A Fast Method for Reconstruction of Total-Variation MR Images With a Periodic Boundary Condition

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***Abstract*—We use a small positive parameter to change the total-variation function for unconstrained MR image reconstruction to a strictly convex perturbed function. Bregman iteration is applied to solve the modified total-variation MR image (TVMRI) reconstruction problem. A lagged diffusivity fixed-point algorithm is applied to solve the minimization problem in the Bregman iteration. We use the periodic boundary condition and a Fourier transform to accelerate TVMRI reconstruction. Real MR images are used to test the approach in numerical experiments. The experimental results demonstrate that the proposed method is very efficient for TVMRI reconstruction.**

***Index Terms*—Bregman iterative regularization, compressed sensing, fixed-point iteration, total variation.**

I. Introduction

COMPRESSED SENSING (CS) has enormous potential for significantly reducing scan time in magnetic resonance imaging (MRI) research, as proposed by Candes, Romberg and Tao [1] and Donoho [2]. Sparse signals can be reconstructed from a very limited number of samples if the measurements satisfy an incoherence property [3]. For CS-MRI, it is possible to accurately reconstruct MR images from undersampled K-space data, that is, the partial Fourier data, by solving a nonlinear optimization problem that exploits the sparsity of the MR image in a transform domain such as wavelet and gradient domains. Total variation (TV) regularization was first proposed for image denoising by Rudin, Osher and Fatemi [4]. TV regularizer can yield piecewise smooth signals while preserving sharp edges or boundaries.

Let $N = m \times n$. Suppose $u \in R^N$ is a vector formed by stacking the columns of a two-dimensional MRI array $(u_{i,j})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. When only the TV sparsifying transform is considered, the optimization problem in CS-MRI can be written as

$$\min_u \left\{ \|u\|_{TV}, \|PFu - b\|^2 \leq \sigma^2 \right\}, \tag{1}$$

where PF is a partial Fourier matrix, $P \in R^{M \times N}$ consists of $M \ll N$ rows of the identity matrix, F is a two-dimensional discrete Fourier matrix that can be obtained by the Kronecker tensor product of two one-dimensional discrete Fourier matrices, and b is an observed data vector that is contaminated by noise such as Gaussian noise of variance σ^2. The unconstrained version of (1) is

$$\min_u \|u\|_{TV} + \frac{\mu}{2}\|PFu - b\|_2^2. \tag{2}$$

$\|u\|_{TV} = \sum_{i=1}^m \sum_{j=1}^n |\nabla_{i,j} u|$, where we use the notation $\nabla_{i,j} u = (\nabla_{i,j}^x u, \nabla_{i,j}^y u)$ with

$$\nabla_{i,j}^x u = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < m \\ u_{1,j} - u_{m,j} & \text{if } i = m \text{ (periodic boundary)} \end{cases}$$

$$\nabla_{i,j}^y u = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < n \\ u_{i,1} - u_{i,n} & \text{if } j = n \text{ (periodic boundary)} \end{cases}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$, where $|\nabla_{i,j} u| = \sqrt{(\nabla_{i,j}^x u)^2 + (\nabla_{i,j}^y u)^2}$. Since (2) is not differentiable, this difficulty can be overcome by using the perturbed TV norm

$$\|u\|_{TV,\beta} = \sum_{i=1}^m \sum_{j=1}^n \sqrt{|\nabla_{i,j} u|^2 + \beta},$$

where β is a small positive parameter. Thus, we usually solve the convex perturbed form of (2) as follows:

$$\min_u \|u\|_{TV,\beta} + \frac{\mu}{2}\|PFu - b\|_2^2. \tag{3}$$

The objective function in this modified model is strictly convex and its global minimizer is unique. According to the discussion in [5], we can regard the solution of (2) as the limit of the solution of (3) when $\beta \to 0$. We can apply Version 2 of the Bregman iterative formulation in [6] to solve (3). Assume $J(u) = \|u\|_{TV,\beta}$; according to [6], the Bregman iterative algorithm for solving (3) is as follows.

Algorithm 1

$b^{(0)} \leftarrow 0, u^{(0)} \leftarrow 0$

for $k = 0, 1, \ldots,$ **do**

$b^{(k+1)} \leftarrow b + (b^{(k)} - PFu^{(k)})$

$$u^{(k+1)} \leftarrow \arg\min_u J(u) + \frac{\mu}{2}\left\|PFu - b^{(k+1)}\right\|_2^2 \tag{4}$$

end for.

