Multi-Pass Parallel Algorithms

Data Structures and Parallelism

Logistics

Exercises 8, 9, 10, 11 are still out

- Due a week from today
- Will know everything you need by end of lecture
- Treat them as practice for P3
- Remember, these are to be done solo

P3 now out as well

- Checkpoint a week from today
- Look at the games handout if you haven't already

Recall: Amdahl's Law

With just some algebra we showed:

$$\frac{T_1}{T_P} \leq \frac{1}{S + \frac{1-S}{P}}$$

and

$$\frac{T_1}{T_\infty} \leq \frac{1}{S}$$

Amdahl's Law and Moore's Law

In the Moore's Law days, 12 years was long enough to get 100x speedup.

Suppose in 12 years, the clock speed is the same, but you have 256 processors.

What portion of your program can you hope to leave unparallelized?

$$100 \leq \frac{1}{S+\frac{1-S}{256}}$$

[wolframalpha says] $S \leq 0.0061$.

Amdahl's Law: Moving Forward

Unparallelized code becomes a bottleneck quickly.

What do we do? Design smarter algorithms!

Consider the following problem:

Given an array of numbers, return an array with the "running sum"

3	7	6	2	4

3	10	16	18	22

Sequential Code

```
output[0] = input[0];
for(int i=1; i<arr.length;i++){
    output[i] = input[i] + output[i-1];
}
```

More clever algorithms

Doesn't look parallelizable…

But it is!

Algorithm was invented by Michael Fischer and Richard Ladner

- Both were in UW CSE's theory group at the time
- Richard Ladner is still around
  - Look for a cowboy hat…

For today: I'm not going to worry at all about constant factors.

Just try to get the ideas across.

Parallelizing

What do we need?

Need to quickly know the “left sum” i.e. the sum of all the elements to my left.

