Ch 30 Convexity, Timing, and Quanto Adjustments

Two-step procedure for valuing a European-style derivative:

1. Calculate expected payoff by assuming the expected value of each underlying variable equals its forward value
2. Discount expected payoff at the risk-free rate applicable for the time period between valuation date and payoff date

For nonstandard interest rate derivatives, this procedure needs to be adjusted.

Three types of adjustments:

- convexity adjustments
- timing adjustments
- quanto adjustments

Convexity Adjustments

Consider an instrument that provides a payoff dependent on a bond yield observed at the time of the payoff.

The forward value of a variable S is usually calculated with reference to a forward contract that pays off $S_T - K$ at time T. The value of K causes the contract to have zero value.

Forward interest rates vs forward yields

- forward interest rate is the rate implied by a forward zero-coupon bond
- forward bond yield is the yield implied by the forward bond price

Suppose that B_T is the bond price at time T and let y_T be its yield.
The relationship between B_T and y_T is

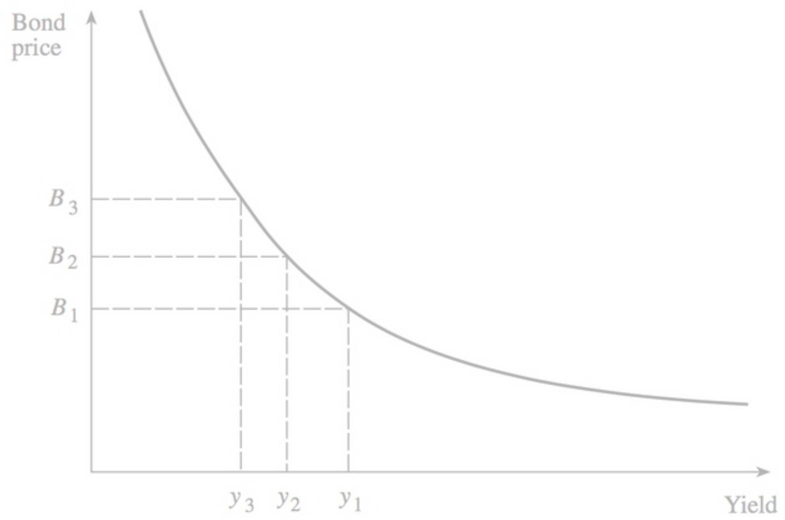
$$B_T = G(y_T)$$

Let F_0 be the forward bond price at time zero for a transaction maturing at time T.
Let y_0 be the forward bond yield at time zero.

By definition of forward bond yield

$$F_0 = G(y_0)$$

The function G is nonlinear. This means that when the expected future bond price equals the forward bond price (a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T), the expected future bond yield does not equal the forward bond yield.



In this relationship between bond prices and bond yields, the bond prices are equally spaced but the corresponding bond yields are not. If the forward bond yield is B_2 then the forward bond yield is y_2. The expected bond yield is the average of y_1, y_2, and y_3. This is greater than y_2.

Consider a derivative that provides payoff dependent on bond yield at time T.

It can be valued by calculating expected payoff in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T and discounting at the current risk-free rate for maturity T.

The expected bond price equals the forward price in the world being considered.
So the value of the expected bond yield when the expected bond price equals the forward bond price is needed.

An approximate expression for the required expected bond yield is

$$E_T(y_T) = y_0 - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

where G' and G'' denote the first and second partial derivatives of G, E_T denotes expectations in a world that is forward risk neutral with respect to $P(t, T)$, and σ_y is the forward yield volatility.

It follows that the expected payoff can be discounted at the current risk-free rate for maturity T provided the expected bond yield is assumed to be $E_T(y_T)$ rather than y_0.

The second term in the expression is the difference between the expected bond yield and the forward bond yield. This is the **convexity adjustment**. It corresponds to the difference between y_2 and the expected yield in the figure shown above. In that case, the convexity adjustment is positive because $G'(y_0) < 0$ and $G''(y_0) > 0$.

Timing Adjustments

Consider the situation where a market variable V is observed at time T and its value is used to calculate a payoff that occurs at a later time T^*.

- Let V_T be the value of V at time T
- Let $E_T(V_T)$ be the expected value of V_T in a world that is forward risk-neutral with respect to $P(t,T)$
- Let $E_{T^*}(V_T)$ be the expected value of V_T in a world that is forward risk-neutral with respect to $P(t,T^*)$

The numeraire ratio when moving from the $P(t,T)$ numeraire to the $P(t,T^*)$ numeraire is

$$W = \frac{P(t,T^*)}{P(t,T)}$$

This is the forward price of a zero-coupon bond lasting between times T and T^*.

- Let σ_V be the volatility of V
- Let σ_W be the volatility of W
- Let ρ_{VW} be the correlation between V and W

The change of numeraire increases the growth rate of V by α_V where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W$$

This can be expressed in terms of the forward interest rate between times T and T^*.

- Let R be the forward interest rate for period between T and T^* expressed with a compounding frequency of m
- Let σ_R be the volatility of R

The relationship between W and R is

$$W = \frac{1}{(1+R/m)^{m(T^*-T)}}$$

The relationship between the volatility of W and the volatility of R can be calculated from Ito's lemma as

$$\sigma_W W = \sigma_r R \frac{\partial W}{\partial R} = -\frac{\sigma_R R(T^*-T)}{(1+R/m)^{m(T^*-T)+1}}$$

so that

$$\sigma_W = -\frac{\sigma_R R(T^*-T)}{1+R/m}$$

So the equation above becomes

$$\alpha_V = -\frac{\rho_{VW}\sigma_V\sigma_R R(T^* - T)}{1 + R/m}$$

where $\rho_{VR} = -\rho_{VW}$ is the instantaneous correlation between V and R.

