Ch 20 Volatility Smiles

volatility smile - plot of the implied volatility of an option as a function of its strike price

Why the Volatility Smile is the Same for Calls and Puts

Put-call parity (when dividend yield on underlying asset is q)

$$p + S_0 e^{-qT} = c + Ke^{-rt}$$

Note that the options must have the same strike price (K) and time to maturity (T).

Put-call parity does not make any assumption about the probability distribution of the asset price in the future. It is true when asset price distribution is lognormal and when it is not.

Let p_{BS} and c_{BS} be the values of option from the Black-Scholes model.
Let p_{mkt} and c_{mkt} be the market values of these options.

Put-call parity holds for the Black-Scholes model

$$p_{\text{BS}} + S_0 e^{-qT} = c_{\text{BS}} + Ke^{-rT}$$

Put-call parity holds for the market prices in the absence of arbitrage opportunities

$$p_{\text{mkt}} + S_0 e^{-qT} = c_{\text{mkt}} + Ke^{-rT}$$

The two equations together give

$$p_{\text{BS}} - p_{\text{mkt}} = c_{\text{BS}} - c_{\text{mkt}}$$

This shows that the dollar pricing error when using Black-Scholes model is the same for European call options and European put options.

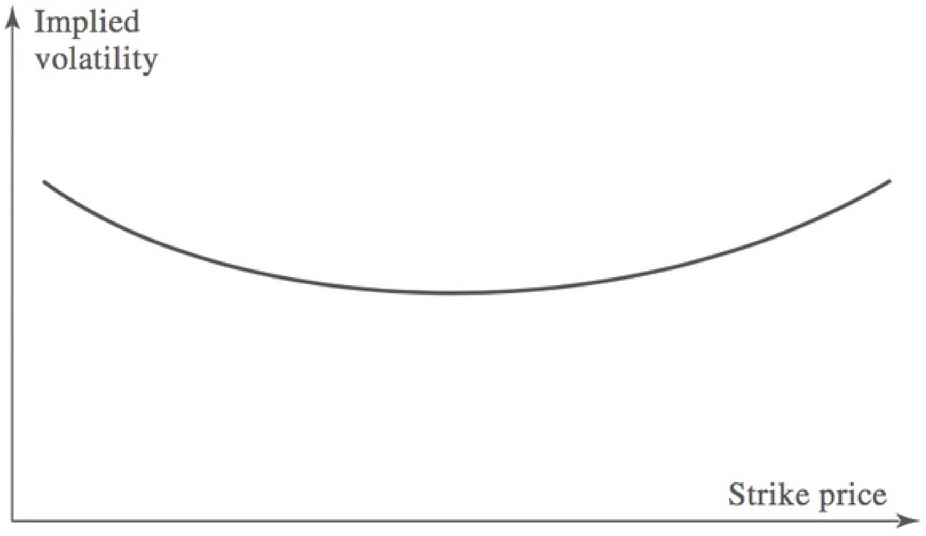
Let implied volatility of the put option be 22%. This means $p_{\text{BS}} = p_{\text{mkt}}$ when a volatility of 22% is used in the Black-Scholes model. The equation above suggests that $c_{\text{BS}} = c_{\text{mkt}}$ when this volatility is used. So the implied volatility for the call is also 22%.

The volatility smile is the same for European calls and European puts.
More generally, the volatility surface is the same for European calls and European puts.

volatility surface - implied volatility as a function of strike price and time to maturity

