CSE 332: Data Structures & Parallelism

Lecture 25: P, NP, NP-Complete (part 2)

Slides from

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Winter 2019

Today’s Agenda

- A Few Problems:
  - Euler Circuits
  - Hamiltonian Circuits
- Intractability: P and NP
- NP-Complete
- What now?

A Glimmer of Hope

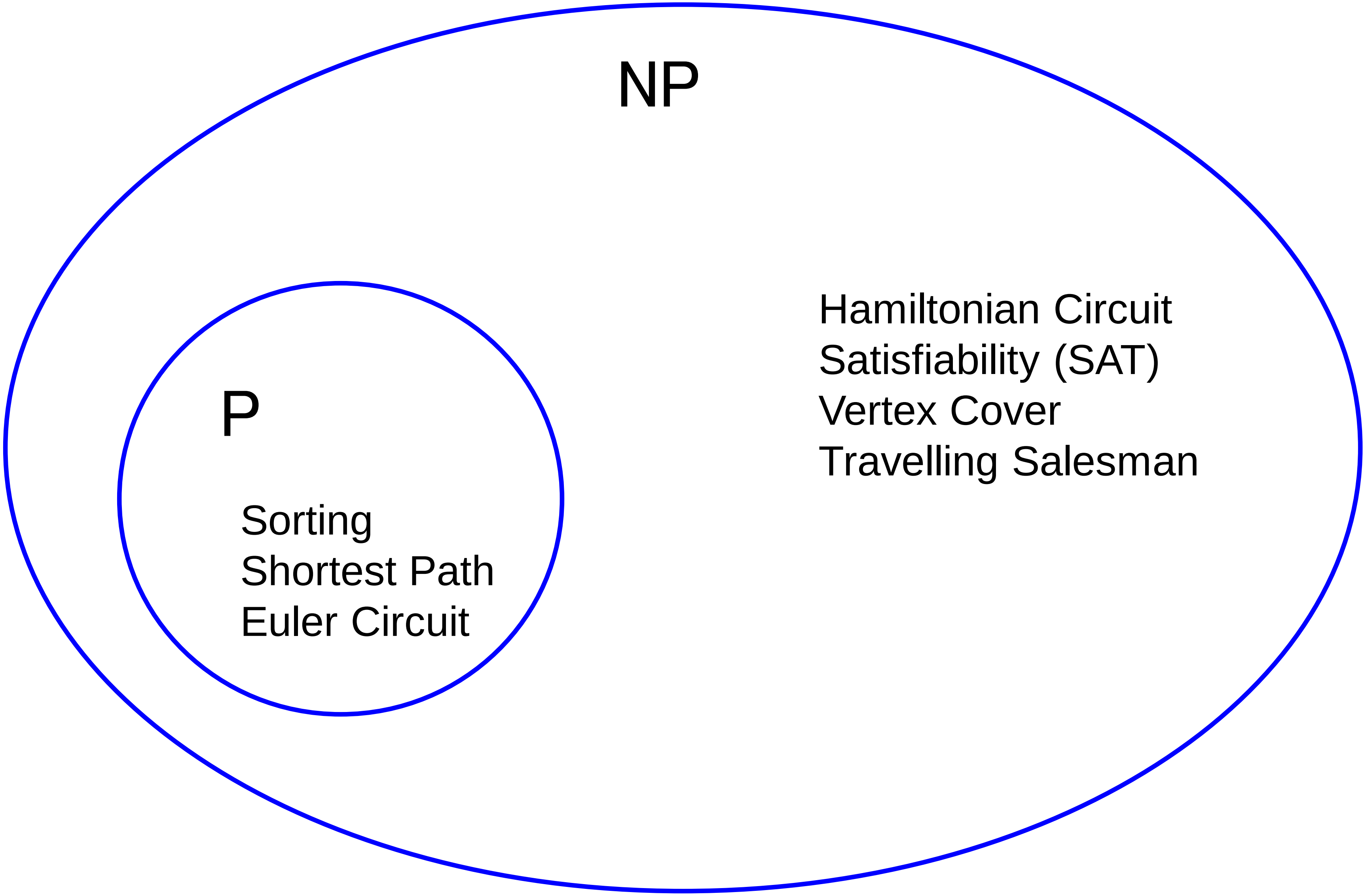
- If given a candidate solution to a problem, we can **check if that solution is correct in polynomial-time**, then **maybe** a polynomial-time solution exists?

