Complex systems research in Psychology

Han van der Maas

2024-01-01

Table of contents

Foreword

This book is intended for psychologists and social scientists interested in the modeling of psychological processes using the tools of complex systems research.

The book has three primary objectives. The first is to provide a comprehensive overview of complex systems research, with a particular emphasis on its applications in psychology and the social sciences. The second is to provide skills for complex systems research. Lastly, it strives to foster critical thinking regarding the potential applications of complex systems in psychology.

For many decades, scientists have been studying all kinds of complex systems made up of many smaller subsystems interacting locally on fast time scales. Well-known examples include lasers, tornados, chemical oscillations, ant nests and flocks of birds. Scientists have built mathematical models of these complex systems and developed techniques to study them. The application of these techniques requires a great deal of mathematical and technical knowledge, but also a deep understanding of the nature of the system. You don't just create a mathematical model off the top of your head. In addition, testing such models requires extensive and reliable quantitative data. The application of complex systems theory to the behavioral and social sciences is therefore not straightforward. Theories are often verbal, and quantitative measurement in these sciences is a longstanding issue. While there has been some reasonable progress over the past 150 years, it is fair to say that the behavioral and social sciences are less mature than the "hard" sciences.

The application of complex systems theory to the behavioral sciences is challenging but, in my view, also essential. Whether we consider humans in isolation, the billions of interacting neurons in the brain, or the social networks in which we find ourselves, complexity is everywhere. Humans, with their complex brains embedded in various hierarchies of social systems, are the ultimate complex systems.

I believe that we can only succeed in exploring this system by understanding its complexity. We need to apply the tools of complexity science to our field of science, which is in desperate need of breakthroughs. After all, the modern world revolves around human beings who, through language and thought, have created an unimaginably complex world. The greatest danger now is man himself, and progress in the field of psychology is necessary and urgent.

This book requires study. Running the simulations, studying these examples, and solving the exercises will contribute to a deeper understanding of the material. I have used the content of the book in a master's level course for students in the research-oriented field of psychology.

I expect quite some background in psychology and its research methods. I assume only pre-university knowledge of mathematics and refer to basic sources where there is quite a bit of mathematics.

A lot of theoretical ideas will be illustrated with real examples. This is a theory and a practice book at the same time. An important prerequisite is a basic knowledge of the programming languages R or Python. **The book is originally written using R, but Python code is also provided.** Many exercises in the book require R or Python. There are many online resources for learning the basics of these languages. I also assume some knowledge of statistics and data analysis, at the level of a Master's in psychology. **In addition to R and Python, we will use NetLogo, but no prior knowledge of NetLogo is expected. NetLogo is a multi-agent programmable modeling environment for the simulation of complex natural and social phenomena.**

It is also good to say that this book is more a book for psychologists who have very limited knowledge of complex systems research than the other way around. Experts in complex systems who wonder how it can be applied in psychology may have to wait for another text.

I have written this book based on 35 years of scientific work in collaboration with fantastic colleagues and co-authors of the many papers. I'm part of the ecosystem of the Psychology department, especially the wonderful methods section, of the University of Amsterdam. Also important is the Institute of Advanced Study in Amsterdam, which has complex systems research as a central theme. In recent years I've also been an external faculty member at the renowned Santa Fe Institute in New Mexico. I am indebted to all of them and to many other colleagues around the world.

Han van der Maas, Amsterdam 2024

1 Introduction

1.1 What are complex systems?

Some things in life are simple. When you push a block, it moves. Pushing harder makes it move faster, and stopping the push stops its movement. When you open the tap, water starts to flow. If you open it further, it flows faster, and if you close it, it stops. Cause-and-effect relationships like this are clear and roughly linear. Such relationships are rare in psychology. Let's take fear as an example. A fear stimulus, for example a barking dog, can lead to fear and flight, but also to anger and attack. Whether the stimulus is perceived as fearful might depend on subtle differences in context. It has also been debated whether the flight response precedes the feeling of fear or vice versa. Fear could also suddenly change into a panic attack. In psychology, cause and effect relationships are rarely simple, and effects are often non-linear.

These difficulties are not unique to psychology . Many systems studied in physics, chemistry, and biology show such complex behavior (Weaver 1948). They are complex systems. There is no full consensus on the definition of a complex system (Ladyman, Lambert, and Wiesner 2013; Heylighen 2009), I believe the core aspects can be summarized as follows.

Complex systems are made up of many smaller interacting subsystems, such as atoms, molecules, cells, neurons, and even entire organisms. I like the term subsystems because the lower level elements can themselves be complex systems[1]. The interactions between subsystems can be of different kinds, but they are usually local, fast, and nonlinear. Complex systems exhibit emergent behavior, meaning that these interactions result in global patterns or properties that do not occur in the subsystems themselves. The emergent processes usually operate on a slower time scale. A typical example, which will be discussed in more detail later, is the traffic jam. Cars react mainly to cars in their vicinity, which can lead to global patterns of congestion. Another example is magnetism that is not present in any of the atoms of the magnet.

In general, these patterns emerge through self-organization. Self-organization is a process in which some form of overall order or coordination develops from the local interactions between the parts of an initially disordered system. An example would be ants building an ant nest. It

[1] As Simon (1962a) noted, atoms were once considered elementary particles, whereas in modern nuclear physics they are themselves complex systems.

is important to note that no one ant oversees or directs this process; it emerges from the local interactions between many ants.[2]

Complex systems are open systems, meaning that they use energy that they have absorbed for the environment. I will explain this in more detail in a later chapter, but in a completely closed system, self-organization would not be possible. The emergent patterns in a complex system may be stable for some time, but often change suddenly. The study of phase transitions or tipping points is therefore central to the study of complex systems. They may also exhibit chaotic behavior, implying that they can be fundamentally unpredictable, the weather being a notorious example.

Let us look at one famous case, the flocking of birds (Figure 1.1). Flocks of birds move in a beautiful choreography as they glide through the air, their formations shifting and morphing as they twist and turn across the sky. Flocks are well understood and easy to simulate, as we will see in Chapter 5. Flocks fulfill all the criteria of a complex system (see Parisi 2023 for an extended analysis). They are open systems as birds use energy to fly. Next, each bird responds only to birds in its local neighborhood. They follow roughly three rules. They try to fly in the same direction as their neighbors, stay close to their neighbors and avoid collisions. These are fast and local interactions. I suggest you watch some videos of flocks of birds on the internet. What you see is globally organized behavior on a much slower time scale than the local interactions. This is a prime example of self-organization. There is evidently no one bird in control ordering other birds to change direction. The globally organized behavior of a flock is a form of spontaneous order. What you can also see in these movies is that stable patterns, say an oval shape, can suddenly change. The birds may change direction or split up. Such bifurcations or catastrophes (to be explained in chapter 3) are very typical of complex systems. You can also see that the behavior of these flocks is rather unpredictable. As said, flocks, and swarms in general, are well understood and can be easily simulated on a computer, but this does not mean that we can always predict these systems, an issue that will be discussed further in the next chapter.

Similar examples can be found in any of the natural sciences. Tornadoes, for example, are made up of air molecules that also interact locally. Tornadoes are unpredictable, self-organizing, global weather phenomena. A famous chemical example is the Belousov-Zhabotinsky reaction, a chemical oscillator (Kuramoto 1984). This reaction involves a mixture of chemicals. As the reaction proceeds, the solution exhibits strikingly colorful oscillations between clear and opaque or between different colors, depending on the specific reactants used. We will see many more examples in later chapters.

[2] It is also argued that emergence is a consequence of symmetry breaking (Krakauer 2023). Symmetry breaking occurs when a system transitions from a symmetric state to an asymmetric state, resulting in the emergence of distinct properties or behaviors. An example of symmetry breaking can be observed in the formation of snowflakes. Initially, ice crystals have a symmetrical hexagonal shape due to the underlying molecular structure of water. However, as the crystal continues to grow, environmental factors such as temperature and humidity influence its growth pattern. Minute variations in these factors lead to the breaking of initial symmetry and the formation of diverse and beautiful snowflake structures.

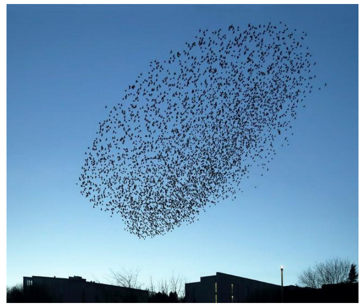

Figure 1.1: A flock of birds.

The prime example from psychology is the brain. About a hundred billion neurons interact with thousands of other neurons in their neighborhood. Compared to computers, brains are extremely energy efficient, but as all open systems, they do consume energy (the equivalent of a light bulb according to (Attwell and Iadecola 2002). Fast local interactions somehow form global waves of electrical activity that make up thought processes and even consciousness. The letters you are reading activate retinal neurons that initiate a cascade of electrical waves across billions of neurons that somehow create your understanding of this text (Roberts et al. 2019; Schöner 2020). How is this possible? For me, this is the most fascinating scientific question of all time. It's the main reason why I'm a psychologist and not a physicist. I view the brain as the ultimate complex system.

1.2 Emergentism

What is the relation between complexity and reductionism? According to reductionism, complex phenomena can be explained by reducing them to the interactions of their individual parts or components. This raises two questions, one related to weak emergence and one related to strong emergence.

The first question is why it is possible to do science in any field other than physics, since ultimately chemistry, biology, and even psychology, are all about interacting elementary particles. Should we not first finish the study of physics before starting to think about complex molecules, cells, neurons, or higher-order human cognition?

Philip Anderson's renowned paper on "More is Different" convincingly argues that the answer to this question is a resounding no (Anderson 1972). Science is possible at many different levels of description without fully understanding the lower levels. There is much to be said for reductionism, but somehow the laws of quantum mechanics are irrelevant when studying interactions between neurons or people. I don't think that emergence in complex systems is inconsistent with a reductionist view of science (Bechtel and Abrahamsen 2005). One could say that complex systems theory explains why emergent phenomena such as atoms or neurons can be used as entities at a higher level of description to explain new higher order phenomena, without being a dualist. This fundamental principle of emergence is what allows disciplines like psychology to exist as distinct and independent fields of science (Fodor 1974). In the words of Herbert Simon: “In the face of complexity, an in-principle reductionist may be at the same time a pragmatic holist” (Simon 1962b). The concept of level is central to Simon’s architecture of complexity, in which each subsystem is itself a complex structure made up of smaller parts, and this pattern is repeated at multiple levels. According to Simon, these nested structures are ubiquitous in the natural world and in human-made systems because they are robust and adaptable. For in in-depth discussion of this level concept i refer to Wimsatt (1994).

The second, more controversial, question is whether emergent phenomena have an independent causal role (strong emergence) or mainly have descriptive value (weak emergence). Strong emergence is often associated with downward causation (Chalmers 2006; Flack 2017; Kim 2006). Downward or circular causation is the idea that higher-level entities or properties can influence the behavior of lower-level entities or properties. I like to link this to the flocking example. Flocks of birds are emergent phenomena that do not determine the behavior of the individual birds. The birds only follow the local rules. Flocking is an example of weak emergence. However, when predators enter the scene things change. Predators get confused by flocks of prey, not by the behavior of individual birds. So, the flock has some causal power. Moreover, the birds react to the movements of the predator. This could be seen as an example of downwards causation and thus strong emergence (Figure 1.2). Recent work attempts to quantify such causal emergence effects (Hoel, Albantakis, and Tononi 2013).

I believe that both weak and strong emergence are essential to understanding psychological phenomena. Our minds, encompassing conscious thought, self-awareness, reasoning abilities, natural language comprehension, emotions, and attitudes, are not mere artifacts and cannot be simply reduced to intricate patterns of neural activity. In my view, these mental constructs, such as consciousness, possess their own causal influence, and this is one of the reasons that psychology stands as a scientific discipline in its own right.

Of course, the relationship between the mind and the brain is one of the most debated topics in psychology. The neuro-reductionist view is popular in the field of psychology, both when it concerns the explanation of higher cognition (Schwartz et al. 2016) and psychological disorders (for a critical review see Denny Borsboom, Cramer, and Kalis (2019)).

1.3 The field of psychology

The study of complex systems in the natural sciences (**?@fig-ch1-img3**) is highly technical. I like to think of the field of complex systems as a toolbox of empirical paradigms and mathematical models and techniques (Grauwin et al. 2012). Models are often formulated in the form of difference or differential equations and subjected to, for example, bifurcation analysis. These are mathematical ways of describing the behavior of complex systems. Additionally, advanced numerical analysis, commonly in the form of computer simulation, is a standard approach. However, educational programs in psychology do not usually include courses in

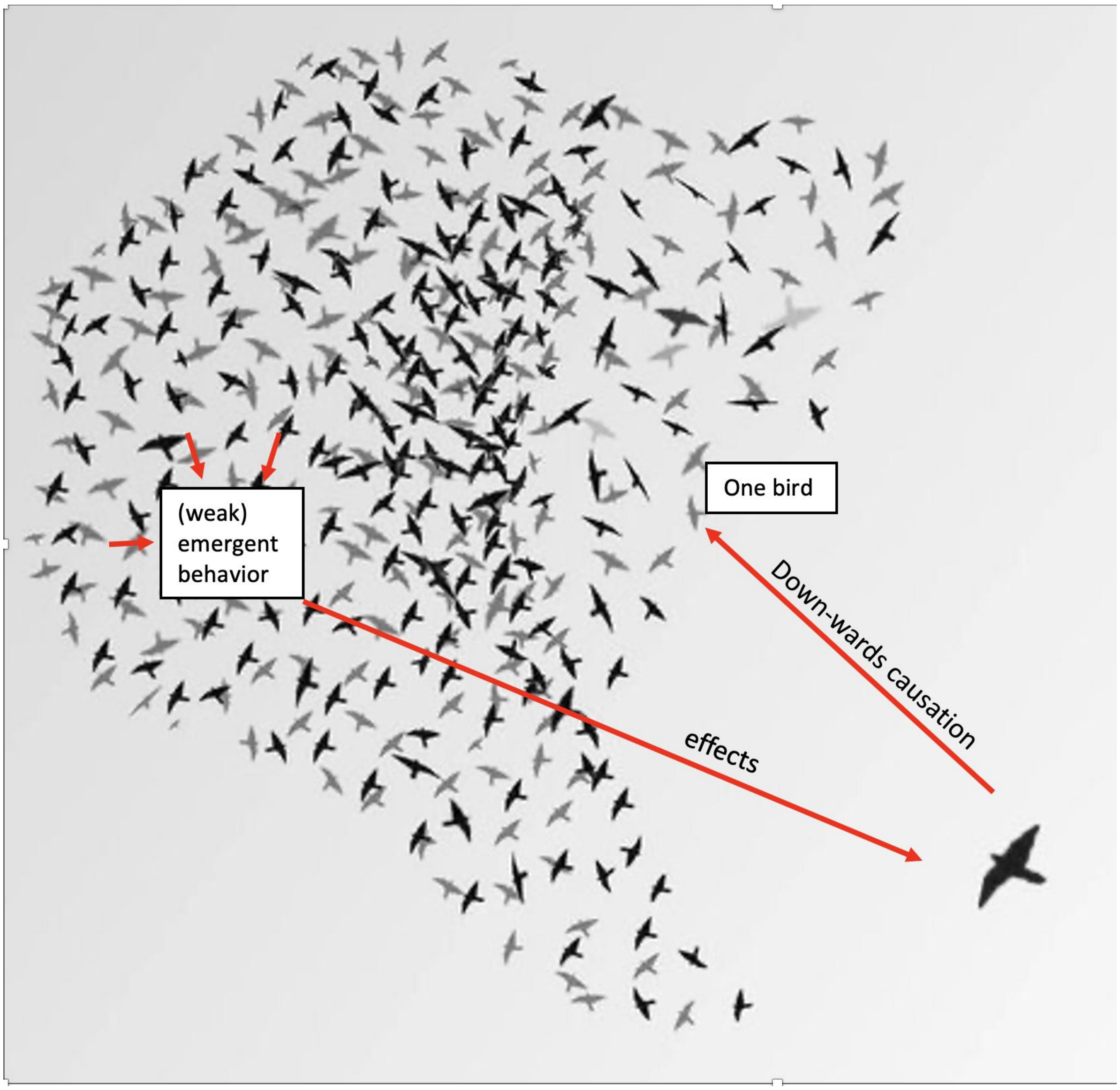

Figure 1.2: An illustration of downwards or circular causation in flocks due to a predator responding to the emerging patterns of the flock and subsequently influencing the flight of individual birds. (adapted from https://arxiv.org/abs/1108.1682).

algebra, calculus and programming. Many psychologists lack the basic knowledge and skills to apply the toolbox of complex systems theory, as these are not ordinarily part of the psychology curriculum. Complex systems research simply seems too complex for psychologists and social scientists. One goal of this book is to provide psychologists with a first introduction to this technical toolbox.

Moreover, there are additional complications in applying the toolbox to our field. First, our subjects are much more complex than flocks of birds or tornadoes and display astonishing behavior. They can do science! They can also walk out of the lab because they find the experiment boring. This does not happen with lasers. Second, we have to deal with the ethical constraints of experimenting on our subjects. We cannot take them apart, a very successful approach in the natural sciences. Finally, there is the measurement issue (Lumsden 1976; Michell 1999). We tend to forget how incredibly precise the natural sciences, especially physics, are. In 1985, Richard Feynman famously claimed that the accuracy of calculating the size of the magnetic moment of the electron was equivalent to measuring the distance from Los Angeles to New York, a distance of over 3,000 miles, to the width of a human hair. I find that shocking. Less famously, I would argue that psychologists have not yet discovered America and have no idea where New York is. Our instruments generally fail to meet elementary requirements of reliability and validity, we are plagued by replication failures, and our theories are often imprecise (Eronen and Bringmann 2021). Navigating the behavioral and social sciences and knowing which data to trust and which empirical phenomena to model is an art in itself.

This is all unfortunate because not only our brains, but every subject in our field seems to have the characteristics of a complex system. Social systems are complex systems made up of individuals interacting to produce emergent phenomena such as cultures and economics. The human brain, the most complex system we know, is embedded in different hierarchies of very complex social systems such as families, education, economies, and cultures. We need the toolbox!

Despite all these problems, I'm not pessimistic. I would also argue that tangible progress in the behavioral and social sciences is possible. It is not that these sciences are completely unsuccessful. We know a lot about people's attitudes, addictive behavior, cognition, and the social systems in which they interact. We study these, with some success, using advanced experimental designs, and we have developed (mainly) verbal theories about almost everything.

We also have no choice; we must make progress. Personally, I feel a strong tension between our struggle to elevate the behavioral and social sciences as a science on the one hand, and the enormous expectations of society to deliver on the other. Our most pressing global problems - climate change, overpopulation, war and violence, poverty, inequality, infectious diseases, addiction, to name but a few - are unsolvable without breakthroughs in the behavioral and social sciences. To quote J. Doyne Farmer: " We have an increasing need to model ourselves" (Thurner 2016).

The realization that the human mind in its social context is an amazingly complex system also offers opportunities. Despite their obvious differences, complex systems show remarkable similarities. A predecessor of complex system theory, general systems theory (Bertalanffy 1969), explicitly assumed that all systems share important characteristics. Certain mechanisms and phenomena seem to operate and to occur in similar ways at all possible levels of description (Simon 1962a). This is the primary reason for providing numerous modeling examples in this book that originate from disciplines beyond psychology.

An inspiring example for me comes from the study of shallow lakes (Scheffer 2004). Shallow lakes tend to be either in a 'healthy' state, with clear water and a diverse population of fish and plants, or in an 'unhealthy' turbid state. I like to compare these complex lake systems in the turbid state to a patient suffering from depression. This turbid state usually occurs suddenly. There is a critical phosphorus load at which the system "turns over" from being healthy to complete dominance by algae and bream. Typical of this type of transition is the hysteresis effect (Figure 1.3). This means that the turning point from clear to turbid and from turbid to clear does not occur at the same phosphorus load. The turning point to clear water only occurs at much lower phosphorus loads. These tipping points may be so far apart that reducing the cause, the phosphorus load, is not a viable option. Of course, all sorts of interventions have been studied, such as supplemental oxygen, chemicals, sunscreens and stocking predatory fish. These interventions have not been very successful, or only in specific cases. The fact that they had some level of success brings to mind the partial effectiveness of clinical interventions, such as those used in the treatment of major depressive disorder.

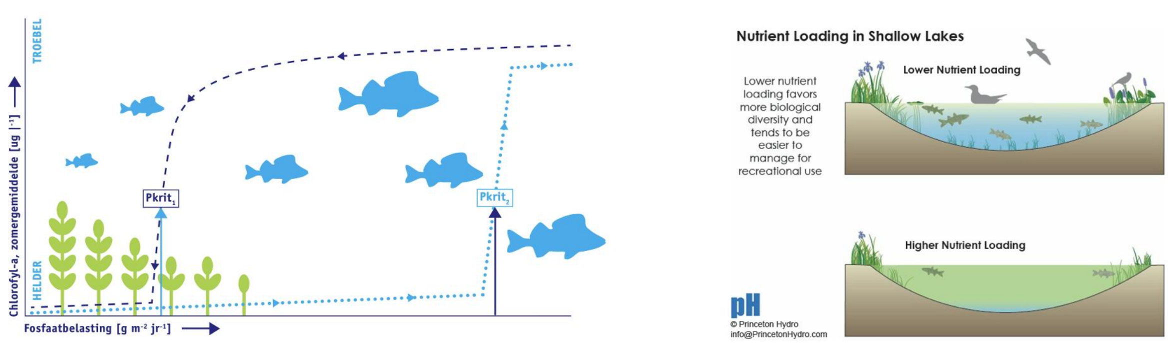

Figure 1.3: The transitions between clear and turbid states of shallow lakes do not occur at the same phosphorus load. This delay in jumps is called hysteresis. Hysteresis explains why transitions are often difficult to reverse. This concept is discussed in detail in Chapter 3.

A breakthrough occurred in the 1980s. Catching all the fish proved to be a very effective intervention. The ecologists caught almost all the fish with nets during the winter. In the spring, a new, healthy equilibrium emerged, characterized by aquatic plants, other fish species and clear water. This new state is often stable for long periods of time. Remarkably, the analysis of the cause, the phosphorus load, was not part of the solution: although the increasing phosphorus load is the primary cause of the transition towards a turbid state, decreasing the phosphorus load does not cause the system to transition back into the clear state. The dogma of intervention, that the cause of the problem is the key to the solution, does not necessarily apply to complex systems. What this means for our thinking about depressed patients will be discussed in chapters 5 and 6.

In the following, I will discuss three reasons to be somewhat optimistic, based on three key observations about complex systems. The first key observation has to do with simplification, the second with the tendency of complex systems to be characterized by a limited number of stable states, and the third key observations is that all complex systems seem to be describable as some kind of network. Simplification is perhaps the most important one.

1.4 The art of simplification

Einstein supposedly said that everything should be made as simple as possible, but not one bit simpler. A fascinating and instructive example is the traffic jam, which is made up of many people, with their amazingly complex brains, in modern cars full of advanced technology. Where to start modeling such a complex phenomenon? The answer is astonishing. It seems that we can reduce people in cars to simple blocks in a lane, speeding up when there is space in front of the artificial car and braking when they get too close to the car in front. All lower levels of modelling are ignored. This is even simpler than a flock of birds.

It is not difficult to set up a computer simulation for this case. I recommend that you spend some time playing around with an example (https://www.traffic-simulation.de/). It does not take long to see that traffic jams can easily form and have an unexpected property: while cars move forward, traffic jams move backwards! Another interesting observation is that variance in speed causes congestion. But the variance is not in any of the cars. Variance and congestion are properties at a higher level of description. With this simulation, you can study different types of traffic situations and interventions. This type of simulation seems to be very useful for the design of motorways (Barceló 2010; Treiber, Hennecke, and Helbing 2000). There are actually different ways to model traffic jams. Jusup et al. (2022) distinguish between fluid-dynamical , kinetic, car-following , coupled-map lattice, and cellular automata models. They all reproduce many phenomena of real traffic jams.

Thus in complex systems, the qualitative properties of large-scale phenomena do not depend on microscopic details. Only higher-level properties are relevant to global behavior. A large part of the art of science is the finding of the right level of simplification. Suppose we are studying smoking. Do we model the effects of nicotine on blood vessels, how the hand with the cigarette moves from the mouth to the ashtray, or the number of cigarettes smoked per day? Do we include the effects of marketing and the smoker's social network?

What is relevant and what can be ignored? It can be challenging to provide a definitive answer for specific cases. Nevertheless, in general, it can be stated that there is a limit to the lower levels that must be considered. When examining traffic jams, it is necessary to incorporate certain characteristics of individuals and vehicles, but delving deeper into topics like neuronal firing, DNA replication, or the intricate workings of car batteries becomes irrelevant. At that level of modeling, there is no relevant information that would alter the explanation of a traffic jam.

The traffic example shows that extreme simplification is sometimes possible and necessary. But finding the right level of simplification is not a simple task at all. In Denny Borsboom et al. (2021) we propose a *theory construction methodology* (TCM) consisting of five steps. The first step is to identify the empirical phenomena that become the target of explanation. The second step is to formulate a set of theoretical principles that putatively explain these phenomena. Third, this set or prototheory is used to construct a formal model, a set of model equations that encode the explanatory principles. Fourth, we analyze the explanatory adequacy of the model, i.e., whether it actually reproduces the phenomena identified in step one. Fifth, we determine whether the explanatory principles are sufficiently parsimonious and substantively plausible. The article explains these steps in detail and provides an example, the mutualism model of general intelligence, which is explained in chapter 6 of this book.

I will add a few comments to this list of steps. First of all, step 1 is key. It is crucial to be precise about what the phenomena to be explained are. Phenomena are not the same as data. Data are particular empirical patterns (a concrete data set), whereas phenomena refer to general empirical patterns, stable and general features of the world (Haig 2014). As noted above, in the behavioral and social sciences it is not always clear which data patterns can be trusted. In the last 10 years, the replication crisis has led to a revolution in psychological methods, but many results are collected using potentially biased methods. One problem is publication bias. Negative results are still harder to publish than results that support hypotheses. In other cases, the results of different studies contradict each other, and meta-analyses show weak effects at best. Drawing up a list of the most important phenomena on a topic, such as depression, forgetting or discrimination, is often a challenge.

The second observation is that taking these steps is not a linear process. Often, when you are developing a model, you realize that some important information is missing from the list of phenomena. For example, you might be modelling addiction, but suddenly you need information about the combination of addictive substances that people use. And such simple questions are often impossible to answer. I spent days searching the literature for information that I expected to be readily available, only to find that many basic things are simply unknown.

A third observation is that formal modelling is mostly a matter of analogical reasoning. You have to study many examples of complex system models to understand how to construct such a model. Indeed, in my own work I often use established models developed in physics and biology as a base model. We will see many examples of this later.

Fourth, good models do not build in phenomena, but explain them from basic principles. What I mean by building in is that a phenomenon should not be an assumption of the model. An example would be a model that says that variance in the speed of cars causes traffic jams. Such a model may explain other things, but not the role of speed variance, because that effect is part of the assumptions. Models that make such assumptions are called phenomenological models. We will see examples of phenomenological models of complex systems, as well as explanatory models, where the latter are based on fundamental principles. Building real explanatory models in our fields is extremely difficult.

Fifth, it is my conviction that a metaphorical use of the complex systems approach should be avoided by using concrete formal models. It is crucial to strive for the highest level of scientific rigor. There are no special, more lenient, methodological rules for complex systems research (H. L. J. van der Maas 1995).

1.5 A limited number of equilibria

That complex systems can be simplified was the first key observation. The second key observation is that complex systems tend to be characterized by a limited number of equilibria. An important example is water. Water normally exists in either a solid, liquid or gaseous state (leaving aside the plasma state). These are stable states over wide ranges of temperature and pressure.

A biological example is the life stages of a butterfly (egg, caterpillar, chrysalis and butterfly). Most of the time these insects are in one of these four relatively stable states. Another example is the horse, which is either standing still, walking, trotting or galloping. I am convinced that we must always start by identifying the equilibria of a complex system. This also applies to psychological and social science applications. A bipolar disorder seems to be characterized by two stable states (depressive and manic). In case of addiction we may think of a state of non-use, recreational use, and heavy use (Epskamp et al. 2022). Similarly, we should identify the stages in falling in love, in understanding of calculus, in sleeping and in radicalization.