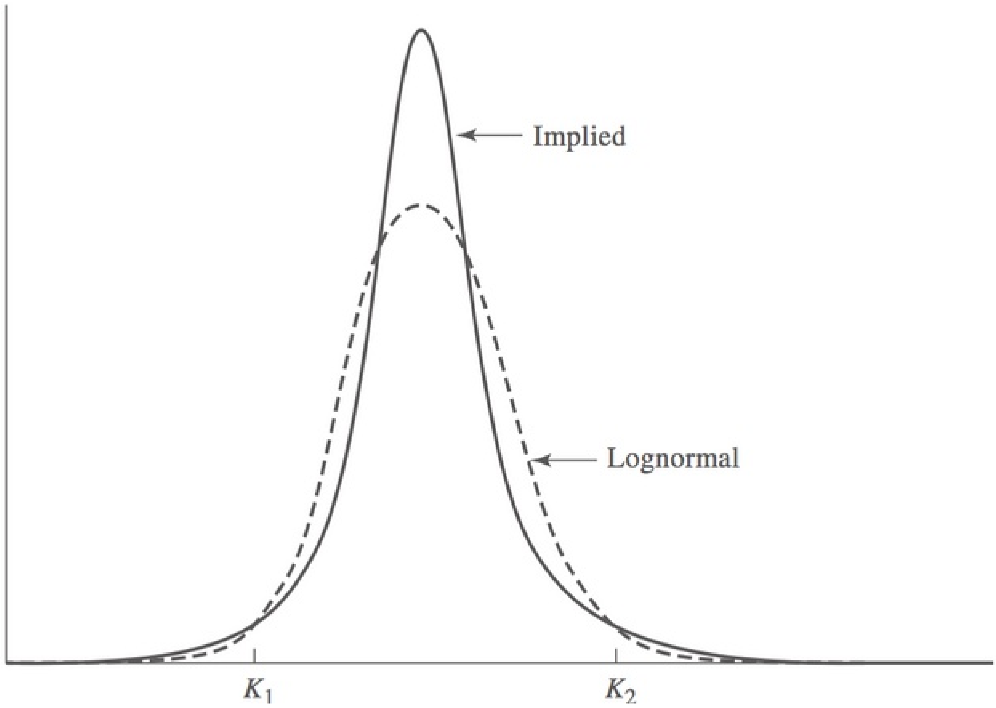
Foreign Currency Options

The implied volatility is relatively low for at-the-money options.
It becomes higher as an option moves either into the money or out of the money.



implied distribution - risk-neutral probability distribution for an asset price at a future time

The implied distribution can be determined from the volatility smile.
The implied distribution has a higher kurtosis compared to the lognormal distribution.



Note that the volatility smile and the implied distribution are consistent. The implied distribution suggests a high probability of high or low prices. High prices lead to high implied volatility, and low prices lead to low implied volatility. This is shown in the volatility smile.

Reasons for the Smile in Foreign Currency Options

Two conditions for an asset price to have a lognormal distribution:

1. The volatility of the asset is constant
2. The price of the asset changes smoothly with no jumps

Neither of these are true for an exchange rate

- volatility is far from constant
- exchange rates frequently exhibit jumps

The result is that extreme outcomes becomes more likely.

Impact of jumps and nonconstant volatility depends on option maturity

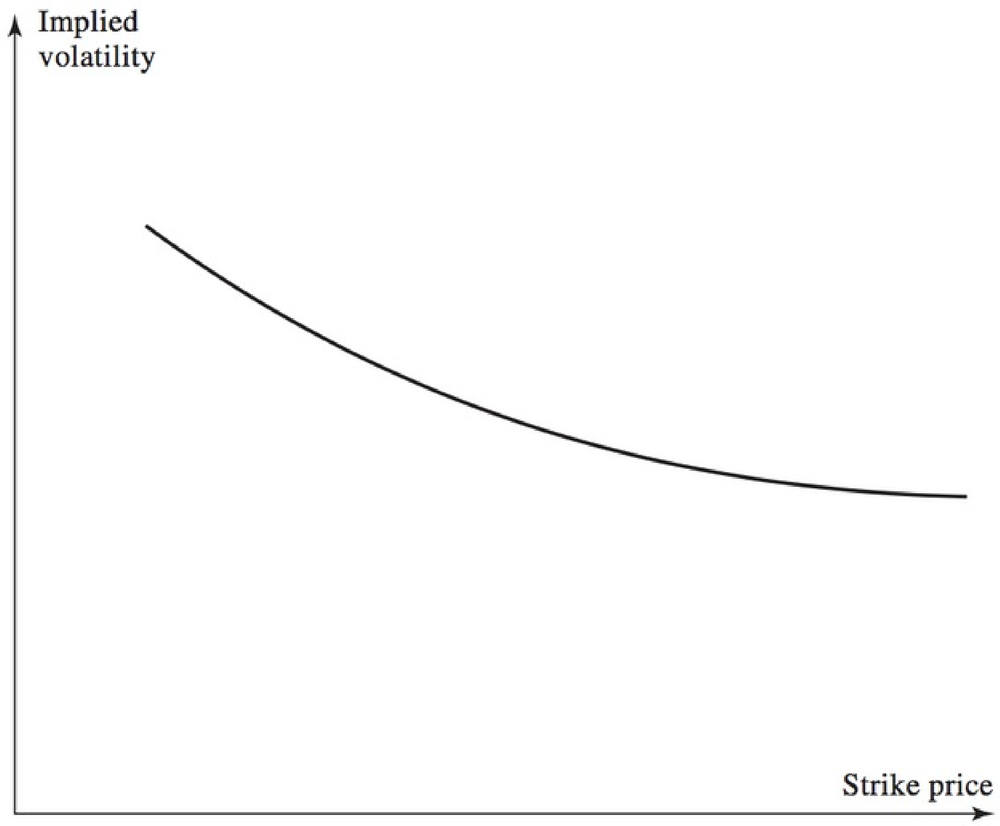
- as maturity increases, the impact of nonconstant volatility
  – becomes more pronounced on prices
  – becomes less pronounced on implied volatility
- as maturity increases, the impact of jumps
  – becomes less pronounced on prices
  – becomes less pronounced on implied volatility

