Ch 31 Interest Rate Derivatives: Models of the Short Rate

The assumption that the probability distribution of market variables at a future point in time being lognormal can be used for valuing various instruments. However, since they do not provide a description of how interest rates evolve through time, they cannot be used for valuing interest rate derivatives that are American-style or structured notes.

term structure model - a model describing the evolution of all zero-coupon interest rates

Background

The risk-free short rate r at time t is the rate that applies to an infinitesimally short period of time t. It is called the *instantaneous short rate*.

Recall that the value at time t of an interest rate derivative that provides a payoff of f_T at time T is

$$\hat{E}[e^{[\bar{r}(T-t)} f_T]$$

where $\bar{r}$ is the average value of r in the time interval between t and T, and $\hat{E}$ denotes expected value in the traditional risk-neutral world.

Define $P(t,T)$ as price at time t of a risk-free zero-coupon bond that pays off \$1 at time T.

$$P(t,T) = \hat{E}[e^{[\bar{r}(T-t)}]$$

If $R(t,T)$ is the continuously compounded risk-free rate time t for a term of $T-t$ then

$$P(t,T) = e^{-R(t,T)(T-t)}$$

so that

$$R(t,T) = -\frac{1}{T-t} \ln P(t,T)$$

$$R(t,T) = -\frac{1}{T-t} \ln \hat{E}[e^{[\bar{r}(T-t)}]$$

This enables the term structure of interest rates at any given time to be obtained from the value of r at that time and the risk-neutral process for r.

Suppose r follows the process

$$dr = m(r,t)\,dt + s(r,t)\,dz$$

From Ito's lemma, any derivative dependent on r follows the process

$$df = \left(\frac{\partial f}{\partial t} + m \frac{\partial f}{\partial r} + \frac{1}{2} s^2 \frac{\partial f}{\partial r} \right) dt + s \frac{\partial f}{\partial r}\, dz$$

In the traditional risk-neutral world, the derivative provides no income.
So the process must have the form

$$df = rf\, dt + \dots$$

so that

$$\frac{\partial f}{\partial t} + m \frac{\partial f}{\partial r} + \frac{1}{2} s^2 \frac{\partial f}{\partial r^2} = rf$$

This is the equivalent of the Black-Scholes differential equation for interest rate derivatives.
One solution must be the zero-coupon bond price $P(t, T)$.

Equilibrium Models

In a one-factor equilibrium model, the process for r involves only one source of uncertainty.

Let the risk-neutral process for short rate be described by an Ito process of the form

$$dr = m(r)\, dt + s(r)\, dz$$

The instantaneous drift m and instantaneous standard deviation s are assumed to be functions of r and independent of time.

The implication of a single factor is that all rates move in the same direction over any short time interval, but not that they all move by the same amount.

The Rendleman and Bartter Model

The risk neutral process for r is

$$dr = \mu r\, dt + \sigma r\, dz$$

where μ and σ are constants.

This means r follows geometric Brownian motion.
The process is of the same type assumed for stock prices.

One difference between interest rates and stock prices is that interest rates exhibit *mean reversion*. When r is high, mean reversion causes negative drift. When r is low, mean reversion causes positive drift.

This model does not incorporate mean reversion.

Economic arguments for mean reversion

- when rates are high, economy tends to slow down, low demand for funds, rates decline
- when rates are low, high demand for funds, rates rise

The Vasicek Model

The risk-neutral process for r is

$$dr = a(b - r)\,dt + \sigma\,dz$$

where a, b, and σ are nonnegative constants.

This model incorporates mean reversion. The short rate is pulled to a level b at rate a.

$$\frac{\partial f}{\partial t} + a(b - r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial f}{\partial r^2} = rf$$

The Cox, Ingersoll, and Ross Model

The risk-neutral process for r is

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dz$$

where a, b, and σ are nonnegative constants.

This has the same mean-reverting drift as Vasicek, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This means that as the rate increases, the standard deviation increases.

$$\frac{\partial f}{\partial t} + a(b - r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial f}{\partial r^2} = rf$$

No-Arbitrage Models

The disadvantage of equilibrium models is they do not fit today's term structure of interest rates. The *no-arbitrage model* is a model designed to be consistent with today's term structure.

- in equilibrium model, today's term structure is an output
- in no-arbitrage model, today's term structure is an input

No-arbitrage models have drift of the short rate be a function of time. Equilibrium models can be converted to no-arbitrage models by including a function of time in the drift of the short rate.

The Ho-Lee Model

The process for r in the traditional risk-neutral world is

$$dr = \theta(t)\,dt + \sigma\,dz$$

where σ is the instantaneous standard deviation of the short rate, and $\theta(t)$ is a function of time chosen to ensure the model fits the initial term structure.