This turns out to be more difficult than it first appears. There is an ongoing discussion about the number of stages, even for something like sleep stages (Boostani, Karimzadeh, and Nami 2017; de Mooij et al. 2020). It is often possible to come up with more substages. For instance, in the case of horse movement people tend to further subdivide trot into three forms (working, medium, and collected). Subdivisions are also made in the case of heavy alcohol consumption (Leggio et al. 2009). It is possible to use objective statistical methods to support such classifications using modern machine learning techniques (automatic clustering) as well as more traditional means (finite mixture models, latent class analysis). I will say more about this in Chapter 3.

A further complication is that equilibria come in different forms. The simplest form consists of fixed points or point attractors, an example being a ball lying in a valley. Under undisturbed conditions, the ball could also be resting on top of a hill, which is an unstable equilibrium. An equilibrium could also be a limit cycle or oscillator. For example, two pendulums could swing in phase or out of phase. It gets even weirder when we get to strange attractors, which often take the form of fractals. This will be explained in more detail in the next two chapters.

Finally, it has also been argued that many complex systems, especially living systems, never reach equilibrium because they are constantly perturbed (Groot and Mazur 2013). I see this distinction between equilibrium and non-equilibrium complex systems as gradual. Some complex psychological systems are clearly stable over the long term. Unfortunately, this is true of many psychological disorders. In contrast, my understanding of the world, psychological science, and complex systems research is better described as a continuously perturbed non-equilibrium system with just enough stability to write this book (once).

I would claim that many psychological complex systems tend to be in one attractor state most of the time, but they occasionally change states. If certain control parameters slowly change their values, the current equilibrium can become unstable and a transition to another equilibrium can occur. This is what happens when we lower the temperature of water to below zero. Transitions can occur in many ways, also depending on the types of equilibria involved. The family of transition models is described by bifurcation theory. This is explained in more detail in Chapter 3, where we focus on a very important transition model, the cusp catastrophe, and in Chapter 4, which considers dynamical systems models.

1.6 Networks are everywhere

The third key observation of great relevance to the attempt to use complex systems modelling in psychology is that complex systems are networks, as they consist of interacting sub-elements. For me, the network is the most interdisciplinary research topic in modern science. Magnets, ecosystems, the brain, the Internet, and social networks are prime examples. Network science is a huge area of research with many fundamental insights and an important tool in modern psychological science.

Two applications in psychology are well known: the first is the study of neural networks, which started 70 years ago and has become the main foundation of the AI revolution of the last 10 years. In Chapter 5 I will discuss neural networks. The second is social networks, the simplest example being dyadic interactions. Social media such as Facebook are large examples. Key ideas relate to concepts such as weak and strong ties, central hubs and homophily, which are discussed in Chapter 6 and 7. The analysis of social network data is an exciting area of research (Scott 2011). It focuses on understanding how social entities are connected and how these connections influence various outcomes and behaviors. Connections between nodes (e.g., individuals, organizations, communities) can be based on different dimensions, such as friendship, communication, collaboration, information flow, or any other form of social interaction. These interactions may also change over time, which is studied in social network dynamics (Snijders 2001). The statnet.org website provides an overview of R packages for social network analysis.

Chapter 6 focuses on a novel use of networks, which I call network psychology. This is a level of description between neural networks and social networks. It involves modelling intelligence, attitudes, and psychological disorders at the individual level. Intelligence, for example, is modeled as an ecosystem of cooperating cognitive functions. This is radically different from the standard view that general intelligence is due to g, a single underlying source. In the mutualism model of general intelligence (Van Der Maas et al. 2006), the observed positive correlations between scores on sub-tests of IQ test batteries are due to cumulative reciprocal developmental interactions between cognitive subsystems such as working memory, spatial cognition, and language.

Similarly, depression can be thought of as a network of mutually reinforcing symptoms. For example, sleep problems, a symptom of depression, can lead to increased fatigue and difficulty concentrating, which in turn can affect a person's ability to manage daily tasks and engage in social activities. This can then lead back to poorer sleep quality, creating a cyclical pattern in which each symptom reinforces the others. This new view of mental disorders originated in our research group and is now very popular (Robinaugh et al. 2020). One reason for this is that many statistical techniques have been developed to investigate this network approach.

The latest line of research is the integrated study of psychological and social networks (Maas, Dalege, and Waldorp 2020). Chapter 7 deals with models in which psychological network models of attitudes are nested within social networks of opinion change. This model provides a new explanation of polarization.

1.7 Methods for Investigating complex systems

Complex systems are studied in various ways across different disciplines. We utilize computer simulations to examine the emergence in complex system models, analyze their unpredictable behavior, categorize the types of tipping points involved, derive equations that describe the overall behavior of complex systems, collect and analyze time series data, or experimentally disrupt the system to test its resilience. Following Sayama (2015), I categorize the methods and tools into two groups: those for systems with a small number of variables and those for systems with many variables. The first category is referred to as nonlinear dynamical system theory, which encompasses chaos theory, and catastrophe or bifurcation theory. The second category includes tools for studying multi-element systems, such as agent-based modeling and network theory. One might assume that the first category is irrelevant to complex systems, which by definition have many variables. However, it has been found that the global behavior of complex systems can often be described with a small number of variables, often just one, that behaves in a highly nonlinear manner. Therefore, I consider nonlinear dynamical system theory an essential part of complex system research.

This categorization is reflected in the setup of the book. The next three chapters will be devoted to systems with a small number of variables. Chapter 2 discusses chaos theory, Chapter 3 addresses sudden transitions as studied in catastrophe and bifurcation theory, and Chapter 4 provides an introduction to modeling dynamical systems.

In the second part of the book, we shift our focus to tools for studying systems with many variables, particularly agent-based modeling of self-organization in Chapter 5, network modeling in Chapter 6, and the application of both to psycho-social systems in Chapter 7.

1.8 Other work and sources

The complex systems approach is often introduced as the next new thing, but those days are gone. Even in psychology it can no longer be considered a new approach. Many different research groups have used the toolbox of complex systems research in all areas of psychology. This book will give many examples. One could even argue that a lot of work has been done that could be considered complex systems research but has not been published under that heading. For example, most neural network models of psychological processes are complex systems models because they investigate emergent computational properties of the interaction of neural units. This is also true of much work in mathematical psychology, for example when differential equations are used to study dynamical systems. Older work in complex systems research has often been published with reference to nonlinear dynamical systems. Other related approaches are computational social science and agent-based modeling.

Today, there are many interdisciplinary centers or hubs for complexity research. The Santa Fe Institute in Santa Fe, New Mexico, is the pioneer of complexity science. Its summer schools are highly recommended. Other examples are the Complexity Science Hub in Vienna and the Centre for Complexity Science at the University of Warwick. In my own country, the Netherlands, we have at least four of these centers. I'm a principal investigator at the Institute for Advanced Study in Amsterdam and an external faculty member at the Santa Fe Institute.

It is impossible to give a balanced review of all past and ongoing work on complex systems. I'm naturally somewhat biased towards our own work and contributions, but I did my best to point out relevant work. As a general resource to complex systems research with a bit less technical approach I recommend Mitchell's book (Mitchell 2009), for a bit more mathematical approach I recommend the books of Serra and Zanarini (Serra and Zanarini 1990) and (Sayama 2015). Overviews of work in psychology are provided by Guastello, Koopmans, and Pincus (2008) and Port and Gelder (1995). Other great books are written by Heath (2000) and Kelso (1995).

1.9 Exercises

1) Visit https://www.traffic-simulation.de/. In what direction do traffic jams move? For roundabouts: what is a bad priority rule? Do traffic jams appear and disappear for the same values of critical parameters? Take for instance the 'ring' road and vary Politeness. (*)

2) Give your own example of a psychological process or theory where different stable stage or states are distinguished. (*)

3) Could consciousness be seen as a process of downwards causation. Explain your answer. (**)

2 Chaos and unpredictability

2.1 Introduction

Suppose we have immense amounts of genetic, biological and psychological data on millions of participants and knowledge of all relevant environmental factors. Suppose also that these huge amounts of data are of fantastic quality. Using state-of-the-art machine learning models and powerful computing resources, we could build advanced statistical models that include main and higher order interaction effects of all variables, even incorporating non-linear transformations. Even then, prediction may not be possible. The reason is a phenomenon called deterministic chaos.

Chaos is one of the three most spectacular phenomena in complex systems and as psychologists we should know the basic results of chaos theory. It is also great fun to learn about chaos and it allows me to introduce many key concepts that we need in later chapters.

In my opinion, the applicability of chaos theory to psychology and social science is somewhat limited. For a long time, researchers have tried to show chaos in time series of psychophysiological measures, but this seems to be difficult. I will briefly review this work at the end of the chapter. The relevance of chaos theory may lie not in its application, but in its fundamental implication for prediction. What chaos theory basically shows is that even in the best of circumstances, where we have very accurate models and data, long-term prediction might be impossible. This is known as the butterfly effect: A butterfly flaps its wings in India and that tiny change in air pressure could eventually cause a tornado in Iowa. How this works will become clear in this chapter.

2.2 The population growth of rabbits

Chaos theory consists of many deep mathematical results, but understanding the basics of chaos is not so hard. Below I will explain chaos in difference equations at a very basic level of mathematics and programming. The elementary example is the famous logistic map, usually introduced as a model of population growth, for instance, of rabbits. Suppose we have rabbits on an island, and they start to multiply, what would be a mathematical model for such a process?

Population growth is a typical example of a dynamical system, it is a model of change. In general, in a dynamical system, the change or growth of a variable (say X) depends on the current state and some parameters. Time plays a very special role. We can use discrete or continuous time steps. In the first case, which is the focus of this chapter, we use difference equations; in the second case, we use differential equations. In the logistic map, time is discrete (population growth takes place in generations). The simplest model for the population growth of rabbits is:

$$X_{t+1} = rX_t \tag{2.1}$$

This simply says that the new value of X is determined by the previous value of X. In this equation r is the growth rate. We can simulate this model by choosing a value for r, $r = 2$, for instance. We also need an initial value, say $X_0 = 1$. If this is completely new to you, enter some values repeatedly. You will see exponential growth ($X_1 = 2,\ X_2 = 4, X_3 = 8, X_4 = 16,\ etc.$). In R we can simulate this using a for loop.

```
n <- 15
r <- 2
x <- rep(0,n)
x[1] <- 2 # initial state X0 = 1 and thus X1 = 2
for (i in 1:(n - 1))
  x[i + 1] <- r * x[i]
plot(x, type = 'b', xlab = 'time', bty = 'n')
```

Note that we can find any X_t given X_0 by iterating the model as we do in the for loop. X_t is called the solution. Simulation is a bit odd in this case. We can compute the solution analytically. It is $X_t = X_0 r^t$. Thus for $X_{15} = 1 \times 2^{15} = 32768$. However, for more complex models the analytical solution is often not available, and we have to use simulation (the numerical solution).

Note that the exponential model ignores the fact that population growth is limited by resources. At some point food will become scarce. One way of making the model, introduced by Verhulst in 1838, more realistic is to add a growth-limiting term:

$$X_{t+1} = f\left(X_t\right) = rX_t\left(1 - \frac{X_t}{K}\right) \tag{2.2}$$

What is the effect of this addition to the equation? If X is much smaller than the resource K then the second term, $\left(1 - \frac{X_t}{K}\right)$, is close to 1 and we will see exponential growth. But as X approaches K, this term becomes very small, reducing the effect of exponential growth. X does not actually grow up to K, but to a lower value, if it converges at all. We are going to see this in a moment. It also turns out that the actual value of K is not of interest. Changing K does not change the qualitative behavior. Therefore, K is usually set to 1, scaling the

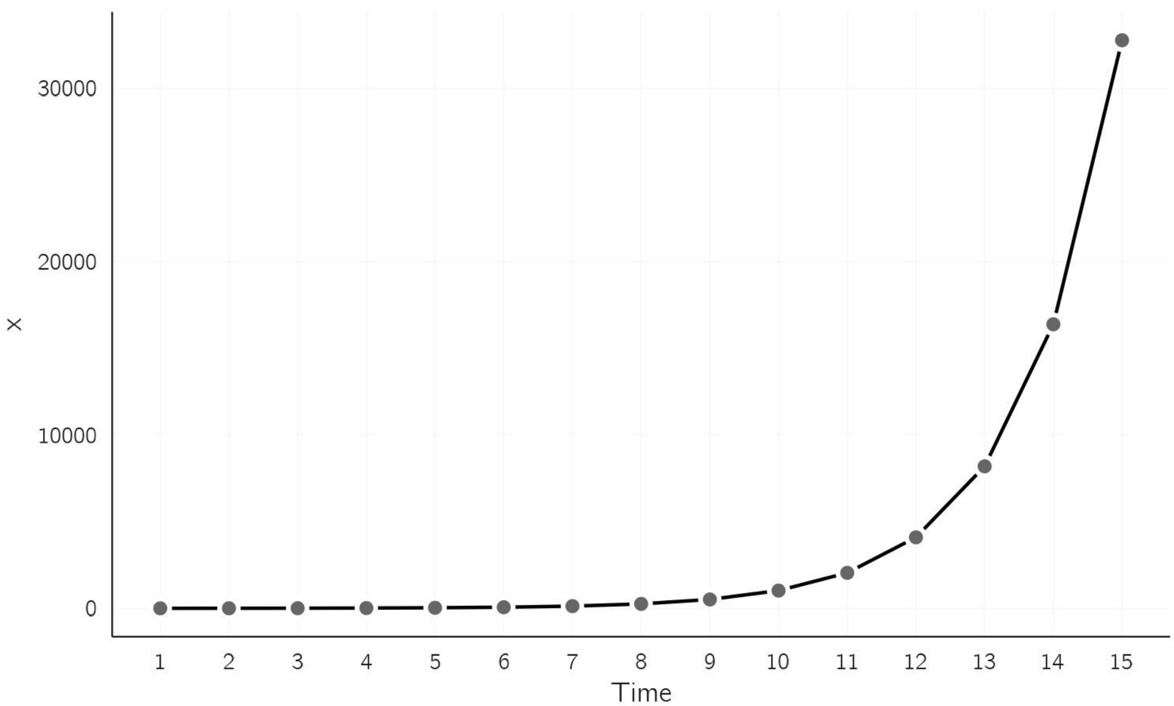

Figure 2.1: Exponential growth.

population X between 0 and 1. The only remaining parameter is r. Changing r, however, leads to a number of surprising behaviors.[1]

2.3 Stable and unstable fixed points

Let us first study a simple 'boring' case, $r = 2$ (Figure 2.2).

```
n <- 15; r <- 2; x <- rep(0,n)
x[1] <- .01 # initial state
for (i in 1:(n - 1))
  x[i + 1] = r * x[i] * (1 - x[i])
plot(x, type = 'b', xlab = 'time', bty = 'n')
```

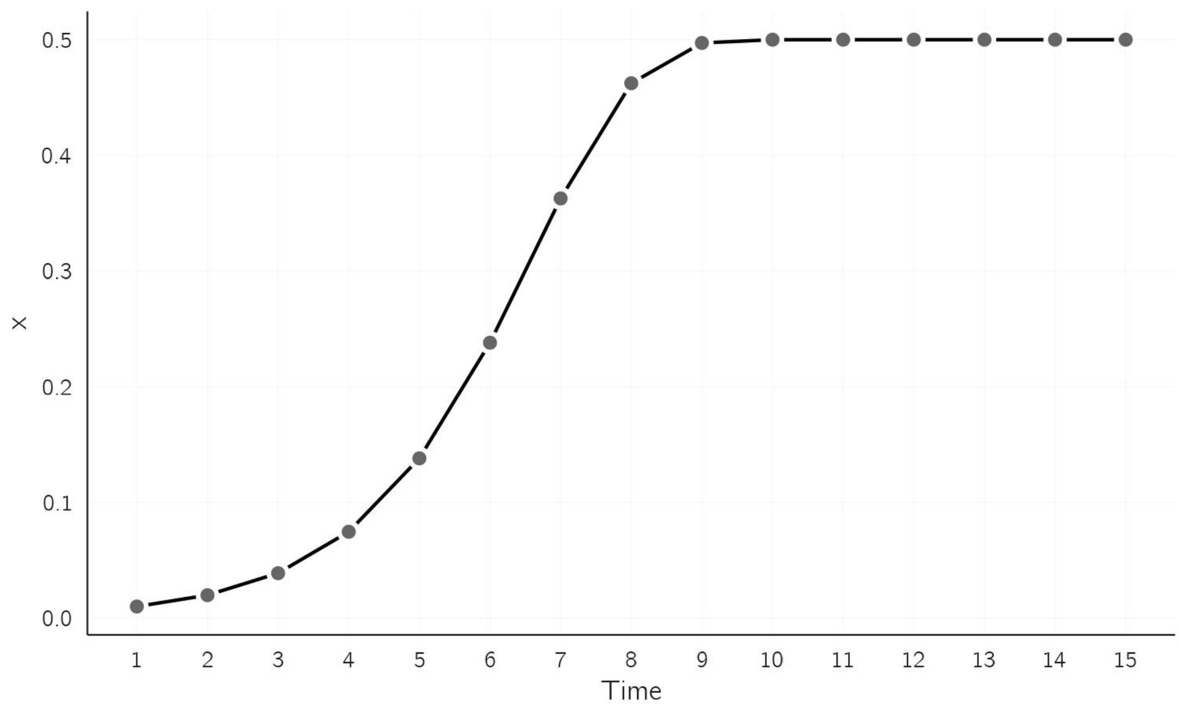

Figure 2.2: The $r = 2$ case.

This is the simple case. The population initially develops exponentially but then levels off and reached a stable state at $X = .5$. We need to understand a bit more about it. What you see here is that we have gone from an unstable initial state to a stable state, a point attractor.

[1] Verhulst proposed this model in the form of a differential equation in continuous time. We will discuss this type of model in Chapter 5. In continuous time, nothing particularly spectacular happens and we only see the kind of behavior displayed in Figure 2.2.

The next code shows that this point attractor attracts from a wide range of initial values, but not all. If we start exactly at zero, X stays at zero. So, zero is an equilibrium too, but a special one. It is an unstable fixed point. A small perturbation will cause X to move to .5, the stable fixed point. All initial values in close proximity of 0 will move away from 0 (repellent), but if $X = 0$ exactly, then it remains 0 for all time. So, $X = 0$ is a fixed point but unstable.

```
n <- 30; r <- 2; x <- rep(0,n)
for (init in seq(0, .7, by = .01))
  # start form different initials values
{
  x[1] <- init
  for (i in 1:(n - 1))
    x[i + 1] <- r * x[i] * (1 - x[i])
  if (x[i] == 0)
    plot(x,type = 'l',xlab = 'time',bty = 'n',ylim = c(0, .8),col = 'red')
  else
    lines(x)
}
```

This concept of equilibrium, stable or unstable, is crucial for later chapters. The essence of the next chapter is to change a control parameter, here r, and study how the pattern of equilibria (the equilibrium landscape) changes. You can easily do this yourself by setting $r = .9$. For $r < 1$, there is only stable attractor (zero).

Simulating this is again not really necessary. One has to realize that a fixed point (X^*) is found when $X_{t+1} = X_t = X^*$. See for yourself that:

$$X_{t+1} = X_t = X^* X^* = rX^*(1 - X^*)X^* = 0 \ or \ 1 = r - rX^*X^* = 0 \ or \ X^* = \frac{r-1}{r}$$

So 0 and $(r-1)/r$ are fixed points. Indeed for $r = 2$, we have seen that 0 and .5 are equilibria, one stable and one unstable. To determine whether fixed points are stable we look at the derivative of the function, $f^{'}(x)$, which, as you can easily check, is $r - 2rX$.

I will not explain why, however, the rule is that the fixed point is stable if the absolute value of the derivative in the fixed-point value is less than 1.[2] For $r = 2$ the fixed points are 0 and .5. $\left|f^{'}(X^* = 0)\right| = |2 - 0| = 2$, which is greater than 1 and thus $X^* = 0$ is unstable. $\left|f^{'}(X^* = .5)\right| = |2 - 2 \times 2 \times .5| = 0$, which is less than 1 and thus $X^* = .5$ is stable. You can check for yourself that $X^* = \frac{r-1}{r}$ is stable for $1 < r < 3$, both with the r-code and with the absolute value of the derivative.

[2]Why this is, is not too difficult to understand. If you google 'fixed points of difference equations', you will quickly arrive at stackexchange.com, where several insightful explanations are given.

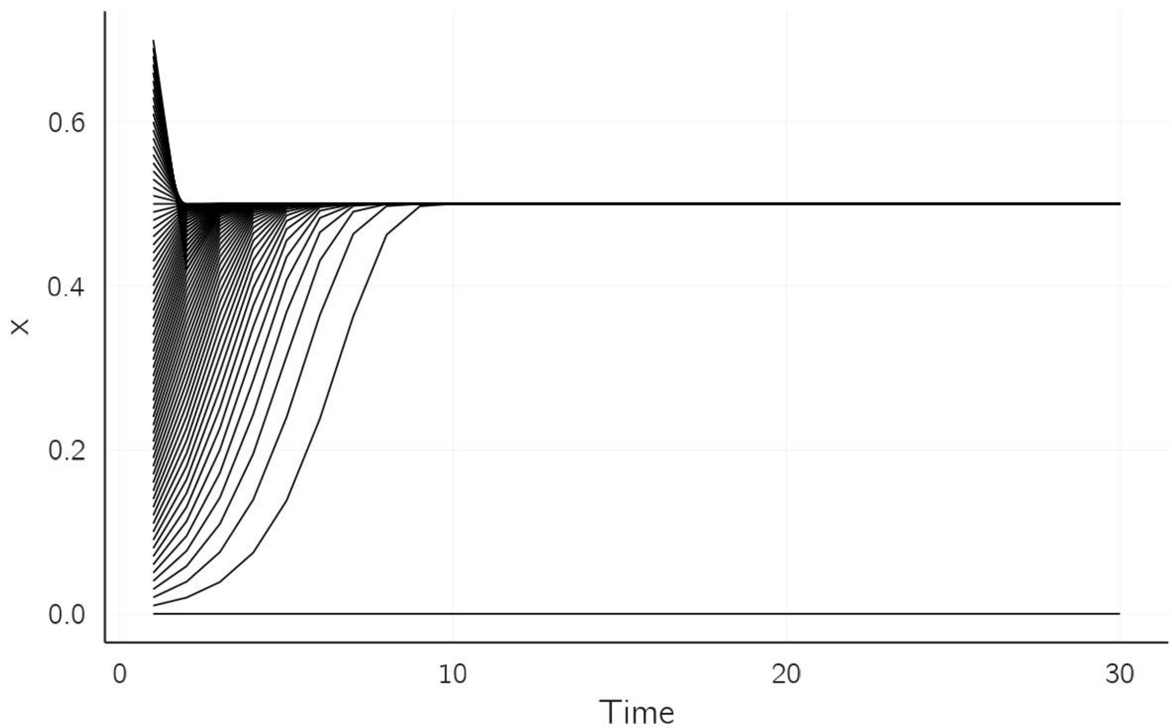

Figure 2.3: Illustration of stable and unstable fixed points. For many initial values, $X = .5$ is an attractor. $X = 0$ is an unstable fixed point. Only if we start exactly at 0 do we stay there.

2.4 Limit cycles

So at $r = 3$ the fixed point or $(r-1)/r$ becomes unstable. Let's study some cases. Plot are made with the code of Figure 2.2.

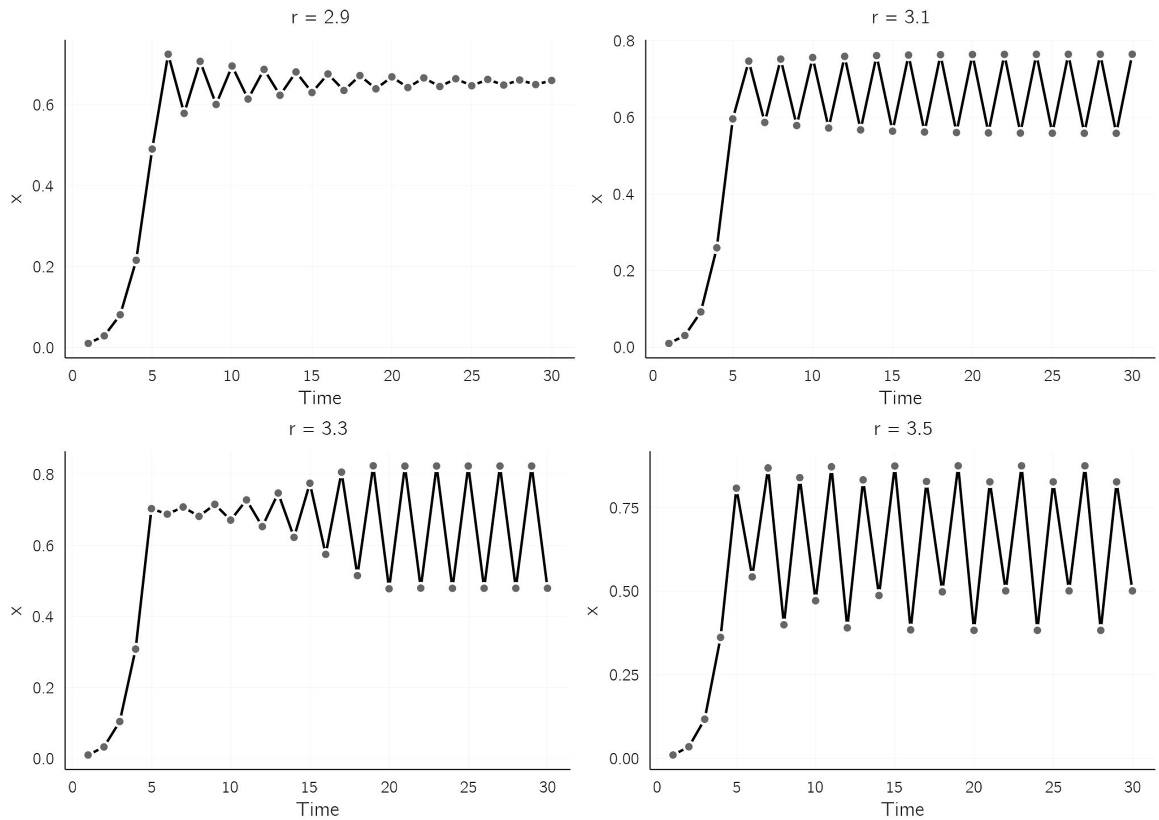

Figure 2.4: Qualitative different behavior of the logistic map for different values of r.

For $r = 2.9$ we see that the series converges to the fixed point $\frac{1.9}{2.9} = .66$, but in a process of over- and undershooting. For $r = 3.1$ and $r = 3.3$ a limit cycle of period 2 arises. The population oscillates between two values. For $r = 3.5$ this becomes even more remarkable, we see a limit cycle of period 4. For slightly larger values we could get cycles with even higher periods.

It has been claimed that these limit cycles occur in real population dynamics. Intuitively, it can be understood as a process of over- and undershooting, which dampens out for r a little below 3, but not for $r > 3$.

2.5 Chaos

If we increase r even further, the doubling of the periods changes to even stranger behavior. This is what the time series looks like for $r = 4$.

