Probability review II:
Random variables, expectation, and variance

Topics we'll cover

1. What is a random variable?
2. Expected value
3. Variance and standard deviation

Random variables

Roll two dice. Let X be their sum.

$$\begin{aligned} \text{outcome} = (1,1) &\quad \Rightarrow \quad X = 2 \\ \text{outcome} = (1,2) \text{ or } (2,1) &\quad \Rightarrow \quad X = 3 \end{aligned}$$

Probability space:

- Sample space: $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.
- Each outcome equally likely.

Random variable X lies in $\{2,3,4,5,6,7,8,9,10,11,12\}$.

A **random variable (r.v.)** is a defined on a probability space.
It is a mapping from Ω (outcomes) to $\mathbb{R}$ (numbers).
We'll use capital letters for r.v.'s.

The distribution of a random variable

Roll a die.

Define $X = 1$ if die is ≥ 3, otherwise $X = 0$.

Expected value, or mean

Expected value of a random variable X:

$$\mathbb{E}(X) = \sum_x x \Pr(X = x).$$

Roll a die. Let X be the number observed.
What is $\mathbb{E}(X)$?

Another example

A biased coin has heads probability p.
Let X be 1 if heads, 0 if tails. What is $\mathbb{E}(X)$?

A property of expected values

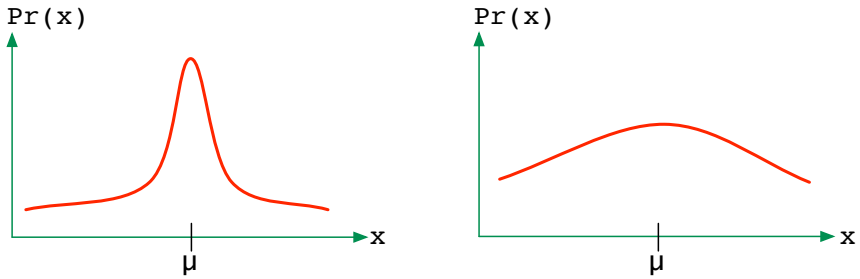
How is the average of a set of numbers affected if:

- You double the numbers?
- You increase each number by 1?

Summary: Let X be any random variable.
If $V = aX + b$ (any constants a, b), then $\mathbb{E}(V) = a\mathbb{E}(X) + b$

Variance

Can summarize an r.v. X by its mean, μ. But this doesn't capture the **spread** of X:



A measure of spread: average distance from the mean, $\mathbb{E}(|X - \mu|)$?

- **Variance:** $\text{var}(X) = \mathbb{E}((X - \mu)^2)$, where $\mu = \mathbb{E}(X)$
- **Standard deviation** $\sqrt{\text{var}(X)}$:
  Roughly, the average amount by which X differs from its mean.

Variance: example

Choose X uniformly at random from $\{1, 2, 3, 4, 5\}$.

Variance: properties

Variance: $\text{var}(X) = \mathbb{E}((X - \mu)^2)$, where $\mu = \mathbb{E}(X)$

- Variance is always ≥ 0
- How is the variance affected if:
  - You increase each number by 1?
  - You double each number?
- Summary: If $V = aX + b$ then $\text{var}(V) = a^2 \, \text{var}(X)$

Alternative formula for variance

Variance: $\text{var}(X) = \mathbb{E}((X-\mu)^2)$, where $\mu = \mathbb{E}(X)$

Another way to write it: $\text{var}(X) = \mathbb{E}(X^2) - \mu^2$

Example: Choose X uniformly at random from $\{1, 2, 3, 4, 5\}$.