Ch 29 Interest Rate Derivatives: The Standard Market Models

Interest rate derivatives are instruments whose payoffs are dependent on the level of interest rates.

A key challenge was to find procedures for pricing and hedging these products

1. The behavior of an individual interest rate is more complicated than that of a stock price or exchange rate.
2. Valuation of many products requires a model describing the behavior of the entire zero-coupon yield curve.
3. Volatilities of different points on the yield curve are different.
4. Interest rates are used for discounting the derivative as well as defining its payoff.

Three most popular over-the-counter interest rate option products

- bond options
- interest rate caps/floors
- swap options

Bond Options

An option to buy or sell a particular bond by a particular date for a particular price.

- trade in OTC market
- also can be embedded in bonds when they are issued
  – make bond more attractive to issuer and purchasers

Embedded Bond Options

Callable bond

- bond that contains provisions allowing the issuer to buy back the bond at a predetermined price at certain times in the future
- the holder has sold a call option to the issuer
- strike price is the predetermined price paid by issuer to holder
- usually have a lock-out period
- call price is usually a decreasing function of time
- value of call option reflected in the quoted yields on bonds
  – bonds with call features offer higher yields than bonds with no call features

Puttable bond

- bond that contains provisions allowing the holder to demand early redemption at a predetermined price at certain times in the future
- the holder has purchased a put option on the bond in addition to the bond itself
- put option increases the value of the bond to the holder
  – bonds with put features provide lower yields than bonds with no put features

Loan and deposit instruments often contain embedded bond options.

- prepayment privileges on mortgages are call options on bonds
- loan commitment made by a bank is a put option on a bond

European Bond Options

Many OTC bond options and some embedded bond options are European.

The assumption is that the forward bond price has volatility σ_B. This allows Black's model to be used. Let F_B be the forward bond price.

$$c = P(0,T)[F_B N(d_1) - KN(d_2)]$$

$$p = P(0,T)[KN(-d_2) - F_B N(-d_1)]$$

where

$$d_1 = \frac{\ln(F_B/K) + \sigma_B^2 T/2}{\sigma_B \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_B \sqrt{T}$$

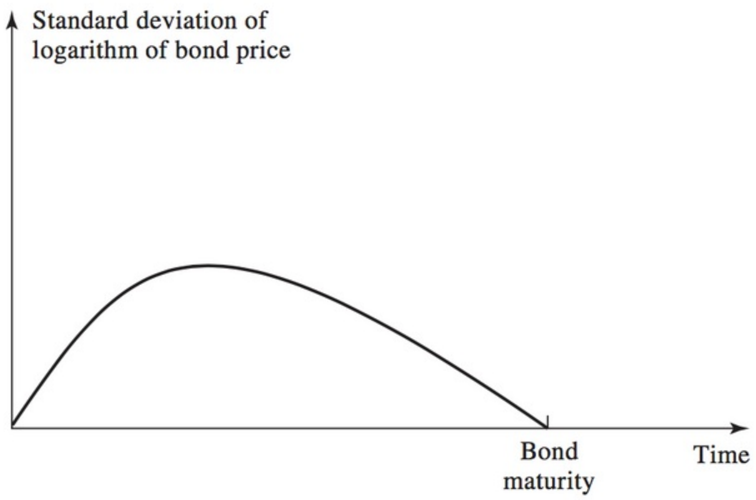
where K is the strike price of the bond option, T is the time to maturity, and $P(0,T)$ is the risk-free discount factor for maturity T.

Note that F_B can be calculated as

$$F_B = \frac{B_0 - I}{P(0,T)}$$

where B_0 is the bond price at time zero and I is the present value of the coupons that will be paid during the life of the option.

The standard deviation of the logarithm of bond price is shown here.



The standard deviation is zero today because there is no uncertainty about the price today.
The standard deviation is zero at maturity because the price will equal face value at maturity.
In between, the standard deviation first increases and then decreases.