The result is that the volatility smile becomes less pronounced as option maturity increases.

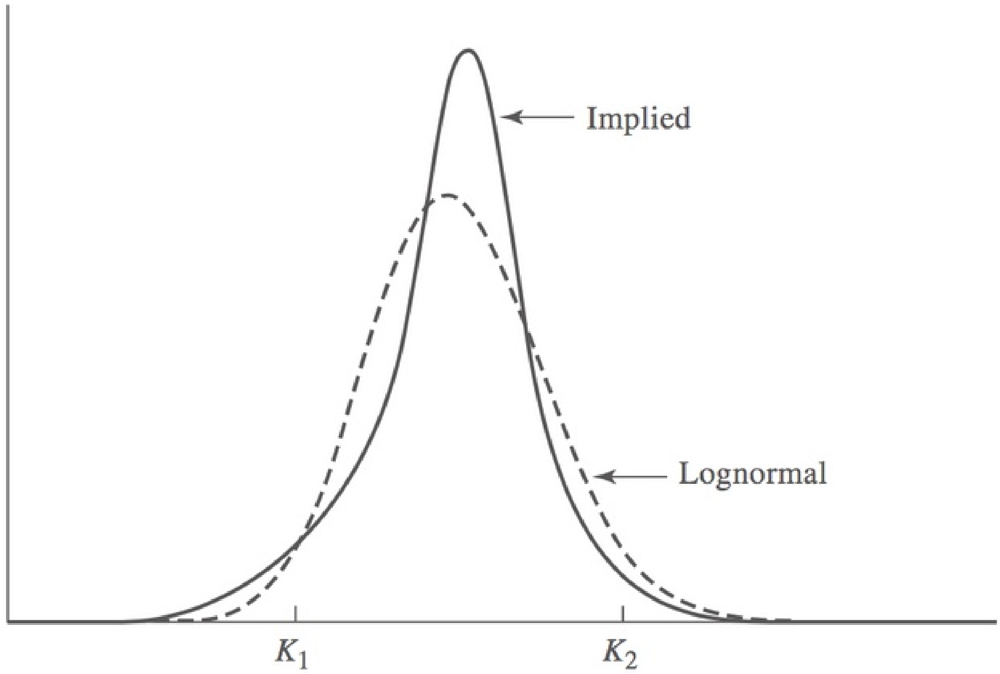
Equity Options

Volatility smile for equity options sometimes called the *volatility skew*

- volatility decreases as strike price increases



It corresponds to the following implied probability distribution.



Note that the implied distribution has a heavier left tail and less heavy right tail than the lognormal distribution.

The volatility smile (skew) and the implied distribution are consistent. An option with high strike price has a lower price when using the implied distribution than when using the lognormal distribution because the probability of payoff is lower (option pays off only if stock price is above strike price). A low price leads to a low implied volatility. On the other hand an option with low strike price has a higher price when using the implied distribution, so the implied volatility is high.

The Reason for the Smile in Equity Options

Leverage

- as company equity declines in value, the company leverage increases
  – equity becomes more risk and its volatility increases
- as company equity increases in value, the company leverage decreases
  – equity becomes less risky and its volatility decreases

This suggests that volatility of a stock is a decreasing function of the stock price.

Alternative Ways of Characterizing the Volatility Smile

Volatility smile has been defined as relationship between implied volatility and strike price.
The relationship depends on current price of the asset.

- if exchange rate or equity price increase, volatility smile/skew moves right
- if exchange rate or equity price decrease, volatility smile/skew moves left

Therefore it is often calculated as the relationship between implied volatility and K/S_0, rather than between implied volatility and K. Then the smile is much more stable.

A refinement is to calculate the volatility smile as the relationship between implied volatility and K/F_0, where F_0 is the forward price of the asset for a contract maturing at the same time as the options under consideration.