- Can we do this with Hamiltonian Circuit?
  - Given a candidate path, is it a Hamiltonian Circuit?

The Complexity Class NP

- *Definition*: NP is the set of all problems for which a given *candidate solution* can be *tested* in polynomial time

- Examples of problems in NP:
  - *Hamiltonian circuit:* Given a candidate path, can test in linear time if it is a Hamiltonian circuit
  - *Vertex Cover:* Given a subset of vertices, do they cover all edges?
  - *All problems that are in P (why?)*


NP
P
Sorting
Shortest Path
Euler Circuit
Hamiltonian Circuit
Satisfiability (SAT)
Vertex Cover
Travelling Salesman

Why do we call it "NP"?

- NP stands for *Nondeterministic Polynomial time*
  - Why "nondeterministic"? Corresponds to algorithms that can guess a solution (if it exists), the solution is then verified to be correct in polynomial time
  - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each branch point.
  - Nondeterministic algorithms don't exist – purely theoretical idea invented to understand how hard a problem could be

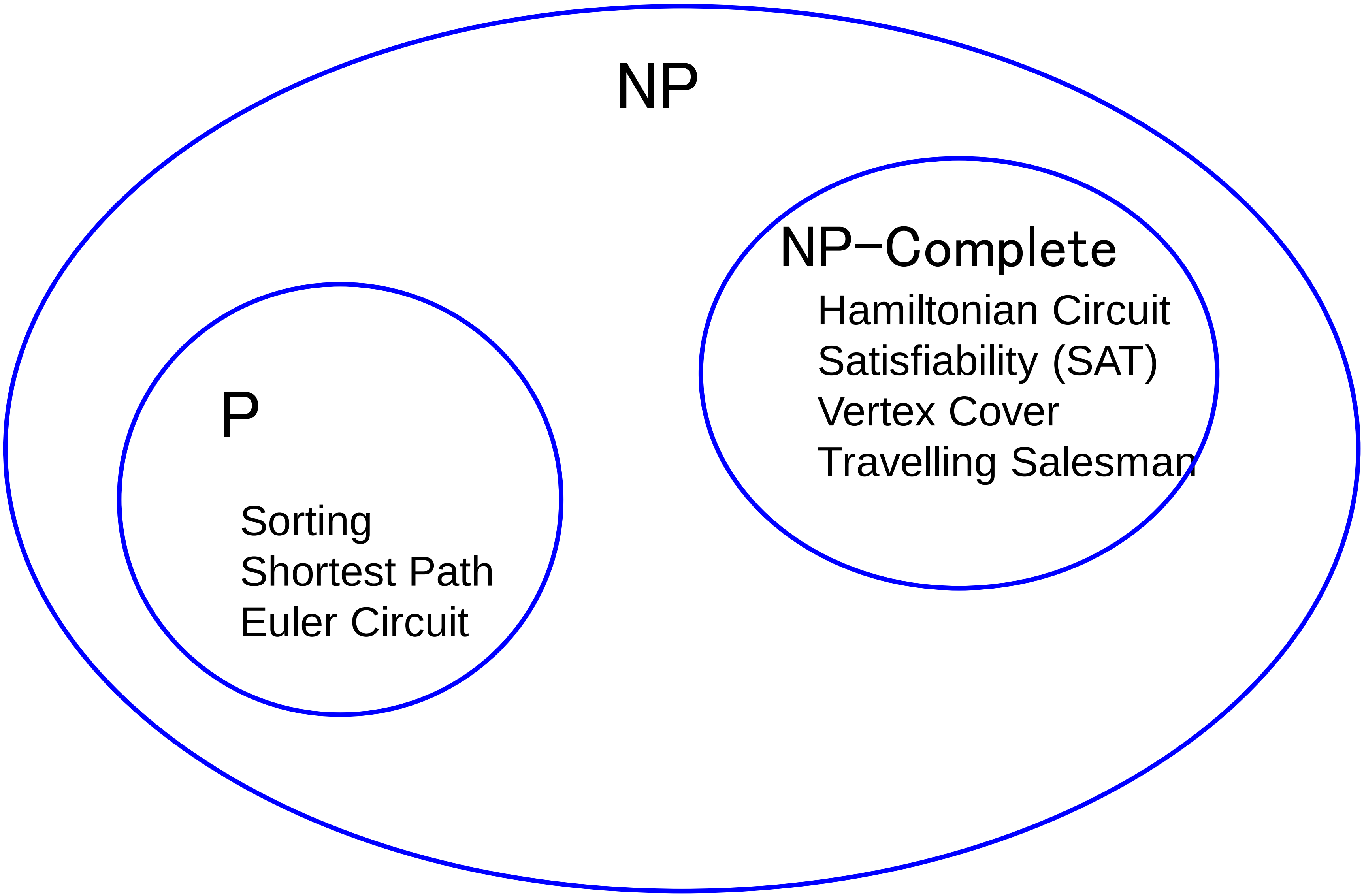
Your Chance to Win a Turing Award!

