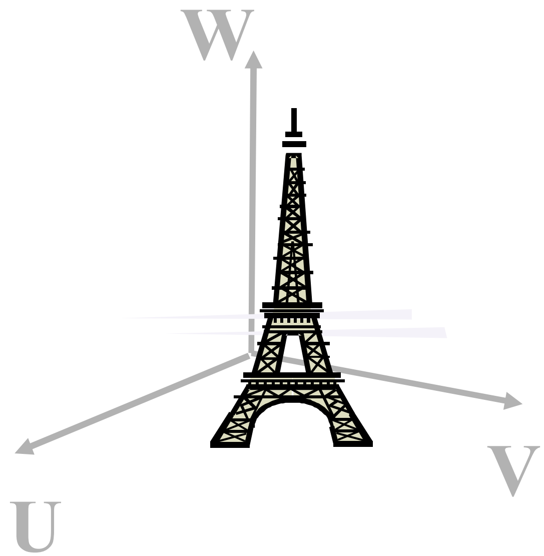
Lecture 12:
Camera Projection

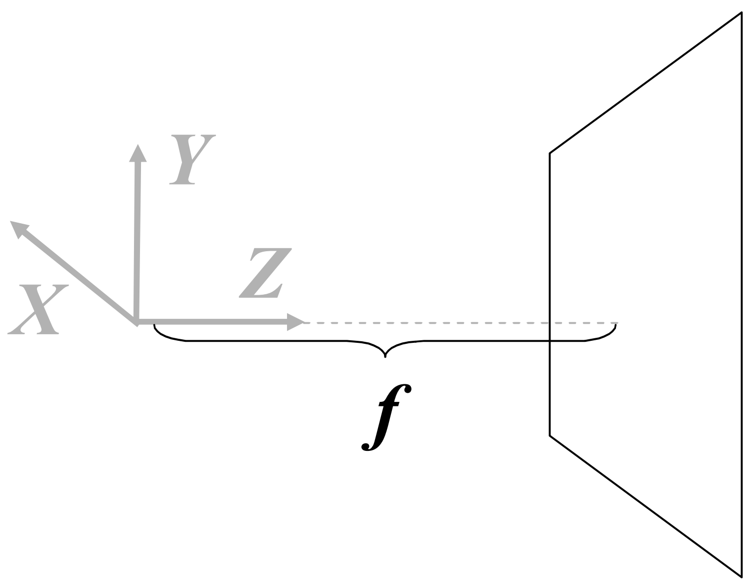
Reading: T&V Section 2.4

Imaging Geometry

Object of Interest in World Coordinate System (U,V,W)



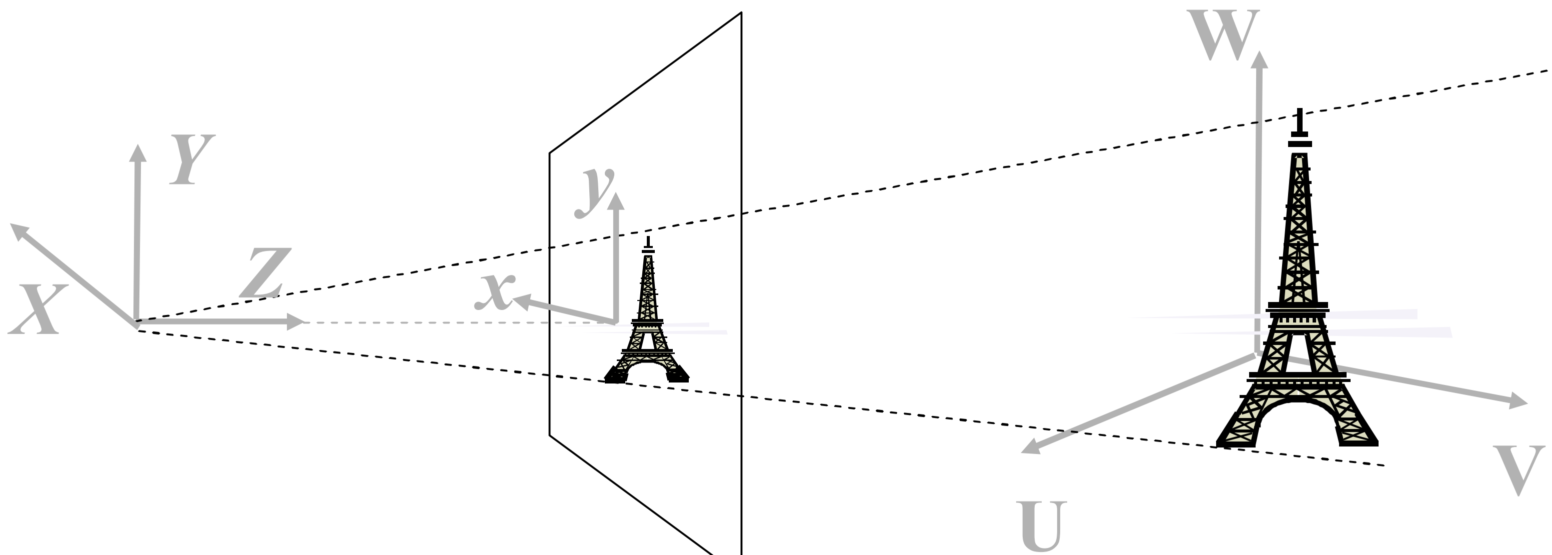
Imaging Geometry



Camera Coordinate System (X,Y,Z).

- Z is optic axis
- Image plane located f units out along optic axis
- f is called focal length

