Ch 21 Basic Numerical Procedures

Three numerical procedures:

1. Trees, such as binomial trees
2. Monte Carlo simulation
3. Finite difference methods

- Monte Carlo used for derivatives where payoff is dependent on history of underlying variable or where there are several underlying variables

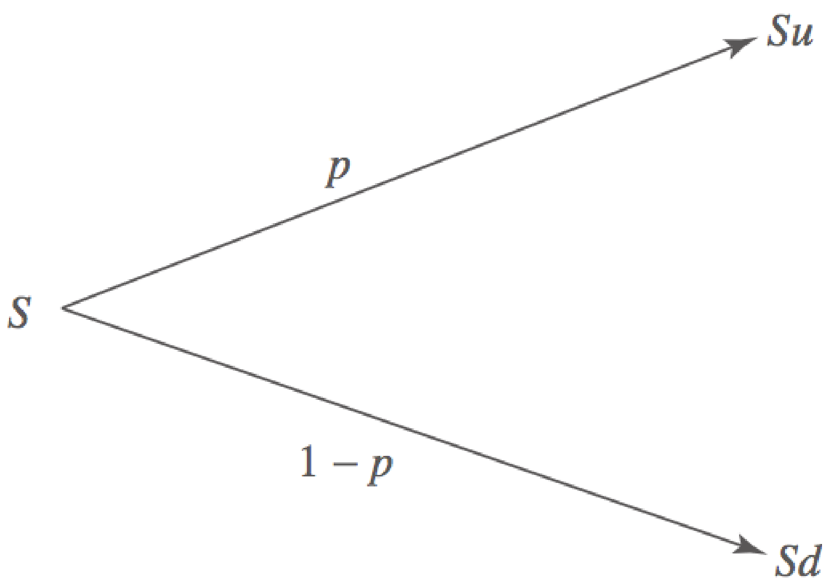
- Trees and finite difference methods used for American options and other derivatives where the holder has decisions to make prior to maturity

- all procedures can be used to calculate Greek letters

Binomial Trees

- binomial trees used to value European or American options
- Black-Scholes gives analytic valuation for European options
  – there are no analytic valuations for American options

Binomial tree valuation involves dividing life of option into small time intervals of length Δt.
In each time interval the price of the underlying asset moves from S to either Su or Sd.



In general $u > 1$ and $d < 1$. Probability of up movement is p and down movement is $1 - p$.

Risk-Neutral Valuation

An option can be valued on the assumption that the world is risk neutral.

Valuation procedure:

1. Assume expected return from all traded assets is the risk-free interest rate
2. Value payoffs from the derivative by calculating expected values and discounting at the risk-free interest rate

Determination of p, u, and d

Suppose asset provides yield q, so expected return in the form of capital gains must be $r - q$.
The expected value of the asset price at the end of interval Δt must be $Se^{(r-q)\Delta t}$, where S is the initial asset price.

To match the mean return with the tree

$$Se^{(r-q)\Delta t} = pSu + (1-p)Sd$$

$$e^{(r-q)\Delta t} = pu + (1-p)d$$

The variance of some variable Q is defined as $E(Q^2) - [E(Q)]^2$.

Let R be percentage change in asset price in time Δt.

$$P([1 + R = u]) = p$$

$$P([1 + R = d]) = 1 - p$$

Then calculate the variance of $1 + R$.

$$E[(1+R)^2] = pu^2 + (1-p)d^2$$

$$[E(1+R)]^2 = (pu + (1-p)d)^2 = (e^{(r-q)\Delta t})^2 = e^{2(r-q)\Delta t}$$

$$\text{Var}(1+R) = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$$

Adding a constant to a variable does not affect the variance.

$$\text{Var}(R) = \text{Var}(1+R)$$

Recall that $\text{Var}(R) = \sigma^2 \Delta t$.

$$pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t} = \sigma^2 \Delta t$$

Note that based on the above

$$e^{(r-q)\Delta t}(u+d) = pu^2 + (1-p)d^2 + ud$$

so that

$$e^{(r-q)\Delta t}(u+d) - ud - e^{2(r-q)\Delta t} = \sigma^2 \Delta t$$

Consider a third condition

$$u = 1/d$$

A solution to the equations (ignoring terms of higher order that Δt) is

$$p = \frac{a-d}{u-d} \qquad u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}} \qquad a = e^{(r-q)\Delta t}$$

The variable a is called the **growth factor**.

Tree of Asset Prices

This is a tree of asset prices using the binomial model with four time steps.