It is generally believed that P $\neq$ NP,
i.e. there are problems in NP that are **not** in P

- But no one has been able to show even one such problem!
- This is the fundamental open problem in theoretical computer science
- Nearly everyone has given up trying to prove it. Instead, theoreticians prove theorems about what follows once we assume P $\neq$ NP !

NP-completeness

- Set of problems in NP that (we are pretty sure) ***cannot*** be solved in polynomial time.
- These are thought of as the **hardest** problems in the class NP.
- **Interesting fact:** If any one NP-complete problem could be solved in polynomial time, then ***all*** NP-complete problems could be solved in polynomial time.
- Also: If any NP-complete problem is in P, then all of NP is in P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P
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Saving Your Job

- Try as you might, every solution you come up with for the Hamiltonian Circuit problem runs in exponential time…..
- You have to report back to your boss.
- Your options:
  - Keep working
  - Come up with an alternative plan…

In general, what to do with a Hard Problem

- Your problem seems really hard.
- If you can transform a known NP-complete problem into the one you're trying to solve, then you can stop working on your problem!

Your Third Task

- Your boss buys your story that others couldn’t solve the last problem.
- Again, your company has to send someone by car to a set of cities. There is a road between every pair of cities.
- The primary cost is distance traveled (which translates to fuel costs).
- Your boss wants you to figure out *how to drive to each city exactly once*, then return to the first city, while *staying within a fixed mileage budget k*.

Travelling Salesman Problem (TSP)

- Your third task is basically TSP:
  - Given complete weighted graph G, integer k.
  - Is there a cycle that visits all vertices with cost <= k?
- One of the canonical problems.

- Note difference from Hamiltonian cycle:
  - graph is complete
  - we care about weight.

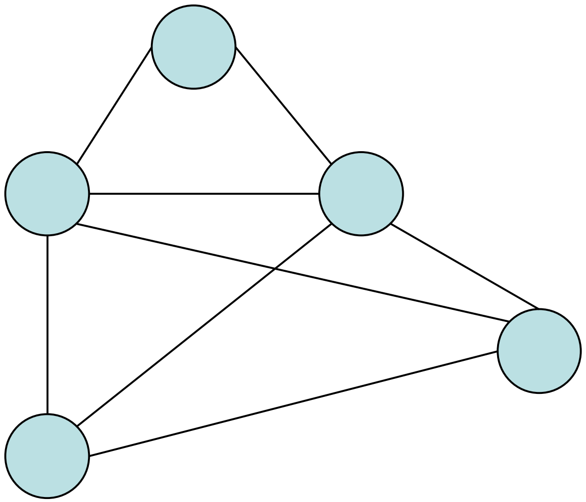
Transforming Hamiltonian Cycle to TSP

- We can "reduce" Hamiltonian Cycle to TSP.
- Given graph G=(V, E):
  - Assign weight of 1 to each edge
  - Augment the graph with edges until it is a complete graph G'=(V, E')
  - Assign weights of 2 to the new edges
  - Let k = |V|.

Notes:

- The transformation must take polynomial time
- You reduce the known NP-complete problem into your problem (not the other way around)
- In this case we are assuming Hamiltonian Cycle is our known NP-complete problem (in reality, both are known NP-complete)

