Comparisons Sorts

Data Structures and Parallelism

Logistics

Project 2 Checkpoint Due Tonight

- Each person should answer separately from your partner
- Points just for answering
- Don't forget about the project writeup, it will take time

Exercises 5, 6, 7 out now

- Will know everything you need to do them after lecture today

Midterm Grades out

- Regrades open through tomorrow

Sorting

Data structures and algorithms up to this point returned one element at a time

What if we want all of them at once?

Very common data requirement

- Alphabetical list of people
- Countries sorted by population
- Sorted search results by relevance
- Opens up other operations (binary search)

A good representation of algorithm design principles.

The main problem, stated carefully

For now we will assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record
- A comparison function (consistent and total)
  - Given keys a & b, what is their relative ordering? <, =, >?
  - Ex: keys that implement Comparable or have a Comparator that can handle them

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**,
  if **`i < j`** then **`A[i] ≤ A[j]`**
- Usually unspoken assumption: **`A`** must have all the same data it started with
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Three goals

Three things you might want in a sorting algorithm:

In-Place

- Only use $O(1)$ extra memory.
- Sorted array given back in the input array.

Stable

- If a appears before b in the initial array and a.compareTo(b) == 0
- Then a appears before b in the final array.
- Example: sort by first name, then by last name.

Fast

Insertion Sort

How you sort a hand of cards.

Maintain a sorted subarray at the front.

Start with one element.

While(your subarray is not the full array)

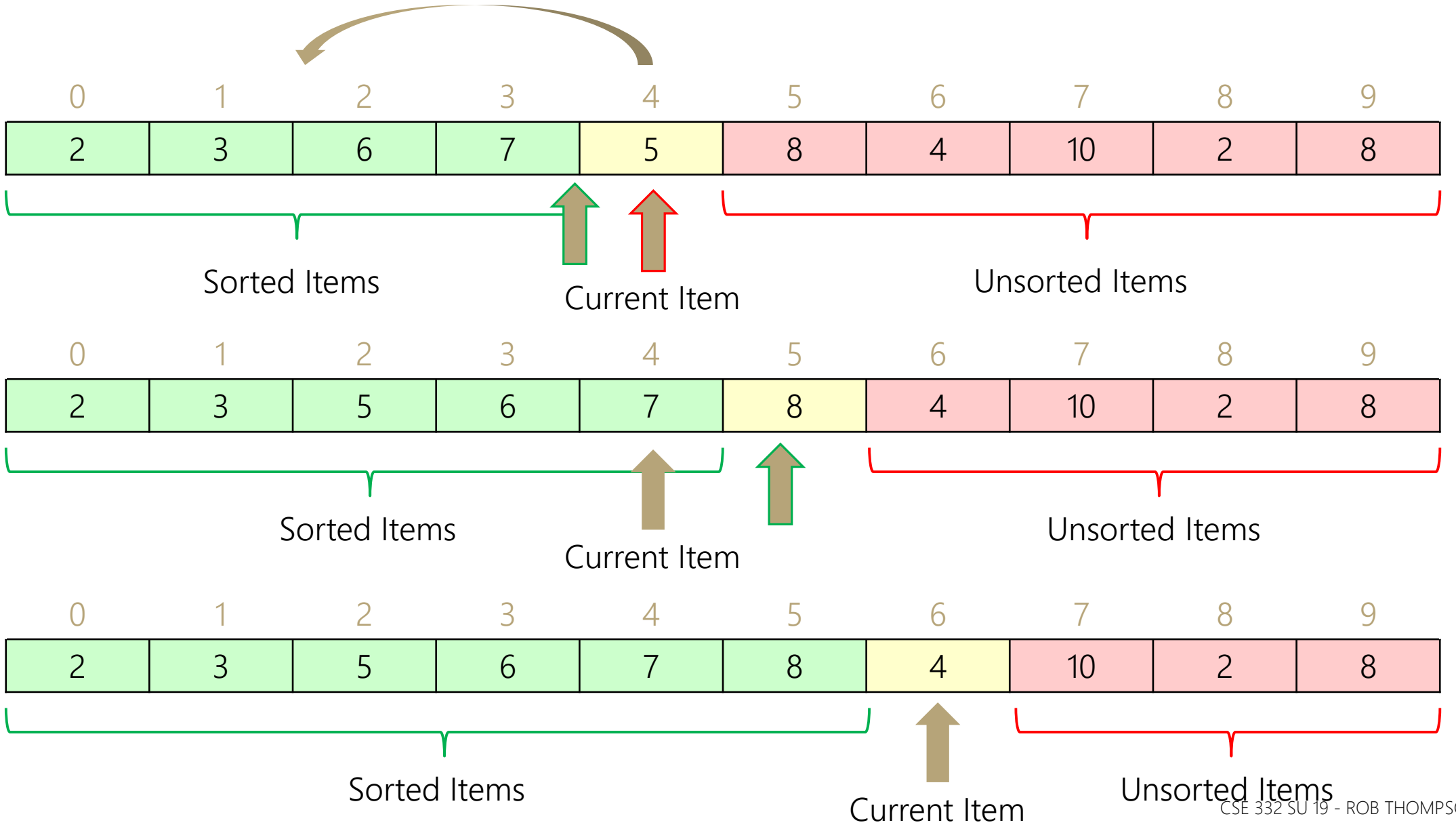
- Take the next element not in your subarray
- Insert it into the sorted subarray

