# *Lecture 13: Applications of Linearity of Expectation*

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We use linearity of expectation in several applications.

## *Expectation*

Recall that the *expected value* of a real valued random variable is defined:

$$\mathbb{E}[X] = \sum_x p(X = x) \cdot x. \tag{1}$$

**Fact 1.** *If $X$ and $Y$ are real valued random variables in the same probability space, then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.*

The amazing thing is that linearity of expectation even works when the random variables are dependent. This allows us to calculate expectation very easily.

The first example we saw of this is Buffon's needle problem.

See the notes at: `http://www.math.leidenuniv.nl/~hfinkeln/seminarium/stelling_van_Buffon.pdf`.

## *Expectation of Geometric Random Variables*

Suppose you toss a fair coin until you see heads. How many times will you toss it in expectation?

Let $T$ be the number of coin tosses. The expected value here can be expressed as

$$\mathbb{E}[T] = 1 \cdot 1/2 + 2 \cdot (1/2)^2 + 3 \cdot (1/2)^3 + \cdots.$$

This can be calculated using identities from calculus, but there is another trick you can do, use conditional expectation.

Given an event $E$, the conditional expectation is exactly what you would expect after conditioning:

$$\mathbb{E}[X|E] = \sum_{x \in E} p(X = x|E) \cdot x.$$

**Fact 2.** *If $E$ is an event and $X$ is a random variable, then*

$$\mathbb{E}[X] = p(E) \cdot \mathbb{E}[X|E] + p(E^c) \cdot \mathbb{E}[X|E^c].$$

*Proof.*

$$\begin{aligned}
\mathbb{E}[X] &= \sum_x p(X = x) \cdot x \\
&= \sum_x (p(E) \cdot p(X = x|E) + p(E^c) \cdot p(X = x|E^c)) \cdot x \\
&= p(E) \sum_x p(X = x|E) \cdot x + p(E^c) \sum_x p(X = x|E^c) \cdot x \\
&= p(E) \cdot \mathbb{E}[X|E] + p(E^c) \cdot \mathbb{E}[X|E^c].
\end{aligned}$$

□

Returning to the problem of determining the number of tosses until the first heads, let $E$ denote the event that the first coin toss is heads. Observe that if the first coin toss is tails, the value of $T$ is 1. If not, the remaining experiment looks identical to the original experiment with one more coin toss. Then we have

$$\begin{aligned}
\mathbb{E}[T] &= p(E)\,\mathbb{E}[T|E] + p(E^c)\,\mathbb{E}[T|E^c] \\
&= (1/2) \cdot 1 + (1/2) \cdot (\mathbb{E}[T] + 1).
\end{aligned}$$

After rearranging, this shows that $\mathbb{E}[T] = 2$. Similarly, you if the coin gives heads with probability $p$, then the number of tosses is $1/p$, because we would get:

$$\begin{aligned}
\mathbb{E}[T] &= p(E)\,\mathbb{E}[T|E] + p(E^c)\,\mathbb{E}[T|E^c] \\
&= p \cdot 1 + (1 - p) \cdot (\mathbb{E}[T] + 1),
\end{aligned}$$

which gives $\mathbb{E}[T] = 1/p$, after rearranging. The number of tosses before the first heads is called a *geometric random variable* with parameter $p$, and its distribution is called the *geometric distribution*.

## *Balls and Bins Processes*

MANY PROBLEMS IN COMPUTER SCIENCE boil down to understanding *balls and bins* processes. Suppose you have $n$ bins, and you throw $m$ balls at them, where each ball goes to a completely random bin. How many balls will be in the most full bin? How many balls will you need to throw to make sure that each bin has at least one bin? These kinds of problems all have applications in computer science.

For example, such processes are used to solve the *leader-election* problem in distributed computing, which allows a distributed system to achieve consensus, and in load balancing when managing a large collection of servers.

Let us try to calculate the expected value of the basic statistics of this process.

### *Expected number of balls in each bin?*

To calculate the expected number of balls we see in a particular bin, let $X_i$ be the $0/1$ random variable that is 1 if and only if the $i$'th ball goes to the bin. Then the number of balls that are in that bin is exactly $X_1 + X_2 + \ldots + X_m$. Thus we have

$$\mathbb{E}\left[X_1 + \ldots + X_m\right] = \mathbb{E}\left[X_1\right] + \mathbb{E}\left[X_2\right] + \ldots + \mathbb{E}\left[X_m\right].$$

Now the expected value of $X_i$ is $1 \cdot (1/n) + 0 \cdot (1 - 1/n) = 1/n$. So the expected number of balls in each bin is $m/n$.

We could have calculated the same quantity using symmetry and linearity of expectation. Let $Y_j$ be the number of balls that end up in the $j$'th bin. Then $Y_1 + \ldots + Y_n = m$ always, so we have

$$m = \mathbb{E}\left[Y_1 + \cdots + Y_n\right] = \mathbb{E}\left[Y_1\right] + \cdots + \mathbb{E}\left[Y_n\right].$$

But by symmetry, all $n$ of these random variables should have the same expectation. So, the expected value should be $m/n$.

### *Number of Throws before every Bin has a Ball*

Suppose we keep throwing balls into the bins until every bin gets a ball. Let $T$ be the number of balls that need to be thrown before every bin gets a ball. What is the expected value of $T$?

To calculate this, let us write $T_i$ to be the number of throws after which exactly $i$ bins have a ball in them. We want to calculate $\mathbb{E}\left[T_n\right]$. By linearity of expectation, we have:

$$\begin{aligned}\mathbb{E}\left[T_n\right] &= \mathbb{E}\left[T_1 + (T_2 - T_1) + \cdots + (T_n - T_{n-1})\right] \\ &= \mathbb{E}\left[T_1\right] + \mathbb{E}\left[T_2 - T_1\right] + \cdots + \mathbb{E}\left[T_n - T_{n-1}\right].\end{aligned}$$

Now $\mathbb{E}\left[T_1\right] = 1$. For the other terms, $T_i - T_{i-1}$ is the number of tosses from the time that $i - 1$ bins have a ball, to the time that $i$ bins have a ball. When $i - 1$ bins have a ball, the probability that the next toss goes to a new bin is exactly $1 - (i-1)/n = (n - i + 1)/n$. Thus, the expected number of tosses before $i$ bins have a ball is a geometric random variable, with parameter $(n - i + 1)/n$. Its expectation is $n/(n - i + 1)$. So, we conclude that

$$\begin{aligned}\mathbb{E}\left[T\right] &= 1 + n/(n-1) + n/(n-2) + \cdots + n/(1) \\ &= n(1 + 1/2 + 1/3 + \cdots + 1/n) \\ &\approx n \ln n.\end{aligned}$$

The sum $H_n = 1 + 1/2 + \cdots + 1/n$ is called a Harmonic sum, and it is known that $\ln n \leq H_n \leq 1 + \ln n$.