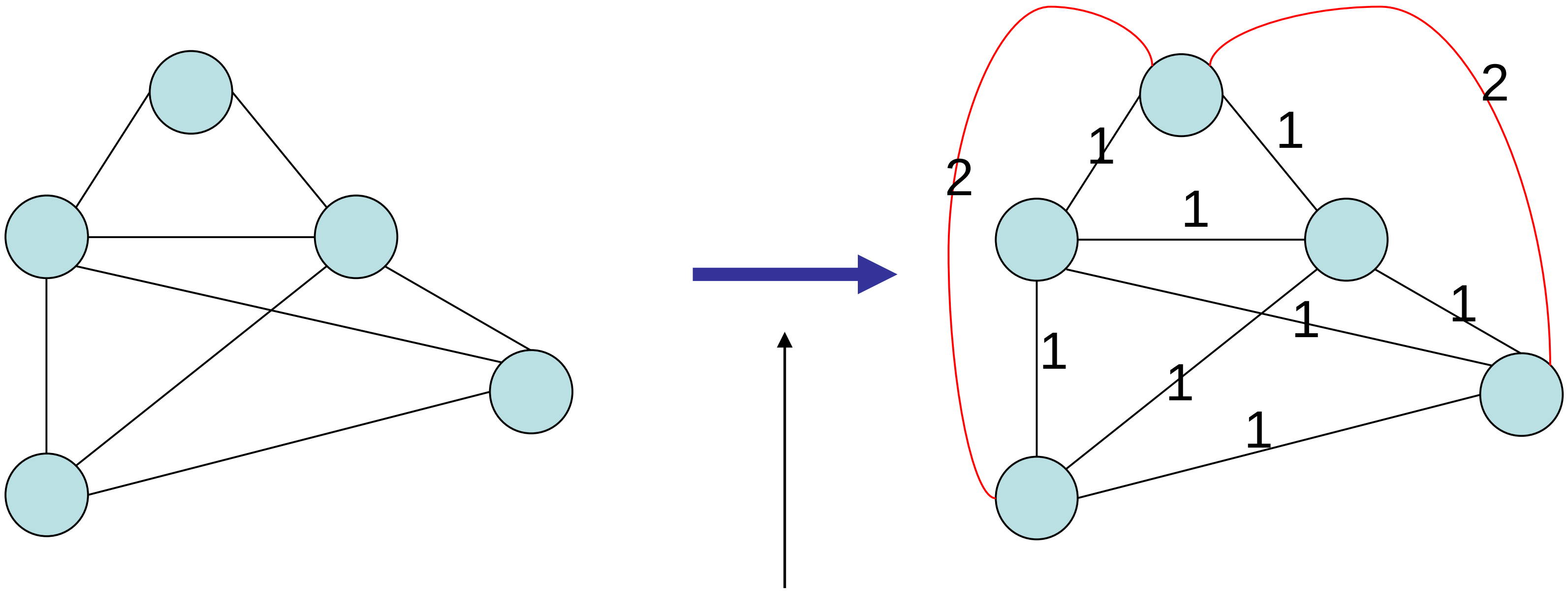
Example



G

Input to Hamiltonian
Circuit Problem

Example



G

Polynomial time transformation

G'

Input to Hamiltonian Circuit Problem

Input to Traveling Salesman Problem

Polynomial-time transformation

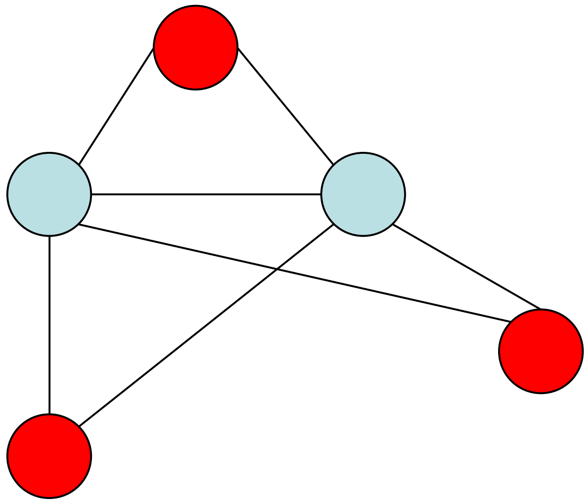
- G' has a TSP tour of weight |V| iff
  G has a Hamiltonian Cycle.
- What was the cost of transforming HC into TSP?

- In the end, because there is a polynomial time transformation from HC to TSP, we say *TSP is "at least as hard as" Hamiltonian cycle.*

Another Example

Independent Set:

For a graph G=(V,E) a subset of vertices S is an independent set if there are no edges connecting two vertices in S



Decision Version

Does a graph G=(V,E) have an independent set of size *g*?

Is this problem in NP?

Is it NP-Complete?