Sometimes an at-the-money option is defined as an option where $K = F_0$ rather than where $K = S_0$ because F_0 is the expected stock price on the option's maturity date in a risk-neutral world.

Another way to define the volatility smile is as the relationship between the implied volatility and the delta of the option.

This makes it possible to apply volatility smiles to options other than European and American calls and puts.

An at-the-money option is then defined as a "50-delta option"

- a call option with a delta of 0.5

- a put option with a delta of -0.5

The Volatility Term Structure and Volatility Surfaces

Implied volatility can depend on time to maturity as well as strike price

- implied volatility tends to be increasing function of maturity when short-dated volatilities are low – expectation that volatilities will increase
- implied volatility tends to be decreasing function of maturity when short-dated volatilities are high – expectation that volatilities will decrease

volatility surface - combine volatility smile with volatility term structure

- used to price options with any strike price and any maturity

Volatility smile becomes less pronounced as option maturity increases.

The volatility smile is sometimes defined as the relationship between implied volatility and

$$\frac{1}{\sqrt{T}} \ln\left(\frac{K}{F_0}\right)$$

where T is time to maturity and F_0 is forward price of the asset for a contract maturing at the same time as the option.

The volatility smile is then usually much less dependent on the time to maturity.

Greek Letters

The volatility smile complicates the calculation of Greek letters.

Assume relationship between implied volatility and K/S for an option with a certain time to maturity remains the same. As the price of the underlying asset changes, the implied volatility of the option changes to reflect the option's “moneyness”.

The formulas for Greek letters then change.

The delta of a call option was given by

$$\Delta = \frac{\partial c}{\partial S}$$

The delta of a call option is now given by

$$\Delta = \frac{\partial c_{\text{BS}}}{\partial S} + \frac{\partial c_{\text{BS}}}{\partial \sigma_{\text{imp}}} \frac{\partial \sigma_{\text{imp}}}{\partial S}$$

where c_{BS} is the Black-Scholes price of the option expressed as a function of the asset price S and the implied volatility σ_{imp}.

Consider an equity call option, where volatility is a decreasing function of K/S (volatility skew).

The implied volatility increases as asset price increases, which means

$$\frac{\partial \sigma_{\text{imp}}}{\partial S} > 0$$

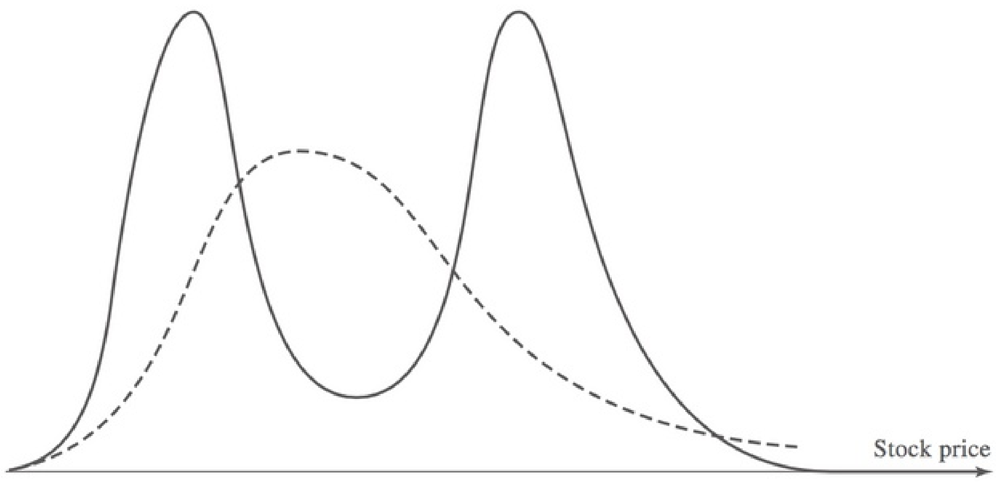
The result is that the delta is higher than that given by the Black-Scholes assumptions.

The Role of the Model

Models have most effect on the pricing of derivatives when similar derivatives do not trade actively in the market. The pricing of many nonstandard exotic derivatives is model-dependent.

When a Single Large Jump is Anticipated

Suppose there is important upcoming news that will either increase or decrease a stock price by a large amount. The probability distribution of the future stock price would consist of a mixture of two lognormal distributions. The first corresponds to favorable news (stock price increases) and the second corresponds to unfavorable news (stock price decreases).

