MAT188H1 - Linear Algebra: Detailed Notes

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Contents

1 Vectors and Vector Spaces

A vector is an element of a vector space, and can be represented as a column or row of numbers. The set of all vectors forms a vector space, which satisfies the following properties:

- **Closure under addition:** If $\boldsymbol{u}, \boldsymbol{v} \in V$, then $\boldsymbol{u} + \boldsymbol{v} \in V$.
- **Closure under scalar multiplication:** If $\boldsymbol{u} \in V$ and $c \in \mathbb{R}$, then $c\boldsymbol{u} \in V$.
- **Zero vector:** There exists a zero vector $\mathbf{0} \in V$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$ for all $\boldsymbol{u} \in V$.

1.1 Example: Vector Addition and Scalar Multiplication

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The vector sum is:

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Scalar multiplication with $c = 2$ is:

$$2\boldsymbol{u} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

1.2 Vector Spaces in $\mathbb{R}^n$

The set $\mathbb{R}^n$ is a vector space, where vectors are n-tuples of real numbers:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

2 Linear Combinations and Span

A linear combination of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is an expression of the form:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$$

where $c_1, c_2, \ldots, c_k$ are scalars.

The **span** of a set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, denoted as $\text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$, is the set of all possible linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

2.1 Example: Span

Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The span of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is:

$$\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

which is the entire space $\mathbb{R}^2$.

3 Systems of Linear Equations

A system of linear equations is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This can be written in matrix form as:

$$A\boldsymbol{x} = \boldsymbol{b}$$

where A is the coefficient matrix, $\boldsymbol{x}$ is the vector of unknowns, and $\boldsymbol{b}$ is the vector of constants.

3.1 Example: Solving a System of Equations

Consider the system:

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$

This system has infinitely many solutions because the second equation is a scalar multiple of the first.

4 Matrix Operations

Matrices can be added, subtracted, and multiplied. For matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, matrix addition and subtraction are defined element-wise:

$$(A+B)_{ij} = A_{ij} + B_{ij}, \quad (A-B)_{ij} = A_{ij} - B_{ij}$$

4.1 Matrix Multiplication

For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, the product $AB \in \mathbb{R}^{m\times p}$ is defined as:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Matrix multiplication is associative but not commutative, i.e., in general, $AB \neq BA$.

4.2 Example: Matrix Multiplication

Let:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then:

$$AB = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

5 Determinants and Inverses

The determinant of a square matrix A, denoted $\det(A)$, is a scalar value that encodes information about the matrix, such as whether it is invertible.

5.1 Properties of the Determinant

- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- If $\det(A) = 0$, then A is not invertible.

The inverse of a matrix A, denoted A^{-1}, is defined such that:

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix.

5.2 Example: Inverse of a Matrix

Let:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

6 Eigenvalues and Eigenvectors

For a square matrix A, a scalar λ and a nonzero vector $\boldsymbol{v}$ are called an eigenvalue and eigenvector, respectively, if:

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

This equation says that multiplying $\boldsymbol{v}$ by A stretches or compresses $\boldsymbol{v}$ by a factor of λ, but does not change its direction.

6.1 Example: Finding Eigenvalues

Let:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

To find the eigenvalues, solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

For this example:

$$\det \begin{bmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{bmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.