Ch 32 HJM, LMM, and Multiple Zero Curves

Interest rate models are used for pricing instruments such as nonstandard interest rate derivatives when simpler models are not appropriate.

Two limitations of the interest rate models are:

1. Most involve only one factor (one source of uncertainty)
2. They do not give the user complete freedom in choosing the volatility structure

The Heath, Jarrow, and Morton Model

HJM described the no-arbitrage conditions that must be satisfied by a model of the yield curve.

- Let $P(t,T)$ be price at time t of a risk-free zero-coupon bond with principal $1 maturing at time T
- Let Ω_t be vector of past and present values of interest rates and bond prices at time t that are relevant for determining bond price volatilities at that time
- Let $v(t,T,\Omega_t)$ be volatility of $P(t,T)$
- Let $f(t,T_1,T_2)$ be forward rate as seen at time t for the period between T_1 and T_2
- Let $F(t,T)$ be instantaneous forward rate as seen as time t for a contract maturing at T
- Let $r(t)$ be short-term risk-free interest rate at time t
- Let $dz(t)$ be Wiener process driving term structure movements

Processes for Zero-Coupon Bond Prices and Forward Rates

Assume there is one factor and use the traditional risk-neutral world.

A zero-coupon bond is a traded security providing no income.
So its return in the traditional risk-neutral world must be r.

The stochastic process has the form

$$dP(t,T) = r(t)P(t,T)\,dt + v(t,T,\Omega_t)P(t,T)\,dz(t)$$

The zero-coupon bond's volatility v can be any well-behaved function of past and present interest rates and bond prices. A bond's price volatility declines to zero at maturity, so

$$v(t,t,\Omega_t) = 0$$

The forward rate $f(t,T_1,T_2)$ can be related to zero-coupon bond price

$$f(t,T_1,T_2) = \frac{\ln[P(t,T_1)] - \ln[P(t,T_2)]}{T_2 - T_1}$$

Using Ito's lemma

$$d\ln[P(t,T_1)] = \left[r(t) - \frac{v(t,T_1,\Omega_t)^2}{2}\right] dt + v(t,T_1,\Omega_t)\,dz(t)$$

$$d\ln[P(t,T_2)] = \left[r(t) - \frac{v(t,T_2,\Omega_t)^2}{2}\right] dt + v(t,T_2,\Omega_t)\,dz(t)$$

so substituting into the equation for f gives

$$df(t,T_1,T_2) = \frac{v(t,T_2,\Omega_t)^2 - v(t,T_1,\Omega_t)^2}{2(T_2-T_1)}\,dt + \frac{v(t,T_1,\Omega_t) - v(t,T_2,\Omega_t)}{T_2-T_1}\,dz(t)$$

This shows that the risk-neutral process for f depends solely on the vs.
It depends on r and the Ps only to the extent that the vs themselves depend on them.

Let $T_1 = T$ and let $T_2 = T + \Delta T$ and take the limits as ΔT tends to zero.
Then $f(t,T_1,T_2)$ becomes $F(t,T)$ and the coefficient of $dz(t)$ becomes $-v_T(t,T,\Omega_t)$.
The coefficient of dt becomes

$$\frac{1}{2}\frac{\partial[v(t,T,\Omega_t)^2]}{\partial T} = v(t,T,\Omega_t)v_T(t,T,\Omega_t)$$

where the subscripts to v denotes a partial derivative.

It follows that

$$dF(t,T) = v(t,T,\Omega_t)v_T(t,T,\Omega_t)\,dt - v_T(t,T,\Omega_t)\,dz(t)$$

This shows a link between the drift and standard deviation of an instantaneous forward rate.
This is the key HJM result.

Integrating between t and T leads to

$$v(t,T,\Omega_t) - v(t,t,\Omega_t) = \int_t^T v_\tau(t,\tau,\Omega_t)\,d\tau$$

Because $v(t,t,\Omega_t) = 0$, this becomes

$$v(t,T,\Omega_t) = \int_t^T v_\tau(t,\tau,\Omega_t)\,d\tau$$

Let $m(t,T,\Omega_t)$ and $s(t,T,\Omega_t)$ be the instantaneous drift and standard deviation of $F(t,T)$

$$dF(t,T) = m(t,T,\Omega_t)\,dt + s(t,T,\Omega_t)\,dz$$

so it follows that

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$$m(t, T, \Omega_t) = s(t, T, \Omega_t) \int_t^T s(t, \tau, \Omega_t)\, d\tau$$

This is the HJM result.

The process for short rate r in HJM is non-Markov.

This means the process for r at a future time depends on the path followed by r between now and time t as well as on the value of r at time t.

This means Monte Carlo simulation has to be used.

Extension to Several Factors

The HJM result can be extended to several independent factors.

Suppose

$$dF(t, T) = m(t, T, \Omega_t)\, dt + \sum_k s_k(t, T, \Omega_t)\, dz_k$$

then

$$m(t, T, \Omega_t) = \sum_k s_k(t, T, \Omega_t) \int_t^T s_k(t, \tau, \Omega_t)\, d\tau$$

The Libor Market Model

A drawback of HJM is that it is expressed in terms of instantaneous forward rates and these are not directly observable in the market. Another drawback is that it is difficult to calibrate the model to prices of actively traded instruments.

The LMM model is expressed in terms of the forward rates that traders use in conjunction with LIBOR discounting.

The Model

Let $t_0 = 0$ and let $t_1, t_2, \ldots$ be the reset times for caps that trade in the market today.
Let $\delta_k = t_{k+1} - t_k$.

