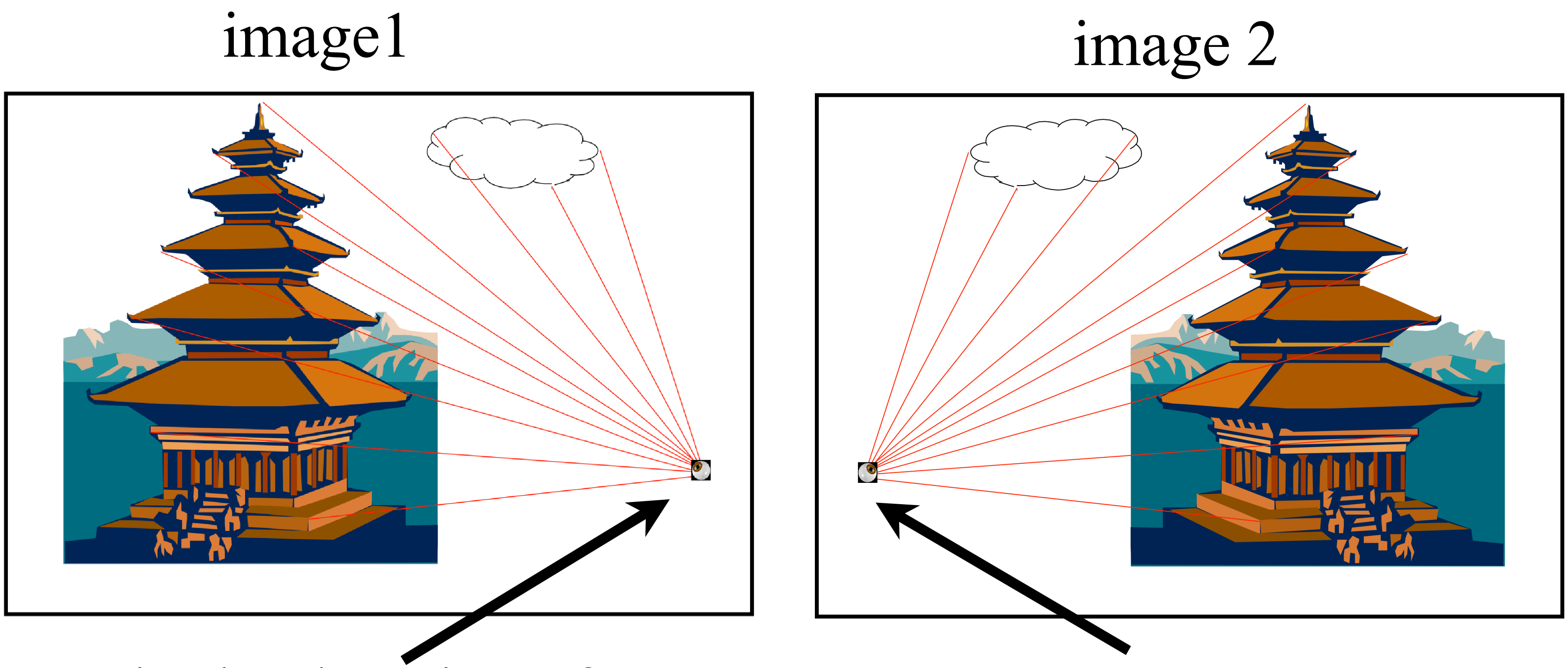
Lecture 19:

