Ch 19 The Greek Letters

Each Greek letter measures a different dimension to the risk in an option position.
Goal of the trader is to manage the Greeks so that all risks are acceptable.

Illustration

Consider a financial institution sells for $300,000 a European call option on 100,000 shares of a non-dividend paying stock.

The stock price is $49, the strike price is $50, the risk-free rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks, and the expected return from the stock is 13%.

$$S_0 = 49, \quad K = 50, \quad r = 0.05, \quad \sigma = 0.2, \quad T = 0.3846, \quad \mu = 0.13$$

The Black-Scholes-Merton price of the option is $240,000.
(Value of an option to buy one share is $2.40)

So the financial institution sold a product for $60,000 more than its theoretical value.

However, it is faced with the problem of hedging the risks.

Naked and Covered Positions

One strategy open to the financial institution is to do nothing.

This is a **naked position**

- works well if stock price is below $50 at the end of the 20 weeks
  – costs nothing and makes profit of $300,000
- works less well if call is exercised
  – need to buy 100,000 shares at market price (> $50) to cover the call
  – cost is 100,000 times amount by which stock price exceeds the strike price

Alternative to a naked position is a **covered position**

- buy 100,000 shares as soon as the option has been sold
- works well if option is exercised
- works less well if option is not exercised

Neither a naked position nor a covered position provides a good hedge.

Based on assumptions underlying the Black-Scholes formula, the cost to the financial institution should always be $240,000 on average for a good hedging approach.

A Stop-Loss Strategy

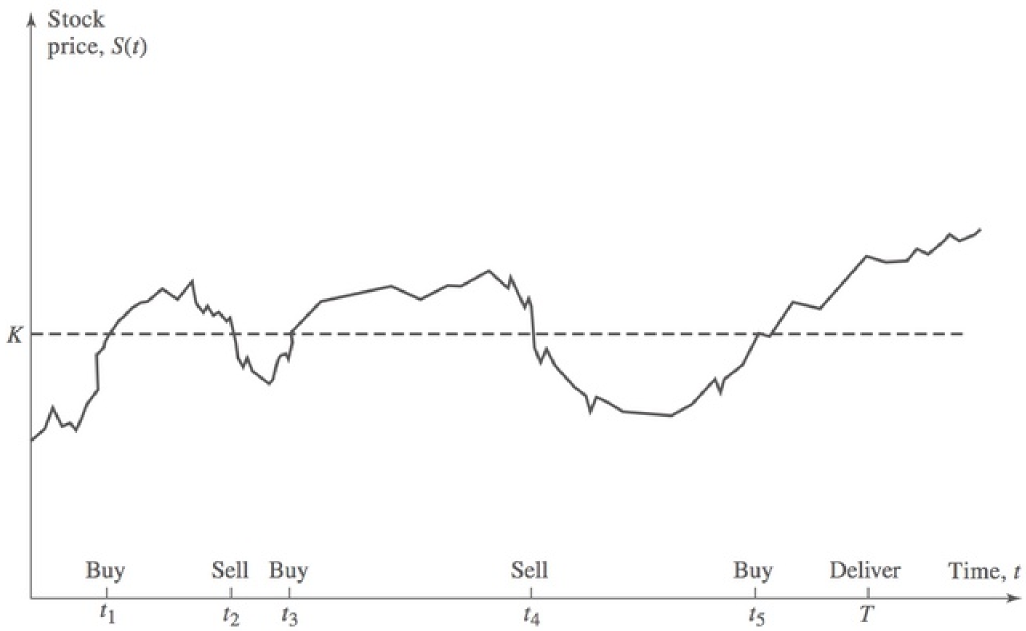
Consider an institution has written a call option with strike price K to buy one unit of stock.

Hedging procedure involves buying one unit of the stock as soon as its price rises above K and selling it as soon as its price falls below K.

- hold a naked position whenever the stock price is less than K
- hold a covered position whenever the stock price is more than K

Ensures that at time T the institution

- owns the stock if the option closes in the money
- does not own the stock if the option closes out of the money



Denote initial stock price as S_0.

The cost of setting up the hedge initially is S_0 if $S_0 > K$ and zero otherwise.

In practice, traders cannot buy or sell at the instant the price is exactly equal to K.