Imaging Geometry



Forward Projection onto image plane.
3D (X,Y,Z) projected to 2D *(x,y)*

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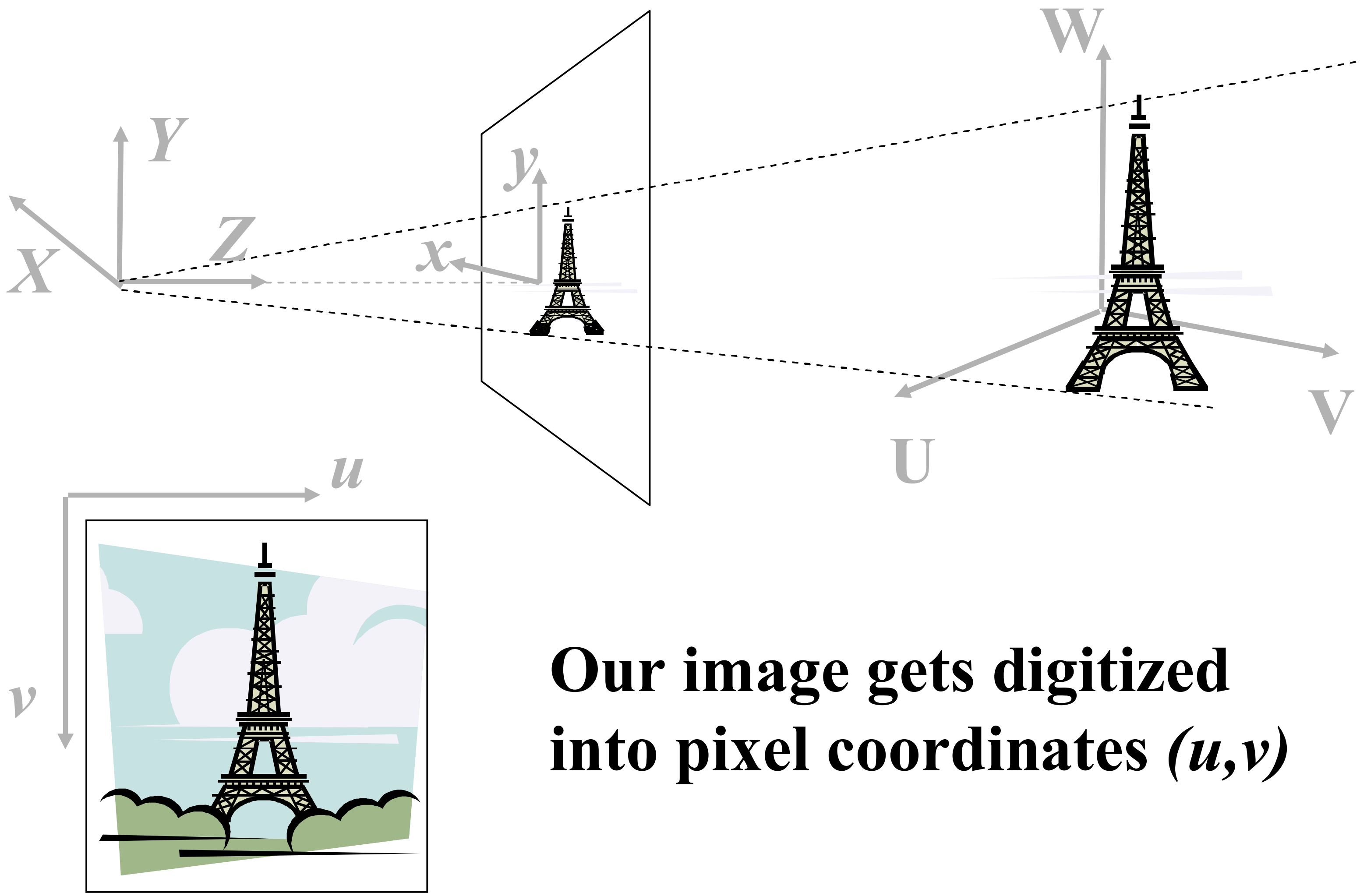
Imaging Geometry



Our image gets digitized into pixel coordinates *(u,v)*

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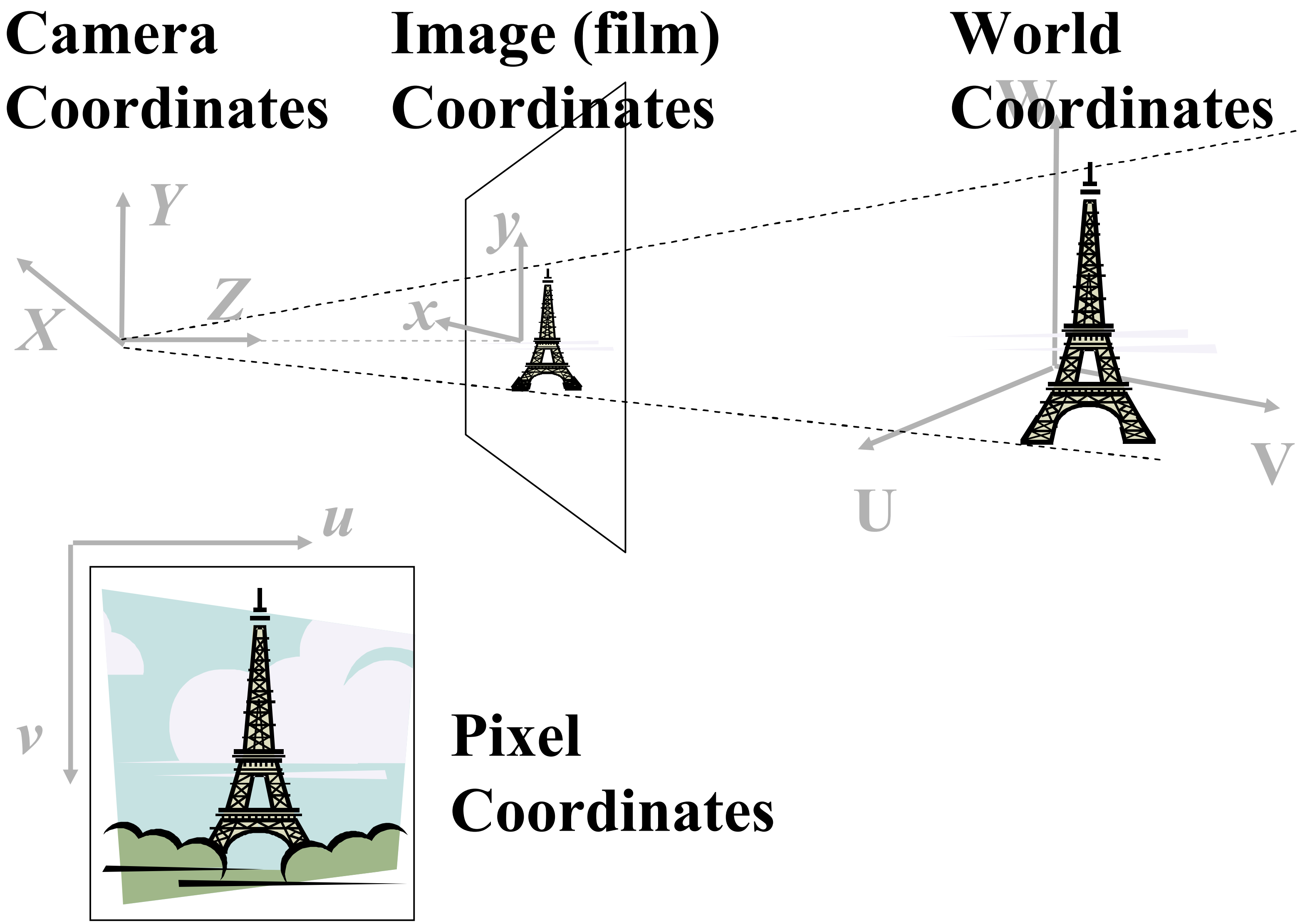
Imaging Geometry



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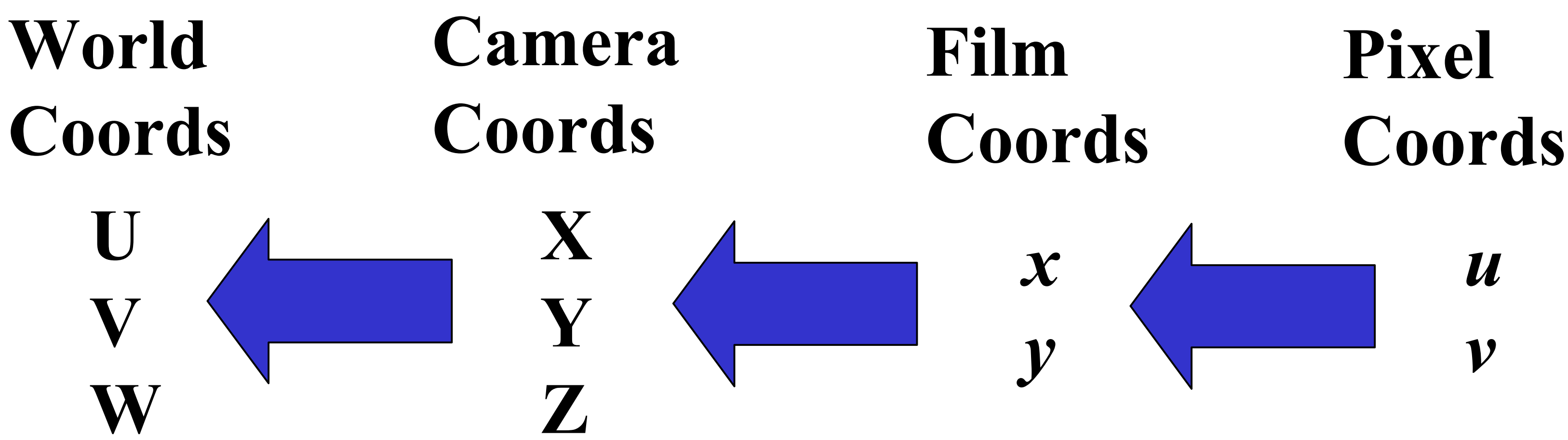
Forward Projection

