CS-E5740 - Complex Networks
Exercise set 2

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1 Properties of Erdős-Rényi networks

1. a) Explain in detail the origin of each of the three factors in the degree distribution $P(k)$ of E-R networks formula

Each node's number of links comes from $N-1$ independent trials with probability p.

- p^k : k links occur with probability p^k
- $(1-p)^{(N-1)-k}$: $(N-1)-k$ failures occur with probability $(1-p)^{N-k}$
- $\binom{(N-1)}{k}$ number of different ways of distributing k successes in a sequence of $(N-1)$ trials.

1. b) Motivate why in E-R networks, the average clustering coefficient C equals p (on expectation).

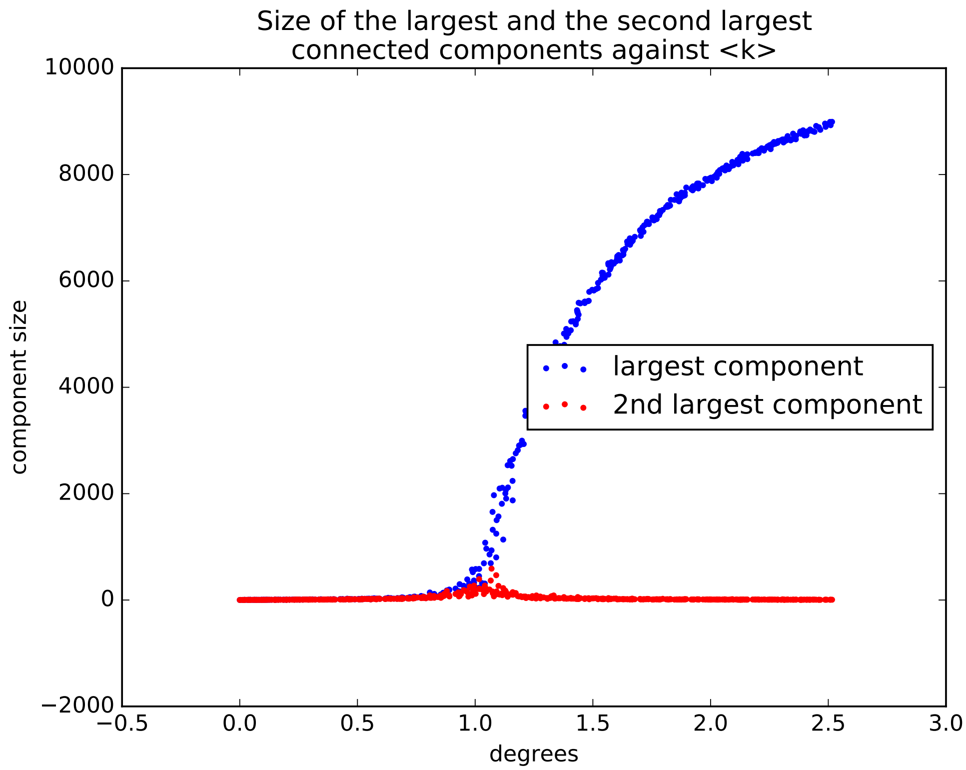
The clustering coefficient of a node is the probability that two randomly selected neighbors of the node are neighbors themselves. In the E-R network, the propability that an edge is present between two nodes is p by definition. Therefore, the clustering coefficient is on average equal to p.

1. c) Explain, what happens to C, if $N \to \infty$ with $\langle k \rangle$ bounded.

The average clustering coefficient becomes very small, indeed

$$C = p = \frac{\langle k \rangle}{N-1} \xrightarrow{N\to\infty} 0, \quad \text{as } \langle k \rangle \text{ is bounded}$$