- in the sequential code that was `output[i-1]`

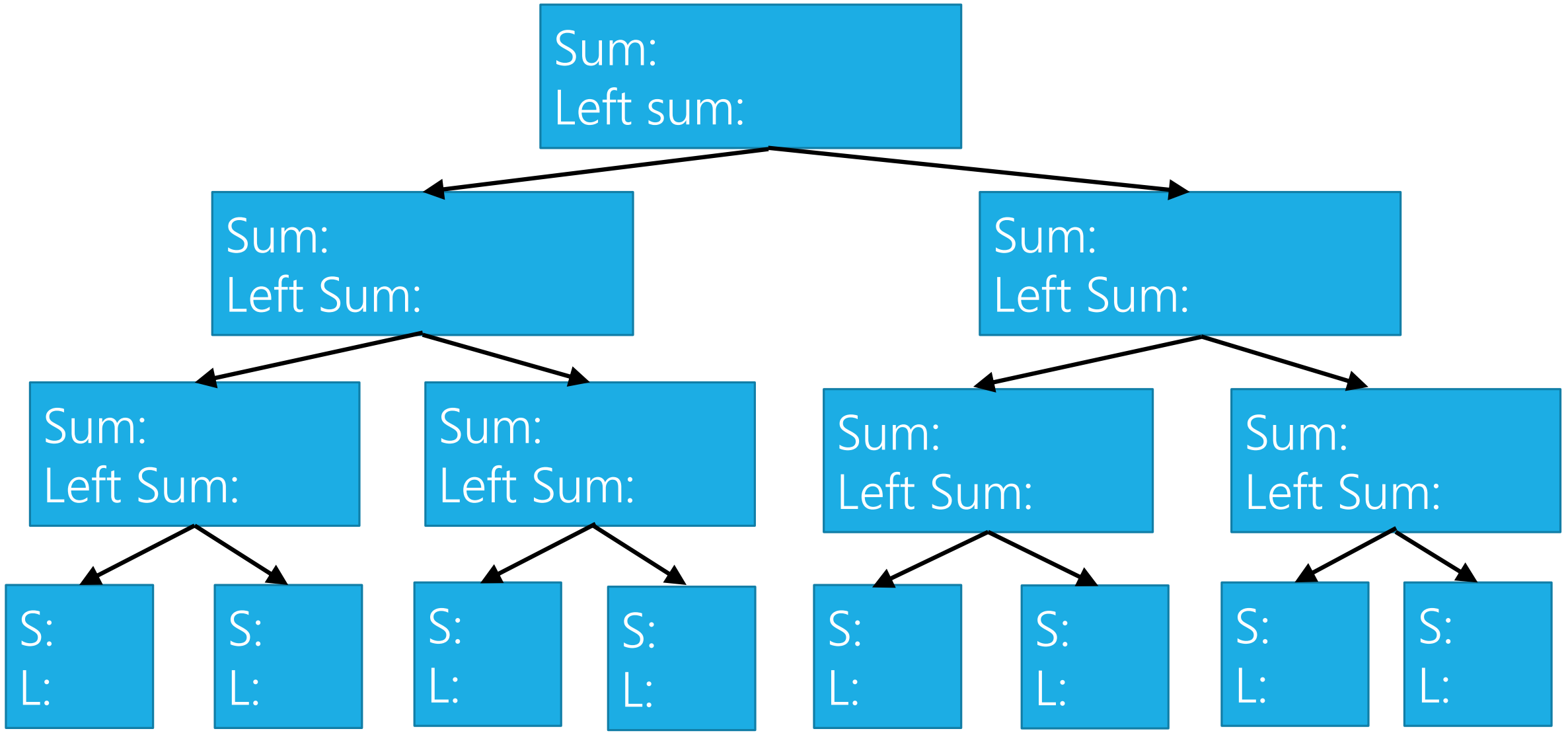
We’ll use three passes,

The first sets up a data structure

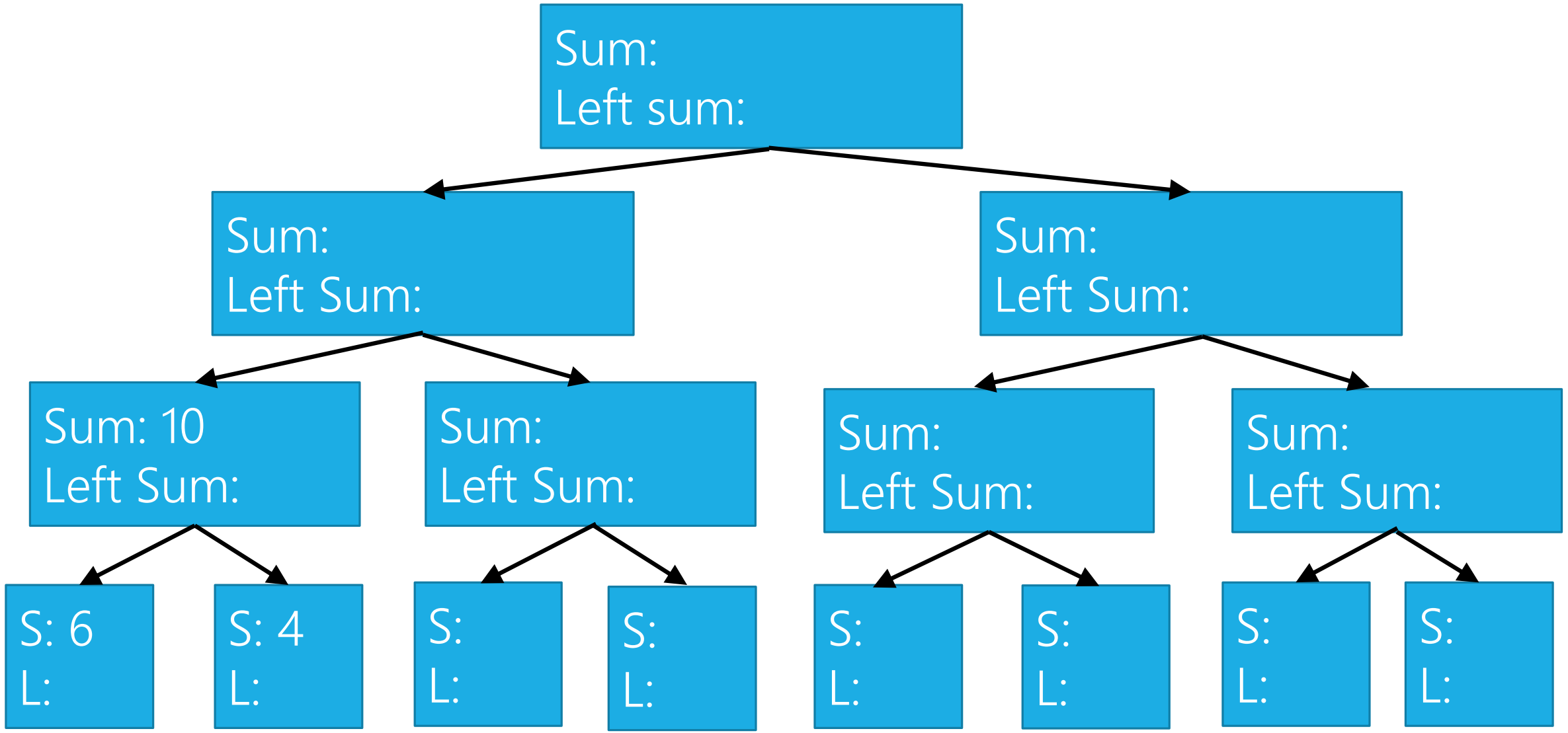
- Which will contain enough information to find left sums quickly

The second will assemble the left sum

The third will do a bit extra to assemble the final output



6	4	16	10	16	14	2	8



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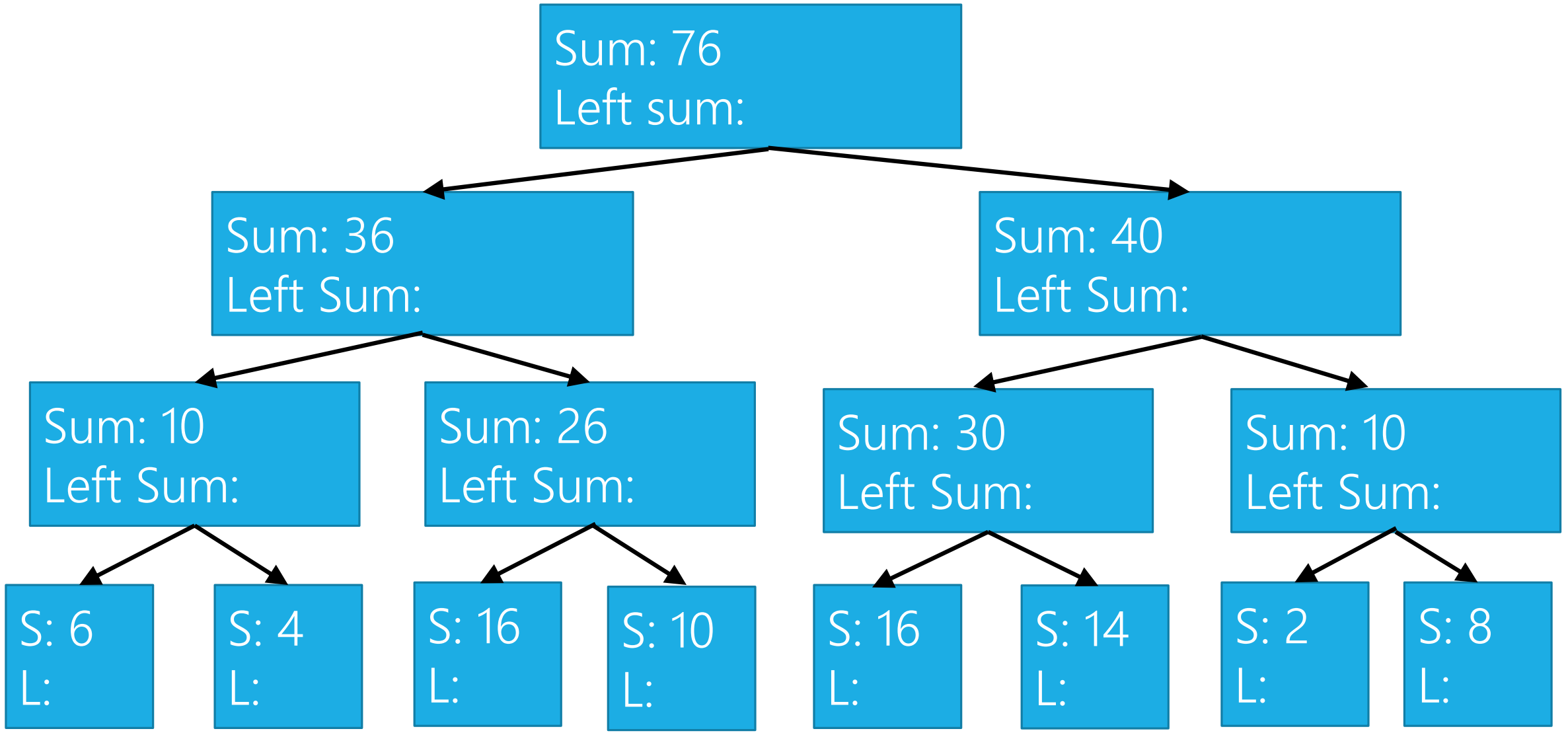

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First Pass

Calculating those sums is the end of the first pass.