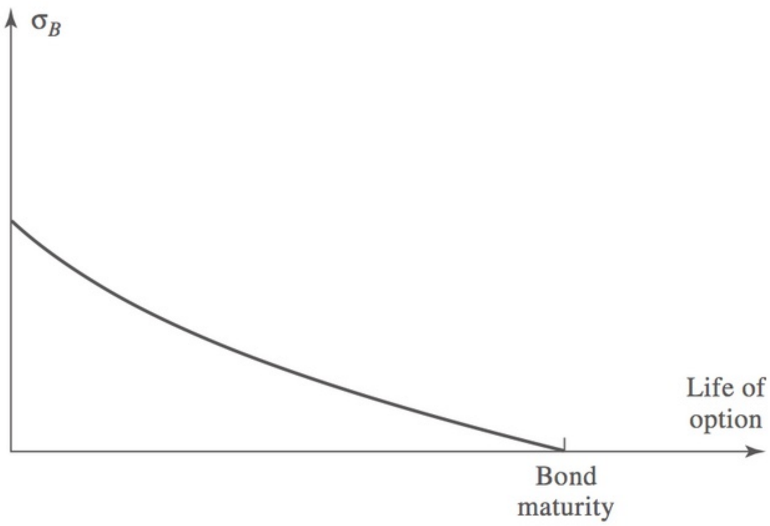
The volatility σB that should be used when a European option on the bond is valued is

$$\frac{\text{Standard deviation of logarithm of bond price at maturity of option}}{\sqrt{\text{Time to maturity of option}}}$$

(Note that maturity of bond is different from maturity of option)

What happens when option life increases (for the same bond)?

This shows a typical pattern for σB as a function of option life.
In this case σB declines as the life of the option increases.



Yield Volatilities

Volatilities quoted for bond options are often yield volatilities rather than price volatilities.
Duration is used to convert a quoted yield volatility into a price volatility.

Suppose that D is the modified duration of the bond underlying the option at the option maturity. The relationship between change ΔF_B in forward bond price F_B and change in Δy_F in forward yield y_F is

$$\frac{\Delta F_B}{F_B} \approx -D\Delta y_F$$

or

$$\frac{\Delta F_B}{F_B} \approx -Dy_F \frac{\Delta y_F}{y_F}$$

Volatility is a measure of the standard deviation of percentage changes in the value of a variable.
So the equation suggests the volatility of the forward bond price σ_B used in Black's model can be approximately related to the volatility of the forward bond yield σ.

$$\sigma_B = Dy_0\sigma_y$$

where y_0 is the initial value of y_F.

Interest Rate Caps and Floors

A popular interest rate option offered in the OTC market is an **interest rate cap**.

Consider a floating-rate note where the interest rate is reset periodically equal to LIBOR.
The time between resets is the **tenor**.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level (the *cap rate*).

Suppose principal is $10m and the tenor is 3 months.
Suppose the 3-month LIBOR is 5%, so the floating rate note would require

$$0.25 \times 0.05 \times \$10{,}000{,}000 = \$125{,}000$$

of interest to be paid in 3 months.

Now suppose the 3-month LIBOR is 4%, so the floating rate note interest payment would be

$$0.25 \times 0.04 \times \$10{,}000{,}000 = \$100{,}000$$

So the cap provides a payoff of $25,000.
The payoff occurs when the payment is due, not when the interest rate is observed.

At each reset date during life of the cap, LIBOR is observed. If LIBOR is less than 4%, there is no payoff from the cap. If LIBOR is greater than 4%, the payoff is the excess applied to the principal.

The Cap as a Portfolio of Interest Rate Options

Consider a cap with total life T, principal L, and cap rate R_K.
Suppose reset dates are $t_1, t_2, \ldots, t_n$, with $t_{n+1} = T$.
Define R_k as LIBOR rate for the period between time t_k and t_{k+1} observed at time t_k.

The cap leads to a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_K, 0)$$

where $\delta_k = t_{k+1} - t_k$.

This is also the payoff from a call option on the LIBOR rate observed at time t_k with the payoff occurring at time t_{k+1}. The cap is a portfolio of n such options. LIBOR rates are observed at multiple times and the corresponding payoffs occur multiple times. The n call options underlying the cap are called *caplets*.

A Cap as a Portfolio of Bond Options

An interest rate can also be characterized as a portfolio of put options on zero-coupon bonds with payoffs on the puts occurring at the time they are calculated.