$\theta(t)$ is the average direction that r moves at time t

$$\theta(t) = F_t(0,t) + \sigma^2 t$$

where $F(0,t)$ is the instantaneous forward rate for a maturity t as seen at time zero.
The subscript t denotes a partial derivative with respect to t.

The Hull-White (One-Factor Model)

This is an extension of the Vasicek model.

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz$$

$$dr = \left[\frac{\theta(t)}{a} - r\right]dt + \sigma\,dz$$

where a and σ are constants.

It can be characterized as the Ho-Lee model with mean reversion at rate a.
It can be characterized as the Vasicek model with a time-dependent reversion level.

At time t, the short rate reverts to $\theta(t)/a$ at rate a.

The Black-Derman-Toy Model

The proposed model is a binomial-tree model for a lognormal short-rate process.
The corresponding stochastic process is

$$d\ln r = [\theta(t) - a(t)\ln r]\,dt + \sigma(t)\,dz$$

with

$$a(t) = -\frac{\sigma'(t)}{\sigma(t)}$$

The advantage of this model is that the interest rate cannot become negative.
The Wiener process dz can cause $\ln(r)$ to be negative, but r is always positive.

The disadvantage of this model is that there are no analytic properties.
Another disadvantage is that it imposes a relationship between volatility parameter $\sigma(t)$ and reversion rate

parameter $a(t)$. The reversion rate is positive only if volatility of the short rate is a decreasing function of time.

When $\sigma(t)$ is constant, the parameter a is zero, so there is no mean reversion.

$$d \ln r = \theta(t)\, dt + \sigma\, dz$$

This can be characterized as a lognormal version of the Ho-Lee Model.

The Black-Karasinski Model

This is an extension of the Black-Derman-Toy model where the reversion rate and volatility are independent of each other.

The general version of the model is

$$d \ln r = [\theta(t) - a(t) \ln r]\, dt + \sigma(t)\, dz$$

This is the same as Black-Derman-Toy except there is no relation between $a(t)$ and $\sigma(t)$.

When $a(t)$ and $\sigma(t)$ are assumed to be constant, the model becomes

$$d \ln r = [\theta(t) - a \ln r]\, dt + \sigma\, dz$$

The Hull-White Two-Factor Model

This model is

$$df(r) = [\theta(t) + u - af(r)]\, dt + \sigma_1\, dz_1$$

where $f(r)$ is a function of r and u has initial value of zero and follows the process

$$du = -bu\, dt + \sigma_2\, dz_2$$

The stochastic variable u is a component of the reversion level of $f(r)$.
It reverts to a level of zero at rate b.

Options on Bonds

Some of the models above allow options on zero-coupon bonds to be valued analytically.

For the Vasicek, Ho-Lee, and Hull-White one-factor models, the price at time zero of a call option that matures at time T on a zero-coupon bond maturing at time s is

$$LP(0, s)N(h) - KP(0, T)N(h - \sigma_P)$$

where L is the principal of the bond, K is its strike price, and

$$h = \frac{1}{\sigma_P} \ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2}$$

The price of a put option on the bond is

$$KP(0,T)N(-h+\sigma_P) - LP(0,s)N(-h)$$

This is essentially the same as Black's model for pricing bond options with the forward bond price volatility equaling $\sigma_P/\sqrt{T}$. Since an interest rate cap or floor can be expressed as a portfolio of options on zero-coupon bonds, it can be valued analytically using the equations above.

Options on Coupon-Bearing Bonds

In a one-factor model of r, all zero-coupon bonds move up in price when r decreases and all zero-coupon bonds move down in price when r increases. So such models allow European options on coupon-bearing bonds to be expressed as the sum of European options on zero-coupon bonds.

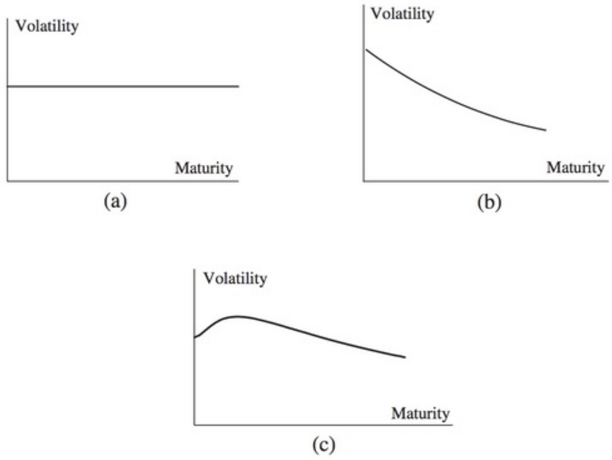
1. Calculate r^*, the critical value of r for which the price of the coupon-bearing bond equals the strike price of the option on the bond at the option maturity T
2. Calculate prices of European options with maturity T on the zero-coupon bonds that comprise the coupon-bearing bond. The strike prices of the options equal the values the zero-coupon bonds will have at time T when $r = r^*$.
3. Set the price of the European option on the coupon-bearing bond equal to the sum of the prices on the options on zero-coupon bonds from (2).

