# Ch 21 Value at Risk

Value at Risk (VaR) is an attempt to provide a single number summarizing the total risk in a portfolio of financial assets.

## The VaR Measure

*I am X percent certain there will not be a loss of more than V dollars in the next N days.*

- *V* is the VaR of the portfolio

VaR is the function of two parameters

- the time horizon (*N* days)
- the confidence level (*X*%)

Problem with VaR: it is tempting to choose a portfolio with a certain distribution such that VaR is the same as another distribution, but is actually much riskier due to larger potential losses

**Figure 22.1** Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is $X\%$. Gains in portfolio value are positive; losses are negative.

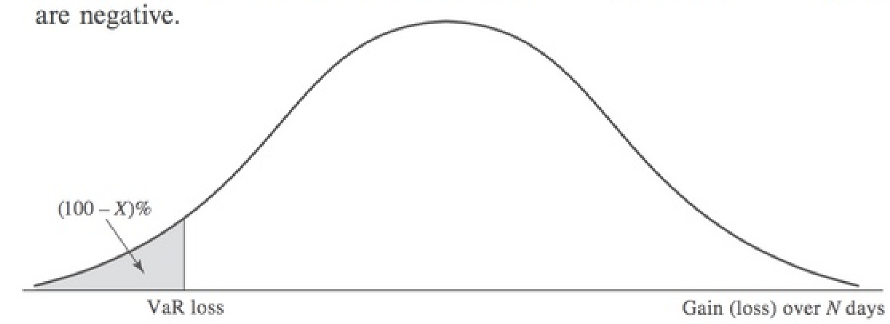
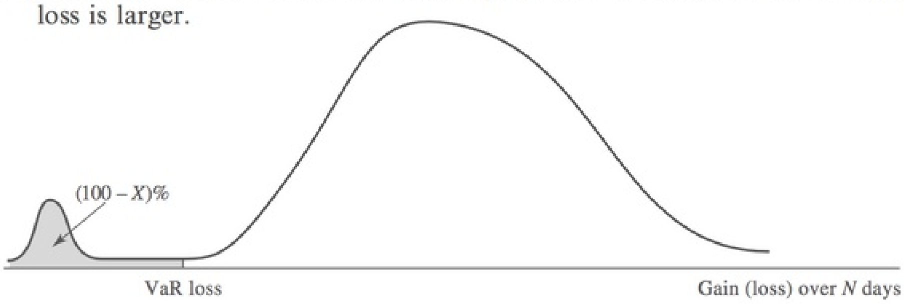



**Figure 22.2** Alternative situation to Figure 22.1. VaR is the same, but the potential loss is larger.




A measure that deals with this problem is **expected shortfall**

- expected loss during an *N*-day period conditional on the loss being worse than the VaR loss

**The Time Horizon**

Typically $N = 1$ (days) when VaR is estimated for market risk

- usually not enough data available to estimate directly the behavior of market variables over longer period of time

The usual assumption is

$$N\text{-day VaR} = \text{1-day Var} \times \sqrt{N}$$

## Historical Simulation

- using past data as a guide to what will happen in the future

First identify market variables affecting the portfolio, typically

- interest rates
- equity pries
- commodity prices
- etc

All prices are measured in the domestic currency.

Then collect data on movements in the market variables over the most recent *N* days.
Suppose $N = 501$, so this provides $500$ scenarios for what can happen between today and tomorrow.

Scenario 1: percentage changes in values of all variables between Day 0 and Day 1
Scenario 2: percentage changes in values of all variables between Day 1 and Day 2
...
Scenario 500: percentage changes in values of all variables between Day 500 and Day 501

For each scenario, the dollar change in the value of the portfolio between today and tomorrow is calculated. This defines a probability distribution for daily loss in the portfolio value.

The 99th percentile is the 5th-highest loss (because 500 scenarios).