Conversion from 3-SAT

If we want to show Independent Set is NP-complete, we need to convert another NP-complete problem into it

3-SAT is what we'll use

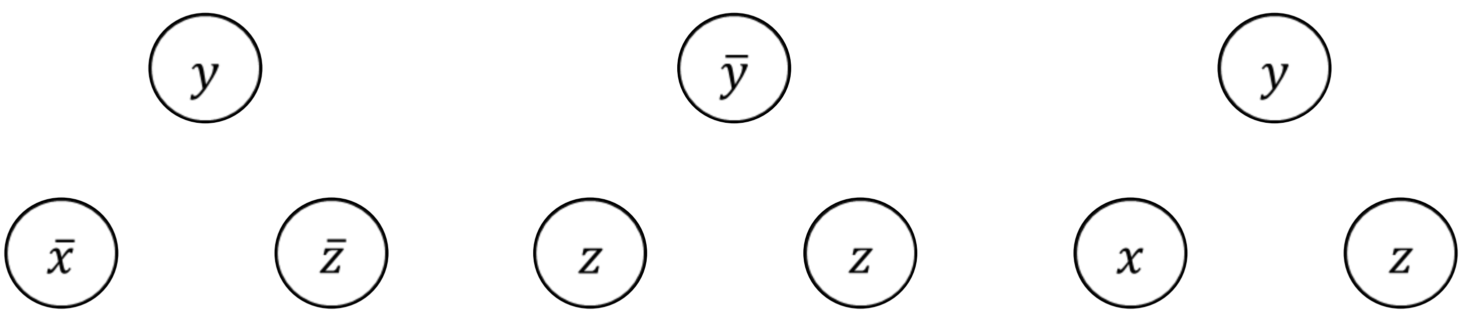
3-SAT

3 variables per clause

$$(\bar{x} \lor y \lor \bar{z}) \land (x \lor \bar{y} \lor z) \land (x \lor y \lor z)$$

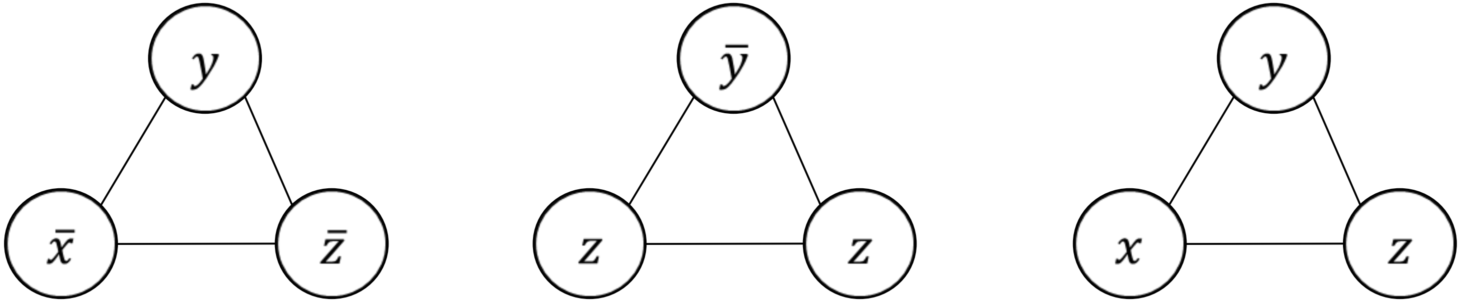
What do we do? Turn it into a graph!