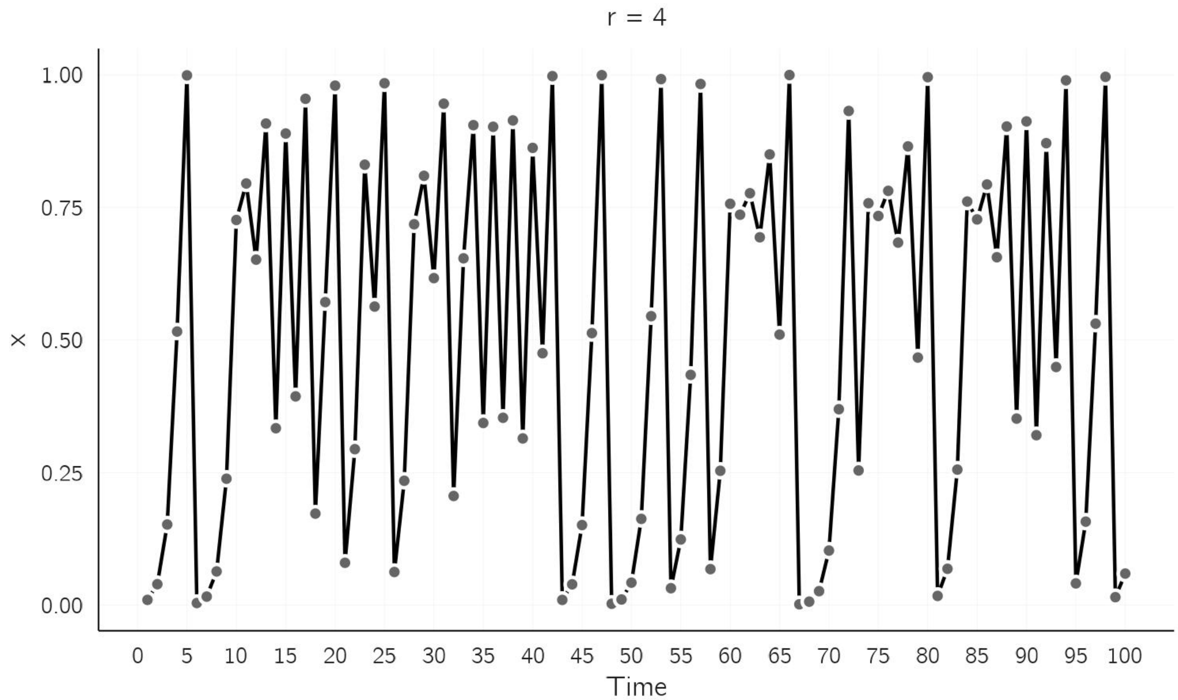

Figure 2.5: Chaos for $r = 4$.

There seems to be no regularity. This is what we call deterministic chaos. This time series is unpredictable, even though we now have the equation, and the system is deterministic. What exactly do we mean by this? Let me illustrate.

```
r <- 4;  n <- 50; x <- rep(0,n)
x[1] <- .001
for (i in 1:(n - 1))
  x[i + 1] <- r * x[i] * (1 - x[i])
plot(x, type = 'l', xlab = 'time', bty = 'n')
x[1] <- .0010001
# restart with sightly different initial state
for (i in 1:(n - 1))
  x[i + 1] <- r * x[i] * (1 - x[i])
lines(x, col = 'red')
```

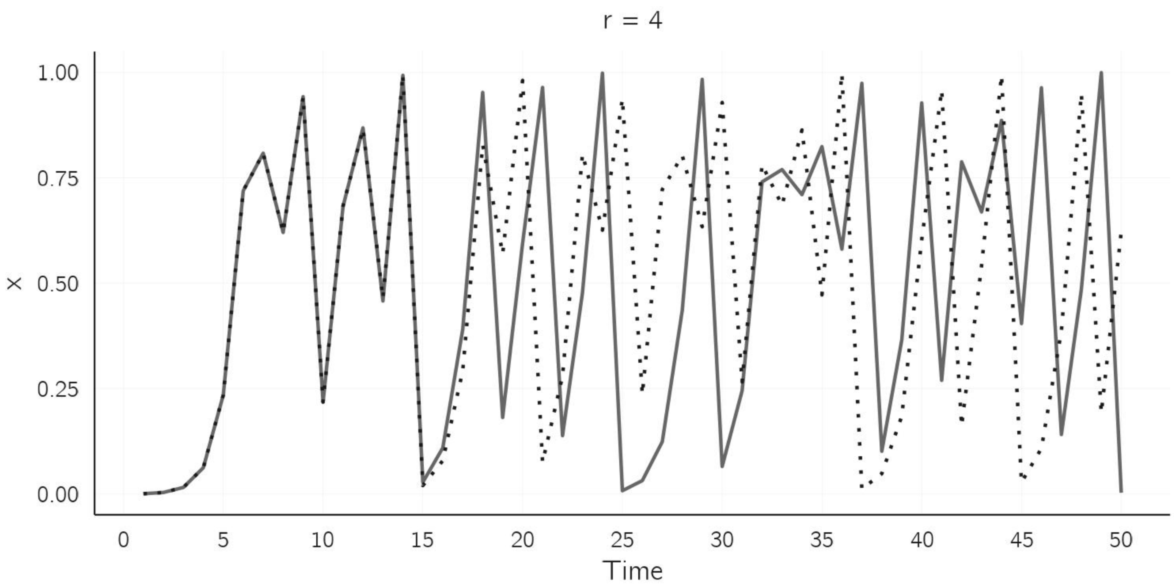

Figure 2.6: The butterfly effect: A small difference in initial state causes divergence in the long run.

We can see that a run with a slightly different initial value will at first follow the same path, but then it will diverge sharply. A tiny perturbation (the butterfly flapping its wings) propagates through the system and dramatically changes the long-term course of the system.

Note that some uncertainty about the exact value of the initial state is always inevitable. Suppose we have an equation like the logistic map for temperature in the weather system, and this equation perfectly describes that system. To make a prediction, we need to feed the current temperature into the computer. But we cannot measure temperature with infinite precision. And even if we could, we do not have a computer that can handle numbers with an infinite number of digits. So, we make a small error in setting the initial state, and this will always mess up our long-term forecast. This is why long-term weather will never be possible, even if we develop much more precise mathematical models, take more intensive and more accurate measurements and use more powerful computers. The weather turns out to be a chaotic system. Sensitivity to initial conditions is a necessary and perhaps sufficient condition for deterministic chaos. For a discussion on the definition of chaos, I refer to (Banks et al. 1992) and (Broer and Takens 2010).

The Lyapunov coefficient quantifies chaos. The idea is to take two very close initial conditions with a difference of ε. In the next iteration this difference might be smaller, the same, or bigger. In the last case the time series diverge, which is typical for chaos. The Lyapunov coefficient is defined as:

$$\lambda_L = \lim_{n\to\infty} \frac{1}{n} \sum_{i}^{n} \ln \left| f^{'}\left(X_i\right)\right| \tag{2.3}$$

where $f^{'}\left(X_i\right) = r - 2rX_i$ for the logistic map and $\lambda_L > 0$ indicates chaos. You may verify in a simulation that $\lambda_L > 0$ for $r = 4$, indicating chaos.

2.6 Phase plot and bifurcation diagrams

Equation 2.2 is very simple. It is just one equation, a deterministic difference equation specifying how X_{t+1} depends on X_t, but the variety of behavior is astonishing. One way to better understand its behavior is to use phase plots. A phase plot is a graphical representation of the relationship between two or more variables that change over time. In one-dimensional systems we plot X_t against X_{t+1}.

```
layout(matrix(1:6,2,3))
r <- 3.3; n <- 200; x <- rep(0,n)
x[1] <- .001
for(i in 1:(n-1)) x[i+1] = r*x[i]*(1-x[i])
x <- x[-1:-100]
plot(x,type='l',xlab='time',bty='n', main=paste('r = ',r),ylim=0:1,cex.main=2)
plot(x[-length(x)],x[-1],xlim=0:1,ylim=0:1,xlab='Xt',ylab='Xt+1',bty='n')
r <- 4; x[1] <- .001;
for(i in 1:(n-1)) x[i+1] <- r*x[i]*(1-x[i])
x <- x[-1:-100]
plot(x,type='l',xlab='time',bty='n',main=paste('r = ',r),cex.main=2)
plot(x[-length(x)],x[-1],xlim=0:1,ylim=0:1,xlab='Xt',ylab='Xt+1',bty='n')
x <- runif(200,0,1)
x <- x[-1:-100]
plot(x,type='l',xlab='time',bty='n',main='random noise',cex.main=2)
plot(x[-length(x)],x[-1],xlim=0:1,ylim=0:1,xlab='Xt',ylab='Xt+1',bty='n')
```

The top figures are time plots and lower figures are phase plots. The first column shows a limit cycle of period 2, the second deterministic chaos and the third noise generated from a uniform distribution. Although the time series of the second and third cases look similar, the phase diagram reveals hidden structure in the chaos time series. Phase plots can help us to distinguish chaos from noise.

The second useful graph is the bifurcation graph. It summarizes the behavior of the logistic map for different values of r in one figure. The idea is to plot the equilibria as y-values for a range of r-values on the x-axes. This means that if we take an$r$ value ($r < 1$), we will only plot zero's, as only $X^* = 0$ is a stable fixed point. Between 1 and 3, we will also see one fixed

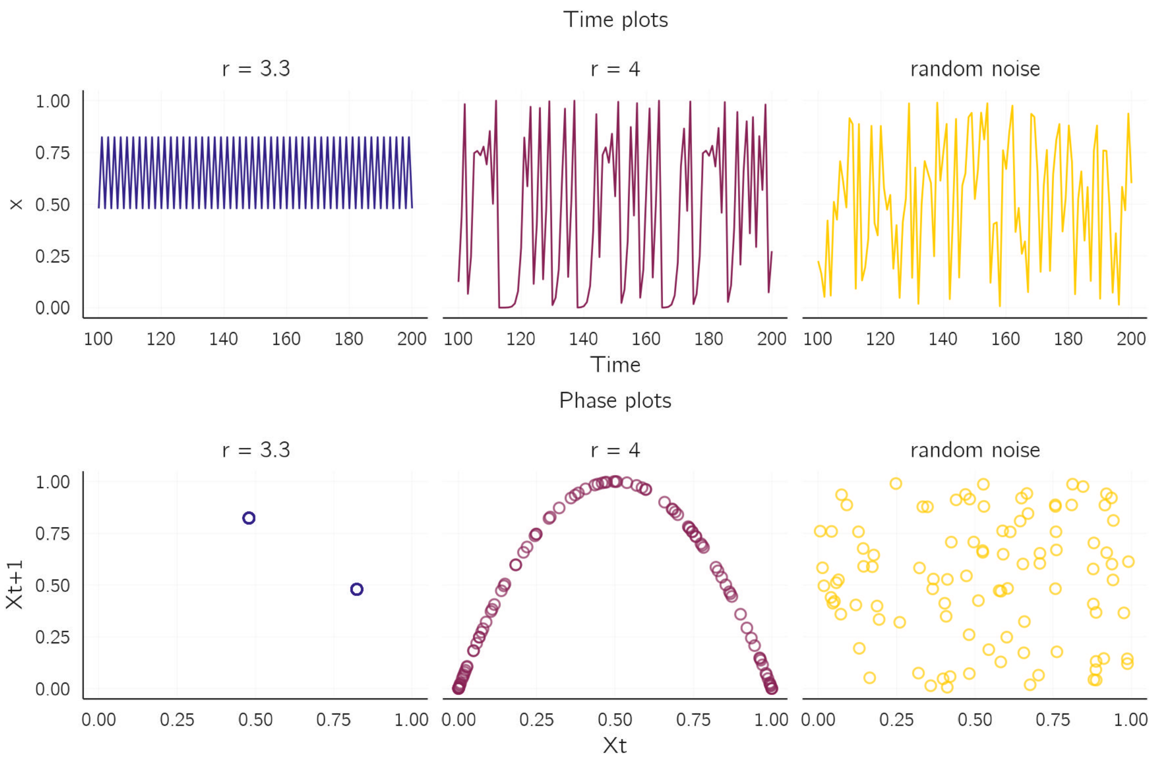

Figure 2.7: Time (top) and phase (bottom) plots for three cases. Chaos and random noise can be distinguished using the phase plots.

point equal to $(r-1)/r$. For $r = 3.3$, we expect to see 2 points as the attractor is a limit cycle with period 2. For higher r we get chaos. How does this all look? It looks amazing!

It is actually a good challenge to program this yourself. The trick is to create time series for a range of values of r, delete the first part of this series (we only want the equilibrium behavior) and plot these as y values. So, if the logistic map has period two ($r = 3.3$), we repeatedly plot only two points. For $r = 4$ we get the whole chaos band.

A clever way to do this is to use the sapply function in r.

```
layout(1)
f <- function(r, x, n, m){
  x <- rep(x,n)
  for(i in 1:(n-1)) x[i+1] <- r*x[i]*(1-x[i])
  x[c((n-m):n)] # only return last m iterations
}
r.range <- seq(0, 2.5, by=0.01)
r.range <- c(r.range,seq(2.5, 4, by=0.001))
n <- 200; m <-100
equilibria <- as.vector(sapply(r.range, f,  x=0.1, n=n, m=m-1))
r <- sort(rep(r.range, m))
plot(equilibria ~ r, pch=19,cex=.01,bty='n')
```

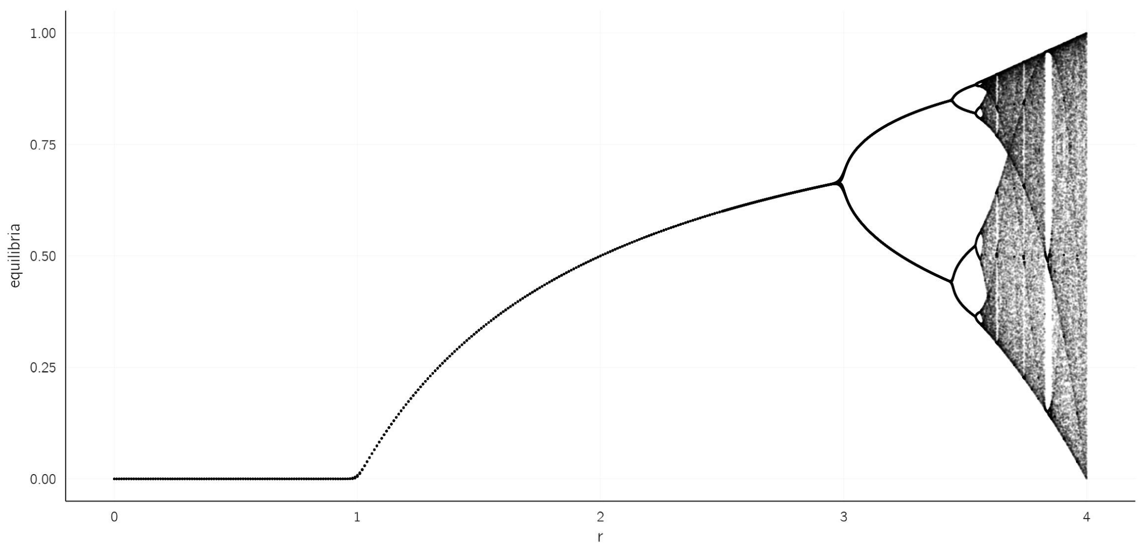

Figure 2.8: The bifurcation diagram of the logistic map.

Here we see indeed fixed stable points for $r < 3$, the period doubling of the limit cycles for

$r > 3$, followed by chaos.

A recurring phenomenon in many chaotic maps are fractals. Fractals are figures in which certain patterns reappear when we zoom in on the figure, and this happens again and again when we zoom in further. You can see this by zooming in on the interval of r between 3.83 and 3.86 (see exercise 2). The three equilibria in the limit cycle split again into period doubling cycles, as we saw in the overall plot between r in 3 and 3.5.

One famous result on this period doubling route to chaos is the Feigenbaum constant. The ratios of distances between consecutive period doubling points (e.g., the distance between first and second divided by the distance between the second and third point), converge to a value of approximately 4.6692. The amazing thing is that this happens for any unimodal map, not only the logistic map.

2.7 What did we learn

I find these results stunning. I note again that the generating function is deceptively simple, but its behavior is utterly complex and beautiful to me. Mathematicians have studied every detail of these plots and most of it is beyond my comprehension. The Wikipedia on the logistic map will introduce you to some more advanced concepts, but for our purposes the present introduction will suffice.

Actually, it is good to realize which concepts we have already learned. The first is the concept of equilibrium. The states of dynamical systems tend to converge to certain values. The simplest of these is the fixed point. Fixed points can be stable or unstable (more on this in the next chapter). If we start a system exactly at its unstable fixed point (and there is no noise in the system), it will stay there. But any small perturbation will cause it to escape and move to the fixed stable point.

The bifurcation diagram summarizes this behavior and also shows how the equilibria change when a control parameter changes. For example, at $r = 1$ we see a bifurcation in the logistic map. Initially 0 was the stable fixed point and $(r - 1)/r$ was unstable. At $r = 1$ this is reversed. At $r = 3$ we see another bifurcation when limit cycles appear.

We have learnt that there are all sorts of equilibria. The strangest ones are called strange attractors, which are associated with deterministic chaos. You can see them by making a phase diagram. Phase diagrams for other famous maps are often stunning. The most famous is the Mandelbrot set (look on the internet). There is an R-blog about the Mandelbrot set. I recommend you check it out. Simulation helps understanding!

The last thing we learned is that even if our world were deterministic (it is not!), and we knew all the laws of motion (say, the logistic map), and knew initial states with enormous precision, the world is unpredictable.

This statement needs some nuance. I have already mentioned that the weather can be chaotic and unpredictable. But the weather is not always so unpredictable. Sometimes longer forecasts are possible. But forecasts beyond, say, 10 days seem out of reach. We also see in the logistic map that when r is close to 4, the forecast suffers from the butterfly effect, but for $r = 2$ the time course is very predictable, even more predictable than in many linear systems. This is because there is only one stable fixed point (.5). The initial state does not matter, we always end up at .5! So the logistic map is extremely predictable or extremely unpredictable depending on r.

2.8 Other maps and fractals

There are many accessible sources on chaos theory. As always, Wikipedia is a great resource. It helps me a lot by actually doing things, i.e., doing computer simulations. One example is the Henon map, which consists of two coupled difference equations:

$$X_{t+1} = 1 - aX_t^2 + Y_t Y_{t+1} = bX_t \tag{2.4}$$

Using the code example from the logistic map, you should be able to generate time series and a phase diagram for this model. Try to reproduce the first image on the Henon map Wikipedia page. The amazing three-dimensional bifurcation diagram may be more challenging.

Fractals are another topic for further study. Another look at Wikipedia is recommended. Making your own fractals in R is made easy by the R blog by Martin Stefan (2020).

2.9 Detecting chaos in psychophysiological data

Chaos theory and the logistic map were popularized about 50 years ago, and since then researchers have been looking for chaos in all kinds of time series (Ayers 1997; Robertson and Combs 2014; Schiepek et al. 2017). One idea behind this work is the hypothesis that chaos might be healthy (Pool 1989) or be helpful. It would be helpful in learning algorithms, such as neutral networks, to prevent getting stuck in local minima (Bertschinger and Natschläger 2004). My very first publication was about chaos in neural networks (Han L. J. van der Maas, Verschure, and Molenaar 1990).

There exist many techniques for chaos detection in times series. This is all but easy because these empirical signals are inevitably contaminated with noise (Rosso et al. 2007). One example is the computation of Lyapunov exponents, quantifying how small differences in initial conditions evolve over time. A positive Lyapunov exponent indicates chaos, signifying exponential divergence of trajectories, which is a hallmark of chaotic systems. This method involves reconstructing the phase space from time series data and calculating the average exponential rate of separation of trajectories. With the Lyapunov function in the package DChaos you can compute the Lyapunov coefficient for times series generated with the logistic map. You may verify that for $r = 4$ you get the Lyapunov coefficient as computed with the derivative earlier.

Chaos detection is an active area of research, with new methods being proposed on a regular basis (Zanin 2022). There are several packages available in R, including new methods based on machine learning techniques (Sandubete and Escot 2021; Toker, Sommer, and D'Esposito 2020).

These methods generally require long time series. Many publications appeared on the detection of chaos in psychophysiological data. Examples are EEG (Pritchard and Duke 1992) heart beat (Freitas et al. 2009), EMG (Lei, Wang, and Feng 2001) and eye movements (Harezlak and Kasprowski 2018). Reviews of these lines of research are provided by Stam (2005), Kargarnovin et al. (2023) and Garc and Pe (2015).

2.10 Exercises

1) For $r = 3.5$, the logistic map iterates between four points. For which value(s) does it iterate between 8 points? (*)
2) In the section on bifurcation diagrams the code to make a bifurcation plot is shown. First, run this code and take a look at the bifurcation plot. In this plot, you can also zoom in by changing the interval between the r's on the x-axis. Adjust the code by changing the r.range to seq(3.4, 4, by=0.0001)), also change 'cex = 0.01' to a lower value. Zoom in on the interval of r between 3.83 and 3.86. In this interval the chaos suddenly disappears and limit cycles with period 3 appear. Check this with a time series plot for a particular value of r. (*)
3) Reproduce the first image from the Henon map Wikipedia page. Provide your R-code and figure (*).
4) Make the bifurcation diagram of the Ricker model (see Wikipedia). Provide your R-code and figure. Why is this model considered a more realistic representation of population growth than the logistic map?
5) Also reproduce the three-dimensional bifurcation diagram of the Henon map (**).
6) Have a look at the definition of the Lyapunov coefficient in Section 2.4. Calculate this Lyapunov coefficient for the logistic map where $r = 4$ using the Dchaos package in R. This coefficient can also be calculated manually using the derivative (Equation 2.3). Do this and check that the coefficients are approximately equal.

7) Use the Rmusic library (installed with devtools::install_github("keithmcnulty/Rmusic", build_vignettes = TRUE)) to create a chaos sound machine. Make one for white noise too. Can you hear the difference? (**)
8) Find a paper on chaos detection in psychology or psychophysiology and summarize it in 300-400 words (*).

3 Transitions in complex systems

3.1 Introduction

My dissertation research was on the theory of Piaget, who proposed a stage theory of cognitive development. These stages were separated by transitions. One such transition should occur between the pre-operational and concrete-operational stages. In the latter stage, children learn logical, concrete physical rules about objects, such as weight, height, and volume. The most famous test to distinguish between the two stages is the conservation task.

There are many conservation tasks, but the setup is always the same. For example, you show a child two equal balls of clay, ask for confirmation that they weigh the same, roll one into a sausage shape, and then ask again for confirmation of equal weight. A non-conserving child will now claim that the longer sausage weighs more. One can **also** do this with two rows of coins (spreading one row out) or two glasses of water (pouring the water from one glass into a smaller longer glass). It is actually a fascinating task, a real fun task to do with children between 5 and 8 years old.

From the 1960's to the 1980's, this was a topic of major interest in developmental psychology, and hundreds of papers were published on the subject . A key question was whether there was **really** a stage transition, and there was a lot of confusion about what a transition actually was. It was my task to clarify this and to prove Piaget's hypothesis. I think I succeeded in clarifying the question, but whether I succeeded in proving the stage theory is debatable.[1]

The idea of Peter Molenaar, my PhD advisor, was to use catastrophe theory to define the concept of a transition in a precise way, to use the so-called catastrophe flags to test the hypothesis of a transition, and also to fit a cusp model to the conservation data. It took me, with the help of many people, more than 20 years to do all these steps (Van der Maas and Molenaar 1992a; Jansen and Van der Maas 2001b; Dolan and Maas 1998; Grasman, Maas, and Wagenmakers 2009).

What is catastrophe theory, what are these flags, and what is the cusp? These are the first questions I will answer in this chapter. But this chapter is also about statistics. You will learn how to fit a cusp model to data. I will present a methodology for studying transitions in areas where we do not have a mathematical description of the underlying system. I will present

[1] Learning a particular conservation task does seem to be rather sudden, but there could easily be two years between learning conservation of number and conservation of volume (Kreitler and Kreitler 1989) . This is inconsistent with the stage theory.

examples from very different subfields of psychology. Finally, I will discuss the criticisms that were made in response to the hype around catastrophe theory about 50 years ago. This is a long chapter, but I have included only what is necessary to make intelligent use of this approach. This requires some basic understanding of the mathematics of catastrophe theory, a good overview of the possibilities for testing cusp models, and knowledge of the controversies from the early days of the popularization of this theory.

3.2 Examples of transitions

In chapter 1, I stated that complex systems tend to be characterized by a limited number of equilibria. As we have seen in the previous chapter, these equilibria can take many different forms, but in this chapter, we consider only stable and unstable fixed points. We are particularly interested in the case where the configuration of stable and unstable points changes due to a smooth change in some external variable, a control variable. In such a case a discontinuous change or a (first order) phase transition can occur. A transition or tipping point is the second of the three intriguing properties of complex systems (chaos being the first).

I call it an intriguing property because in the linear systems we are used to, smooth changes in control variables lead to similar (proportional) changes in behavior variables. We may see a big change in some behavior, but this requires a big change in the controls. An example would be the speed of your bike and the force you apply. But in the case of fear or panic, this process is often non-linear. If a smooth change in an independent or control variable, such as the smell of smoke, leads to a sudden jump in fear (e.g., panic), we are likely to be dealing with a phase transition.

A key physical example is the change in state of water. Between, say, 10 and 80 degrees Celsius, a smooth change in temperature results in only a slight change in the liquid state of water. But if we change the temperature very slowly, close to the thresholds of 0 or 100 degrees Celsius, we see sudden phase transitions.

We saw something similar for the logistic map when r crossed the boundary $r = 1$. However, this did not lead to a sudden change in X^*. This is often called a second order phase transition, meaning that the configuration of stable and unstable points changes, but there is no discontinuous change in behavior.[2]

Discontinuous phase transitions such as melting and freezing occur in many systems. Famous examples from the natural sciences include collapsing bridges, turning over ships, cell division, and climate transitions such as the onset of ice ages. Examples from the social sciences include conflict, war, and revolution. Some examples from psychology are falling asleep, outbursts of aggression, radicalization, falling in love, sudden insights, relapses into depression or addiction, panic, and multistable perception.

[2] **A related and very similar distinction is that between a subcritical bifurcation and a supercritical bifurcation.**

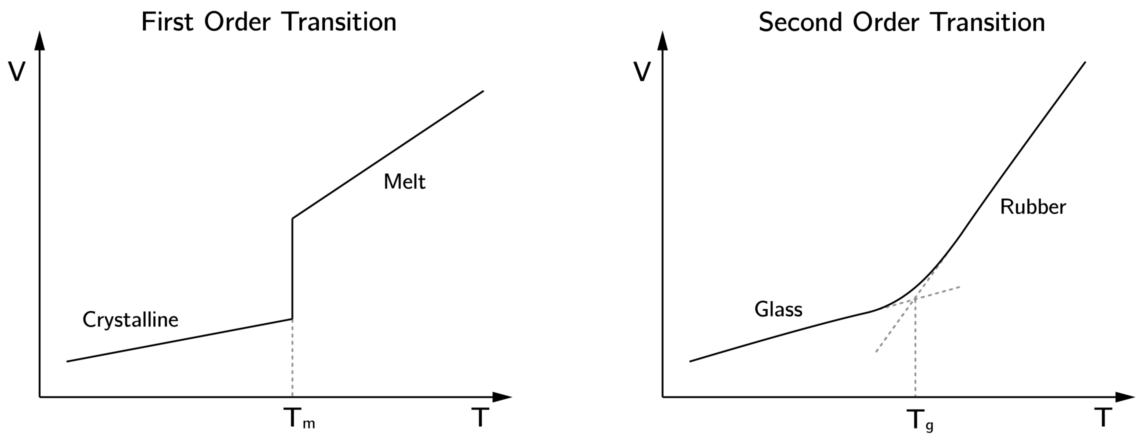

Figure 3.1: A first order (discontinuous) and second order (continuous) phase transition (from: https://polymerdatabase.com/polymer%20physics/ThemalTransitions.html).

Building and testing models of these psychological transitions is challenging but rewarding. These transitions represent significant changes, in contrast to the smaller, often marginally significant effects typically observed in psychological intervention studies. During these transitions, there is a profound change in psychological systems.

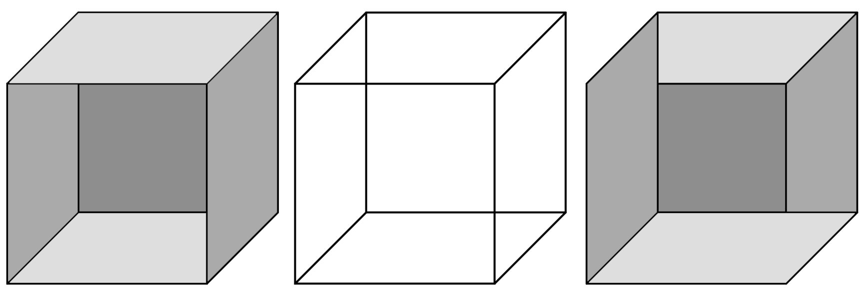

Figure 3.2: Transitions in the perception of the Necker cube. The perception of the middle cube is bistable, and sudden transitions occur between the left ('front') and right ('back') percepts. Multistable perception is a much studied psychological phenomenon that is still not fully understood.

