Ch 13 Binomial Trees

A popular technique for pricing an option involves construction a **binomial tree**.

- represents possible paths that might be followed by the stock price over the life of an option
- assumes the stock price follows a **random walk**

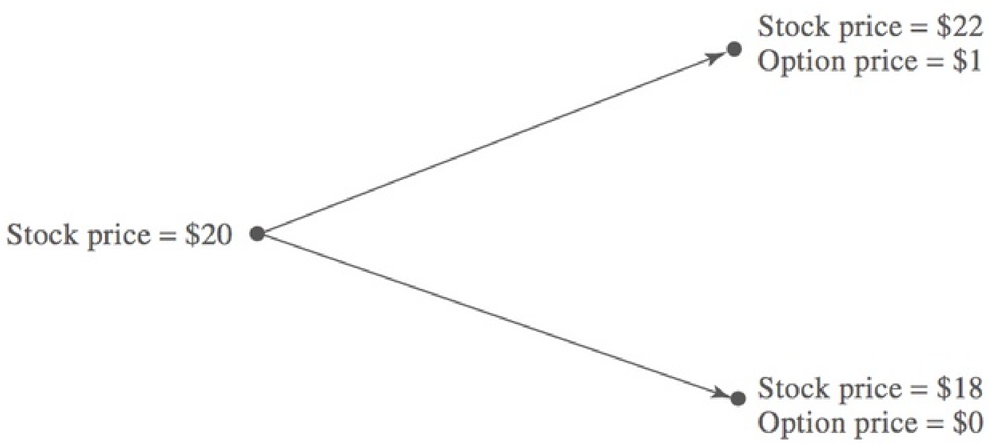
A One-Step Binomial Model and a No-Arbitrage Argument

Consider the situation

- current stock price is $20
- after 3 months it will be either $22 or $18

What is the value of a European call option to buy the stock for $21 in 3 months?

- if stock price is $22, option value is $1
- if stock price is $18, option value is zero



- assume no arbitrage opportunities
- set up portfolio such that there is no uncertainty about the value after 3 months
- since portfolio has no risk, the return must equal the risk-free interest rate

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Consider portfolio of a long position in Δ shares of the stock and a short position in one call option. Calculate the value of Δ that makes the portfolio riskless.

If stock price moves from $20 to $22

- value of shares is 22Δ and

- value of option is 1
- total value is $22\Delta - 1$

If stock price moves from \$20 to \$18

- value of shares is 18Δ
- value of option is zero
- total value is 18Δ

Choose Δ such that the portfolio is riskless (final value is the same)

$$22\Delta - 1 = 18\Delta$$

$$\Delta = 0.25$$

Therefore a riskless portfolio should have

- Long: 0.25 shares
- Short: 1 option

The value of the portfolio today must be the present value of this at the risk-free rate. Let $r = 12\%$, so $4.5e^{-0.12\times\frac{3}{12}} = 4.367$. The value of the stock price today is \$20 and the option price is f.

$$20 \times 0.25 - f = 5 - f = 4.367$$

$$f = 0.633$$

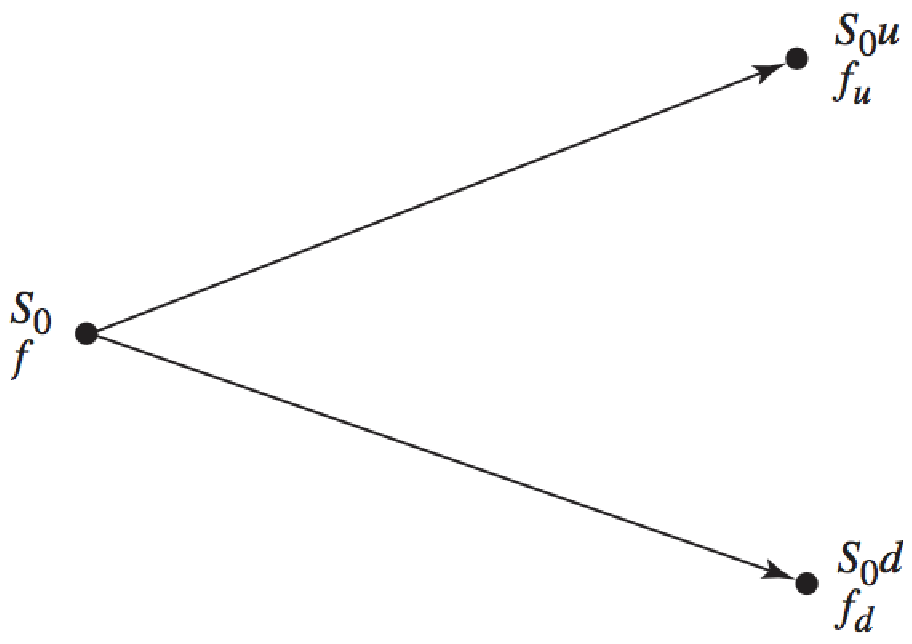
This means that the current value of the option must be 0.633.

