Lecture 8: Working with Bayes' Rule

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We discuss several examples of how one can use Bayes' rule.

There are many obvious identities that probabilities satisfy:

Fact 1 (Bayes' Rule)**.** *If A, B are events, then*

$$p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)}.$$

Fact 2 (Chain Rule)**.** *If $A_1, A_2, \ldots, A_n$ are events, then*

$$\begin{aligned}&p(A_1 \cap A_2 \cap \ldots A_n)\\ &= p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2) \cdot \ldots \cdot p(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).\end{aligned}$$

Fact 3 (Law of Total Probability)**.** *If $A_1, A_2, \ldots, A_n$ are disjoint events that form a partition of the whole sample space, and B is another event, then*

$$p(B) = p(A_1 \cap B) + p(A_2 \cap B) + \ldots + p(A_n \cap B).$$

Exampe: Using Bayes' Rule

Suppose an urn either contains 3 red balls and 3 blue balls with probability $3/4$, or 6 red balls with probability $1/4$. You draw 3 balls at random and come up with 3 red balls. What is the probability that the remaining balls are red?

Let M denote the event that the balls are mixed. Let D denote the event that 3 red balls were drawn. We have

$$p(D|M) = \binom{3}{3} \Big/ \binom{6}{3} = 1/20.$$

and

$$p(M|D) = \frac{p(D|M)p(M)}{p(D)}.$$

We can calculate

$$\begin{aligned}p(D) &= p(D|M)p(M) + p(D|M^c)p(M^c)\\ &= (1/20)(3/4) + 1 \cdot (1/4)\\ &= 23/80.\end{aligned}$$

Then we get

$$p(M|D) = \frac{p(D|M)p(M)}{p(D)} = \frac{(1/20)(3/4)}{23/80} = 3/23.$$

$p(M)$ is often called the *prior*. $p(M|D)$ is called the *posterior*.

Example: Radar

Suppose the airforce designs a new radar system. If an aircraft is present in the range of the radar system, then the aircraft is detected with probability 0.99. If the aircraft is not present, then the radar reports that an aircraft is present with probability 0.1. Suppose the probability than an aircraft is present is 0.05. What is the probability that the system gives a false alarm, meaning that an aircraft is not presented but is detected? What is the probability that an aircraft is present and detected? What is the probability that an aircraft is present given that the radar reports an aircraft?

The first thing to do is to model all the events we care about. Let A be the event that an aircraft is present, and R be the event that the radar detects an aircraft. Then we have $p(R|A) = 0.99$ and $p(R|A^c) = 0.1$. Finally, we know that $p(A) = 0.05$. The probability of a false alarm is $p(A^c \cap R)$. We have:

$$p(A^c \cap R) = p(A^c) \cdot p(R|A^c) = (1 - 0.05) \cdot 0.1 = 0.095.$$

Similarly, we have

$$p(A \cap R) = p(A) \cdot p(R|A) = 0.05 \cdot 0.99 = 0.0495.$$

The probability that there is an aircraft given that the radar reports one can be calculated using Bayes' rule:

$$p(A|R) = \frac{p(R|A)p(A)}{p(R)}.$$

We see that we know all of the quantities on the right hand side except $p(R)$. However, we have

$$\begin{aligned} p(R) &= p(R \cap A) + p(R \cap A^c) \\ &= 0.095 + 0.0495 = 0.1445. \end{aligned}$$

So, we get

$$p(A|R) = \frac{p(R|A)p(A)}{p(R)} = \frac{0.0495}{0.1445} = 0.34256,$$

which is not as high as you might expect. The point is that because the probability of an aircraft being present is so low, the probability of a false alarm from the radar better be extremely low for the radar to be effective.