Equilibrium Path Analysis Including Bifurcations with an Arc-Length Method Avoiding A Priori Perturbations

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Abstract Wrinkling or pattern formation of thin (floating) membranes is a phenomenon governed by buckling instabilities of the membrane. For (post-) buckling analysis, arc-length or continuation methods are often used with a priori applied perturbations in order to avoid passing bifurcation points when traversing the equilibrium paths. The shape and magnitude of the perturbations, however, should not affect the post-buckling response and hence should be chosen with care. In this paper, our primary focus is to develop a robust arc-length method that is able to traverse equilibrium paths and post-bifurcation branches without the need for a priori applied perturbations. We do this by combining existing methods for continuation, solution methods for complex roots in the constraint equation, as well as methods for bifurcation point indication and branch switching. The method has been benchmarked on the post-buckling behaviour of a column, using geometrically non-linear isogeometric Kirchhoff-Love shell element formulations. Excellent results have been obtained in comparison to the reference results, from both bifurcation point and equilibrium path perspective.

1 Introduction

Linear buckling analysis of (maritime) structures is widely used in engineering to estimate the loads for which instabilities or even collapse will occur. Post-buckling analysis is often considered to assess the load carrying capacity after instability or collapse. For (floating) thin membranes-like offshore solar platforms [14], (post-)

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buckling analysis involves the wrinkling phenomenon when loads on the membrane exceed critical values [3, 8, 15, 18–20].

When modelling instabilities like wrinkling, *a priori* perturbations of some shape and magnitude are often applied to initiate post-buckling without passing bifurcation points. Perturbations are required since bifurcation points introduce singularities in the system matrix, meaning that commonly used solution procedures are not able to provide the post-buckling response. However, as previously reported by Taylor et al. [18], the magnitude of the initial perturbations might influence the final solution.

Hence, in this paper our primary focus is to develop a numerical procedure - based on a combination of the conventional and extended arc-length method [5, 6], solution methods for complex roots [12, 24], as well as methods for bifurcation point indication and branch switching [9] (section 2). The performance of the proposed method is illustrated using a benchmark problem (section 3) and conclusions are drawn to complete the work (section 4).

2 The Arc-Length Method

The arc-length method, also known as a path-following algorithm or a continuation method, is a method to advance through a solution space $\mathbf{w}(\mathbf{u}, \lambda)$ of the system

$$\mathbf{G}(\mathbf{u}, \lambda) = \mathbf{N}(\mathbf{u}) - \lambda \mathbf{P} = \mathbf{0}, \tag{1}$$

where $\mathbf{N}(\mathbf{u})$ is a vector function in terms of solution vector $\mathbf{u}$ and $\mathbf{P}$ is a constant vector multiplied by scaling λ. Both $\mathbf{N}$ and $\mathbf{P}$ can follow from a finite element discretization of a system of partial differential equations based on the finite solution vector $\mathbf{u} \in \mathbb{R}^n$. The function $\mathbf{G}$ can thus be used to find the solution $\mathbf{u}$ for a particular scaling λ (i.e. "load control") or vice-versa (i.e. "displacement control") . Alternatively, one can use the function $\mathbf{G}$ and a constraint equation $f(\mathbf{w})$ to find the combination $\mathbf{w} = (\mathbf{u}, \lambda)$ that satisfies $\mathbf{G}(\mathbf{w}) = \mathbf{0}$ and $f(\mathbf{w}) = 0$. This principle is used in the *arc-length method* [4, 16], which will be used to obtain the solution of eq. (1) in the case that the solution is not known to be unique.