A Generalization

Consider a stock with price S_0 and an option on the stock with price f.
The stock price can move up to S_0u where $u > 1$ or down to S_0d where $d < 1$.

- percentage increase after an up movement is $u - 1$
- percentage decrease after a down movement is $1 - d$

Let the payoff from the option after an up movement be f_u.
Let the payoff from the option after a down movement be f_d.



Consider a portfolio with long position in Δ shares of the stock and a short position in one option. Calculate the value of Δ that makes the portfolio riskless.

If stock price moves up, portfolio values is

$$S_0u\Delta - f_u$$

If stock price moves down, portfolio value is

$$S_0d\Delta - f_d$$

Choose Δsuch that the portfolio is riskless (final value is the same)

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Let the risk-free interest rate be r, so the present value of the portfolio is

$$(S_0u\Delta - f_u)e^{-rT}$$

The cost of setting up the portfolio is $S_0\Delta - f$ so it follows that

$$S_0\Delta - f = (S_0u\Delta - f_u)e^{-rT}$$

$$f = S_0\Delta(1 - ue^{-rT}) + f_ue^{-rT}$$

Substitute the previous expression for Δ to obtain

$$f = S_0\left(\frac{f_u - f_d}{S_0u - S_0d}\right)(1 - ue^{-rT}) + f_ue^{-rT}$$

$$f = \frac{f_u(1 - de^{-rt}) + f_d(ue^{-rT} - 1)}{u - d}$$

Finally,

$$f = e^{-rT}[pf_u + (1 - p)f_d]$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

Irrelevance of the Stock's Expected Return

Note that the option pricing formula does not depend on the probability of the up or down price movement. This is because the option is not being valued in absolute terms. The value is calculated in terms of the price of the underlying stock. The probabilities of up or down movements are already incorporated into the stock price.

Risk-Neutral Valuation

risk-neutral valuation - when valuing a derivative, assume investors are **risk-neutral**

risk-neutral - investors do not increase expected return required from an investment to compensate for increased risk

Why does it work (in a world that is not risk-neutral)?

- when pricing an option in terms of the price of the underlying stock, risk preferences are not important
- as investors become more risk-averse, stock prices decline, but the formulas relating option prices to stock prices remain the same

The risk-neutral assumption has two features that simplify derivatives pricing:

1. Expected return on an investment is the risk-free rate
2. Discount rate used for the expected payoff on an option is the risk-free rate

Interpret the parameter p as probability of up move, so $1 - p$ is probability of down move.

Then the expression

$$pf_u + (1 - p)f_d$$

is the expected future payoff from the option in a risk-neutral world.

So the equation for f in the previous section states that the value of the option today is its expected future payoff in a risk-neutral world discounted at the risk-free rate.

To apply risk-neutral valuation to pricing derivatives, calculate the probabilities of different outcomes if the world were risk-neutral. Then calculate the expected payoff from the derivative and discount it at the risk-free interest rate.

The One-Step Binomial Example Revisited

The stock price is \$20 and can move to \$22 or to \$18 after 3 months.
The risk-free rate is 12%.