World Coords		Camera Coords		Film Coords		Pixel Coords
U V W	→	X Y Z	→	*x* *y*	→	*u* *v*

We want a mathematical model to describe how 3D World points get projected into 2D Pixel coordinates.

Our goal: describe this sequence of transformations by a big matrix equation!

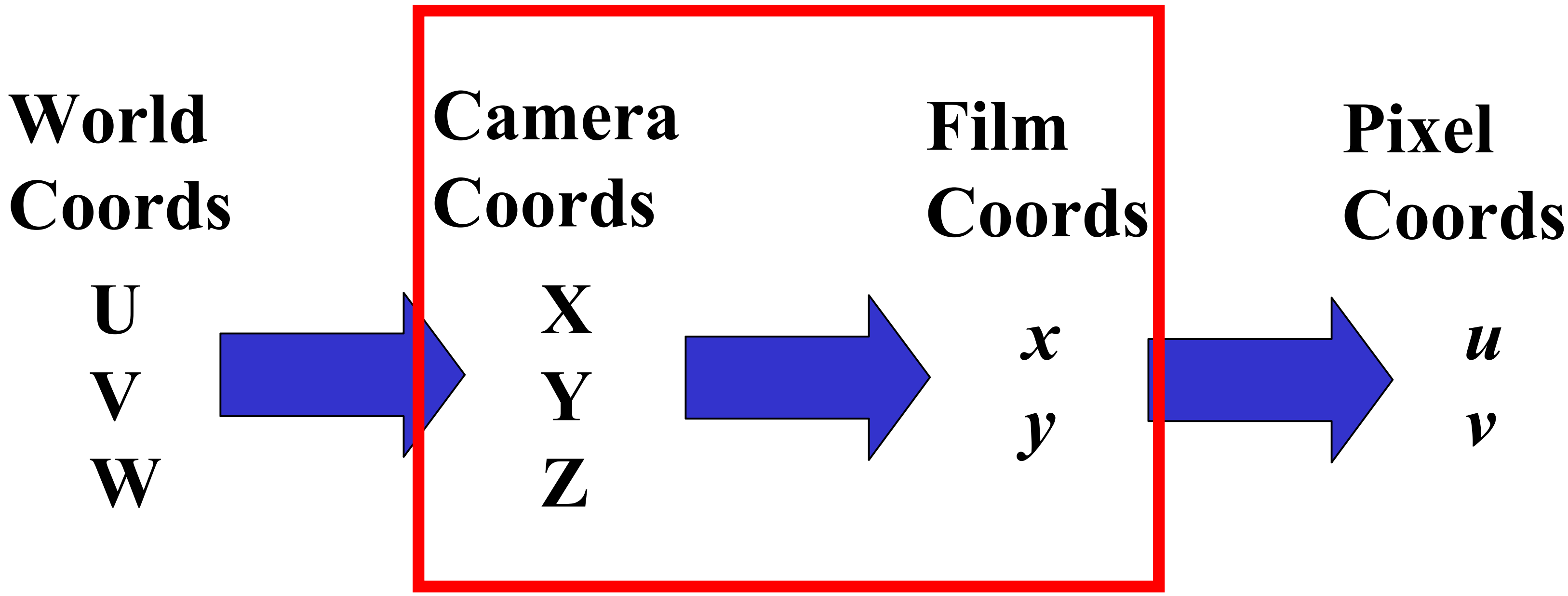
Backward Projection



Note, much of vision concerns trying to derive backward projection equations to recover 3D scene structure from images (via stereo or motion)

But first, we have to understand forward projection…

Forward Projection



3D-to-2D Projection
• perspective projection

We will start here in the middle, since we've already talked about this when discussing stereo.

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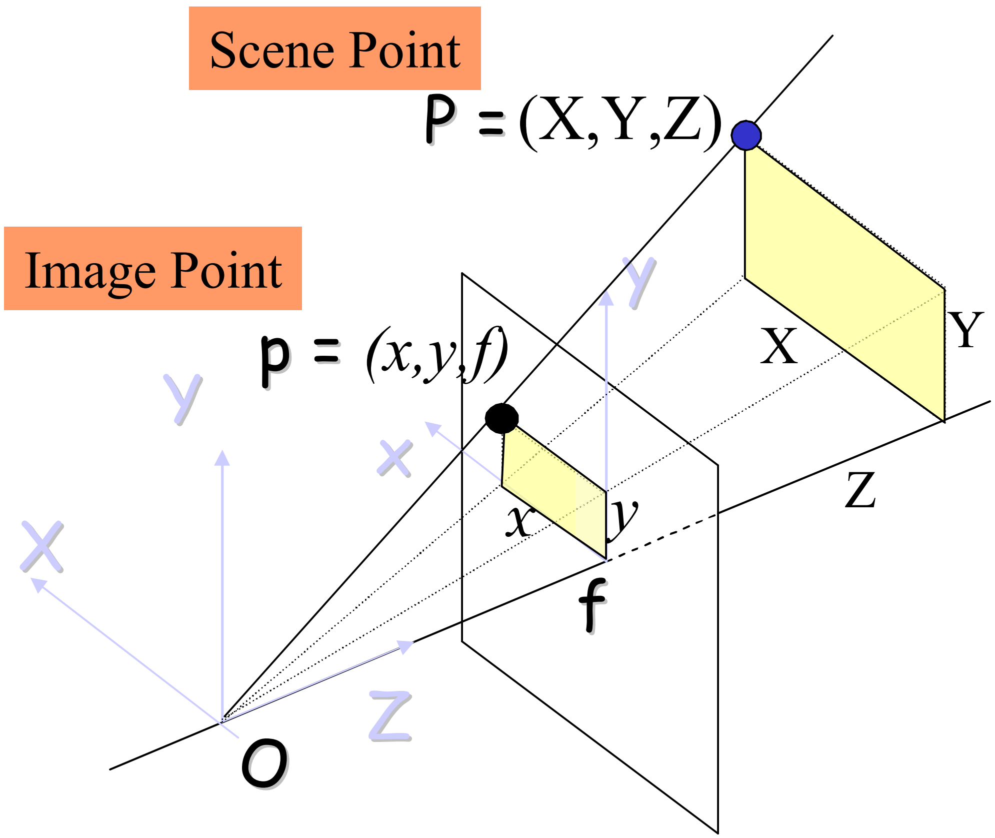
Basic Perspective Projection

