Ch 14 Wiener Processes and Ito's Lemma

A variable whose value changes over time in an uncertain way follows a **stochastic process**.

- stochastic processes can be **discrete time** or **continuous time**
- stochastic processes can be **discrete variable** or **continuous variable**

The Markov Property

A **Markov process** is a type of stochastic process where only the current value of a variable is relevant for predicting the future. The history of the variable and the way the present state has emerged are irrelevant.

- stock prices assumed to follow a Markov process
- predictions for future are uncertain, must be expressed in terms of probability distributions
- Markov property implies the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past

Markov property of stock prices is consistent with the weak form of market efficiency.

- present price of a stock includes all information contained in a record of past prices
- if weak form efficiency were not true, technical analysts could make above-average returns – there is little evidence that they can do this

Competition in the marketplace ensures weak-form efficiency and the Markov property hold.

- many investors watching the market closely
- patterns causing a stock price to rise would lead to high demand for the stock
- this leads to the stock price rising, eliminating the observed effect

Continuous-Time Stochastic Processes

Consider a variable that follows a Markov stochastic process.

Current value is 10. The change in its value during a year is $\phi(0,1)$.

$\phi(m,v)$ denotes a probability that is normally distributed with mean m and variance v.

What is the probability distribution of the change in the value of the variable during 2 years?

The change in 2 years is the sum of two normal distributions, each of which has a mean of zero and a variance of 1. Because the variable is Markov, the two probability distributions are independent.

If ϕ_1 and ϕ_2 are independent normal distributions, then the sum is given by

$$\phi_1(m_1, v_1) + \phi_2(m_2, v_2) = \phi(m_1 + m_2, v_1 + v_2)$$

$$\phi(0,1) + \phi(0,1) = \phi(0,2)$$

So the mean of the change during 2 years in the variable is 0 and the variance of the change is 2.
The standard deviation of the change is $\sqrt{2}$.

What is the change in the variable during 6 months?

The variance of the change during 1 year equals the variance of the change during the first 6 months plus the variance of the change in the second 6 months. Assume these are the same.
So the variance of the change during a 6-month period is 0.5. Then the standard deviation of the change is $\sqrt{0.5}$. The probability distribution is $\phi(0, 0.5)$.

A similar argument shows the probability distribution for the change in the value of the variable during 3 months is $\phi(0, 0.25)$.

Generally, the change during the any time period of length T is $\phi(0, T)$.
The change during a very short time period of length Δt is $\phi(0, \Delta t)$.

Wiener Process

The process followed by the variable above is a **Wiener process**. It is a type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year. It is closely related to **Brownian motion**.

A variable z follows a Wiener process if it has the following two properties:

$\text{Property } 1.$ The change Δz during a small period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution $\phi(0, 1)$.

$\text{Property } 2.$ The values of Δz for any two different intervals of time Δt are independent.

It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$

$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$

$$\text{variance of } \Delta z = \Delta t$$

The second property implies that z follows a Markov process.

Consider the change in the value of z during a long period of time T, $z(T) - z(0)$.
It is the sum of changes in z in N small time intervals of length Δt, where

$$N = \frac{T}{\Delta t}$$

So

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$$

where the ϵ_i are distributed $\phi(0, 1)$.

From the second property of Wiener processes, the ϵ_i are independent of each other.