Essential and Fundamental Matrices

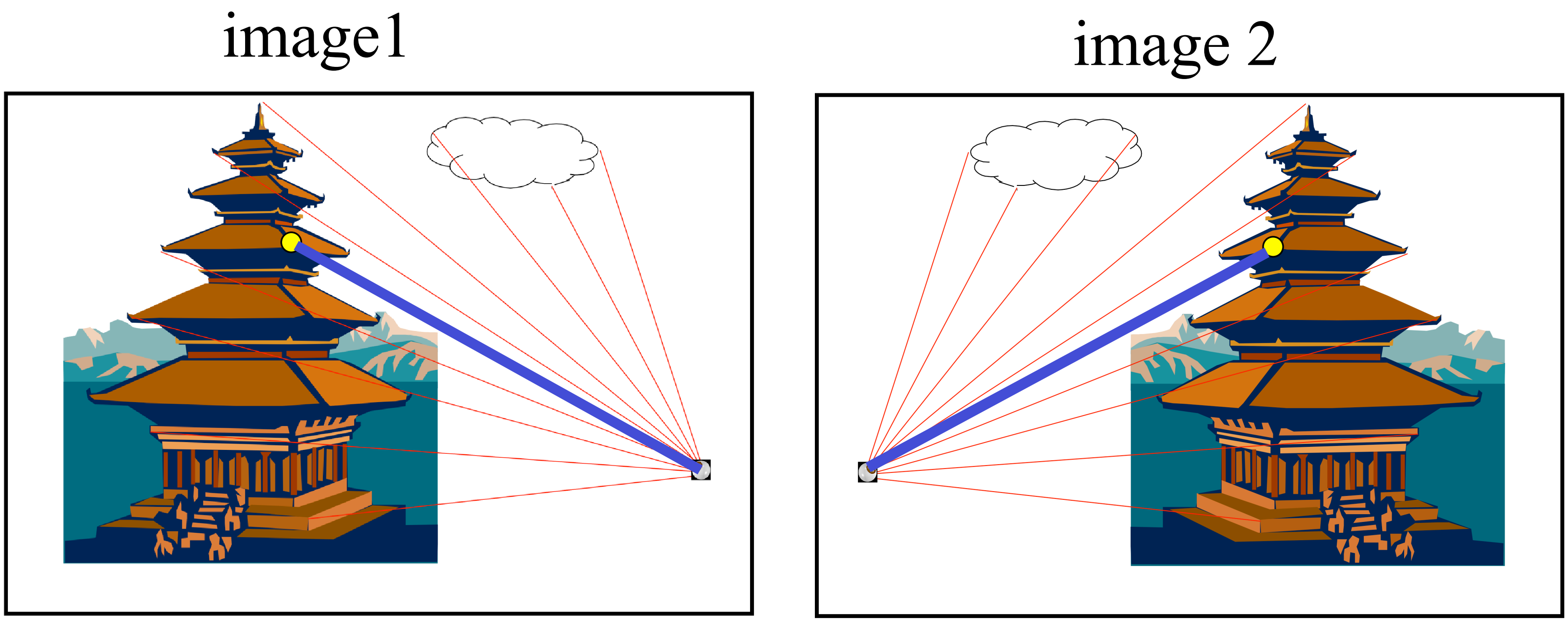
Epipolar Geometry



Epipole : location of cam2 as seen by cam1.

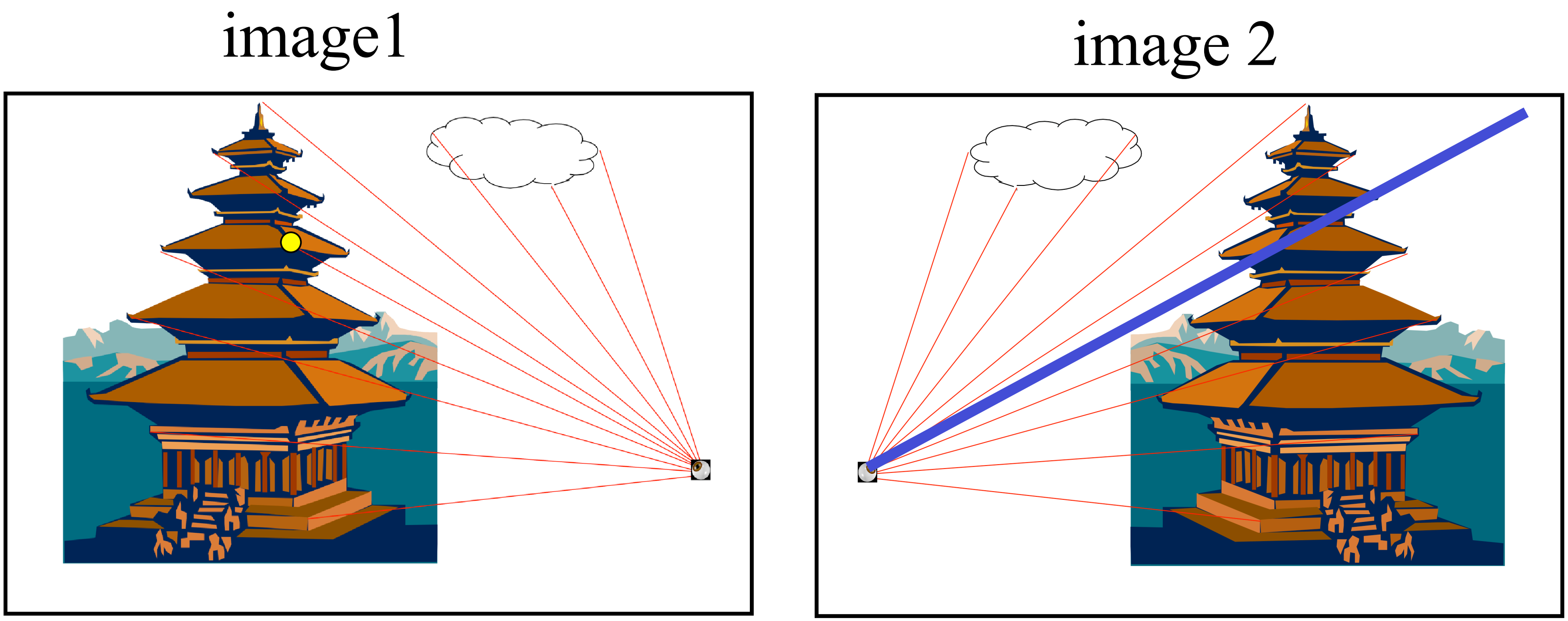
Epipole : location of cam1 as seen by cam2.

