Ch 27 More on Models and Numerical Procedures

Alternatives to Black-Scholes-Merton

The Black-Scholes model assumes asset price changes continuously in a way that produces lognormal distribution for the price at any future time.

Alternative processes

- assume a process other than geometric Brownian motion
- overlay continuous asset price changes with jumps
- assume a process where all asset price changes are jumps

These processes are called **Levy processes**.

The Constant Elasticity of Variance Model (CEV)

This is a diffusion model where the risk-neutral process for a stock price S is

$$ds = (r - q)S\,dt + \sigma S^{\alpha}\,dz$$

where r is the risk-free rate, q is the dividend yield, dz is a Wiener process, σ is a volatility parameter, and α is a positive constant.

When $\alpha = 1$, the CEV model is the geometric Brownian motion model.

When $\alpha < 1$, the volatility increases as the stock price decreases.

- creates probability distribution with heavy left tail and less heavy right tail
  – like that observed for equities

When $\alpha > 1$, the volatility increases as the stock price increases.

- creates probability distribution with heavy right tail and less heavy left tail
  – corresponds to volatility smile where implied volatility is increasing function of time
  – like volatility smile for options on futures

The CEV model is useful for valuing exotic equity options.

Merton's Mixed Jump-Diffusion Model

Merton suggested a model where jumps are combined with continuous changes.

λ : Average number of jumps per year
k : Average jump size, measured as percentage of asset price

The probability of a jump in time Δt is $\lambda \Delta t$.
So the average growth rate in the asset price from the jumps is λk.

The risk-neutral process for the asset price is

$$\frac{dS}{S} = (r - q - \lambda k)\, dt + \sigma\, dz + dp$$

where dz is a Wiener process, dp is the Poisson process generating the jumps, and σ is the volatility of the geometric Brownian motion. Assume dz and dp are independent processes.

In this model, the jumps are superimposed upon the usual lognormal diffusion process assumed for stock prices. When derivatives are valued, it is important to ensure that the overall expected return from the asset is the risk-free rate. This means that the drift rate for the diffusion component in the model is $r - q - \lambda k$.

The Variance-Gamma Model

This is a pure jump model.

Let g be the change over time T in a variable that follows a gamma process with mean rate of 1 and variance rate of v.

gamma process - pure jump process where small jumps occur frequently and large jumps occur occasionally

One way of characterizing the variance-gamma distribution is that g defines the rate at which information arrives during time T. If g is large, a lot of information arrives and the sample taken from a normal distribution has large mean and variance. If g is small, little information arrives and the sample taken has a small mean and variance.

The variance-gamma model tends to produce a U-shaped volatility smile.
The model can be used for equity and foreign currency plain vanilla option prices.

Stochastic Volatility Models

The Black-Scholes model assumes volatility is constant. In practice, volatility varies through time.

The variance-gamma model reflects this with g. An alternative is a model where the process followed by the volatility variable is specified exactly. Suppose the volatility parameter is a known function of time. Then the risk-neutral process followed by the asset price is

$$dS = (r - q)S\, dt + \sigma(t) S\, dz$$

The Black-Scholes formulas are correct provided the variance rate is equal to the average variance rate during the life of the option. The variance rate is the square of the volatility.

The equation assumes that instantaneous volatility is perfectly predictable.
In practice volatility varies stochastically. This leads to models with two stochastic variables.

Stochastic volatility models can be fitted to the prices of plain vanilla options and then used to price exotic options. The impact of stochastic volatility on pricing is small for short-maturity options and increases as option life increases.

The impact of stochastic volatility on delta hedging is large.
This requires traders to monitor exposure to volatility changes by calculating vega.

The IVF Model

The **implied volatility model** (IVF) or the **implied tree** model provides an exact fit to the European option prices observed on any given day, regardless of the shape of the vol surface.

The risk-neutral process for the asset price in the model has the form

$$dS = [r(t) - q(t)]S\,dt + \sigma(S,t)S\,dz$$

where $r(t)$ is the instantaneous forward interest rate for a contract maturing at time t and $q(t)$ is the dividend yield as a function of time. The volatility $\sigma(S,t)$ is a function of both S and t and is chosen so that the model prices all European options consistently with the market.

An alternative approach to the analytical formula this produces involves constructing a tree for the asset price that is consistent with option prices in the market.

In practice the IVF model is recalibrated daily to the prices of plain vanilla options.
It is a tool to price exotic options consistently with plain vanilla options.

However the model does not necessarily get the joint distribution of asset price at two or more times, so it may incorrectly price exotic options such as compound options and barrier options.

Convertible Bonds

Convertible bonds are bonds issued by a company where the holder has the option to exchange the bonds for the company's stock at certain times in the future.

conversion ratio - number of shares of stock obtained for one bond

The bonds are almost always callable (issuer can buy them back at certain times).
The holder has the right to convert the bond once it has been called.

Credit risk plays an important role in the valuation of convertibles.

One model used in practice involves modeling the issuer's stock price.

