SSY75] on the exterior region of M we thus conclude that $\int_M |A_M|^4 < \infty$. It follows from [And84, Theorem A] (cf. [SZ98, Tys89]) that the ends of M are graphical over hyperplanes with bounded slope (see

[And84, p. 22]). Granted this, [Sch83, Proposition 3] gives they are regular at infinity. This completes the proof. \square

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