Epipolar Geometry



Corresponding points
lie on conjugate epipolar lines

This Lecture…



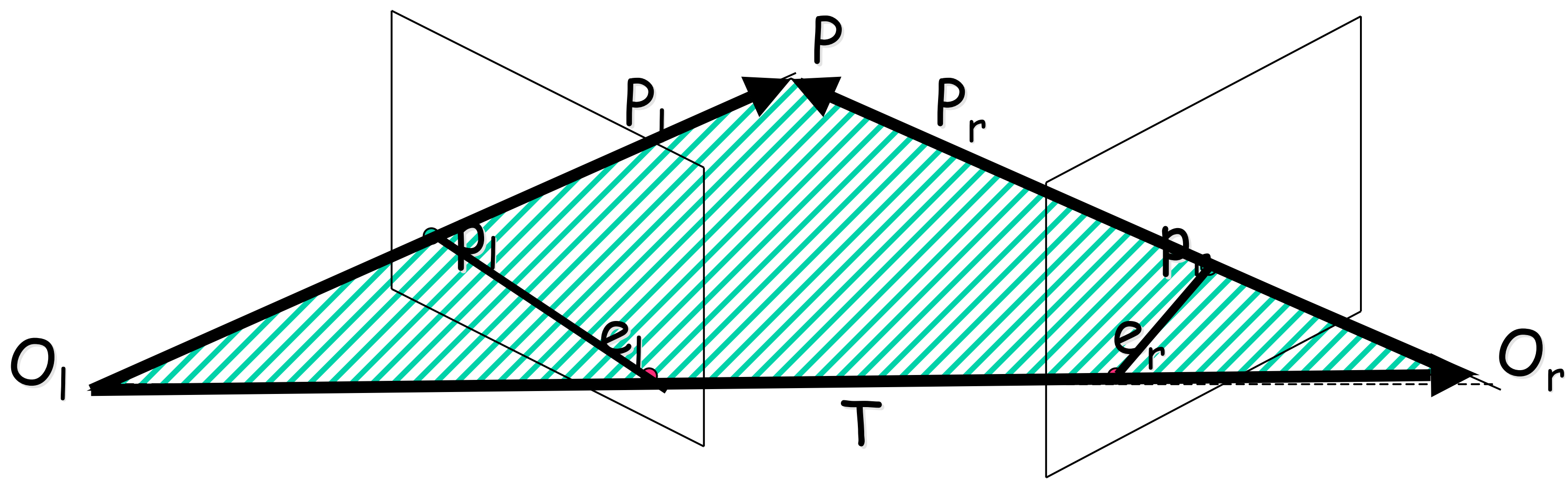
Given a point in one image, how do we determine the corresponding epipolar line to search along in the second image?

Essential Matrix

The essential and fundamental matrices are 3x3 matrices that “encode” the epipolar geometry of two views.

Motivation: Given a point in one image, multiplying by the essential/fundamental matrix will tell us which epipolar line to search along in the second view.

Essential Matrix



$$P_r = R(P_l - T)$$

R,T = rotation, and translation

$$\Rightarrow P_r^T R S P_l = 0$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

$$\Rightarrow P_r^T E P_l = 0$$

E=RS is “essential matrix”

Essential Matrix Properties

$$E = RS$$

- has rank 2
  ➔ has both a left and right nullspace (important!!!!)

- depends only on the EXTRINSIC Parameters (R & T)

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Longuet-Higgins equation

$$P_r^T E P_l = 0$$

$$p_l = \frac{f_l}{Z_l} P_l \qquad p_r = \frac{f_r}{Z_r} P_r$$

$$(\cancel{\frac{Z_r}{f_r}} p_r)^T E (\cancel{\frac{Z_l}{f_l}} p_l) = 0$$

$$\boxed{p_r^T E p_l = 0}$$

Longuet-Higgins equation

$$P_r^T E P_l = 0$$

This relates viewing rays

Importance of Longuet-Higgens ...

This relates 2D film points

$$\boxed{p_r^T E p_l = 0}$$

Longuet-Higgins Makes Sense

- Note, there is nothing magic about Longuet-Higgins equation.
- A film point can also be thought of as a viewing ray. They are equivalent.
    - (u,v) 2D film point
    - (u,v,f) 3D point on film plane
    - k(u,v,f) viewing ray into the scene
    - k(X, Y, Z) ray through point P in the scene
      [hint: k=f/Z, and we have u=fX/Z, v=fY/Z].