How long does it take in parallel?

Work:

Span:

Remember “work” is the running time on one processor.

“span” is the running time on infinitely many processors.

First Pass

Calculating those sums is the end of the first pass.

How long does it take in parallel?

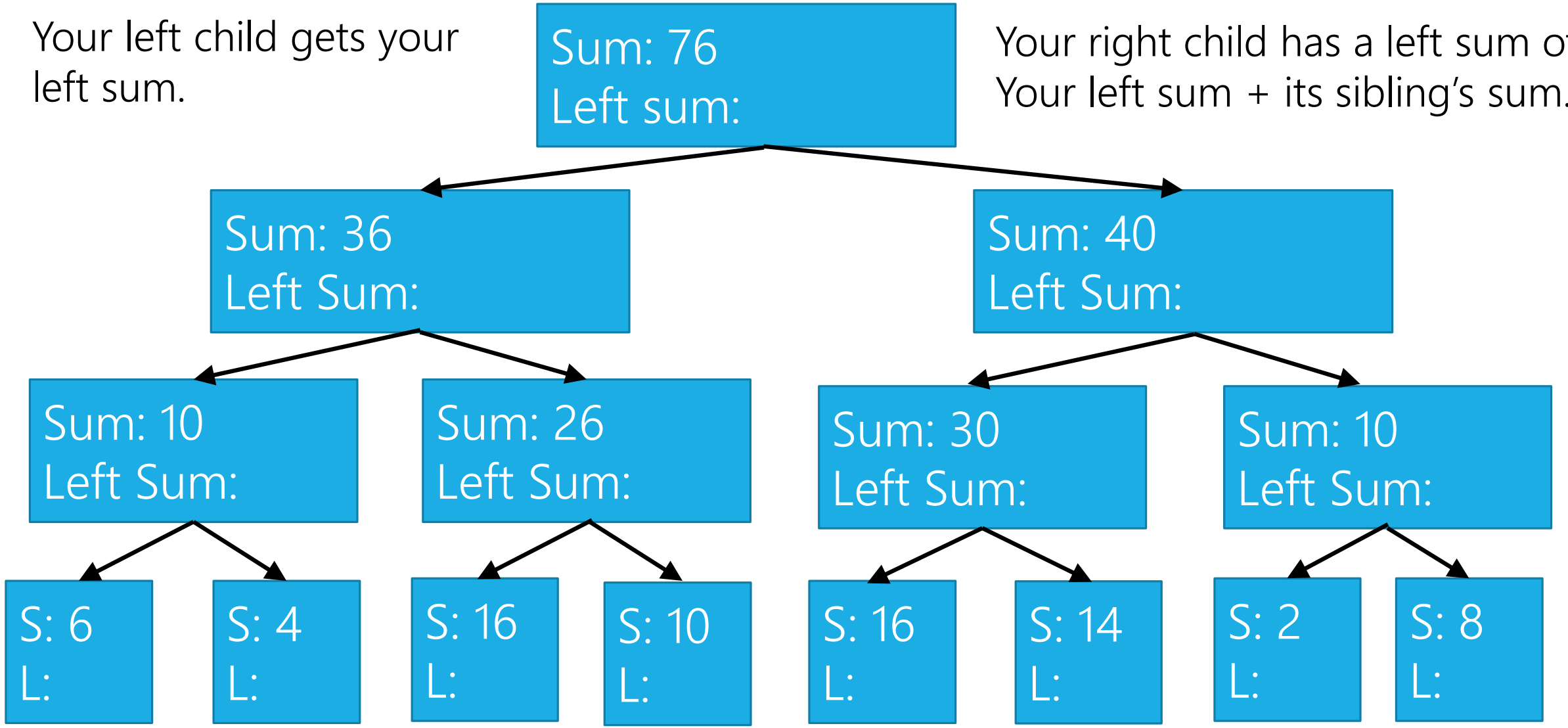
Work: $O(n)$

Span:$O(\log n)$

Just slightly modify our sum reduce code to build the data structure.

Your left child gets your left sum.

Your right child has a left sum of: Your left sum + its sibling's sum.



