Ch 15 The Black-Scholes-Merton Model

Lognormal Property of Stock Prices

Assume the percentage changes in stock price in a short period of time are normally distributed.

Define

μ : Expected return on stock per year
σ : Volatility of the stock price per year

The mean and standard deviation of the return in time Δt are approximately $\mu\Delta t$ and $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t)$$

where ΔS is the change in stock price S in time Δt.

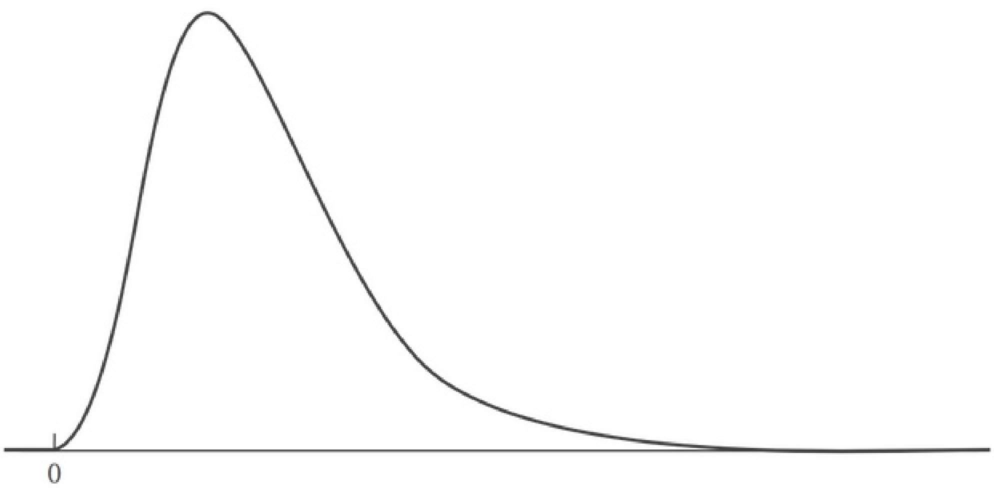
The previous chapter shows the model implies that

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where S_T is the stock price at time T and S_0 is the stock price at time 0.

The variable $\ln S_T$ is normally distributed, so S_T has a lognormal distribution.

A variable that has a lognormal distribution can take any value between zero and infinity.



From the properties of the lognormal distribution, the expected value and variance of S_T are

$$E(S_T) = S_0 e^{\mu T}$$

$$\mathrm{Var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

The Distribution of the Rate of Return

Let x be the continuously compounded rate of return per annum between times 0 and T

$$S_T = S_0 e^{xT}$$

so that

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

A slight modification of a previous equation is given by

$$\ln \frac{S_T}{S_0} \sim \phi \left[\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right]$$

so it follows that

$$x \sim \phi \left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T} \right)$$

So the continuously compounded rate of return (per annum) is normally distributed

- mean $\mu - \sigma^2/2$
- standard deviation $\sigma/\sqrt{T}$

As T increases, the standard deviation of x decreases.

The Expected Return

The model implies that the mean return in a short period of time is $\mu\Delta t$. Note that μ is not the continuously compounded return on the stock. The continuously compounded return x over a period of time of length T is given by

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

as shown above.

Why is the expected return $E(x) = \mu - \sigma^2/2$ instead of μ?

Start with

$$E(S_T) = S_0 e^{\mu T}$$

Take the logarithm

$$\ln\left[E(S_T)\right] = \ln\left(S_0\right) + \mu T$$

Incorrect approach:

$$\ln\left[E(S_T)\right] = E[\ln\left(S_T\right)]$$

$$E[\ln\left(S_T\right)] - \ln\left(S_0\right) = \mu t$$

$$E\left[\ln \frac{S_T}{S_0}\right] = \mu T$$

$$E(x) = \mu$$

This is incorrect because $\ln$ is a nonlinear function. It is the case that

$$\ln\left[E(S_T)\right] > E[\ln\left(S_T\right)]$$

so

$$E[\ln\left(S_T/S_0\right)] < \mu T$$

which leads to

$$E(x) < \mu$$

This is consistent with the previous finding that $E(x) = \mu - \sigma^2/2$.