- purchases are made it price $K + \epsilon$
- sales are made at price $K - \epsilon$

So every purchase and subsequent sale involves a cost (aside from transaction costs) of 2ϵ.

The response is to monitor prices closely to make ϵ as small as possible.
However, reducing ϵ means making trades more frequently.

The cost reduction per trade is offset by the increased number of trades.

The expected number of times a Wiener process equals a particular value in a given time is ∞.
As $\epsilon \to 0$, the expected number of trades tends to ∞.

A stop-loss strategy does not work very well as a hedging procedure.

For an out-of-the-money option, if stock prices never reaches strike price K, the hedging procedure costs nothing. If the path of the stock price crosses the strike price level many times, the hedging procedure is expensive.

Monte Carlo simulation can be used to assess the performance of stop-loss hedging

- one hedge performance measure is the standard deviation of the cost of hedging the option to he Black-Scholes-Merton price
- an effective hedging scheme should have a hedge performance measure close to zero
- the stop-loss strategy has a hedge performance measure above 0.7

This emphasizes the stop-loss strategy is not a good hedging procedure.

Delta Hedging

delta (Δ) - rate of change of the option price with respect to the price of the underlying asset

- slope of the curve that relates the option price to the underlying asset price

$$\Delta = \frac{\partial c}{\partial S}$$

where c is the call price and S is the stock price.

Let stock price be $100 and option price be $10. An investor who sold call options to buy 2000 shares of a stock (sold 20 call option contracts) can hedge by buying 1200 shares (let Δ be 0.6).

Gains or losses on the stock position would then offset losses or gains on the option position.

The delta of the short position in 2000 options is

$$0.6 \times (-2000) = -1200$$

so the investor loses $1200\Delta S$ on the option position when the stock price increases by ΔS.

The delta of one share of the stock is 1, so the long position in 1200 shares has delta of $+1200$.

The delta of the investor's overall position is zero (**delta neutral**).

The delta of an option does not remain constant

- need to adjust the hedge periodically (**rebalancing**)

dynamic hedging - hedge adjusted on a regular basis
static hedging - hedge is set up initially and never adjusted

Delta of European Stock Options

For a European call option on a non-dividend-paying stock

$$\Delta(\text{call}) = N(d_1) \qquad (\text{long})$$

$$\Delta(\text{call}) = -N(d_1) \quad (\text{short})$$

For a European put option on a non-dividend-paying stock

$$\Delta(\text{put}) = N(d_1) - 1 \qquad (\text{long})$$

$$\Delta(\text{put}) = 1 - N(d_1) \qquad (\text{short})$$

Note that delta is negative for a (long) put, so a long position in a put option should be hedged with a long position in the underlying stock. However, a long position in a call option should be hedged with a short position in the underlying stock.

Dynamic Aspects of Delta Hedging

A Monte Carlo simulation can show that delta hedging is a great improvement over the stop-loss strategy. Unlike a stop-loss strategy, the performance of a delta-hedging strategy gets better as the hedge is monitored more frequently.

Goal of delta hedging is to keep the value of the position as close to unchanged as possible.

Delta of a Portfolio

The delta of a portfolio of derivatives dependent on a single asset whose price is S is

$$\frac{\partial \Pi}{\partial S}$$

where Π is the value of the portfolio.

It can be calculated from the individual option deltas

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where w_i is the quantity of option i and Δ_i is the delta of the option i.

Transaction Costs

Derivatives dealers usually rebalance their positions once a day to maintain delta neutrality. This is more feasible for large portfolios than for small portfolios due to bid-offer spreads.

Only one trade in the underlying asset is needed to zero out the delta of the whole portfolio. The bid-offer spread transaction costs are absorbed by the profits on many different trades.

Theta

theta - rate of change of portfolio value with respect to the passage of time

- referred to as the **time decay** of the portfolio

For a European call option on a non-dividend-paying stock

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where $N'(x)$ is the probability density function for a standard normal distribution.

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

For a European put option on a non-dividend-paying stock

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Note that $N(-d_2) = 1 - N(d_2)$ so the theta of a put exceeds the theta of a corresponding call by rKe^{-rT}.