2.1 Conventional and Extended Arc-Length Method

The constraint equation $f(\mathbf{w})$ is often imposed on the solution increment $\Delta \mathbf{w} = (\delta \mathbf{u}_k, \delta \lambda_k)$ and can take different forms, e.g. using Riks' method [16] or Crisfield's method [4]. The latter imposes:

$$f(\mathbf{w}) = \delta \mathbf{u}_k^\top \delta \mathbf{u}_k + \Psi^2 \delta \lambda_k^2 \mathbf{P}^\top \mathbf{P} = \Delta l^2. \tag{2}$$

Here, Δl is the arc-length or the radius of the constraint equation, Ψ is a scaling factor to incorporate the dimensionality of the system in the factor λ. The constraint equation of Crisfield was used because this method always finds a solution, despite the curvature of the equilibrium path. The disadvantage, however, is that two solutions are found per iteration, and hence, that a particular solution needs to be selected. Note that the square root of the constraint equation, $\sqrt{f(\mathbf{w})}$, is a proper norm.

Crisfield [4] originally used $\Psi = 0$, referred to as a spherical constraint, but the elliptical constraint is used to maintain displacement and load steps in the same order of magnitude for different refinements:

$$\Psi^2 = \mathbf{u}_0^\top \mathbf{u}_0 / \lambda_0^2 \mathbf{P}^\top \mathbf{P}. \tag{3}$$

Here, λ_0 and $\mathbf{u}_0$ correspond to the solutions on a previous equilibrium point (i.e. a converged point). In the origin $\mathbf{w}_0 = (\mathbf{u}_0, \lambda_0) = (\mathbf{0}, 0)$, a slightly different procedure is used [12]. As a consequence of the constraint equation, the system matrix, if banded, loses its banded nature hence affecting convergence behaviour of nonlinear solvers [21]. Therefore, the system of equations is solved in a segregated way. To this extent, eq. (1) is considered in terms of the unknown increments $\delta\lambda_k$ and $\delta\mathbf{u}_k$ at iteration k, such that

$$\mathrm{K}\delta\mathbf{u}_k = \mathbf{G}(\mathbf{u}, \delta\lambda_k) = \mathbf{N}(\mathbf{u}) - \delta\lambda_k \mathbf{P}. \tag{4}$$

Where the splitting of the incremental displacement $\delta\mathbf{u}_k$ in terms of a standard load-controlled Newton-Raphson method $\delta\bar{\mathbf{u}}_k$ and a component from the increment $\delta\lambda_k$ being $\delta\hat{\mathbf{u}}_k$ is used:

$$\delta\mathbf{u}_k = \beta\delta\bar{\mathbf{u}}_k + \delta\lambda_k \delta\hat{\mathbf{u}}_k. \tag{5}$$

The line-search parameter β is relevant when dealing with complex roots (see section 2.2) and is equal to 1.0 otherwise. Then, for iteration k,

$$\mathrm{K}\delta\bar{\mathbf{u}}_k = \mathbf{G}(\mathbf{w}_k), \tag{6}$$

$$\mathrm{K}\delta\hat{\mathbf{u}}_k = \mathbf{P}. \tag{7}$$

where K is the Jacobian of the system to be solved and has to be computed once. A disadvantage is that no solutions can be found on limit points, since the Jacobian is singular there [6]. At each iteration, the load and displacement increments are updated using

$$\Delta\mathbf{w}_k = (\Delta\mathbf{u}_k, \Delta\lambda_k) = (\delta\mathbf{u}_{k-1}, \delta\lambda_{k-1}) + (\delta\mathbf{u}_k, \delta\lambda_k) \tag{8}$$

Using the constraint equation from eq. (2) and using the fact that the iterative increment $\delta\mathbf{u}_k$ is depending on the unknown $\delta\lambda_k$, the constraint equation can be written as a polynomial in $\delta\lambda_k$:

