# On the Ramsey number of the double star

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#### Abstract

The double star  $S(m_1, m_2)$  is obtained from joining the centres of a star with  $m_1$  leaves and a star with  $m_2$  leaves. We give a short proof of a new upper bound on the two-colour Ramsey number of  $S(m_1, m_2)$  which holds for all  $m_1, m_2$  with  $\frac{\sqrt{5}+1}{2}m_2 < m_1 < 3m_2$ . Our result implies that for all positive m, the Ramsey number of the double star S(2m, m) is at most  $\lceil 4.275m \rceil + 1$ .

### 1 Introduction

The much studied Ramsey number R(H) of a graph H is defined as the smallest integer n such that every 2-colouring of the edges of  $K_n$  contains a monochromatic copy of H. The case when H is a complete graph is the subject of Ramsey's famous theorem from the 1930's, and determining Ramsey numbers of complete graphs is notoriously difficult. For a recent breakthrough, see [3].

Among the earliest non-complete graphs H to be studied were different kinds of trees. In 1967, Gerencsér and Gyárfás [4] showed that  $R(P_k) = k + \lfloor \frac{k+1}{2} \rfloor$ , where  $P_k$  is the k-edge path. For k-edge stars  $K_{1,k}$ , the Ramsey number is larger: Harary [6] observed in 1972 that  $R(K_{1,k}, K_{1,k}) = 2k$  if k is odd, and  $R(K_{1,k}, K_{1,k}) = 2k - 1$  if k is even.

Burr and Erdős [2] conjectured in 1976 that  $R(T_k) \leq R(K_{1,k}, K_{1,k})$ , for any tree  $T_k$  with k edges. For large k, it is known that  $R(T_k) \leq 2k$ , by the results of [9]. However, this bound far from best possible for paths, which motivated the search for a more fine-tuned conjecture. Note that paths are (almost) completely balanced trees, while stars are the most unbalanced trees. So, it seems natural to suspect that the Ramsey

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number of a tree might be related to its unbalancedness, i.e. the difference in size between the two bipartition classes.

It is easy to see that

$$R_B(T) := \max\{2t_1, t_1 + 2t_2\} - 1$$

is a lower bound for the Ramsey number of any tree T with bipartition classes of sizes  $t_1 \geq t_2 \geq 2$ . This can be seen by considering the *canonical colourings*, which are defined as follows. Take a complete graph G on  $R_B(T) - 1$  vertices. If  $t_1 > 2t_2$ , partition V(G) into two sets of equal size, colour all edges inside each set red and colour all remaining edges blue. If  $t_1 \leq 2t_2$ , take a set of  $t_1 + t_2 - 1$  vertices, colour all edges inside this set red, and colour all remaining edges blue. It is straightforward to see that no monochromatic copy of T is present in this colouring.

Note that if T is a path then  $R_B(T)=R(T)$ , and the same holds if T is a star with an even number of edges. In [1], Burr discusses the canonical colourings and expresses his belief that R(T) may be equal to  $R_B(T)$  unless T is an odd star. In 2002, Haxell, Łuczak, and Tingley [7] confirmed this suspicion asymptotically for all trees with linearly bounded maximum degree. Namely, they proved that for every  $\eta > 0$ , there exist  $t_0$  and  $\delta$  such that  $R(T) \leq (1+\eta)R_B(T)$  for each tree T with  $\Delta(T) \leq \delta t_1$  and  $t_1 > t_0$ , where  $t_1 \geq t_2$  are, as before, the sizes of the bipartition classes of the tree T.

But already in 1979, Grossman, Harary and Klawe [5] found that, contrary to Burr's suspicion, there are values of  $m_1, m_2$  such that  $R(S(m_1, m_2)) > R_B(S(m_1, m_2))$  (where  $S(m_1, m_2)$  is the double star with  $m_i$  leaves in partition class i). However, the examples from [5] still allowed for the possibility that for every tree T we would have that  $R(T) \leq R_B(T) + 1$ . The authors of [5] conjectured this to be the truth for all double stars, which they confirmed for a range of values of  $m_1, m_2$ . Currently, it is known that this holds if  $m_1 \geq 3m_2$  [5] or if  $m_1 \leq 1.699(m_2 + 1)$  [8]. In other words, for  $m_1, m_2 \in \mathbb{N}^+$  it holds that

$$R(S(m_1, m_2)) \le \max\{2m_1, m_1 + 2m_2\} + 2 = R_B(S(m_1, m_2)) + 1 \tag{1}$$

unless

$$1.699(m_2+1) < m_1 < 3m_2. (2)$$

But in general, inequality (1) is not true. Norin, Sun and Zhao [8] showed that  $R(S(m_1, m_2)) \ge 5m_1/3 + 5m_2/6 + o(m_2)$  for all  $m_1 \ge m_2 \ge 0$  and  $R(S(m_1, m_2)) \ge 189m_1/115 + 21m_2/23 + o(m_2)$  for all  $m_1 \ge 2m_2 \ge 0$ . In particular, their results imply that  $R(S(m_1, m_2)) > R_B(S(m_1, m_2)) + 1$  if  $m_1, m_2$  fulfill

$$\frac{7}{4}m_2 + o(m_2) \le m_1 \le \frac{105}{41}m_2 + o(m_2).$$

This range covers the special case that  $m_1 = 2m_2$ . For this case, the results from [8] yield that  $R(S(2m, m)) \ge 4.2m + o(m)$  while  $R_B(S(2m, m)) = 4m + 2$ . This discovery lead the authors of [8] to pose the following question.