The payoff given above for time t_{k+1} can be discounted to time t_k and given as

$$\frac{L\delta_k}{1 + R_k\delta_k} \max(R_k - R_K, 0)$$

This can be reduced to

$$\max\left[L - \frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}, 0\right]$$

where the expression

$$\frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}$$

is the value at time t_k of a zero-coupon bond that pays off $L(1 + R_K\delta_k)$ at time t_{k+1}.

So the expression above is the payoff from a put option with maturity t_k on a zero-coupon bond with maturity t_{k+1} when the face value of the bond is $L(1 + R_K\delta_k)$ and the strike price is L.

It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.

Floors and Collars

Interest rate floors and interest rate collars are defined analogously to interest rate caps.

A **floor** provides a payoff when the interest rate on the underlying floating-rate note falls below a certain rate.

The payoff from a floor at time t_{k+1} is

$$L\delta_k \max(R_K - R_k, 0)$$

An interest rate floor can be regarded as

- a portfolio of put options on interest rate options
- a portfolio of call options on zero-coupon bonds

A **collar** is an instrument designed to guarantee that the interest rate on the underlying LIBOR floating-rate note always lies between two levels. It is a combination of a long position in a cap and a short position in a floor. It is usually constructed so the price of the cap equals the price of the floor, so the cost of entering into the collar is zero.

Valuation of Caps and Floors

As shown above, the caplet corresponding to the rate observed at time t_k provides a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_K, 0)$$

Under the standard market model, the value of the caplet is

$$L\delta_k P(0, t_{k+1})[F_k N(d_1) - R_K N(d_2)]$$

where

$$d_1 = \frac{\ln(F_k/R_K) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} \quad \text{and} \quad d_2 = d_1 - \sigma_k\sqrt{t_k}$$

where F_k is the forward interest rate at time 0 for the period between time t_k and t_{k+1}, and σ_k is the volatility of this forward interest rate.

This is an extension of Black's model.

The value of the corresponding floorlet is

$$L\delta_k P(0, t_{k+1})[R_K N(-d_2) - F_k N(-d_1)]$$

Each caplet of a cap and each floorlet of a floor must be valued separately.

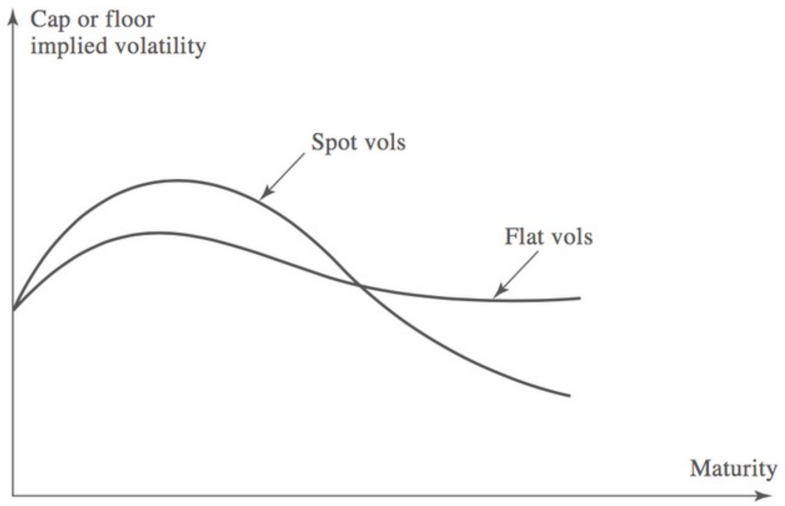
One approach is to use a different volatility for each caplet (or floorlet).
These volatilities are *spot volatilities*.

Another approach is to use the same volatility for all the caplets (floorlets) comprising any particular cap (floor) but to vary this volatility according to the life of the cap (floor).
These volatilities are *flat volatilities*.

Spot Volatilities vs Flat Volatilities

This shows a typical pattern for spot volatilities and flat volatilities as a function of maturity.

- spot volatility - maturity of a caplet or floorlet
- flat volatility - maturity of a cap or floor



The flat vols are akin to cumulative averages of spot vols, so they exhibit less variability.
Volatilities initially rise but then mean reversion of interest rates causes volatilities to fall.

European Swap Options

Swap options (*swaptions*) are options on interest rate swaps.

