Kernel methods II
The kernel trick

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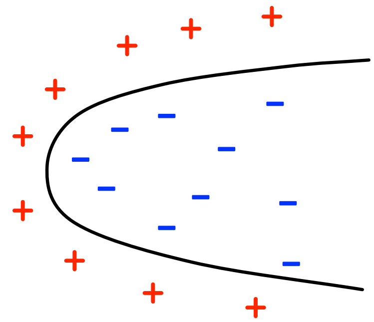
Topics we'll cover

1. The kernel trick for quadratic boundaries
2. The kernel Perceptron

Adding new features



Actual boundary is something like $x_1 = x_2^2 + 5$.

- This is quadratic in $x = (x_1, x_2)$
- But it is linear in $\Phi(x) = (x_1, x_2, x_1^2, x_2^2, x_1 x_2)$

Basis expansion: embed data in a higher-dimensional feature space.
Then we can use a linear classifier!

Perceptron with basis expansion

Learning in the higher-dimensional feature space:

- $w = 0$ and $b = 0$
- while some $y(w \cdot \Phi(x) + b) \leq 0$:
  - $w = w + y\,\Phi(x)$
  - $b = b + y$

Problem: number of features has now increased dramatically.
For MNIST, with quadratic boundary: from 784 to 308504.

The kernel trick: implement this without ever writing down a vector in the higher-dimensional space!

The kernel trick

- $w = 0$ and $b = 0$
- while some $y^{(i)}(w \cdot \Phi(x^{(i)}) + b) \leq 0$:
  - $w = w + y^{(i)}\,\Phi(x^{(i)})$
  - $b = b + y^{(i)}$

1. w is always a linear combination of the $\Phi(x^{(i)})$.

$$w = \sum_{j=1}^{n} \alpha_j y^{(j)} \Phi(x^{(j)})$$

   Represent w in **dual** form: $\alpha = (\alpha_1, \ldots, \alpha_n)$.

2. Compute $w \cdot \Phi(x)$ using the dual representation.

$$w \cdot \Phi(x) = \sum_{j=1}^{n} \alpha_j y^{(j)} (\Phi(x^{(j)}) \cdot \Phi(x))$$

3. Compute $\Phi(x) \cdot \Phi(z)$ without ever writing out $\Phi(x)$ or $\Phi(z)$.

Computing dot products

First, in 2-d.
Suppose $x = (x_1, x_2)$ and $\Phi(x) = (x_1, x_2, x_1^2, x_2^2, x_1 x_2)$.

Actually, tweak a little: $\Phi(x) = (1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, x_2^2, \sqrt{2}x_1 x_2)$

What is $\Phi(x) \cdot \Phi(z)$?

Computing dot products

Suppose $x = (x_1, x_2, \ldots, x_d)$ and

$$\Phi(x) = (1, \sqrt{2}x_1, \ldots, \sqrt{2}x_d, x_1^2, \ldots, x_d^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{d-1}x_d)$$

$$\begin{aligned}\Phi(x) \cdot \Phi(z) &= (1, \sqrt{2}x_1, \ldots, \sqrt{2}x_d, x_1^2, \ldots, x_d^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{d-1}x_d)\cdot \\ &\quad (1, \sqrt{2}z_1, \ldots, \sqrt{2}z_d, z_1^2, \ldots, z_d^2, \sqrt{2}z_1z_2, \ldots, \sqrt{2}z_{d-1}z_d) \\ &= 1 + 2\sum_i x_i z_i + \sum_i x_i^2 z_i^2 + 2\sum_{i \neq j} x_i x_j z_i z_j \\ &= (1 + x_1 z_1 + \cdots + x_d z_d)^2 \quad = \quad (1 + x \cdot z)^2\end{aligned}$$

For MNIST:
We are computing dot products in 308504-dimensional space.
But it takes time proportional to 784, the original dimension!

Kernel Perceptron

Learning from data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathcal{X} \times \{-1, 1\}$

Primal form:

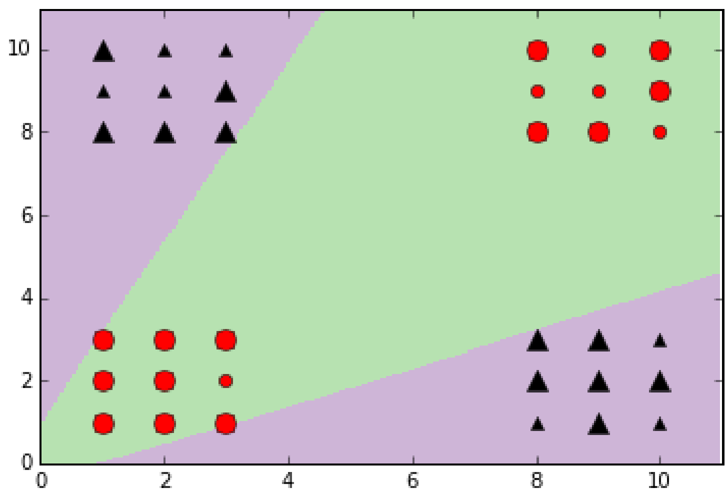
- $w = 0$ and $b = 0$
- while there is some i with $y^{(i)}(w \cdot \Phi(x^{(i)}) + b) \leq 0$:
  - $w = w + y^{(i)}\,\Phi(x^{(i)})$
  - $b = b + y^{(i)}$

Dual form: $w = \sum_j \alpha_j y^{(j)} \Phi(x^{(j)})$, where $\alpha \in \mathbb{R}^n$

- $\alpha = 0$ and $b = 0$
- while some i has $y^{(i)} \left(\sum_j \alpha_j y^{(j)} \Phi(x^{(j)}) \cdot \Phi(x^{(i)}) + b \right) \leq 0$:
  - $\alpha_i = \alpha_i + 1$
  - $b = b + y^{(i)}$

To classify a new point x: sign $\left(\sum_j \alpha_j y^{(j)} \Phi(x^{(j)}) \cdot \Phi(x) + b \right)$.

Kernel Perceptron: examples



Kernel Perceptron: examples