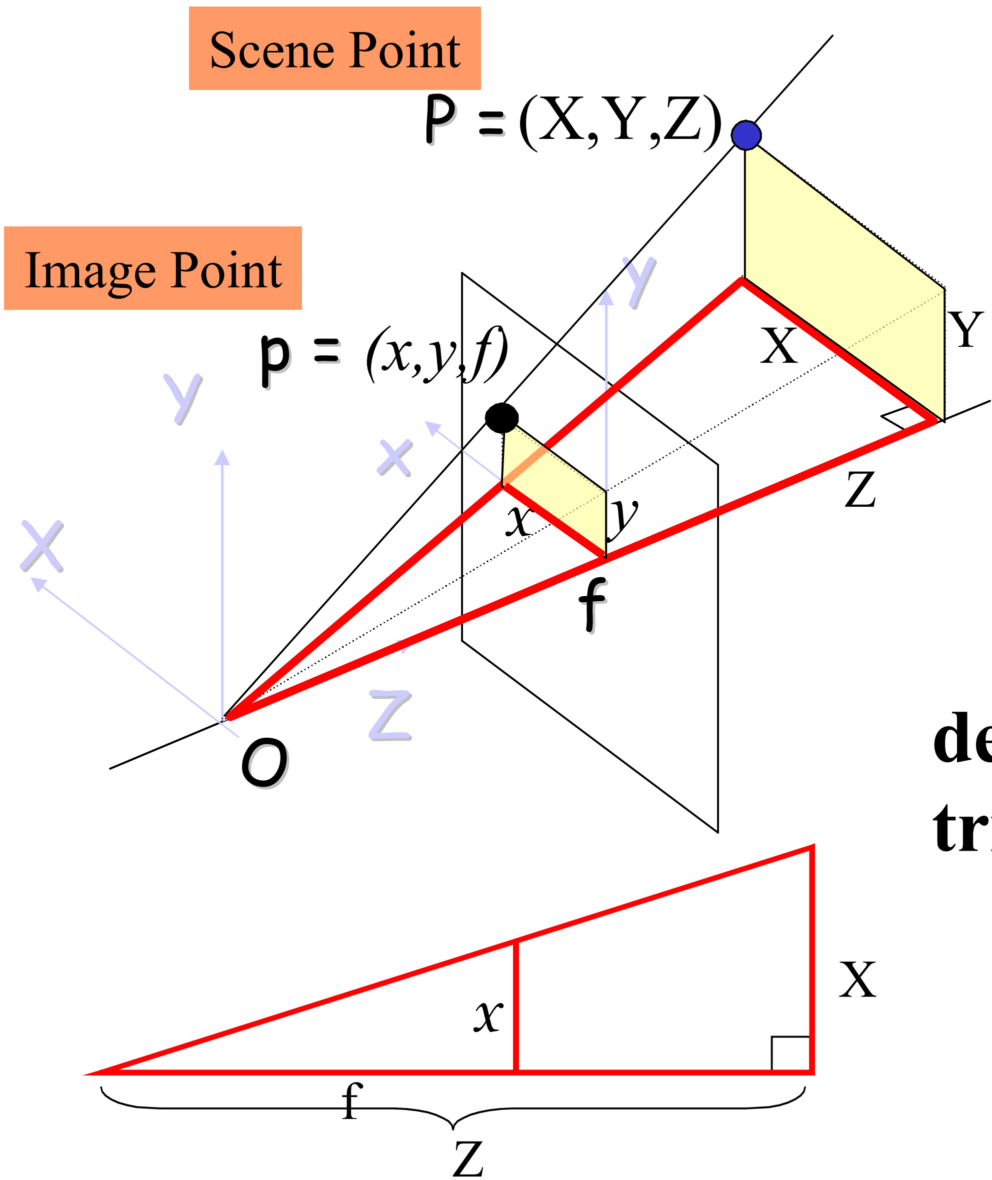


Perspective Projection Eqns

$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

Basic Perspective Projection



Perspective Projection Eqns

$$x = f\,\frac{X}{Z}$$

$$y = f\,\frac{Y}{Z}$$

derived via similar triangles rule

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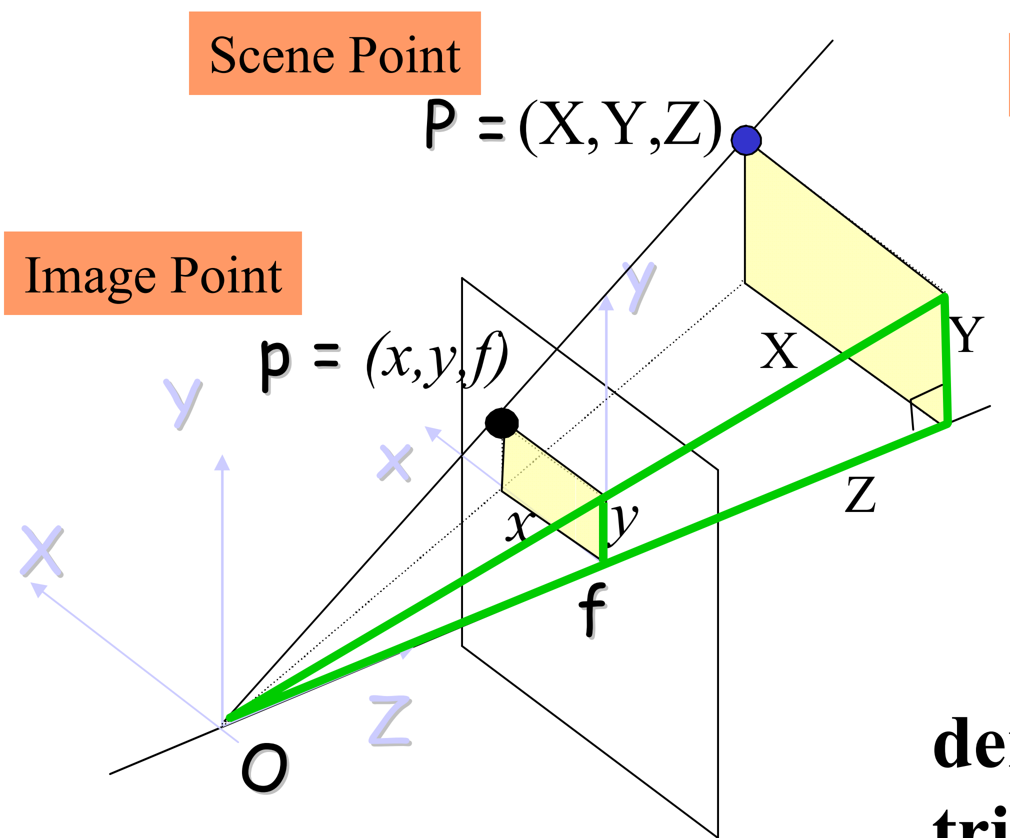
Basic Perspective Projection