Insertion Sort

```
for(i from 1 to n-1){
    int index = i
    while(a[index-1] > a[index]){
        swap(a[index-1], a[index])
        index = index-1
    }
}
```

Insertion Sort

https://www.youtube.com/watch?v=ROalU379l3U



Insertion Sort Analysis

Stable?
- Yes! (If you're careful)

In Place
- Yes!

Running time:
- Best Case: $O(n)$
- Worst Case: $O(n^2)$
- Average Case: $O(n^2)$

Bad asymptotic, but actually pretty good for small n

_____ Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

    Find the smallest element remaining in the unsorted part.

    Insert it at the end of the sorted part.

Selection Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

Find the smallest element remaining in the unsorted part.

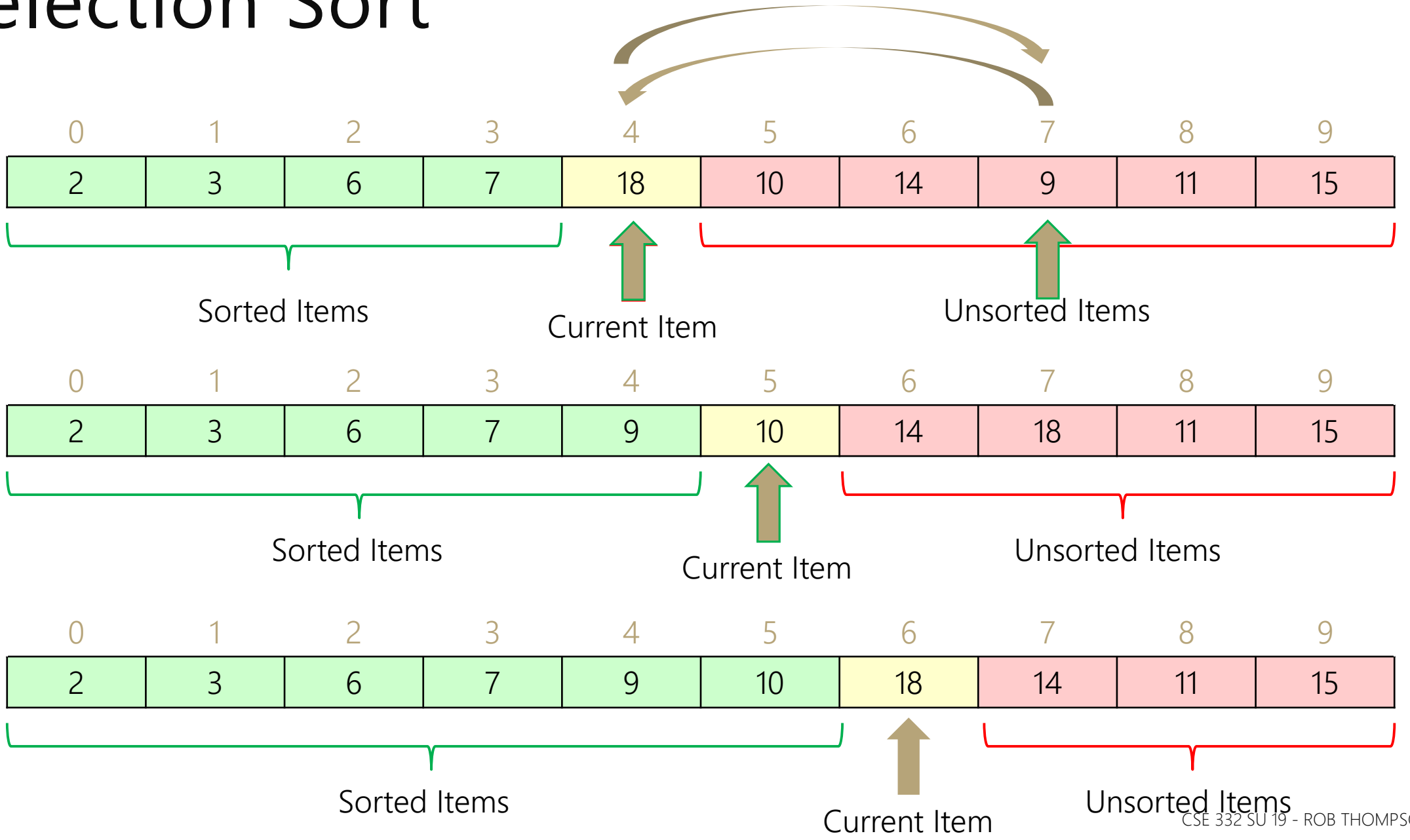
- By scanning through the remaining array