Epipolar Lines

- Let l be a line in the image:

$$au + bv + c = 0$$

- Using homogeneous coordinates:

$$\tilde{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \tilde{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \boxed{\tilde{p}^T \tilde{l} = \tilde{l}^T \tilde{p} = 0}$$

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Epipolar Lines

- Remember:

$$p_r^T E p_l = 0$$

$$\tilde{l}_r = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

p_r belongs to epipolar line in the right image defined by

$$\tilde{l}_r = E p_l$$

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Epipolar Lines

- Remember:

$$p_r^T E p_l = 0$$

$$\tilde{l}_l^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T$$

p_l belongs to epipolar line in the left image defined by

$$\tilde{l}_l = E^T p_r$$

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Epipoles

- Remember: epipoles belong to the epipolar lines

$$e_r^T E p_l = 0 \qquad\qquad p_r^T E e_l = 0$$

- And they belong to <u>all</u> the epipolar lines

$$e_r^T E = 0 \qquad\qquad E e_l = 0$$

We can use this to compute the location of the epipoles.
There will be an example, shortly...

Essential Matrix Summary

Longuet-Higgins equation $$p_r^T E p_l = 0$$

Epipolar lines:

$$\tilde{p_r}^T \tilde{l_r} = 0 \qquad \tilde{p_l}^T \tilde{l_l} = 0$$

$$\tilde{l_r} = E p_l \qquad \tilde{l_l} = E^T p_r$$

Epipoles:

$$e_r^T E = 0 \qquad E e_l = 0$$

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Fundamental Matrix

The essential matrix uses CAMERA coordinates

To use image coordinates we must consider the INTRINSIC camera parameters:

Pixel coord (row,col)

Affine transform matrix

Camera (film) coord

$$\bar{p_l} = M_l p_l \qquad p_l = M_l^{-1} \bar{p_l}$$

$$\bar{p_r} = M_r p_r \qquad p_r = M_r^{-1} \bar{p_r}$$

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Fundamental Matrix

$$p_l = M_l^{-1} \bar{p}_l$$

$$p_r = M_r^{-1} \bar{p}_r$$

$$p_r^T E p_l = 0$$

$$(M_r^{-1} \bar{p}_r)^T E (M_l^{-1} \bar{p}_l) = 0$$

$$\bar{p}_r^{\,T} (M_r^{-T} E M_l^{-1}) \bar{p}_l = 0$$

$$\boxed{\bar{p}_r^{\,T} F \bar{p}_l = 0}$$

short version: The same equation works in pixel coordinates too!

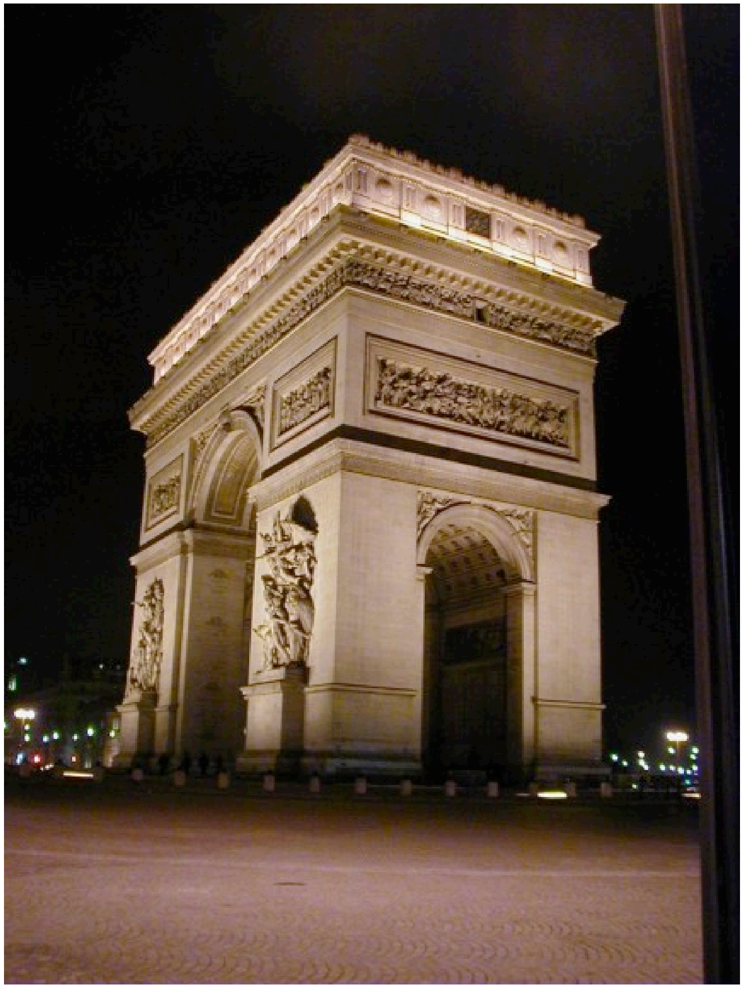
Fundamental Matrix Properties

$$F = M_r^{-T} R S M_l^{-1}$$

- has rank 2
- depends on the INTRINSIC and EXTRINSIC Parameters (f, etc ; R & T)

Analogous to essential matrix. The fundamental matrix also tells how pixels (points) in each image are related to epipolar lines in the other image.