1. d) Figures



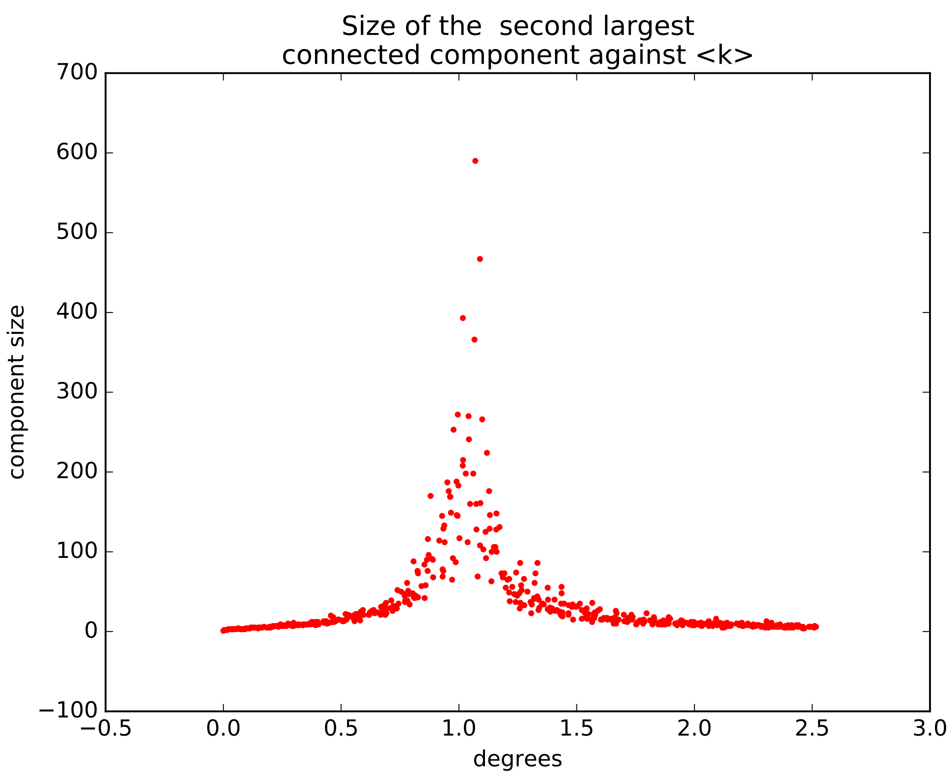


Figure 1.1: Size of the largest and second largest component in an E-R network of size $N = 10^4$ against $\langle k \rangle$