99% certainty that there will not be a loss greater than this if changes in market variables in the last 501 days are representative of what will happen between today and tomorrow.

Let $v_i$ be the value of a market variable on Day $i$ and suppose that today is Day $n$.

The $i$th scenario in the historical simulation approach assumes that the value of the market variable tomorrow will be

$$\text{Value under } i\text{th scenario} = v_n \frac{v_i}{v_{i-1}}$$

# Model-Building Approach

### Daily Volatilities

In option pricing, time is usually measured in years, so volatility of an asset is quoted as volatility per year. When using the model-building approach to calculate VaR, time is measured in days, so volatility of an asset is quoted as volatility per day.

$$\sigma_{\text{year}} = \sigma_{\text{day}}\sqrt{252}$$

This assumes 252 trading days in a year.

Note that $\sigma_{\text{day}}$ is approximately equal to the standard deviation of the percentage change in the asset price in one day. Assume exact equality for VaR calculations.

### Single-Asset Case

Consider a portfolio of a position in a single stock ($10m in shares).

Suppose $N = 10$ and $X = 99$, but initially consider a $1$-day time horizon.

Assume $\sigma_{\text{day}} = 2\%$ (or $32\%$ per year).
The size of the position is $10m so the standard deviation of daily change is $200,000.

Assume the expected change in a market variable over the time period is zero. So the change in value of the portfolio over a 1-day period has standard deviation $200,000 and mean zero. Assume the change is normally distributed.

99% one-tailed z-score is 2.326

There is a 1% probability a normally distributed variable will decrease in value by more than 2.326 standard deviations.

Equivalently, 99% certain that a normally distributed variable will not decrease in value by more than 2.326 standard deviations.

So the 1-day 99% VaR for the portfolio consisting of $10m position is

$$2.326 \times 200{,}000 = \$465{,}300$$

The $10$-day VaR is

$$465{,}300 \times \sqrt{10} = \$1{,}471300$$

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Consider a portfolio of $5m position in some stock. Suppose $\sigma_{\text{day}} = 1\%$.

So the standard deviation of change in value of the portfolio in 1 day is

$$5{,}000{,}000 \times 0.01 = 50{,}000$$

Assuming the change is normally distributed, the 1-day 99% VaR is

$$50{,}000 \times 2.326 = \$116{,}300$$

and the 10-day 99% VaR is

$$116{,}300 \times \sqrt{10} = \$367{,}800$$

**Two-Asset Case**

Consider a portfolio consisting of $10m of shares in some company and $5m in another.

Suppose the returns on the two shares have a bivariate normal distribution with correlation 0.3.

Note that if two variables $X$ and $Y$ have standard deviations $\sigma_X$ and $\sigma_Y$ with correlation $\rho$, then the standard deviation of $X + Y$ is given by

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

Let $X$ be change in value of position in first company over 1-day period.
Let $Y$ be change in value of position in second company over 1-day period.

$$\sigma_X = 200{,}000 \quad \text{and} \quad \sigma_Y = 50{,}000$$

The standard deviation of the change in portfolio value over a 1-day period is

$$\sigma_{X+Y} = 220{,}200$$

The mean change is assumed to be zero and the change is normally distributed.

So the 1-day 99% VaR is

$$220{,}200 \times 2.326 = \$512{,}300$$

The 10-day 99% VaR is

$$512{,}300 \times \sqrt{10} = \$1{,}620{,}100$$

**The Benefits of Diversification**

From the examples above

1. The 10-day 99% VaR for the first portfolio of one company shares is $1,471,300
2. The 10-day 99$ VaR for the second portfolio of other company shares is $367,800
3. The 10-day 99% VaR for the portfolio of both company shares is $1,620,100

$$1{,}471{,}300 + 367{,}800 - 1{,}620{,}100 = \$219{,}000$$

This represents the benefits of diversification.

If the two company shares were perfectly correlated, the VaR for the portfolio of both shares would equal the sum of the VaRs of the single-asset portfolios.