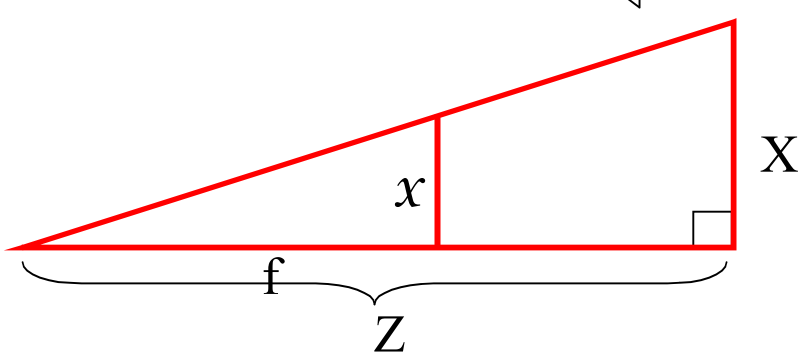


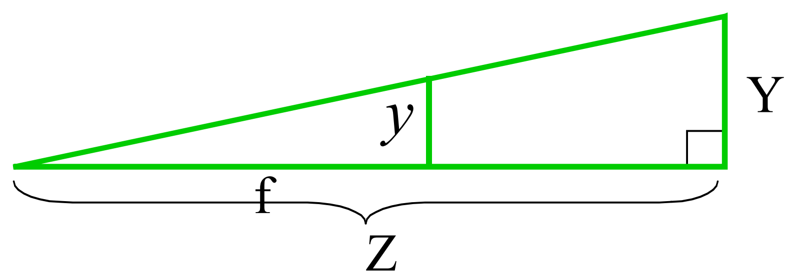
Perspective Projection Eqns

$$x = f\,\frac{X}{Z}$$

$$y = f\,\frac{Y}{Z}$$

derived via similar triangles rule





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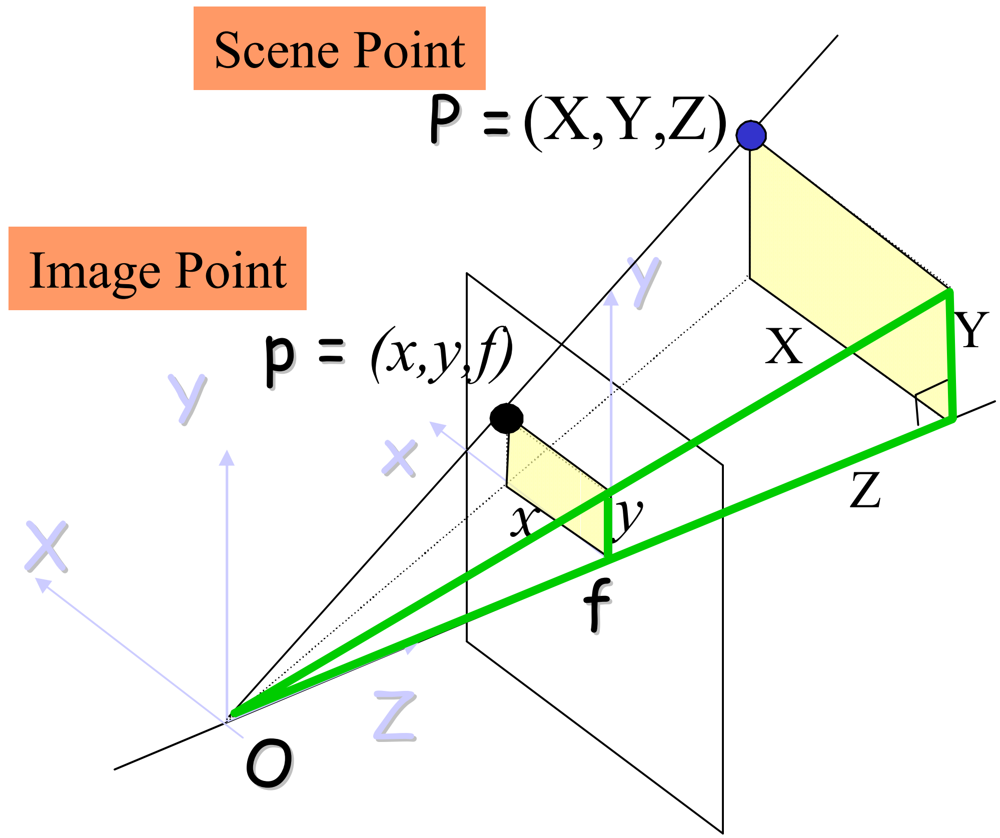
Basic Perspective Projection



Perspective Projection Eqns

$$x = f\,\frac{X}{Z}$$

$$y = f\,\frac{Y}{Z}$$

So how do we represent this as a matrix equation?
We need to introduce homogeneous coordinates.

Homogeneous Coordinates

Represent a 2D point (x,y) by a 3D point (x’,y’,z’) by adding a “fictitious” third coordinate.

By convention, we specify that given (x’,y’,z’) we can recover the 2D point (x,y) as

$$x = \frac{x'}{z'} \qquad y = \frac{y'}{z'}$$

Note: (x,y) = (x,y,1) = (2x, 2y, 2) = (k x, ky, k)
for any nonzero k (can be negative as well as positive)

Perspective Matrix Equation

(in Camera Coordinates)

$$x = f\frac{X}{Z} \qquad y = f\frac{Y}{Z} \quad \Longleftrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

