Rosenzweig-MacArthur Predator-Prey Model
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December 16, 2016

Introduction:

The Predator-Prey model is a system of differential equations that models the predator-prey relationship of an ecosystem. Let the following system of differential equations be a closed ecosystem in which no migration is allowed in or out of the system.

$$\{\frac{dx}{dt} = ax - \alpha xy \ \frac{dy}{dt} =- cy + \gamma xy$$

1. *$x(t)$ denotes the population of the prey species*
2. *$y(t)$denotes the population of the predator species*

In absence of the predators, the prey population will grow exponentially. When the preys are absent the predator population will decay exponentially to zero from starvation. The system of equations is the Lotka–Volterra model. Predator–prey interactions are ubiquitous in nature and the dynamical behaviors of predator–prey system are very complex. The model was later extended to include density dependent prey growth by Rosenzweig- MacArthur model.

Rosenzneig-MacArthur predator-prey model with Holling type II functional response:

$$\{\frac{dN}{dt} = N(b - d_1 - \alpha N) - \frac{sNP}{1+sh_1N} \ \frac{dP}{dt} = \frac{c_1 sNP}{1+sh_1N} - d_2P \tag{1}$$

1. $N -$ the population densities of prey at time t
2. P - predators at time t
3. $b -$ per capita maximum fertility rate of prey population
4. $d_1, d_2 -$ the per capita death rates of prey and predators respectively
5. $\alpha -$ the strength of intra-competition of prey population
6. $s -$ the effective search rate
7. $h_1 -$ *handling time of predators*
8. $c_1 -$ the conversion efficiency of ingested prey into new predators

All parameters assumed positive.

To summarize the possible interactions as follows:

1. If the carrying capacity of prey is low ($0 < \frac{b-d_1}{\alpha} < \frac{d_2}{s(c_1-h_1d_1)}$), then the predator population goes extinct at the equilibrium $(\frac{b-d_1}{\alpha},\ 0)$.

2. For intermediate value of the carrying capacity, when $\frac{d_2}{s(c_1-h_1d_1)} < \frac{b-d_1}{\alpha} \leq \frac{d_2+\frac{c_1}{h_1}}{s(c_1-h_1d_2)}$, it there is a coexistence at the positive equilibrium $(\frac{1}{(c_1-1)},\ -\frac{((b-1)(1-c_1)+\alpha)c_1}{(c_1-1)^2})$.

3. For high value of the carrying capacity of prey, ($\frac{b-d_1}{\alpha} > \frac{d_2+\frac{c_1}{h_1}}{s(c_1-h_1 d_1)}$) , predator and prey population coexist on a limit cycle and the limit cycle is globally stable.

<u>Analysis and Interpretation</u>:

In the following sections I will be analyzing (1). We first simplify the system by letting $d_i,\ h_1, s = 1$. The equation becomes:

$$\{\frac{dN}{dt} = N(b-1-\alpha N) - \frac{NP}{1+N} \ \frac{dP}{dt} = \frac{c_1 NP}{1+N} - P$$

Fixed points:

1. $(N,\ P) = (0,\ 0)$ (2)
2. $(N,\ P) = (\frac{b-1}{\alpha},\ 0)$
3. $(N,\ P) = (\frac{1}{(c_1-1)},\ -\frac{((b-1)(1-c_1)+\alpha)c_1}{(c_1-1)^2})$

The Jacobian of the matrix:

$$J(N,P) = [-2\alpha N + b - 1 - \frac{P}{(N+1)^2} \quad -\frac{N}{N+1} \quad \frac{c_1 P}{(N+1)^2} \quad \frac{c_1 N}{N+1} - 1\ ]$$

At (0,0) the linearized system has coefficient matrix:

$$J(0,0) = \ [b-1\ 0\ 0\ -1\ ]$$

$$\lambda_1 = b-1, \lambda_1 =- 1$$

This equilibrium describes the extinctions of the prey and predator.
If b <1 the system at that point is a locally stable node, implying that both prey and predators will become extinct.