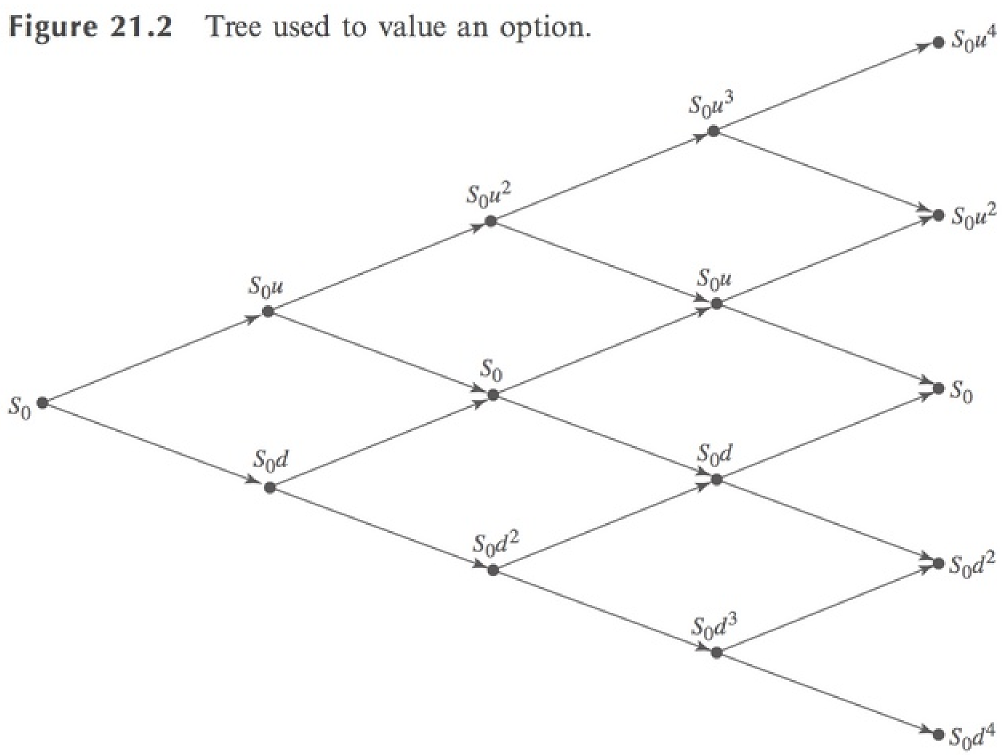


Figure 21.2 Tree used to value an option.

At time zero the asset price is S_0.

At time $i\Delta t$, there are $i+1$ possible asset prices

$$S_0 u^j d^{i-j}, \quad j = 0, 1, \ldots, i$$

The relationship $u = 1/d$ is used. For example, $S_0 u^2 d = S_0 u$. Also, an up movement followed by a down movement is the same as a down movement followed by an up movement.

Working Backward through the Tree

Options are evaluated by starting at the end of the tree (time T) and working backward.

Value of option is known at time T. For example, a put option is worth $\max(K - S_T, 0)$ and a call option is worth $\max(S_T - K, 0)$, where S_T is asset price at time T and K is strike price.

Assuming a risk-neutral world, the value at each node at time $T - \Delta t$ can be calculated as the expected value at time T discounted at rate r for time period Δt. For American options it is necessary to check at each node whether early exercise is preferable to holding the option for a further time period Δt.

Work backwards through all nodes to obtain value of option at time zero.

Estimating Delta and Other Greek Letters

Define $f_{i,j}$ as the value of the option at the (i, j) node.
The price of the underlying asset at the (i, j) node is $S_0 u^j d^{i-j}$.

Delta

Recall that the delta (Δ) of an option is the rate of change of its price with respect to the underlying stock price.

$$\Delta = \frac{\Delta f}{\Delta S}$$

At time Δ estimate option price $f_{1,1}$ for asset price $S_0 u$ and $f_{1,0}$ for asset price $S_0 d$.
So $\Delta S = S_0 u - S_0 d$ and $\Delta f = f_{1,1} - f_{1,0}$.

$$\Delta = \frac{f_{1,1} - f_{1,0}}{S_0 u - S_0 d}$$

Gamma

There are two estimates of Δ and time $2\Delta t$.