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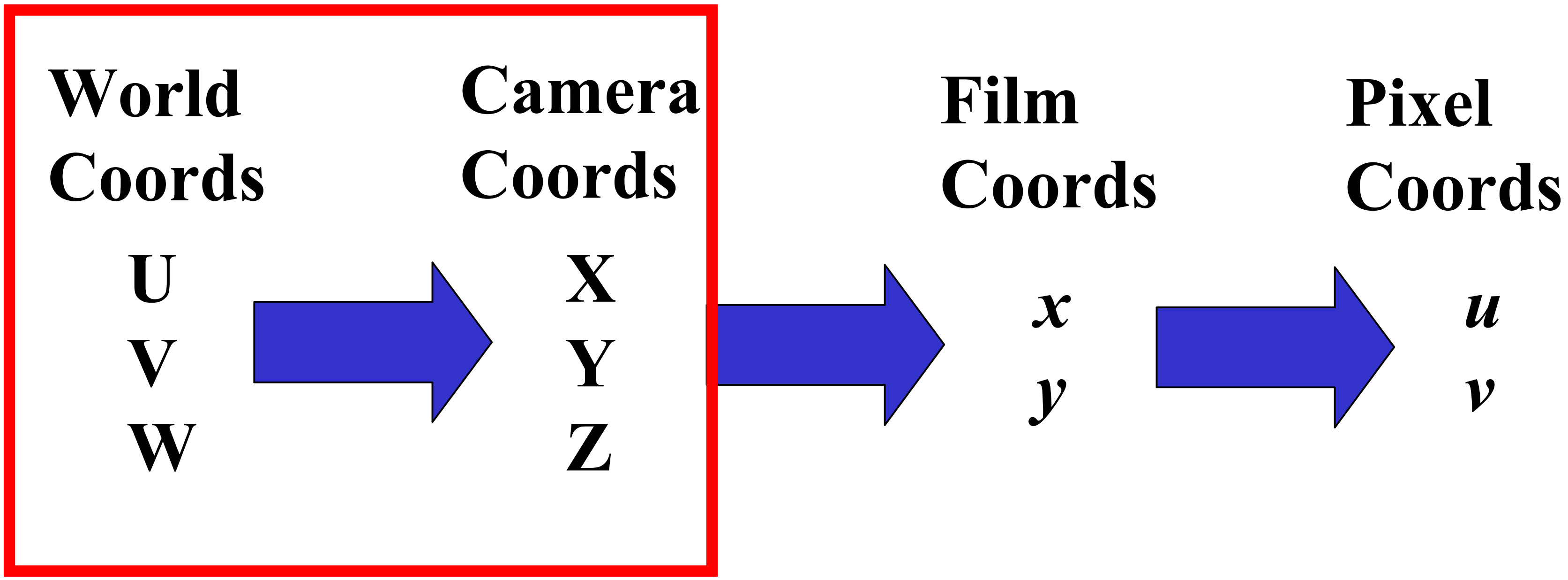
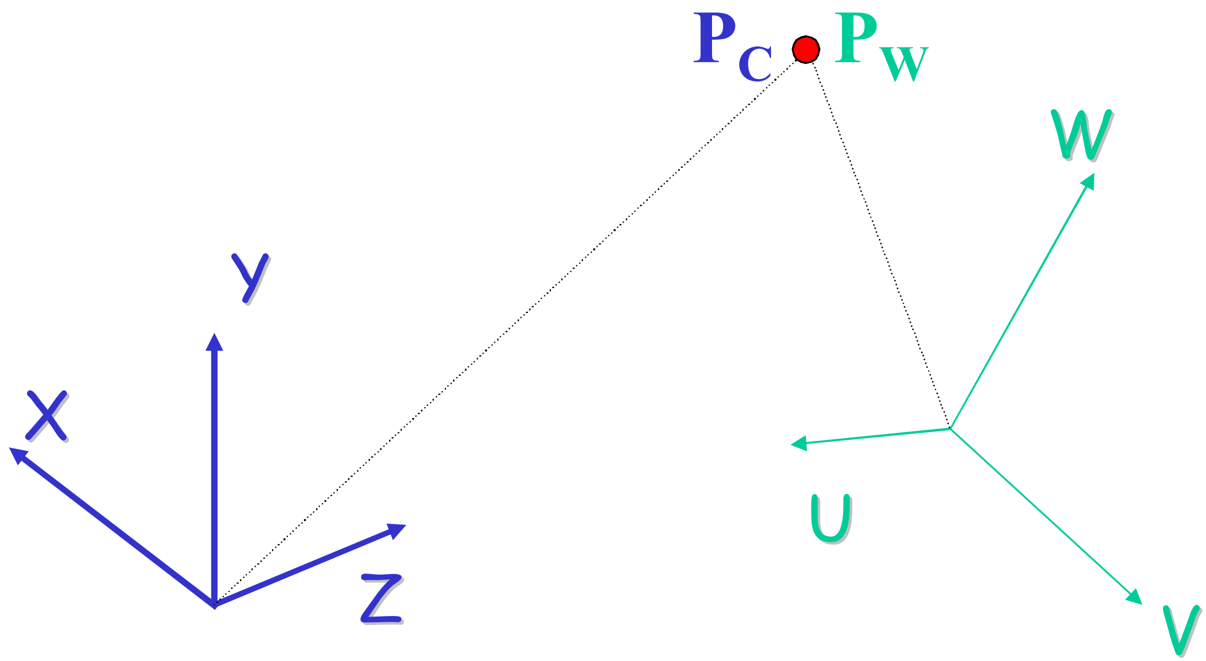
Forward Projection



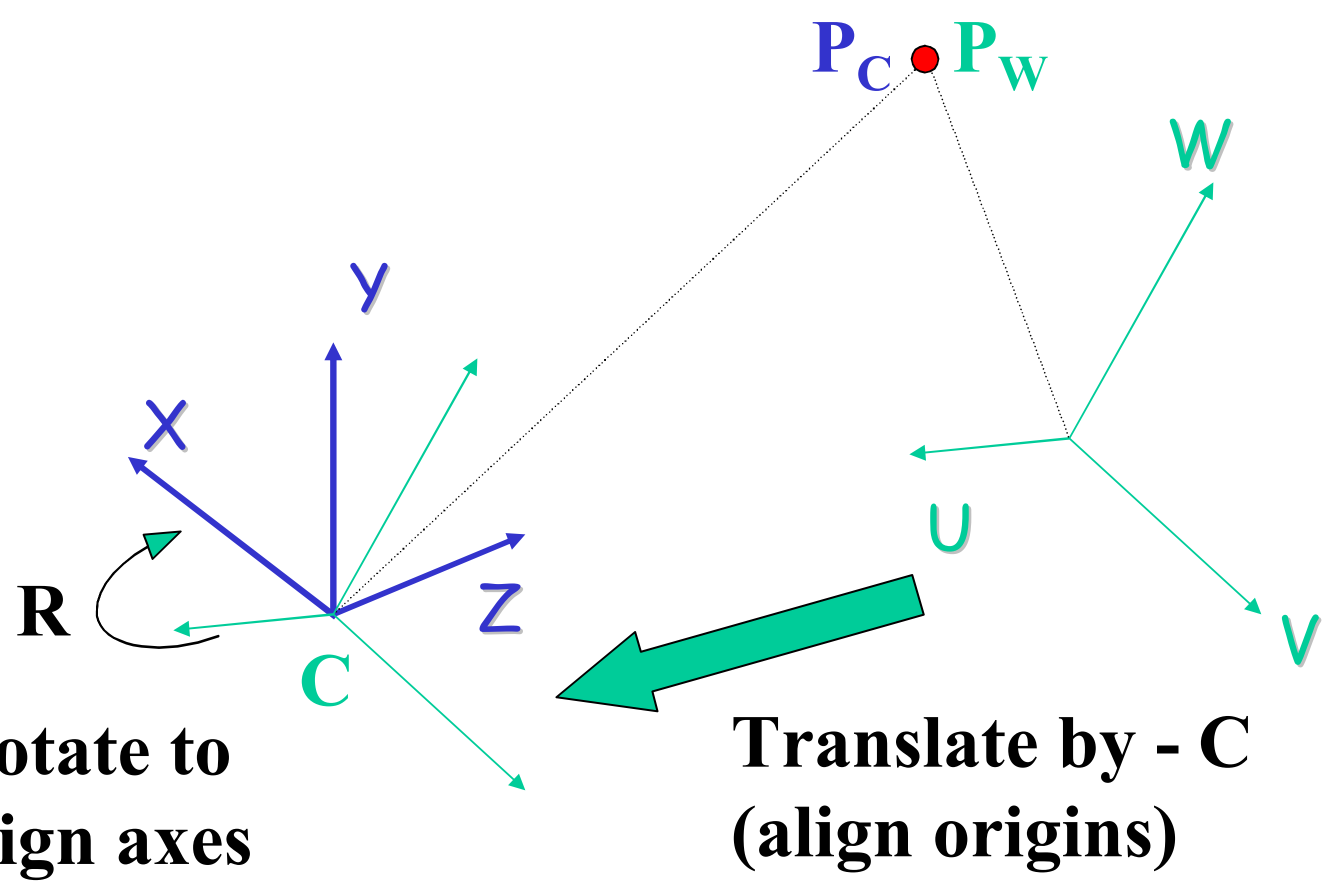
Rigid Transformation (rotation+translation) between world and camera coordinate systems

World to Camera Transformation



Avoid confusion: Pw and Pc are not two different points. They are the same physical point, described in two different coordinate systems.