The formulas measure time in years, but a quoted theta is usually measured in days

- divide formula for theta by 365 to obtain theta per calendar day
- divide formula for theta by 252 to obtain theta per trading day

Theta is usually negative for an option because the option becomes less valuable as time passes.

For a call option

- when stock price is low, theta is close to zero
- for an at-the-money option, theta is large and negative
- as stock price rises, theta tends to $-rKe^{-rT}$

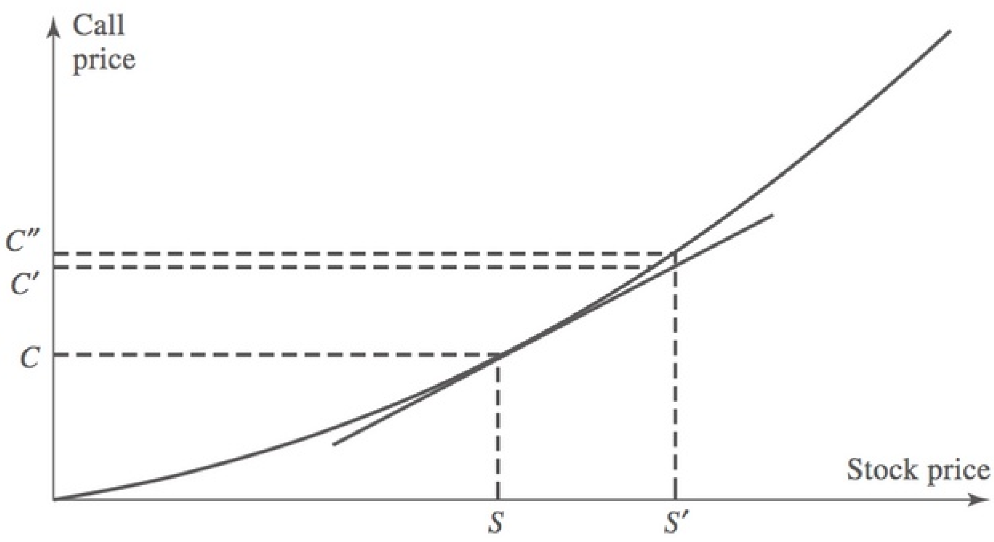
Theta is different from delta because there is uncertainty about future stock price but no uncertainty that time will pass. So it makes sense to hedge against changes in the price of the underlying asset but not to hedge against the passage of time. However, theta is still a useful descriptive statistic for a portfolio.

Gamma

gamma - rate of change of portfolio delta with respect to the price of the underlying asset

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

- if gamma is small, delta changes slowly
  – adjustments to keep portfolio delta neutral need to be made only infrequently
- if gamma is large (negative or positive), delta is very sensitive to price of underlying asset
  – risky to leave a delta-neutral portfolio unchanged for any length of time



There is a hedging error introduced by nonlinearity. Stock price moves from S to S'. The delta suggests the option price moves from C to C'. However it actually moves to C'', so the difference $C'' - C'$ leads to a hedging error. The size of the error depends on the curvature of the relationship between option price and stock price. Gamma measures the curvature.

Let ΔS be the price change of the underlying asset during a time interval Δt.
Let $\Delta\Pi$ be the corresponding price change in the portfolio.

Ignoring terms of order higher than Δt

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2$$

for a delta-neutral portfolio.

When gamma is positive, theta tends to be negative

- portfolio declines in value if no change in S
- portfolio increases in value if large change in S

When gamma is negative, theta tends to be positive

- portfolio increases in value if no change in S
- portfolio decreases in value if large change in S

As $|\Gamma|$ increases, the sensitivity of the value of the portfolio to S increases.

Making a Portfolio Gamma Neutral

A position in the underlying asset has zero gamma. To change gamma of the portfolio, a position in an option that is not linearly dependent on the underlying asset is needed.

Suppose a delta-neutral portfolio has gamma equal to Γ, a traded option has gamma equal to Γ_T, and the number of traded options in the portfolio is w_t.

The gamma of the portfolio is

$$w_t \Gamma_t + \Gamma$$

So the position in the traded option needed to make the portfolio gamma neutral is $-\Gamma / \Gamma_T$.

Including the option is likely to change the portfolio delta, so the position in the underlying asset has to be rebalanced to maintain delta neutrality.