Example



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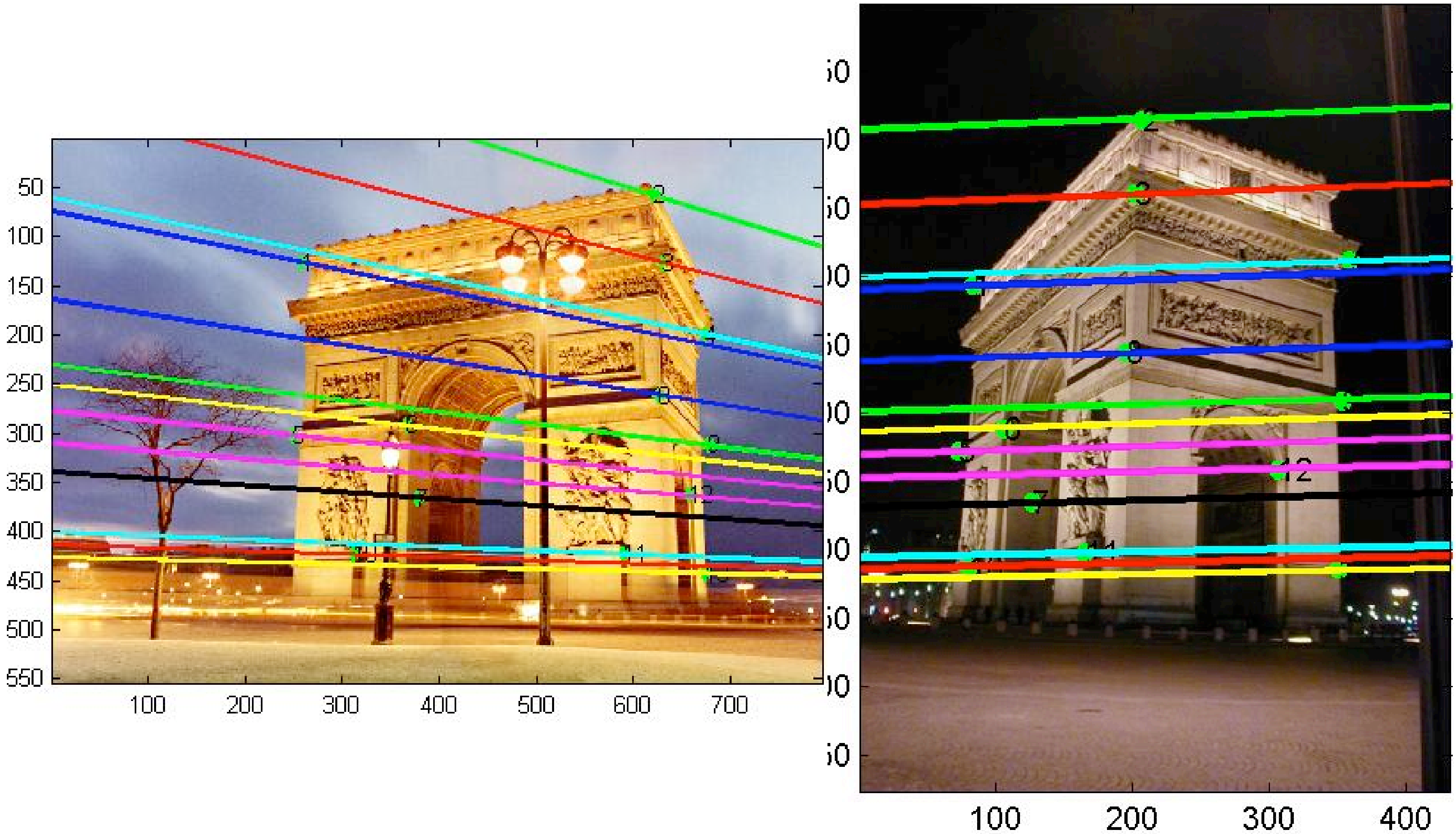
Example



Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix}$$

Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \begin{pmatrix} 343.53 \\ 221.70 \\ 1.0 \end{pmatrix}$$

x = 343.5300 y = 221.7005

$$\begin{matrix} 0.0001 \\ 0.0045 \\ -1.1942 \end{matrix} \rightarrow \begin{matrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{matrix}$$