Volatility

The volatility σ of a stock is a measure of uncertainty about the returns provided by the stock.

Volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding.

When Δt is small, the variance of the percentage change in stock price in time Δt is approximately $\sigma^2 \Delta t$. So the standard deviation of the percentage change in stock price in time Δt is $\sigma\sqrt{\Delta t}$.

Estimating Volatility from Historical Data

Estimate volatility empirically by observing stock price at fixed intervals of time.

Let $n+1$ be number of observations, S_i be stock price at end of ith interval, and τ be length of time interval in years.

Let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \text{ for } i = 1, 2, \ldots, n$$

From previous, the standard deviation of the u_i is given by $\sigma\sqrt{\tau}$.
The estimate s of the standard deviation is therefore an estimate of $\sigma\sqrt{\tau}$.
It follows that σ itself can be estimated as

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate is approximately $\hat{\sigma}/\sqrt{2n}$.

Trading Days vs Calendar Days

Research shows volatility is much higher when the exchange is open for trading than when it is closed. Practitioners tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option.

$$\text{Vol per annum} = \text{Vol per trading day} \times \sqrt{\text{Number of trading days per annum}}$$

Number of trading days is usually assumed to be 252 for stocks.

The life of an option is also usually measured using trading days rather than calendar days.

$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

The Idea Underlying the Black-Scholes-Merton Differential Equation

The Black-Scholes-Merton differential equation must be satisfied by the price of any derivative dependent on a non-dividend-paying stock.

When stock price movements were assumed to be binomial, valuation involved setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. This can be set up because the stock price and derivative price are both affected by the same underlying source of uncertainty: stock price movements.

In a short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. In the riskless portfolio, the gain/loss from the stock position is offset by the gain/loss from the derivative position. The value of the portfolio at the end of the short period of time is known with certainty.

Consider a point in time when the change in stock price and the resultant change in European call option price is given by

$$\Delta c = 0.4\,\Delta S$$

A riskless portfolio consists of:

1. A long position in 40 shares
2. A short position in 100 call options

Increasing stock price by 10 cents will increase option price by 4 cents.
The gain on 40 shares is $4 and the loss on the short position in 100 call options is also $4.

The difference between Black-Scholes-Merton model and binomial model is that the position in the stock and derivative is riskless for only an instantaneously short period of time. The portfolio must be *rebalanced* frequently to remain riskless. It remains true that the return from the riskless portfolio in any short period of time must be the risk-free interest rate.

Assumptions

1. Stock price follows the process developed in the previous chapter with μ and σ constant
2. Short selling of securities with full use of proceeds is permitted
3. No transaction costs or taxes
4. No dividends during life of derivative
5. No riskless arbitrage opportunities
6. Security trading is continuous
7. Risk-free interest rate r is constant and the same for all maturities

Derivation of the Black-Scholes-Merton Differential Equation

The stock price process assumed is the following (continuous and discrete versions)

$$dS = \mu S\,dt + \sigma S\,dz$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta z$$

Let f be the price of a call option or other derivative contingent on S.
It must be some function of S and t.

From Ito's lemma

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S\,dz$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z$$

The Wiener processes underlying f and S are the same. This means the Δz (equal to $\epsilon\sqrt{\Delta t}$) are the same in the equations for ΔS and Δf.

It follows that a portfolio of the stock and the derivative can be constructed to eliminate the Wiener process. The portfolio is:

1. -1 derivative
2. $+\partial f/\partial S$ shares

Let Π be the value of the portfolio, so by definition

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in value of the portfolio in time interval Δt is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

Substitute Δf and ΔS using the discrete-time equations above

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

Since the equation does not involve Δz, the portfolio must be riskless during time Δt. Due to the assumption of no-arbitrage, this portfolio must earn the same rate as other short-term risk-free securities.