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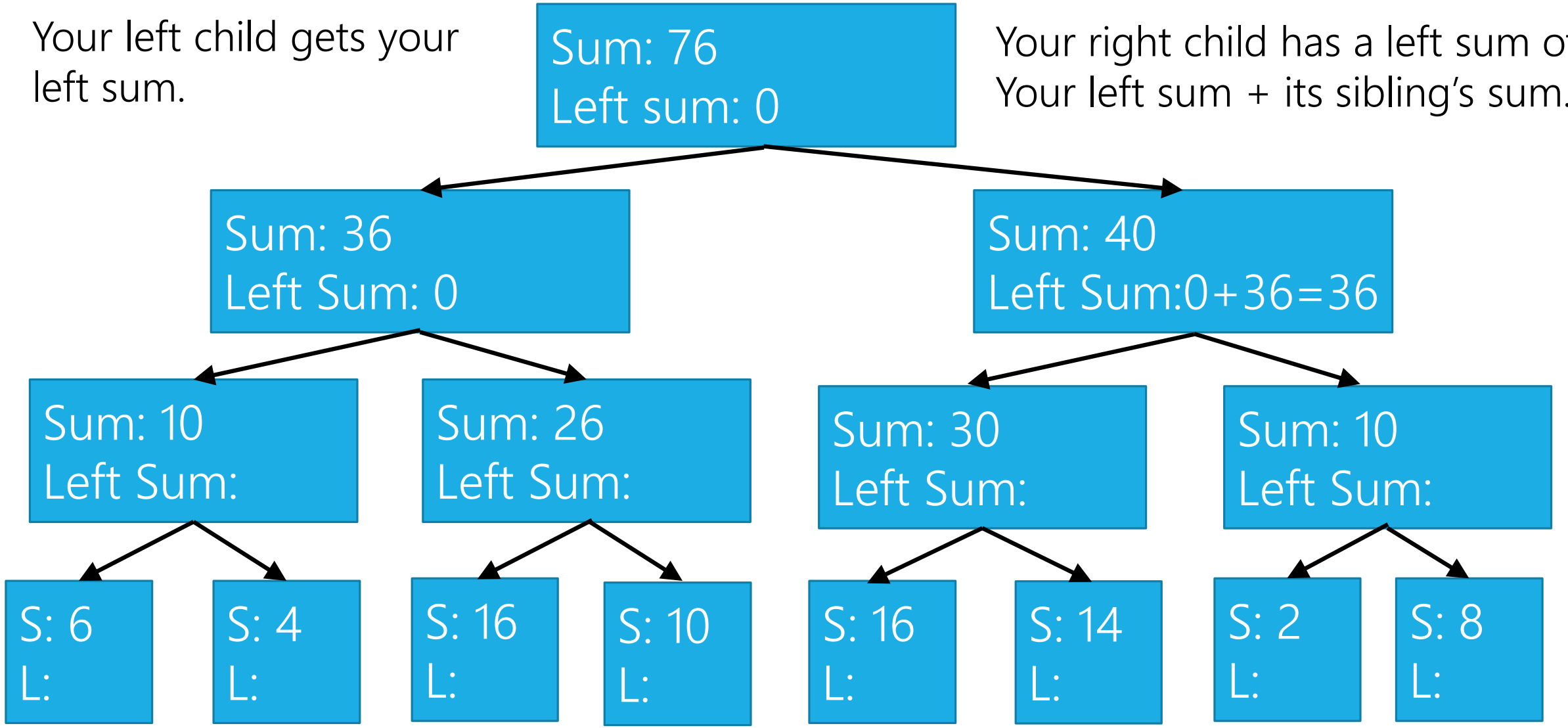

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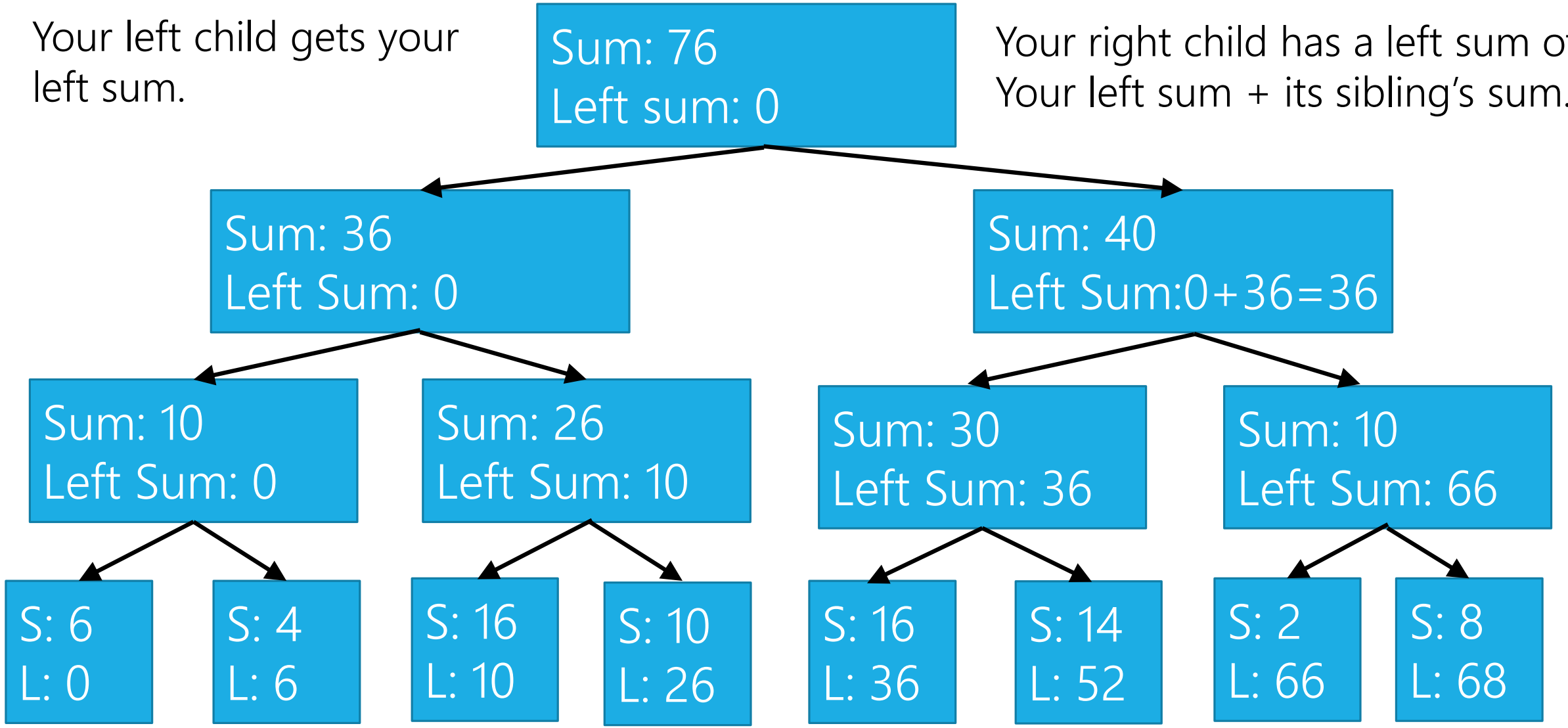

6	4	16	10	16	14	2	8

Second Pass

Once we've finished calculating the sums, we'll start on the left sums. Can we do that step in parallel?

YES!

Why are we doing two separate passes?
Those sum values have to be stored and ready.