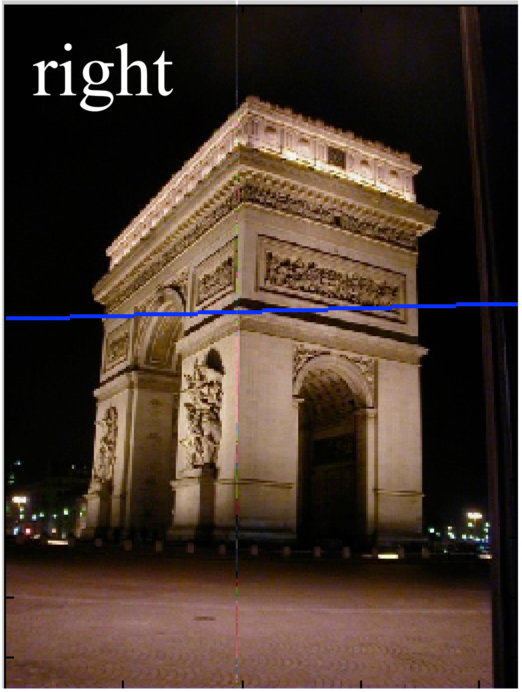
normalize so sum of squares
of first two terms is 1 (optional)

Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \begin{pmatrix} 343.53 \\ 221.70 \\ 1.0 \end{pmatrix}$$

x = 343.5300 y = 221.7005

$$\begin{matrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{matrix}$$



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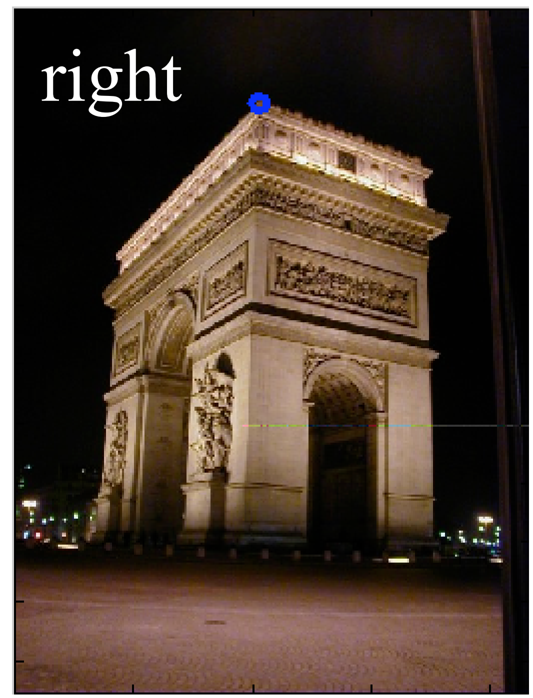
Example

$$
\begin{pmatrix} 205.5526 & 80.5 & 1.0 \end{pmatrix}
\begin{pmatrix}
-0.00310695 & -0.0025646 & 2.96584 \\
-0.028094 & -0.00771621 & 56.3813 \\
13.1905 & -29.2007 & -9999.79
\end{pmatrix}
$$

