Ch 18 Futures Options

spot options - when exercised, the sale or purchase of asset takes place immediately

futures options - exercise of option gives holder a position in a futures contract

Nature of Futures Options

A **futures option** is the right, but not obligation, to enter into a futures contract at a certain futures price by a certain date.

call futures option - right to enter into a long futures contract at a certain price
put futures option - right to enter into a short futures contract at a certain price

Futures options are generally American options.

If call futures option is exercised, the holder acquires

1. long position in the underlying futures contract
2. cash amount equal to most recent settlement futures price minus strike price

If put futures option is exercised, the holder acquires
3. short position in the underlying futures contract
4. cash amount equal to strike price minus most recent settlement futures price

Effective payoff from call futures option: $\max(F - K, 0)$
Effective payoff from put futures option: $\max(K - F, 0)$

where F is futures price at time of exercise and K is strike price.

Expiration Months

Futures options are referred to by the delivery month of the underlying futures contract, not by the expiration month of the option. This is usually a short period of time before the last trading day of the underlying futures contract.

Options on Interest Rate Futures

Most actively traded interest rate options in the US (exchange-traded):

- Treasury bond futures
- Treasury note futures

- Eurodollar futures

Treasury bond futures option - option to enter a Treasury bond futures contract
Eurodollars futures option - option to enter into a Eurodollar futures contract

Interest rate futures prices increase when bond prices increase (when interest rates fall)

- investors can speculate by buying call options on Eurodollar futures

Interest rate futures prices decrease when bond prices decrease (when interest rates rise)

- investors can speculate by buying put options on Eurodollar futures

Reasons for the Popularity of Futures Options

Why trade options on futures rather than options on underlying asset?

- futures contract is often more liquid and easier to trade
  – especially for commodities
- futures price is known immediately from trading on the futures exchange
  – spot price of underlying asset may not be readily available

Note that exercising a futures option does not usually lead to delivery of the underlying asset

- underlying futures contract is closed out prior to delivery

So futures options are normally settled in cash

Futures and futures options are traded side by side in the same exchange

- facilitates hedging, arbitrage, speculation
- tends to make the markets more efficient

Futures options entail lower transaction costs than spot options

European Spot and Futures Options

Payoff from a European call option with strike price K on the spot price of an asset

$$\max(S_T - K, 0)$$

where S_T is the spot price at the option's maturity.

Payoff from a European call option with the same strike price on the futures price of the asset

$$\max(F_T - K, 0)$$

where F_T is the futures price at the option's maturity.

If the futures contract matures at the same time as the option, then $F_T = S_T$.

The above is similar for European put options.

Put-Call Parity

Consider European call and put futures options, both with strike price K and time to exp T.

Portfolio A: a European call futures option plus an amount of cash equal to Ke^{-rT}
Portfolio C: a European put futures option plus a long futures contract plus an amount of cash equal to F_0e^{-rT}, where F_0 is the futures price

In portfolio A, the cash invested at the risk-free rate r grows to K at time T.
Let F_T be the futures price at maturity of the option.

- if $F_T > K$ the call option is exercised and the portfolio is worth F_T
- if $F_T \leq K$ the call option is not exercised and the portfolio is worth K

Value of portfolio A at time T is

$$\max(F_T, K)$$

In portfolio B, the cash invested at the risk-free rate r grows to F_0 at time T.
The put option provides payoff of $\max(K - F_T, 0)$.
The futures contract (assume settled end of life, not daily) provides payoff of $F_T - F_0$.

Value of portfolio B at time T is

$$F_0 + (F_T - F_0) + \max(K - F_T, 0) = \max(F_T, K)$$

The two portfolios have the same value at time T, and European options cannot be exercised early, so it follows that they are worth the same today.

Value of portfolio A today: $c + Ke^{-rT}$
Value of portfolio B today: $p + F_0e^{-rT}$

(The long futures contract in portfolio B is worth zero today due to daily settlement process)

Put-call parity relationship

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

This is the same as the relationship for a non-dividend-paying stock, but with the stock price S_0 replaced by the discounted futures price F_0e^{-rT}.

Recall that when underlying futures contract matures at the same time as the option, European futures and spot options are the same. So this equation gives a relationship between the price of a call option on the spot price, the price of a put option on the spot price, and the futures price when both options mature at the same time as the futures contract.