3.3 Bifurcation and Catastrophe theory

Bifurcation theory is a branch of mathematics that studies changes in the qualitative or topological structure of a given family of dynamical systems as parameters are smoothly varied. Such changes are called bifurcations when, for example, equilibria disappear, appear, or split. Simply put, it studies how small changes in parameters or conditions can lead to large changes in outcomes in mathematical systems.

Catastrophe theory can be viewed of as a branch of bifurcation theory, describing a subclass of bifurcations. It was developed by Rene Thom (1977) and popularized by Zeeman (1976). The reason I chose to focus on catastrophe theory in this chapter is fourfold: Firstly, it provides one of the few systematic treatments of bifurcations. A systematic treatment is more effective than simply listing all types of bifurcations. Secondly, once you have a grasp of the basics of catastrophe theory, it becomes easier to learn about other bifurcations not encompassed by this theory. Thirdly, it is most widely used approach in psychology and the social sciences. Finally, the field has developed an empirical program and statistical procedures for the practical application of catastrophe theory.

Catastrophe theory is concerned with gradient systems, in which some quantity is minimized or maximized. These are dynamic systems that can be described by a potential function. Potential functions can be thought of as landscapes with minima and maxima in which we throw a ball and see where it ends up. The simplest case, discussed in the next section, is the quadratic minimum. We can also study what happens to the ball if the landscape changes shape smoothly and a minimum disappears. Then sudden jumps can occur.

Minima and maxima are called critical points, points where the first derivative of the potential function is zero. Catastrophe theory analyzes so-called degenerate critical points of the potential function — points where not only the first derivative, but also the second derivative of the potential function is zero. Phase transitions can occur at these bifurcation points. Thom proved that there are only seven fundamental types of catastrophes (given a limited set of control parameters). I will start with a mathematical introduction and, after explaining the main concepts, give some psychological examples. An in depth discussion of the role of potential functions in catastrophe theory can be found in the introduction of Chapter 1 of Gilmore (1993).

3.3.1 The quadratic case

Thom's theorems are known to be highly complicated, but the basic concepts are actually not that difficult to grasp. The simplest potential function is

$$V(X) = X^2 \tag{3.1}$$

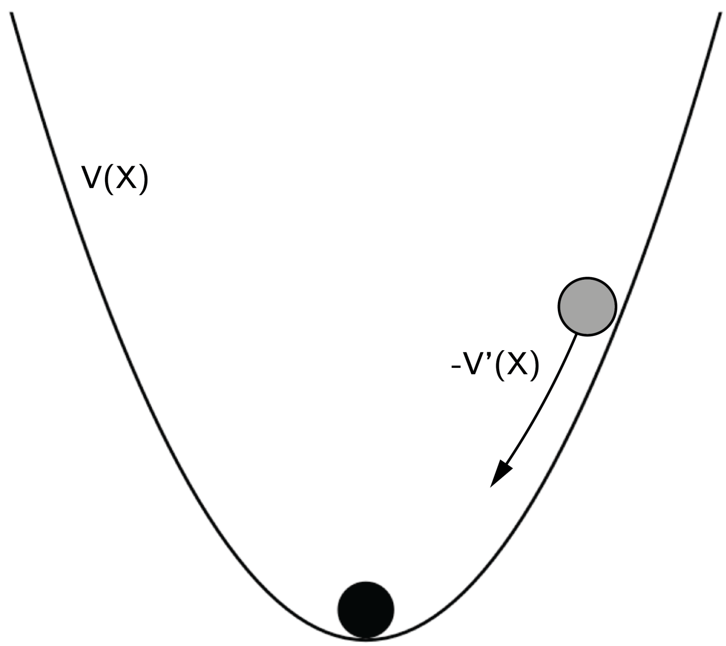

Figure 3.3: The quadratic potential function. **A boll rolls to the minimum value of $V(X)$. It's change is defined by negative of the derivative, $-V'(X)$, of the potential function.**

You can imagine a ball in a landscape. The ball will roll to the minimum of the potential function. We learned in school that this is the point where the first derivative is zero and the second derivative is positive. The first and second derivatives are $V^{'}(X) = 2X$ and $V^{''}(X) = 2$, respectively. At $X = 0$ we find the minimum.

The potential function describes a dynamical system defined by

$$\frac{dX}{dt} = -V^{'}(X). \tag{3.2}$$

This makes sense. When the ball is in $(1, 1)$, $-V^{'}(X) = -2$ and the ball will move towards $X = 0$. But if $X = 0$, $-V^{'}(X) = 0$, and the ball will not move anymore. In the case of the quadratic potential function, there is only one fixed point. By adding parameters and lower order terms to V, i.e., $aX + X^2$, we can move its location, but the qualitative form (one stable fixed point) will not change. Also note that the second derivative is positive, which tells us that we are dealing with a minimum and not a maximum (the so-called second derivative test).

Many dynamical systems behave according to this potential function. Nothing spectacular happens: no bifurcations and no jumps. This is different when we consider potential functions with higher order terms.

3.3.2 The fold catastrophe

The fold catastrophe is defined by the potential function

$$V(X) = -aX + X^3. \tag{3.3}$$

This function has a degenerate critical (bifurcation) point at $X = 0, \quad a = 0$, because at this point $V^{'}(X) = -a + 3X^2 = 0$ and $V^{''}(X) = 6X = 0$, so both the first and second derivative are zero. What makes this point so special? This is illustrated in Figure 3.4.

```
layout(t(1:3))
v <- function(x,a) -a * x + x^3
curve(v(x,a=-2),-3,3,bty='n')
curve(v(x,a=0),-3,3,bty='n')
curve(v(x,a=2),-3,3,bty='n')
```

In the left plot, $a < 0$ and there is no fixed point, the ball rolls away to minus infinity. This can be checked by setting the first derivative to zero, which gives $X = \pm\sqrt{\frac{a}{3}}$. For negative a there is no solution. A positive value of a gives two solutions, as shown on the right for $a = 2$. The positive solution $X = \sqrt{\frac{2}{3}}$ is a stable fixed point because the second derivative in

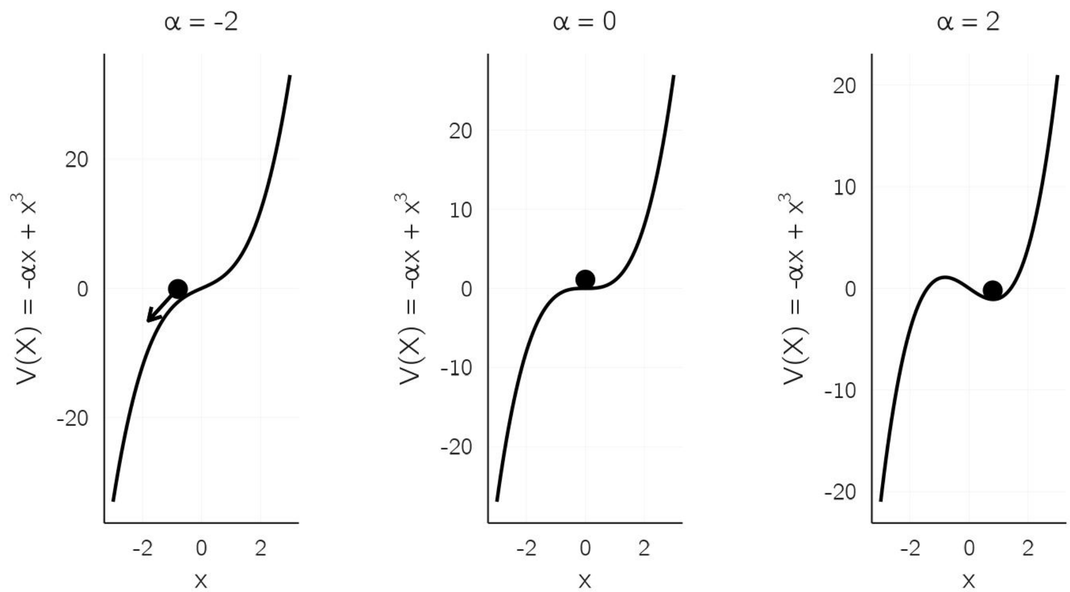

Figure 3.4: A bifurcation at $a = 0$: the equilibria change qualitatively. **For $a < 0$ there is no equilibrium, for $a > 0$ we have a minimum and a maximum.**

this point is positive. The negative solution $X = -\sqrt{\frac{2}{3}}$ is an unstable fixed point because the second derivative in this point is negative.

The middle figure depicts the case just in between these two cases. Here the equilibrium is an inflection point, a degenerate critical point. The bifurcation occurs at this point as we go from a landscape with no fixed points to one with two, one stable and one unstable.

Another way to visualize this is making a bifurcation diagram as we did for the logistic map. On the x-axis we put a, from -1 to 2. On the y-axis we plot X^*, the fixed points of Equation 3.3. We use lines for stable fixed points and dashed lines for unstable points. The diagram is shown in Figure 3.5.

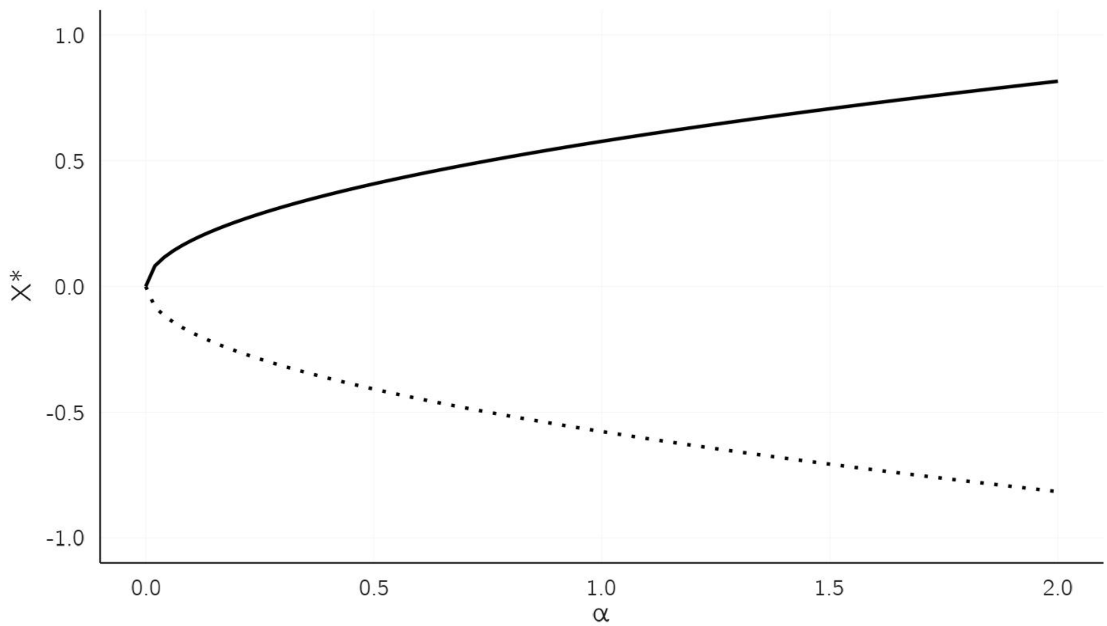

Figure 3.5: The bifurcation diagram of the fold catastrophe. **Similar to what's shown in the preceding figure, when $a = 0$, there is a dramatic alteration in the equilibrium landscape. Suddenly, both a stable and an unstable equilibrium emerge, seemingly from nowhere.**

This bifurcation diagram may not look as spectacular as the logistic map, but its importance cannot be overstated. The fold is everywhere! In a fascinating book, The Seduction of Curves (McRobie 2017), McRobie shows that whenever we see an edge, we see a fold. Figure 3.6 is from the book where he demonstrates how different catastrophes appear in art. I also recommend his YouTube lecture.

The fold catastrophe has been studied in fields ranging from evolution theory (Dodson and

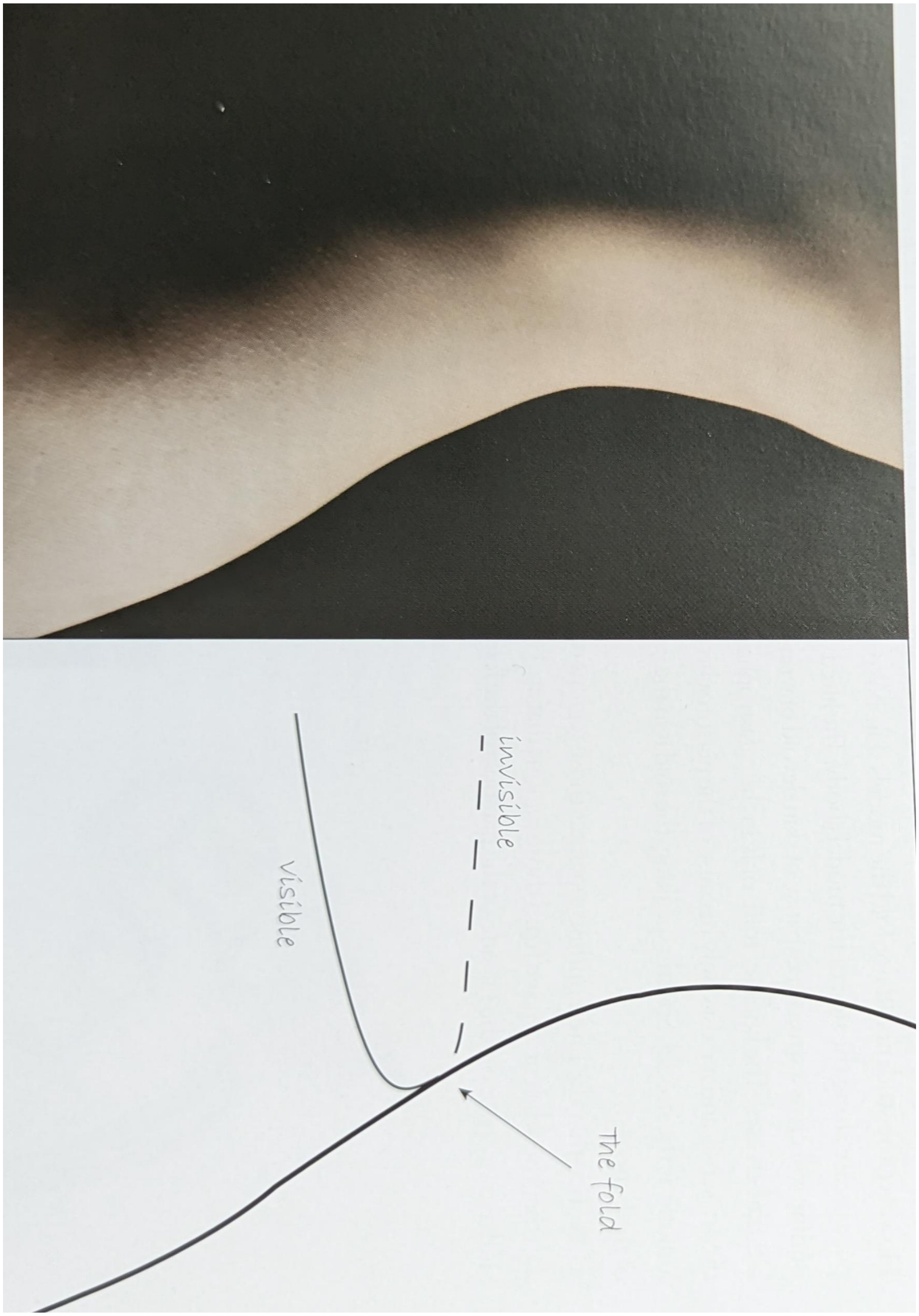

Figure 3.6: The fold in drawings (from, McRobie 2017). **The fold line separates the parts that can be seen from the parts that are hidden.**

Hallam 1977) to buoyancy in diving (Güémez, Fiolhais, and Fiolhais 2002). In addition, higher order catastrophes are composed of folds. The fold catastrophe is also known as a saddle node, tangential or blue-sky bifurcation. It is an example of a second order phase transition. **It is not accompanied by sudden jumps in behavior.**

3.3.3 The cusp catastrophe

Sudden jumps between stable states are associated with first order phase transitions. The cusp, the best-known catastrophe, is the simplest catastrophe with this behavior. The potential function of the cusp is

$$V(X) = -aX - \frac{1}{2}bX^2 + \frac{1}{4}X^4. \tag{3.4}$$

The half and quarter are added to make later derivations a little easier. The highest power is now 4. The first two terms contain the control variables a and b, known as the normal and splitting variables. You might ask why there is no third order term. The non-technical answer is that such a term would not change the qualitative behavior of the bifurcation. Catastrophe theory studies bifurcations that are structurally stable, meaning that perturbing the equations (and not just the parameters) does not fundamentally change the behavior (see Section 3.2.6 and Stewart 1982 for further explication).

It is advisable to do some minimal research on this equation yourself, using an online graphic claculator tool like Desmos or GeoGebra (paste `f(X)=-a X-(1/2) b X^2+(1/4) X^4`). For example, set $a = 1$ and $b = 3$ and look at the graph of the potential function. Think in terms of the ball moving to a stable fixed point. What you should see is that there are three fixed points, of which the middle one is unstable. This bistability is important. Again, there is a relationship to unpredictability. Although you know the potential function and the values of a and b, you are still not sure where the ball is. It could be in either of the minima.

Other typical behavior occurs when we slowly vary a (up and down from -2 to 2), for a positive b value ($b = 2$). This is shown in Figure 3.7. At about $a = 1.5$ we see the sudden jump. The left fixed point loses its stability and the ball rolls to the other minimum.

Now consider what will happen if we decrease a from 2 to -2. In this case the ball will stay in the right minimum until $a = -1.5$. Where the jump takes place depends on the direction of the change in a, the normal variable. This lagging behind, or resistance to change, is called hysteresis. Hysteresis is of great importance in understanding change or lack of change in complex systems. The state in which the ball is the less deep minimum (for $a = 0.5$ in Figure 3.7) is often called a metastable or locally stable state. Metastable states appear to be stable for some time but are not in their globally stable state.

In his classic paper on the Psychophysical Law (Stevens 1957), Stevens reports hysteresis effects in perceptual judgments when properties such as brightness and

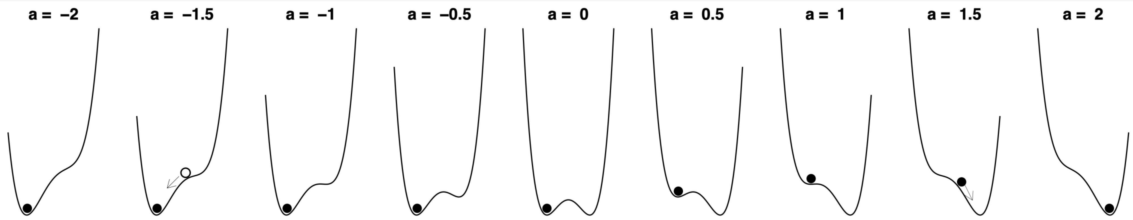

Figure 3.7: The change in the potential function of the cusp by varying a. **Note that the jump to the other state does not happen at $a = 0$ but is delayed and depends on the direction of the change in a. This delay is called hysteresis.**

loudness are systematically varied from low to high and vice versa. In this paper he says: "I'm trying to describe it, not explain it. I am not sure I know how to explain it." I'm not sure he would have agreed, but to me the cusp at least partially explains why hysteresis occurs.

Gilmore (1993) made an important point about noise in the system. If there is a lot of noise, the jumps occur earlier and we see less or no hysteresis effect. This is called the Maxwell convention as opposed to the 'delay' convention. Demonstrating hysteresis therefore requires precise experimental control.

Another very interesting pattern occurs when $a = 0$ and b is increased.

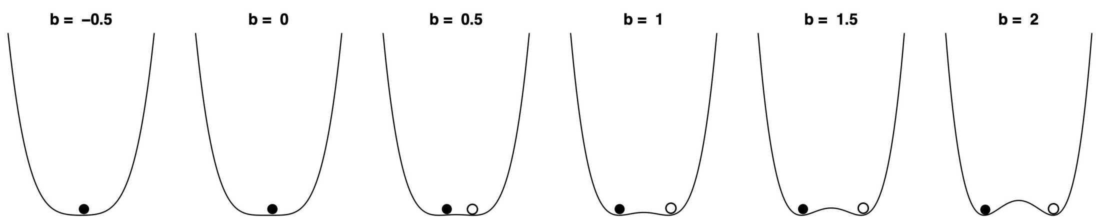

Figure 3.8: The change in the potential function of the cusp by varying b. **One minimum splits up in two.**

For low b there is one stable fixed point that becomes unstable. It splits up in two new attractors. As we did for the fold, we can make bifurcation diagrams showing the equilibria of X as a function of a and b. Along the a-axis we see hysteresis and along the b-axis we see divergence or what is often called a pitchfork bifurcation.

To depict the combined effects of a and b, we require a three-dimensional plot, which combines the hysteresis and pitchfork diagrams.

The cusp diagram can be expressed mathematically by setting the first derivative to zero:

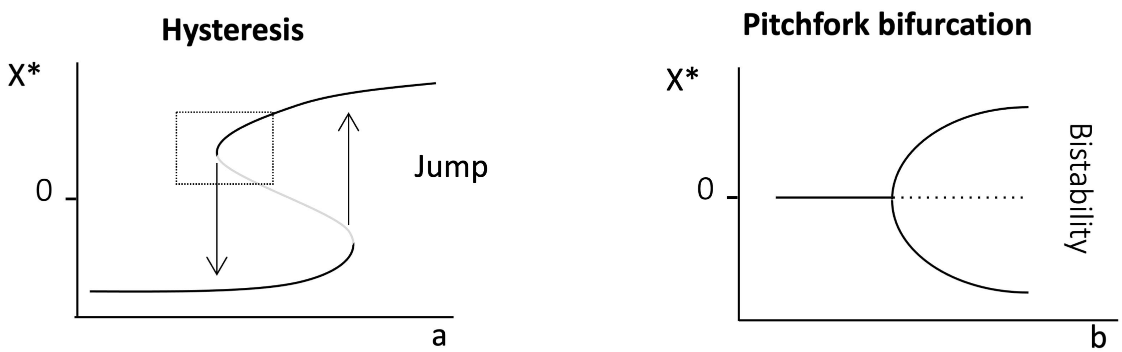

Figure 3.9: Bifurcation diagrams along the a and b axes. The area in the dotted box is a fold.

$$V^{'}(X) = -a - bX + X^3 = 0. \tag{3.5}$$

This type of equation is called a cubic equation.[3] The degenerate critical points of the cusp can be found by setting the first and second derivative to zero. This is just within reach of your high school mathematics training and I leave this as an exercise. The result is:

$$27a^2 = 4b^3. \tag{3.6}$$

This equation defines the bifurcation lines where the first and second derivatives are both zero and sudden jumps occur (see Figure 3.10). The region between the bifurcation lines is the bifurcation set. In this region, the bifurcation has three fixed points, the middle of which is unstable. These unstable states in the middle are called the inaccessible area, colored gray in the cusp diagram. The bifurcation lines meet at (0,0,0). At this point, the third derivative is also zero. This is the cusp point.

3.3.4 Examples of cusp models

To illustrate the cusp, I always use the business card (Figure 3.11). I strongly recommend that you test this example (not with your credit card). You can play with two forces. Fv is the vertical force and the splitting control variable (b) in the cusp. Fh is the horizontal force and the normal variable (a) in the cusp. Note that you will only get smooth changes when $Fv = 0$, but sudden jumps and hysteresis when you employ vertical force. One very important phenomenon is that the card has no 'memory' when the $Fv = 0$. You can push the card to

[3]The cubic equation that cannot solved easily. This is due to the fact that the cusp is not a function of the form $y = f(x)$. Functions assign to each element of x exactly one element of y. But in bistable systems we assign two values of y to one value of x.

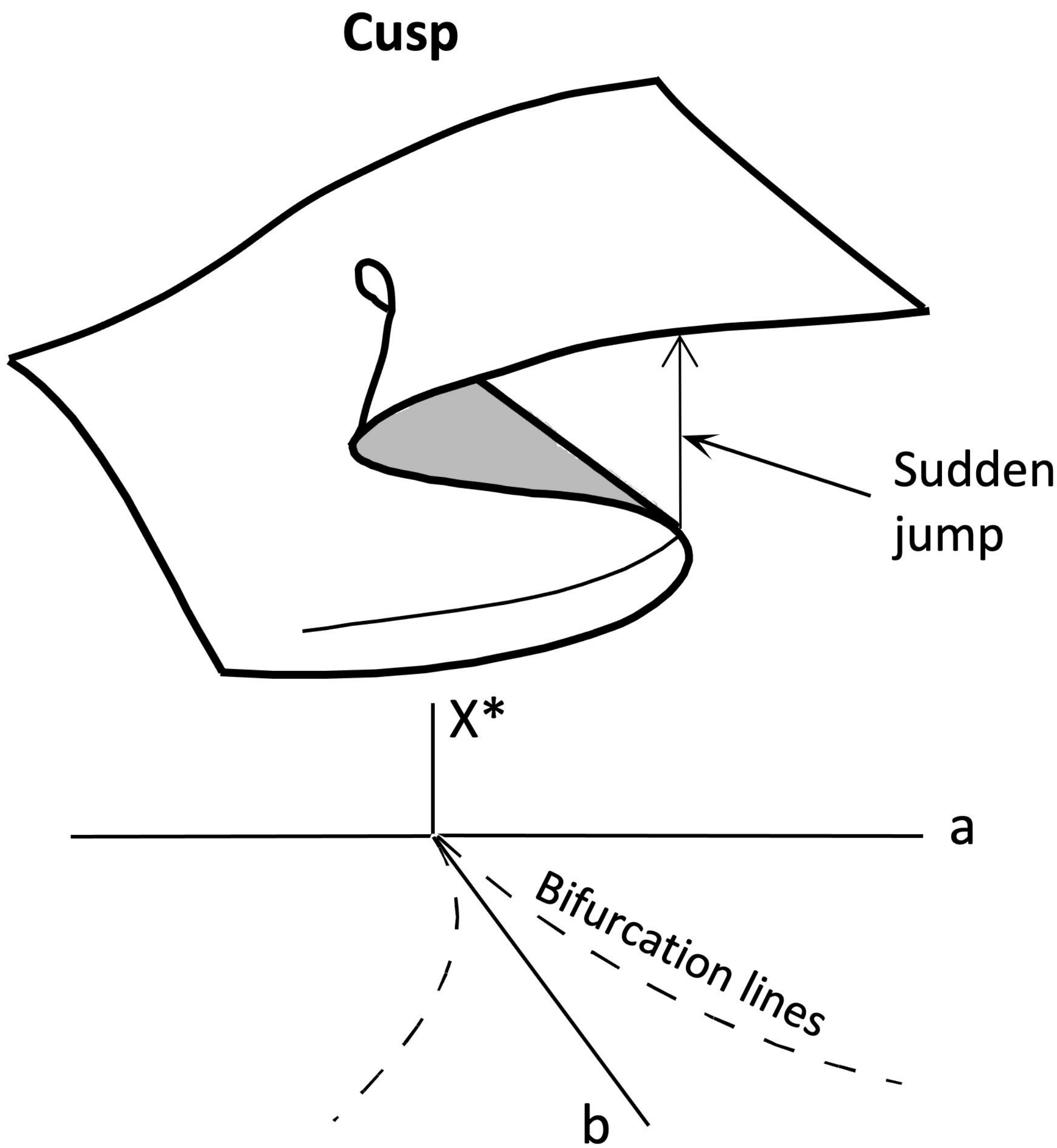

Figure 3.10: The cusp catastrophe combines hysteresis along the normal axis (a), and the pitchfork along the splitting axis (b). At the back of the cusp, changes in a only lead to smooth changes in the equilibrium behavior X^*. At the front, sudden jumps occur when we cross the bifurcation lines. These jumps are typical of first order phase transitions. The area between the bifurcation lines is called the bifurcation set. In this area there are two stable and one unstable equilibrium (colored grey).

a position, but as soon as you release this force (Fh back to zero), the card moves back to the center position. This is not the case with vertical pressure. If we force the card to the left or right position it will stay there, even if we remove the horizontal force. The card has a 'memory'.