We use the periodic boundary condition and a Fourier transform to produce an acceleration method to solve minimization problem (4) in Algorithm 1. The details for solving (4) are discussed in Section II.

A number of numerical methods have been proposed for solving the linear combination of total-variation regularization and wavelet sparse regularization. So these methods can also solve total-variation MRI (TVMRI) reconstruction model (2).

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The conjugate gradient method [7] is a common approach. Zhu *et al.* applied an alternating minimization method to solve (2) [8]. Ma *et al.* proposed an operator-splitting algorithm to solve the MRI reconstruction problem [9]. The alternating direction method [10] and EdgeCS reconstruction method [11] are also efficient approaches in the CS-MRI reconstruction. Huang *et al.* presented a fast composite splitting algorithm (FCSA) can also be used to solve (2) [12]. FCSA is formed by combining a composite splitting denoising method [12] with a fast iterative shrinkage-thresholding algorithm [13]. FCSA is a state-of-the-art method for CS-MRI reconstruction. Instead of the TV term with the nonlocal total variation (NLTV) in the FCSA, NLTV-FCSA algorithm was proposed for MR image reconstruction [14]. In Section IV, we compare the proposed fast method with FCSA.

The remainder of the paper is organized as follows. Section II presents a fast method for TVMRI reconstruction. In Section III, we use real MR images in numerical experiments to demonstrate effectiveness of our method in sparse MRI reconstruction. Finally, some concluding remarks are presented in Section IV.

II. A Fast Method for TVMRI Reconstruction

The first-order condition for (4) is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\nabla_{i,j}^{T}\left(\frac{\nabla_{i,j}u}{|\nabla_{i,j}u|_{\beta}}\right)+\mu F^{*}P^{*}\left(PFu-b^{(k+1)}\right)=0, \tag{5}$$

where $\nabla_{i,j}^{T}$ is the transpose of the operator $\nabla_{i,j}$, superscript $*$ denotes the transpose of the matrix conjugate, and $|\nabla_{i,j}u|_{\beta} = \sqrt{|\nabla_{i,j}u|^{2}+\beta}$. Numerical solution of (5) is not easy because of the presence of the diffusivity coefficient $1/|\nabla_{i,j}u|_{\beta}$. Here we adopt the lagged diffusivity fixed-point iteration proposed by Vogel and Oman [15] to change (5) to a linear equation:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\nabla_{i,j}^{T}\left(\frac{\nabla_{i,j}u^{(k+1)}}{\left|\nabla_{i,j}u^{(k)}\right|_{\beta}}\right)+\mu F^{*}P^{*}\left(PFu^{(k+1)}-b^{(k+1)}\right)=0. \tag{6}$$

Since $1/|\nabla_{i,j}u^{(k)}|_{\beta}$ is a scalar, we can rewrite (6) as

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\nabla_{i,j}^{T}\left(\nabla_{i,j}u^{(k+1)}\right)}{\left|\nabla_{i,j}u^{(k)}\right|_{\beta}}+\mu F^{*}P^{*}\left(PFu^{(k+1)}-b^{(k+1)}\right)=0. \tag{7}$$

Under the periodic boundary condition for u, the coefficient matrix associated with $u^{(k+1)}$ in $\sum_{i=1}^{m}\sum_{j=1}^{n}\nabla_{i,j}^{T}(\nabla_{i,j}u^{(k+1)}/|\nabla_{i,j}u^{(k)}|_{\beta})$ is block circulant with circulant blocks (BCCB) [16]. Thus, we can apply a Fourier transform to (7) to obtain the solution of (6). We now describe this approach in detail.

Assuming that the coefficient matrix associated with $u^{(k+1)}$ in $\sum_{i=1}^{m}\sum_{j=1}^{n}\nabla_{i,j}^{T}(\nabla_{i,j}u^{(k+1)}/|\nabla_{i,j}u^{(k)}|_{\beta})$ is C, we can obtain

$$Cu^{(k+1)}+\mu F^{*}P^{*}\left(PFu^{(k+1)}-b^{(k+1)}\right)=0. \tag{8}$$

That is,

$$Cu^{(k+1)}+\mu F^{*}P^{*}PFu^{(k+1)}=\mu F^{*}P^{*}b^{(k+1)}. \tag{9}$$

We multiply both sides of (9) by the Fourier matrix F and obtain

$$FCu^{(k+1)}+\mu P^{*}PFu^{(k+1)}=\mu P^{*}b^{(k+1)}. \tag{10}$$

Here, we note $FF^{*}=I$, because F is a unitary matrix. Since C is BCCB, $FCu^{(K+1)}=DFu^{(k+1)}$, where D is a diagonal matrix [17, Proposition 5.31]. Therefore, (10) becomes