Second pass has:
Work:

Span:

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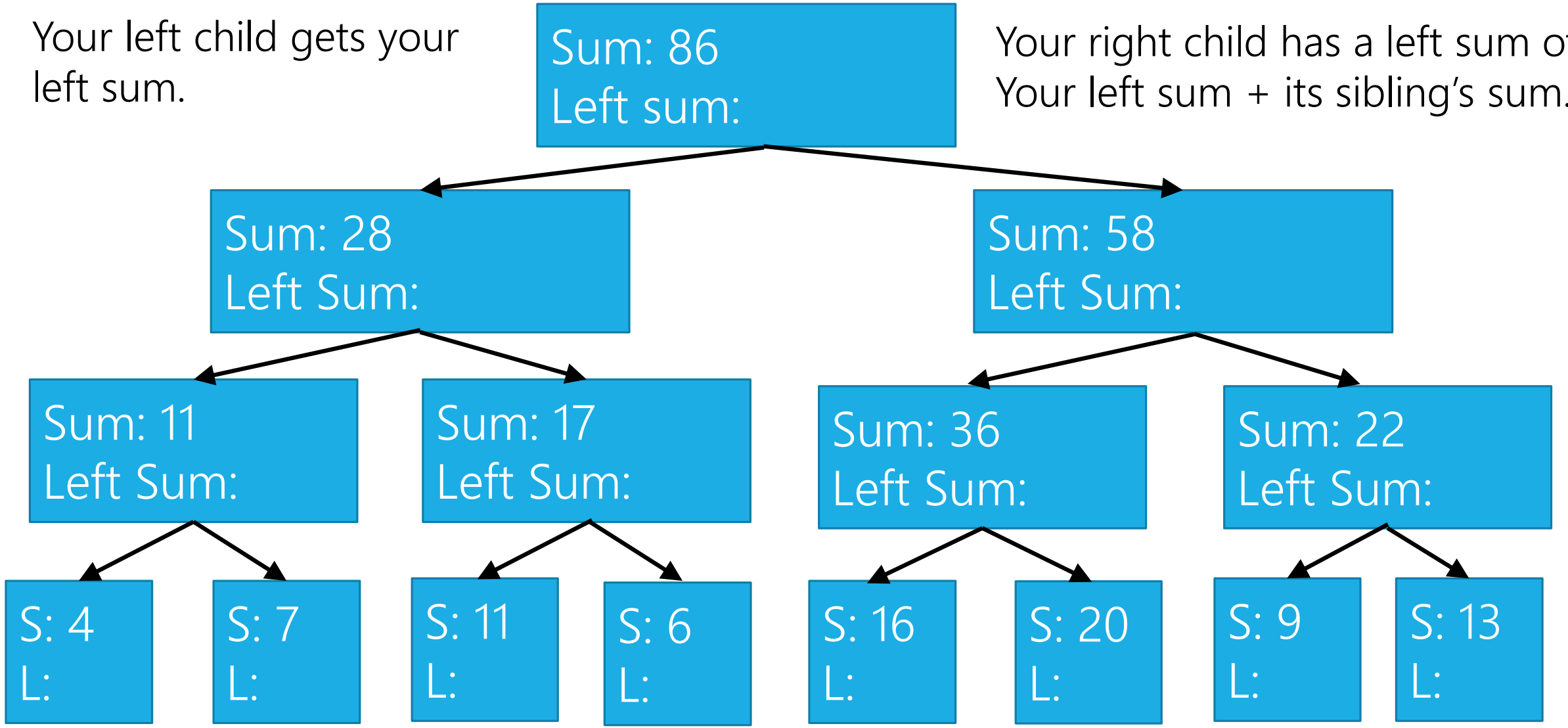

4	7	11	6	16	20	9	13

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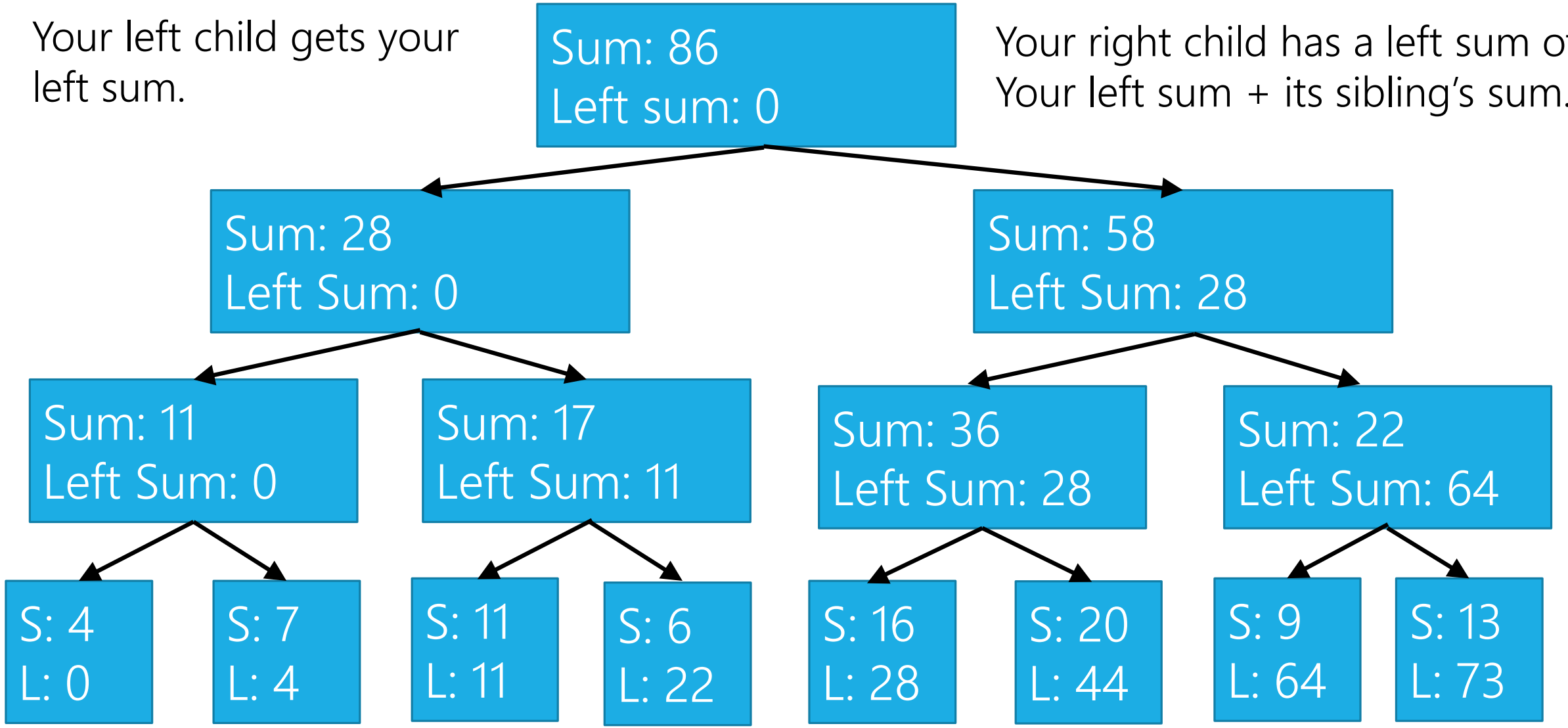

4	7	11	6	16	20	9	13

Third Pass

What's our final answer?