Note that the portfolio is gamma neutral only for a short period of time. Maintain gamma neutrality by adjusting the position in the traded option so that it is always equal to $-\Gamma / \Gamma_T$.

Making a portfolio gamma neutral and delta neutral can be regarded as a correction for the hedging error caused by non-linearity.

- delta neutrality provides protection against small stock prices moves
- gamma neutrality provides protection against large stock price moves

Calculation of Gamma

For a European call or put option on a non-dividend-paying stock

$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

The gamma of a long position is always positive.

Relationship Between Delta, Theta, and Gamma

The price of a single derivative dependent on a non-dividend-paying stock must satisfy

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

which is the Black-Scholes differential equation.

It follows the value Π of a portfolio of such derivatives must satisfy the differential equation

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Note that

$$\Theta = \frac{\partial \Pi}{\partial t} \qquad \Delta = \frac{\partial \Pi}{\partial S} \qquad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

so it follows that

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

For a delta-neutral portfolio

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

- when theta is large and positive, gamma is large and negative
- when theta is large and negative, gamma is large and positive

So theta can to some extent be regarded as a proxy for gamma in a delta-neutral portfolio.

Vega

Volatilities change over time. The value of derivative can change due to movements in volatility.

vega - rate of change of the portfolio value with respect to the volatility of the underlying asset

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$$

If vega is highly positive or highly negative, the portfolio value is highly sensitive to small changes in volatility. If vega is close to zero, volatility changes have little impact on the value of the portfolio.

A position in the underlying asset has zero vega. The vega of a portfolio can be changed by adding a position in a traded option (same as changing gamma). If $\mathcal{V}$ is the vega of a portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position $-\mathcal{V}/\mathcal{V}_T$ in the traded option makes the portfolio vega neutral.

A portfolio that is gamma neutral will generally not be vega neutral, and vice versa.
If a portfolio must be both gamma and vega neutral, at least two traded derivatives are needed.

For a European call or put option on a non-dividend-paying stock

$$\mathcal{V} = S_0\sqrt{T}N'(d_1)$$

The vega of a long position in a European or American option is always positive.

Black-Scholes assumes volatility is constant, so it is theoretically more correct to calculate vega from a model in which volatility is a stochastic process. However, the vega calculated in that way is similar to the Black-Scholes vega, so this works reasonably well.

Gamma neutrality protects against large changes in price of underlying asset.
Vega neutrality protects against a variable volatility.

Implied volatilities of short-dated options tend to change by more than the implied volatilities of long-dated options. So the vega of a portfolio is often calculated by changing the volatilities of long-dated options by less than that of short-dated options.

Rho

rho - rate of change of the portfolio value with respect to the interest rate

$$\rho = \frac{\partial \Pi}{\partial r}$$

It measures the sensitivity of the value of a portfolio to a change in the interest rate.

For a European call option on a non-dividend-paying stock

$$\rho = KTe^{-rT}N(d_2)$$

For a European put option on a non-dividend-paying stock

$$\rho = -KTe^{-rT}N(-d_2)$$

The Realities of Hedging

In an ideal world, traders can rebalance portfolios very frequently to maintain all Greeks at zero.

When managing a large portfolio dependent on a single underlying asset, traders make delta close to zero, but zero gamma and zero vega are less easy to achieve. This is because it is difficult to find options or other nonlinear derivatives that can be traded in the volume required at competitive prices.

There are big economies of scale in trading derivatives

- maintaining delta neutrality for a small portfolio is not economically feasible (trading costs)
- maintaining delta neutrality for a large portfolio is likely to be much more reasonable

Scenario Analysis

Scenario analysis involves calculating the gain or loss on a portfolio over a specified period under a variety of different scenarios. The time period is based on the liquidity of the instruments. The scenarios can be chosen or generated by a model.

Extension of Formulas

Consider a stock that pays a continuous dividend yield at rate q.

Greek letter	*Call option*	*Put option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1)-1]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $-qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

- obtain Greeks for European options on indices by setting q equal to the dividend yield on an index
- obtain Greeks for European options on currencies by setting q equal to the foreign risk-free rate
- obtain Greeks for European options on a futures contract by setting $q=r$
  - ρ for call futures option is $-cT$
  - ρ for put futures option is $-pT$

Delta of Forward Contracts

Consider a forward contract on a non-dividend-paying stock.