$$(D+\mu P^{*}P)Fu^{(k+1)}=\mu P^{*}b^{(k+1)}. \tag{11}$$

Since $P^{*}P=P^{T}P$, the matrix $D+\mu P^{*}P$ is a diagonal matrix. Therefore, $Fu^{(k+1)}$ is easily obtained by (11). Then we apply the operator F^{*} to obtain $u^{(k+1)}=F^{*}(Fu^{(k+1)})$. Computing $u^{(k+1)}$ involves two FFTs and one inverse FFT, so solution of (6) using this approach is fast. Using this method to solve (4) in Bregman iterative Algorithm 1 yields a fast algorithm for solving (3). The fast method for TVMRI with CS can be described as follows.

Algorithm 2

Step 1) Input $b, P, F, \mu > 0$, and $\beta > 0$.
Step 2) Change unconstrained MRI reconstruction model (2) into the convex perturbed MRI reconstruction model (3).
Step 3) Initialization:

$$k=0, b^{(0)}=\mathbf{0}, u^{(0)}=\mathbf{0};$$

Iterations

When (the stopping criterion is not satisfied)

{

$b^{(k+1)}=b+(b^{(k)}-PFu^{(k)});$

Compute $\mu P^{*}b^{(k+1)}, \mu P^{*}P$, and D;

Compute $Fu^{(k+1)}$ by (11);

Compute $u^{(k+1)}$ by $F^{*}(Fu^{(k+1)})$;

$k=k+1.$

}

Note that Step 3 in Algorithm 2 is the acceleration step in MRI reconstruction. When the observed vector b is contaminated by strong noise, the diagonal property of matrix $D+\mu P^{*}P$ and the fast Fourier transform can increase the speed of the MRI reconstruction algorithm compared to other iterative reconstruction methods. In the next section, we describe numerical experiments that demonstrate that our fast method is very efficient for TVMRI reconstruction.

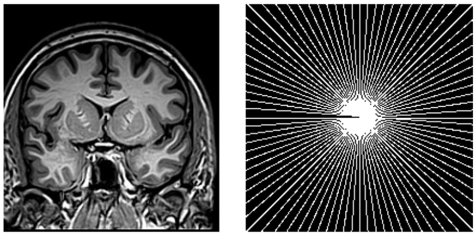
III. Numerical Experiments

In this section, we evaluate the performance of Algorithm 2 in solving model (2) for CS-MRI. We compare our acceleration method with FCSA [12], the state-of-the-art method for CS-MRI.

The signal to noise ratio (SNR) is used to estimate the quality of reconstructed images. SNR is defined as

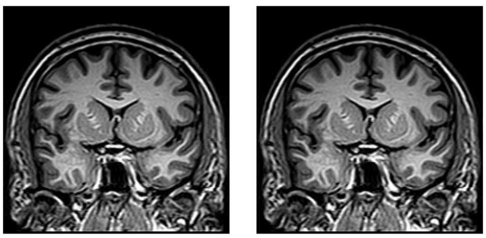
$$SNR=20\log 10\left(\frac{\|u_{o}\|_{2}}{\|u_{o}-u\|_{2}}\right), \tag{12}$$

where u and u_{o} are the reconstructed and original images, respectively. We use the CPU time to evaluate the speed of MRI



(a) (b)

Fig. 1. (a) Original image; (b) 44 views in the frequency space.



(a) (b)

Fig. 2. (a) Proposed method; (b) FCSA.

reconstruction. All experiments were performed in MATLAB on a laptop with an Intel Core Duo P8400 processor and 2 GB of memory.

We first assume that the mean and standard deviation for additive Gaussian noise are 0 and 0.01, respectively. In reconstruction tests, we choose $\mu = 0.001$ and set $\beta = 10^{-3}$. The sampling ratio is M/N, where M and N are as defined in Section I. The stopping criterion is the relative difference between successive iteration for the reconstructed image and should satisfy the following inequality:

$$\frac{\left\| u^{(k)} - u^{(k-1)} \right\|_2}{\left\| u^{(k)} \right\|_2} < 10^{-3}. \tag{13}$$

According to the parameters and stopping criterion above, we applied our algorithm to a reconstruction experiment for a brain MR image. Fig. 1(a) is an original 210 × 210 brain MR image and Fig. 1(b) shows 44 radial lines in the frequency space for the image. If the MR image is sampled with 44 views in the frequency space, its sampling ratio is 22.6%.