Let p be probability of an upward movement in stock price in a risk-neutral world.
The expected return in a risk-neutral world must equal the risk-free rate.

$$22p + 18(1-p) = 20e^{0.12\times\frac{3}{12}}$$

$$p = 0.6523$$

The expected value of the call option after 3 months is

$$0.6523 \times 1 + (1 - 0.6523) \times 0 = 0.6523$$

In a risk-neutral world this should be discounted at the risk-free rate.
The value of the option today is

$$0.6523e^{-0.12\times\frac{3}{12}} = 0.633$$

This is the same value obtained from the no-arbitrage argument.

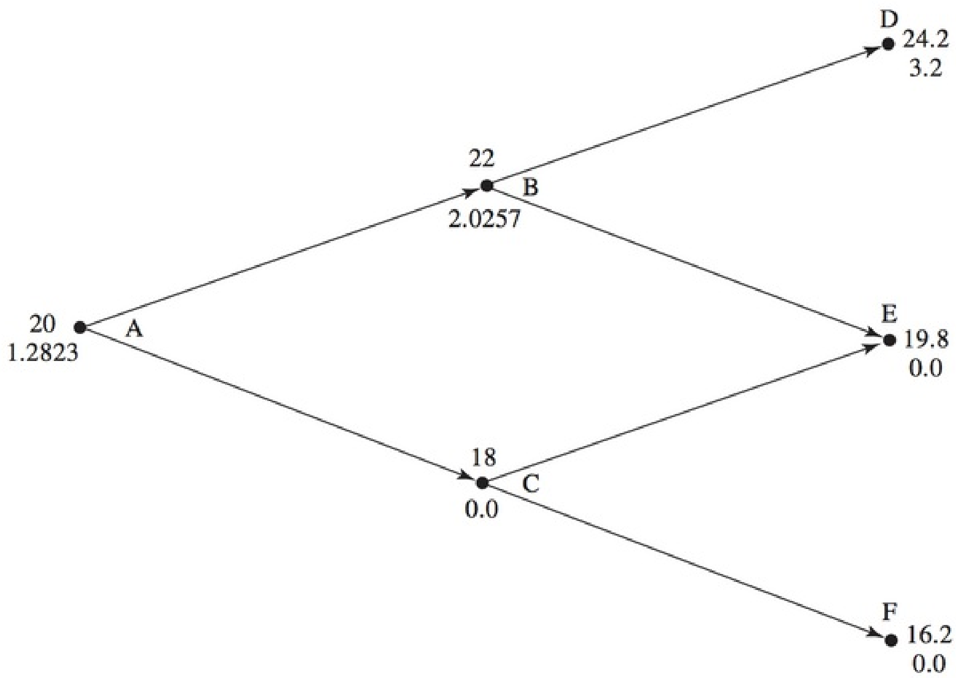
Real World vs Risk-Neutral World

Note that p is the probability of an up movement in a risk-neutral world.
It is not the probability of an up movement in the real world.

In the real world it is difficult to know what discount rate to apply to the expected payoff.
Using risk-neutral valuation solves this problem because in a risk-neutral world the expected return on all assets is the risk-free rate.

Two-Step Binomial Trees

Consider a stock with price \$20. In each time step it may go up by 10% or down by 10%. Each time step is 3 months and the risk-free rate is 12%. Consider a 6-month option with strike price \$21.