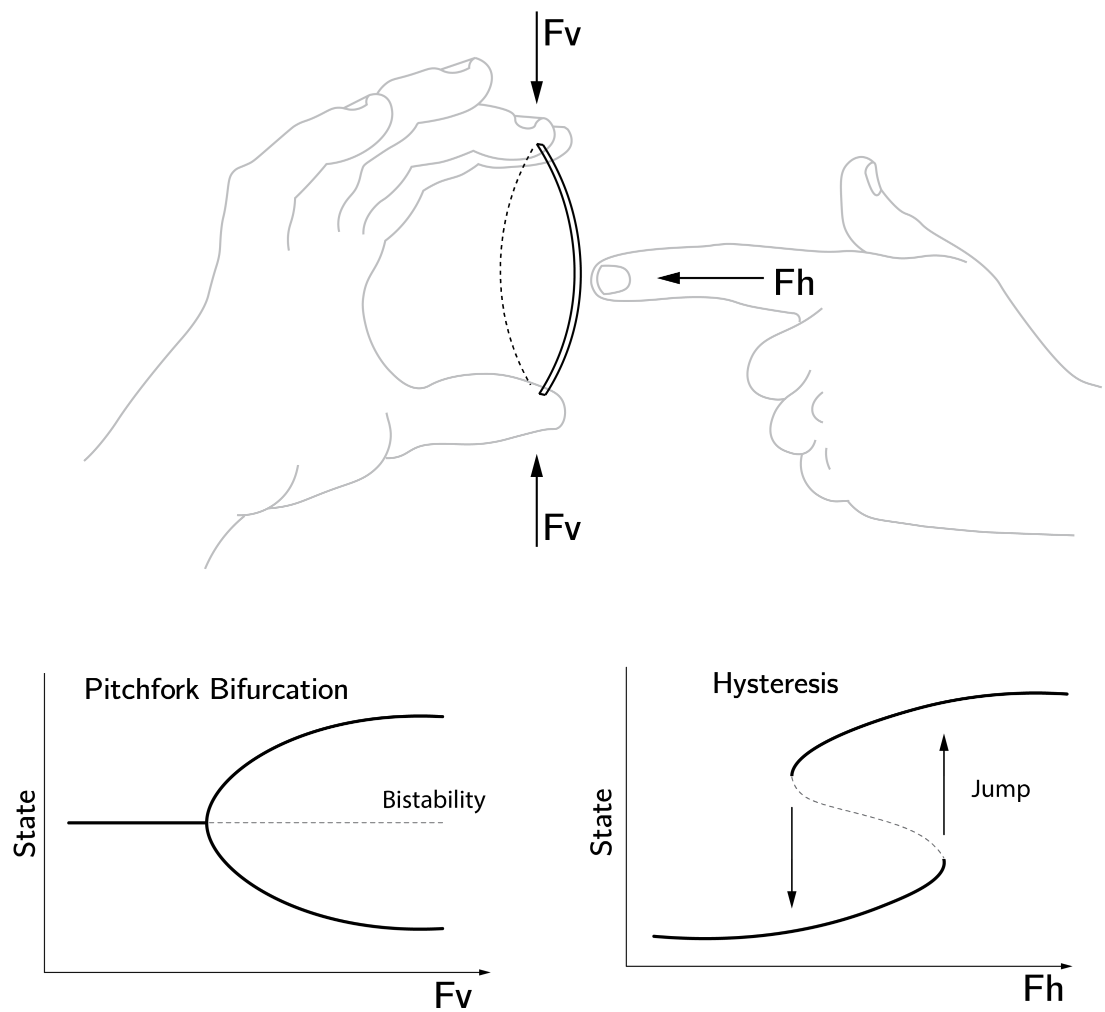

Figure 3.11: A simple business card can be used to illustrate all the properties of the cusp (see text).

This seems simple but the mathematical analysis of such elastic bending structures is a huge topic in itself (Poston and Stewart 2014). The freezing of water is also a cusp. As an approximation, we could say that the density of water is the behavioral variable, temperature is the normal variable, and pressure acts as a splitting variable (see chapter 14 of Poston and Stewart 2014, for a more nuanced analysis). It is very instructive to study the full phase diagram of water. We do not have to know what a triple point is. What we do need to understand is that this can be viewed as a map of the equilibria. This type of mapping would be extremely useful in psychology and the social sciences.

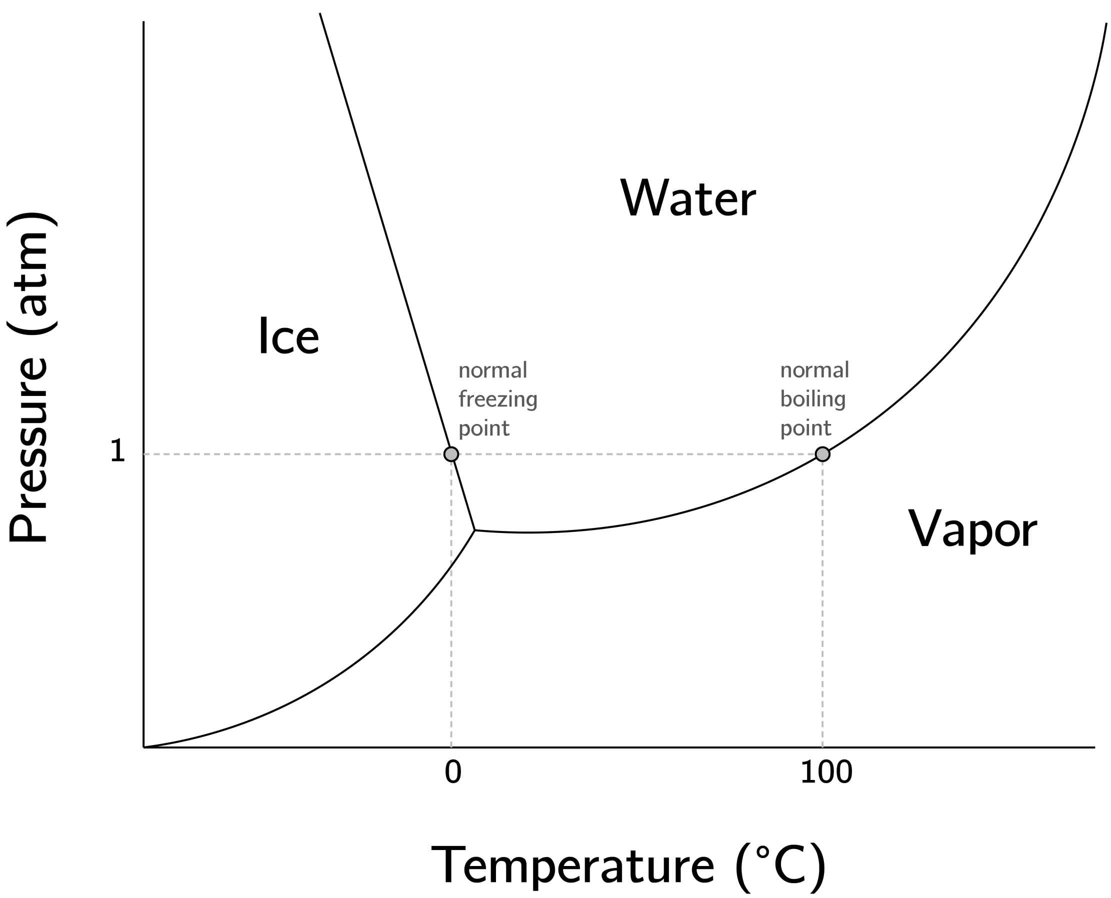

Figure 3.12: The phase diagram of water, which summarizes the equilibria and transitions in the state of water as a function of temperature and pressure.

A psychological example of a cusp concerns sudden jumps in attitudes (Latané and Nowak 1994; Han L. J. van der Maas, Kolstein, and van der Pligt 2003). Attitudes will be discussed in much more detail in later chapters. In general, we have relatively stable attitudes toward many things in life (politics, snakes, hamburgers, and sports), but sometimes they change, and in rare cases they change radically. For example, you may suddenly become a conspiracy theorist, an atheist, or a vegetarian. One example is the attitude toward abortion.

Cusp modeling begins by the definition of the states of the behavioral variable. In this case

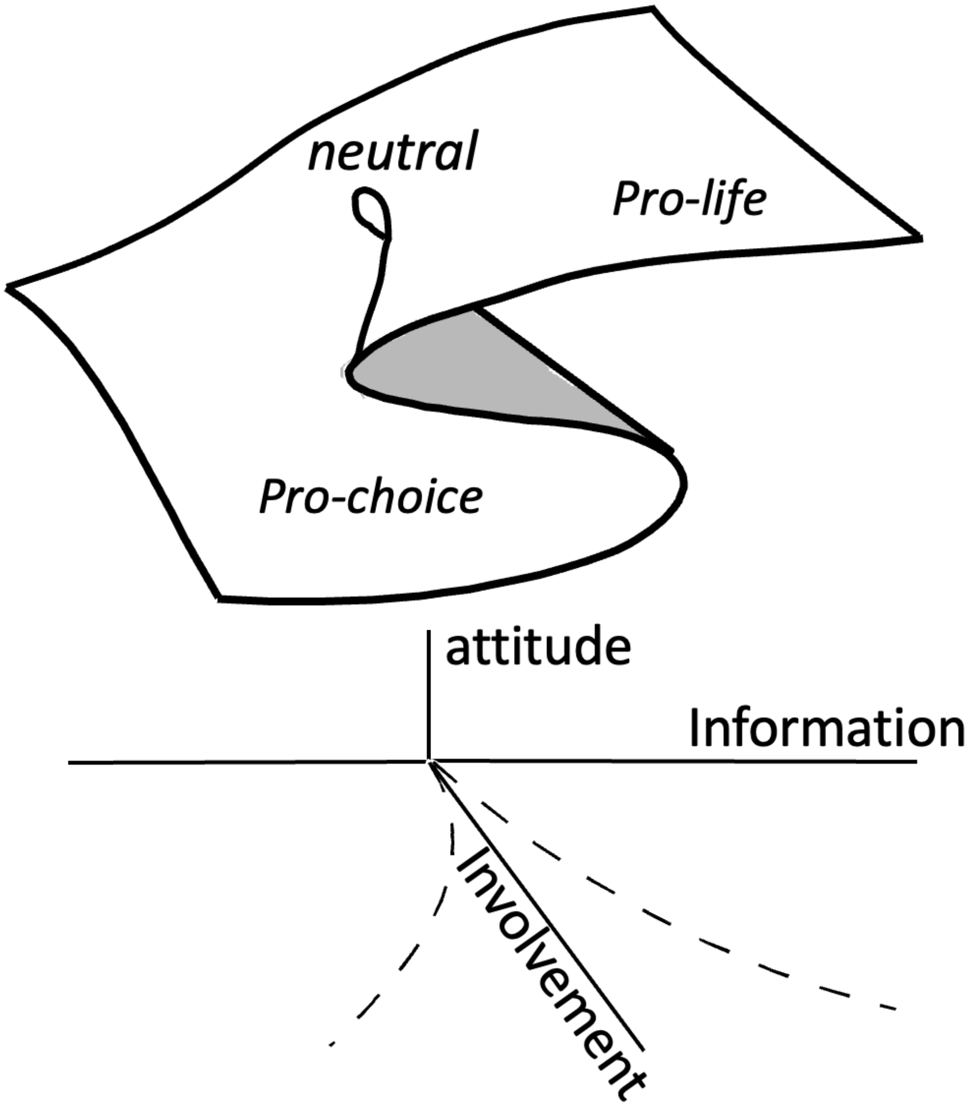

Figure 3.13: The cusp model of attitudes (toward abortion). Because of hysteresis, it is very difficult to persuade highly involved people with new information, but if they change it will be a sudden jump.

the two states of the bistable cusp are the two opposing positions, pro-life[4] and pro-choice. The other state associated with $a = 0, b = 0$ is the neutral state, an 'I don't know' or 'I don't care' position.

The normal (a) and splitting (b) variable are interpreted as information and involvement. Information is a collection of factors that influence whether people tend to be in the pro-life and pro-choice position. Political and religious orientation as well as personal experiences add to this overall factor. One way to construct this information variable is through a factor analysis or principal component analysis.

The splitting factor, involvement, also combines a number of effects (importance, attention). The main idea is that there are two types of independent variables. Some will work (mainly) along the normal axis and some will (mainly) impact the splitting axis.

The implications of this model are that for low involvement change is continuous (Figure 3.13). Presenting people with some new information supporting the pro-life and pro-choice position will have a moderate effect. One problem, as demonstrated with the business card, is that the uninvolved person has "no memory". As soon as you stop influencing this person, he drifts to the neutral 'I don't care' position. We have another problem when people are highly involved. Because of the hysteresis effect, it will be very difficult to persuade people with new information. When this hysteresis effect is high, persuasion just does not work. If you have been involved in political discussions, you probably have experienced that yourself.

But if the underlying change in information is large enough, attitudes can show a sudden jump. There is a lot of anectodical evidence for this, but it is very hard to capture such an effect in actual time series of attitude measures. Another effect that is consistent with the cusp model is ambivalence. In the cusp, ambivalence is not the same thing as being neutral. The neutral point is at the back of the cusp and is associated with low involvement. Ambivalence is associated with high involvement. Highly involved people with balanced information ($a = 0$), may oscillate between extreme positions (see **?@fig-ch4n-img2-old-50** in Chapter 4). Finally, the pitchfork bifurcation can explain issue or political polarization. When involvement increases in a group of neutral people, e.g., due to discussion, they may split into two extreme positions.

Another psychological example of the cusp-like behavior is multistable perception. The necker cube is a famous example. Stewart and Peregoy (1983) proposed a model in which the perception of male phase or female figure is used as a behavioral variable, the splitting variable is the amount of detail, and the normal variable is a change in detail related to the male/female distinction. The results are shown in Figure 3.14.

[4] Perhaps better to call this view anti-choice

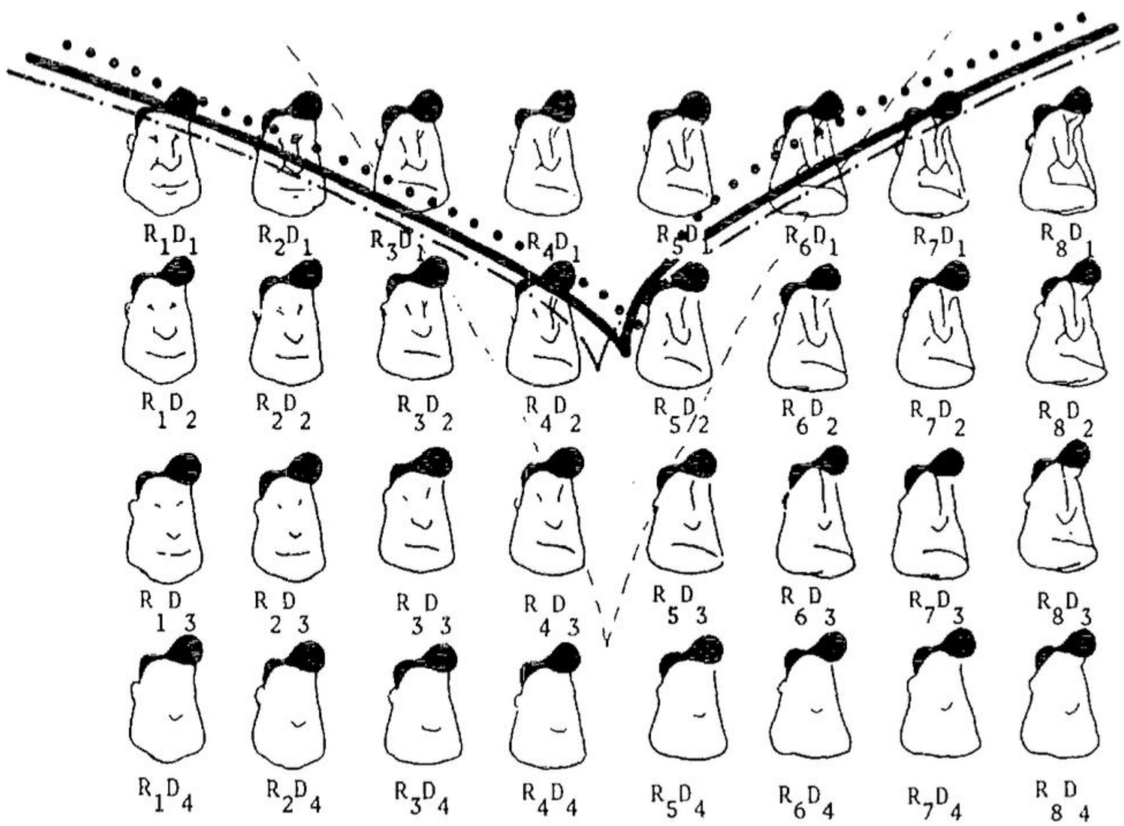

Figure 3.14: From Stewart and Peregoy (1983). The fitted bifurcation lines were calculated using Cobb's method, which is explained in section 3.3.3.

3.3.5 Higher order catastrophes

Note that the cusp is made up of folds. This is best seen in the hysteresis diagram in Figure 3.9 (see the dotted rectangle). Higher order catastrophes yield elements of cusps and folds. The swallowtail catastrophe with potential function $V(X) = -aX - \frac{1}{2}bX^2 - \frac{1}{3}cX^3 + \frac{1}{5}X^5$ consists of three surfaces of fold bifurcations meeting in two lines of cusp bifurcations, which in turn meet in a single swallowtail bifurcation point. We need a four-dimensional space to visualize this, which is difficult. The Wikipedia page on Catastrophe theory has some rotating graphs that may help. The butterfly catastrophe has X^6 as the highest term (and four control variables).

The butterfly catastrophe is of interest when we observe trimodal behavior. We will discuss this catastrophe in Chapter 6 in relation to the modeling of attitudes. I note that the butterfly catastrophe and the butterfly effect in chaos theory are two completely unrelated concepts. Other catastrophes have two instead of one behavioral variable. However, the vast majority of applications of catastrophe theory focus on the cusp, which will also be the focus of the remaining of this chapter. There are many good (but not easy) books that present the full scope of catastrophe theory (Gilmore 1993; Poston and Stewart 2014).

3.3.6 Other bifurcations

Catastrophe theory is limited to structurally stable, local bifurcations. Bifurcation theory also deals with non-structurally stable bifurcations and so-called global bifurcations.

Examples of nonstructural stable local bifurcations are the transcritical bifurcation ($\frac{dX}{dt} = aX - X^2$) and pitchfork bifurcation ($\frac{dX}{dt} = bX - X^3$). The pitchfork is part of the cusp and is not structurally stable because it can be perturbed by an additional term a, which, if unequal to 0, will distort the pitchfork (see Figure 3.15).

Another one we have already seen is the period doubling bifurcation. This happened in the logistic map when the fixed point changed in a limit cycle of period 2. Finally, global bifurcations cannot be localized to a small neighborhood in the phase space, such as when a limit cycle diverges (Guckenheimer and Holmes 1983). However, I don't know of any applications of global bifurcations in psychology or the social sciences.

3.4 Building catastrophe models

3.4.1 Mechanistic models

The model of the attitude towards abortion is called a phenomenological model, as opposed to a mechanistic model. In the former one, we assume the presence of a cusp, and make hypotheses about the involved variables. In the latter, the cusp is derived from basic assumptions or first

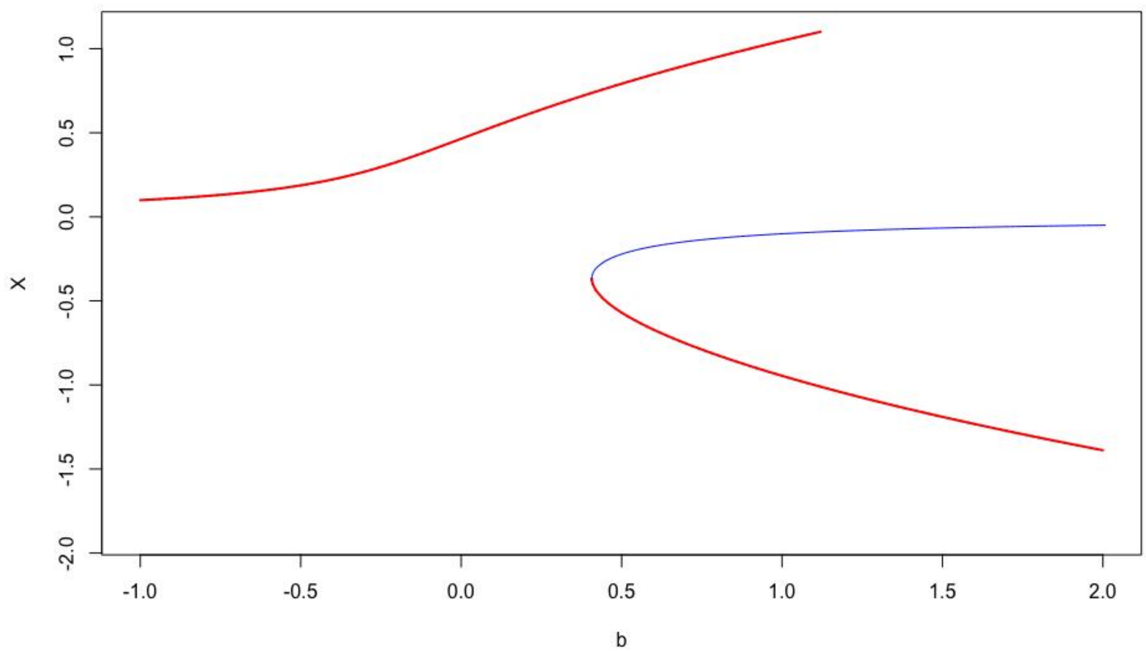

Figure 3.15: Perturbed pitchfork bifurcation ($a = .1$). For $a = 0$ we would get the pitchfork bifurcation as shown in Figure 3.9. Thus, a perturbation in a model parameter leads to qualitative change in this bifurcation, and this is the reason why it is not considered structurally stable.

principles. The mechanistic approach is much more common in the physical and life sciences. An example is the phase transition in water described by the van der Waals equation. Poston and Stewart (2014) show how the van der Waal equation can be reparametrized to take the form of the cusp equation. The advantage is that we learn how exactly temperature and pressure are related to the control variables of the cusp. This gives us a full understanding of the dynamics of this phase transition.

One model that I will use as a psychological model in Chapter 4 is the model of the spruce budworm outbreak, which occurs every 30 to 40 years and results in the defoliation of tens of millions of hectares of trees. The model is

$$\frac{dN}{dt} = r_b N \left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}. \tag{3.7}$$

Where N is the size of budworm population, r_b is the growth rate, K is the carrying capacity, B is the upper limit of predation, and $1/A$ is the responsiveness of the predator.

The first part is the logistic growth equation. N will grow to K at a rate r_b. Note that this is a differential equation, not a difference equation. There is no chaos in logistic growth in continuous time. The second part is the predation function and has a concave shape flattening out at B. The curvature of this function is determined by A. High A makes the concave shape less steep, meaning that predation rather slowly reacts to the increase in budworms (more about the construction of this model later).

The analytical approach to this model is to reparametrize the model so that it takes the form of a cusp. Such reparameterizations are not so easy to do yourself. The idea is to create a smaller set of new variables that are functions of the model parameters. For this model a convenient reparameterization is

$$r = \frac{A\ r_b}{B}\ and\ q = \frac{K}{A}. \tag{3.8}$$

Using these two 'constructed' control variables, we can depict the bifurcation lines of the cusp as:

In later chapters we will discuss psychological examples of a mechanistic approach, **but as far as models of transitions are concerned,** these are rare. The phenomenological approach is much more common.

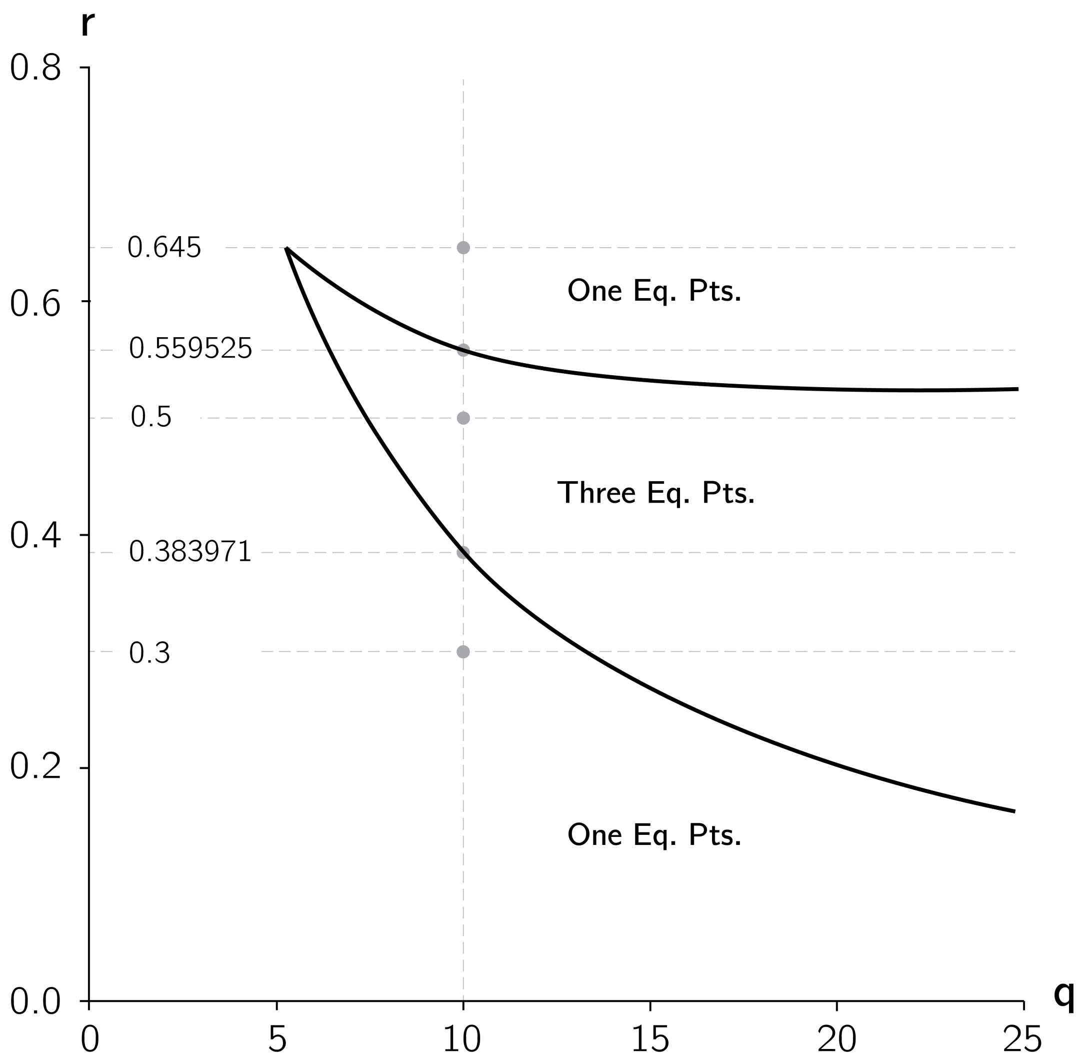

Figure 3.16: The bifurcation diagram of the Spruce Budworm model (Copied from Modeling the Spruce Budworm, Jacai deNeveu, 2015).

3.4.2 Phenomenological models

The cusp model of attitude is a typical phenomenological model. We simply assume that the cusp is a model of the attitude. Phenomenological models are less convincing than mechanistic models because they do not provide a deep understanding of the underlying mechanisms that drive the system. But in psychology and the social sciences we cannot be too picky. Compared to many other, verbally stated, attitude models, the cusp attitude model is quite precise. It implies a number of phenomena and is testable. This will be the subject of the next section.

Setting up a phenomenological model is not a trivial task either. I suggest some guidelines for setting up a model. First, define the behavioral variable. It is important to think about the bistable modes. What are they? What is the inaccessible state in-between? Can you have jumps between these states? What is neutral state at the back of the cusp? If you cannot answer these questions, you should reconsider whether a cusp is an appropriate model.

Second, select the control variables. What could be a normal variable and what could be a splitting variable. These are not easy questions. Sometimes there are too many candidates. For the cusp model of attitudes, instead of involvement, we could suggest interest, importance, emotional value, etc. In this case, I think of the splitting axis as a common factor of all these slightly different variables. In other cases, we have no good candidates. In the example in Figure 3.14, it is not clear exactly what is being manipulated along the normal axis. If you made a choice, it is good to check whether, at high values of the splitting values, variation of the normal variable may lead to sudden jumps and hysteresis. Also check whether the pitchfork bifurcation makes sense.

