Ch 17 Options on Stock Indices and Currencies

Options on Stock Indices

Stock indices

- some track the movement of the market as a whole
- others are based on performance of a particular sector

Several exchanges trade options on stock indices

- one index option contract is on 100 times the index
- usually settled in cash

Portfolio Insurance

Portfolio managers can use index options to limit their downside risk

- buy put option contracts to obtain insurance against portfolio value dropping

When the Portfolio Beta is Not 1.0

β put options must be purchased for each $100S_0$ dollars in the portfolio where S_0 is the current value of the index.

Use the capital asset pricing model to calculate the appropriate strike price.

Cost of hedging increases as portfolio beta increases for two reasons:

1. more put options are required
2. the put options have a higher strike price

Currency Options

Currency options primarily traded in OTC market

- large trade possible
- contract features tailored to meet needs of parties

Foreign currency options are an alternative to forward contracts for hedging a foreign exchange exposure

- company due to receive sterling in the future can hedge risk by buying puts on sterling
- hedging strategy guarantees exchange rate will not be less than the strike price

  - company can still benefit from favorable exchange-rate movements
- company due to pay sterling in the future can hedge by buying calls on sterling
- hedging strategy guarantees cost of sterling will not be greater than a certain amount
  - company can still benefit from favorable exchange-rate movements

Whereas a forward contract locks in the exchange rate for a future transaction, an option provides a type of insurance. However, it costs nothing to enter into a forward transaction. Options require a premium to be paid up front.

Range Forwards

range forward contract - variation on a standard forward contract for hedging FX risk

Consider a company that will receive 1m pounds in 3 months.
Let the FX rate by 1.52 dollars per pound.

The company can lock in the exchange rate for dollars received by entering into a short forward contract to sell 1m pounds in 3 months. This ensures the amount received for the 1m pounds is equal to the 3-month forward exchange rate ($1.52m).

Alternatively, company can buy a European put option with a strike price K_1 and sell a European call option with strike price K_2, where $K_1 < 1.52 < K_2$. This is known as a short position in a range forward contract.

- if exchange rate in 3 months is less than K_1, put option is exercised
  - company can sell 1m pounds at exchange rate K_1
- if exchange rate is between K_1 and K_2$, neither option is exercised
  - company gets the current exchange rate for the 1m pounds
- if exchange rate is greater than K_2, the call option is exercised against the company
  - company sells the 1m pounds at exchange rate K_2

If the company were due to pay 1m pounds in 3 months, it could sell a European put option with strike price K_1 and buy a European call option with strike price K_2. This is a long position in a range forward contract.

- if exchange rate in 3 months is less than K_1, put option is exercised against the company
  - company buys 1m pounds at exchange rate K_1
- if exchange rate is between K_1 and K_2, neither option is exercised
  - company buys 1m pounds at the current exchange rate
- if exchange rate is greater than K_2, the call option is exercised
  - company can buy 1m pounds at exchange rate K_2

Range forward contract is typically set up so that the price of the put equals the price of the call. Then there is no cost to set up the range forward contract, just like a regular forward contract.

Options on Stocks Paying Known Dividend Yields

Dividends cause stock prices to reduce on the ex-dividend date by the amount of the dividend. The payment of a dividend yield at rate q causes growth rate in the stock price to reduce by q.

Let dividend yield be q and stock price grow from S_0 today to S_T at time T.

Then, in the absence of dividends, it would grow

$$S_0 \longrightarrow S_T e^{qT}$$

Equivalently, in the absence of dividends, it would grow

$$S_0 e^{-qT} \longrightarrow S_T$$

This shows the following two cases have the same probability distribution for S_T:

1. Stock starts at price S_0 and provides dividend yield rate q
2. Stock starts at price $S_0 e^{-qT}$ and pays no dividends

When valuing a European option lasting for time T on a stock paying a known dividend yield rate q, reduce the current stock price from S_0 to $S_0 e^{-qT}$. Then value the option as though the stock pays no dividends.

Lower Bounds for Option Prices

Recall lower bound for a call option on non-dividend-paying stock

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$

Substitute for $S_0 e^{-qT}$ for S_0 which gives

$$c \geq \max(S_0 e^{-qT} - Ke^{-rT}, 0)$$

This could also be proven directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}
Portfolio B: e^{-qT} shares with dividends being reinvested in additional shares

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Recall lower bounds for a put option on non-dividend-paying stock

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

Substitute for $S_0 e^{-qT}$ for S_0 which gives

$$p \geq \max(Ke^{-rT} - S_0 e^{-qT}, 0)$$

This could also be proven directly by considering the following two portfolios:

Portfolio C: one European put option plus e^{-qT} shares with dividends reinvested in more shares
Portfolio D: an amount of cash equal to Ke^{-rT}

Put-Call Parity

Recall the put-call parity relationship

$$c + Ke^{-rT} = p + S_0$$

Substitute for $S_0 e^{-qT}$ for S_0 which gives

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

This could also be proven directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}
Portfolio C: one European put option plus e^{-qT} shares with dividends reinvested in more shares

Both portfolios are worth $\max(S_T, K)$ at time T, so they must be worth the same today.
The equation above is the result.

For American options, the put-call parity relationship is

$$S_0 e^{-qT} - K \leq C - P \leq S_0 - Ke^{-rT}$$

Pricing Formulas

Black-Scholes-Merton formulas for European options on a stock paying a dividend yield rate q

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

Note that

$$\ln \frac{S_0 e^{-qT}}{K} = \ln \frac{S_0}{K} - qT$$

so it follows that

$$d_1 = \frac{\ln (S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln (S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

If the dividend yield rate is known but not constant, the formulas for c and p are still valid.
The value of q should be the average annualized dividend yield during the option's life.