In the final nodes

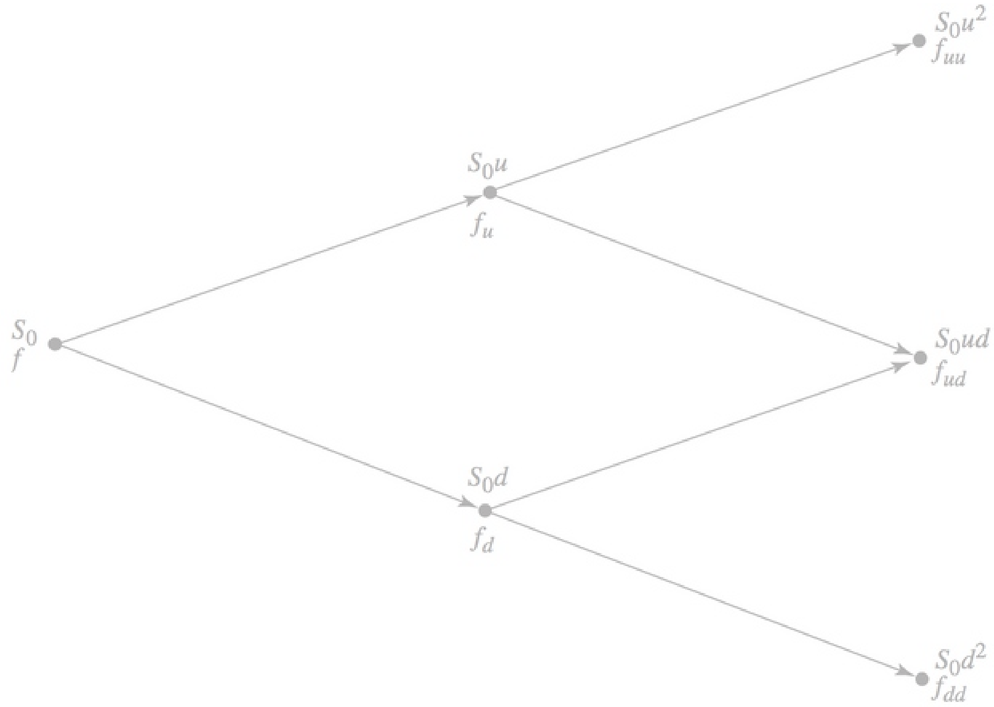
- node D has option price 24.2 - 21 = 3.2
- node E is out of the money, so option value is 0
- node F is out of the money, so option value is 0

In the middle nodes

- node C has option price 0 because it leads to final prices that are out of the money
- node B calculated with $u = 1.1$, $d = 0.9$, $r = 0.12$, and $T = 0.25$
  so $p = 0.6523$ and $f = e^{-0.12 \times \frac{3}{12}}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$

The option value at node A can be calculated similarly to node B, $f = 1.2823$.

A Generalization



Let initial stock price be S_0. At each time step it either moves to up to u times its value or down to d times its value. The risk-free interest rate is r and the length of the time step is Δt years.

The equation for option value becomes

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d]$$

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

Apply the equation to later time steps

$$f_u = e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}]$$

$$f_d = e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}]$$

Substitute into the equation for the initial option value

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}]$$

This is consistent with the principle of risk-neutral valuation. The values p^2, $2p(1-p)$, and $(1-p)^2$ are the probabilities that the upper, middle, and lower final nodes will be reached. The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

This continues to hold as more steps are added to the binomial tree.

A Put Example

The procedures can be used to price puts as well as calls.