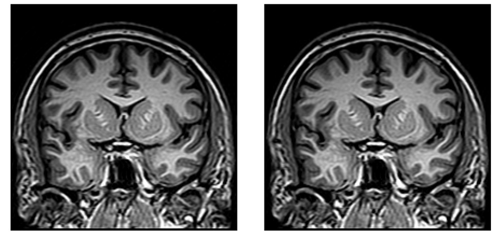
The reconstructed image when the stopping criterion is satisfied is shown in Fig. 2(a). The SNR is 19.8533 dB and the CPU time 0.3130 s.

We then applied FCSA for brain MR image reconstruction using the same sampling ratio, as shown in Fig. 2(b). The SNR is 19.8123 dB and the CPU time is 0.6125 s. Therefore, the proposed method is faster than FCSA and the SNR is almost the same.

We then increased the level of noise and let its standard deviation be 0.1 and performed the reconstruction tests again. The results are shown in Fig. 3. The SNR is 19.7064 and 19.6789 dB and the CPU time is 0.3440 and 0.7813 s for the proposed method and FCSA, respectively.

The results reveal that the proposed method is twice as fast as FCSA with almost the same SNR when the standard deviation for noise is 0.1.

Let the standard deviation for noise be $\sigma = 1, 10, 100, 1000$. We kept the same sampling ratio and parameters as above and used the proposed method and FCSA in reconstruction tests. The results are shown in Fig. 4. Using the proposed method, the SNR is 19.6822, 19.6463, 19.6041 and 18.9804 dB and the CPU



(a) (b)

Fig. 3. (a) Proposed method; (b) FCSA.

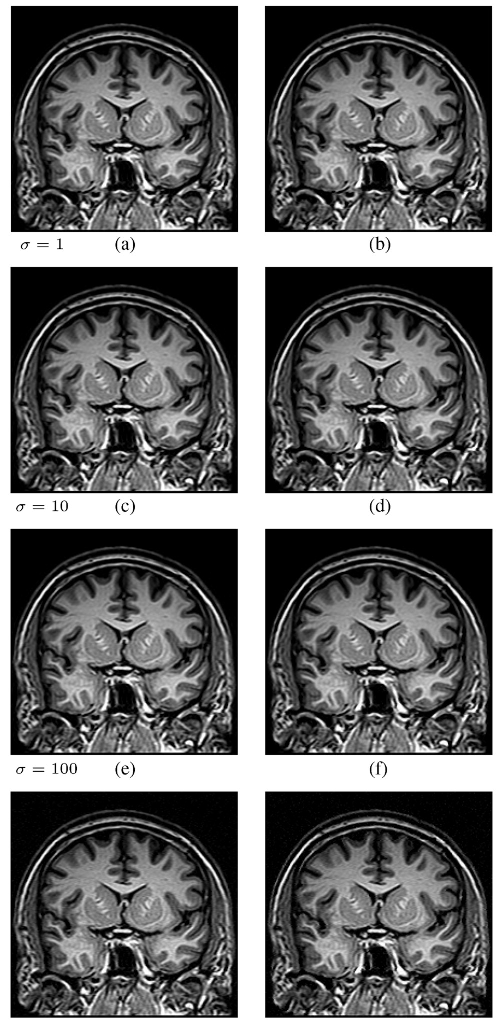


Fig. 4. Images reconstructed using the proposed method and FCSA; $\sigma = 1$ (a) Proposed method; (b) FCSA; $\sigma = 10$ (c) Proposed method; (d) FCSA; $\sigma = 100$ (e) Proposed method; (f) FCSA; $\sigma = 1000$ (g) Proposed method; (h) FCSA.

time is 0.3750, 0.4060, 0.422 and 0.4370 s for $\sigma = 1, 10, 100$ and 1000, respectively. The corresponding values for FCSA are 19.6607, 19.5872, 19.5666 and 18.9728 dB and 0.7813, 1.3906, 1.4531 and 2.6875 s.