Insert it at the end of the sorted part.

Running time $O(n^2)$

Selection Sort

https://www.youtube.com/watch?v=Ns4TPTC8whw



Selection Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

Find the smallest element remaining in the unsorted part.

- By scanning through the remaining array

Insert it at the end of the sorted part.

Running time $O(n^2)$

Can we do better? With a data structure?

Heap Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray; **Make the unsorted part a min-heap**

While(subarray is not full array)

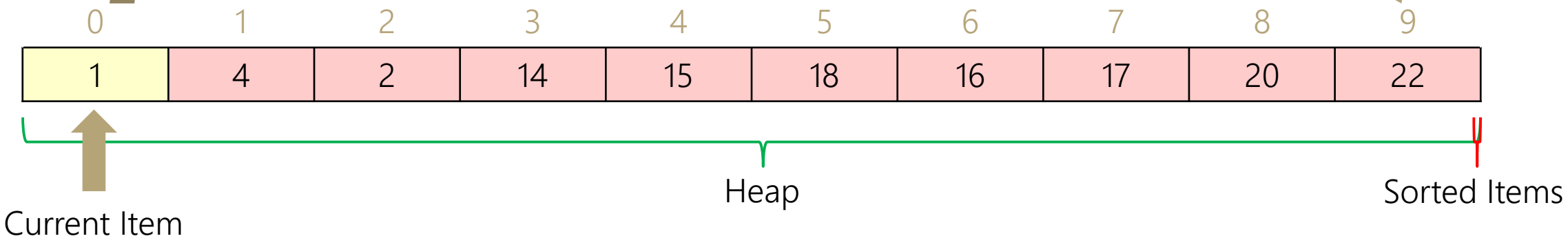
Find the smallest element remaining in the unsorted part.
- By calling `removeMin` on the heap

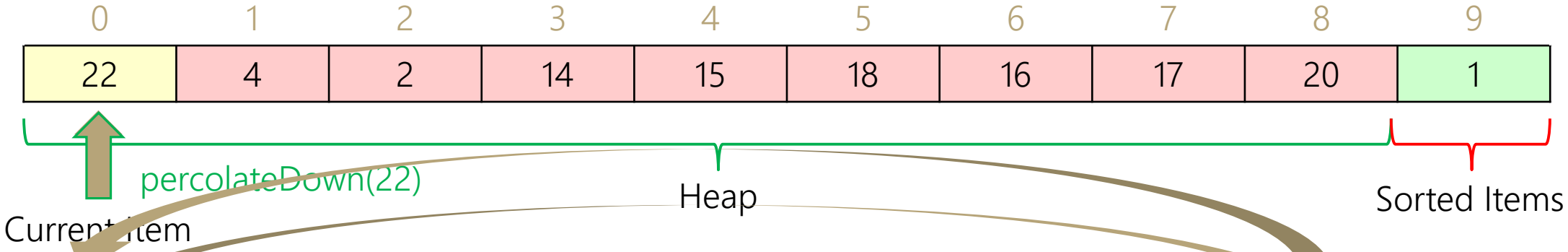
Insert it at the end of the sorted part.