$$a\delta\lambda_k^2 + b\delta\lambda_k + c = 0, \tag{9}$$

With,

$$
\begin{aligned}
a &= \delta\hat{\mathbf{u}}_k^\top \delta\hat{\mathbf{u}}_k + \Psi^2 \mathbf{P}^\top \mathbf{P} = a_0, \\
b &= 2\left(\delta\hat{\mathbf{u}}_k^\top \Delta\mathbf{u} + \Delta\lambda\Psi^2 \mathbf{P}^\top \mathbf{P}\right) + 2\beta\delta\hat{\mathbf{u}}_k^\top \delta\bar{\mathbf{u}}_k = b_0 + \beta b_1, \\
c &= \beta^2 \delta\bar{\mathbf{u}}_k^\top \delta\bar{\mathbf{u}}_k + 2\beta\delta\bar{\mathbf{u}}_k^\top \Delta\mathbf{u} + \Delta\mathbf{u}^\top \Delta\mathbf{u} + \Delta\lambda^2 \Psi^2 \mathbf{P}^\top \mathbf{P} - \Delta l^2 \\
&= c_0 + \beta c_1 + \beta^2 c_2.
\end{aligned}
\tag{10}
$$

where $\Delta\mathbf{u} = (\Delta\mathbf{u}, \Delta\lambda)$ (indices omitted) denotes the increment in the previous load step. Since $\mathbf{u}_t$ and $\bar{\mathbf{u}}$ are known from eq. (6) and eq. (7), the only unknown in eq. (9) is the load increment $\delta\lambda$. Therefore, eq. (9) is a scalar quadratic equation that is easily solved for $\delta\lambda_k$ and has two solutions. The choice of the solution is based on the 'angle' between the arc-length increment $\Delta\mathbf{w}$ of the previous load step and the current $\Delta\mathbf{w}_k$. Since this term is minimised for the increment $\delta\lambda_k$, it is sufficient to look at the following roots [17]:

$$
\Theta_r = \delta\lambda_r \left(\Delta\mathbf{u}^\top \delta\hat{\mathbf{u}}_k + \Psi^2 \Delta\lambda\right) \quad r = 1, 2. \tag{11}
$$

The root $\delta\lambda_r$ for which Θ_r is largest is the selected root. In the original work of Crisfield [4] a different method was proposed, where the increment $\Delta\mathbf{u}_k$ is computed for both values of $\Delta\lambda_r$ and the largest inner-product is taken. Both methods were implemented and no major changes in the robustness of the methods were observed. By comparing the current increment with the previous load increment, both methods are robust as long as no sharp snap-back behaviour is present with respect to the chosen arc-length Δl.

In the first iteration of a new load step, the vector $\delta\mathbf{u}_{k-1}$ and the scalar $\delta\lambda_{k-1}$ are equal to zero. Hence, the trivial solution is found for eq. (9). Therefore, the following method is used to initialize the method in a new load step. Note that by eqs. (6) and (7) $\delta\hat{\mathbf{u}}_k$ is non-zero and $\bar{\mathbf{u}}$ is zero since the residual in the first iteration $\mathbf{G}(\mathbf{w}_0)$ is zero. Therefore, the load increment in the first iteration is defined as [5, 12]:

$$
\Delta\lambda_0 = \begin{cases} \Delta l / \sqrt{2\delta\hat{\mathbf{u}}_k^\top \delta\hat{\mathbf{u}}_k}, & \text{if } (\mathbf{u}_0, \lambda_0) = (\mathbf{0}, 0) \\ \Delta l / \sqrt{\delta\hat{\mathbf{u}}_k^\top \delta\hat{\mathbf{u}}_k + \Psi^2 \mathbf{P}^\top \mathbf{P}} & \text{otherwise.} \end{cases} \tag{12}
$$

Its sign is determined by the previous load increment $\Delta\mathbf{w}$ [7]

$$
\operatorname{sign}(\Delta\lambda_0) = \operatorname{sign}(\Delta\mathbf{u}^\top \delta\hat{\mathbf{u}}_k + \Delta\lambda\Psi^2 \mathbf{P}^\top \mathbf{P}). \tag{13}
$$

2.2 Solution Methods for Complex Roots

In the case of complex roots for eq. (9), i.e. when $b^2 - 4ac < 0$, the numerical procedure as discussed in the previous section fails [2]. Complex roots occur when the equilibrium path is strongly curved in the region that is covered by one step. As a solution to complex roots, the arc-length can simply be bisected until real roots are found [1] or by utilising a pseudo line-search technique [12, 24]. The methods in the latter works are slightly different in the choice of the line-search parameter as will be detailed later.