It follows that $z(T) - z(0)$ is normally distributed, with

$$\text{mean of } [z(T) - z(0)] = 0$$

$$\text{variance of } [z(T) - z(0)] = N\Delta t = T$$

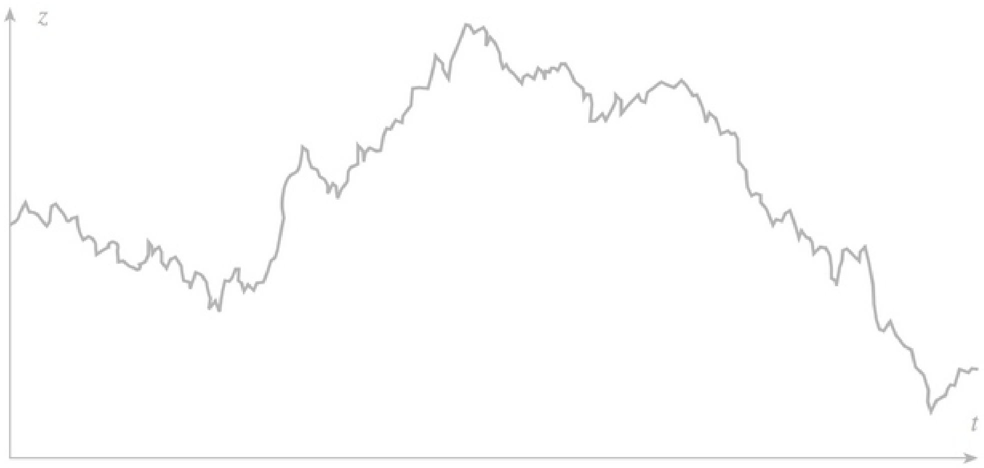
$$\text{standard deviation of } [z(T) - z(0)] = \sqrt{T}$$

Note that the standard deviation of the movement in z in time Δt equals $\sqrt{\Delta t}$.
As $\Delta t \to 0$, $\sqrt{\Delta t} > \Delta t$. This results in the two properties below.

Two properties of Wiener processes:

1. The expected length of the path followed by z in any time interval is infinite.
2. The expected number of times z equals any particular value in any time interval is infinite.

This is because z has nonzero probability of equaling any value v in the time interval.
It it equals v in time t, the expected number of times it equals v in the vicinity of t is infinite.



The true process obtained as $\Delta t \to 0$

Generalized Wiener Process

drift rate - the mean change per unit time for a stochastic process
variance rate - the variance per unit time for a stochastic process

The Wiener process considered so far has drift rate zero and variance rate 1.

- drift rate zero means expected value of z at any future time equals its current value
- variance rate 1 means the variance of the change in z in time interval of length T equals T

A *generalized Wiener process* for a variable x can be defined in terms of dz as

$$dx = a\,dt + b\,dz$$

where a and b are constants.

The $a\,dt$ terms implies that x has an expected drift rate of a per unit of time.
Without the second term, the equation is $dx = a\,dt$, which implies $\frac{dx}{dt} = a$.

Integrate with respect to time

$$x = x_0 + at$$

where x_0 is the value of x at time 0. In period of length T, variable x increases by amount aT.

The $b\,dz$ term adds noise or variability to the path followed by x. The amount of this noise or variability is b times a Wiener process. A Wiener process has variance rate of 1, so b times a Wiener process has a variance rate per unit time of b^2.

In a small time interval Δt, the change in the value of x is given by

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$$

where $\epsilon \sim \phi(0, 1)$.

This means Δx has a normal distribution with

$$\text{mean of } \Delta x = a\Delta t$$

$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$

$$\text{variance of } \Delta x = b^2 \Delta t$$

Similar arguments can show that the change in the value of x in any time interval T is normally distributed with

$$\text{mean of } x = aT$$

$$\text{standard deviation of } x = b\sqrt{T}$$

$$\text{variance of } x = b^2 T$$

So the generalized Wiener process given above has expected drift rate a and variance rate b^2.

Ito Process

An **Ito process** is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and time t.

$$dx = a(x, t)\, dt + b(x, t)\, dz$$

Both the expected drift rate and variance rate are liable to change over time.

In a small interval between t and $t + \Delta t$, the variable changes from x to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

This assumes the drift and variance rate of x do not change during the interval t to $t + \Delta t$.

Note that the Ito process as written above is a Markov process because the change in x and time t depends only on the value of x at time t and not on its history. A non-Markov Ito process can be defined by letting a and b depend on the values of x prior to time t.

The Process for a Stock Price

Stock price does not follow a generalized Wiener process (constant expected drift rate and constant variance rate) because the expected return required by investors is independent of the stock's price.

The assumption of constant expected drift rate must be replaced by the assumption that the expected return (expected drift divided by stock price) is constant.