When $S = (S_0 u^2 + S_0)/2$, delta is $(f_{2,2} - f_{2,1})/(S_0 u^2 - S_0)$
When $S = (S_0 + S_0 d^2)/2$, delta is $(f_{2,1} - f_{2,0})/(S_0 - S_0 d^2)$

The difference between the two values of S is h.

$$h = 0.5(S_0 u^2 - S_0 d^2)$$

Gamma is the change in delta divided by h

$$\Gamma = \frac{[(f_{2,2} - f_{2,1})/(S_0 u^2 - S_0)] - [(f_{2,1} - f_{2,0})/(S_0 - S_0 d^2)]}{h}$$

Theta

This is the rate of change of option price with time.

The value of the option at time zero is $f_{0,0}$ and at time $2\Delta t$ is $f_{2,1}$.

So an estimate of theta is

$$\Theta = \frac{f_{2,1} - f_{0,0}}{2\Delta t}$$

Vega

Calculate by making a small change $\Delta\sigma$ in the volatility and constructing a new tree to obtain a new value of the option.

The estimate of vega is

$$\mathcal{V} = \frac{f^* - f}{\Delta\sigma}$$

where f and f^* are estimates of the option price from the original tree and the new tree.

Rho

Calculate similarly to vega.

Using the Binomial Tree for Options on Indices, Currencies, and Futures Contracts

These can be considered as assets providing known yields

- stock index: relevant yield is the dividend yield on the stock portfolio underlying the index
- currency: relevant yield is the foreign risk-free interest rate
- futures contract: relevant yield is the domestic risk-free interest rate

Use the binomial tree approach to value options on these provided that q is interpreted appropriately.

Binomial Model for a Dividend-Paying Stock

Known Dividend Yield

If the time $i\Delta t$ is prior to the stock going ex-dividend, the nodes on the tree correspond to stock prices

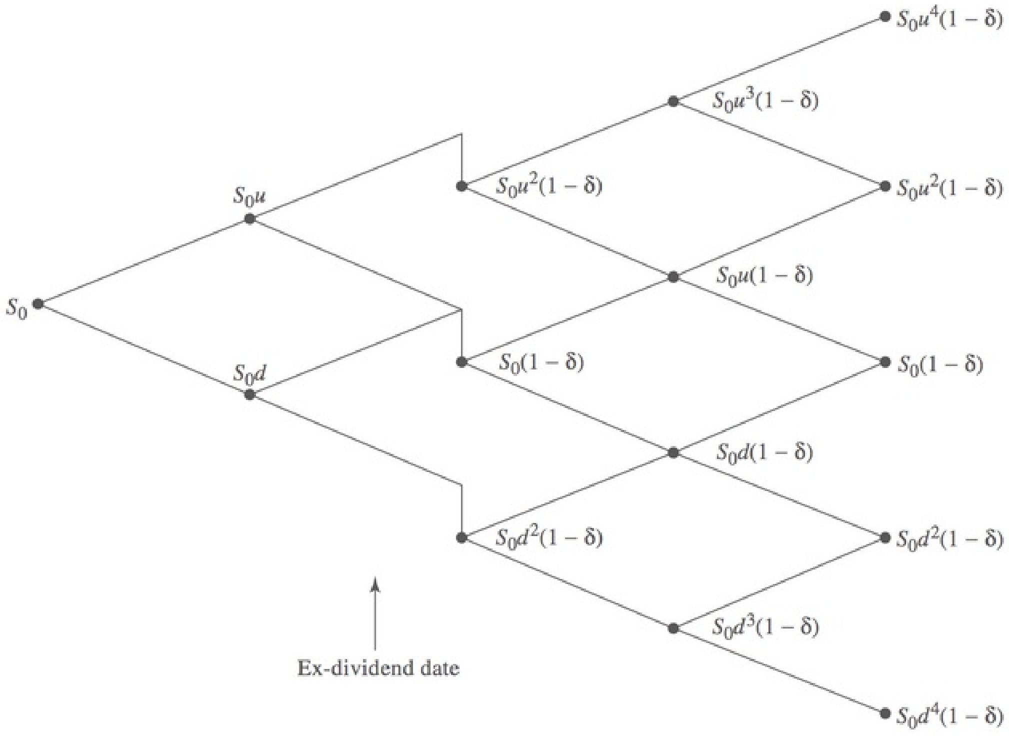
$$S_0 u^j d^{i-j} \qquad j = 0, 1, \ldots, i$$

If the time $i\Delta t$ is after the stock goes ex-dividend, the nodes correspond to stock prices

$$S_0(1 - \delta_i) u^j d^{i-j} \qquad j = 0, 1, \ldots, i$$

where δ_i is the total dividend yield associated with all ex-dividend dates between time zero and time $i\Delta t$.