$$L = (0.0010 \quad -0.0030 \quad -0.4851)$$

$$\rightarrow (0.3211 \quad -0.9470 \; -151.39)$$

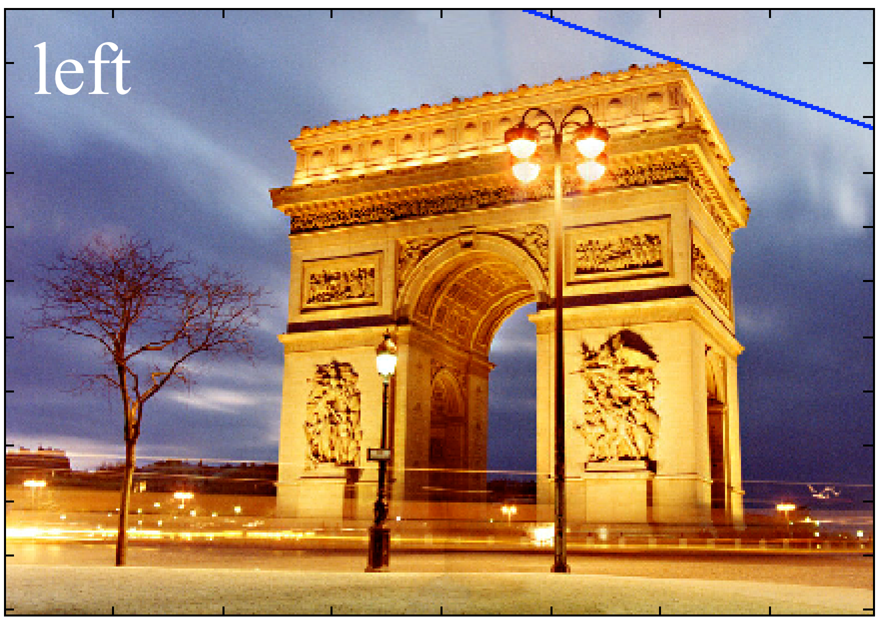


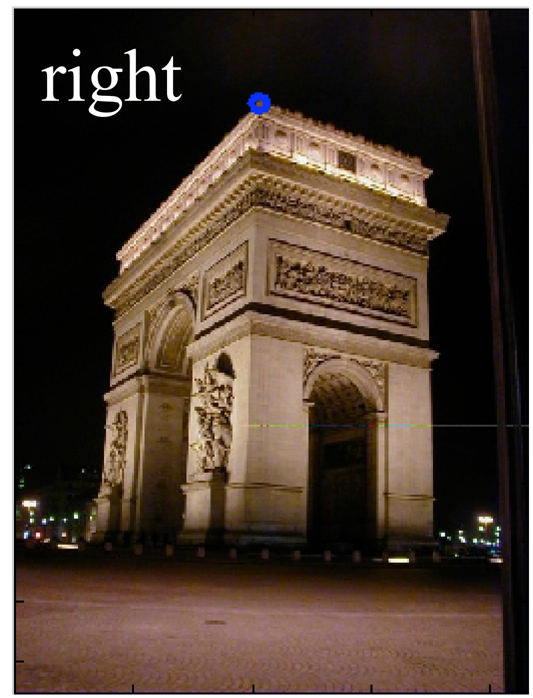
x = 205.5526 y = 80.5000

Example

$$\begin{pmatrix} 205.5526 & 80.5 & 1.0 \end{pmatrix} \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix}$$

$$L = \begin{pmatrix} 0.3211 & -0.9470 & -151.39 \end{pmatrix}$$

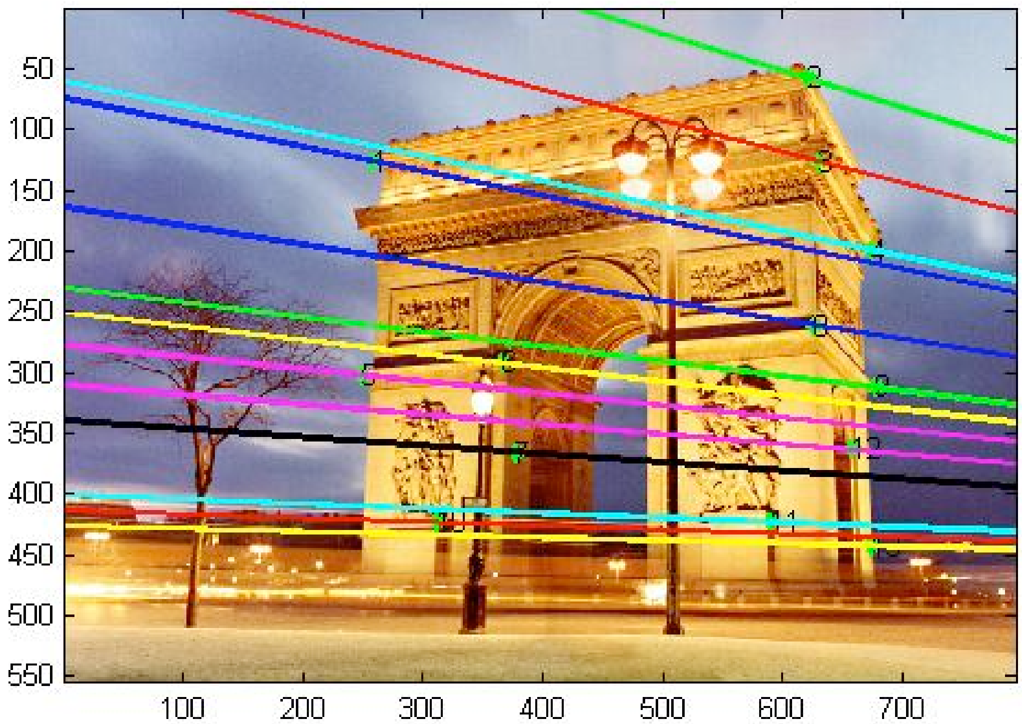




x = 205.5526 y = 80.5000

Example

where is the epipole?



$$F * e_L = 0$$

vector in the right nullspace of matrix F

However, due to noise, F may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of F

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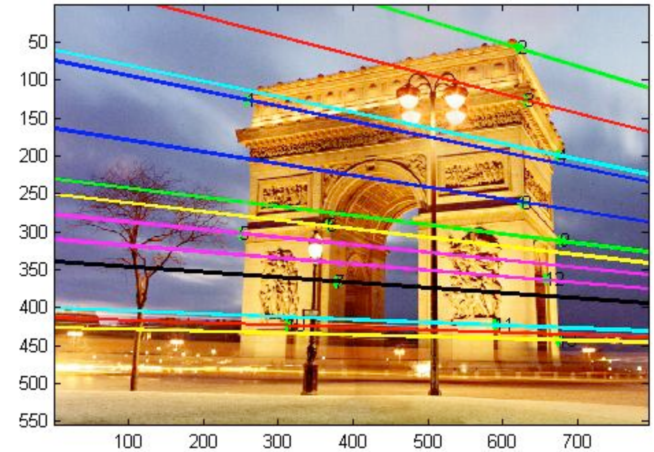
Example