For American futures options, the put-call relationship is

$$F_0 e^{-rT} - K < C - P < F_0 - K e^{-rT}$$

Bounds for Futures Options

The price of a put cannot be negative ($p \geq 0$) so it follows that

$$c + K e^{-rT} \geq F_0 e^{-rT}$$

so that

$$c \geq \max\left((F_0 - K)e^{-rT}, 0\right)$$

Similarly, since price of a call cannot be negative ($c \geq 0$) it follows that

$$K e^{-rT} \leq F_0 e^{-rT} + p$$

so that

$$p \geq \max\left((K - F_0)e^{-rT}, 0\right)$$

These bounds are similar to those for European stock options.

Prices of European call and put options are close to the lower bounds when the options are deep in the money.

- when a call is deep in the money, the corresponding put option is deep out of the money
- so p is close to zero
- the difference between c and its lower bound equals p
- so c must be close to its lower bound

A similar argument can be applied to put options.

American options can be exercised at any time so

$$C \geq \max(F_0 - K, 0)$$

$$P \geq \max(K - F_0, 0)$$

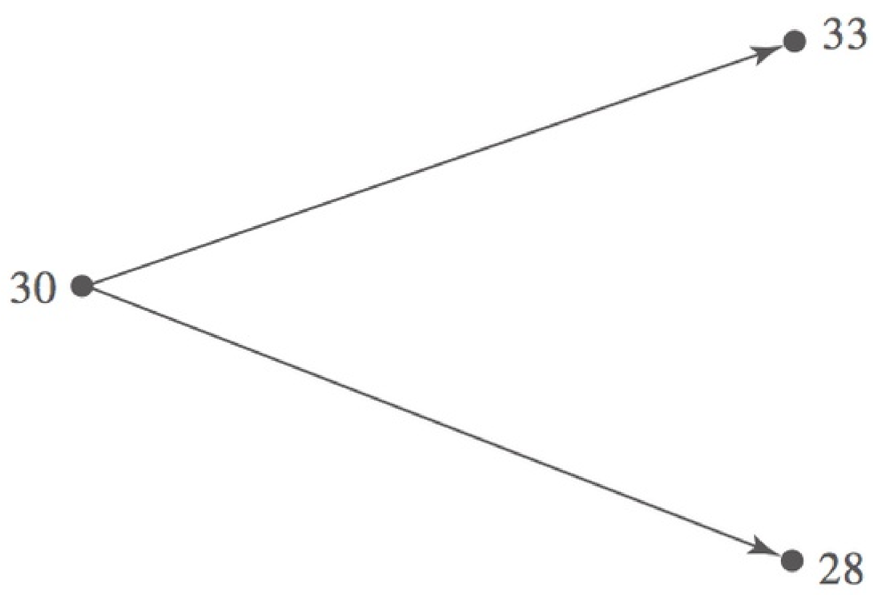
Assuming interest rates are positive, the lower bound for an American option price is always higher than the lower bound for the corresponding European option price. There is always some chance than an

American futures option will be exercised early.

Valuation of Futures Options using Binomial Trees

A key difference between futures options and stock options is that there are no up-front costs when a futures contract is entered into.

Let current futures price be 30. It will move up to 33 or down to 28 over the next month.
Consider a 1-month call option on the futures with a strike price of 29. Ignore daily settlement.



Suppose futures price moves up to 33

- payoff from option is 4
- value of futures contract is 3

Suppose futures price moves down to 28

- payoff from option is 0
- value of futures contract is -2

Set up a riskless hedge: short position in 1 options contract, long position in Δ futures contracts

- futures price of 33 gives portfolio value $3\Delta - 4$
- futures price of 28 gives portfolio value of -2Δ

The portfolio is riskless when

$$3\Delta - 4 = -2\Delta$$

so $\Delta = 0.8$.

For this value of Δ, the portfolio will be worth $3 \times 0.8 - 4 = -1.6$ in 1 month.