As complex roots occur when $b^2 - 4ac < 0$ in eq. (9), a line-search parameter $\tilde{\beta} \neq 1$ exists such that $b^2 - 4ac \geq 0$ is satisfied. Substitution of the coefficients from eq. (10) in this condition provides a quadratic equation in terms of the unknown line-search parameter $\tilde{\beta}$:

$$a_s \tilde{\beta}^2 + b_s \tilde{\beta} + c_s \geq 0,$$

with [17]:

$$a_s = b_1^2 - 4a_0 c_2, \quad b_s = 2b_0 b_1 - 4a_0 c_1 \quad \text{and} \quad c_s = b_0^2 - 4a_0 c_0.$$

and which can be solved for the equality. When the parameter $\tilde{\beta}$ is obtained, eq. (9) can again be solved to find the roots for $\delta\lambda_k$. Selection of $\tilde{\beta}$ can be done using $0 < \tilde{\beta} \leq \tilde{\beta}_{\text{max}}$, where $\tilde{\beta}_{\text{max}} = \min(1, \tilde{\beta}_2)$, since the solutions $\tilde{\beta}_{1,2}$ ($\tilde{\beta}_1 < \tilde{\beta}_2$) are of opposite sign and if $\tilde{\beta}$ is between those roots (i.e. if $a_s c_s > 0$), the constraint equation is satisfied. If $\tilde{\beta}$ is close to zero, the iterative method becomes inefficient and it is recommended to cut the arc-length [17, 24].

2.3 Methods for Bifurcation Point Indication and Branch Switching

When applying the arc-length method on buckling analysis, singular points indicate a transition between stability and instability. The singular points can be characterised as either limit points or bifurcation points. The tangential stiffness matrix K is singular, i.e. the determinant of this matrix is equal to zero. Additionally, the first eigenvector $\boldsymbol{\phi}_1$ of the tangential stiffness matrix on a singular point represents the buckling mode shape in case of a bifurcation point. Limit points and bifurcation points are distinguished by considering the inner product $\boldsymbol{\phi}_1^\top \mathbf{P}$. If this product is non-zero, a limit point is found [22].

When passing a singular point, the determinant of this matrix becomes negative, or equivalently, the product of the diagonal entries of the diagonal matrix D of the $\text{LDL}^\top$ Cholesky decomposition changes sign. Unless a bifurcation point is exactly passed - which rarely occurs in practice - the matrix K is symmetric positive-definite. In this case, the $\text{LDL}^\top$ decomposition can be used to factorise and solve eqs. (6) and (7) and bifurcation points can be pinpointed by considering the sign of the lowest values of the diagonal matrix D. These determine the sign of the determinant of K and thus the stability of the system [21].

The bifurcation points are approached using the extended arc-length [23], which provides the solution $\mathbf{w}$ and the first eigenvector $\boldsymbol{\phi}_1$ of corresponding to the bifurcation point. This method converges quadratically since it is based on a Newton-Raphson method for solving the equilibrium equations $\mathbf{G}(\mathbf{u}, \lambda) = \mathbf{0}$, the singularity condition $\mathrm{K}(\mathbf{u}, \lambda)\boldsymbol{\phi}_1 = \mathbf{0}$ and a constraint equation to prevent the trivial solution $\boldsymbol{\phi}_1 = \mathbf{0}$ to be found [21–23].

