Ch 28 Martingales and Measures

The risk-neutral valuation principle states that a derivative can be valued by:

1. calculating the expected payoff on the assumption that the expected return from the underlying asset equals the risk-free interest rate
2. discounting the expected payoff at the risk-free interest rate

traditional risk-neutral world - assumes that all market prices of risk are zero

martingale - a zero-drift stochastic process
measure - the unit in which security prices are valued

Equivalent martingale measure result

- if the price of a traded security is used as the unit of measurement then there is a market price of risk for which all security prices follow martingales

The Market Price of Risk

Consider the properties of derivatives dependent on the value of a single variable θ.

Assume the process followed by θ is

$$\frac{d\theta}{\theta} = m\,dt + s\,dz$$

where dz is a Wiener process, m is the expected growth rate in θ, and s is the volatility of θ.

Let f_1 and f_2 be prices of two derivatives dependent only on θ and t.
Assume that during the time period under consideration the two derivatives provide no income.

Suppose the processes followed by f_1 and f_2 are

$$\frac{df_1}{f_1} = \mu_1\,dt + \sigma_1\,dz$$

$$\frac{df_2}{f_2} = \mu_2\,dt + \sigma_2\,dz$$

where μ_1, μ_2, σ_1, and σ_2 are functions of θ and t. The dz term is the same in all the equations above because it is the only source of uncertainty in the prices of f_1 and f_2.

The discrete versions of these processes are

$$\Delta f_1 = \mu_1 f_1 \Delta t + \sigma_1 f_1 \Delta z$$

$$\Delta f_2 = \mu_2 f_2 \Delta t + \sigma_2 f_2 \Delta z$$

Eliminate the Δz by forming an instantaneously riskless portfolio consisting of $\sigma_2 f_2$ of the first derivative and $-\sigma_1 f_1$ of the second derivative.

If Π is the value of the portfolio then

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$

$$\Delta\Pi = \sigma_2 f_2 \Delta f_1 - \sigma_1 f_1 \Delta f_2$$

Substituting Δf_1 and Δf_2 from above gives

$$\Delta\Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\Delta t$$

The portfolio is instantaneously riskless so it must earn the risk-free rate

$$\Delta\Pi = r\Pi\Delta t$$

$$\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2 = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)$$

$$(f_1 f_2)(\mu_1 \sigma_2 - \mu_2 \sigma_1) = (r)(f_1 f_2)(\sigma_2 - \sigma_1)$$

$$\mu_1 \sigma_2 - \mu_2 \sigma_1 = r\sigma_2 - r\sigma_1$$

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$$

This shows that if f is the price of a derivative dependent only on θ and t with

$$\frac{df}{f} = \mu\, dt + \sigma\, dz$$

then

$$\frac{\mu - r}{\sigma} = \lambda$$

The parameter λ is the **market price of risk** of θ (or **Sharpe ratio**).

The variable σ is the quantity of θ-risk present in f.
The absolute value $|\sigma|$ is the volatility of f.

Alternative Worlds

The process followed by derivative price f is

$$df = \mu f\, dt + \sigma f\, dz$$

The value of μ depends on the risk preferences of investors.

In a world where market price of risk is zero, $\lambda = 0$, so $\mu = r$.

Then the process followed by f is

$$df = rf\,dt + \sigma f\,dz$$

This is the **traditional risk-neutral world**.

Other assumptions about the market price of risk λ enable other worlds that are internally consistent to be defined.

The market price of risk of a variable determines the growth rates of all securities dependent on the variable.

Girsanov's Theorem

- moving from one market price of risk to another changes the expected growth rates of security prices but volatilities remain the same

Choosing a particular market price of risk is referred to as defining the **probability measure**.

Some value of the market price of risk corresponds to the "real world".

Several State Variables

Suppose that n variables $\theta_1, \theta_2, \ldots, \theta_n$ follow stochastic processes of the form

$$\frac{d\theta_i}{\theta_i} = m_i\,dt + s_i\,dz_i$$

where the dz_i are Wiener processes, and m_i and s_i are the expected growth rates and volatilities and may be functions of the θ_i and time.

The process for the price f of a security that is dependent on the θ_i has n stochastic components and can be written

$$\frac{df}{f} = \mu\,dt + \sum_{i=1}^{n} \sigma_i\,dz_i$$

where μ is the expected return from the security and $\sigma_i\,dz_i$ is the components of the risk of this return attributable to θ_i. Both μ and the σ_i are potentially dependent on the θ_i and time.

It can be shown that

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

where λ_i is the market price of risk for θ_i.

The $\lambda_i \sigma_i$ terms measures the extent that the excess return required by investors on a security is affected by the dependence of the security on θ_i.

- if $\lambda_i \sigma_i = 0$ then there is no effect
- if $\lambda_i \sigma_i > 0$ then investors require a higher return to compensate for risk arising from θ_i
- if $\lambda_i \sigma_i < 0$ then investors require a lower return due to dependence of the security on θ_i

This last scenario occurs when the variable has the effect of reducing rather than increasing the risks in the portfolio of a typical investor.

The capital asset pricing model (CAPM) argues that an investor requires excess returns to compensate for any risk that is correlated to the risk in the return from the stock market, but requires no excess return for other risks.

- systematic risk - risks that are correlated with the return from the stock market
- nonsystematic risk - other risks

If CAPM is true than λ_i is proportional to the correlation between changes in θ_i and the return from the market. When θ_i is uncorrelated with the return from the market then $\lambda_i = 0$.

Martingales

martingale - a zero-drift stochastic process

A variable θ follows a martingale if its process has the form

$$d\theta = \sigma\, dz$$

where dz is a Wiener process. The variable σ may itself be stochastic.