1. e) Averages for ER networks

- **For** $N = 3$

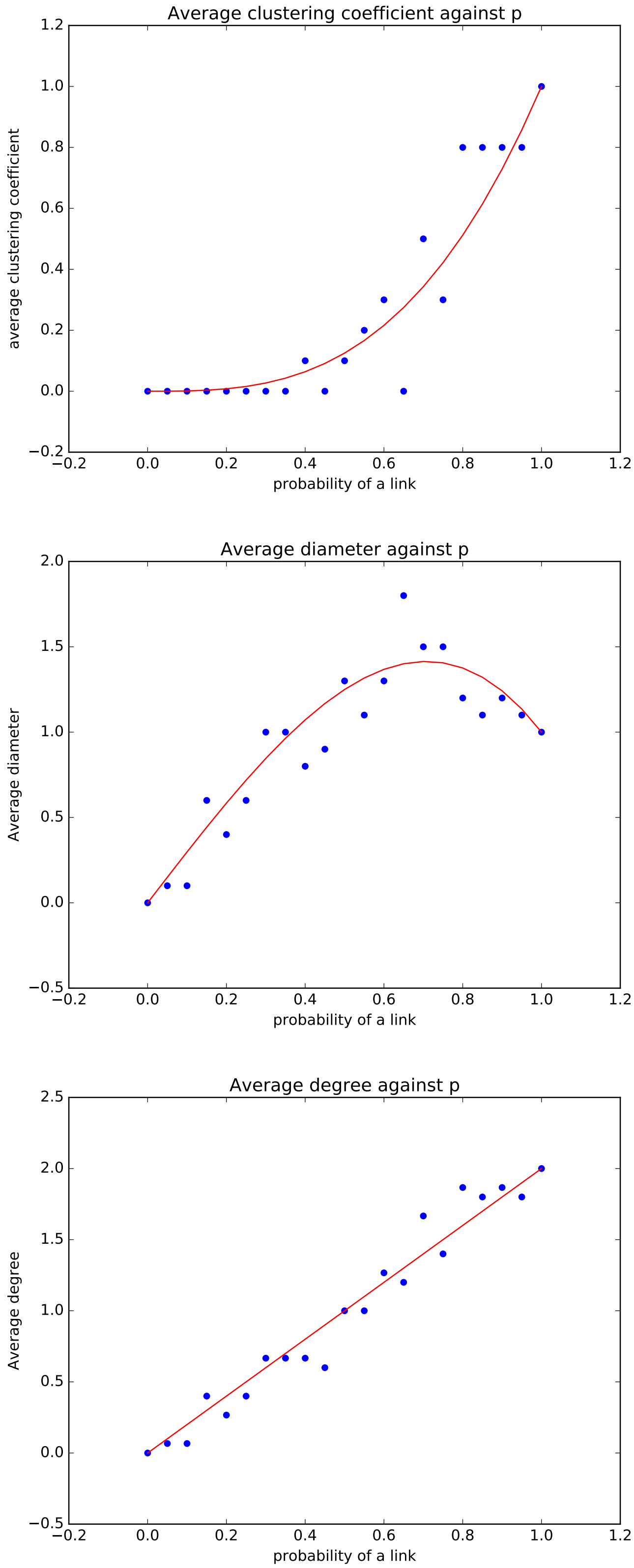


Figure 1.2: Average against the probability of link in a ER-network of size $N = 3$ (Analytical solution plotted in red)

- **For** $N = 100$

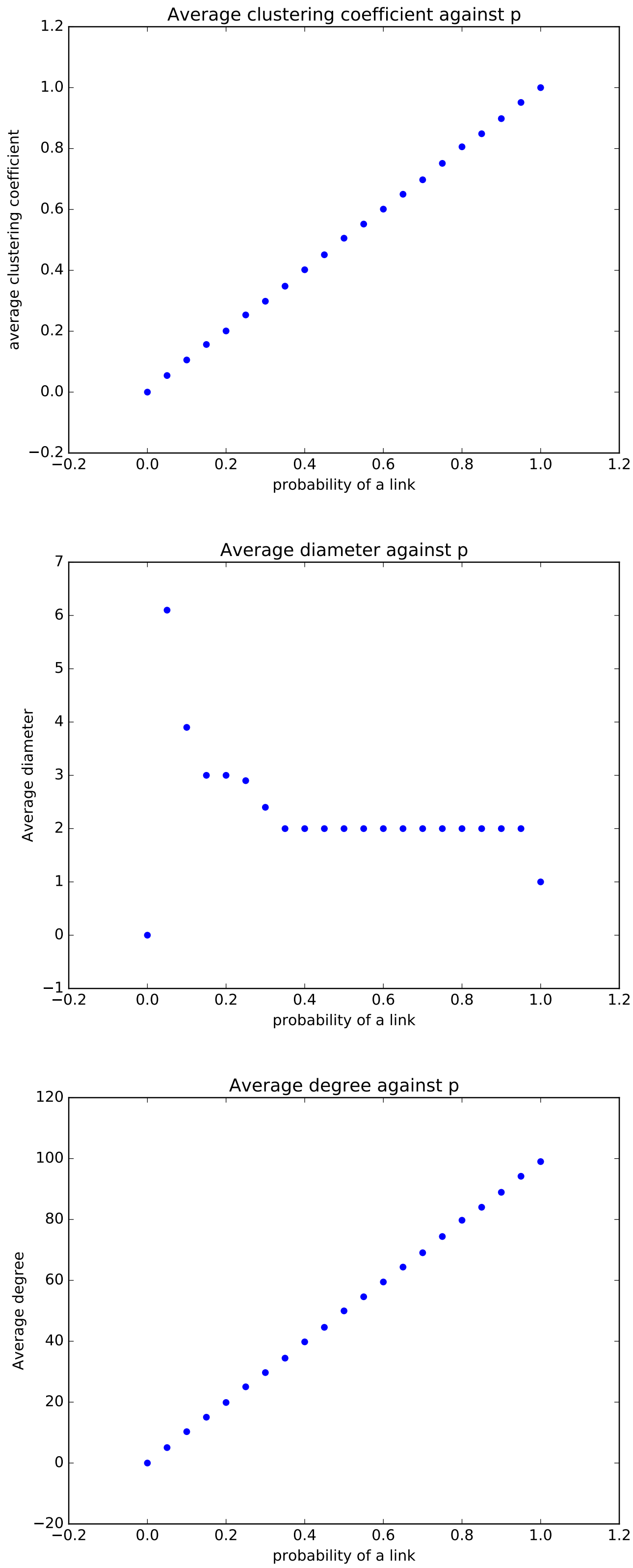


Figure 1.3: Average against the probability of link in a ER-network of size $N = 100$

- **Explain the benefits and downsides of this method as compared to the analytical method**
  - This method allows us to have a real behavior of the network, and to get the laws for multiple values of N quickly.
  - The analatycal method allows us to get the real result instead of an approximation.