Assume the stock follows geometric Brownian motion except that there is a probability $\lambda \Delta t$ that there will be a default in each short period of time Δt. In the event of default, the stock price falls to zero and there is a recovery on the bond. The λ parameter is the risk-neutral hazard rate.

The stock price process can be represented by varying the usual binomial tree so that at each node there is:

1. A probability p_u of a percentage up movement of size u over the next time period Δt
2. A probability p_d of a percentage down movement of size d over the next time period Δt
3. A probability $\lambda \Delta t$ (more accurately $1 - e^{-\lambda \Delta t}$) that there will be a default with the stock price moving to zero over the next time period Δt

Choose parameter values matching the first two moments of the stock price distribution:

$$p_u = \frac{a - de^{-\lambda \Delta t}}{u - d} \qquad p_d = \frac{ue^{-\lambda \Delta t} - a}{u - d} \qquad u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}} \qquad d = \frac{1}{u}$$

where $a = e^{(r-q)\Delta t}$, r is the risk-free rate, and q is the dividend yield on the stock.

At each node, if conversion is allowed, test whether conversion is optimal. Also test whether the position of the issuer is improved by calling the bonds. If so, then assume the bonds are called and test again whether conversion is optimal.

The value at a node is

$$\max[\min(Q_1, Q_2), Q_3]$$

where Q_1 is the value given by no conversion and no call, Q_2 is the call price, and Q_3 is the value if the conversion takes place.

Interest paid on the debt must be taken into account. At each node, when valuing the bond on the assumption that it is not converted, the present value of any interest payable on the bond in the next time step should be included. The risk-neutral hazard rate λ can be estimated from either bond prices or credit default swap spreads.

A disadvantage of this model is that the probability of default is independent of the stock price.
An extension of the model is to have the hazard rate λ be a function of stock price and time.

Path-Dependent Derivatives

A path-dependent derivative is a derivative where the payoff depends on the path followed by the price of the underlying asset, not just its final value.

One valuation approach is Monte Carlo simulation. The main problem is that computation time can be high. Also, American-style derivatives (those with exercise opportunities or other decisions) cannot be easily handled.

Binomial tree methods can also be extended for valuation of path-dependent derivatives. This is computationally more efficient and can handle American-style path-dependent derivatives.

Two conditions must be satisfied:

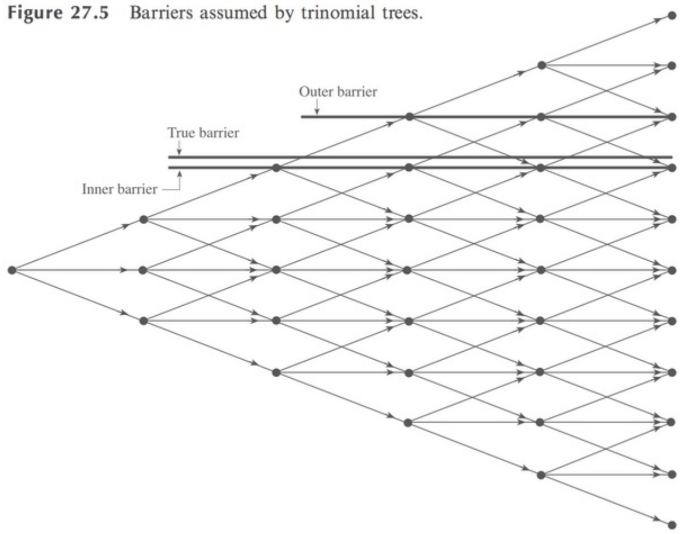
1. The payoff from the derivative must depend on a single function F of the path followed by the underlying asset
2. It must be possible to calculate the value of F at time $\tau + \Delta t$ from the value of F at time τ and the value of the underlying asset at time $\tau + \Delta t$

Barrier Options

Many barrier options can be valued using binomial trees. Trinomial trees can work even better. However, convergence is slow for trinomial trees. A large number of time steps are required.
This is because the barrier being assumed by the tree is different from the true barrier.

inner barrier - barrier formed by nodes just on the inside of the true barrier
outer barrier - barrier formed by nodes just outside the true barrier

Figure 27.5 Barriers assumed by trinomial trees.



Valuation approach:

1. Calculate price using inner barrier as the true barrier
2. Calculate price using outer barrier as the true barrier
3. Interpolate between the two prices

Another approach is to ensure that nodes lie on the barrier.

The Adaptive Mesh Model

This model is useful when the initial asset price is close to a barrier.

The idea behind the model is that computational efficiency can be improved by grafting a fine tree onto a coarse tree to achieve a more detailed modeling of the asset price in the regions of the tree where it is needed most. It is useful to have a fine tree close to barriers.

The tree is arranged so that nodes lie on the barriers. The probabilities on branches are chosen, as usual, to match the first two moments of the process followed by the underlying asset.

Options on Two Correlated Assets

Another problem is valuing American options dependent on assets with correlated prices.