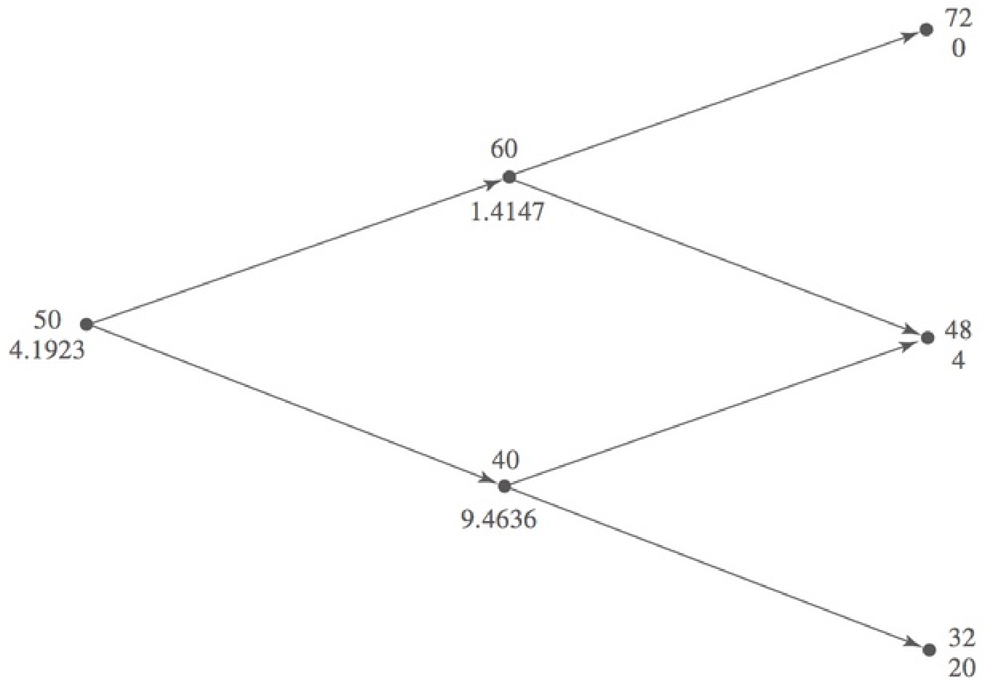
Consider a 2-year European put with a strike price of $52 on a stock with current price $50. There are two time steps of 1 year. In each step the price moves up 20% or down 20%. The risk-free interest rate is 5%.

$u = 1.2$, $d = 0.8$, $\Delta t = 1$, and $r = 0.05$, so $p = \frac{e^{0.05\times1} - 0.8}{1.2 - 0.8} = 0.6282$.

The option values for final prices $72, $48, and $32 are $f_{uu} = 0$, $f_{ud} = 4$, and $f_{dd} = 20$.

The equation for f from the previous section yields $f = 4.1923$.

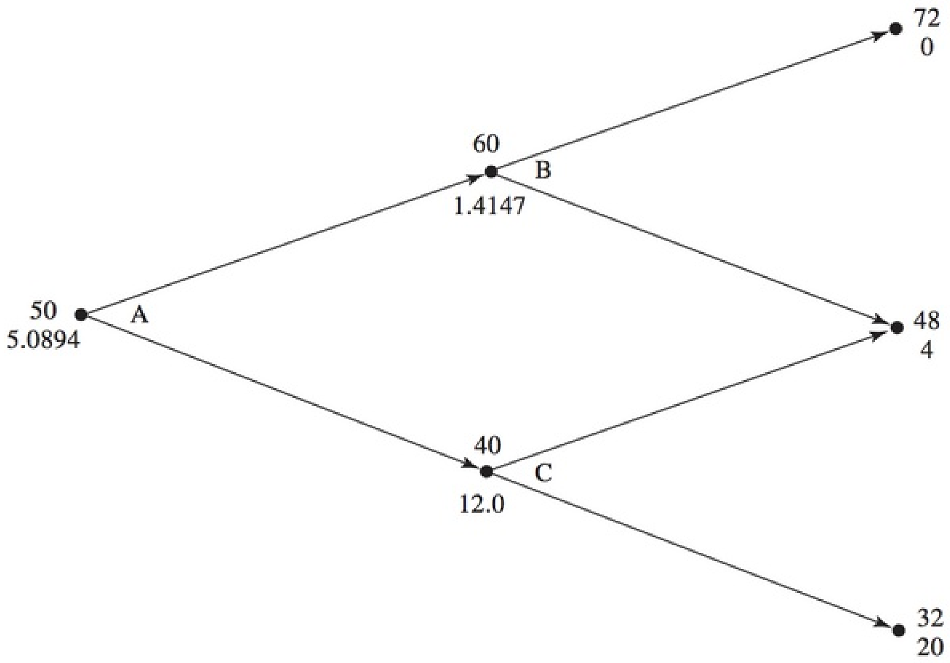
The same option price can be obtained using the binomial tree, as shown below.



American Options

The procedure to value American options is similar, but involves testing at teach node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of

1. The value given by the equation for f
2. The payoff from early exercise



The stock prices and probabilities are unaffected by the option being American.
The values for the option at the final nodes are unchanged as well.