The tree shows a stock that pays a known dividend yield at one particular time.



Known Dollar Dividend

When volatility σ is constant, the tree does not recombine, so many nodes have to be evaluated. Let the ex-dividend date τ be between $k\Delta t$ and $(k + 1)\Delta t$ and let D be the dividend amount.

When $i \leq k$, the nodes on the tree at time $i\Delta t$ correspond to stock prices

$$S_0 u^j d^{i-j} \qquad j = 0, 1, 2, \ldots, i$$

When $i = k + 1$ the nodes on the tree correspond to stock prices

$$S_0 u^j d^{i-j} - D \qquad j = 0, 1, 2, \ldots, i$$

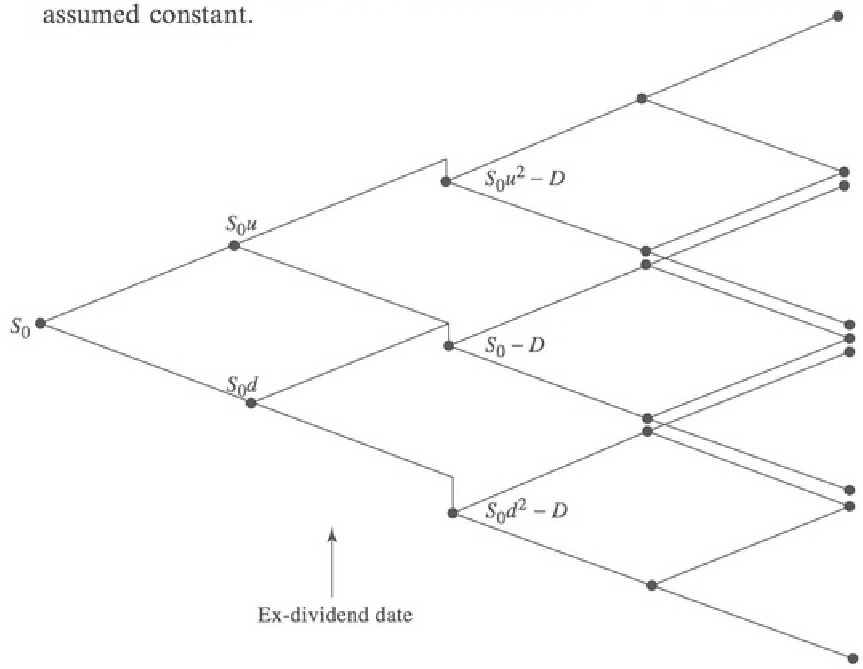
When $i = k + 2$ the nodes on the tree correspond to stock prices

$$(S_0u^jd^{i-1-j} - D)u \qquad \text{and} \qquad (S_0u^jd^{i-1-j} - D)d$$

So when $i = k + 2$ there are $2i$ nodes rather than $i + 1$ nodes.
So when $i = k + m$ there are $m(k + 2)$ nodes rather than $k + m + 1$ nodes.

The number of nodes expands even faster when there are several ex-dividend dates.

Figure 21.8 Tree when dollar amount of dividend is assumed known and volatility is assumed constant.



Control Variate Technique

A technique to improve the accuracy of pricing American options.

- use the same tree to calculate value of both American option f_A and corresponding European option f_E
- also calculate the Black-Scholes price of the European option f_{BSM}
- the error when the tree is used to price the European option is $f_{\text{BSM}} - f_E$
- assume this is equal to the error when using the tree to price the American option

So the estimate of the price of the American option is

$$f_A + (f_{\text{BSM}} - f_E)$$

Alternative Procedures for Constructing Trees

The change in $\ln S$ in time Δt in a risk-neutral world has mean $(r - q - \sigma^2/2)\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$.

These can be matched by setting $p = 0.5$ and

$$u = e^{(r-q-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}} \qquad d = e^{(r-q-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

The advantage of this procedure is that the probabilities are always 0.5 regardless of the value of σ or the number of time steps.

The disadvantage of this procedure is that it is not as straightforward to calculate the Greek letters from the tree because the tree is not centered at the initial stock price.

Time-Dependent Parameters

The parameters r, q, rf, and σ have been assumed to be constants.
In practice, they are assumed to be time dependent.

The values between times t and $t + \Delta t$ are assumed to be equal to their forward values.