It follows that

$$\Delta \Pi = r\Pi\Delta t$$

where r is the risk-free interest rate.

Substituting for Π and $\Delta\Pi$ from above

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S} S\right)\Delta t$$

which leads to the **Black-Scholes-Merton differential equation**

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

There are many different solutions, corresponding to the different derivatives that can be defined with S as the underlying variable. The particular derivative obtained when the equation is solved depends on the *boundary conditions* that are used.

Boundary conditions specify the values of the derivative at the boundaries of possible values of S and t.

For European call options, the key boundary condition is

$$f = \max(S - K, 0) \quad \text{when } t = T$$

For European put options, the key boundary condition is

$$f = \max(K - S, 0) \quad \text{when } t = T$$

Consider a forward contract on a non-dividend-paying stock. The value of the contract f at a general time t in terms of the stock price S at this time is given by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price.

This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)} \qquad \frac{\partial f}{\partial S} = 1 \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

Substitute these into the Black-Scholes equation to get

$$-rKe^{-r(T-t)} + rS$$

This equals rf, so the Black-Scholes equation is satisfied.

A Perpetual Derivative

Consider a perpetual derivative that pays a fixed amount Q when the stock price equals H for the first time. The value of the derivative for a particular S has no dependence on t. The $\partial f/\partial t$ terms vanishes and the Black-Scholes PDE becomes an ODE.

Suppose $S < H$. The boundary conditions are $f = 0$ when $S = 0$ and $f = Q$ when $S = H$. The solution $f = QS/H$ satisfies both boundary conditions and the differential equation, so it must be the value of the derivative.

Suppose $S > H$. The boundary conditions are $f = 0$ as S goes to infinity and $f = Q$ when $S = H$. The derivative price

$$f = Q\left(\frac{S}{H}\right)^{-\alpha}$$

where $\alpha > 0$ satisfies the boundary conditions. It also satisfies the differential equation when

$$-r\alpha + \frac{1}{2}\sigma^2\alpha(\alpha+1) - r = 0$$

which means $\alpha = 2r/\sigma^2$. The value of the derivative is therefore

$$f = Q\left(\frac{S}{H}\right)^{-2r/\sigma^2}$$

The Prices of Tradeable Derivatives

Any function $f(S,t)$ that is a solution of the Black-Scholes differential equation is the theoretical price of a derivative that could be traded. A derivative with that price does not create arbitrage opportunities.

A function $f(S,t)$ that does not satisfy the Black-Scholes differential equation cannot be the price of a derivative without creating arbitrage opportunities for traders.

Risk-Neutral Valuation

Risk-neutral valuation arises from the property of the Black-Scholes-Merton differential equation that the equation does not involve any variables that are affected by the risk preferences of investors.

The variables that do appear in the equation are:

- current stock price
- time
- price volatility
- risk-free rate of interest

All of these are independent of risk preferences.

Steps to value a derivative using risk-neutral valuation:

1. Assume the expected return from the underlying asset is the risk-free interest rate r
2. Calculate the expected payoff from the derivative
3. Discount the expected payoff at the risk-free interest rate

Risk-neutral valuation is merely a device for obtaining solutions to the Black-Scholes equation. However, the solutions obtained are valid in all worlds.

When moving from a risk-neutral world to a risk-averse world, two things happen:

1. The expected growth rate in the stock price changes
2. The discount rate applied to payoffs from the derivatives changes

The above two changes always offset each other exactly.

Application to Forward Contracts on a Stock

Consider a long forward contract that matures at time T with delivery price K. The value of the contract at maturity is

$$S_T - K$$

where S it the stock price at time T.

From the risk-neutral valuation argument, the value of the contract at time 0 is its expected value at time T discounted at the risk-free interest rate.

Let the value of the contract at time 0 be f, which means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world.

Since K is a constant, this becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT}$$

The expected return μ on the stock becomes r in a risk-neutral world, so

$$\hat{E}(S_T) = S_0 e^{rT}$$

Substitute this into the equation for f to get

$$f = S_0 - Ke^{-rT}$$

as expected.