Less than perfect correlation leads to some of the risk being *diversified away*.

## The Linear Model

The examples above use the linear model for calculating VaR.

Suppose a portfolio worth $P$ consists of $n$ assets with an amount $\alpha_i$ being invested in asset $i$.

Let $\Delta x_i$ be the return on asset $i$ in 1 day.

The dollar change in the value of the investment in asset $i$ in 1 day is $\alpha_i \Delta x_i$.

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

where $\Delta P$ is the dollar change in the portfolio value in 1 day.

In the examples above, $10m was invested in first company and $5m in second company, so

$$\Delta P = 10\Delta x_1 + 5\Delta x_2$$

Assume the $\Delta x_i$ are multivariate normal, so $\Delta P$ is normally distributed.
Then calculating VaR requires only the mean and standard deviation of $\Delta P$.

As before, assume the expected value of each $\Delta x_i$ is zero, so the mean of $\Delta P$ is zero.

Let $\sigma_i$ be daily volatility of $i$th asset.
Let $\rho_{ij}$ be correlation coefficient between returns on asset $i$ and asset $j$.

This means $\sigma_i$ is the standard deviation of $\Delta x_i$ and $\rho_{ij}$ is the correlation coefficient between $\Delta x_i$ and $\Delta x_j$.

The variance of $\Delta P$ is given by

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

The standard deviation of the change over $N$ days is $\sigma_P \sqrt{N}$.

The 99% VaR for an $N$-day time horizon is $2.326 \sigma_P \sqrt{N}$.

The portfolio return in 1 day is $\Delta P / P$. The variance of this is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \omega_i \omega_j \sigma_i \sigma_j$$

where $\omega_i = \alpha_i / P$ is the weight of the $i$th investment in the portfolio.

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From the example above, $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $\rho_{12} = 0.3$, $\alpha_1 = 10$, and $\alpha_2 = 5$.

$$\sigma_P^2 = 10^2 \times 0.02^2 + 5^2 \times 0.01^2 + 2 \times 10 \times 5 \times 0.3 \times 0.02 \times 0.01 = 0.0485$$

$$\Rightarrow \sigma_P = 0.2202$$

The 10-day 99% VaR is

$$2.326 \times 0.2202 \times \sqrt{10} = \$1.62\,\text{million}$$

This is consistent with the previous calculation.

**Correlation and Covariance Matrices**

Variances and covariances are often used instead of correlations and volatilities.

The daily variance of variable $i$ is the square of its daily volatility

$$\text{var}_i = \sigma_i^2$$

The covariance between variable $i$ and variable $j$ is

$$\text{cov}_{ij} = \sigma_i \sigma_j \rho_{ij}$$

So the equation for the variance of the portfolio (given above) can be written

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}_{ij} \alpha_i \alpha_j$$

Using a covariance matrix the equation for portfolio variance becomes

$$\sigma_P^2 = \alpha^\intercal C \alpha$$

where $\alpha$ is the column vector whose $i$th element is $\alpha_i$ and $C$ is the covariance matrix.

**Handling Interest Rates**

Simplifications are needed in the model-building approach, because it is unreasonable to define a separate market variable for every price or rate a company is exposed to.

One simplification is to assume that only parallel shifts in the yield curve occur. Then it is necessary to define only one market variable: the size of the shift.

Changes in the value of a bond portfolio can be calculated using the duration relationship

$$\Delta P = -DP\Delta y$$

where $P$ is portfolio value, $\Delta P$ is change in $P$ in 1 day, $D$ is modified duration of the portfolio, and $\Delta y$ is the parallel shift in 1 day.

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The above approach is not very accurate.

Another procedure is to choose as market variables the prices of zero-coupon bonds with standard maturities: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years.

Cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates.

Consider a $1m position in a Treasury bond lasting 1.2 years that pays 6% coupon semiannually.