Transforming Variables

It is easy to construct a tree in three dimensions to represent movements of two uncorrelated variables

1. Construct a two-dimensional tree for each variable
2. Combine these trees into a single three-dimensional tree

The probabilities on the branches of the three-dimensional tree are the product of corresponding probabilities on the two-dimensional trees.

Consider prices of two stocks S_1 and S_2. The three-dimensional tree probabilities are:

$S_1 \text{ increases}, S_2 \text{ increases} : p_1 p_2$

$S_1 \text{ increases}, S_2 \text{ decreases} : p_1(1 - p_2)$

$S_1 \text{ decreases}, S_2 \text{ increases} : (1 - p_1)p_2$

$S_1 \text{ decreases}, S_2 \text{ decreases} : (1 - p_1)(1 - p_2)$

Consider the situation where S_1 and S_2 are correlated. The risk-neutral processes are:

$$dS_1 = (r - q_1)S_1\, dt + \sigma_1 S_1\, dz_1$$

$$dS_2 = (r - q_2)S_2\, dt + \sigma_2 S_2\, dz_2$$

and the instantaneous correlation between the Wiener processes dz_1 and dz_2 is ρ.

This means that

$$d \ln S_1 = (r - q_1 - \sigma_1^2/2)\, dt + \sigma_1\, dz_1$$

$$d \ln S_2 = (r - q_2 - \sigma_2^2/2)\, dt + \sigma_2\, dz_2$$

Two new uncorrelated variables can be defined:

$$x_1 = \sigma_2 \ln S_1 + \sigma_1 \ln S_2$$

$$x_2 = \sigma_2 \ln S_1 - \sigma_1 \ln S_2$$

These variables can be modeled using two separate binomial trees and combined into a single three-dimensional tree (because they are uncorrelated). At each node of the tree, S_1 and S_2 can be calculated from x_1 and x_2 using the inverse relationships.

$$S_1 = \exp\left[\frac{x_1 + x_2}{2\sigma_2}\right] \quad \text{and} \quad S_2 = \exp\left[\frac{x_1 - x_2}{2\sigma_1}\right]$$

Using a Nonrectangular Tree

A way of building a three-dimensional tree for two correlated stock prices is by using a nonrectangular arrangement of the nodes. From a node (S_1, S_2), there is a 0.25 chance of moving to each of the following:

$$(S_1u_1, S_2A) \qquad (S_1u_1, S_2B) \qquad (S_1d_1, S_2C) \qquad (S_1d_1, S_2D)$$

where

$$u_1 = \exp[(r - q_1 - \sigma_1^2/2)\Delta t + \sigma_1\sqrt{\Delta t}]$$
$$d_1 = \exp[(r - q_1 - \sigma_1^2/2)\Delta t - \sigma_1\sqrt{\Delta t}]$$

and

$$A = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}(\rho + \sqrt{1-\rho^2})]$$
$$B = \exp[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2\sqrt{\Delta t}(\rho - \sqrt{1-\rho^2})]$$
$$C = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}(\rho - \sqrt{1-\rho^2})]$$
$$D = \exp[(r - q_2 - \sigma_2^2/2)\Delta t - \sigma_2\sqrt{\Delta t}(\rho + \sqrt{1-\rho^2})]$$

When the correlation is zero, this is equivalent to constructing separate trees for S_1 and S_2.

Adjusting the Probabilities

Another approach to building a three-dimensional tree for S_1 and S_2 involves first assuming no correlation and then adjusting the probabilities at each node to reflect the correlation.

S_1 increases, S_2 increases : $0.25 \longrightarrow 0.25(1+\rho)$

S_1 increases, S_2 decreases : $0.25 \longrightarrow 0.25(1-\rho)$

S_1 decreases, S_2 increases : $0.25 \longrightarrow 0.25(1-\rho)$

S_1 decreases, S_2 decreases : $0.25 \longrightarrow 0.25(1+\rho)$

Monte Carlo Simulation and American Options

Monte Carlo simulation is well suited to valuing path-dependent options and options where there are many stochastic variables. Trees and finite difference methods are well suited to valuing American-style options.

If an option is both path-dependent and American, there are Monte Carlo simulation approaches to value these.

The Least-Squares Approach

Valuing an American-style option requires choosing between exercising and continuing at each early exercise point.

The value of exercising is usually easy to determine.

The value of continuing can be determined using Monte Carlo simulation. The approach involves using a least-squares analysis to determine the best-fit relationship between the value of continuing and the values of relevant variables at each time an early exercise decision has to be made.

The Exercise Boundary Parameterization Approach

An alternative approach is where the early exercise boundary is parameterized and the optimal values of the parameters are determined iteratively by starting at the end of the life of the option and working backward.

A Monte Carlo simulation may be used to determine the early exercise boundary itself.
It may be necessary to make assumptions about how the early exercise boundary should be parameterized.

Upper Bounds

The two approaches above tend to underprice American-style options because they assume a suboptimal early exercise boundary. This led to a procedure that provides an upper bound to the price. This procedure can be used with any algorithm that generates a lower bound and pinpoints the true value of an American-style option more precisely than the algorithm does by itself.