- give holder right to enter into a certain interest rate swap at a certain time in the future
- provide companies with guarantee that the fixed rate of interest they paid on a loan at some future time will not exceed some level
- alternative to forward swaps

The difference between a swaption and a forward swap is analogous to the difference between an option on a foreign currency and a forward contract on the currency.

- forward swaps involve no up-front cost
- in a forward swap, the company is obligated to enter into the swap agreement
- swaptions allow company to benefit from favorable interest rate movements while acquiring protection from unfavorable interest rate movements

call swaption - gives holder right to pay floating and receive fixed
put swaption - gives holder right to pay fixed and receive floating

Valuation of European Swaptions

Consider a swaption where the holder has the right to pay a rate s_K and receive LIBOR on a swap that will last n years starting in T years. Suppose there are m payments per year under the swap and that the notional principal is L.

Each fixed payment on the swap is the fixed rate times L/m.

Suppose that the swap rate for an n-year swap starting at time T proves to be s_T.
The payoff from the swaption consists of a series of cash flows equal to

$$\frac{L}{m}\max(s_T - s_K, 0)$$

The cash flows are received m times per year for the n years of the life of the swap.

Suppose swap payment dates are $T_1, T_2, \ldots, T_{mn}$, measured in years from today.
Each cash flow is the payoff from a call option on s_T with strike price s_K.

Whereas a cap is a portfolio of options on interest rates, a swaption is a single option on the swap rate with repeated payoffs. The standard market model gives the value of a swaption where the holder has the right to pay s_K as

$$\sum_{i=1}^{mn} \frac{L}{m} P(0, T_i)[s_0 N(d_1) - s_k N(d_2)]$$

where

$$d_1 = \frac{\ln(s_0/s_K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

where s_0 is the forward swap rate at time zero and σ is the volatility of the forward swap rate. (So $\sigma\sqrt{T}$ is the standard deviation of $\ln s_T$)

This is an extension of Black's model.

If the swaption gives the holder the right to received a fixed rate of s_K instead of paying it, the payoff from the swaption is

$$\frac{L}{m}\max(s_K - s_T, 0)$$

The standard market model gives the value of the swaption as

$$\sum_{i=1}^{mn} \frac{L}{m} P(0,T_i)[s_K N(-d_2) - s_0 N(-d_1)]$$

OIS Discounting

For cap/floor and swaption arguments above, LIBOR has been used to determine cash flows as well as to determine risk-free discount rates.

When OIS discounting is used, forward LIBOR rate for the period between t_k and t_{k+1} is $E_{k+1}(R_k)$, where R_k is the realized LIBOR rate for this period and E_{k+1} denotes the expectation in a world that is forward risk-neutral with respect to a risk-free (OIS) zero-coupon bond maturing at time t_{k+1}.

Hedging Interest Rate Derivatives

Delta

For interest rate derivatives, delta risk is the risk associated with a shift in the zero curve.
There are many ways in which the zero curve can shift, so many deltas can be calculated.

Alternatives:

1. Calculate the impact of a 1-basis-point parallel shift in the zero curve (**DV01**).
2. Calculate the impact of small changes in the quotes for each of the instruments used to construct the zero curve.
3. Divide the zero curve into sections and calculate the impact of shifting the rates in on section by 1 basis point, keeping the rest of the term structure unchanged.
4. Carry out a principal components analysis by calculating delta with respect to the changes in each of the first few factors. The first delta measures the impact of a small parallel shift, the second delta measures the impact of a small twist in the curve, etc.

Gamma

With several deltas, there are several gammas. If 10 instruments are used to compute the zero curve, there can be 10 deltas (using the 2nd approach above). Then there are 55 gammas.

One approach is to ignore cross-gammas.
Another approach is to calculate a single gamma measure as the second partial derivative of the value of the portfolio with respect to a parallel shift in the zero curve.
A third approach is to calculate gammas with respect to the first two factors in a principal components analysis.

Vega

The vega of a portfolio of interest rate derivatives measures its exposure to volatility changes.

One approach is to calculate the impact on the portfolio of making the same small change to the Black volatilities of all caps and European swap options. This assumes that one factor drives all volatilities.

Another approach is to carry out a principal components analysis on the volatilities of caps and swap options and calculate vega measures corresponding to the first 2 or 3 factors.