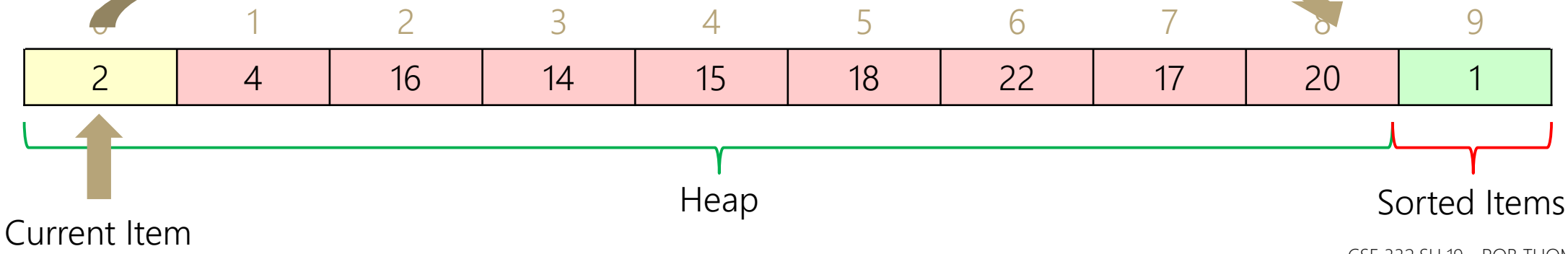
Running time $O(n \log n)$

Heap Sort

https://www.youtube.com/watch?v=Xw2D9aJRBY4







Heap Sort (Better)

We're sorting in the wrong order!

- Could reverse at the end.

Our heap implementation will implicitly assume that the heap is on the left of the array.

Switch to a max-heap, and keep the sorted stuff on the right.

What's our running time? $O(n \log n)$

Heap Sort

Our first step is to make a heap. Does using `buildHeap` instead of inserts improve the running time?

Not in a big-O sense (though we did by a constant factor).

Exercise 7 will show some sorting problems where `buildHeap` does give you a better $O()$ bound.

In place:

- Yes

Stable:

- No

AVL sort?

One old data structure worked, how about the rest?

We can also use a balanced tree to:

- **`insert`** each element: total time $O(n \log n)$
- Do an in-order traversal $O(n)$

But this cannot be made in-place and has worse constant factors than heap sort

- both are $O(n \log n)$ in worst, best, and average case
- neither parallelizes well
- heap sort is better

Don't even think about trying to sort with a hash table…

A Different Idea

So far our sorting algorithms:

- Start with an (empty) sorted array
- Add something to it.

Different idea: Divide And Conquer:

Split up array (somehow)

Sort the pieces (recursively)