There is another issue here. In some phenomenological models the control variables are rotated by 45 degrees. The most famous example is Zeeman's (1976) model dog aggression.

The control variables are fear and rage (see Figure 3.17). In such a rotation the normal variable is the difference between fear and rage, while the splitting variable is the sum of fear and rage. Another example can be found in our model of the speed-accuracy trade- off in reaction time tasks (Dutilh et al. 2011). When constructing a phenomenological model, these two options for setting the control variable should be considered.

To explain catastrophic drops in performance in work and sports, Hardy and Parfitt (1991) proposed a cusp model with cognitive anxiety as the splitting factor and physiological arousal as the normal factor. The idea is that at high levels of cognitive anxiety, increases and decreases in arousal lead to sudden changes, including a hysteresis effect. Hardy (1996) present further tests of this model. It has been criticized by Cohen, Pargman, and Tenenbaum (2003). Extension of to the butterfly model are presented in Guastello (1984b) and Hardy, Woodman, and Carrington (2004).

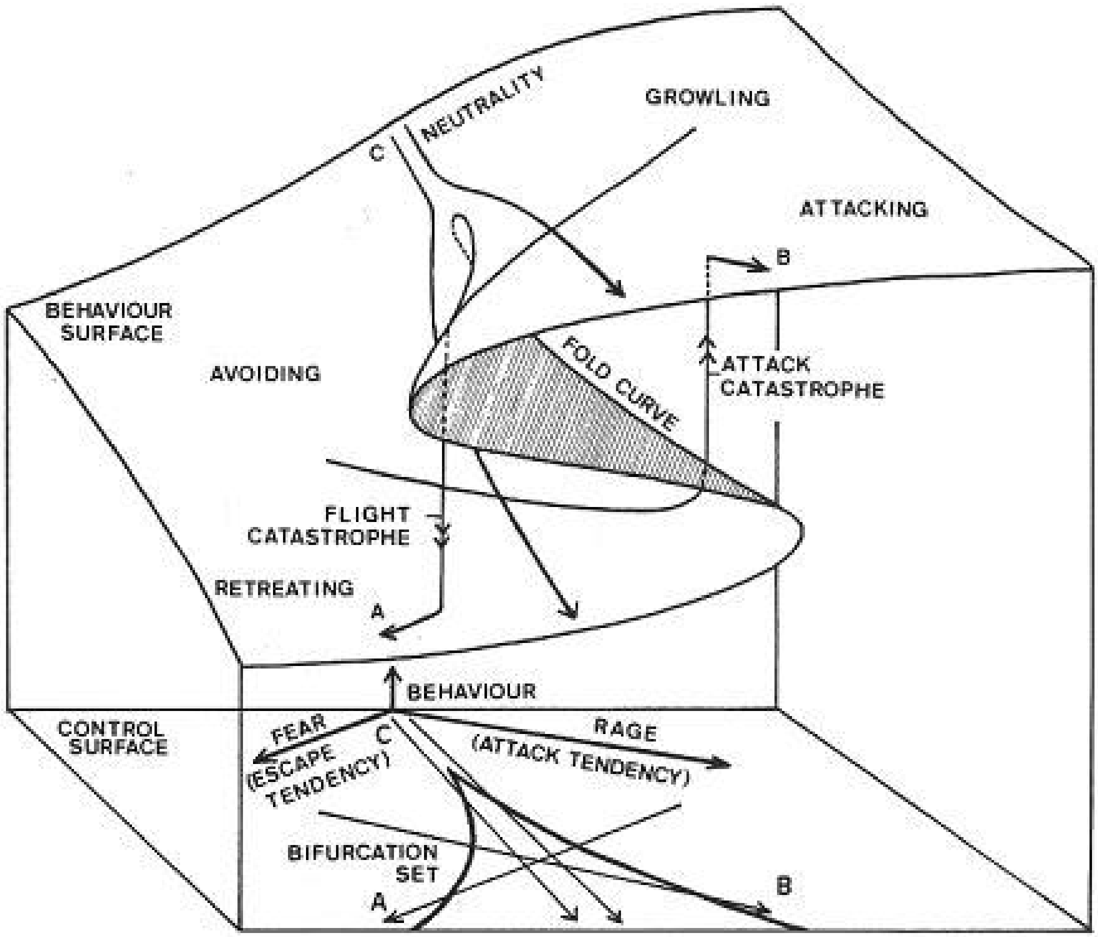

Figure 3.17: Zeeman (1976) dog aggression model with rage and fear as rotated control variables.

Cusp models have also been developed for addiction (Guastello 1984a; Mazanov and Byrne 2006). Witkiewitz et al. (2007)] propose using distal risk as the split axis and proximal risk as the normal axis. The model is tested using the renowned dataset from Project MATCH, an 8-year, multisite investigation of the effectiveness of various treatments for alcoholism.

As a final example I mention the model for humor presented by Paulos (2008) in his fascinating book on mathematics and humor.[5] Paulos explains his model in the context of puns. His example is "do you consider clubs appropriate for young children?" with the punchline "only when kindness fails," which is probably only funny to people with children. Paulos uses the rotated control axis as in the dog aggression model. Interpretation of the pun is the behavioral axis. One axis represents the first meaning of "clubs," the other axis represents the second meaning. The bifurcation set represents the ambiguous region. The punch line forces a catastrophic change in interpretation, accompanied by a release of tension through laughter. Paulos claims that this cusp model combines cognitive incongruity theory, various psychological theories of humor, and the release theory of laughter. Tschacher and Haken (2023) propose a related complexity account of humor.

3.5 Testing catastrophe models

3.5.1 The catastrophe flags

A question I often get is: How sudden is sudden? It is claimed that climate changes are transitions between stages (e.g., ice ages), but these transitions can take hundreds of years. Even when the ball is rolling towards its new minimum, it takes time to roll. So sudden transitions are not instantaneous.

But then what is the difference with an acceleration, such as we see in a logistic growth pattern? The time course of an acceleration and a sudden jump will look something like this:

Thus, in terms of time series data, they may look exactly the same. The difference is that in the continuous case the intermediate values are stable. It can be understood as a quadratic minimum that changes its position quickly. If we stop the process by freezing the manipulated control variable in the process, the state will remain at an intermediate value. These intermediate values are all stable values. If we freeze the manipulated variable in a discontinuous process, it will continue to move to a stable state. In this case, the intermediate state is unstable. The ball keeps rolling.

In practice using data, this is a difficult distinction to make. It means that simple time series are not sufficient to distinguish accelerations from phase transitions. So how do we distinguish

[5]Given these guidelines and examples it is an interesting exercise to develop one's own cusp model, for example, for falling in love. This is a tricky exercise.

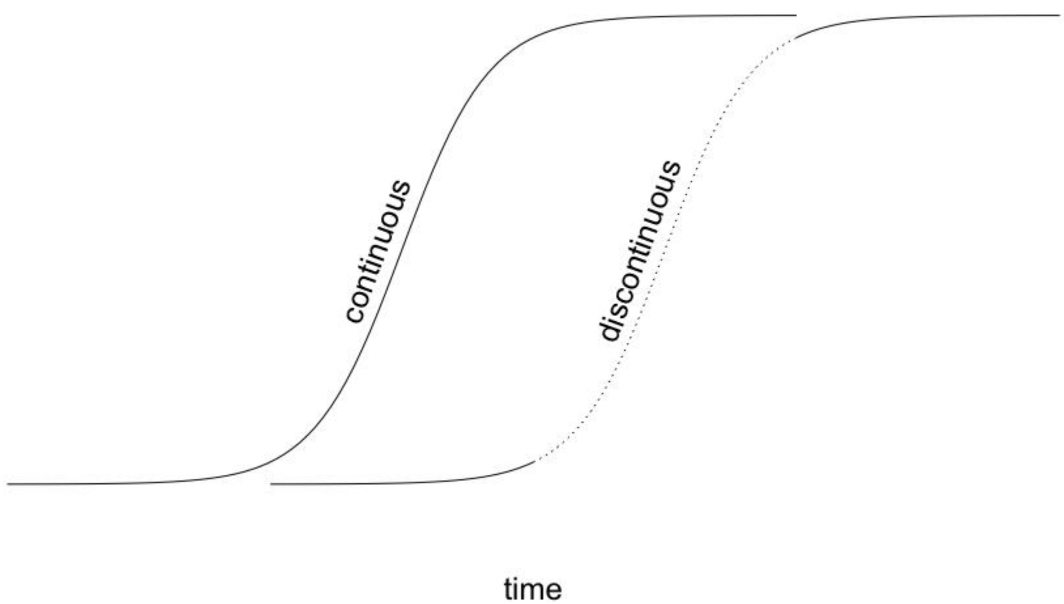

Figure 3.18: Continuous and discontinuous growth curves my look very similar.

between the two processes? In the context of catastrophe theory, Gilmore (1993) proposed the catastrophe flags. These are cusp related phenomena that can be seen in the data. While no single one of these is enough to indicate the cusp, when considered together they provide compelling evidence for its existence.

In the upcoming subsections, I will provide definitions for the flags and illustrate their applications in psychology using examples. The first flag is the sudden jump.

3.5.1.1 Sudden jump

The sudden jump is large fast change in equilibrium behavior. Although the sudden jump is not sufficient (it could be due to an acceleration), demonstrating a sudden jump in time series is useful (also in relation to other flags). Statistical detection of sudden jumps is possible using a number of techniques. Figure 3.19 presents raw weekly measurements of depressive symptoms using the SCL-90-R depression subscale of a patient who gradually quitted antidepressant medication during the study. The participant and researchers were blind to the dose reduction scheme (Wichers, Groot, and Psychosystems 2016). One question was whether this reduction led to a sudden jump to the depressed state. Using a change point detection method (James and Matteson 2014), we found a jump at 18 weeks with a bootstrapped p-value of .005 (with the null hypothesis of no change point).

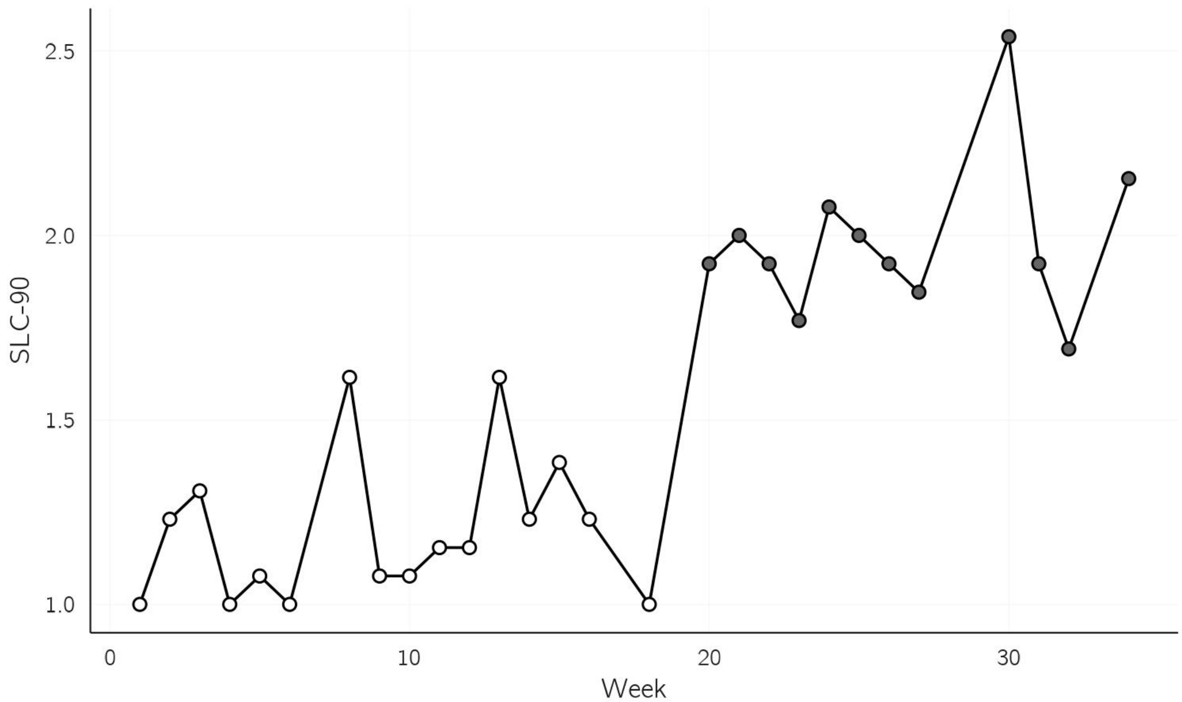

Figure 3.19: A sudden jump to depression (score at the SLC-90) in a patient who gradually quitted antidepressant medication during the study.

Many methods for change point analysis have been developed and compared in Burg and Williams (2022).

The code for this figure is:

```
layout(t(1)); par(mar=c(4,4,1,1))
x <- read.table('data/PNAS_patient_data.txt',header=T)
library(ecp) # if error: install.packages('ecp')
e1 <- e.divisive(matrix(x$dep,,1),sig=.01,min.size=10)
plot(x$week,x$dep,type='b',pch=(e1$cluster-1)*16+1,xlab='Week',ylab='SLC-90',bty='n',main=
```

3.5.1.2 Multimodality

Multimodality (in the case of the cusp bimodality) is an important and easy-to-use flag, as it can be tested with cross-sectional data. Finite mixture models have been developed to test for multimodality in frequency distributions (McLachlan, Lee, and Rathnayake 2019).

An example is shown in Figure 3.20. These data come from a conservation anticipation task, where children have to predict the level of water in the second glass when it is poured over. The resulting data and the fit of a mixture of two normal distributions are shown on the right. The data are clearly bimodal supporting the hypothesis of a transition in conservation learning (Van der Maas and Molenaar 1992b). These data were used in Dolan and van der Maas (1998) to fit multivariate normal mixture distributions subject to a structural equation model.

The code is:

```
x=unlist(read.table('data/conservation_anticipation_item3.txt'))
library(mixtools) # if error: install.packages('mixtools')
result=normalmixEM(x)
plot(result,whichplot=2,breaks=30)
```

There is a whole field in statistics focused on multimodality, mixtures and clustering. There are some blogs that present overviews of the relevant R-packages (Arnaud 2021). Several detailed examples from psychology, using hidden Markov models, are presented in Visser and Speekenbrink (2022).

The advantage of multimodality over the sudden jump is that we can use it with cross-sectional data. To capture a sudden jump in a development process, you need a lot of high-frequency data. Sudden shifts in opinion are also rare. But it is easy to collect data on large numbers of people who are asked to make judgments about statements on an issue such as abortion. If these judgments are bimodally distributed, this is consistent with a phase transition. Bimodal data may also be produced by a process of acceleration, with time series consisting mainly of data values before and after the acceleration. So, bimodality is not sufficient. It can be

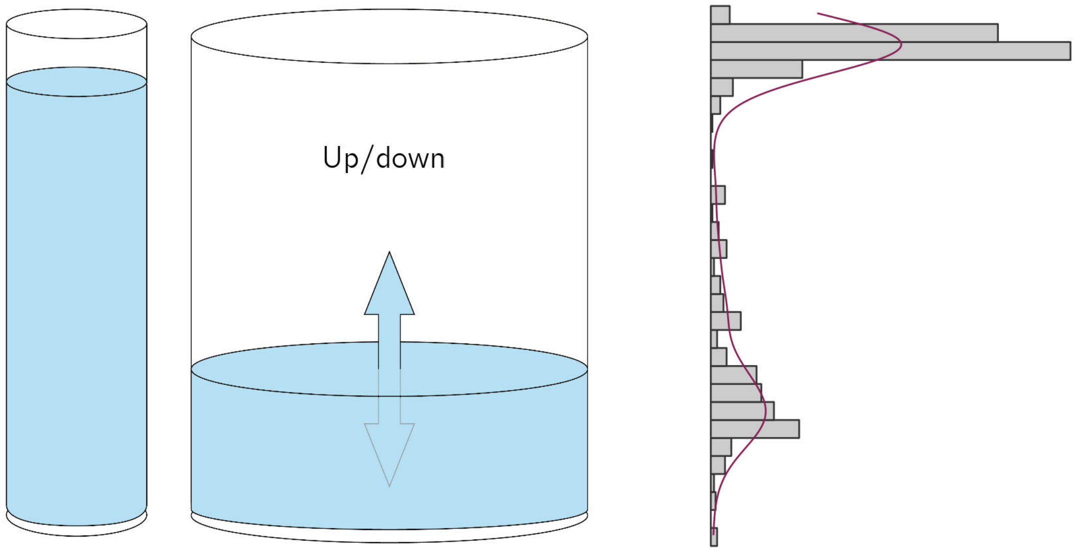

Figure 3.20: Bimodality in the expected heights of water when it is poured into a wider glass. This variation of the Piagetian maintenance task is used with children ages 5-8.

considered necessary, so I always suggest starting with cross-sectional multimodal studies. If they fail, you might reconsider your hypothesis. I have often looked for multimodality in measures of arithmetic learning and never found anything convincing, which made me rethink my hypothesis.

3.5.1.3 Inaccessibility

Inaccessibility means that certain values of the behavioral variable are unstable. The business card is a good example. Given some vertical pressure, we can try what we want but we cannot force the card to stay in the middle position, it is unstable. Inaccessibility is relevant to reject the alternative hypothesis that the sudden jump and bimodality are due to an acceleration.

In Experiment 2 of Dutilh et al. (2011) we focused on this flag. Our hypothesis was that in simple choice response tasks there is a phase transition between a fast-guessing state and a slower stimulus-driven response state. The idea is that if we force subjects to speed up, there will be a catastrophic jump in performance (from almost 100% correct to 50% correct).

We created a game in which subjects responded to a series of simple choice items (a lexical decision task). The length of the series was not known to the subject. At the end of a series, they were rewarded according to how close their percentage correct was to 75%. Speed was also rewarded, but much less. So, we asked the subject to be in the inaccessible state. The alternative hypothesis, based on information accumulation models, was that there was no phase transition and that responding with 75% accuracy required the correct setting of a boundary (see Section 5.2.1).

It appeared that subjects solved the task by switching between the fast-guessing mode and the slower stimulus-controlled mode, even when instructed according to the alternative model. Thus, the 75% intermediate state appears to be unstable.

3.5.1.4 Divergence

Divergence or the pitchfork bifurcation requires the manipulation of the splitting variable. In the case of attitudes, we hypothesize this to be involvement or some related variable. In Han L. J. van der Maas, Kolstein, and van der Pligt (2003) we re-analyzed a dataset from Stouffer et al. (1949), which Latané and Nowak (1994) presented as evidence for the cusp model. The attitude concerned demobilization (from 0, unfavorable, to 6, favorable) and respondents were asked to indicate how strongly they felt about their answer (from intensity 0 to intensity 5). For low intensities of feeling the data are normally distributed whereas for higher intensities data are bimodally distributed (see Figure 3.21).

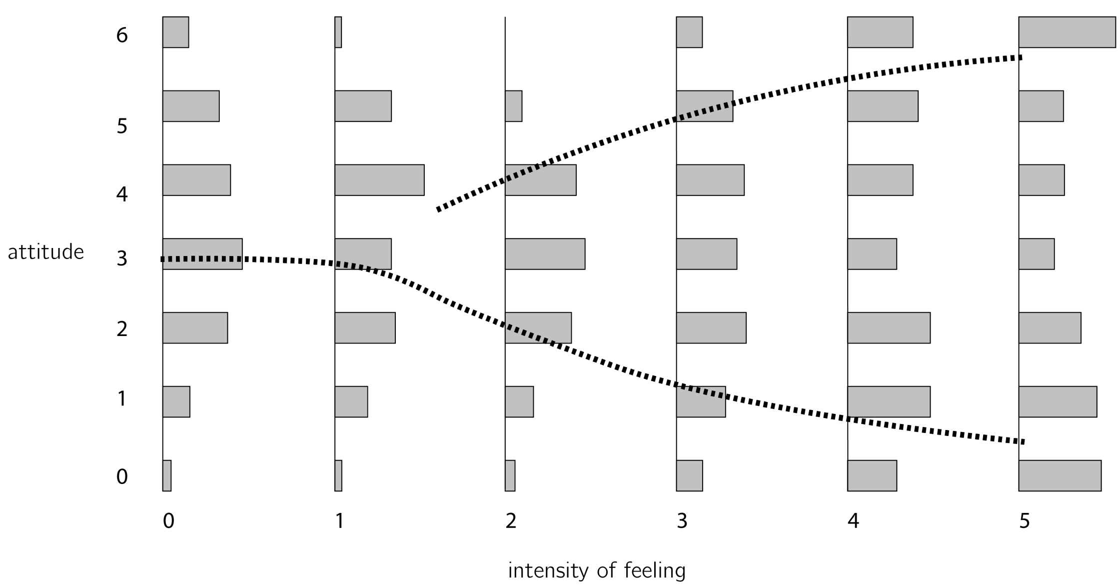

Figure 3.21: The pitchfork bifurcation in attitudes. The dotted lines represent the fit of the cusp model to these data. This technique will be discussed in the final section in this chapter.

3.5.1.5 Hysteresis

Hysteresis, **the lagging behind of the sudden jump,** requires sophisticated manipulation of the normal control variable. We need to slowly increase and decrease this variable and test whether sudden jumps occur with a delay. **We have demonstrated hysteresis in proportional reasoning using Piaget's balance scale test in which a specific dimension (distance from the fulcrum) was systematically varied (Jansen and Van der Maas 2001a). We also hypothesized that speeding up subjects in response time tasks would eventually lead to a catastrophe in accuracy. To support this claim, we demonstrated bimodality in response times and hysteresis in the speed-accuracy trade-off (Dutilh et al. 2011).** To support the cusp model of multistable perception we used the quartet motion paradigm (Ploeger, Van Der Maas, and Hartelman 2002). In this perceptual paradigm two lights are presented simultaneously, first a pair from two of the diagonally opposite corners of the rectangle, and then a second pair from the other two diagonally opposite corners of the rectangle. Usually, either vertical or horizontal apparent motion is perceived. By gradually increasing or decreasing the aspect ratio (i.e., the ratio of height to width of the quartet), hysteresis in the jumps between the two percepts was demonstrated (see Figure 3.22).

In Ploeger, Van Der Maas, and Hartelman (2002) we used a special design, the method of modified limits, to rule out the alternative explanation that hysteresis is simply due to delayed responses. It could be that the switches always occur in the middle (at an aspect ratio of 1), but the self-report is delayed. In the modified limits method, subjects do not respond during a trial, but only after the entire trial. By varying the length of the trials, it is possible to determine at which parameter value the subject perceives a switch.

3.5.1.6 Anomalous variance, divergence of linear response and critical slowing down

Gilmore's last three flags, anomalous variance, divergence of linear response and critical slowing down, are indicators that occur near the bifurcation lines. They are also known as early warning systems and are a popular line of research (Dakos et al. 2012).

Anomalous variance occurs because near a bifurcation point the second derivative diminishes, meaning that the minimum becomes less deep. Assuming there is always some perturbation of the state, this will lead to larger fluctuations in the state.

Divergence of linear response is the size of the effect of a small perturbation of the state, which will be greater near a bifurcation point. It will also take longer to return to equilibrium. The latter is known as critical slowing down and is also studied in other approaches to nonlinear dynamical systems (e.g., synergetics (Haken 1977). Examples of applications in psychology can be found in Leemput et al. (2014) and Olthof et al. (2020). A somewhat critical review is provided in Dablander et al. (2023).

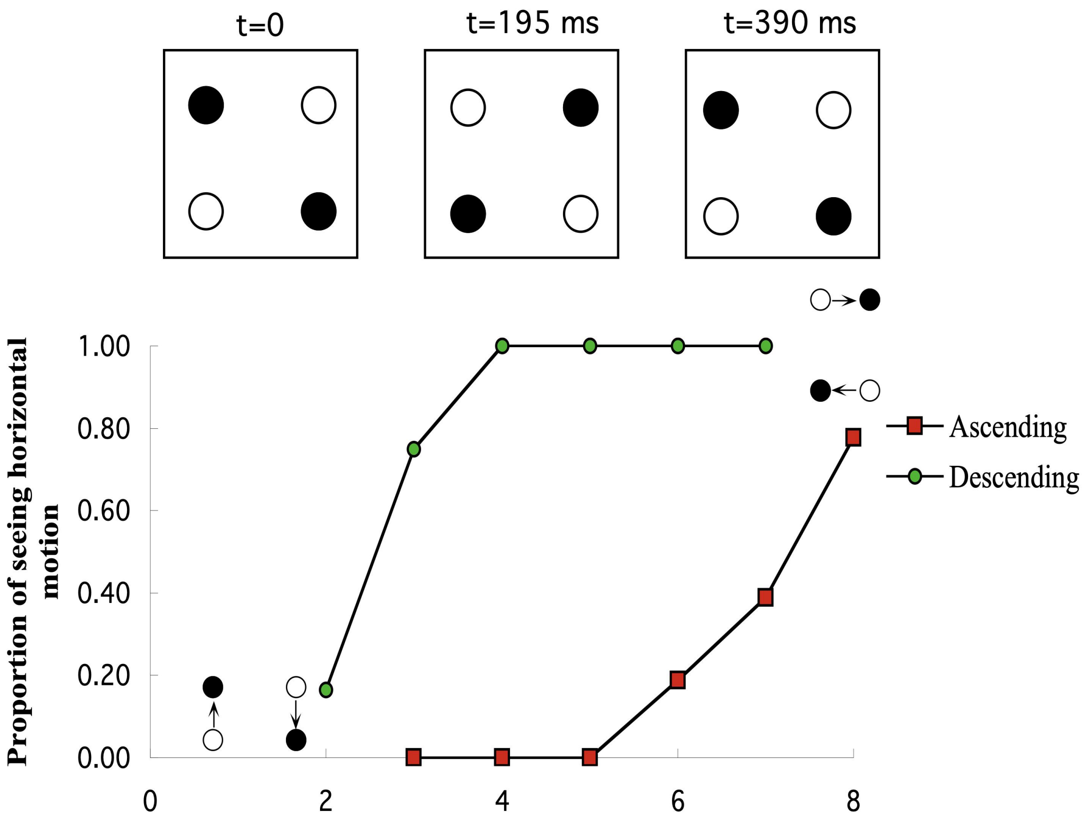

Figure 3.22: Hysteresis in the perception of apparent motion. Switches between the perception in of vertical or horizontal apparent motion occur when the aspect ratio (horizontal axis) is varied. The aspect ratio is the ratio of height to width of the quartet.

In my experience, the problem with early warning signals is that both type 1 and type 2 errors should be low for predicting transitions. This is challenging even in simulations, let alone in noisy psychological data. It would be fantastic if these early warnings really worked. For example, being able to predict a relapse into depression or addiction would be of great therapeutic value.

The catastrophe flags together provide a research methodology for phase transition research in psychology. A single flag may not be sufficient, but the combination is. For example, the combination of evidence for inaccessibility and hysteresis is convincing. I have given psychological examples of most of the flags. Which flags to use in a particular case depends on the knowledge and experimental control of the control variables. Another approach is to fit the cusp model directly to the data. This is the subject of the next section.

3.5.2 Fitting the cusp to cross-sectional data

3.5.2.1 Cobb's maximum likelihood approach

In a series of papers Loren Cobb an colleagues (Cobb and Zacks 1985; Cobb 1978) developed a maximum likelihood approach to fit the cusp catastrophe to data consisting of cross-sectional measurements of X, a and b. We have implemented this approach in a cusp R package described in Grasman, van der Maas, and Wagenmakers (2009).

The basic idea is to make catastrophe theory, a deterministic theory, stochastic by adding a stochastic term, called Wiener noise (with variance σ^2), to Equation 3.2[6]:

$$dX = -V^{'}(X)dt + \sigma dW(t). \tag{3.9}$$

We will discuss this type of stochasticity later, but it is important to note that it is not the same as measurement noise. Measurement noise, i.e., ε in $Y = X + \varepsilon$, does not affect the dynamics of X. Wiener noise does, it is part of the updating equation of X itself. This stochastic differential equation is associated with a probability distribution of the form

$$f(X) = \frac{1}{Z\sigma^2} e^{\frac{-V(X)}{\sigma^2}}, \tag{3.10}$$

where Z is a normalizing constant[7] necessary to ensure that the area under $f(X)$ is 1. This may look complicated but for the quadratic case $V(X) = \frac{1}{2}X^2$, this results in the standard normal distribution, with $Z = \frac{\sqrt{2\pi}}{\sigma}$.

