Linear algebra V
Eigenvectors and eigenvalues

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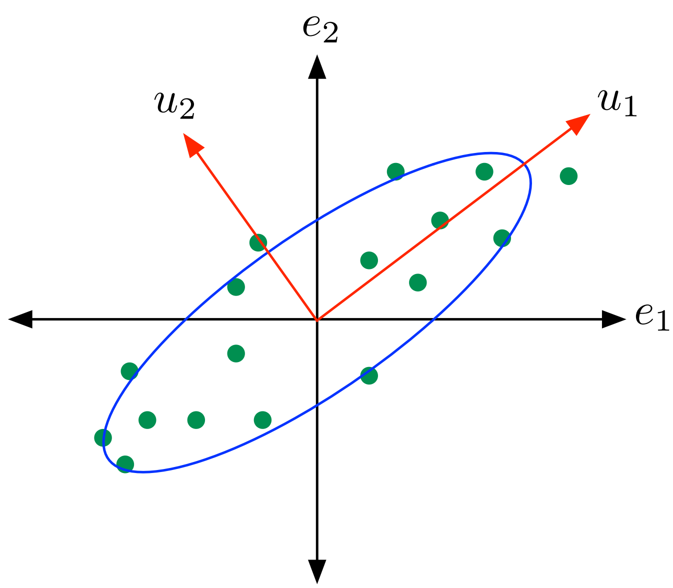
Topics we'll cover

1. Moving between coordinate systems
2. Eigenvectors and eigenvalues of a square matrix
3. Orthonormal basis of eigenvectors for symmetric matrix

Moving between coordinate systems



The linear function defined by a matrix

- Any matrix M defines a linear function, $x \mapsto Mx$.
  If M is a $d \times d$ matrix, this maps $\mathbb{R}^d$ to $\mathbb{R}^d$.

- This function is easy to understand when M is **diagonal**:

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 10 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 2x_1 \\ -x_2 \\ 10x_3 \end{pmatrix}}_{Mx}$$

  In this case, M simply scales each coordinate separately.

- General symmetric matrices also just scale coordinates separately...
  but in a **different coordinate system!**

Eigenvector and eigenvalue: definition

Let M be a $d \times d$ matrix. We say $u \in \mathbb{R}^d$ is an **eigenvector** of M if

$$Mu = \lambda u$$

for some scaling constant λ. This λ is the **eigenvalue** associated with u.

Key point: M **maps eigenvector** u **onto the same direction**.

Question: What are the eigenvectors and eigenvalues of:

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 10 \end{pmatrix} ?$$

Eigenvectors of a real symmetric matrix

Fact: Let M be any real symmetric $d \times d$ matrix. Then M has

- d eigenvalues $\lambda_1, \ldots, \lambda_d$
- corresponding eigenvectors $u_1, \ldots, u_d \in \mathbb{R}^d$ that are orthonormal

Can think of $u_1, \ldots, u_d$ as the axes of the natural coordinate system for M.

Example

$$M = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \text{ has eigenvectors } u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

1. Are these orthonormal?
2. What are the corresponding eigenvalues?