- coupons are paid in 0.2 years, 0.7 years, 1.2 years
- principal is paid in 1.2 years

The bond is regarded as

- $30,000 position in 0.2-year zero-coupon bond
- $30,000 position in 0.7-year zero-coupon bond
- $1.03m position in 1.2-year zero-coupon bond

The positions are then replaced by approximately equivalent positions

- positions in 1-month and 3-month zero-coupon bonds
- positions in 6-month and 1-year zero-coupon bonds
- positions in 1-year and 2-year zero-coupon bonds

The result is that the position in the original bond is regarded as a position in several zero-coupon bonds with standard maturities in order to calculate VaR.

This is called **cash-flow mapping**.

**Applications of the Linear Model**

Simplest application is to a portfolio with no derivatives consisting of positions in stocks and bonds. Cash-flow mapping converts bonds to zero-coupon bonds with standard maturities. The change in portfolio value is linearly dependent on returns on stocks and the zero-coupon bonds.

One derivative that can be handled by the linear model is a forward contract to buy a foreign currency. Suppose contract matures at time $T$. Regard it as the exchange of a foreign zero-coupon bond maturing at time $T$ for a domestic zero-coupon bond maturing at time $T$. So to calculate VaR, treat the forward position as a long position in the foreign bond combined with a short position in the domestic bond.

Consider an interest rate swap. Regard it as the exchange of a floating-rate bond for a fixed-rate bond. The fixed-rate bond is a regular coupon-bearing bond. The floating-rate bond is worth par just after the next payment date, so regard it as a zero-coupon bond with maturity date equal to the next payment date. So the swap reduces to a portfolio of long and short positions in bonds.

**The Linear Model and Options**

Consider a portfolio consisting of options on a single stock with current price $S$.

The delta of the position is $\delta$, so by definition it is approximately true that

$$\delta = \frac{\Delta P}{\Delta S}$$

$$\Delta P = \delta \Delta S$$

where $\Delta S$ is dollar change in stock price and $\Delta P$ is dollar change in portfolio value in 1 day.

Let $\Delta x$ be the percentage change in stock price in 1 day

$$\Delta x = \frac{\Delta S}{S}$$

so it follows that

$$\Delta P = S\delta\Delta x$$

When the position has several underlying market variables and includes options, the approximate linear relationship is given by

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i$$

where $S_i$ is the value of the $i$th market variable and $\delta_i$ is the delta of the portfolio with respect to the $i$th market variable.

This corresponds to the linear model equation

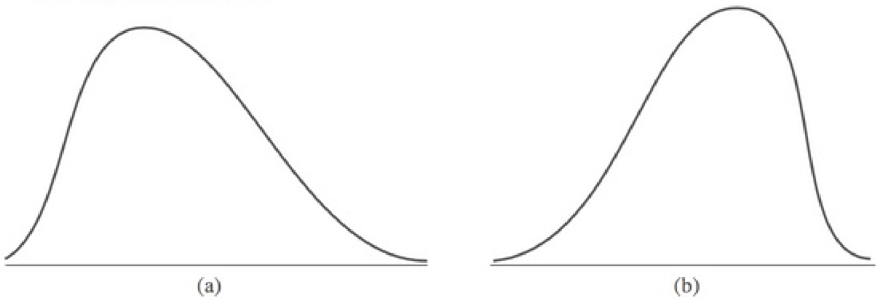
$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

where $\alpha_i = S_i \delta_i$.

# The Quadratic Model

The linear model is only an approximation when the portfolio includes options because it does not include the gamma of the portfolio.