2 Implementing the Watts-Strogatz small-world model

2. a) Watts-strogatz visualizations

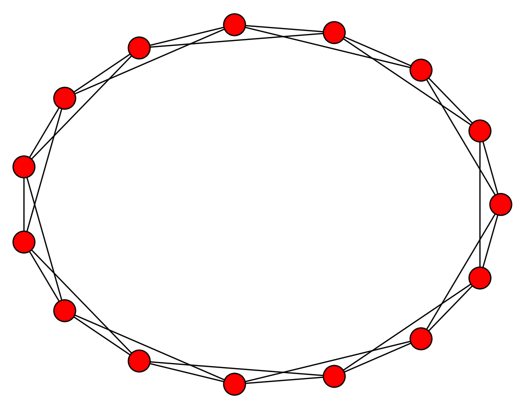


Figure 2.1: Watts-strogatz using $N = 15$, $m = 2$ and $p = 0$

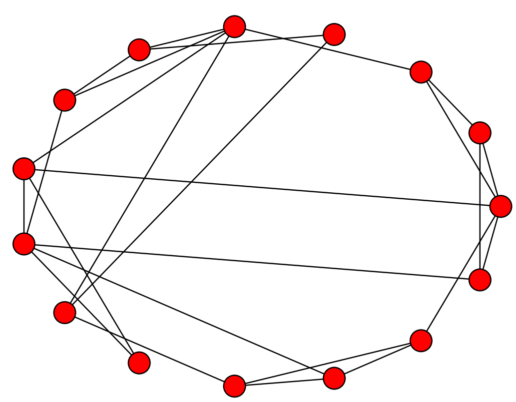


Figure 2.2: Watts-strogatz using $N = 15$, $m = 2$ and $p = 0.5$

2. b) Relative averages

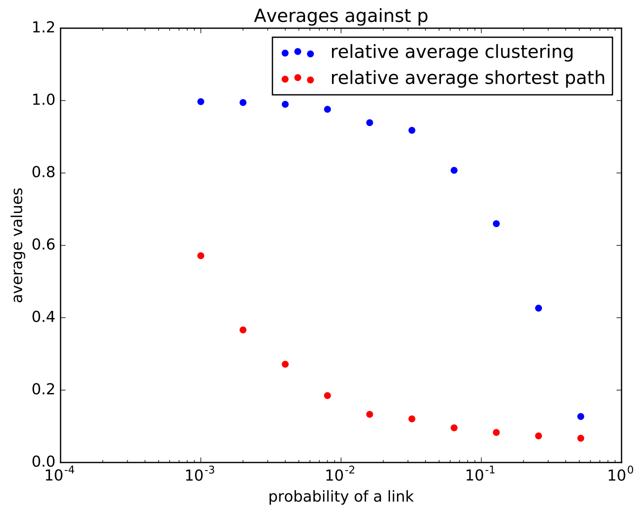


Figure 2.3: Relative averages in a Watts-strogatz network using $N = 1000$, $m = 5$ and $p \in [0.001, 0.512]$