As an approximation, assume R remains constant at its initial value R_0.
Also assume that the volatilities and correlation are constant.

$$E_{T^*}(V_T) = E_T(V_T)\exp\left[-\frac{\rho_{VR}\sigma_V\sigma_R R_0(T^* - T)}{1 + R_0/m}T\right]$$

Quantos

A **quanto** or **cross-currency derivative** is an instrument where two currencies are involved.

The payoff is defined in terms of a variable that is measured in one of the currency and the payoff is made in the other currency.

Example: CME futures contract on the Nikkei

- market variable underlying the contract is the Nikkei 225 index (measured in yen)
- the contract is settled in US dollars

Consider a quanto that provides a payoff in currency X at time T.
Assume the payoff depends on value V of a variable that is observed in currency Y at time T.

- Let $P_X(t, T)$ be value at time t in currency X of a zero-coupon bond paying off 1 unit of currency X at time T
- Let $P_Y(t, T)$ be value at time t in currency Y of a zero-coupon bond paying off 1 unit of currency Y at time T
- Let V_T be value of V at time T
- Let $E_X(V_T)$ be expected value of V_T in a world that is forward risk neutral with respect to $P_X(t, T)$
- Let $E_Y(V_T)$ be expected value of V_T in a world that is forward risk neutral with respect to $P_Y(t, T)$

The numeraire ratio moving from $P_Y(t, T)$ to $P_X(t, T)$ is

$$W(t) = \frac{P_X(t, T)}{P_Y(t, T)}S(t)$$

where $S(t)$ is the spot exchange rate (units of Y per unit of X) at time t.

It follows that the numeraire ratio $W(t)$ is the forward exchange rate (units of Y per unit of X) for a contract maturing at time T.

- Let σ_W be volatility of W

- Let σ_V be volatility of V
- Let ρ_{VW} be instantaneous correlation between V and W

The change of numeraire increases the growth rate of V by α_V where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W$$

Assume volatilities and correlation are constant so

$$E_X(V_T) = E_Y(V_T)e^{\rho_{VW}\sigma_V\sigma_W T}$$

or as an approximation

$$E_X(V_T) = E_Y(V_T)(1 + \rho_{VW}\sigma_V\sigma_W T)$$

This is used for the valuation of diff swaps.

Using Traditional Risk-Neutral Measures

The forward risk-neutral measure works well when payoffs occur at only one time.
Other situations should use the traditional risk-neutral measure.

Suppose the process followed by a variable V in the traditional currency-Y risk-neutral world is known and the goal is to estimate its process in the traditional currency-X risk-neutral world.

- Let S be spot exchange rate (units of Y per unit of X)
- Let σ_S be volatility of S
- Let σ_V be volatility of V
- Let ρ be instantaneous correlation between S and V

The change of numeraire ratio from the money market account in currency Y to the money market account in currency X (with both accounts denominated in currency X) is

$$g_X S / g_Y$$

where g_X is the value of the account in currency X and g_Y is the value of the account in currency Y.

The variables $g_X(t)$ and $g_Y(t)$ have a stochastic drift but zero volatility.
From Ito's lemma it follows that the volatility of the numeraire ratio is σ_S.

So the change of numeraire involves increasing the expected growth rate of V by

$$\rho\sigma_V\sigma_S$$

The market price of risk changes from zero to $\rho\sigma_S$.
This also leads to Siegel's Paradox.

Siegel's Paradox

Consider two currencies, X and Y. Suppose the interest rates in each are constant, r_X and r_Y.
Let S be number of units of currency Y per unit of currency X.

Recall that a currency is an asset that provides a yield at the foreign risk-free rate.
So the traditional risk-neutral process for S is

$$dS = (r_Y - r_X)S\,dt + \sigma_S S\,dz$$

From Ito's lemma this implies the process for $1/S$ is

$$d(1/S) = (r_X - r_Y + \sigma_S^2)(1/S)\,dt - \sigma(1/S)\,dz$$

This leads to the paradox.

Expected growth rate of S is $r_Y - r_X$ in a risk-neutral world.
Symmetry suggests that the expected growth rate of $1/S$ should be $r_X - r_Y$ rather than $r_X - r_Y + \sigma_S^2$.

Why?

The process given for S is the risk-neutral process for S in a world where the numeraire is the money market account in currency Y.

The process for $1/S$ also assumes this numeraire.

However $1/S$ is the number of units of X per unit of Y. To be symmetrical, the process for $1/S$ should be measured in a world where the numeraire is the money market account in currency X.

Applying a change in numeraire accordingly, the growth rate of a variable V increases by $\rho\sigma_V\sigma_S$, as shown above. Since $V = 1/S$ in this case, $\rho = -1$ and $\sigma_V = \sigma_S$. It follows that the change of numeraire causes growth rate of $1/S$ to increase by $-\sigma_S^2$. This neutralizes the $+\sigma_S^2$ in the process given above.

So the process for $1/S$ in a world where the numeraire is the money market account in currency X is

$$d(1/S) = (r_X - r_Y)(1/S)\,dt - \sigma_S(1/S)\,dz$$

which is consistent with the process for S.

So the paradox is resolved.