Black-Scholes-Merton Pricing Formulas

Solution to Black-Scholes equation for price of European call option

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Solution to Black-Scholes equation for price of European put option

$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability distribution function for a variable with a standard normal distribution (probability a variable with a standard normal distribution will be less than x).

When the Black-Scholes formula is used in practice, the interest rate r is set to equal the zero-coupon risk-free interest rate for a maturity T. This is theoretically correct when r is a known function of time.

Understanding $N(d_1)$ and $N(d_2)$

The term $N(d_2)$ is the probability that a call option will be exercised in a risk-neutral world.

The expression $S_0N(d_1)e^{rT}$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero. The strike price is paid only if stock price exceeds K, which has probability $N(d_2)$.

The expected payoff is therefore

$$S_0N(d_1)e^{rT} - KN(d_2)$$

The present value of this (from time T to time 0) gives the Black-Scholes equation above

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

Note that this can be written as

$$c = e^{-rT}N(d_2)[S_0e^{rT}N(d_1)/N(d_2) - K]$$

The terms have the following interpretations:

- e^{-rT} is the present value factor
- $N(d_2)$ is the probability of exercise
- $e^{rT}N(d_1)/N(d_2)$ is the expected percentage increase in stock price if option is exercised
- K is the strike price paid if option is exercised

Properties of the Black-Scholes-Merton Formulas

When the stock price S_0 becomes very large, a call option is almost certain to be exercised.
It becomes very similar to a forward contract with delivery price K.

Then the expected call price is

$$S_0 - Ke^{-rT}$$

This is in fact the call price given by the equation for c above.
When S_0 becomes very large, d_1 and d_2 become very large, so $N(d_1)$ and $N(d_2)$ become close to 1.

When the stock price becomes very large, the price of a European put option p approaches zero. This is consistent with the equation for p above because $N(-d_1)$ and $N(-d_2)$ are both close to zero in this case.

Consider when volatility σ approaches zero. Since the stock is virtually riskless, the price will grow at rate r to $S_0 e^{rT}$ at time T.

The payoff from a call option is

$$\max(S_0 e^{rT} - K, 0)$$

Discounting at rate r, the value of the call option today is

$$e^{-rT} \max(S_0 e^{rT} - K, 0) = \max(S_0 - Ke^{-rT}, 0)$$

The two cases below show that this is consistent with the formula for c above.

1. $S_0 > Ke^{-rT}$ implies $\ln(S_0/K) + rT > 0$. As $\sigma \to 0$, $d_1 \to +\infty$ and $d_2 \to +\infty$, so $N(d_1) \to 1$ and $N(d_2) \to 1$. Then the equation becomes $c = S_0 - Ke^{-rT}$.

2. $S_0 < Ke^{-rT}$ implies $\ln(S_0/K) + rT < 0$. As $\sigma \to 0$, $d_1 \to -\infty$ and $d_2 \to -\infty$, so $N(d_1) \to 0$ and $N(d_2) \to 0$. Then the equation becomes $c = 0$.

This shows the call price is always $\max(S_0 - Ke^{-rT}, 0)$ as σ tends to zero.

It can be similarly shown that the put price is always $\max(Ke^{-rT} - S_0, 0)$ as σ tends to zero.

Warrants and Employee Stock Options

The exercise of a regular call option on a company has no effect on the number of the company's outstanding shares.

The exercise of warrants and employee stock options leads to the company issuing more shares and then selling them to the option holder for the strike price. Strike price is less than market price, so this dilutes the interest of the existing shareholders.

Consider company contemplating a new issue of employee stock options.
Assume company has N shares worth S_0 each.
Assume the number of new options contemplated is M with strike price K.

The value of the company today is NS_0. This does not change as a result of the new issue.

Without the warrant issue, share price will be S_T at warrant's maturity.
So with or without the issue, the total value of the equity and warrants at time T is NS_T.