[6] Many different t notations exist for his. Perhaps clearer is $dX(t) = -V^{'}(X(t))\,dt + \sigma dW(t)$, as dX and dW depend on time.

[7] For consistency with later chapters, I define Z differently from the notation in (Grasman, van der Maas, and Wagenmakers 2009). It is the inverse of Z in that paper.

As in the case of the normal distribution we want to allow for some transformations of the variables. To simplify the necessary statistical notation, we write the cusp as $V(y) = -\alpha y - \frac{1}{2}\beta y^2 + \frac{1}{4}y^4$. The probability distribution for the cusp is

$$f(y) = \frac{1}{Z\sigma^2} e^{\frac{\alpha y + \frac{1}{2}\beta y^2 - \frac{1}{4}y^4}{\sigma^2}}. \tag{3.11}$$

As in regression models, the cusp variables are modelled as linear function of measured variables. That is, the dependent variables $Y_{i1}, \ Y_{i2}, \ldots, Y_{ip}$ and the independent variables $X_{i1}, \ X_{i2}, \ldots, X_{iq}$, for subjects $i = 1, \ldots, n$, are related to the cusp variables as follows:

$$y_i = w_0 + w_1 Y_{i1} + w_2 Y_{i2} + \ldots + w_p Y_{ip} \alpha_i = a_0 + a_1 X_{i1} + a_2 X_{i2} + \ldots + a_q X_{iq} \beta_i = b_0 + b_1 X_{i1} + b_2 X_{i2} + \ldots + b_q X \tag{3.12}$$

By estimating these regression parameters, we fit the cusp model to empirical data. The cusp package in R makes this possible. I will first demonstrate this using simulated data. This is my number one rule when using statistical techniques: never use a statistical technique on real data before you have tested it on simulated data. First, it forces you to understand what the statistical technique actually does, and second, it gives you a way to test the power and investigate violations of the technique's assumptions.

```
library(cusp) # if error: install.packages('cusp')
set.seed(1)
X1 <- runif(1000) # independent variable 1
X2 <- runif(1000) # independent variable 2
# to be estimated parameters
w0 <- 2; w1 <- 4; a0 <- -2; a1 <- 3; b0 <- -2; b1 <- 4
# sample Y1 according to cusp using rcusp and the chosen parameter values
Y1 <- -w0/w1 + (1/w1) * Vectorize(rcusp)(1, a0+a1*X1, b0+b1*X2)
data <- data.frame(X1, X2, Y1) # collect ‘measured’ variables in data
```

I recommend doing some descriptive analysis first. With `hist(data$Y1)` we can inspect whether there is some indication of bimodality. X2 is the splitting variable so perhaps we see stronger bimodality with `hist(data$Y1[data$X2>mean(data$X2)])`. The function pairs in R, `pairs(data)`, is also always recommended. In this perfect simulated case, you will already see strong indications of the cusp. Now we fit the full model with alpha and beta both as function of X1 and X2.

```
fit <- cusp(y ~ Y1, alpha ~ X1+X2, beta ~ X1+X2, data)
summary(fit)
```

Summary provides the following table:

Table 3.1: Table 1: The parameter estimates including standard errors and p-values generated by the cusp package.

Coefficients:	Estimate	Std.Error	z-value	Pr(>|z|)	
a[(Intercept)]	-2.13	0.19	-11.0	< 2e-16	***
a[X1]	3.11	0.22	14.2	< 2e-16	***
a[X2]	0.15	0.17	0.9	0.39	
b[(Intercept)]	-2.29	0.34	-6.7	2.66e-11	***
b[X1]	-0.09	0.33	-0.3	0.79	
b[X2]	4.40	0.27	16.5	< 2e-16	***
w[(Intercept)]	1.98	0.07	27.6	< 2e-16	***
w[Y1]	3.97	0.10	38.0	< 2e-16	***

Note that we fit a model with too many parameters. We also estimated a_2 and b_1 (because the model was specified as alpha ~ X1+X2, beta ~ X1+X2). These estimates are not significantly different from 0. The other parameters are estimated reasonably close to their true values, since the true values fall within the confidence interval of the estimates (defined by twice the standard error on either side). We expect a better fit in terms of AIC and BIC if we fit a reduced model without a_2 and b_1. **These fit measures penalize goodness of fit (e.g., the log-likelihood) for the number of parameters used.**

```
fit_correct_model <- cusp(y ~ Y1, alpha ~ X1, beta ~ X2, data)
summary(fit_correct_model)
```

Table 3.2: Table 2: The comparative fit measures, AIC, AICc, and BIC indicate that the reduced model should be the model of choice.

	R.Squared	logLik	npar	AIC	AICc	BIC
Full model	0.428	-1058.7	8	2133.3	2133.5	2172.6
Reduced model	0.426	-1059.0	6	2130.0	2130.1	2159.5

The next simulation demonstrates that we can detect hysteresis using this approach. We simulate data with $-2 < \alpha < 2$, and fixed β. If $\beta < 0$ we have no hysteresis, but if $\beta > 0$ we do have hysteresis. With the code below we simulate datasets for different β, and compare the goodness of fit between the linear and cusp model. The figure summarizes the results. Note that a lower BIC indicated the better fitting model.

```
set.seed(10)
n <- 500
X1 <- seq(-1,1,le=n) # rnorm(n) #runif(1000) # independent variable 1
```

```r
a0 <- 0; a1 <- 2; b0 <- 2 # to be estimated parameters
b0s <- seq(-1,2,by=.25)
i <- 0
dat <- matrix(0,length(b0s),7)
for (b0 in b0s)
{
  i <- i + 1
  Y1 <- Vectorize(rcusp)(1, a1 * X1, b0)
  data <- data.frame(X1, Y1) # collect ‘measured’ variables in data
  fit <- cusp(y ~ Y1, alpha ~ X1, beta ~ 1, data)
  sf <- summary(fit)
  dat[i, ] <- c(b0, sf$r2lin.r.squared[1], sf$r2cusp.r.squared[1],sf$r2lin.bic[1], sf$r2cu
}
par(mar=c(4,5,1,1))
matplot(dat[,1],dat[,4:5],ylab='Bic',xlab='b0',bty='n',type='b',pch=1:2,cex.lab=1.5)
legend('right',legend=c('linear','cusp'),lty=1:2,pch=1:2,col=1:2,cex=1.5)
abline(v=0,lty=3)
text(-.5,800, 'no hysteresis',cex=1.5)
text(.5,800, 'hysteresis',cex=1.5)
```

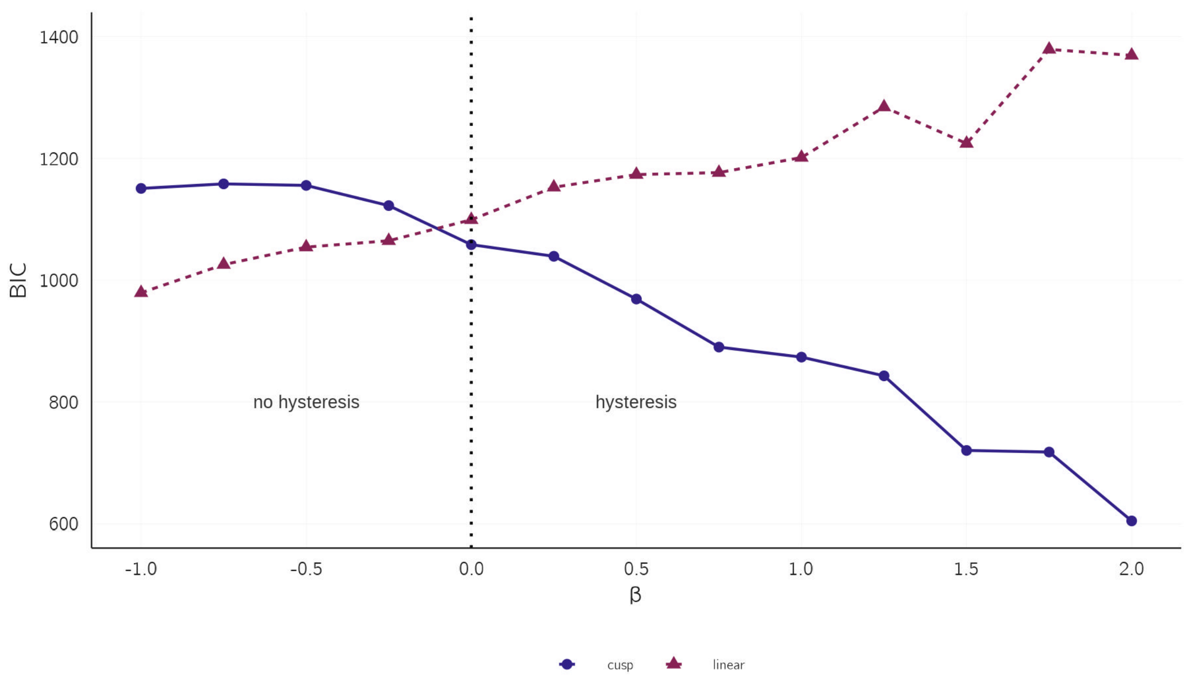

Figure 3.23: At the back of the cusp (low $b0$), the cusp is approximately linear and cuspfit prefers this simpler model over the cusp model.

3.5.2.2 Empirical examples

In Grasman, van der Maas, and Wagenmakers (2009) we present several examples with real data. As another example, we use Stoufer's data, which we used as an example of divergence before (see Figure 3.21).

```
x <- read.table('data/stoufer.txt')
colnames(x) <- c('IntensityofFeeling','Attitude')
fit <- cusp(y ~ Attitude, alpha ~ IntensityofFeeling, beta ~ IntensityofFeeling, x)
summary(fit)
```

Inspection of the parameter estimates shows that, as expected, Intensity of Feeling only loads on the splitting axis and not on the normal axis. This is the location of the data in the bifurcation set (`plot(fit)`).

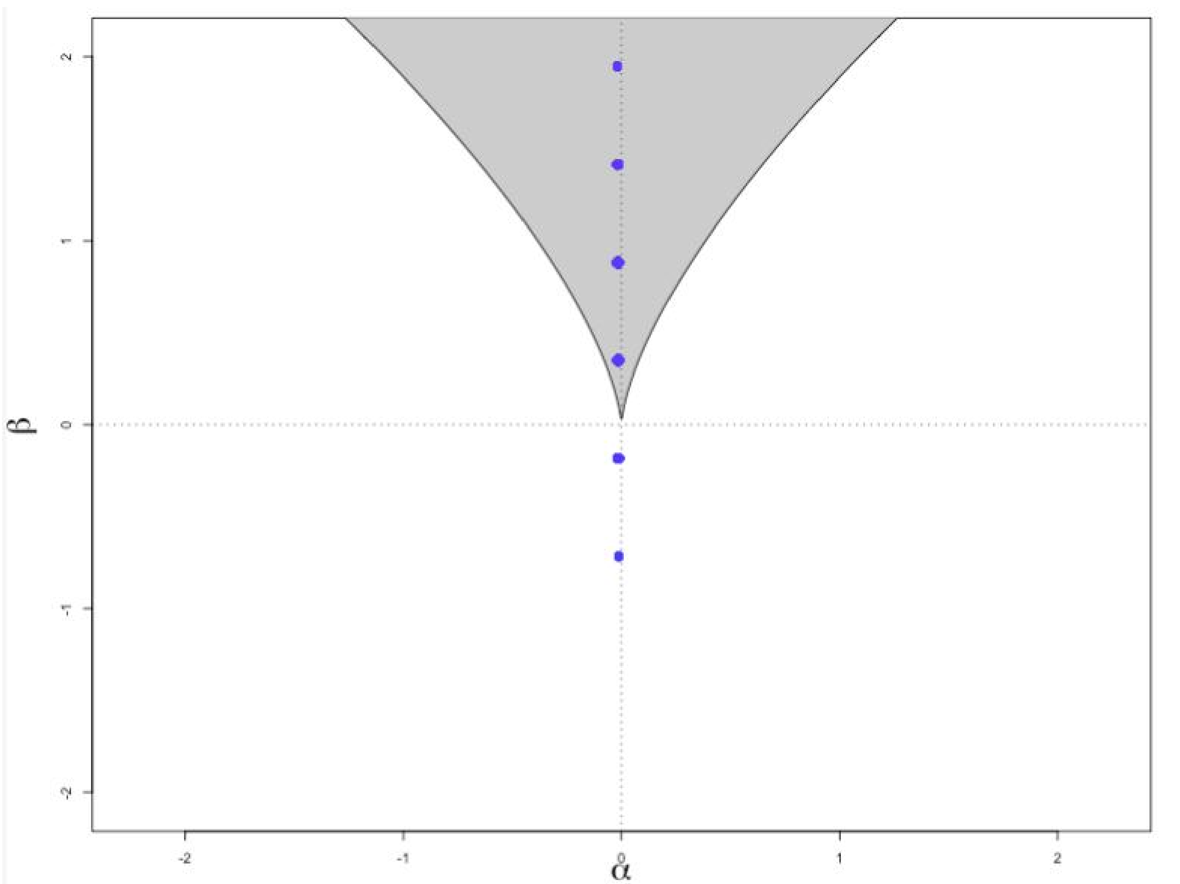

Figure 3.24: Placement of the data of Figure 3.21 in the bifurcation set.

Another example is the conservation data set of Bentler (1970), which contains the scores on a 12 item test from a conservation test of 560 children from 8 different age groups. These data are expected to be bimodal and to move along the normal axis (Van der Maas and Molenaar 1992b).

```
x <- read.table('data/bentler.txt',header=T)
layout(t(1:8))
age <- c('age 4 to 4.5','age 4.5 to 5','age 5 to 5.5','age 5.5 to 6','age 6 to 6.5','age 6
for(i in 1:8)
{
  if(i==1) {par(mar=c(4,3,2,1));names=0:12} else {names='';par(mar=c(4,1,2,1))}
  barplot(table(factor(x[x[,1]==i,2],levels=0:12)),horiz=T,axes=F,main=age[i],xlab='',name
}
fit <- cusp(y ~ score, alpha ~ age_range, beta ~ age_range, x)
summary(fit)
plot(fit)
```

This is supported by results of cuspfit. You can verify that a model with beta ~ 1, fits better according to the AIC and BIC.

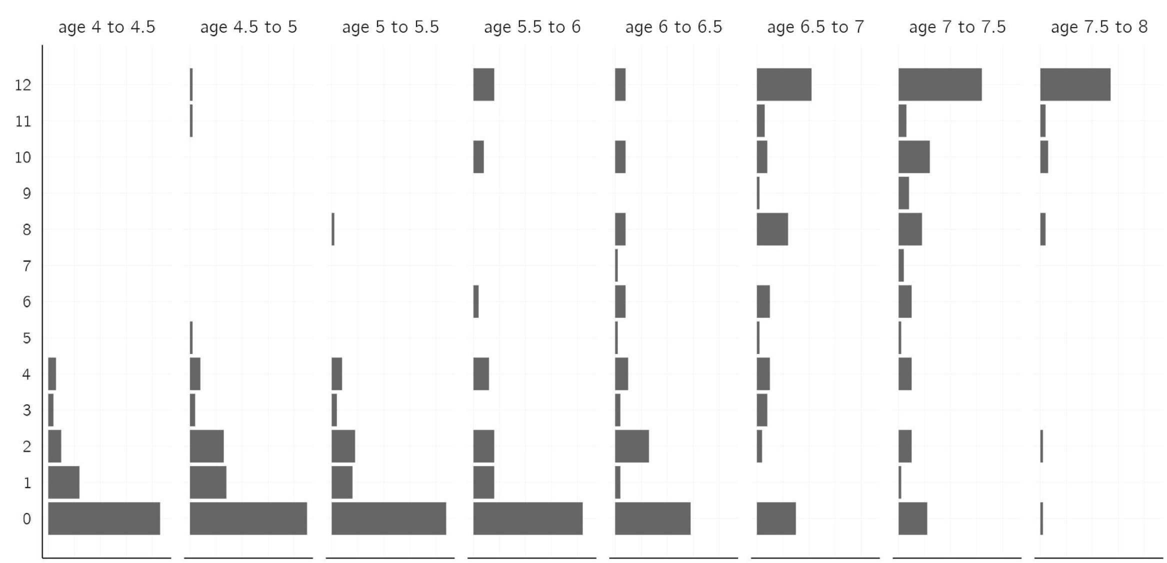

Figure 3.25: The fit of Bentler conservation data.

A great exercise we have often used in the classroom is to build a Zeeman machine, collect data with the machine and fit the cusp model to the data (see Grasman, van der Maas, and Wagenmakers (2009), for details). This machine was invented by Zeeman to demonstrate the properties of the cusp. Our students were rewarded for the quality of the model and the artistic value of their Zeeman machine.

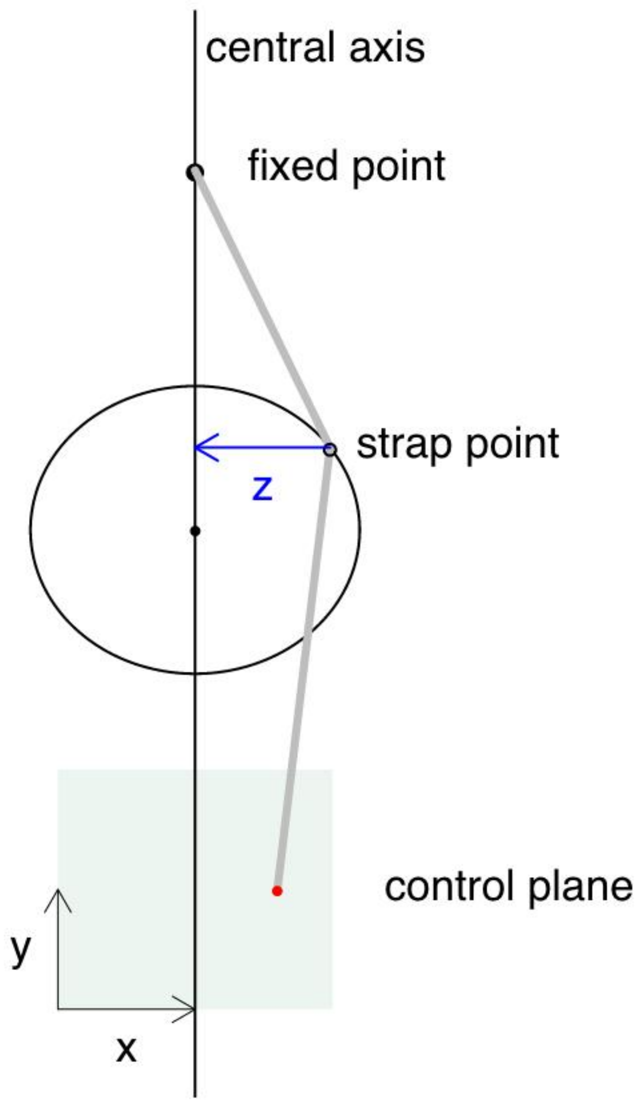

Figure 3.26: Zeemans's catastrophe machine.**It consists of a rotating disk and two elastic bands. The first elastic band is attached to a fixed point and the strap point. The end of the other elastic (red dot) is moved by hand through the control plan.** The strap point moves according to the cusp catastrophe. Data is gathered by collecting a set of X, Y and Z values. Typically, 50 to 100 data points are sufficient to run Cuspfit.

3.5.2.3 Evaluation

I make a few final remarks. First, Cobb's method is not valid for time series. Data points should be independent. To test for hysteresis in time series other approaches are required. One option is to use hidden Markov models as in Dutilh et al. (2011).

The second remark is that there are some issues with the approach of Cobb that are due to fundamental differences between probability distributions and potential functions. The latter can be transformed in many ways (so called local diffeomorphisms) without changing the qualitative properties of the cusp. With the added constraint on probability distributions (area = 1), the same transformations can lead to qualitative effects, such as a change in the number of modes. Wagenmakers et al. (2005) suggest a solution to this problem for time series.

The third remark is that two alternative approaches have been proposed. Both Guastello's (1982) change score least square regression approach and the Gemcat approach (1987) use the first derivative of the cusp as point of departure. A problem with both approaches is that they do not distinguish between stable and unstable equilibrium states. Data points in the inaccessible region improve the fit of the model, whereas they should decrease the fit. Alexander et al. (1992) provide a detailed critique.

3.6 Criticism of catastrophe theory

One often speaks of the rise and the fall of catastrophe theory (Rosser 2007). The hype following Zeeman's famous Scientific American paper[8] (Zeeman 1976), **in which he introduced the phenomenological application of catastrophe theory in the behavioral and social sciences,** led to a strongly worded reply by Zahler and Sussmann (1977) in Nature.

Because people still refer to this paper when we use catastrophe theory in our work, let us briefly respond to the main points of criticism made by Zahler and Sussmann. I note that in the introduction to their paper they state that there may be legitimate uses of catastrophe theory in physics and engineering. Moreover, they do not question the correctness or importance of catastrophe theory as a purely mathematical subject.

They raise ten points, some of which we have already addressed. For example, their first point is about how sudden a jump actually is, but they call this a less serious criticism. As I explained earlier, it is not the suddenness that matters, but whether or not the intermediate states are unstable.

A number of points are about inferring a cusp from data, which was indeed done rather superficially in Zeeman's earlier work. They pointed out that there are no testable predictions,

[8] I recommend the BBC documentary "Case Study Catastrophe Theory Maths Foundation Course" (https://www.youtube.com/watch?v=myDvcvox1V4&t=1435s) to see Zeeman at work.

that the location of the cusp can be shifted, and that there is no way to decide whether the data fit the cusp. We hope to have shown that these problems are largely solved. The catastrophe flags allow us to make new testable predictions, and with Cobb's maximum likelihood approach we can fit the model as we would any statistical model in modern science. Of course, one can be critical of the use of statistics in psychology and the social sciences, but these criticisms are not specific to catastrophe theory.

Another somewhat inconsistent line of criticism is that many catastrophe models in psychology and the social sciences are just wrong and inconsistent with the data (which could be true), while it is also claimed that Zeeman's models are not falsifiable. But you cannot have it both ways, if it is wrong or inconsistent with the data, it is falsifiable. Nevertheless, I agree that it is important to think about falsifiability. Theories in psychology tend to be moving targets. As soon as someone finds an empirical result that contradicts the theory, the theory is quickly modified.

Then they point out that catastrophe theory theorists often try to make a discrete variable into a continuous one. Their example is aggression, which they believe is inherently discrete. They call Zeeman's interpretation of aggression as a continuous family of behaviors absurd and utterly meaningless. This may be a bit strong. We can think of situations in which aggression can vary from mild to severe. Aggression can vary from verbal to physical, directed at a person's belongings, mild physical directed at the person, to severe physical. **A rich ordering of aggressive acts is very useful for describing domestic violence. Sometimes the change along these acts or variants is gradual, while in other circumstances the change is sudden. Whether such an ordering can be treated as a quantitative continuum is one of the most difficult questions in our field** [Chapter 8; D. Borsboom et al. (2016); Michell (2008)].

The last point of Zahler and Sussmann is that there are better alternatives, such as quantum mechanics, discrete mathematics, and bifurcation theory. There is work on quantum mechanics in psychology (especially in the context of consciousness), but whether this will lead to breakthroughs in this field remains to be seen. Discrete mathematics may be an alternative in some cases (e.g., to model symbolic thinking). I see catastrophe theory as a special branch of bifurcation theory, especially useful when the system under study is difficult to describe in terms of mathematical equations. This goes back to the distinction between phenomenological and mechanistic models. I think we should put more effort into developing mechanistic models based on first principles. More on this in the next chapters.

Loehle (1989) present an excellent discussion on the usefulness of catastrophe theory in the context of modelling ecosystems. He concluded that "an unresolved problem in applying catastrophe models is that of testing the goodness of fit of the model to data", but this problem has now been largely solved.

3.7 Conclusion

Psychologists are often concerned with psychological types and classes, stages and phases, and the transitions between them. Our thinking about transitions becomes much clearer and more advanced when we know the basics of bifurcations.

Catastrophe theory comes with a toolbox for behavioral and social sciences. We can build phenomenological models, test for catastrophe flags, and even fit cusp models to data. With the development of this toolbox, most of the criticisms of catastrophe theory lose their relevance.

However, there is room for improvement. Phenomenological models have limited explanatory power. As explained in the next chapter on dynamical system models, it is possible to create more mechanistic models that support the use of phenomenological models. In Chapter 6, another option is introduced using networks. It will be demonstrated that the behavior of the Ising network model for attitudes is governed by the cusp model, which is very similar to the cusp model proposed for the attitude toward abortion.

3.8 Exercises

1) The equilibria of the fold are $X = \pm\sqrt{\frac{a}{3}}$. This can can be checked by setting the first derivative to zero. Show this. (*)
2) In Zeeman's dog aggression model fear and rage are 'rotated' control variables. How can we translate this to a model with unrotated axes? Provide the equations that specify the normal and splitting axis as function of fear and rage. (*)
3) Derive the equation for the bifurcation lines of the cusp ($27a^2 = 4b^3$), by setting the first and second derivatives to zero. Plot the bifurcation lines in GeoGebra or Desmos. (**)
4) Some insight into the butterfly catastrophe $V(X) = -aX - bX^2 - cX^3 - dX^4 + X^6$ can be gained by entering the equation in free online graphing calculators such as DesMos or GeoGebra. Set a, b, c, d to 0, -5, 0, 5. Then start varying a and c. What is the difference in the effect of these two parameters **on the appearance and disappearance** of attractors? (*)
5) Set up a phenomenological cusp for falling in love. Follow my guidelines (see Section 3.3.1). (**)
6) Check whether indeed the Bentler data fit better with when age_range only loads on the normal axis (according to the AIC and BIC). What is the correct specification of beta in cusp() in this case? (*)

7)
```
n=500
z = Vectorize(rcusp)(1, .7*rnorm(n), 2+2*rnorm(n)) # sample z
x = rnorm(n)
y = rnorm(n)
data <- data.frame(z,x,y) # collect variables in data
```

What is the best fitting cusp model (according to the BIC) for this tricky data set? Why? (**)

8) Build a Zeeman machine, collect data and fit the cusp (see Example III of Grasman, van der Maas, and Wagenmakers 2009). What is your best fitting model? Provide a plot of the data in the bifurcation set and a picture of your Zeeman machine. (**)

References

Alexander, Ralph A., Glenn R. Herbert, Richard P. DeShon, and Paul J. Hanges. 1992. "An Examination of Least-Squares Regression Modeling of Catastrophe Theory." *Psychological Bulletin* 111: 366–74. https://doi.org/10.1037/0033-2909.111.2.366.

Anderson, P. W. 1972. "More Is Different." *Science* 177 (4047): 393–96. https://doi.org/10.1126/science.177.4047.393.

Arnaud, M. 2021. "Mixture Modelling from Scratch, in R." *Medium*. https://towardsdatascience.com/mixture-modelling-from-scratch-in-r-5ab7bfc83eef.

Attwell, David, and Costantino Iadecola. 2002. "The Neural Basis of Functional Brain Imaging Signals." *Trends in Neurosciences* 25 (12): 621–25. https://doi.org/10.1016/S0166-2236(02)02264-6.

Ayers, Susan. 1997. "The Application of Chaos Theory to Psychology." *Theory & Psychology* 7 (June): 373. https://doi.org/10.1177/0959354397073005.

Banks, J., J. Brooks, G. Cairns, G. Davis, and P. Stacey. 1992. "On Devaney's Definition of Chaos." *The American Mathematical Monthly* 99 (4): 332–34. https://doi.org/10.1080/00029890.1992.11995856.

Barceló, Jaume, ed. 2010. *Fundamentals of Traffic Simulation*. Vol. 145. International Series in Operations Research & Management Science. New York, NY: Springer New York. https://doi.org/10.1007/978-1-4419-6142-6.

Bechtel, William, and Adele Abrahamsen. 2005. "Explanation: A Mechanist Alternative." *Studies in History and Philosophy of Science Part C: Studies in History and Philosophy of Biological and Biomedical Sciences*, Mechanisms in biology, 36 (2): 421–41. https://doi.org/10.1016/j.shpsc.2005.03.010.