A martingale has the property that its expected value at any future time is equal to its value today.

$$E(\theta_t) = \theta_0$$

Why?

Note that over a very small time interval the change in θ is normally distributed with zero mean.
So the expected change in θ over any very small time interval is zero.
The change in θ between time 0 and time T is the sum of its changes over many small time intervals. It follows that the expected change in θ between time 0 and time T must also be zero.

The Equivalent Martingale Measure Result

Suppose f and g are the prices of traded securities dependent on a single source of uncertainty. Assume the securities provide no income during the time period under consideration.

Define $\phi = f/g$, the relative price of f with respect to g.
It can be thought of as measuring the price of f in units of g rather than dollars.
The security price g is the **numeraire**.

The **equivalent martingale measure result** shows that when there are no arbitrage opportunities, ϕ is a martingale for some choice of the market price of risk. Also, for a given numeraire security g, the same choice of the market price of risk makes ϕ a martingale for all securities f. This choice of the market price of risk is the volatility of g.

In other words, when the market price of risk is set equal to the volatility of g, the ratio f/g is a martingale for all security prices f.

A world where the market price of risk is the volatility σ_g of g is **forward risk neutral** with respect to g.

Because f/g is a martingale in a world that is forward risk neutral with respect to g, it follows, from the Martingale property given above, that

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

$$f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right)$$

where E_g is the expected value in a world that is forward risk neutral with respect to g.

Alternative Choices for the Numeraire

Money Market Account as the Numeraire

The dollar money market account is a security that is worth $1 at time zero and earns the instantaneous risk-free rate r at any given time. The variable r may be stochastic.

Set g equal to the money market account so it grows at rate r so that

$$dg = rg\,dt$$

The drift of g is stochastic but its volatility is zero. It follows that f/g is a martingale in a world where the market price of risk is zero.

From before

$$f_0 = g_0 \hat{E}\left(\frac{f_T}{g_T}\right)$$

where $\hat{E}$ is expectation in the traditional risk-neutral world.

In this case $g_0 = 1$ and

$$g_T = e^{\int_0^T r\,dt}$$

so this gives

$$f_0 = \hat{E}(e^{-\bar{r}T} f_T)$$

where $\bar{r}$ is the average value of r between time 0 and time T.

When the short-term interest rate r is assumed to be constant, this becomes

$$f_0 = e^{-rT}\hat{E}(f_T)$$

which is consistent with risk-neutral valuation.

Zero-Coupon Bond Price as the Numeraire

Let $P(t,T)$ be price at time t of a risk-free zero-coupon bond that pays off $1 at time T.

Suppose $g = P(t,T)$.

Let E_T be expectations in a world that is forward risk neutral with respect to $P(t,T)$.

$$g_T = P(T,T) = 1$$

$$g_0 = P(0,T)$$

$$\Rightarrow f_0 = P(0,T)E_T(f_T)$$

Note that the discounting is outside the expectations operator rather than inside, as it is with the money market account as numeraire. This simplifies things for a security that provides payoff solely at time T.

Consider any variable θ that is not an interest rate. A forward contract on θ with maturity T is defined as a contract that pays off $\theta_T - K$ at time T. Let f be the value of this contract.

$$f_T = \theta_T - K$$

$$E_T(f_T) = E_T(\theta_T - K) = E_T(\theta_T) - K$$

$$\Rightarrow f_0 = P(0,T)[E_T(\theta_T) - K]$$

The forward price F of θ is the value of K for which $f_0 = 0$.

$$P(0,T)[E_T(\theta_T) - F] = 0$$

$$\Rightarrow F = E_T(\theta_T)$$

This shows that the forward price of any variable (except an interest rate) is its expected future spot price in a world that is forward risk neutral with respect to $P(t, T)$.

Recall that the futures price of a variable is the expected future spot price in the traditional risk-neutral world.

Black's Model Revisited

Recall that Black's model is a tool for pricing European options in terms of forward or futures price of the underlying asset when interest rates are constant. The constant interest rate assumption can be relaxed to show that Black's model can be used to price European options in terms of the forward price of the underlying asset when interest rate are stochastic.

Consider a European call option on an asset with strike price K that lasts until time T.

$$c = P(0, T)E_T(\max(S_T - K, 0)]$$

where S_T is the asset price at time T and E_T denotes expectations in a world that is forward risk neutral with respect to $P(t, T)$.

Define F_0 and F_T as forward price of the asset at time 0 and time T for a contract maturing at time T.

Note that $S_T = F_T$ which means

$$c = P(0, T)E_T[\max(F_T - K, 0)]$$

Assume F_T is lognormal with standard deviation of $\ln(F_T)$ equal to $\sigma_F\sqrt{T}$.
This could be because the forward price follows a stochastic process with volatility σ_F.

$$E_T[\max(F_T - K, 0)] = E_T(F_T)N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{\ln[E_T(F_T)/K] + \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

$$d_2 = \frac{\ln[E_T(F_T)/K] - \sigma_F^2 T/2}{\sigma_F\sqrt{T}}$$

Note that $E_T(F_T) = E_T(S_T) = F_0$ so

$$\Rightarrow c = P(0, T)[F_0 N(d_1) - KN(d_2)]$$

$$\Rightarrow p = P(0, T)[KN(-d_2) - F_0 N(-d_1)]$$

with

$$d_1 = \frac{\ln[F_0/K] + \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

$$d_2 = \frac{\ln[F_0/K] - \sigma_F^2 T/2}{\sigma_F \sqrt{T}}$$

This is Black's model. It applies to both investment and consumption assets. It is true when interest rates are stochastic provided that F_0 is the forward asset price. The variable σ_F can be interpreted as the volatility of the forward asset price.