Our sequential code said element `i` of the new array should be

```
arr[i] + output[i-1]
```

Or equivalently

```
arr[i] + left_sum[i]
```

Just need one more map using the data structure.

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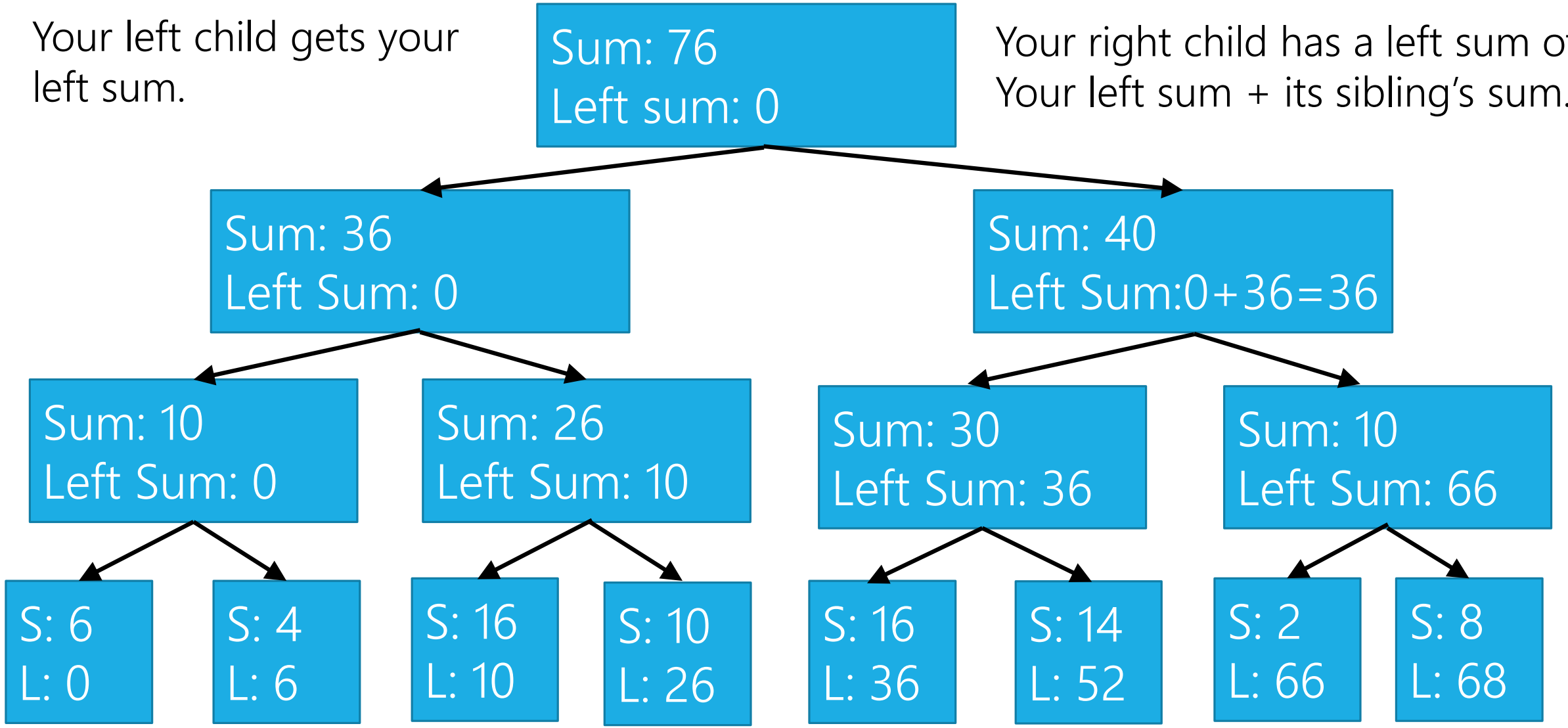

6	4	16	10	16	14	2	8
6	10	26	36	52	66	68	76

Analyzing Parallel Prefix

What's the

Work?

Span?

First pass was a slightly modified version of our sum reduce code.

Second pass had a similar structure

Third pass was a map

Analyzing Parallel Prefix

What's the

Work $O(n)$

Span $O(\log n)$

First pass was a slightly modified version of our sum reduce code.

Second pass had a similar structure.

Third pass was a map.

Parallel Pack (aka Filter)

You want to find all the elements in an array meeting some property.

And move ONLY those into a new array.

Input:

6	4	16	10	16	14	2	8

Want every element >= 10

Output:

16	10	16	14

Parallel Pack

Easy – do a map to find the right elements…

Hard – How do you copy them over?

Parallel Pack

Easy – do a map to find the right elements…

Hard – How do you copy them over?

I need to know what array location to store in,

i.e. how many elements to my left will go in the new array.

Parallel Pack

Easy – do a map to find the right elements…

Hard – How do you copy them over?

I need to know what array location to store in,

i.e. how many elements to my left will go in the new array.

- Use Parallel Prefix!

Parallel Pack

Step 1: Parallel Map – produce bit vector of elements meeting property

6	4	16	10	16	2	14	8
0	0	1	1	1	0	1	0

Step 2: Parallel prefix sum on the bit vector

0	0	1	2	3	3	4	4

Step 3: Parallel map for output.

16	10	16	14

Step 3

How do we do step 3?

i.e. what's the map?

```
if(bits[i] == 1)
     output[ bitsum[i] - 1] = input[i];
```

Parallel Pack

We did 3 passes:

A map

A prefix

And another map.

Work:

Span:

Remark: You could fit this into 2 passes instead of 3. Won't change O().

Parallel Pack

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A map

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And another map.

Work: $O(n)$

Span: $O(\log n)$

Remark: You could fit this into 2 passes instead of 3. Won't change O().