If warrants are exercised, there is cash inflow from strike price, increasing this to $NS_T + MK$.
This is distributed among $N + M$ shares, so the share price immediately after exercise is

$$\frac{NS_T + MK}{N + M}$$

So the payoff to an option holder if the option is exercised is

$$\frac{NS_T + MK}{N + M} - K$$

which is equivalent to

$$\frac{N}{N + M}(S_T - K)$$

So the value of each option is

$$\frac{N}{N + M}$$

regular call options on the company's stock.

The total cost of the options is M times this. Assuming no benefits to the company from the warrant issue, the total value of the company's equity will decline by the total cost of the options.

So the reduction in the stock price is

$$\frac{M}{N + M}$$

times the value of a regular call option with strike price K and maturity T.

Implied Volatilities

The one parameter in the Black-Scholes pricing formulas that cannot be observed is volatility.

- traders work with **implied volatilities**
- volatilities implied by option prices observed in the market

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Calculating Implied Volatility

Suppose that the value of a European call option a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$.

The implied volatility is the value of σ such that $c = 1.875$ in the call option pricing formula

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

It is impossible to invert the formula to obtain a formula for σ, so use an iterative search, or some numerical method such as Newton-Raphson search.

In this case the implied volatility is 0.235.

Implied volatilities are used to monitor the market's opinion about the volatility of a stock.

- historical volatility is backward looking
- implied volatility is forward looking

The VIX Index

There are indices of implied volatility

- VIX for S&P 500
- VXN for NASDAQ 100
- VXD for DJIA

VIX is an index of implied volatility of 30-day options on the S&P 500.

- calculated from a wide range of calls and puts
- index value of 15 indicates the implied volatility of 30-day options on the S&P 500 is 15%
- one contract is 1000 times the index

A trade involving futures or options on the S&P 500 is a bet on the future level of the S&P 500 and the volatility of the S&P 500.

A futures or options contract on the VIX is a bet only on volatility.

Dividends

Assume the amount and timing of dividends during the life of an option can be predicted with certainty. The date on which the dividend is paid is assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.

European Options

Assume the stock price is the sum of two components:

1. a riskless component that corresponds to the known dividends during the life of the option
2. a risky component

The riskless component is the present value of all dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate.

At option maturity, all dividends will have been and the riskless component will no longer exist.

So the Black-Scholes formula is correct if S_0 equals the risky component of the stock price and σ is the volatility of the process followed by the risky component. This means the formulas can be used provided the stock price is reduced by the present value of all the dividends during the life of the option.

American Call Options

With no dividends it is never optimal to exercise American call options early.
With dividends, it can only be optimal to exercise just before the stock goes ex-dividend.

Assume that n ex-dividend dates are anticipated at times $t_1, t_2, \ldots, t_n$.
The corresponding dividends are $D_1, D_2, \ldots, D_n$.

Consider the possibility of early exercise just prior t_n.

If the option is exercised at time t_n, then the investor receives

$$S(t_n) - K$$

where $S(t)$ denotes the stock price at time t.

If the option is not exercised, the stock price drops to $S(t_n) - D_n$.

Note that the value of a call option paying dividends is

$$c \geq \max\left(S_0 - D - Ke^{-rT}, 0\right)$$

so the value of the option is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

There are two cases.

1. If it is true that

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geq S(t_n) - K$$

$$D_n \leq K[1 - e^{-r(T-t_n)}]$$

   then it cannot be optimal to exercise at time t_n.

2. If it is true that

$$S(t_n) - D_n - Ke^{-r(T-t_n)} < S(t_n) - K$$

$$D_n > K[1 - e^{-r(T-t_n)}]$$

   then it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$.

Case (2) tends to occur when t_n is close to T and D_n is large.

The most likely time for early exercise of an American call option is immediately before the final ex-dividend date. It many cases, early exercise is never optimal, so the American option can be treated as a European option.

Black's Approximation

Calculate the prices of European options that mature at times T and t_n and then set the American price equal to the greater of the two. This in effect assumes the option holder decides to time zero whether the option will be exercised at time T or t_n.