Let S be the stock price at time t. Then the expected drift rate is μS for some constant μ.
This means in a short interval of time Δt, the expected increase in S is $\mu S \Delta t$.
The a parameter μ is the expected rate of return on the stock.

If the coefficient of dz is zero (no uncertainty), then the model implies that

$$\Delta S = \mu S \Delta t$$

In the limit $\Delta \to 0$

$$ds = \mu S\, dt$$

$$\frac{dS}{S} = \mu\, dt$$

Integrate between time 0 and time T

$$ln(S_T) - ln(S_0) = ln\left(\frac{S_T}{S_0}\right) = \mu T$$

$$S_T = S_0 e^{\mu T}$$

where S_0 and S_T are the stock price at time 0 and time T.

This shows that when there is no uncertainty, the stock price grows at a continuously compounded rate of μ per unit of time.

In practice, there is uncertainty. Assume the variability of the return in a short period of time Δt is the same regardless of the stock price. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price.

$$dS = \mu S\, dt + \sigma S\, dz$$

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz$$

This is a widely used model of stock price behavior.

- μ is the stock's expected rate of return
- σ is the volatility of the stock price
- σ^2 is the variance rate

In a risk-neutral world, μ equals the risk-free rate r.

Discrete-Time Model

The model above (also shown below) is known as **geometric Brownian motion**.

$$dS = \mu S\, dt + \sigma S\, dz \quad \text{or} \quad \frac{dS}{S} = \mu\, dt + \sigma\, dz$$

The discrete-time version of the model is

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t} \quad \text{or} \quad \frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

The variable ΔS is the change in the stock price S in a small time interval Δt, and $\epsilon \sim \phi(0,1)$.

- μ is the stock's expected rate of return
- σ is the volatility of the stock price

The parameters μ and σ will be assumed to be constant.

The quantity $\frac{\Delta S}{S}$ is the discrete approximation to the return provided by the stock in Δt.

- $\mu \Delta t$ is the expected value of this return
- $\sigma \epsilon \sqrt{\Delta t}$ is the stochastic component of the return.

The variance of the stochastic component (and the whole return) is $\sigma^2 \Delta t$.

The quantity $\frac{\Delta S}{S}$ is approximately normally distributed

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma^2 \Delta t)$$

Monte Carlo Simulation

A Monte Carlo simulation of a stochastic process is a procedure for sampling random outcomes for the process.

The Parameters

The parameter μ is the annualized expected return earned in a short period of time.

- most investors require higher expected returns to induce them to take higher risks
- so μ should depend on the risk of the return from the stock
- it should also depend on the level of interest rates in the economy
  – higher interest rates means higher the expected return required on a given stock

However, the determinants of μ are not a concern because the value of a derivative dependent on a stock is in general independent of μ.

On the other hand, the parameter σ (stock price volatility) is important to the determination of the value of many derivatives.

The standard deviation of the proportional change in the stock price in Δt is $\sigma\sqrt{\Delta t}$.
The standard deviation of the proportional change over a long period T is roughly $\sigma\sqrt{T}$.

This means volatility can be approximated as the standard deviation of the change in the stock price in 1 year. The volatility of the stock price is actually exactly equal to the standard deviation of the continuously compounded return provided by the stock in 1 year.

Correlated Processes

Now extend the analysis to two or more variables following correlated stochastic processes.

Suppose the processes followed by two variables x_1 and x_2 are

$$dx_1 = a_1\,dt + b_1\,dz_1 \quad \text{and} \quad dx_2 = a_2\,dt + b_2\,dz_2$$

where dz_1 and dz_2 are Wiener processes.

The discrete-time approximations for these processes are

$$\Delta x_1 = a_1\Delta t + b_1\epsilon_1\sqrt{\Delta t} \quad \text{and} \quad \Delta x_2 = a_2\Delta t + b_2\epsilon_2\sqrt{\Delta t}$$

where ϵ_1 and ϵ_2 are samples from a standard normal distribution $\phi(0, 1)$.