- Let $F_k(t)$ be the forward rate between times t_k and t_{k+1} as seen at time t
- Let $m(t)$ be index for the next reset date at time t

- Let $\zeta_k(t)$ be volatility of $F_k(t)$ at time t

Assume there is only one factor.

Recall that in a world that is forward risk neutral with respect to $P(t, t_{k+1})$, $F_k(t)$ is a martingale and follows the process

$$dF_k(t) = \zeta_k(t)F_k(t)\,dz$$

where dz is a Wiener process.

The process for $P(t, t_k)$ has the form

$$\frac{dP(t, t_k)}{P(t, t_k)} = \cdots + v_k(t)\,dz$$

where $v_k(t)$ is negative because bond prices and interest rates are negatively related.

A rolling forward risk-neutral world is a world that is always forward risk neutral with respect to a bond maturing at the next reset date.

The process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$dF_k(t) = \zeta_k(t)[v_{m(t)}(t) - v_{k+1}(t)]F_k(t)\,dt + \zeta_k(t)F_k(t)\,dz$$

The relationship between forward rates and bond prices is

$$\frac{P(t, t_i)}{P(t, t_{i+1})} = 1 + \delta_f F_i(t)$$

or

$$\ln P(t, t_i) - \ln P(t, t_{i+1}) = \ln[1 + \delta_i F_i(t)]$$

Ito's lemma can be used to calculate the process by both sides of this equation.
Equating the coefficients of dz gives

$$v_i(t) - v_{i+1}(t) = \frac{\delta_i F_i(t)\zeta_i(t)}{1 + \delta_i F_i(t)}$$

so the process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)\zeta_k(t)}{1 + \delta_i F_i(t)}\,dt + \zeta_k(t)\,dz$$

The HJM result is the limiting case of this as the δ_i tend to zero.

Implementation of the Model

The LMM can be implemented using Monte Carlo simulation.

Extension to Several Factors

The LMM can be extended to incorporate several independent factors.

Bermudan Swap Options

Bermudan swap option

- popular interest rate derivative
- can be exercised on some or all of the payment dates of the underlying swap
- difficult to value using LMM because it the model relies on Monte Carlo simulation
  – difficult to evaluate early exercise decisions with Monte Carlo simulation

These can be valued via least-squares approach or optimal early exercise boundary approach.
Most traders value these by using one of the one-factor no-arbitrage models.

Handling Multiple Zero Curves

The models assume that only one yield curve is required to value an interest rate derivative.

It is now usual to use the OIS zero curve as the risk-free zero curve for discounting.
This means more than one zero curve must be modeled for derivatives such as swaps, interest rate caps, and swaptions whose payoffs depend on LIBOR. A LIBOR zero curve is necessary to calculate payoffs and the OIS zero curve is used for discounting.

If modeling both the OIS zero curve and the LIBOR/swap zero curve and assuming that banks can risklessly borrow or lend at the rates given by either curve, it is not possible to assume that there is no arbitrage in financial markets.

The alternative is to model credit risk and liquidity risk so that the spread between LIBOR and OIS is explained. However this adds a huge amount of complexity and makes the model hard to use.

Practitioners decided to model LIBOR and OIS separately without explicitly considering default risk and liquidity risk and ignoring the arbitrage opportunities created by the use of more than one zero curve.

Agency Mortgage-Backed Securities

An agency MBS is similar to an ABS except that payments are guaranteed by a government-related agency such as GNMA or FNMA so that investors are protected against defaults.

So it is similar to a regular fixed-income security issued by the government.
However, there is a critical difference: mortgages in an agency MBS pool have prepayment privileges. This means the householder has an American-style put option on the mortgage.

Reasons for prepayments

- interest rates fall so home owner decides to refinance at a lower rate
- house is being sold
- etc

Valuing an agency MBS involves determining the *prepayment function*.
This is a function describing expected prepayments on the underlying pool of mortgages at a time t in terms of the yield curve at time t and other relevant variables.

Prepayments are more accurately predicted from historical data rather than from prepayment functions.
Prepayments are not always motivated by pure interest rate considerations.
However there is a tendency for prepayments to be more likely when interest rates are low.

So investors require a higher rate of interest on an agency MBS than on other fixed-income securities for the prepayment options they have written.

Collateralized Mortgage Obligations

The simplest type of agency MBS is a *pass-through*

- all investors receive the same return and bear the same prepayment risk

A *collateralized mortgage obligation* works differently

- investors are divided into classes and rules are developed for determining how principal repayments are channeled to different classes
- this is analogous to how an ABS creates classes of securities bearing different amounts of credit risk

Valuing Agency Mortgage-Backed Securities

- model the behavior of Treasury rates via Monte Carlo simulation
- HJM or LMM approach can be used

Procedure

- expected prepayments are calculated from current yield curve and history of yield curve movements
- expected cash flow is determined from the prepayments
- cash flows are discounted at the Treasury rate plus a spread to time zero
- average of the sample values over many trials gives the estimate of the agency MBS value

Option-Adjusted Spread

- measure of the spread over the yields on government Treasury bonds provided by the instrument when all options have been taken into account

Calculation

- price the instrument as described above using Treasury rates plus a spread for discounting
- price of the instrument given by the model is compared to the price in the market
- iteratively determine the value of the spread that causes model price to equal market price
- the result is the OAS

IOs and POs

stripped MBS - principal payments are separated from interest payments

- principal payments are channeled to one class of security known as *principal only* (PO)
- interest payments are channeled to another class of security known as *interest only* (IO)

Both IOs and POs are risky instruments

- as prepayments increase
  – PO becomes more valuable
  – IO becomes less valuable
- as prepayments decrease
  – PO becomes less valuable
  – IO becomes more valuable