Let risk-free interest rate be 6%. The value of the portfolio today is

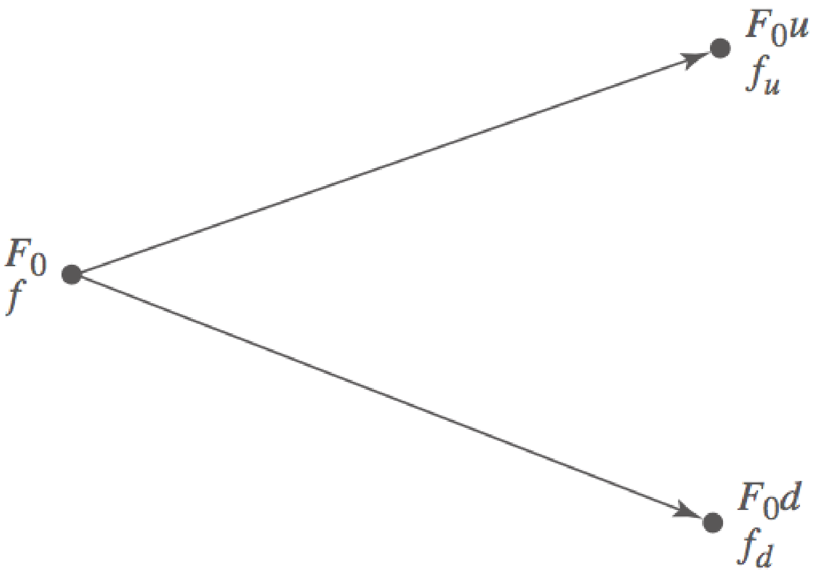
$$-1.6e^{-0.06\times 1/12} = -1.592$$

The portfolio consists of 1 short option and Δ futures contracts. The value of the futures contracts today is zero, so the value of the option today must be 1.592.

A Generalization

Let futures price start at F_0. It may rise to F_0u or move down to F_0d over time period T.

Consider an option maturing at time T and suppose its payoff is f_u if the futures price moves up and f_d if the futures price moves down.



The riskless portfolio is a short position in 1 option and a long position in Δ futures contracts

$$(F_0u - F_0)\Delta - f_u = (F_0d - F_0)\Delta - f_d$$

$$\Delta = \frac{f_u - f_d}{F_0u - F_0d}$$

The value of the portfolio at time T is then always

$$(F_0u - F_0)\Delta - f_u$$

Let the risk-free interest rate be r so the value of the portfolio today is

$$[(F_0u - F_0)\Delta - f_u]e^{-rT}$$

The present value of the portfolio is also $-f$ where f is the value of the option today

$$-f = [(F_0u - F_0)\Delta - f_u]e^{-rT}$$

Substitute Δ and simplify

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

where

$$p = \frac{1-d}{u-d}$$

This is the risk-neutral probability of an up movement.
It is consistent with the result found previously for the probability of an up movement.

Multistep Trees

Multistep binomial trees are used to value American-style futures options in the same way that they are used to value options on stocks.

The parameter u defining up movements in the futures price is $e^{\sigma\sqrt{\Delta t}}$ where σ is the volatility of the futures price and Δt is the length of one time step.

The probability of an up movement in the futures price is

$$p = \frac{1-d}{u-d}$$

as before.

Drift of a Futures Price in a Risk-Neutral World

Key Result

In a risk-neutral world, a futures price behaves in the same way as a stock paying a dividend yield at the domestic risk-free interest rate r.

First clue for the key result

The equation for probability p in a binomial tree for a futures price is the same as that for a stock paying a dividend yield q when $q = r$.

Recall the equation for p for a stock paying a known dividend yield is

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d}$$

so setting $q = r$ gives

$$p = \frac{1-d}{u-d}$$

as stated above.

Second clue for the key result

The put-call parity relationship for futures options prices is the same as that for options on a stock paying a dividend yield at rate q when the stock price is replaced by the futures price and $q = r$.

Recall the put-call parity relationship for a stock paying a known dividend yield

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

Replacing S_0 with F_0 and setting $q = r$ gives

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

as stated above.

Formal Proof

Calculate the drift of a futures price in a risk-neutral world.

Let F_t be the futures price at time t and suppose the settlement dates are $0, \Delta t, 2\Delta t, \ldots$

Enter into a long futures contract at time 0. It's value is 0.
At time Δt, it provides a payoff of $F_{\Delta t} - F_0$.

Let r be the short-term interest rate at time 0, so risk-neutral valuation gives the contract value

$$e^{-r\Delta t}\hat{E}[F_{\Delta t} - F_0]$$

at time 0, where $\hat{E}$ denotes expectations in a risk-neutral world.

This means

$$e^{-r\Delta t}\hat{E}(F_{\Delta t} - F_0) = 0$$

which shows that

$$\hat{E}(F_{\Delta t}) = F_0$$

Similarly, $\hat{E}(F_{2\Delta t}) = F_{\Delta t}$, $\hat{E}(F_{3\Delta t}) = F_{2\Delta t}$, and so on. Putting these together shows that

$$\hat{E}(F_T) = F_0$$

for any time T.