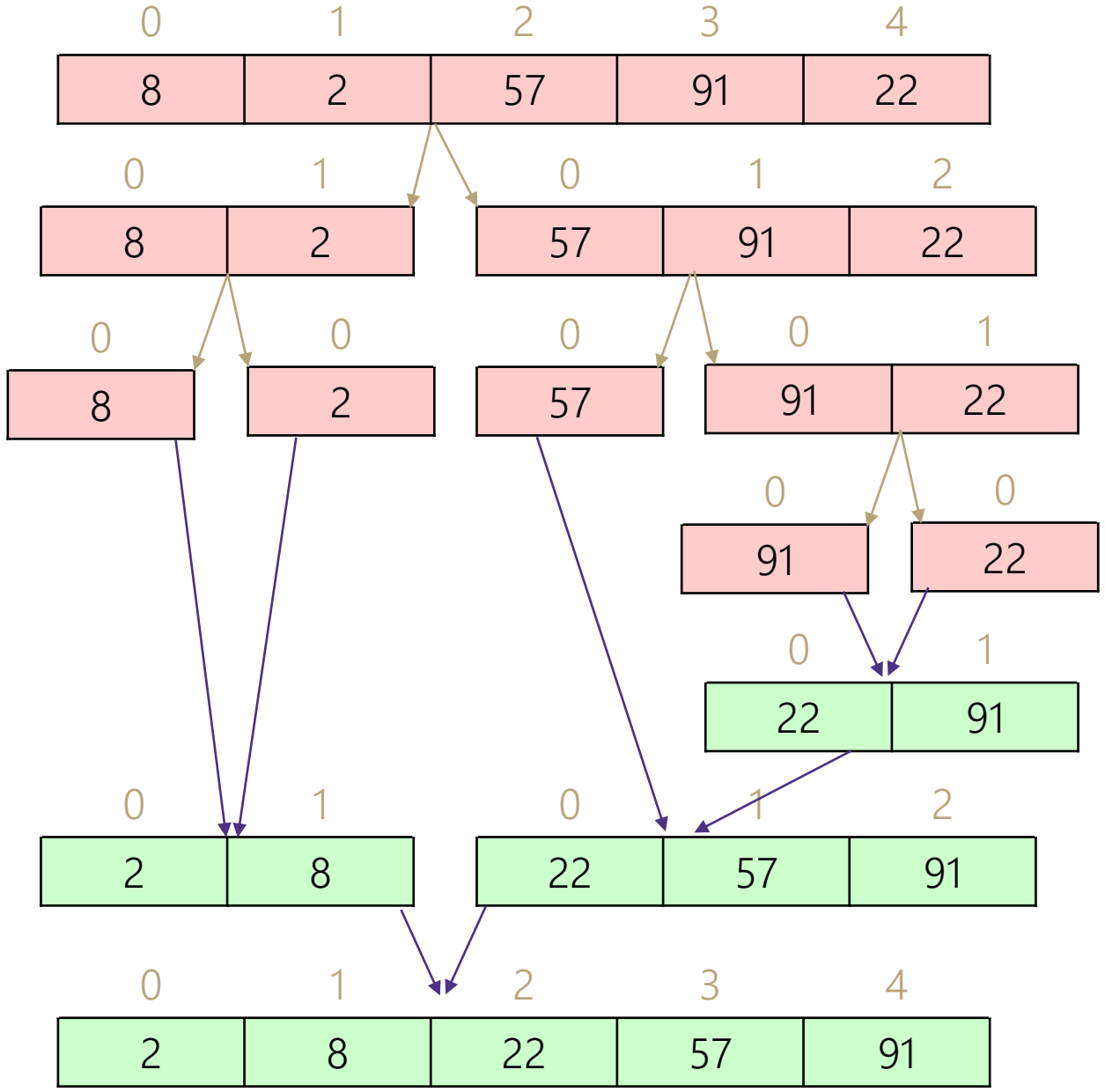
Combine the pieces

Merge Sort

Split array in the middle

Sort the two halves

Merge them together



Merge Sort Pseudocode

https://www.youtube.com/watch?v=XaqR3G_NVoo

```
mergeSort(input) {
    if (input.length == 1)
        return
    else
        smallerHalf = mergeSort(new [0, ..., mid])
        largerHalf = mergeSort(new [mid + 1, ...])
        return merge(smallerHalf, largerHalf)
}
```