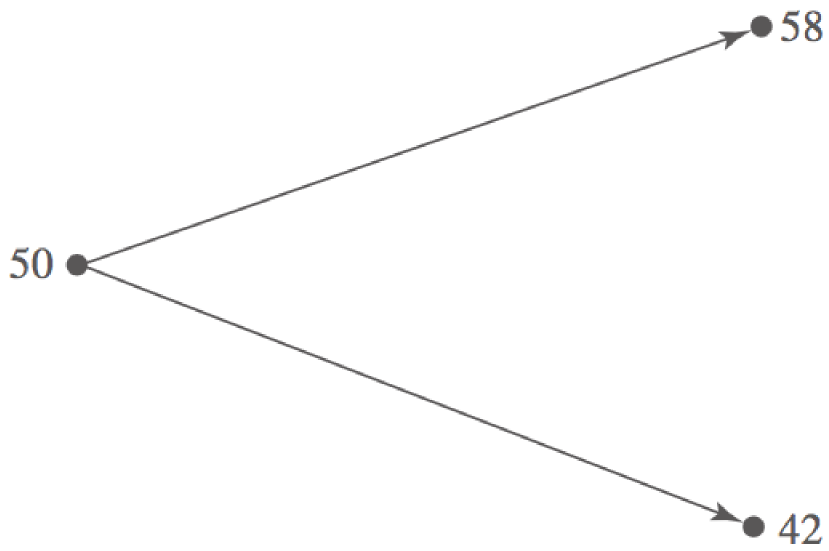


The solid line shows the mixture of lognormal distributions.

The dashed line shows a lognormal distribution with the same mean and standard deviation.

The true probability distribution (solid line) is bimodal. Investigate the effect of a bimodal stock price distribution by considering the extreme case where there are only two possible future stock prices.

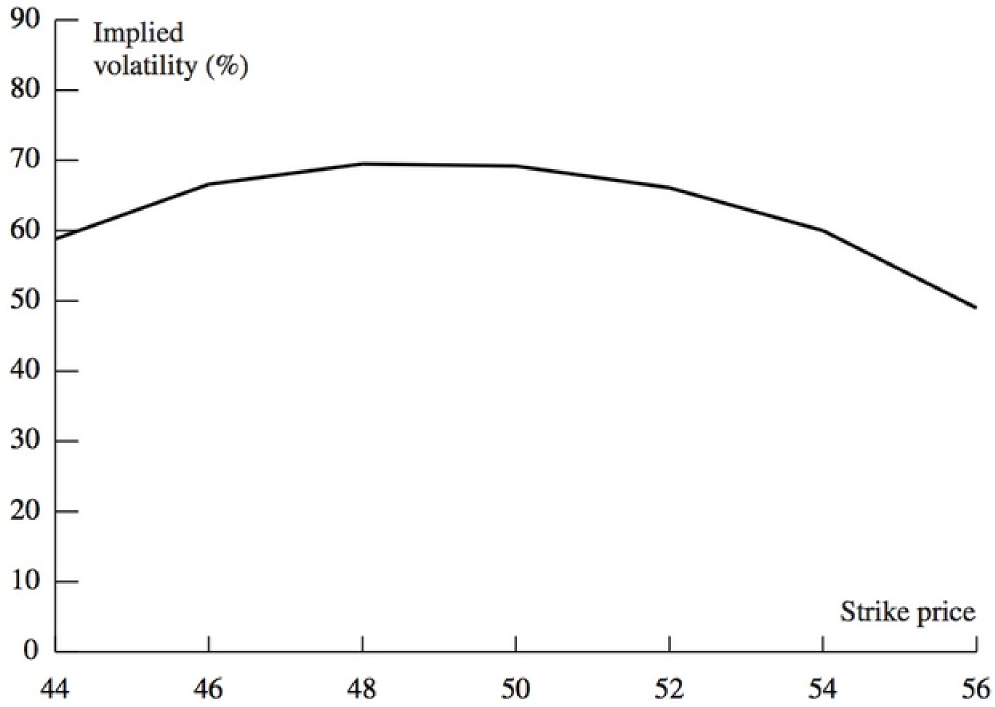
Suppose stock price is $50. In 1 month it will be either $42 or $58. Let risk-free rate by 12%.



$$u = 1.16 \qquad d = 0.84 \qquad a = 1.0101 \qquad p = 0.5314$$

The call and put prices of 1-month European options can be calculated for various strike price between $42 and $58. The implied volatility can be calculated (same for calls and puts).

The volatility smile is actually a “frown”.


Implied volatility (%)
90
80
70
60
50
40
30
20
10
0
Strike price
44
46
48
50
52
54
56