At node B, the equation gives 1.4147 and the payoff from early exercise is -8.
Early exercise is not optimal, so value is 1.4147 (as with the European option).

At node C, the equation gives 9.4636 and the payoff from early exercise is 12.
Early exercise is optimal, so value is 12 (unlike the European option).

The value at node A becomes

$$f = e^{-0.05\times 1}(0.6282 \times 1.4147 + 0.3718 \times 12.0) = 5.0894$$

The payoff at node A from early exercise is 2, so early exercise is not optimal.
The value of the option is therefore $5.0894.

Delta

The **delta** of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

It is the number of units of the stock to hold for each option shorted in order to create a riskless portfolio. This is called **delta hedging**.

The delta of a call option is positive.
The delta of a put option is negative.

Delta can change over time. In to maintain a riskless hedge, the holdings in the stock need to be adjusted periodically.

Matching Volatility with u and d

It was previously shown that

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

The parameters u and d should be chosen to match the volatility.

The volatility σ is defined so that the standard deviation of the asset's return in a short period of time Δt is $\sigma\sqrt{\Delta t}$. Equivalently, the variance of the return in time Δt is $\sigma^2 \Delta t$.

(Note that variance $\mathrm{Var}(X)$ is defined as $E(X^2) - [E(X)]^2$ where E is expected value.)

During time step of length Δt, there is probability p the stock provides a return $u - 1$ and probability $1 - p$ the stock provides a return $d - 1$.

$$E(X) = p(u-1) + (1-p)(d-1)$$

$$E(X^2) = p(u-1)^2 + (1-p)(d-1)^2$$

$$[E(X)]^2 = [p(u-1) + (1-p)(d-1)]^2$$

So volatility is matched if

$$\mathrm{Var}[X] = E(X^2) - [E(X)]^2 = \sigma^2 \Delta t$$

$$p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = \sigma^2 \Delta t$$

Substitute the value of p and simplify to get

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t$$

Next apply the series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Ignore terms in Δt^2 and higher power of Δt.

Then a solution to the equation is

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

Girsanov's Theorem

When moving from the risk-neutral world to the real world, the expected return from the stock price changes, but its volatility remains the same.

More generally, when moving from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rate in variable change, but their volatilities remain the same.

Moving from one set of risk preferences to another is called *changing the measure*.

Real world measure: *P-measure*
Risk-neutral measure: *Q-measure*

The Binomial Tree Formulas

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

Increasing the Number of Steps

In practice, the life of the option is typically divided into 30 or more steps.
At each time step there is a binomial stock price movement, leading to 2^{30} possible paths.

The equations in the previous section define the tree regardless of the number of time steps.

Options on Other Assets

Options on Stocks Paying a Continuous Dividend Yield

Consider a stock paying a known dividend yield at rate q. The total return from dividends and capital gains in a risk-neutral world is r. The dividends provide a return of q so capital gains must provide a return of $r - q$.

If initial stock price is S_0, expected value after one time step of length Δt must be $S_0 e^{(r-q)\Delta t}$.

This means

$$p(S_0 u) + (1 - p)(S_0 d) = S_0 e^{(r-q)\Delta t}$$

so that

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

Match volatility by setting

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

This is the same as for non-dividend-paying stocks for u and d.
The difference is the first term in the numerator for p includes the dividend yield rate q.

Options on Stock Indices

The assumption is that the index provides a dividend yield at rate q. Then the valuation of an option on a stock index is similar to the valuation of an option on a stock paying a known dividend yield (previous section).

Options on Currencies

A foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest rf. Therefore the equations are the same as already shown, substituting rf for dividend yield rate q.

$$p = \frac{e^{(r-rf)\Delta t} - d}{u - d}$$

Options on Futures

There is no cost to taking a long or short position in a futures contract, so the risk-neutral world futures price should have expected growth rate of zero. If F_0 is the initial futures price, the expected futures price after a time step of length Δt should also be F_0.

This means that

$$p(F_0u) + (1-p)(F_0d) = F_0$$

so that

$$p = \frac{1-d}{u-d}$$