One node per term



3-SAT to Indep. Set

Add edges between variables in the same clause



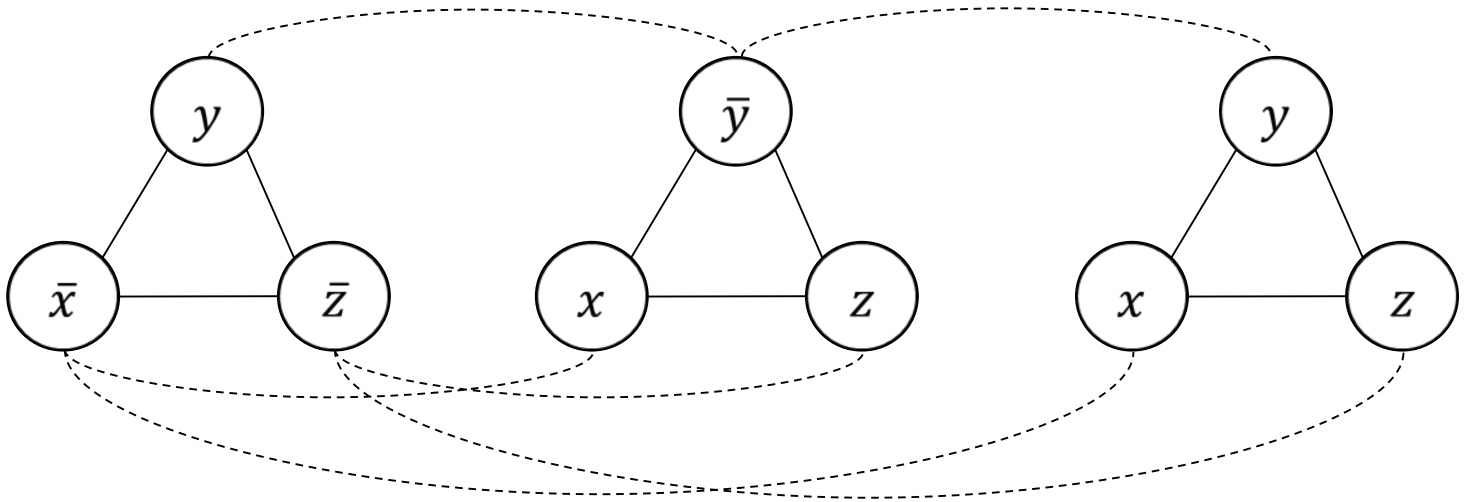
Set *g* = # of clauses

Now every clause has to have a true variable

Almost Done

What about repeated/negated variables?

Add edges between them too



Another Successful Conversion

We converted 3-SAT to Independent set

The set of clauses C is only true together if the converted graph C' has an independent set of size |C|

C' cost us polynomial time to produce

This shows us Independent Set is also NP-Complete

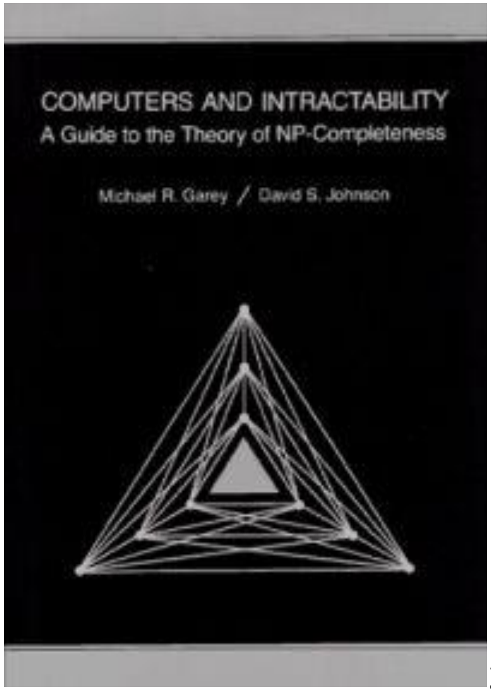
Not a full proof, but enough to give you the idea

How do we handle NP-Complete Problems?

- Approximation Algorithm:
  - Can we get an efficient algorithm that guarantees something *close* to optimal? (e.g. Answer is guaranteed to be within 1.5x of Optimal, but solved in polynomial time).
- Restrictions:
  - Many hard problems are easy for restricted inputs (e.g. graph is always a tree, degree of vertices is always 3 or less).
- Heuristics:
  - Can we get something that seems to work well (good approximation/fast enough) *most* of the time? (e.g. In practice, n is small-ish)

Great Quick Reference

- *Computers and Intractability: A Guide to the Theory of NP-Completeness*, by Michael S. Garey and David S. Johnson



- For the following problems, circle **ALL** the sets they belong to:

Determining if a chess move is the best move on an N x N board	NP	P	NP-complete	None of these
Finding the maximum value in an array	NP	P	NP-complete	None of these
Finding a cycle that visits each vertex in a graph exactly once	NP	P	NP-complete	None of these
Finding a cycle that visits each edge in a graph exactly once	NP	P	NP-complete	None of these
Determining if a program will ever stop running	NP	P	NP-complete	None of these

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• Finding a cycle that visits each edge in a graph exactly once	**NP**	**P**	NP-complete	None of these
• Determining if a program will ever stop running	NP	P	NP-complete	**None of these**

Fun What-If

You (somehow) manage to prove P=NP

What happens?

You still win the Turing award and the millennium prize…

Fun What-If

- Your packages come faster (Traveling Salesman)
- In fact, basically all transportation and production becomes optimal
- Another $5,000,000 from the Clay Math Institute
- Put mathematicians out of work.
- Decrypt (essentially) all current internet communication.
- No more secure online shopping or online banking or online messaging…or online *anything.*
- Machine learning becomes optimal
- Maybe find the cure for cancer?
- A world where P=NP is a very very different place from the world we live in now.