If the correlation $\rho(x_1, x_2) = 0$, then the random samples ϵ_1 and ϵ_2 should be independent.

If the correlation $\rho(x_1, x_2) \neq 0$, then ϵ_1 and ϵ_2 should be sampled from a bivariate normal distribution. Each variable in the bivariate normal distribution has a standard normal distribution and the correlation between the variables if ρ. The Wiener processes dz_1 and dz_2 are referred to as having a correlation ρ.

To sample standard normal variables ϵ_1 and ϵ_2 with correlation ρ

$$\epsilon_1 = u \quad \text{and} \quad \epsilon_2 = \rho u + \sqrt{1-\rho^2 v}$$

where u and v are sampled as uncorrelated variables with standard normal distributions.

Note that the parameters a_1, a_2, b_1, and b_2 can be functions of x_1, x_2, and t.
In fact, a_1 and b_1 can be functions of x_2 as well as x_1 and t.
And, a_2 and b_2 can be functions of x_1 as well as x_2 and t.

These results can be generalized to more than two variables as well. For three variables following correlated stochastic processes, sample three different ϵ's from a trivariate normal distribution. For n correlated variables, there are n different ϵ's sampled from an appropriate multivariate normal distribution.

Ito’s Lemma

The price of a stock option is a function of the underlying stock’s price and time. More generally, the price of a derivative is a function of the stochastic variables underlying the derivative and time.

Suppose the value of a variable x follows the Ito process

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

where dz is a Wiener process and a and b are functions of x and t.

The variable x has drift rate a and variance rate b^2.

Ito's Lemma. A function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\,dz$$

where dz is the same Wiener process as in the above Ito process.

This means that G also follows an Ito process with a drift rate of

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x} \right)^2 b^2$$

It was previous shown that

$$dS = \mu S\,dt + \sigma S\,dz$$

with μ and σ constant. It is a reasonable model of stock price movements.

From Ito’s lemma, it follows that the process followed by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\,dz$$

Note that both S and G are affected by the same underlying source of uncertainty: dz.

Application to Forward Contracts

Consider a forward contract on a non-dividend-paying stock.
Assume the risk-free interest rate is constant and equal to r for all maturities.

$$F_0 = S_0 e^{rT}$$

where F_0 is the forward price at time zero, S_0 is the spot price at time zero, and T is the time to maturity of the forward contract.

Let F and S be forward price and stock price at time t, with $t < T$.

$$F = Se^{r(T-t)}$$

Assume the process for S is given by $dS = \mu S\,dt + \sigma S\,dz$, so Ito's lemma can be used to determine the process for F.

Differentiating the above equation for F gives

$$\frac{\partial F}{\partial S} = e^{r(T-t)} \qquad \frac{\partial^2 F}{\partial S^2} = 0 \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

Using Ito's lemma, the process for F is given by

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right] dt + e^{r(T-t)}\sigma S\,dz$$

Substituting F for $Se^{r(T-t)}$ gives

$$dF = (\mu - r)F\,dt + \sigma F\,dz$$

Like S, the forward price F follows geometric Brownian motion. The expected growth rate is $\mu - r$, rather than μ. The growth rate of F is the excess return of S over the risk-free rate.

The Lognormal Property

Use Ito's lemma to derive the process followed by $\ln S$ when S follows the process

$$dS = \mu S\,dt + \sigma S\,dz$$

Define

$$G = \ln S$$

Differentiating gives

$$\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \qquad \frac{\partial G}{\partial t} = 0$$

From Ito's lemma, it follows that the process followed by G is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\,dz$$

Since μ and σ are constant, the equation indicates that $G = \ln S$ follows a generalized Wiener process. It has constant drift rate $\mu - \frac{\sigma^2}{2}$ and constant variance rate σ^2.

The change in $\ln S$ between time 0 and some future time T is therefore normally distributed with a mean of $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$.

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

This shows that $\ln S_T$ is normally distributed. A variable has a *lognormal distribution* if the natural logarithm of the variable is normally distributed. The model of stock price behavior developed here implies that a stock's price at time T, given its price today, is lognormally distributed.