**Figure 22.4** Probability distribution for value of portfolio: (a) positive gamma; (b) negative gamma.




- positive gamma (e.g. long call option) leads to positive skew
- negative gamma (e.g. short call option) leads to negative skew

The VaR for a portfolio is dependent on the left tail of the probability distribution

- positive gamma portfolio has less heavy left tail than the normal distribution
  – if distribution of $\Delta P$ is normal, the calculated VaR tends to be too high
- negative gamma portfolio has heavier left tail than the normal distribution
  – if distribution of $\Delta P$ is normal, the calculated VaR tends to be too low

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Consider a portfolio dependent on a single asset of price $S$.
Let $\delta$ and $\gamma$ be the delta and gamma of the portfolio.

The equation

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

is an improvement over the linear approximation ($\Delta P = \delta \Delta S$).

Generally, for a portfolio with $n$ underlying market variables, with each instrument in the portfolio being dependent on only one of the market variables, the equation becomes

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i^2 \gamma_i (\Delta x_i)^2$$

where $S_i$ is the value of the $i$th market variable, and $\delta_i$ and $\gamma_i$ are the delta and gamma of the portfolio with respect to the $i$th market variable.

When individual instruments may depend on more than one market variable, the equation is

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where $\gamma_{ij}$ is the cross gamma defined as

$$\gamma_{ij} = \frac{\partial^2 P}{\partial S_i \, \partial S_j}$$

## Monte Carlo Simulation

The model-building approach can also be implemented via Monte Carlo simulation to generate the probability distribution for $\Delta P$.

Procedure to calculate the 1-day VaR for a portfolio:

1. Value the portfolio today in the usual way using the current values of market variables
2. Sample from the multivariate normal probability distribution of the $\Delta x_i$
3. Use the values of the $\Delta x_i$ that are sampled to determine the value of each market variable at the end of 1 day
4. Revalue the portfolio at the end of the day
5. Calculate the change in portfolio value for the day $\Delta P$
6. Repeat many times to build a probability distribution for $\Delta P$

The VaR is the appropriate percentile of the probability distribution of $\Delta P$.
The $N$-day VaR is the 1-day VaR multiplied by $\sqrt{N}$.

Monte Carlo simulation tends to be slow because the portfolio has to be revalued many times.

## Comparison of Approaches

Model-building approach

- advantage: results can be produced quickly
- advantage: can easily be used in conjunction with volatility updating schemes
- disadvantage: assumes market variables have a multivariate normal distribution
- disadvantage: tends to give poor results for low-delta portfolios

Historical simulation approach

- advantage: historical data determine the joint probability distribution of market variables
- advantage: avoids need for cash-flow mapping

- disadvantage: computationally slow
- disadvantage: does not easily allow volatility updating schemes to be used

## Stress Testing and Back Testing

stress testing - estimating how company's portfolio would perform under extreme market moves

- way of taking into account extreme events that occur from time to time but are virtually impossible according to the probability distributions assumed for market variables

stressed VaR - VaR based on historical simulation during a period of stressed market conditions

back testing - testing how well the VaR estimates would have performed in the past

## Principal Components Analysis

- an approach to handling the risk arising from groups of highly correlated market variables
- takes historical data on movements of market variables and attempts to define a set of components or factors that explain the movements

factor loading - the market variable move for a particular factor
factor score - quantity of a particular factor in the market variable changes on a particular day

- importance of a factor measured by standard deviation of its factor score

Properties

- factor loadings have the property that the sum of their squares for each factor is 1.0
- factors have the property that the factor scores are uncorrelated across the data
- factor scores have the property that their variances add up to the total variance of the data

#### Using Principal Components Analysis to Calculate VaR

Suppose a financial institution has exposures to a number of different stock indices. A principal components analysis can be used to identify factors describing movements in the indices. The most important factors can be used to replace that market indices in a VaR analysis.

Effectiveness of a principal components analysis for a group of market variables depends on how closely correlated they are.

VaR is usually calculated by relating the actual changes in a portfolio to percentage changes in market variables (the $\Delta x_i$). For a VaR calculation, it may be appropriate to carry out a principal components analysis on percentage changes in market variables rather than actual changes.