Four Patterns

We've now seen four common patterns in parallel code

1. Map
2. Reduce
3. Prefix
4. Pack (a.k.a. Filter)

Parallelizing Quick Sort

```
Quicksort(){
    pick a pivot
    partition array such that:
        left side of the array is less than pivot
        pivot in middle
        right side of array is greater than pivot
    recursively sort left and right sides.
}
```

Quick Analysis Note

For all of our quick sort analysis, we'll do best case.

The average case is the same as best case.

Worst case is still going to be the same (bad) $\Theta(n^2)$ with parallelism or not.

Parallelizing Quick Sort

Step 1: Make the recursive calls forks instead of recursion.

What is the new (need some recurrences)

Work?

Span?

Parallelizing Quick Sort

Step 1: Make the recursive calls forks instead of recursion.

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Work? $T_1(n) = \begin{cases} 2T_1\left(\frac{n}{2}\right) + c_1 \cdot n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

Span? $T_\infty(n) = \begin{cases} T_\infty\left(\frac{n}{2}\right) + c_1 \cdot n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

Parallelizing Quick Sort

Step 1: Make the recursive calls forks instead of recursion.

What is the new (need some recurrences)

Work? $T_1(n) = \Theta(n \log n)$

Span? $T_\infty(n) = \Theta(n)$

Parallel Quick Sort

With infinitely many processors, we can speed up quicksort from $\Theta(n \log n)$ to…

$\Theta(n)$.

So…yeah….

We can do better! (if we sacrifice in-place)

What about the partition step?

Parallel Quick Sort

The bottleneck of the code isn't the number of recursive calls

It's the amount of time we spend doing the partitions.

Can we partition in parallel?

What is a partition?

It's moving all the elements smaller than the pivot into one subarray

And all the other elements into the other

Better Parallel Quick Sort

Sounds like a pack! (or two)

Step 1: choose pivot

Step 2: parallel pack elements smaller than pivot into the auxiliary array

Step 3: parallel pack elements greater than pivot into the auxiliary array

Step 4: Recurse! (in parallel)

Better Parallel Quick Sort

What is (a recurrence for)

The work?

The span?

Better Parallel Quick Sort

What is (a recurrence for)

The work: $\mathrm{T}_1(\mathrm{n}) = \begin{cases} 2T_1\left(\frac{n}{2}\right) + c_1 \cdot n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

The span: $T_\infty(n) = \begin{cases} T_\infty\left(\frac{n}{2}\right) + c_1 \cdot \log n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

Better Parallel Quick Sort

What is (a recurrence for)

The work: $\Theta(n \log n)$

The span: $\Theta(\log^2 n)$

Parallel Merge Sort?

Sequential: $T(\mathrm{n}) = \begin{cases} 2T\left(\frac{n}{2}\right) + c_1 \cdot n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

Can turn recursive calls into parallel calls again

Merge costs us: $O(n)$, can we parallelize that?

Parallel Merge

None of our 4 parallel operations apply here

What about just splitting the work between two parallel threads?

Parallel Merge

How do we merge?

Find median of one array. (in $O(1)$ time)

Binary search in the other to find where it would fit.

Merge the two left subarrays and the two right subarrays

- In parallel!

Only need one auxiliary array

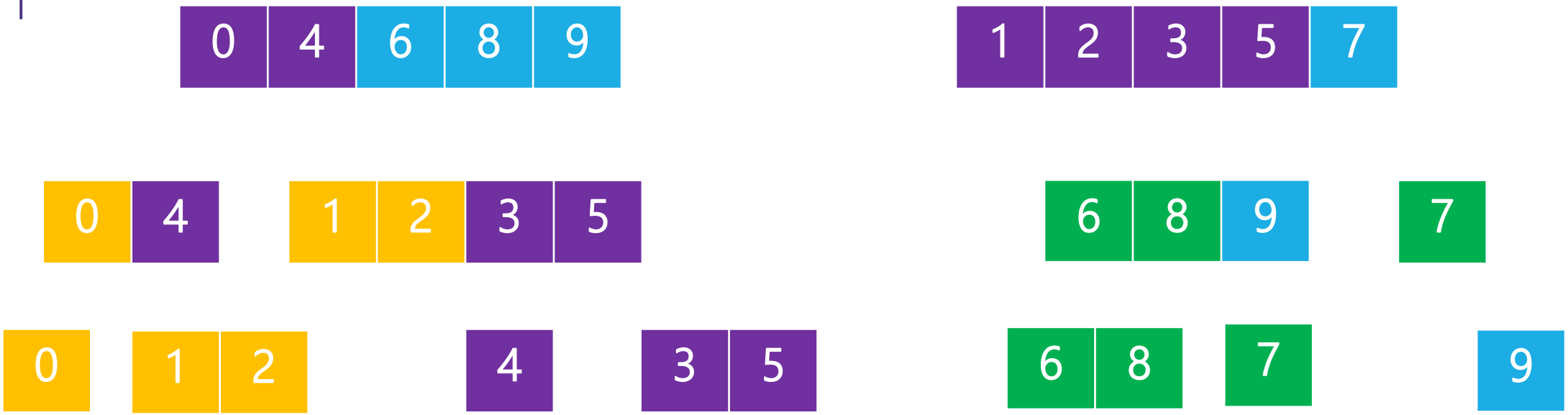
Each recursive call knows which range of the output array it's responsible for.

Key for the analysis: find the median in the bigger array, and binary search in the smaller array.

Parallel Merge

Find median of larger subarray (left chosen arbitrarily for tie)
Binary search to find where 6 fits in other array
Parallel recursive calls to merge