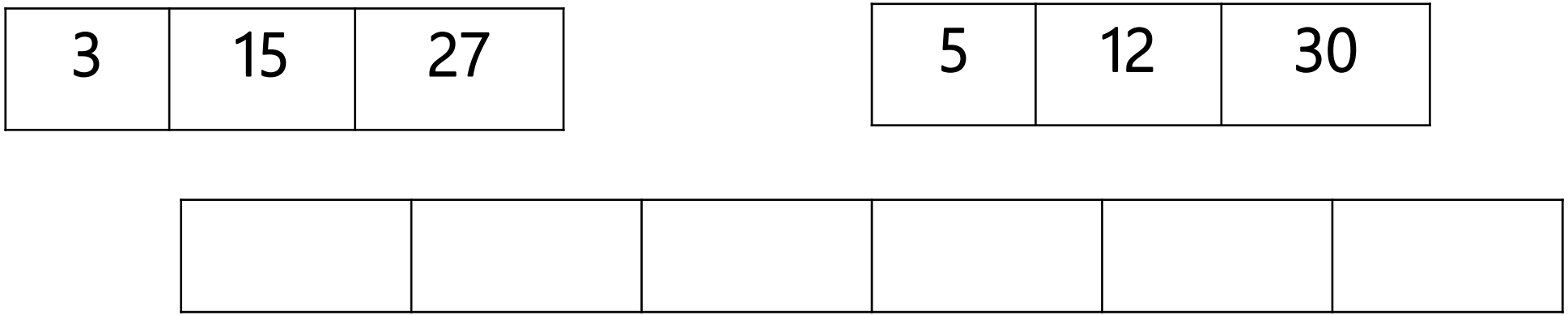
How Do We Merge?

Turn two sorted lists into one sorted list:

Start from the small end of each list.

Copy the smaller into the combined list

Move that pointer one spot to the right.



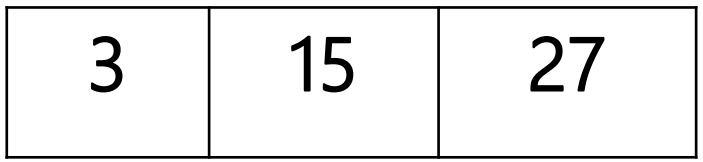
How Do We Merge?

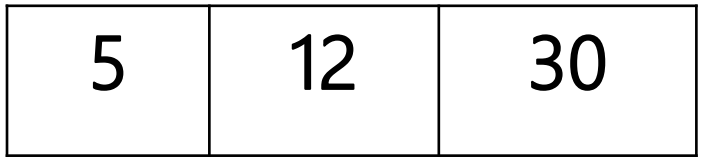
Turn two sorted lists into one sorted list:

Start from the small end of each list.

Copy the smaller into the combined list

Move that pointer one spot to the right.





Merge Sort Analysis

Running Time:

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + c_1 n & \text{if } n \geq 1 \\ c_2 & \text{otherwise} \end{cases}$$

This is a closed form you should have memorized by the end of the quarter.

The closed form is $\Theta(n \log n)$.

Stable:

- yes! (if you merge correctly)

In place:

- no.

Some Optimizations

We need extra memory to do the merge

It's inefficient to make a new array every time

Instead have a single auxiliary array

- Keep reusing it as the merging space

Even better: make a single auxiliary array

- Have the original array and the auxiliary array "alternate" being the list and the merging space.

Quick Sort

Still Divide and Conquer, but a different idea:

Let's divide the array into "big" values and "small" values

- And recursively sort those

What's "big"?

- Choose an element ("the pivot") anything bigger than that.

How do we pick the pivot?

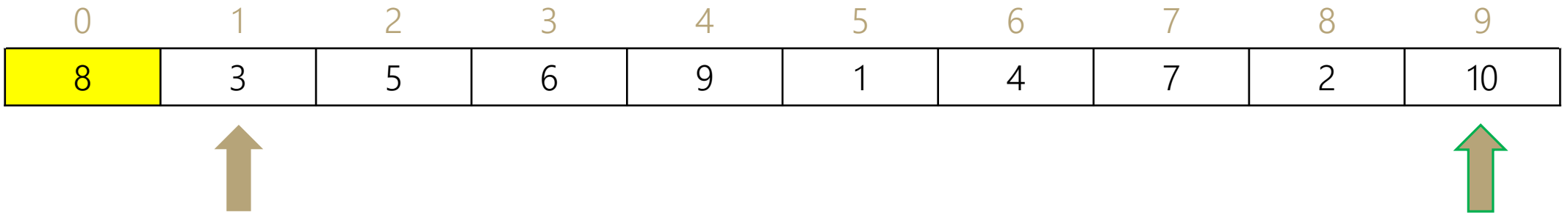
For now, let's just take the first thing in the array:

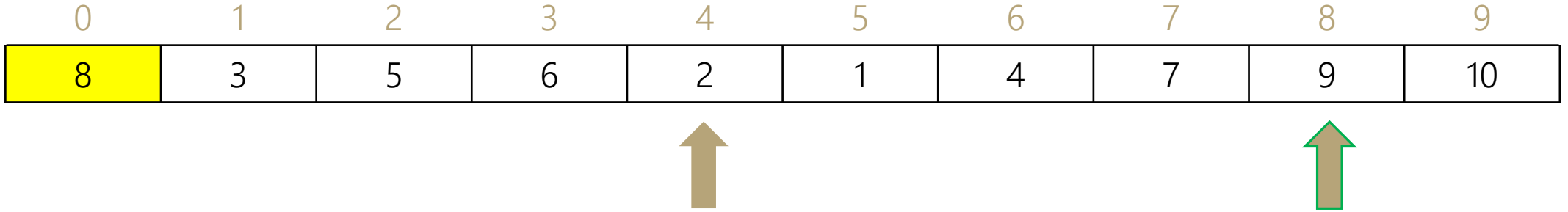
Swapping

How do we divide the array into "bigger than the pivot" and "less than the pivot?"

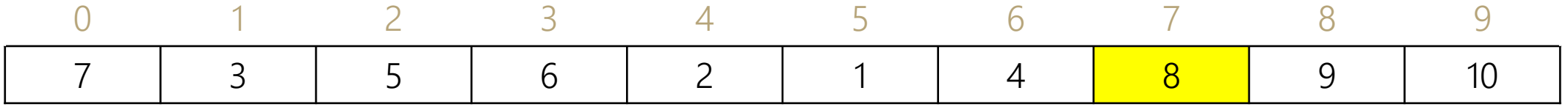
1. Swap the pivot to the far left.
2. Make a pointer i on the left, and j on the right
3. Until i, j meet
   - While $A[i] < \textbf{pivot}$ move i left
   - While $A[j] > \textbf{pivot}$ move j right
   - Swap $A[i], A[j]$
4. Swap A[i] or A[i-1] with pivot.