Volatility Structures

The different models give rise to different volatility environments.

- for Ho-Lee, volatility of the forward rate is the same for all maturities.
- for one-factor Hull-White, volatility of the forward rate is a declining function of maturity – this is the effect of mean reversion
- for two-factor Hull-White, volatility of the forward rate rises then falls

Figure 31.5 Volatility of 3-month forward rate as a function of maturity for (a) the Ho–Lee model, (b) the Hull–White one-factor model, and (c) the Hull–White two-factor model (when parameters are chosen appropriately).



Interest Rate Trees

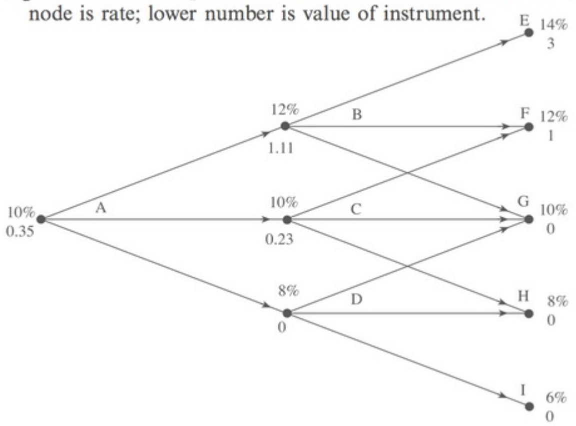
An interest rate tree is a discrete-time representation of the stochastic process for short rate.
It is analogous to the stock price tree which is a representation of the process for stock price.

The main difference is how discounting is done

- in stock price tree, discount rate is assumed to be the same at each node
  – or a function of time
- in interest rate tree, discount rate varies from node to node

A trinomial tree is more convenient than a binomial tree for interest rates. The advantage of a trinomial tree is that it provides an extra degree of freedom, so it is easier to represent features of the interest rate process such as mean reversion.

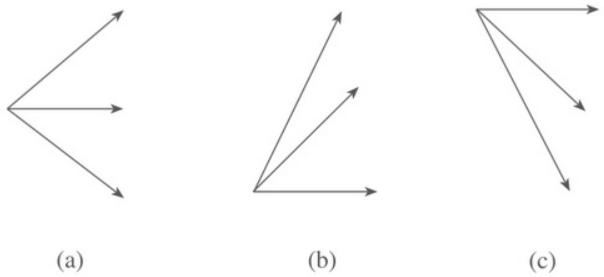
Figure 31.6 Example of the use of trinomial interest rate trees. Upper number at each node is rate; lower number is value of instrument.



Nonstandard Branching

Alternative branching possibilities are useful for incorporating mean reversion when interest rates are very low or very high.

Figure 31.7 Alternative branching methods in a trinomial tree.



A General Tree-Building Procedure

Handling Low Interest Rate Environments

When interest rates are very low, it is not easy to choose a model.

The probability of negative interest rates in Hull-White is no longer negligible.

Black-Karasinski does not work well since the same volatility is not appropriate for both low and high rates.

One way to avoid negative rates is to choose $f(r)$ as proportional to $\ln r$ when r is low and proportional to r when it is high.

Another way is to choose the short rate as the absolute value of the rate given by a Vasicek-type model.

Another way is to construct a model where both the reversion rate and the volatility of r are functions of r estimated from empirical data.

Calibration

Calibrating the model refers to determining the volatility parameters a and σ.

The volatility parameters are determined from market data on actively traded options.
These are *calibrating instruments*.

Suppose there are n calibrating instruments. A popular goodness-of-fit measure is

$$\sum_{i=1}^{n}(U_i - V_i)^2$$

where U_i is the market price of the ith calibrating instrument and V_i is the price given by the model for this instrument.

The goal of calibration is to choose the model parameters so that this measure is minimized.

Hedging Using a One-Factor Model

The calculation of deltas, gammas, and vegas involves making small changes to either the zero curve or the volatility environment and recomputing the value of the portfolio.

Although one factor may be assumed when pricing interest rate derivatives, it is not appropriate to assume only one factor when hedging. Taking into account changes that cannot happen under the model being considered is *outside model hedging*.

Simple models used carefully can give reasonable prices for instruments.
However, good hedging procedures must explicitly or implicitly assume many factors.