To make r and q a function of time in the binomial tree model set

$$a = e^{[f(t)-g(t)]\Delta t}$$

for nodes at time t, where $f(t)$ is the forward interest rate between times t and $t + \Delta t$ and $g(t)$ is the forward value of q between these times.

The tree is used in the same way as before, except discounting between times t and $t + \Delta t$ uses $f(t)$.

Suppose $\sigma(t)$ is the volatility used to price an option with maturity t.

One approach is to make the length of each time step inversely proportional to the average variance rate during the time step. (The variance rate is the square of the volatility.)

Define $V = \sigma(T)^2T$ where T is the life of the tree.
Define t_i as the end of the ith time step.

For N time steps, choose t_i to satisfy $\sigma(t_i)^2t_i = iV/N$ and set $u = e^{\sqrt{V/N}}$ and $d = 1/u$.

The two procedures above can be combined so that both interest rates and volatilities are time-dependent.

Monte Carlo Simulation

For option valuation, Monte Carlo simulation uses the risk-neutral valuation result

- sample paths to obtain expected payoff in a risk-neutral world
- discount this payoff at the risk-free rate

Consider a derivative dependent on a single market variable S that provides a payoff at time T.

Assuming interest rates are constant, value the derivative as follows:

1. Sample a random path for S in a risk-neutral world
2. Calculate payoff from derivative
3. Repeat first two steps to get many sample payoff values
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff
5. Discount this expected payoff at the risk-free rate to get an estimate of derivative value

Suppose the process followed by the underlying market variable in a risk-neutral world is

$$dS = \hat{\mu}S\,dt + \sigma S\,dz$$

where dz is a Wiener process, $\hat{\mu}$ is the expected return, and σ is the volatility.

To simulate the path followed by S, divide the life of the derivative into N short intervals of Δt. Approximate the equation as

$$S(t+\Delta t) - S(t) = \hat{\mu}S(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t}$$

where $S(t)$ denotes value of S at time t and ϵ is a random sample from a normal distribution with mean zero and standard deviation 1.

In practice it is more accurate to simulate $\ln S$ rather than S.

From Ito's lemma the process followed by $\ln S$ is

$$d\ln S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$

so that

$$\ln S(t+\Delta t) - \ln S(t) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

or equivalently

$$S(t+\Delta t) = S(t)\exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}\right]$$

This equation is used to construct a path for S.

Working with $\ln S$ rather than S gives more accuracy.

Also, if $\hat{\mu}$ and σ are constant, then

$$\ln S(T) - \ln S(0) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{\Delta t}$$

is true for all T.

It follows that

$$S(T) = S(0)\exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right]$$

This equation can be used to value derivatives that provide a nonstandard payoff at time T.

Advantage of Monte Carlo simulation is that it can be used when the payoff depends on the path followed by the underlying variable S as well as when it depends only on the final value of S. Any stochastic process for S can be accommodated.

Disadvantages of Monte Carlo simulation are that it is computationally intensive and cannot easily handle situations where there are early exercise opportunities.

Derivatives Dependent on More than One Market Variable

Consider the situation where the payoff from a derivative depends on n variables θ_i.
Let s_i be the volatility of θ_i, $\hat{m}_i$ be the expected growth rate of θ_i, and ρ_{ik} be the correlation between the Wiener process driving θ_i and θ_k.

As before, divide the life of the derivative into N subintervals of length Δt.

The discrete version of the process for θ_i is then

$$\theta_i(t + \Delta t) - \theta_i(t) = \hat{m}_i\theta_i(t)\Delta t + s_i\theta_i(t)\epsilon_i\sqrt{\Delta t}$$

where ϵ_i is a random sample from a standard normal distribution.

The coefficient of correlation between ϵ_i and ϵ_k is ρ_{ik}.

Generating the Random Samples from Normal Distributions

When two correlated samples ϵ_1 and ϵ_2 from standard normal distributions are required, use the following procedure.

Obtain independent samples x_1 and x_2 from a univariate standardized normal distribution.

Then the required samples are calculated as follow:

$$\epsilon 1 = x1$$

$$\epsilon 2 = \rho x1 + x2\sqrt{1-\rho^2}$$

where ρ is the coefficient of correlation.

More generally, use *Cholesky decomposition* for n correlation samples.

Number of Trials

The accuracy of the result from Monte Carlo simulation depends on the number of trials.

Let μ be the mean and ω be the standard deviation.

The variable μ is the simulation's estimate of the value of the derivative.
The standard error of the estimate is

$$\frac{\omega}{\sqrt{M}}$$

where M is the number of trials.