Bentler, P. M. 1970. "Evidence Regarding Stages in the Development of Conservation." *Perceptual and Motor Skills* 31 (3): 855–59. https://doi.org/10.2466/pms.1970.31.3.855.

Bertalanffy, Ludwig Von. 1969. *General System Theory: Foundations, Development, Applications*. Revised edition. New York, NY: George Braziller Inc.

Bertschinger, Nils, and Thomas Natschläger. 2004. "Real-Time Computation at the Edge of Chaos in Recurrent Neural Networks." *Neural Computation* 16 (7): 1413–36. https://doi.org/10.1162/089976604323057443.

Boostani, Reza, Foroozan Karimzadeh, and Mohammad Nami. 2017. "A Comparative Review on Sleep Stage Classification Methods in Patients and Healthy Individuals." *Computer Methods and Programs in Biomedicine* 140 (March): 77–91. https://doi.org/10.1016/j.cmpb.2016.12.004.

Borsboom, Denny, Angélique O. J. Cramer, and Annemarie Kalis. 2019. "Brain Disorders? Not Really: Why Network Structures Block Reductionism in Psychopathology Research."

Behavioral and Brain Sciences 42: e2. https://doi.org/10.1017/S0140525X17002266.

Borsboom, Denny, Han L J van der Maas, Jonas Dalege, Rogier A Kievit, and Brian D Haig. 2021. "Theory Construction Methodology: A Practical Framework for Building Theories in Psychology." *Perspectives on Psychological Science* 16 (4): 756–66. https://doi.org/10.1177/1745691620969647.

Borsboom, D., M. Rhemtulla, A. O. J. Cramer, H. L. J. van der Maas, M. Scheffer, and C. V. Dolan. 2016. "Kinds *Versus* Continua: A Review of Psychometric Approaches to Uncover the Structure of Psychiatric Constructs." *Psychological Medicine* 46 (8): 1567–79. https://doi.org/10.1017/S0033291715001944.

Broer, Henk, and Floris Takens. 2010. *Dynamical Systems and Chaos*. 2011th edition. New York: Springer.

Burg, Gerrit J. J. van den, and Christopher K. I. Williams. 2022. "An Evaluation of Change Point Detection Algorithms." arXiv. https://doi.org/10.48550/arXiv.2003.06222.

Chalmers, David J. 2006. "Strong and Weak Emergence." https://philpapers.org/rec/chasaw.

Cobb, Loren. 1978. "Stochastic Catastrophe Models and Multimodal Distributions." *Behavioral Science* 23 (4): 360–74. https://doi.org/10.1002/bs.3830230407.

Cobb, Loren, and Shelemyahu Zacks. 1985. "Applications of Catastrophe Theory for Statistical Modeling in the Biosciences." *Journal of the American Statistical Association* 80 (392): 793–802. https://doi.org/10.1080/01621459.1985.10478184.

Cohen, Alexander, David Pargman, and Gershon Tenenbaum. 2003. "Critical Elaboration and Empirical Investigation of the Cusp Catastrophe Model: A Lesson for Practitioners." *Journal of Applied Sport Psychology* 15 (2): 144–59. https://doi.org/10.1080/10413200305393.

Dablander, Fabian, Anton Pichler, Arta Cika, and Andrea Bacilieri. 2023. "Anticipating Critical Transitions in Psychological Systems Using Early Warning Signals: Theoretical and Practical Considerations." *Psychological Methods* 28 (4): 765–90. https://doi.org/10.1037/met0000450.

Dakos, Vasilis, Stephen R. Carpenter, William A. Brock, Aaron M. Ellison, Vishwesha Guttal, Anthony R. Ives, Sonia Kéfi, et al. 2012. "Methods for Detecting Early Warnings of Critical Transitions in Time Series Illustrated Using Simulated Ecological Data." *PloS One* 7 (7): e41010. https://doi.org/10.1371/journal.pone.0041010.

de Mooij, Susanne M. M., Tessa F. Blanken, Raoul P. P. P. Grasman, Jennifer R. Ramautar, Eus J. W. Van Someren, and Han L. J. van der Maas. 2020. "Dynamics of Sleep: Exploring Critical Transitions and Early Warning Signals." *Computer Methods and Programs in Biomedicine* 193 (September): 105448. https://doi.org/10.1016/j.cmpb.2020.105448.

Dodson, M. M., and A. Hallam. 1977. "Allopatric Speciation and the Fold Catastrophe." *The American Naturalist* 111 (979): 415–33. https://doi.org/10.1086/283176.

Dolan, Conor V., and Han L. J. van der Maas. 1998. "Fitting Multivariage Normal Finite Mixtures Subject to Structural Equation Modeling." *Psychometrika* 63 (3): 227–53. https://doi.org/10.1007/BF02294853.

Dolan, Conor V., and Han L. J. van der Maas. 1998. "Fitting Multivariage Normal Finite Mixtures Subject to Structural Equation Modeling." *Psychometrika* 63 (3): 227–53. https://doi.org/10.1007/BF02294853.

Dutilh, Gilles, Eric-Jan Wagenmakers, Ingmar Visser, and van der Han L. J. Maas. 2011. "A Phase Transition Model for the Speed-Accuracy Trade-Off in Response Time Experiments." *Cognitive Science* 35 (2): 211–50. https://doi.org/10.1111/j.1551-6709.2010.01147.x.

Epskamp, Sacha, Han L. J. van der Maas, Roseann E. Peterson, Hanna M. van Loo, Steven H. Aggen, and Kenneth S. Kendler. 2022. "Intermediate Stable States in Substance Use." *Addictive Behaviors* 129 (June): 107252. https://doi.org/10.1016/j.addbeh.2022.107252.

Eronen, Markus I., and Laura F. Bringmann. 2021. "The Theory Crisis in Psychology: How to Move Forward." *Perspectives on Psychological Science* 16 (4): 779–88. https://doi.org/10.1177/1745691620970586.

Flack, Jessica C. 2017. "Coarse-Graining as a Downward Causation Mechanism." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 375 (2109): 20160338. https://doi.org/10.1098/rsta.2016.0338.

Fodor, J. A. 1974. "Special Sciences (or: The Disunity of Science as a Working Hypothesis)." *Synthese* 28 (2): 97–115. https://doi.org/10.1007/BF00485230.

Freitas, Ubiratan, Elise Roulin, Jean-François Muir, and Christophe Letellier. 2009. "Identifying Chaos from Heart Rate: The Right Task?" *Chaos: An Interdisciplinary Journal of Nonlinear Science* 19 (2): 028505. https://doi.org/10.1063/1.3139116.

Garc, P., and Callegari Pe. 2015. "Analysis of EEG Signals Using Nonlinear Dynamics and Chaos: A Review." In. https://www.semanticscholar.org/paper/Analysis-of-EEG-Signals-using-Nonlinear-Dynamics-A-Garc-Pe/7c7bae15b868a68049d017edd27031c2746e20e4.

Gilmore, Robert. 1993. *Catastrophe Theory for Scientists and Engineers*. Courier Corporation.

Grasman, Raoul, Han L. J. van der Maas, and Eric-Jan Wagenmakers. 2009. "Fitting the Cusp Catastrophe in R: A Cusp Package Primer." *Journal of Statistical Software* 032 (i08). https://ideas.repec.org/a/jss/jstsof/v032i08.html.

Grasman, Raoul, Han L. J. van der Maas, and Eric-Jan Wagenmakers. 2009. "Fitting the Cusp Catastrophe in R: A Cusp Package Primer." *Journal of Statistical Software* 032 (i08).

Grauwin, Sebastian, Guillaume Beslon, Éric Fleury, Sara Franceschelli, Celine Robardet, Jean-Baptiste Rouquier, and Pablo Jensen. 2012. "Complex Systems Science: Dreams of Universality, Interdisciplinarity Reality." *Journal of the American Society for Information Science and Technology* 63 (7): 1327–38. https://doi.org/10.1002/asi.22644.

Groot, S. R. De, and P. Mazur. 2013. *Non-Equilibrium Thermodynamics*. Courier Corporation.

Guastello, Stephen J. 1982. "Moderator Regression and the Cusp Catastrophe: Application of Two-Stage Personnel Selection, Training, Therapy, and Policy Evaluation." *Behavioral Science* 27 (3): 259–72. https://doi.org/10.1002/bs.3830270305.

———. 1984a. "Cusp and Butterfly Catastrophe Modeling of Two Opponent Process Models: Drug Addiction and Work Performance." *Behavioral Science* 29 (4): 258–62. https://doi.org/10.1002/bs.3830290405.

———. 1984b. "Cusp and Butterfly Catastrophe Modeling of Two Opponent Process Models: Drug Addiction and Work Performance." *Behavioral Science* 29 (4): 258262. https://www.proquest.com/docview/1301285594/citation/C898CBB773DF4B72PQ/1.

Guastello, Stephen J., Matthijs Koopmans, and David Pincus. 2008. *Chaos and Complexity in Psychology: The Theory of Nonlinear Dynamical Systems*. Cambridge University Press.

Guckenheimer, John, and Philip Holmes. 1983. "Global Bifurcations." In *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, edited by John Guckenheimer and Philip Holmes, 289–352. Applied Mathematical Sciences. New York, NY: Springer. https://doi.org/10.1007/978-1-4612-1140-2_6.

Güémez, J., C. Fiolhais, and M. Fiolhais. 2002. "The Cartesian Diver and the Fold Catastrophe." *American Journal of Physics* 70 (7): 710–14. https://doi.org/10.1119/1.1477433.

Haig, Brian D. 2014. *Investigating the Psychological World: Scientific Method in the Behavioral Sciences*. MIT Press.

Haken, Herman. 1977. "Synergetics." *Physics Bulletin* 28 (9): 412. https://doi.org/10.1088/0031-9112/28/9/027.

Hardy, Lew. 1996. "Testing the Predictions of the Cusp Catastrophe Model of Anxiety and Performance." *The Sport Psychologist* 10 (2): 140–56. https://doi.org/10.1123/tsp.10.2.140.

Hardy, Lew, and Gaynor Parfitt. 1991. "A Catastrophe Model of Anxiety and Performance." *British Journal of Psychology* 82 (2): 163–78. https://doi.org/10.1111/j.2044-8295.1991.tb02391.x.

Hardy, Lew, Tim Woodman, and Stephen Carrington. 2004. "Is Self-Confidence a Bias Factor in Higher-Order Catastrophe Models? An Exploratory Analysis." *Journal of Sport and Exercise Psychology* 26 (3): 359–68. https://doi.org/10.1123/jsep.26.3.359.

Harezlak, Katarzyna, and Pawel Kasprowski. 2018. "Searching for Chaos Evidence in Eye Movement Signals." *Entropy* 20 (1): 32. https://doi.org/10.3390/e20010032.

Heath, Richard A. 2000. *Nonlinear Dynamics: Techniques and Applications in Psychology*. 1st edition. Mahwah, N.J: Psychology Press.

Heylighen, Francis. 2009. "Complexity and Self-Organization." In, 3rd ed. CRC Press.

Hoel, Erik P., Larissa Albantakis, and Giulio Tononi. 2013. "Quantifying Causal Emergence Shows That Macro Can Beat Micro." *Proceedings of the National Academy of Sciences* 110 (49): 19790–95. https://doi.org/10.1073/pnas.1314922110.

James, Nicholas A., and David S. Matteson. 2014. "**Ecp** : An *R* Package for Nonparametric Multiple Change Point Analysis of Multivariate Data." *Journal of Statistical Software* 62 (7). https://doi.org/10.18637/jss.v062.i07.

Jansen, Brenda R. J., and Han L. J. Van der Maas. 2001a. "Evidence for the Phase Transition from Rule I to Rule II on the Balance Scale Task." *Developmental Review* 21 (4): 450–94. https://doi.org/10.1006/drev.2001.0530.

———. 2001b. "Evidence for the Phase Transition from Rule I to Rule II on the Balance Scale Task." *Developmental Review* 21 (4): 450494. https://doi.org/10.1006/drev.2001.0530.

Jusup, Marko, Petter Holme, Kiyoshi Kanazawa, Misako Takayasu, Ivan Romić, Zhen Wang, Sunčana Geček, et al. 2022. "Social Physics." *Physics Reports*, Social physics, 948 (February): 1–148. https://doi.org/10.1016/j.physrep.2021.10.005.

Kargarnovin, Shaida, Christopher Hernandez, Farzad V. Farahani, and Waldemar Karwowski. 2023. "Evidence of Chaos in Electroencephalogram Signatures of Human Performance: A Systematic Review." *Brain Sciences* 13 (5): 813. https:

//doi.org/10.3390/brainsci13050813.

Kelso, J. A. S. 1995. *Dynamic Patterns: The Self-Organization of Brain and Behavior*. Dynamic Patterns: The Self-Organization of Brain and Behavior. Cambridge, MA, US: The MIT Press.

Kim, Jaegwon. 2006. "Emergence: Core Ideas and Issues." *Synthese* 151 (3): 547–59. https://doi.org/10.1007/s11229-006-9025-0.

Krakauer, David C. 2023. "Symmetry–simplicity, Broken Symmetry–complexity." *Interface Focus* 13 (3): 20220075. https://doi.org/10.1098/rsfs.2022.0075.

Kreitler, Shulamith, and Hans Kreitler. 1989. "Horizontal Decalage: A Problem and Its Solution." *Cognitive Development* 4 (1): 89–119. https://doi.org/10.1016/0885-2014(89)90006-3.

Kuramoto, Yoshiki. 1984. "Chemical Turbulence." In *Chemical Oscillations, Waves, and Turbulence*, edited by Yoshiki Kuramoto, 111–40. Springer Series in Synergetics. Berlin, Heidelberg: Springer. https://doi.org/10.1007/978-3-642-69689-3_7.

Ladyman, James, James Lambert, and Karoline Wiesner. 2013. "What Is a Complex System?" *European Journal for Philosophy of Science* 3 (1): 33–67. https://doi.org/10.1007/s13194-012-0056-8.

Latané, Bibb, and Andrzej Nowak. 1994. "Attitudes as Catastrophes: From Dimensions to Categories with Increasing Involvement." In *Dynamical Systems in Social Psychology*, 219–49. San Diego, CA, US: Academic Press.

Leemput, van de Ingrid A., Marieke Wichers, Angélique O. J. Cramer, Denny Borsboom, Francis Tuerlinckx, Peter Kuppens, van Egbert H. Nes, et al. 2014. "Critical Slowing down as Early Warning for the Onset and Termination of Depression." *Proceedings of the National Academy of Sciences* 111 (1): 87–92. https://doi.org/10.1073/pnas.1312114110.

Leggio, Lorenzo, George A. Kenna, Miriam Fenton, Erica Bonenfant, and Robert M. Swift. 2009. "Typologies of Alcohol Dependence. From Jellinek to Genetics and Beyond." *Neuropsychology Review* 19 (1): 115–29. https://doi.org/10.1007/s11065-008-9080-z.

Lei, Min, Zhizhong Wang, and Zhengjin Feng. 2001. "Detecting Nonlinearity of Action Surface EMG Signal." *Physics Letters A* 290 (5): 297–303. https://doi.org/10.1016/S0375-9601(01)00668-5.

Loehle, Craig. 1989. "Catastrophe Theory in Ecology: A Critical Review and an Example of the Butterfly Catastrophe." *Ecological Modelling* 49 (1): 125–52. https://doi.org/10.1016/0304-3800(89)90047-1.

Lumsden, James. 1976. "Test Theory." *Annual Review of Psychology* 27: 251–80. https://doi.org/10.1146/annurev.ps.27.020176.001343.

Maas, Han L J van der, Jonas Dalege, and Lourens Waldorp. 2020. "The Polarization Within and Across Individuals: The Hierarchical Ising Opinion Model." *Journal of Complex Networks* 8 (cnaa010). https://doi.org/10.1093/comnet/cnaa010.

Mazanov, Jason, and D. G. Byrne. 2006. "A Cusp Catastrophe Model Analysis of Changes in Adolescent Substance Use: Assessment of Behavioural Intention as a Bifurcation Variable." *Nonlinear Dynamics, Psychology, and Life Sciences* 10: 445–70.

McLachlan, Geoffrey J., Sharon X. Lee, and Suren I. Rathnayake. 2019. "Finite Mixture Models." *Annual Review of Statistics and Its Application* 6 (1): 355–78. https://doi.org/

10.1146/annurev-statistics-031017-100325.

McRobie, Allan. 2017. *The Seduction of Curves: The Lines of Beauty That Connect Mathematics, Art, and the Nude*. Princeton University Press.

Michell, Joel. 1999. *Measurement in Psychology: A Critical History of a Methodological Concept*. Cambridge University Press.

———. 2008. "Is Psychometrics Pathological Science?" *Measurement: Interdisciplinary Research and Perspectives* 6 (January): 7–24. https://doi.org/10.1080/15366360802035489.

Mitchell, Melanie. 2009. *Complexity: A Guided Tour*. Oxford University Press.

Oliva, Terence A., Wayne S. Desarbo, Diana L. Day, and Kamel Jedidi. 1987. "Gemcat: A General Multivariate Methodology for Estimating Catastrophe Models." *Behavioral Science* 32 (2): 121–37. https://doi.org/10.1002/bs.3830320205.

Olthof, Merlijn, Fred Hasselman, Guido Strunk, Marieke van Rooij, Benjamin Aas, Marieke A. Helmich, Günter Schiepek, and Anna Lichtwarck-Aschoff. 2020. "Critical Fluctuations as an Early-Warning Signal for Sudden Gains and Losses in Patients Receiving Psychotherapy for Mood Disorders." *Clinical Psychological Science* 8 (1): 25–35. https://doi.org/10.1177/2167702619865969.

Parisi, Giorgio. 2023. *In a Flight of Starlings: The Wonders of Complex Systems*. Penguin Publishing Group.

Paulos, John Allen. 2008. *Mathematics and Humor*. University of Chicago Press.

Ploeger, Annemie, Han L. J. Van Der Maas, and Pascal A. I. Hartelman. 2002. "Stochastic Catastrophe Analysis of Switches in the Perception of Apparent Motion." *Psychonomic Bulletin & Review* 9 (1): 26–42. https://doi.org/10.3758/BF03196255.

Pool, Robert. 1989. "Is It Healthy to Be Chaotic?" *Science* 243 (4891): 604–7. https://doi.org/10.1126/science.2916117.

Port, Robert F., and Timothy Van Gelder. 1995. *Mind as Motion: Explorations in the Dynamics of Cognition*. MIT Press.

Poston, Tim, and Ian Stewart. 2014. *Catastrophe Theory and Its Applications*. Courier Corporation.

Pritchard, Walter s., and Dennis w. Duke. 1992. "Measuring Chaos in the Brain: A Tutorial Review of Nonlinear Dynamical Eeg Analysis." *International Journal of Neuroscience* 67 (1-4): 31–80. https://doi.org/10.3109/00207459208994774.

Roberts, James A., Leonardo L. Gollo, Romesh G. Abeysuriya, Gloria Roberts, Philip B. Mitchell, Mark W. Woolrich, and Michael Breakspear. 2019. "Metastable Brain Waves." *Nature Communications* 10 (1): 1056. https://doi.org/10.1038/s41467-019-08999-0.

Robertson, Robin, and Allan Combs, eds. 2014. *Chaos Theory in Psychology and the Life Sciences*. New York: Psychology Press. https://doi.org/10.4324/9781315806280.

Robinaugh, Donald J., Ria H. A. Hoekstra, Emma R. Toner, and Denny Borsboom. 2020. "The Network Approach to Psychopathology: A Review of the Literature 2008–2018 and an Agenda for Future Research." *Psychological Medicine* 50 (3): 353–66. https://doi.org/10.1017/S0033291719003404.

Rosser, J. Barkley. 2007. "The Rise and Fall of Catastrophe Theory Applications in Economics: Was the Baby Thrown Out with the Bathwater?" *Journal of Economic Dynamics and Control* 31 (10): 3255–80. https://doi.org/10.1016/j.jedc.2006.09.013.

Rosso, O. A., H. A. Larrondo, M. T. Martin, A. Plastino, and M. A. Fuentes. 2007. "Distinguishing Noise from Chaos." *Physical Review Letters* 99 (15): 154102. https://doi.org/10.1103/PhysRevLett.99.154102.

Sandubete, Julio, E., and Lorenzo Escot. 2021. "DChaos: An R Package for Chaotic Time Series Analysis." *The R Journal* 13 (1): 232. https://doi.org/10.32614/RJ-2021-036.

Sayama, Hiroki. 2015. "Introduction to the Modeling and Analysis of Complex Systems." *Milne Open Textbooks*, January. https://knightscholar.geneseo.edu/oer-ost/14.

Scheffer, Marten. 2004. *Ecology of Shallow Lakes*. Dordrecht: Springer Netherlands. https://doi.org/10.1007/978-1-4020-3154-0.

Schiepek, Günter K., Kathrin Viol, Wolfgang Aichhorn, Marc-Thorsten Hütt, Katharina Sungler, David Pincus, and Helmut J. Schöller. 2017. "Psychotherapy Is Chaotic—(not Only) in a Computational World." *Frontiers in Psychology* 8. https://www.frontiersin.org/articles/10.3389/fpsyg.2017.00379.

Schöner, Gregor. 2020. "The Dynamics of Neural Populations Capture the Laws of the Mind." *Topics in Cognitive Science* 12 (4): 1257–71. https://doi.org/10.1111/tops.12453.

Schwartz, Seth J., Scott O. Lilienfeld, Alan Meca, and Katheryn C. Sauvigné. 2016. "The Role of Neuroscience Within Psychology: A Call for Inclusiveness over Exclusiveness." *American Psychologist* 71 (1): 52–70. https://doi.org/10.1037/a0039678.

Scott, John. 2011. "Social Network Analysis: Developments, Advances, and Prospects." *Social Network Analysis and Mining* 1 (1): 21–26. https://doi.org/10.1007/s13278-010-0012-6.

Serra, Roberto, and Gianni Zanarini. 1990. *Complex Systems and Cognitive Processes*. Berlin, Heidelberg: Springer. https://doi.org/10.1007/978-3-642-46678-6.

Simon, Herbert A. 1962b. "The Architecture of Complexity." *Proceedings of the American Philosophical Society* 106 (6): 467–82. https://www.jstor.org/stable/985254.

———. 1962a. "The Architecture of Complexity." *Proceedings of the American Philosophical Society* 106 (6): 467–82. https://www.jstor.org/stable/985254.

Snijders, Tom A. B. 2001. "The Statistical Evaluation of Social Network Dynamics." *Sociological Methodology* 31 (1): 361–95. https://doi.org/10.1111/0081-1750.00099.

Stam, C. J. 2005. "Nonlinear dynamical analysis of EEG and MEG: review of an emerging field." *Clinical Neurophysiology: Official Journal of the International Federation of Clinical Neurophysiology* 116 (10): 2266–2301. https://doi.org/10.1016/j.clinph.2005.06.011.

Stefan, Martin. 2020. "RPubs - Fractals with R." https://rpubs.com/mstefan-rpubs/fractals.

Stevens, S. S. 1957. "On the Psychophysical Law." *Psychological Review* 64 (3): 153181. https://doi.org/10.1037/h0046162.

Stewart, Ian. 1982. "Catastrophe Theory in Physics." *Reports on Progress in Physics* 45 (2): 185–221. https://doi.org/10.1088/0034-4885/45/2/002.

Stewart, Ian, and P. L. Peregoy. 1983. "Catastrophe Theory Modeling in Psychology." *Psychological Bulletin* 94: 336–62. https://doi.org/10.1037/0033-2909.94.2.336.

Stouffer, Samuel A., Edward A. Suchman, Leland C. Devinney, Shirley A. Star, and Robin M. Williams Jr. 1949. *The American Soldier: Adjustment During Army Life. (Studies in Social Psychology in World War II), Vol. 1.* The American Soldier: Adjustment During Army Life. (Studies in Social Psychology in World War II), Vol. 1. Oxford, England: Princeton Univ. Press.

Thom, René. 1977. "Structural Stability, Catastrophe Theory, and Applied Mathematics." *SIAM Review* 19 (2): 189–201. https://doi.org/10.1137/1019036.

Thurner, Stefan. 2016. *43 Visions For Complexity*. World Scientific.

Toker, Daniel, Friedrich T. Sommer, and Mark D'Esposito. 2020. "A Simple Method for Detecting Chaos in Nature." *Communications Biology* 3 (1): 1–13. https://doi.org/10.1038/s42003-019-0715-9.

Treiber, Martin, Ansgar Hennecke, and Dirk Helbing. 2000. "Congested Traffic States in Empirical Observations and Microscopic Simulations." *Physical Review E* 62 (2): 1805–24. https://doi.org/10.1103/PhysRevE.62.1805.

Tschacher, Wolfgang, and Hermann Haken. 2023. "A Complexity Science Account of Humor." *Entropy* 25 (2): 341. https://doi.org/10.3390/e25020341.

van der Maas, H. L. J. 1995. "Beyond the Metaphor?" *Cognitive Development* 10.

Van Der Maas, Han L. J., Conor V. Dolan, Raoul P. P. P. Grasman, Jelte M. Wicherts, Hilde M. Huizenga, and Maartje E. J. Raijmakers. 2006. "A Dynamical Model of General Intelligence: The Positive Manifold of Intelligence by Mutualism." *Psychological Review* 113 (4): 842–61. https://doi.org/10.1037/0033-295X.113.4.842.

van der Maas, Han L. J., Rogier Kolstein, and Joop van der Pligt. 2003. "Sudden Transitions in Attitudes." *Sociological Methods & Research* 32 (2): 125–52. https://doi.org/10.1177/0049124103253773.

van der Maas, Han L. J., Paul F. M. J. Verschure, and Peter C. M. Molenaar. 1990. "A Note on Chaotic Behavior in Simple Neural Networks." *Neural Networks* 3 (1): 119–22. https://doi.org/10.1016/0893-6080(90)90050-U.

Van der Maas, Han L., and Peter C. Molenaar. 1992b. "Stagewise Cognitive Development: An Application of Catastrophe Theory." *Psychological Review* 99 (3): 395–417. https://doi.org/10.1037/0033-295X.99.3.395.

———. 1992a. "Stagewise Cognitive Development: An Application of Catastrophe Theory." *Psychological Review* 99 (3): 395–417. https://doi.org/10.1037/0033-295X.99.3.395.

Visser, Ingmar, and Maarten Speekenbrink. 2022. *Mixture and Hidden Markov Models with R*. Springer International Publishing.

Wagenmakers, Eric-Jan, Peter C. M. Molenaar, Raoul P. P. P. Grasman, Pascal A. I. Hartelman, and Han L. J. van der Maas. 2005. "Transformation Invariant Stochastic Catastrophe Theory." *Physica D: Nonlinear Phenomena* 211 (3): 263–76. https://doi.org/10.1016/j.physd.2005.08.014.

Weaver, Warren. 1948. "Science and Complexity." *American Scientist* 36 (4): 536–44. https://www.jstor.org/stable/27826254.

Wichers, Marieke, Peter C. Groot, and ESM Group Psychosystems. 2016. "Critical Slowing Down as a Personalized Early Warning Signal for Depression." *Psychotherapy and Psychosomatics* 85 (2): 114–16. https://doi.org/10.1159/000441458.

Wimsatt, William C. 1994. "The Ontology of Complex Systems: Levels of Organization, Perspectives, and Causal Thickets." *Canadian Journal of Philosophy Supplementary Volume* 20 (January): 207–74. https://doi.org/10.1080/00455091.1994.10717400.

Witkiewitz, Katie, Han L. J. van der Maas, Michael R. Hufford, and G. Alan Marlatt. 2007. "Nonnormality and Divergence in Posttreatment Alcohol Use: Reexamining the Project

MATCH Data "Another Way."." *Journal of Abnormal Psychology* 116: 378–94. https://doi.org/10.1037/0021-843X.116.2.378.

Zahler, Raphael S., and Hector J. Sussmann. 1977. "Claims and Accomplishments of Applied Catastrophe Theory." *Nature* 269 (October): 759–63. https://doi.org/10.1038/269759a0.

Zanin, Massimiliano. 2022. "Can Deep Learning Distinguish Chaos from Noise? Numerical Experiments and General Considerations." *Communications in Nonlinear Science and Numerical Simulation* 114 (November): 106708. https://doi.org/10.1016/j.cnsns.2022.106708.

Zeeman, E. C. 1976. "Catastrophe Theory." *Scientific American* 234 (4): 65–83.