**Question 1** (Norin, Sun and Zhao [8]). Is it true that R(S(2m, m)) = 4.2m + o(m)?

There are few results giving upper bounds on the Ramsey number of the double star for the range of  $m_1, m_2$  where (1) does not hold. The inequality  $R(S(m_1, m_2)) \leq 2m_1 + m_2 + 2$  for all  $m_1 \geq m_2 \geq 0$  was established in [5], where it is described as a 'weak upper bound'. In the preprint [8], very good asymptotic bounds for  $R(S(m_1, m_2))$  are obtained from a computer-assisted proof using the flag algebra method, but as these are not quick to state, we refer the reader to [8]. We remark that Theorem 4.5 from [8], used with the invalid pair number 5 from Table 1 of [8], implies that  $\lim_{m\to\infty} R(S(2m, m))/m$  is bounded from above by 4.21526.

Our main result is a short elementary proof of a new upper bound on  $R(S(m_1, m_2))$  which holds for all values of  $m_1, m_2 \in \mathbb{N}^+$  fulfilling  $\frac{\sqrt{5}+1}{2}m_2 < m_1 < 3m_2$ . Observe that  $\frac{\sqrt{5}+1}{2} > 1.618$ , and thus our result covers the whole range of values of  $m_1, m_2$  from (2).

**Theorem 2.** Let  $m_1, m_2 \in \mathbb{N}^+$ , with  $\frac{\sqrt{5}+1}{2}m_2 < m_1 < 3m_2$ . Then

$$R(S(m_1, m_2)) \le \left\lceil \sqrt{2m_1^2 + (m_1 + \frac{m_2}{2})^2} + \frac{m_2}{2} \right\rceil + 1.$$

As an immediate corollary of our theorem, we obtain for the double star S(2m, m) the following bound.

Corollary 3.  $R(S(2m, m)) \leq \lceil 4.27492m \rceil + 1 \text{ for all } m \in \mathbb{N}^+.$ 

### 2 Preliminaries

In this section we prepare the proof of the main result, Theorem 2, by proving some auxiliary results. We start with a very simple lemma for recurrent later use. A similar lemma appears in [8].

**Lemma 4.** Let  $m_1, m_2 \in \mathbb{N}$ , let G be a graph and let  $vw \in E(G)$  such that  $d(v) > m_1$ ,  $d(w) > m_2$ , and  $|N(v) \cup N(w)| \ge m_1 + m_2 + 2$ . Then  $S(m_1, m_2) \subseteq G$ .

*Proof.* To form the double star with central edge vw, first choose  $m_1$  neighbours of v, as many as possible outside  $N(w) \cup \{w\}$ , the others in N(w). Then, choose  $m_2$  neighbours of w in N(w), different from v and from the previously chosen neighbours of v. This concludes the proof.

Next we show a useful statement about vertex degrees when no double star is present.

**Lemma 5.** Let  $m_1, m_2 \in \mathbb{N}$ , and let G be a graph on  $n \geq m_1 + m_2 + 2$  vertices such that  $S(m_1, m_2) \not\subseteq G$ . Let  $v \in V(G)$ , let  $A \subseteq N(v)$  with  $|A| > m_1$  and  $d(u) > m_2$  for each  $u \in A$ . Let  $w \in A$ . Then w has at most  $m_1 + m_2 - |A|$  neighbours in  $V(G) \setminus (A \cup \{v\})$ . Furthermore, there is a vertex  $z \in V(G) \setminus (A \cup \{v\})$  having at most

$$\frac{m_1+m_2-|A|}{n-|A|-1}\cdot |A|$$

neighbours in A.

Proof. Set  $D := V(G) \setminus (A \cup \{v\})$ . If w has  $m_1 + m_2 - |A| + 1$  or more neighbours in D, then  $|N(v) \cup N(w)| \ge |A| + (m_1 + m_2 - |A| + 1) + |\{v\}| = m_1 + m_2 + 2$  (we count v as a neighbour of w), and we can apply Lemma 4 to see that  $S(m_1, m_2) \subseteq G$ , which is a contradiction.

So w has at most  $m_1 + m_2 - |A|$  neighbours in D, which is as desired. Further, as this holds for every  $u \in A$ , the average number of neighbours in A of a vertex from D is at most

$$\frac{(m_1 + m_2 - |A|) \cdot |A|}{|D|} = \frac{m_1 + m_2 - |A|}{n - |A| - 1} \cdot |A|.$$

So any vertex  $z \in D$  having at most the average number of neighbours in A is as desired.