World to Camera Transformation



$$P_C = R\,(\,P_W - C\,)$$

Matrix Form, Homogeneous Coords

$$P_C = R\,(P_W - C)$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

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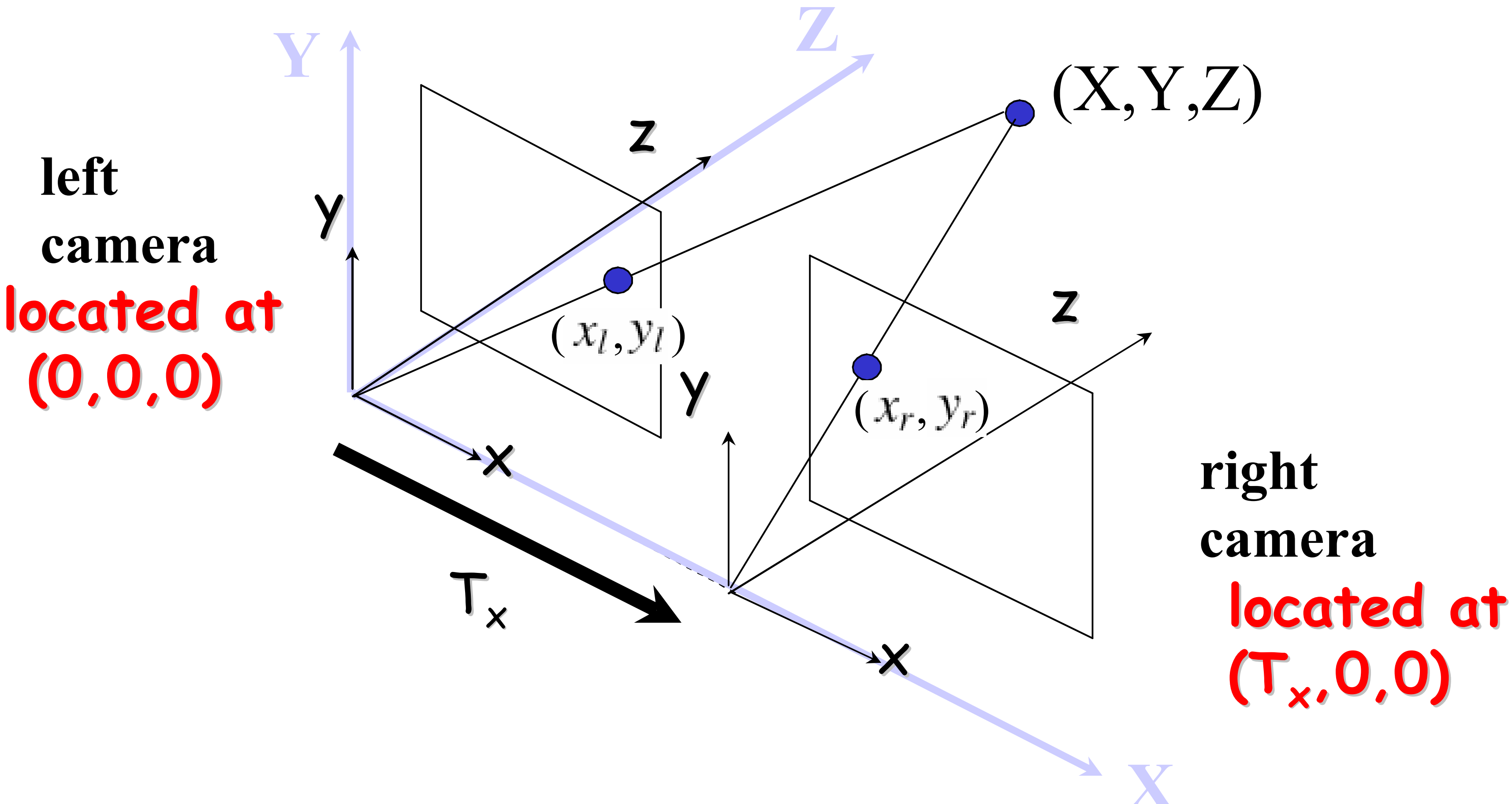
Example: Simple Stereo System



Left camera located at world origin (0,0,0)
and camera axes aligned with world coord axes.

Simple Stereo, Left Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned with world axes

located at world position (0,0,0)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

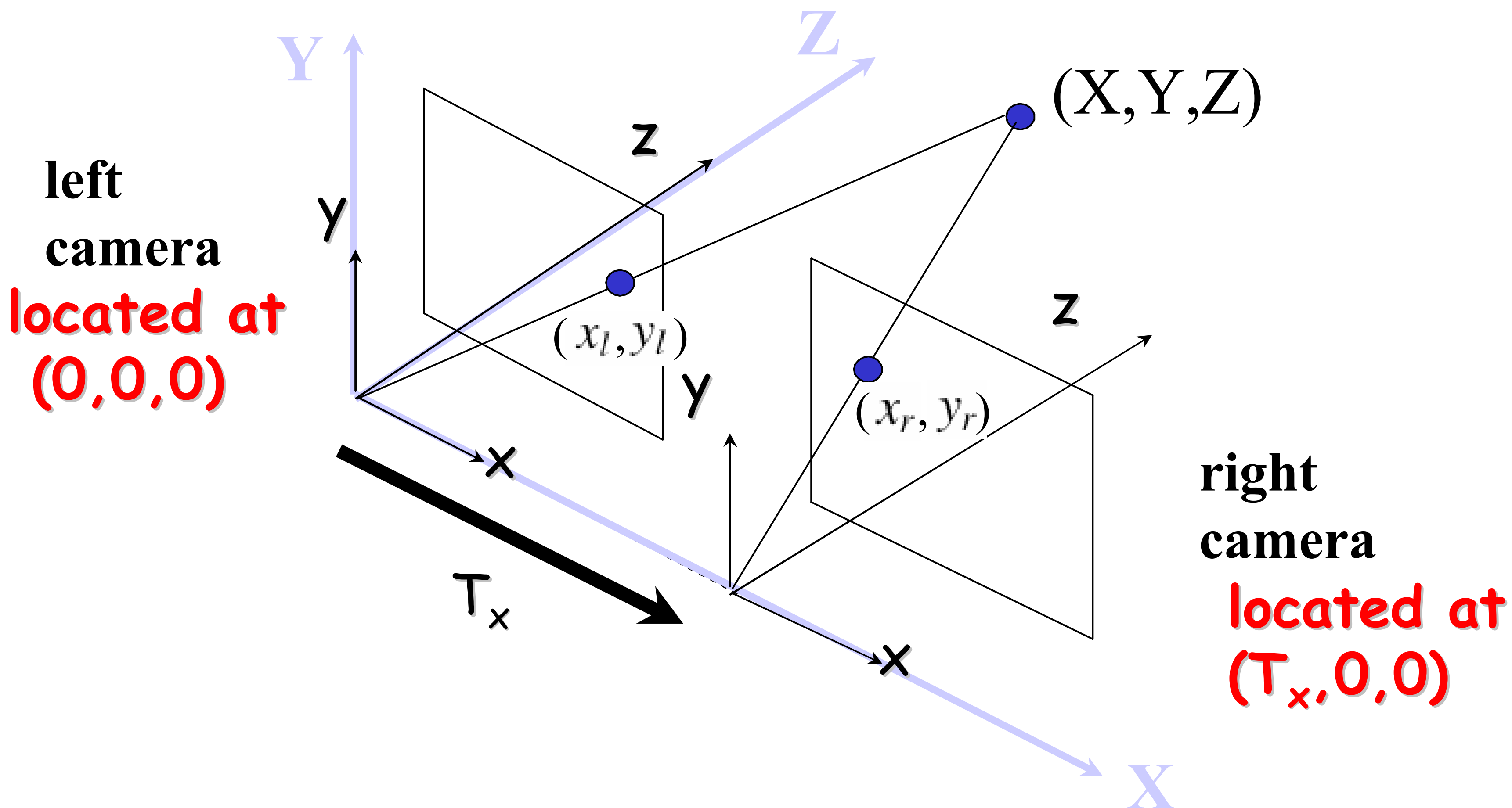
Simple Stereo Projection Equations

Left camera

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_l = f\,\frac{X}{Z} \qquad y_l = f\,\frac{Y}{Z}$$

Example: Simple Stereo System



Right camera located at world location (Tx,0,0)
and camera axes aligned with world coord axes.