Recall the value of a forward contract is $S_0 - Ke^{-rT}$ where K is the delivery price and T is the forward contract's time to maturity. When stock price changes by ΔS, the value of a forward contract also changes by ΔS.

So the delta of a long forward contract on one share of the stock is always 1.

- a long forward contract can be hedged by shorting one share
- a short forward contract can be hedged by purchasing one share

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Consider a forward contract on a stock paying a known dividend yield at rate q.

Recall the value of a forward contract is

$$f = S_0 e^{-qT} - Ke^{-rT}$$

When stock price changes by ΔS, the value of a forward contract changes by $e^{-qT}\Delta S$.

So the delta of a long forward contract on one share of the stock is e^{-qT}.

- for forward contract on a stock index, q is the dividend yield on the index
- for forward contract on a foreign exchange, q is the foreign risk-free rate rf

Delta of a Futures Contract

Recall the futures price for a contract on a non-dividend-paying stock is

$$f = S_0 e^{rT}$$

When the stock price changes by ΔS, the futures price changes by $e^{rT}\Delta S$.
Futures contracts are settled daily so the holder of a long futures position makes an almost immediate gain of this amount. So the delta of a futures contract is e^{rT}.

For a futures position on an asset providing a dividend yield q, the delta is $e^{(r-q)T}$.

Daily settlement makes the deltas of futures and forward contracts slightly different.
This is true even when interest rates are constant and the forward price equals the futures price.

A futures contract can be used to achieve delta neutrality.

H_A : Required position in asset for delta hedging
H_F : Required position in futures contracts for delta hedging

Let T be the maturity of the futures contract.

If the underlying asset is a non-dividend-paying stock, then the above analysis shows that

$$H_F = e^{-rT} H_A$$

When the underlying asset pays a dividend yield q then

$$H_F = e^{-(r-q)T} H_A$$

Portfolio Insurance

Put options provide protection against market declines. They can be bought or created.

Creating an option synthetically involves maintaining a position in the underlying asset so that the delta of the position equals the delta of the required option. The position necessary to create an option synthetically is the reverse of that necessary to hedge it.

Two reasons to create an option rather than buy it:

1. Markets do not always have the required liquidity to absorb trades
2. Markets may not have the required strike prices and exercise dates

Synthetic option can be created from

- trading the portfolio
- trading in index futures contracts

Trading the portfolio

The delta of a European put on the portfolio is

$$\Delta = e^{-qT}[N(d_1) - 1]$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

as before.

The volatility of the portfolio can be assumed to be its beta times the volatility of a well-diversified market index.

To create the put option synthetically, the fund manager should ensure that at any given time a proportion

$$e^{-qT}[1 - N(d_1)]$$

of stocks in the original portfolio has been sold and the proceeds invested in riskless assets.

As value of portfolio declines, the delta of the put becomes more negative, so the proportion of the original portfolio sold must be increased.

As value of portfolio increases, the delta of the put becomes less negative, so the proportion of the original portfolio sold must be decreased (some of it must be repurchased).

Use of Index Futures

Using index futures to create options synthetically can be preferable to using underlying stocks because transaction costs are lower with trades in index futures than with trades in stocks.

The dollar amount of the futures contracts shorted as a proportion of the portfolio value is

$$e^{-qT}e^{-(r-q)T^*}[1 - N(d_1)] = e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]$$

where T^* is the maturity of the futures contract.

Note that this is H_F from above multiplied by the proportion of stocks in the original portfolio to sell to create a synthetic put option.

If the portfolio is worth A_1 times the the index and each index futures contract is on A_2 times the index, the number of futures contracts shorted at any given time should be

$$e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]A_1/A_2$$

The above analysis assumes the portfolio mirrors the index.

When this is not true then it is necessary to

- calculate portfolio beta
- find position in options on the index that gives required protection
- choose position in index futures to create options synthetically

Stock Market Volatility

Portfolio insurance strategies (like those above) have the potential to increase volatility

- when market declines, managers sell stock or index futures contracts
  – these actions may accentuate the decline
- when market rises, managers buy stock or index futures contracts
  – these actions may accentuate the rise

These strategies can affect volatility if they are too large for the market to absorb the trades.