We will also need a lemma from [8], whose elementary proof can be found there.

**Lemma 6** (Lemma 2.3 in [8]). Let  $n \ge \max\{2m_1, m_1 + 2m_2\} + 2$ , and let the edges of  $K_n$  be coloured with red and blue such that there is no monochromatic  $S(m_1, m_2)$ . Then there is a colour  $C \in \{red, blue\}$  such that each vertex of  $K_n$  has degree at most  $m_1$  in colour C.

### 3 Proof of Theorem 2.

The whole section is devoted to the proof of Theorem 2. Let  $m_1, m_2 \in \mathbb{N}^+$  be given, fulfilling

$$\frac{\sqrt{5}+1}{2}m_2 < m_1 < 3m_2. \tag{3}$$

Set

$$m_3 := \left\lceil \sqrt{2m_1^2 + (m_1 + \frac{m_2}{2})^2} - (m_1 + \frac{m_2}{2}) \right\rceil. \tag{4}$$

Using (3) and (4), it is easy to calculate that

$$m_3 > \max\{m_2, m_1 - m_2\},\tag{5}$$

and in particular, we have that  $m_3 \geq 1$ . Set  $n := m_1 + m_2 + m_3 + 1$ , and let a red and blue colouring of the edges of  $K_n$  be given. Let  $G_r$  be the subgraph of  $K_n$  induced by the red edges, and  $G_b$  be the subgraph of  $K_n$  induced by the blue edges. For any  $u \in V(K_n)$ , let  $N_r(u)$  be the set of all neighbours of u in  $G_r$ , and let  $N_b(u)$  be the set of all neighbours of u in  $G_b$ . Set  $d_r(u) := |N_r(u)|$  and  $d_b(u) := |N_b(u)|$ .

For contradiction assume that there is no monochromatic  $S(m_1, m_2)$ . Note that  $n \ge \max\{2m_1, m_1 + 2m_2\} + 2$  because of (5) and since n is an integer. So, we can use Lemma 6 to see that there is a colour, which we may assume to be blue, such that every vertex has degree at most  $m_1$  in that colour. That is,  $d_b(u) \le m_1$  for all  $u \in V(G)$ , and thus,

$$\delta(G_r) \ge m_2 + m_3. \tag{6}$$

Now choose any vertex v and a subset A of  $N_r(v)$  with

$$|A| = m_2 + m_3. (7)$$

By (6), and since  $m_2 + m_3 > m_1$  by (5), we know that  $|A| > m_1$  and  $\delta(G_r) > m_2$ . So, we can use Lemma 5 in  $G_r$  to see that for any  $w \in A$ , we have

$$|N_r(w) \setminus (A \cup \{v\})| \le m_1 + m_2 - (m_2 + m_3) = m_1 - m_3.$$

and therefore,

$$|N_r(w) \cap (A \cup \{v\})| = d_r(w) - |N_r(w) \setminus (A \cup \{v\})|$$

$$\geq m_2 + m_3 - (m_1 - m_3)$$

$$= m_2 + 2m_3 - m_1.$$
(8)

We employ Lemma 5 once more, this time to find a vertex  $z \notin A \cup \{v\}$  such that

$$|N_r(z) \cap A| \le \frac{m_1 + m_2 - |A|}{n - |A| - 1} \cdot |A| = \frac{m_1 - m_3}{m_1} \cdot (m_2 + m_3),$$

where we use (7) for the equality. We deduce that

$$|N_r(z) \setminus A| = d_r(z) - |N_r(z) \cap A|$$

$$\geq (m_2 + m_3) - \frac{m_1 - m_3}{m_1} \cdot (m_2 + m_3)$$

$$= (m_2 + m_3) \frac{m_3}{m_1}.$$
(9)

Further, note that  $d_b(z) \leq m_1 < m_2 + m_3 = |A|$  because of (6), (5) and (7). Therefore, we know that vertex z sends at least one red edge to A. Consider any red edge uz with  $u \in A$ . Using (8) and (9), we get

$$|N_r(u) \cup N_r(z)| \ge |N_r(u) \cap (A \cup \{v\})| + |N_r(z) \setminus A| + |\{u, z\}|$$

$$\ge m_2 + 2m_3 - m_1 + (m_2 + m_3) \frac{m_3}{m_1} + 2$$

$$\ge m_1 + m_2 + 2,$$

where for the last inequality we use the fact that  $2m_1m_3 + m_2m_3 + m_3^2 \ge 2m_1^2$  which can be calculated from (4). So, we can apply Lemma 4 to find a red double star with central edge uz, and are done.

## Acknowledgment

The second author would like to thank C. Karamchedu, M. Karamchedu and M. Klawe for pointing out a missing ceiling in the bound of Theorem 2 in an earlier version of this paper.

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