Differential Equation and Risk-Neutral Valuation

Proving the results for c and p above

Recall that the Black-Scholes differential equation is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

When including a dividend yield q, the differential equation becomes

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

It does not involve variables affected by risk preferences, so risk-neutral valuation can be used.

In a risk-neutral world, the total return from the stock must be r.
The dividends provide a return of q.
So the expected growth rate in the stock price must be $r - q$.

The risk-neutral process for the stock price is then

$$dS = (r-q)S\,dt + \sigma S\,dz$$

The expected payoff for a call option in a risk-neutral world is

$$e^{(r-q)T}S_0N(d_1) - KN(d_2)$$

where d_1 and d_2 are defined above.

Discounting this at rate r for time T leads to the formula for c given above.

Valuation of European Stock Index Options

Assume the index can be treated as an asset paying a known yield

- use the above results for lower bound for European index options
- use the above put-call parity result for European index options
- use the above formulas for c and p to value European options on an index
- use binomial tree approach to value American options

In all cases

- S_0 is equal to the value of the index
- σ is equal to the volatility of the index
- q is equal to the average annualized dividend yield on the index during the option's life

If the absolute amount of the dividend is known instead of the dividend yield, the Black-Scholes formulas can be used with the initial stock price being reduced by the present value of the dividends. This requires knowledge of the dividends expected on every stock underlying the index.

Forward Prices

Define F_0 as the forward price of the index for a contract with maturity T

$$F_0 = S_0 e^{(r-q)T}$$

The equations for the European call price c and European put price p can be written

$$c = F_0 e^{-rT} N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - F_0 e^{-rT} N(-d_1)$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(F_0/k) - \sigma^2 T/2}{\sigma\sqrt{T}}$$

The put-call parity relationship can be written

$$c + K e^{-rT} = p + F_0 e^{-rT}$$

$$F_0 = K + (c - p) e^{rT}$$

This can be used to estimate forward price of the index for particular maturity dates, if pairs of puts and calls with the same strike price are actively traded for that maturity date. Forward prices of the index for various maturity dates can be used to estimate the term structure of forward prices. Then other options can be valued using the formulas for c and p given above.

The advantage of this approach is that the dividend yield on the index does not have to be explicitly estimated.

Implied Dividend Yields

If estimates of the dividend yield are required (e.g. for American options) then calls and puts with the same strike price and time to maturity can be used.

Solving the put-call parity relationship for q

$$q = -\frac{1}{T} \ln \frac{c - p + K e^{-rT}}{S_0}$$

For a particular strike price and time to maturity, estimates of q are likely to be unreliable.
However, the results from many matched pairs of calls and puts combined can paint a clearer picture of the term structure of dividend yields being assumed by the market.

Valuation of European Currency Options

Let S_0 be the spot exchange rate (value of one unit of foreign currency in US dollars).

- foreign currency is analogous to stock paying known dividend yield
- owner of foreign currency receives a yield equal to the risk-free interest rate rf in the foreign currency

Lower bounds for European call price and European put price, with q replaced by rf

$$c \geq \max(S_0e^{-rfT} - Ke^{-rT}, 0)$$

$$p \geq \max(Ke^{-rT} - S_0e^{-rfT}, 0)$$

Put-call parity relationship for European currency options, with q replaced by rf

$$c + Ke^{-rT} = p + S_0e^{-rfT}$$

Pricing formulas for European currency options, with q replaced by rf

$$c = S_0e^{-rfT}N(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - S_0e^{-rfT}N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - rf + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - rf - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Both the domestic interest rate r and the foreign interest rate rf are the rates for a maturity T.

Using Forward Exchange Rates

Forward exchange rates are often used for valuing options.

$$F_0 = S_0e^{(r-rf)T}$$

Use this relationship to simplify the equations above

$$c = e^{-rT}[F_0N(d_1) - KN(d_2)]$$

$$p = e^{-rT}[KN(-d_2) - F_0N(-d_1)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(F_0/k) - \sigma^2T/2}{\sigma\sqrt{T}}$$

These are the same equations stated previously.

A European option on the spot price of any asset can be valued in terms of the price of a forward or futures contract on the asset using these equations. The maturity of the forward or futures contract must be the same as the maturity of the European option.

American Options

Binomial trees can be used to value American options on indices and currencies.

Parameters

- size of up movements $u = e^{\sigma\sqrt{\Delta t}}$
- size of down movements $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$
- volatility σ
- length of time steps Δt

For a non-dividend-paying stock, the probability of an up movement is

$$p = \frac{a - d}{u - d}$$

where $a = e^{r\Delta t}$.

For options on indices

$$a = e^{(r-q)\Delta t}$$

where q is the dividend yield on the index.

For options in currencies

$$a = e^{(r-rf)\Delta t}$$

where rf is the foreign risk-free rate.

In some cases it is optimal to exercise American index and currency options prior to maturity. Therefore these are worth more than the European counterparts.

Most likely to be early exercised (index options)

- call options on indices with high-dividend yields
- put options on indices with low-dividend yields

Most likely to be early exercised (currency options)

- call options on high-interest currencies (because currency expected to depreciate)

- put options on low-interest currencies (because currency expected to appreciate)