Swapping



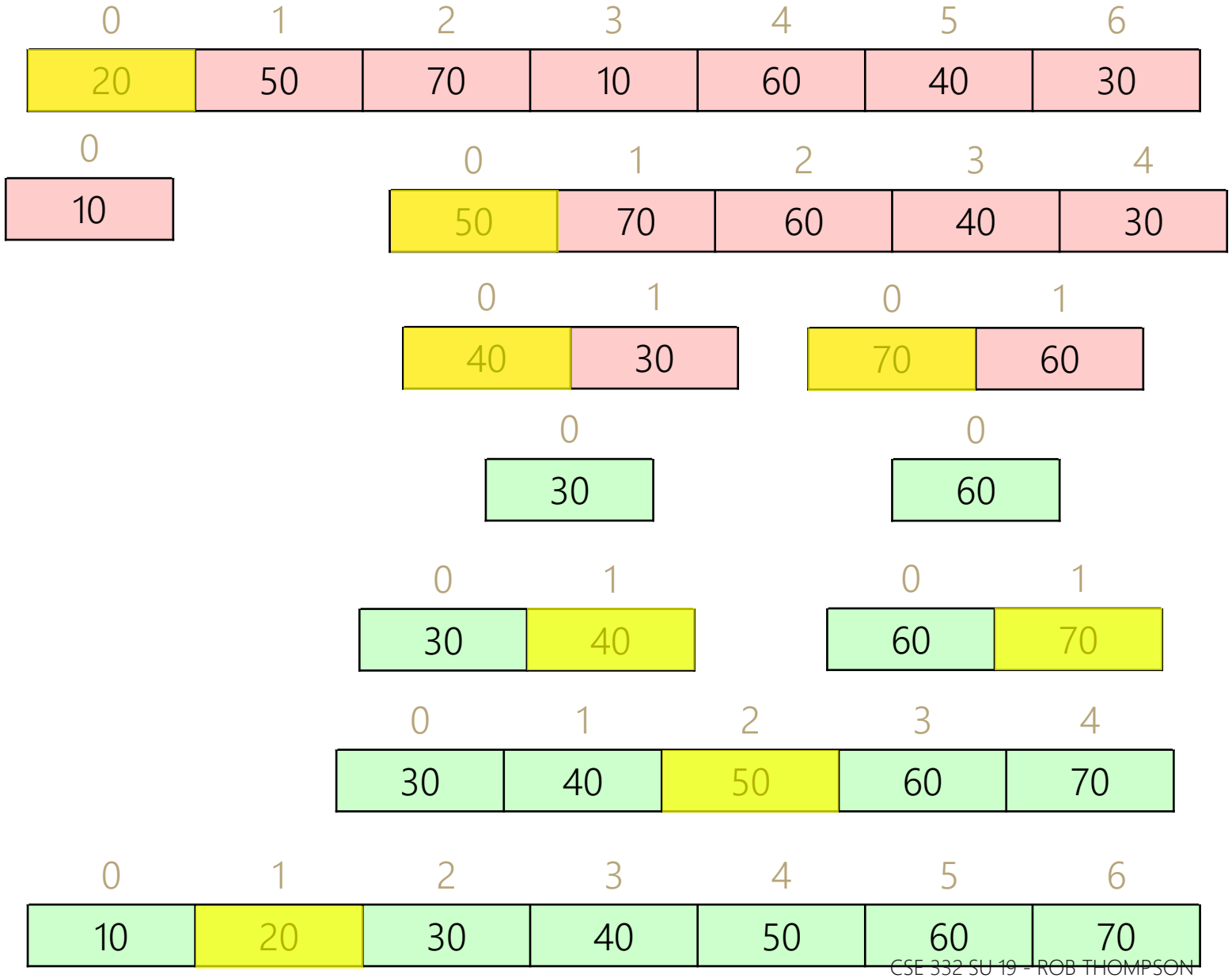


i, j met. $A[i]$ is larger than the pivot, so it belongs on the right, but $A[i-1]$ belongs on the left. Swap pivot and $A[i-1]$.



Quick Sort

https://www.youtube.com/watch?v=ywWBy6J5gz8



Quick Sort Analysis (Take 1)

What is the best case and worst case for a pivot?

- Best case: Picking the median
- Worst case: Picking the smallest or largest element

Recurrences:

Best:
$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + c_1 n & \text{if } n \geq 2 \\ c_2 & \text{otherwise} \end{cases}$$

Worst:
$$T(n) = \begin{cases} T(n-1) + c_1 n & \text{if } n \geq 2 \\ c_2 & \text{otherwise} \end{cases}$$

Running times:

- Best: $O(n \log n)$
- Worst: $O(n^2)$

Choosing a Pivot

Average case behavior depends on a good pivot.

Pivot ideas:

Just take the first element

- Simple. But an already sorted (or reversed) list will give you a bad time.

Pick an element uniformly at random.

- $O(n \log n)$ running time with probability at least $1 - 1/n^2$.
- Regardless of input!
- Probably too slow in practice :(

Find the actual median!

- You can actually do this in linear time
- Definitely not efficient in practice

Choosing a Pivot

Median of Three

- Take the median of the first, last, and midpoint as the pivot.
- Fast!
- Unlikely to get bad behavior (but definitely still possible)
- Reasonable default choice.

Quick Sort Analysis

Running Time:

- Worst $O(n^2)$
- Best $O(n \log n)$
- Average $O(n \log n)$ (not responsible for the proof, talk in OH if you're curious)

In place:

- Yes

Stable:

- No.

Divide and Conquer Sorting Cutoffs

Used n=1 cutoff for mergesort
- and presumably something similar for quicksort

For small *n*, all that recursion tends to cost more than doing a quadratic sort
- Remember asymptotic complexity is for large *n*
- Also, recursive calls add a lot of overhead for small n

Common engineering technique: switch to a different algorithm for subproblems below a cutoff
- Reasonable rule of thumb: use insertion sort for $n < 10$

Notes:
- Cutoffs are also the norm with parallel algorithms
  - switch to sequential algorithm
  - We will come back to this
- None of this affects asymptotic complexity

Lower Bound

We keep hitting $O(n \log n)$ in the worst case.

Can we do better?

Or is this $O(n \log n)$ pattern a fundamental barrier?

Without more information about our data set, we can do no better.

Comparison Sorting Lower Bound

Any sorting algorithm which only interacts with its input by comparing elements must take $\Omega(n \log n)$ time.

We'll prove this theorem on Friday!