Then the 95% confidence interval for the price f of the derivative is given by

$$\mu - 1.96\frac{\omega}{\sqrt{M}} < f < \mu + 1.96\frac{\omega}{\sqrt{M}}$$

So the uncertainty is inversely proportional to the square root of the number of trials.

Sampling through a Tree

Instead of Monte Carlo simulation (randomly sampling from the stochastic process for an underlying variable), an N-step binomial tree can be used (sampling from the 2^N possible paths). Obtain a complete path from the initial node to the end of the tree and calculate a payoff. The mean of the payoffs of many trials is discounted at the risk-free rate to get an estimate of the value of the derivative.

Calculating the Greek Letters

Suppose the goal is to find the partial derivative of f with respect to x, where f is the value of the derivative and x is the value of an underlying variable or parameter.

Use Monte Carlo simulation as usual to calculate an estimate $\hat{f}$ for the value of the derivative.

Then make a small increase Δx in the value of x and compute a new value $\hat{f}^*$.

Then the estimate for the hedge parameter (Greek letter) is given by

^ ^

$$\frac{f^* - f}{\Delta x}$$

To minimize the standard error of the estimate, use the same values in both simulations for

- number of time intervals, N
- number of trials, M
- random samples

Applications

Advantages of Monte Carlo simulation

- more efficient than other numerical procedures when there are many stochastic variables – time taken increases linearly with number of variables, rather than exponentially
- can provide a standard error for the estimates made
- can accommodate complex payoffs and complex stochastic processes
- can be used when payoff depends on some function of the whole path followed by a variable, not just terminal value

Variance Reduction Procedures

The simulations above can require a huge number of trials to get reasonable accuracy, which is computationally expensive. There are procedures to save on computation time.

Antithetic Variable Technique

A simulation trial involves calculating two values of the derivative.

- first value f_1 is calculated in the usual way
- second value f_2 is calculated by changing the sign of all the random samples from standard normal distributions

The sample value of the derivative calculated from a simulation trial is the average of f_1 and f_2.

$$\bar{f} = \frac{f_1 + f_2}{2}$$

The final estimate of the derivative value is the average of all $\bar{f}$.

If $\bar{\omega}$ is the standard deviation of all $\bar{f}$ and M is the number of trials, the standard error is

$$\bar{\omega}/\sqrt{M}$$

This is usually much less than the standard error from $2M$ random trials.

Control Variate Technique

Applicable when there are two similar derivatives, A and B

- A is being valued
- B is similar to A and has an analytic solution

Two simulations using the same random number streams and same Δt are run in parallel.
The first obtains an estimate f_A^* and the second obtains an estimate f_B^*.

Then a better estimate of f_A is obtained using the formula

$$f_A = f_A^* - f_B^* + f_B$$

where f_B is the known true value of B calculated analytically.

Importance Sampling

Choose only important paths.

Suppose F is the unconditional probability distribution function for stock price at time T.
Let q be the probability of the stock price being greater than K at maturity.

Then $G = F/q$ is the probability distribution of stock price conditional on stock price being greater than K. Sample from G rather than F to avoid wasting computation time on paths that contribute very little to determination of derivative value. The estimate of the value is the average discounted payoff multiplied by q.

Stratified Sampling

Sampling representative values rather than random values can give more accuracy.

Moment Matching

Involves adjusting the samples taken from a standard normal distribution so that the first, second, and possibly higher moments are matched.

Suppose samples ϵ_i are sampled from a standard normal distribution.
To match the first two moments, calculate the mean m and standard deviation s of the samples.

Then define adjusted samples ϵ_i^* as

$$\epsilon_i^* = \frac{\epsilon_i - m}{s}$$

Use the adjusted samples for all calculations.

Moment matching saves computation time but can cause memory problems since every number sampled must be stored until the end of the simulation.

Moment matching is also called *quadratic resampling*.

Using Quasi-Random Sequences

quasi-random sequence - sequence of representative samples from a probability distribution

These sequences can have the property that they lead to the standard error of an estimate being proportional to $1/M$ rather than $1/\sqrt{M}$, where M is the sample size.

The objective is to sample representative values for underlying variables. Sampled points are roughly evenly spaced throughout the probability space. Samples are taken in such a way to fill in gaps between existing samples.

Finite Difference Methods

Finite difference methods value a derivative by solving the differential equation that the derivative satisfies. The differential equation is converted into a set of difference equations, and the difference equations are solved iteratively.