The drift of the futures price in a risk-neutral world is therefore zero.

Recall the risk-neutral process for the stock price of a stock paying known dividend yield q is

$$dS = (r - q)S\,dt + \sigma S\,dz$$

The futures price behaves like a stock providing a known dividend yield q equal to r.

The result is true for all futures prices and does not depends on assumptions about interest rates, volatilities, etc.

The usual assumption made for the process followed by a futures price F in the risk-neutral world is

$$dF = \sigma F\,dz$$

where σ is a constant.

Differential Equation

Another way to show that a futures price behaves like a stock paying a dividend yield at rate q is to derive the differential equation satisfied by a derivative dependent on a futures price.

This is done in the same way as deriving the differential equation for a derivative dependent on a non-dividend-paying stock.

The result is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = rf$$

This has the same form as that, with q set equal to r.

This confirms that, for the purposes of valuing derivatives, a futures price can be treated in the same way as a stock providing a dividend yield at rate r.

Black's Model for Valuing Futures Options

European futures options can be valued by extending the results already produced.

Assume the futures price follows the (lognormal) process $dF = \sigma F\,dz$

Recall the pricing formulas for European options on stocks paying dividend yield at rate q

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

The European call and put prices for a futures option are given by these equations with S_0 replaced by F_0 and $q = r$

$$c = e^{-rT}[F_0N(d_1) - KN(d_2)]$$

$$p = e^{-rT}[KN(-d_2) - F_0N(-d_1)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where σ is the volatility of the futures price.

When cost of carry and convenience yield are functions only of time, it can be shown that the volatility of the futures price is the same as the volatility of the underlying asset.

Using Black's Model Instead of Black-Scholes-Merton

European futures options and European spot options are equivalent when the option contract matures at the same time as the futures contract.

So the equations above provide a way of calculating the value of European options on the spot price of an asset.

The advantage of Black's model (over Black-Scholes-Merton) is that it avoids the need to estimate the income (or convenience yield) on the underlying asset. The futures or forward price that is used in the model incorporate the market's estimate of this income.

American Futures Options vs American Spot Options

Traded futures options are usually American.

Assuming risk-free rate r is positive, there is some chance it will be optimal to exercise early.
So American options are worth more than their European counterparts.

It is not true that an American futures option is worth the same as the corresponding American spot option when the futures and options contracts have the same maturity.

normal market - futures prices get increasingly higher with longer expiration times
inverted market - futures prices get increasingly lower with longer expiration times

Suppose there is a normal market with futures prices higher than spot prices prior to maturity

- American call futures option worth more than corresponding American spot call option
- American put futures option worth less than corresponding American spot put option

Suppose there is an inverted market with futures prices lower than spot prices prior to maturity

- American call futures options worth less than corresponding American spot call options
- American put futures options worth more than corresponding American spot put option

The differences become even more pronounced when the futures contract expires later than the options contract.

Futures-Style Options

futures-style options - futures contracts on the payoff from an option

With options on spot price or futures price of an asset, traders pay or receive cash up front.
With futures-style options, traders post margin, as with a regular futures contract.

- contract is settled daily as with any other futures contract
- final settlement price is the payoff from the option

A futures contract is a bet on what the future price of an asset will be.
A futures-style option is a bet on what the payoff from an option will be.

If interest rates are constant, the futures price in a futures-style option is the same as the forward price in a forward contract on the option payoff. This shows that the futures price for a futures-style option is the price that would be paid for the option if payment were made in arrears. It is therefore the value of a regular option compounded forward at the risk-free rate.

Black's model gives price of a European option on an asset in terms of futures price F_0 for a contract maturing at the same time as the option.

So the futures price in a call futures-style option is

$$c = F_0 N(d_1) - K N(d_2)$$

and the futures price in a put futures-style option is

$$p = K N(-d_2) - F_0 N(-d_1)$$

These formulas do not depend on interest rates.

The put-call parity relationship for a futures-style option is

$$p + F_0 = c + K$$

An American futures-style option can be exercised early, in which case there is an immediate final settlement at the option's intrinsic value.

However, it is never optimal to exercise an American futures-style option on a futures contract early because the futures price of the option is always greater than the intrinsic value. Treat this type of American futures-style option as if it were the corresponding European futures-style option.