>> [u,d] = eigs(F' * F)

u =

-0.0013	0.2586	-0.9660
0.0029	-0.9660	-0.2586
1.0000	0.0032	-0.0005

d = 1.0e8*

-1.0000	0	0
0	-0.0000	0
0	0	-0.0000

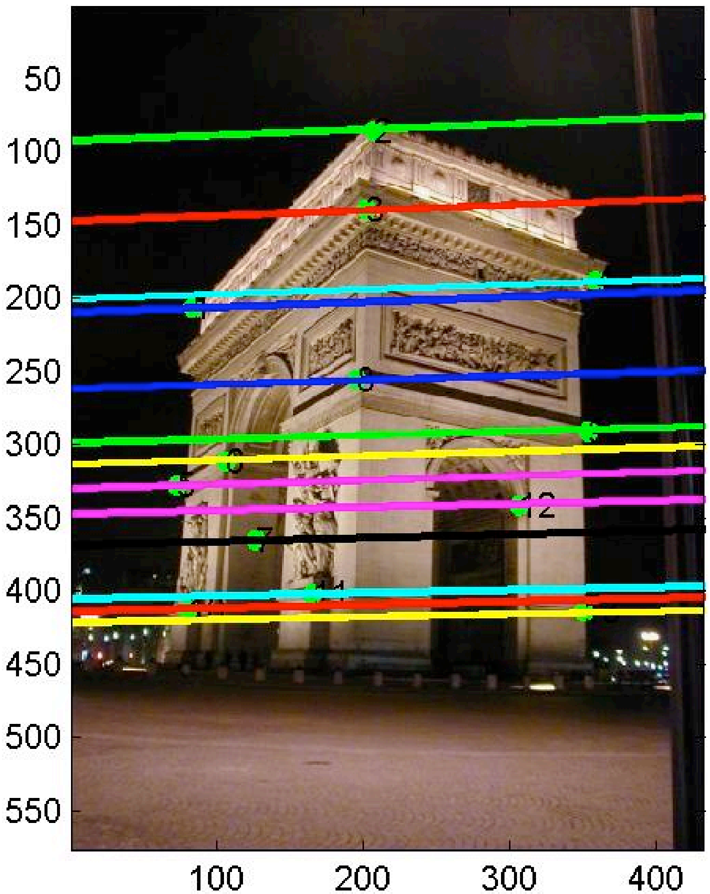
eigenvector associated with smallest eigenvalue

>> uu = u(:,3)
uu = (-0.9660 -0.2586 -0.0005)

>> uu / uu(3) : to get pixel coords
(1861.02 498.21 1.0)

Example

where is the epipole?



$$e'_r * F = 0$$

$$\rightarrow F' * e_r = 0$$

vector in the right nullspace of matrix F’

However, due to noise, F’ may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of F’

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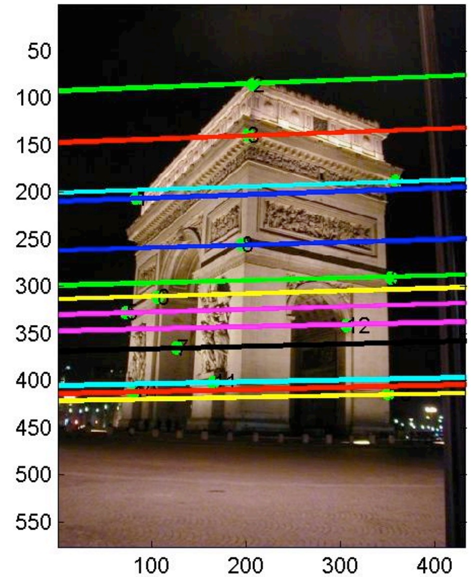
Example

>> [u,d] = eigs(F * F')

$$u = \begin{matrix} -0.0003 & -0.0618 & \boxed{-0.9981} \\ -0.0056 & -0.9981 & \boxed{0.0618} \\ 1.0000 & -0.0056 & \boxed{0.0001} \end{matrix}$$

$$d = 1.0e8* \begin{matrix} -1.0000 & 0 & 0 \\ 0 & -0.0000 & 0 \\ 0 & 0 & -0.0000 \end{matrix}$$



eigenvector associated with smallest eigenvalue

>> uu = u(:,3)
uu = (-0.9981 0.0618 0.0001)

>> uu / uu(3) : to get pixel coords
(-19021.8 1177.97 1.0)