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Simple Stereo, Right Camera

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mathbf{-T_x} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

camera axes aligned with world axes

located at world position $(T_x,0,0)$

$$= \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Simple Stereo Projection Equations

Left camera

$$\begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_l = f\,\frac{X}{Z} \qquad y_l = f\,\frac{Y}{Z}$$

Right camera

$$\begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -T_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x_r = f\,\frac{X - T_x}{Z} \qquad y_r = f\,\frac{Y}{Z}$$

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Bob's sure-fire way(s) to figure out the rotation

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cancel{\begin{pmatrix} 1 & 0 & 0 & -c_x \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

forget about this while thinking about rotations

$$\mathbf{P_C = R\ P_W}$$

This equation says how vectors in the world coordinate system (including the coordinate axes) get transformed into the camera coordinate system.

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Figuring out Rotations

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \qquad \mathbf{P_C = R\, P_W}$$

what if world x axis (1,0,0) corresponds to camera axis (a,b,c)?

$$\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = \begin{pmatrix} a & r_{12} & r_{13} & 0 \\ b & r_{22} & r_{23} & 0 \\ c & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we can immediately write down the first column of R!

Figuring out Rotations

and likewise with world Y axis and world Z axis...

same axis in camera coords

axis is world coords

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

world X axis (1,0,0)
in camera coords

world Y axis (0,1,0)
in camera coords

world Z axis (0,0,1)
in camera coords

Figuring out Rotations

Alternative approach: sometimes it is easier to specify what camera X,Y,or Z axis is in world coordinates. Then do rearrange the equation as follows.

$$\mathbf{P_C = R\,P_W} \;\Rightarrow\; \mathbf{R^{-1}P_C = P_W} \;\Rightarrow\; \mathbf{R^{T}P_C = P_W}$$

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

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Figuring out Rotations

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \qquad \mathbf{R^T P_C = P_W}$$

what if camera X axis (1,0,0) corresponds to world axis (a,b,c)?

$$\begin{pmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{a} & r_{21} & r_{31} & 0 \\ \mathbf{b} & r_{22} & r_{32} & 0 \\ \mathbf{c} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ 1 \end{pmatrix}$$

we can immediately write down the first column of R^T,
(which is the first row of R).

Figuring out Rotations

and likewise with camera Y axis and camera Z axis...

same axis in camera coords

axis is world coords

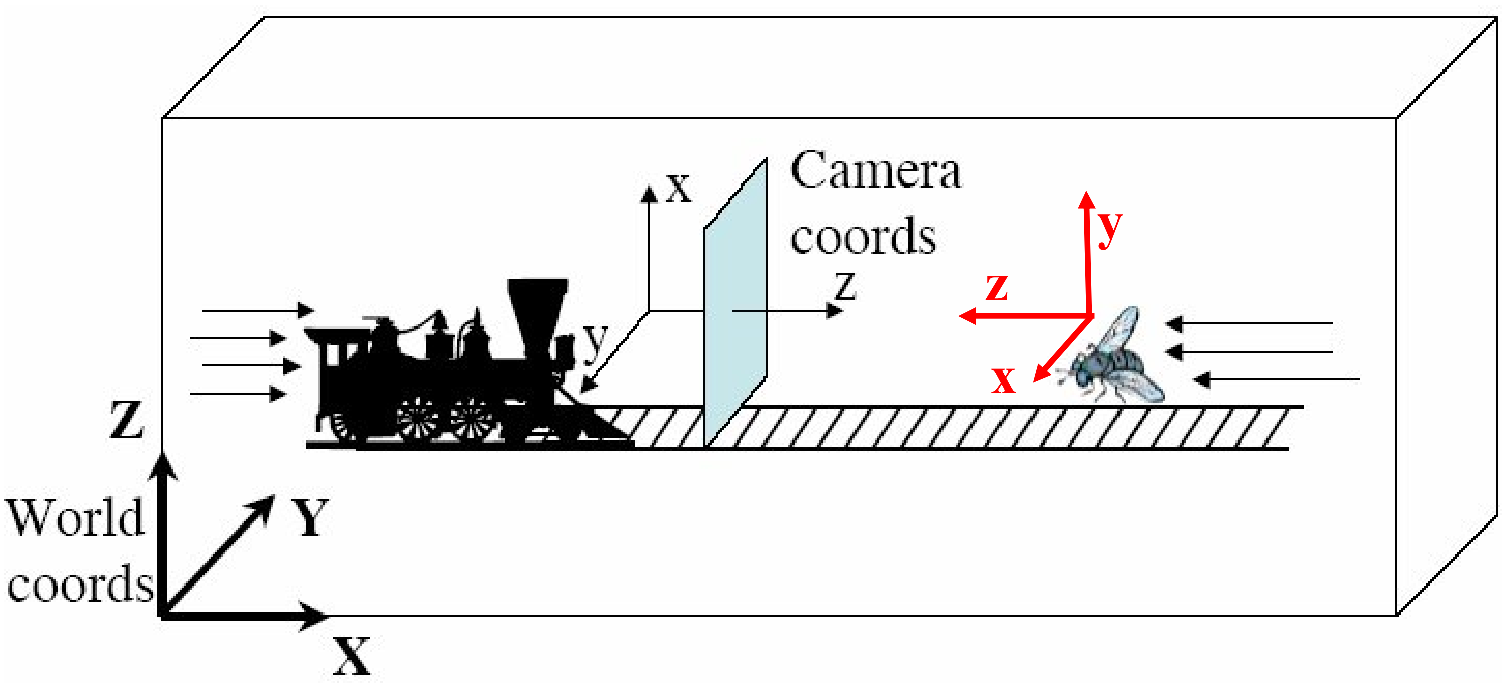
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

**camera X axis (1,0,0)
in world coords**

**camera Y axis (0,1,0)
in world coords**

**camera Z axis (0,0,1)
in world coords**

Example



$$R_{train} \quad \begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$R_{fly} \quad \begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{matrix}$$

Note: External Parameters also often written as R,T

$$\underbrace{\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

$$\mathbf{R}\,(\,\mathbf{P_W} - \mathbf{C}\,)$$

$$= \mathbf{R}\,\mathbf{P_W} - \mathbf{R}\,\mathbf{C}$$

$$= \mathbf{R}\,\mathbf{P_W} + \mathbf{T}$$

Summary

World Coords		Camera Coords		Film Coords		Pixel Coords
U V W	$\Rightarrow$	X Y Z	$\Rightarrow$	x y	$\Rightarrow$	u v

We now know how to transform 3D world coordinate points into camera coords, and then do perspective project to get 2D points in the film plane.

Next time: pixel coordinates