At $(\frac{b-1}{\alpha},\ 0)$ the linearized system has coefficient matrix:

$$J(\frac{b-1}{\alpha},\ 0) = [-b+1 \ -\frac{N}{sN+1}\ 0\ \frac{c_1(b-1)}{\alpha+(b-1)} - 1\ ]$$

$$\lambda_1 =- b+1, \lambda_1 = \frac{c_1(b-1)}{\alpha+(b-1)} - 1$$

The equilibriums describe only the prey population existing.
It is stable if b > 1 and $(c_1) < \frac{\alpha}{b-1} + 1$.

At $(\frac{1}{(c_1-1)},\ \frac{((b-1)(c_1-1)-\alpha)c_1}{(c_1-1)^2})$ the linearized system has coefficient matrix:

$$J\,(\frac{1}{(c_1-1)},\ -\frac{((b-1)(1-c_1)+\alpha)c_1}{(c_1-1)^2}) = [\frac{\alpha+b-1}{c_1} - \frac{2\alpha}{c_1-1} \ -\frac{1}{c_1}\ ((b-1)(c_1-1)-\alpha)\ 0\ ]$$

Finally, at this point there is a coexistence equilibrium because both species are alive. Furthermore, the populations are balanced between stationary and cyclic coexistence. We will determine the stability through the trace and determinant of this matrix.

$$tr(J) = \frac{\alpha+b-1}{c_1} - \frac{2\alpha}{c_1-1}$$
$$det\,(J) = \frac{((b-1)(c_1-1)-\alpha)}{c_1}$$

The point is only stable if the $tr(J) < 0$ $and det\,(J) > 0$

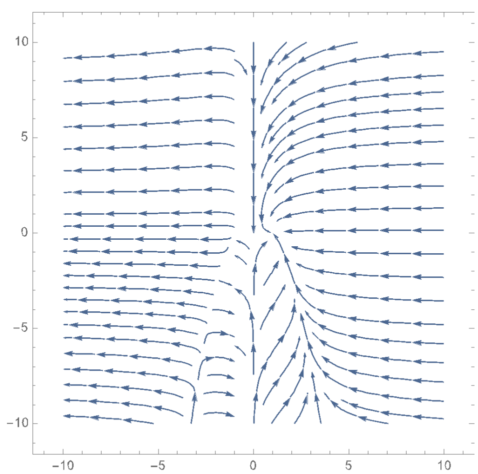
Nullclines are curves where the rate of change of the equations are 0, that is where $N' = 0$ or $P' = 0$. There are two sets of nullclines.
The prey nullcines:

$$N' = 0 \rightarrow N = 0\ ,\ N = \pm\frac{\sqrt{\alpha^2-2\alpha b-4\alpha P-2\alpha+b^2+2b+1}+(\alpha+b+1)}{2\alpha}$$

The predator nullcines:
$P' = 0 \rightarrow P = 0,\ N = \frac{1}{c_1-1}$

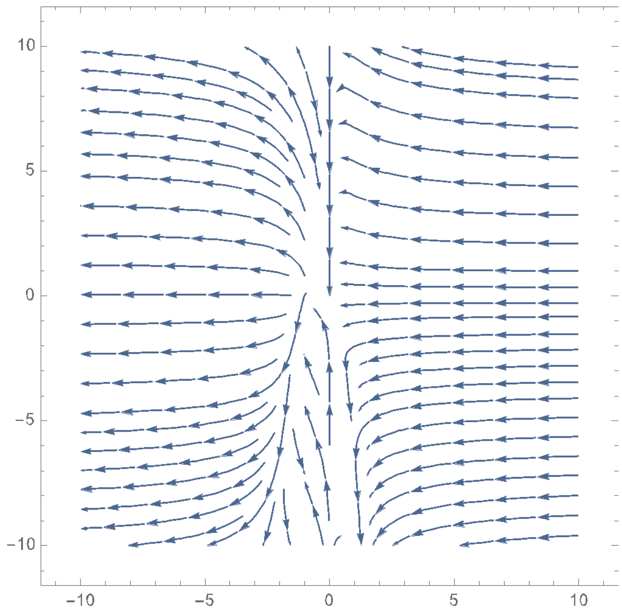
Again, all parameters are positive constants, therefore, there are two cases to consider. Last, the horizontal axis will be N variable and the vertical axis will be for P variable.

case $0 < c_1 < 1$:



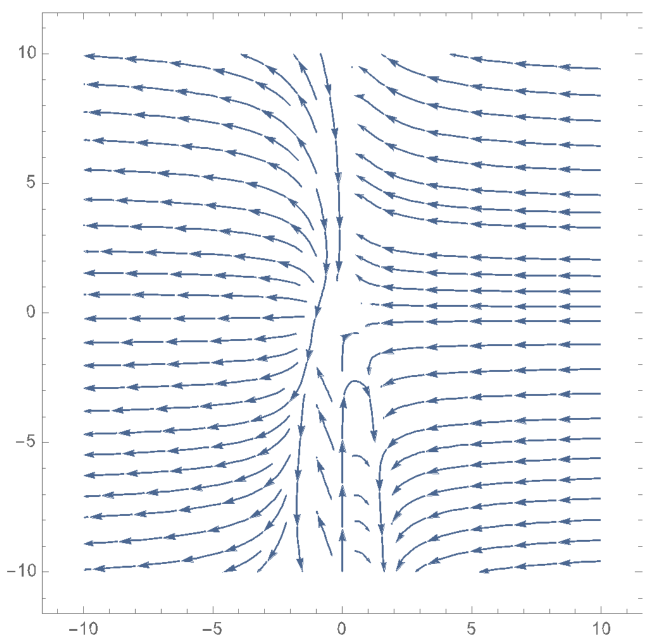
We can see that on the vertical axis, it converges to zero or the population of the predator reaches zero. From this simulation, both population converges to zero.

case $c_1 > 0$:
For this this case, the analysis has three sub-cases that corresponds to the three carrying capacities mentioned.



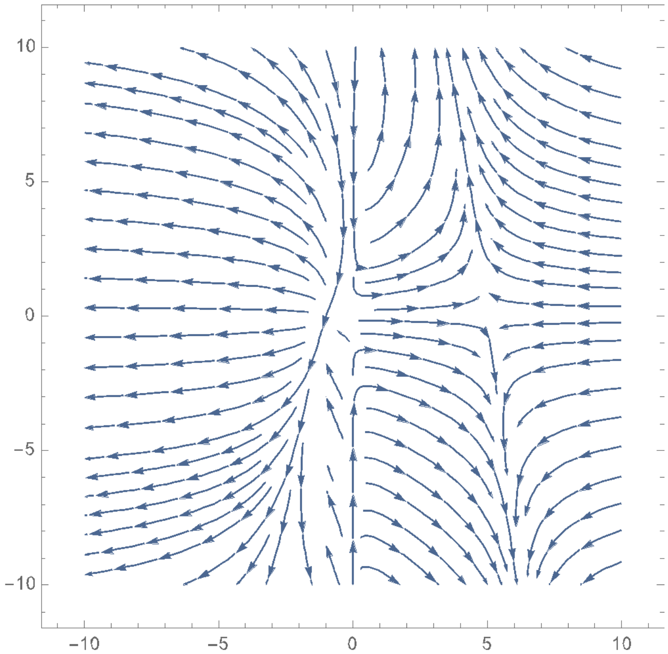
(Low)

Again, we see that the population of the predator reaches zero. The simulation confirms this. We see that the predator population reaches zero first.



We see that the two species are co-existing at the equilibrium point. Again, the simulation confirms this.

Done on Mathematica



Finally, the predator and prey populations coexist.

The simulation shows that the populations are osculating. When the prey populations increase, it is followed by an increase in the predator population because the predator has more to eat. However, the prey population goes down because there is less to eat and the predator population also goes down.

Conclusion:

The predator-prey models describe the dynamics of a biological system. The simplest model is the Lotka-Volterra model. Unfortunately, this model does not fully describe the actual behavior found in nature. One problem of this equation is that that the prey population can grow arbitrarily large. The Rosenzwig-MacArthur model extends the Lotka-Volterra model to include density dependent prey growth. In this paper, I used Mathematica and Python to analyze how the two species' population changes over time depending on multiple parameters. I found that depending on the parameters, the two species can coexist or one specie dies out. Computers play a large role in analyzing mathematical equations. More specifically, nonlinear equations will not yield nice solutions, however, computers can assist in analyzing equations such as the Rosenzwig-MacArthur model.