Parallel Merge



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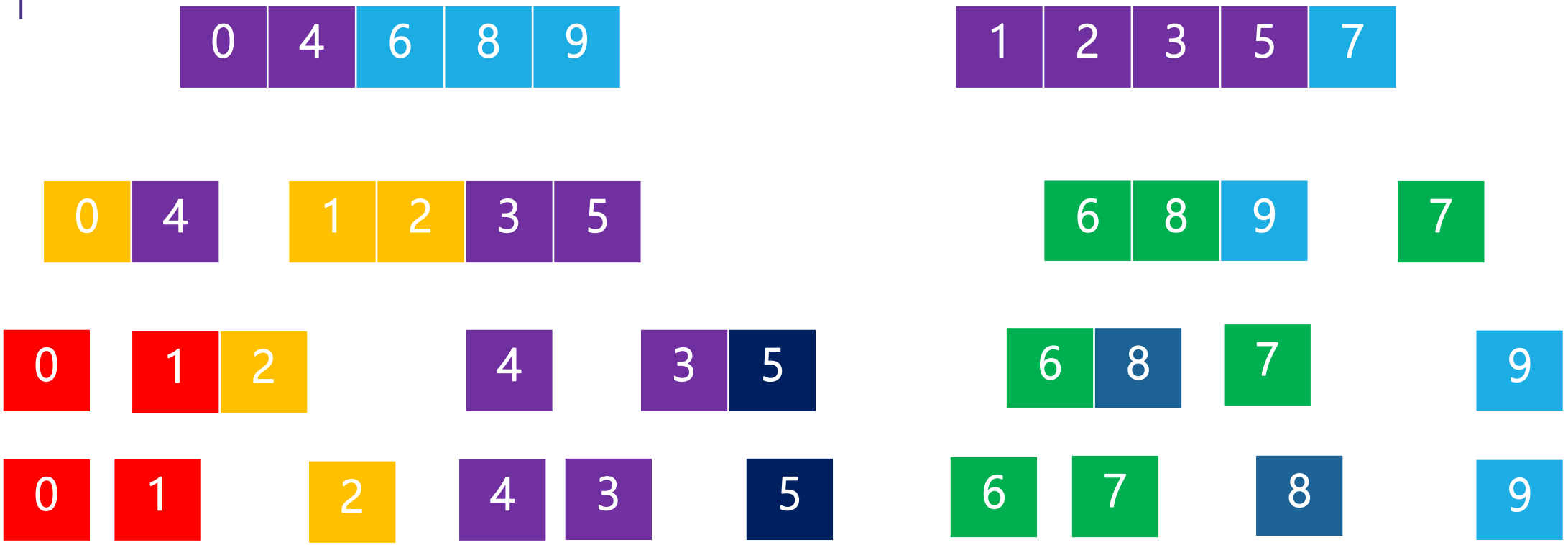

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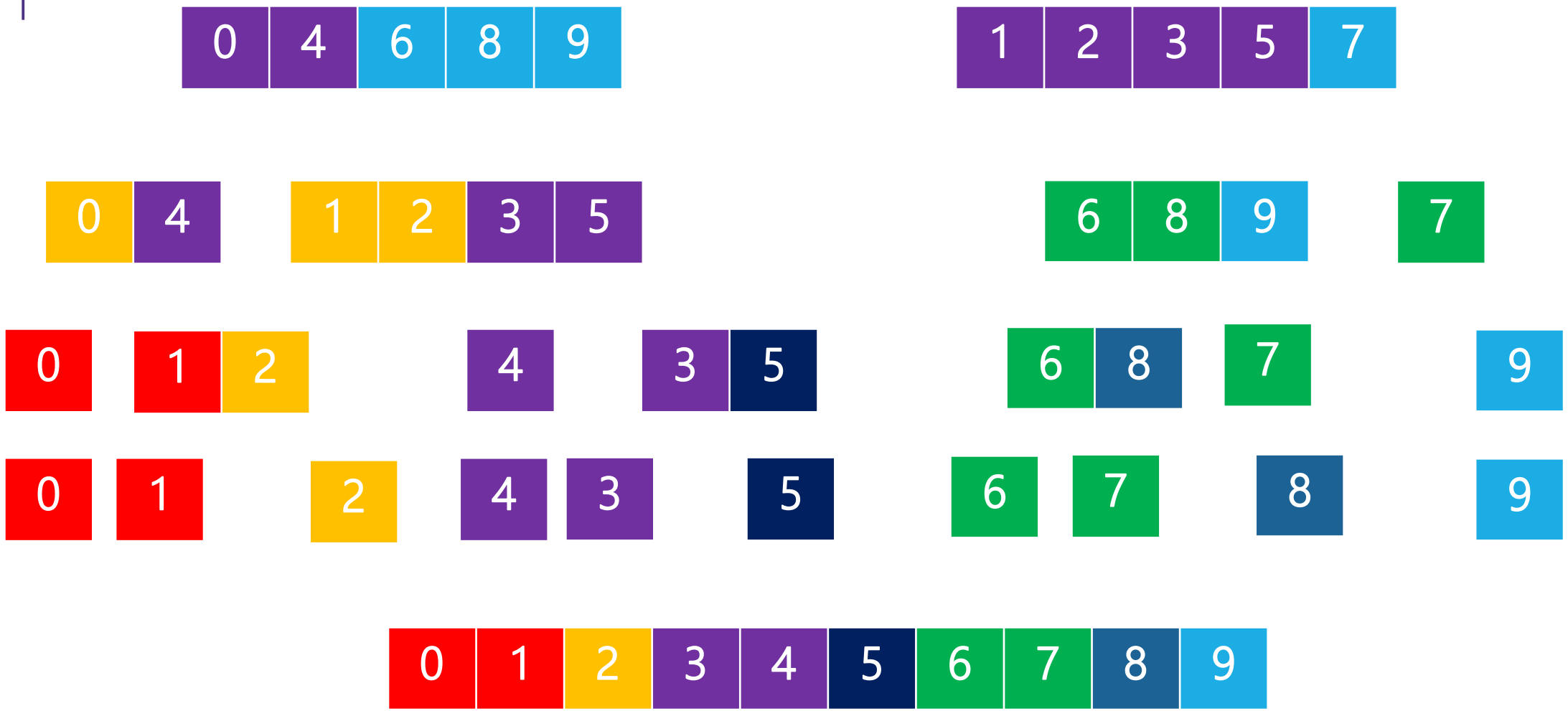

Parallel Merge

Let's just analyze the merge:

What's the worst case?

One subarray has ¾ of the elements, the other has ¼ .

This is why we start with the median of the larger array.

Work: ?

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Parallel Merge

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This is why we start with the median of the larger array.

Work: $\mathrm{T}_1(\mathrm{n}) = O(n)$

Span: $T_\infty(n) = O(\log^2 n)$

Parallel Merge Sort

Now the full mergesort algorithm:

Work: $\mathrm{T}_1(\mathrm{n}) = \begin{cases} 2T_1\left(\frac{n}{2}\right) + c \cdot n & \text{if } n \geq \text{cutoff} \\ c_2 & \text{if } n < \text{cutoff} \end{cases}$

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Parallel Merge Sort

Now the full mergesort algorithm:

Work: $\mathrm{T}_1(\mathrm{n}) = \Theta(n \log n)$

Span: $T_\infty(n) = \Theta(\log^3 n)$