The reconstructed results demonstrate that the proposed method is much faster than FCSA in the case of stronger noise. Table I presents SNR and CPU time results when the stopping criterion is satisfied under the 22.6% sampling ratio for standard deviation between 1100 and 2000. The data show that the SNR is slightly greater for the proposed method than for FCSA. In addition, the CPU time is much shorter for the proposed method compared to FCSA for MRI reconstruction in the presence of strong noise. Fig. 5 shows SNR and CPU time

TABLE I
SNR AND CPU TIME FOR THE RESULTS RECONSTRUCTED BY OURS AND FCSA

Standard Deviation	Method	SNR (dB)	CPU (s)
1100	Proposed	18.8555	0.4850
	FCSA	18.7160	2.7344
1200	Proposed	18.7164	0.5160
	FCSA	18.4425	2.7969
1300	Proposed	18.5650	0.5470
	FCSA	18.1569	2.9688
1400	Proposed	18.4028	0.5940
	FCSA	17.8624	2.9844
1500	Proposed	18.2314	0.6250
	FCSA	17.5634	3.0625
1600	Proposed	18.0520	0.6570
	FCSA	17.2633	3.1719
1700	Proposed	17.8656	0.6870
	FCSA	16.9618	3.4219
1800	Proposed	17.6735	0.7030
	FCSA	16.6571	3.4844
1900	Proposed	17.4765	0.7500
	FCSA	16.3531	3.5938
2000	Proposed	17.2763	0.7970
	FCSA	16.0484	3.6250

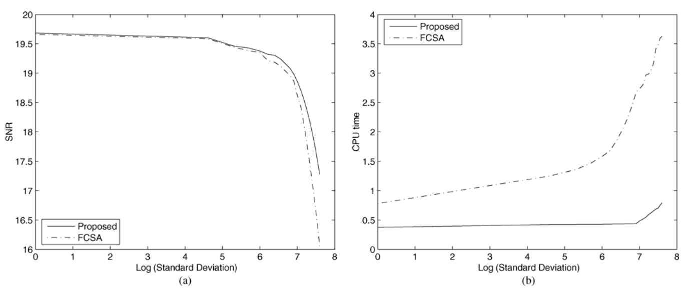


Fig. 5. SNR and CPU time versus Log of standard deviation for images reconstructed using the proposed method and FCSA. (a) SNR; (b) CPU time.

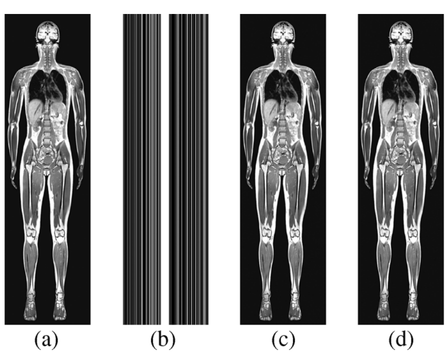


Fig. 6. (a) is original image. (b), (c) and (d) are sampling mask for 1280 $\times$ 400, images reconstructed using the proposed method and FCSA, respectively.

versus Log of standard deviation for σ between 1 and 2000 for images reconstructed using the proposed method and FCSA.

Fig. 6(a) shows a 1280 $\times$ 400 MR body image sampled according to the Cartesian mask (sampling ratio of 15.957%) shown in Fig. 6(b). The standard deviation (σ) is set to 100. The results reconstructed by the proposed method and FCSA are shown in Fig. 6(c) and (d). The SNR is 29.7692 and 29.0121 dB and the CPU time is 4.7341 and 10.0782 s for the proposed method and FCSA, respectively. The data reveal that the proposed method yields better reconstruction results than FCSA does. Under the different sampling ratios, the performance results for the proposed method and FCSA are shown in Fig. 7. From Fig. 7 we can see that the proposed method is also very efficient under different sampling ratios.

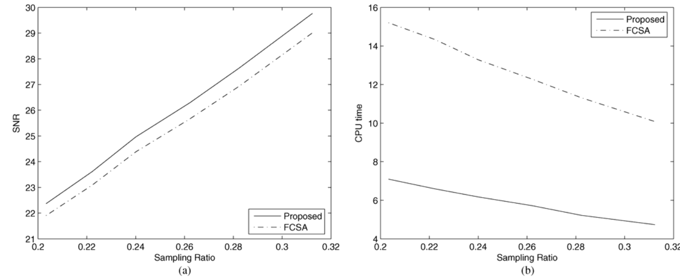


Fig. 7. SNR and CPU time for images reconstructed using the proposed method and FCSA under different sampling ratios.

IV. CONCLUSION

We proposed a convex perturbed reconstruction model using a positive parameter β to solve the TV sparsifying MRI reconstruction problem. Bregman iteration was used to solve the convex perturbed TV model, and Lagged diffusivity fixed-point iteration was applied to solve the minimization problem in the Bregman iteration. The periodic boundary condition and a Fourier transform were used to accelerate TVMRI reconstruction. This method has been validated on real MR images and compared with FCSA. The experimental results showed that the fast method is very efficient for TVMRI reconstruction.

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