When a bifurcation point $\mathbf{w}_P = (\mathbf{u}_P, \lambda_P)$ is found within a specified tolerance of the extended arc-length method, the eigenvector $\boldsymbol{\phi}_1$ is known from this method and the method can switch to the bifurcation branch by applying perturbation using the buckling mode shape, i.e. using $\boldsymbol{\phi}_1$. Branch switching is simply done by perturbing the displacements $\mathbf{u}_P$ at the bifurcation point by the normalized eigenvector $\bar{\boldsymbol{\phi}}_1$ multiplied by a factor τ. This factor can be chosen arbitrarily small [21].

3 Benchmark Problem

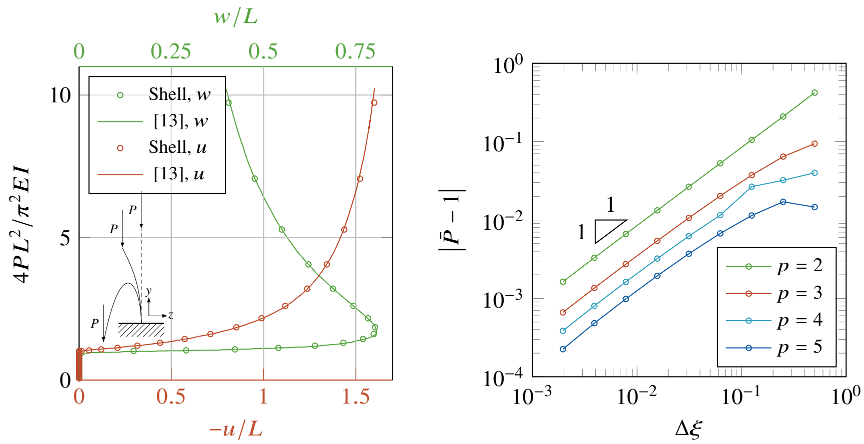
The geometrically linear isogeometric Kirchhoff-Love shell [11] formulation in the open-source Geometry+Simulation Modules (G+Smo[1]) [10] are used to model a thin shell. The benchmark is a column, i.e. a beam fixed at one side and loaded in-plane at the other side [13]. The column has length 1 $[m]$, thickness 0.01 $[m]$ and Young's modulus of 75 $[MPa]$. In both models, 32 elements of order 2 over the length and one element of order 2 in other directions are used. The eigenvector perturbation factor τ is 10^{-3}.

The results obtained with the arc-length method (fig. 1a) show excellent agreement with the reference results [13] for both bifurcation point prediction and post-buckling behaviour. Furthermore, fig. 1b shows the convergence of the extended iterations to the buckling point for both models with respect to $\bar{P} = 4\lambda P_{\text{ref}} L^2/\pi^2 EI$, where P_{ref} is the applied reference load. Convergence of the first order to the analytical solution is observed irrespective of the B-spline order. Hence the speed of convergence is not depending on the spline order p, but the magnitude of the error is.

4 Conclusions

In this paper, an arc-length method that does not require a priori perturbations was presented. The procedure is based on the Crisfield arc-length method with extensions for complex roots in the constraint equation for more robustness, and is able to find bifurcation branches without the need for a priori applied perturbations.

[1] The source of G+Smo can be found on `github.com/gismo`



(a) Horizontal (u) and vertical (w) displacement (bottom and top axis, resp.) of the end-point versus the applied load. The inset represents the undeformed (dashed) and deformed (solid) configuration.

(b) Convergence of the present arc-length method to the buckling load, for different knot vector spacings $\Delta\xi$ and B-spline orders p.

Fig. 1: Deformation of a column subject to a vertical end load.

For benchmarking, the model was applied on buckling and post-buckling analysis of a column with a compressive end load, modelled using isogeometric Kirchhoff-Love shell elements. The benchmark results show that the present method is able to provide accurate results in both path following as well as bifurcation point prediction. In future work, we will apply the present model on modelling wrinkles in thin (supported) sheets subject to large strains for validation and verification with previous studies [3, 8, 15, 18–20].

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