Linear algebra VI
Spectral decomposition

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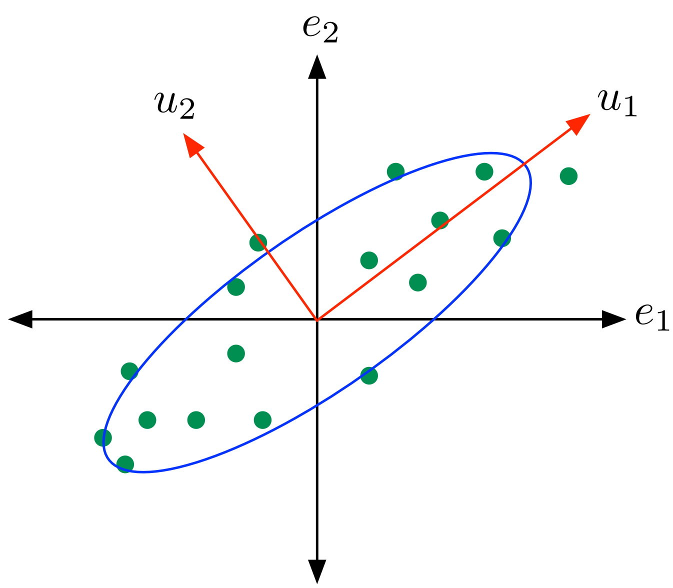
Topics we'll cover

1. The eigenbasis as a natural coordinate system for a matrix
2. Spectral decomposition of a symmetric matrix
3. Principal component analysis revisited

Moving between coordinate systems



Eigenvectors and eigenvalues

Let M be a $d \times d$ matrix. We say $u \in \mathbb{R}^d$ is an **eigenvector** of M if

$$Mu = \lambda u$$

for some scaling constant λ. This λ is the **eigenvalue** associated with u.

Key point: M **maps eigenvector** u **onto the same direction**.

Fact: Let M be any real symmetric $d \times d$ matrix. Then M has

- d eigenvalues $\lambda_1, \ldots, \lambda_d$
- corresponding eigenvectors $u_1, \ldots, u_d \in \mathbb{R}^d$ that are orthonormal

Eigenvectors: axes of a natural coordinate system for M.

Spectral decomposition

Fact: Let M be any real symmetric $d \times d$ matrix. Then M has orthonormal eigenvectors $u_1, \ldots, u_d \in \mathbb{R}^d$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_d$.

Spectral decomposition: Another way to write M:

$$M = \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \cdots & u_d \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{U\text{: columns are eigenvectors}} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix}}_{\Lambda\text{: eigenvalues on diagonal}} \underbrace{\begin{pmatrix} \longleftarrow & u_1 & \longrightarrow \\ \longleftarrow & u_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & u_d & \longrightarrow \end{pmatrix}}_{U^T}$$

Thus $Mx = U\Lambda U^T x$:

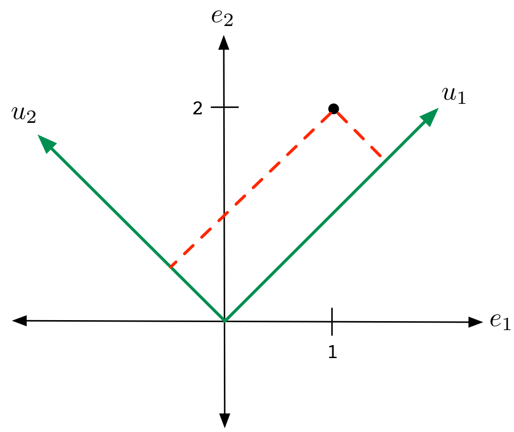
- U^T rewrites x in the $\{u_i\}$ coordinate system
- Λ is a simple coordinate scaling in that basis
- U sends the scaled vector back into the usual coordinate basis

Apply spectral decomposition to the matrix we saw earlier: $M = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

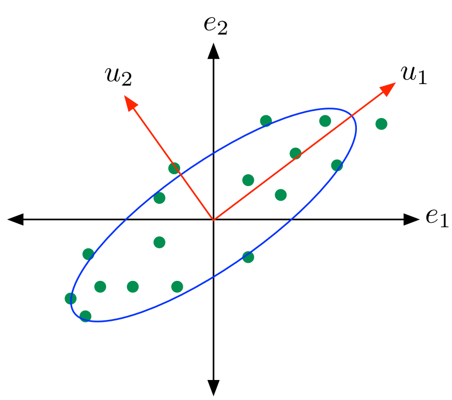
- Eigenvectors $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$.

$$M = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{U^T}$$

$$M \begin{pmatrix} 1 \\ 2 \end{pmatrix} = U \Lambda U^T \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Principal component analysis revisited



Data vectors $X \in \mathbb{R}^d$

- Covariance matrix Σ is a $d \times d$ symmetric matrix.
- Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$
  Eigenvectors $u_1, \ldots, u_d$.
- $u_1, \ldots, u_d$: another basis in which to represent data.
- Variance of X in direction u_i is λ_i.
- Projection to k dimensions: $x \mapsto (x \cdot u_1, \ldots, x \cdot u_k)$.

What is the covariance of the projected data?