3 Implementing the Barabási-Albert (BA) model

3. a) Implement a Python function for generating Barabási-Albert networks

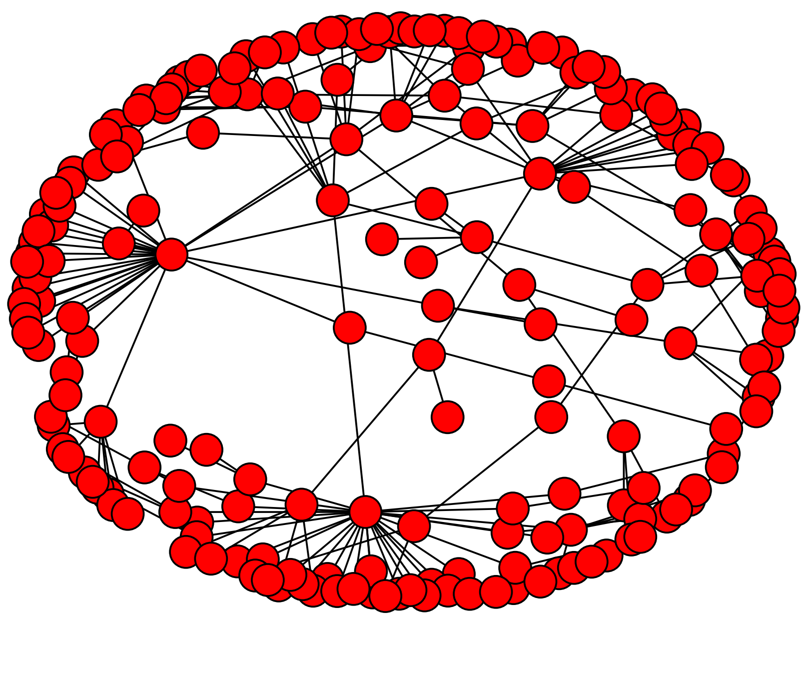


Figure 3.1: Barabási-Albert $N = 200$ and $m = 1$

3. b) Plot both the experimental and theoretical distributions to the same axes

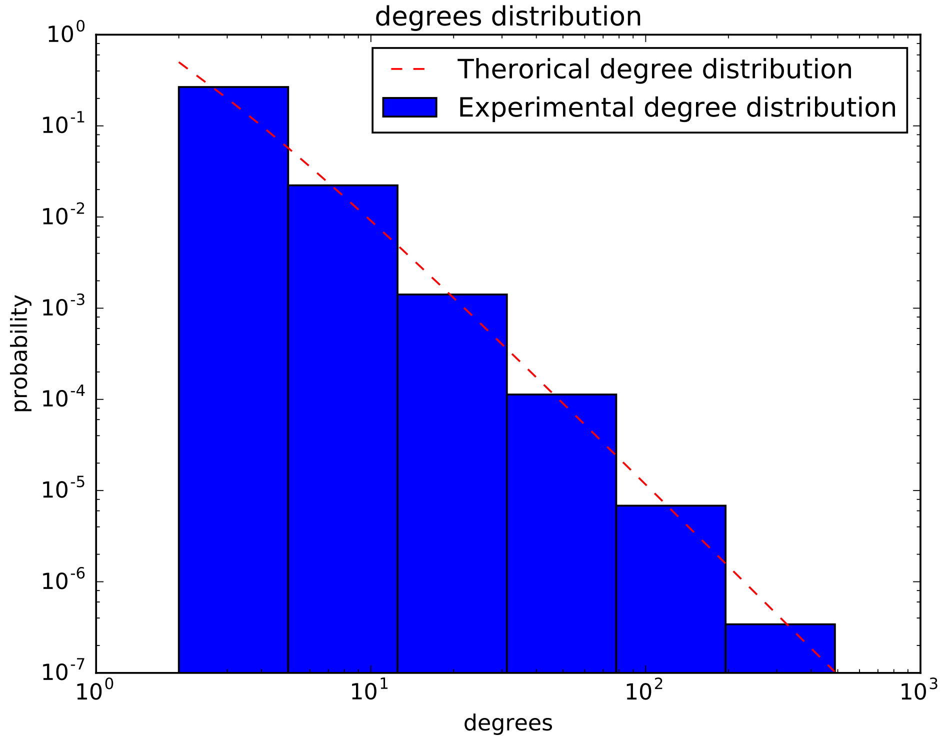
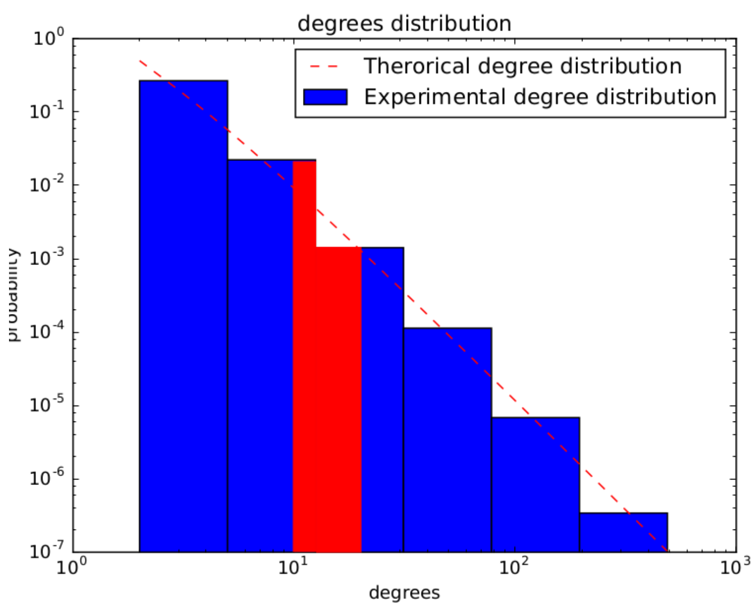


Figure 3.2: Degree distribution of the Barabási-Albert model for $N = 10\,000$

3. c) By reading from the plot of the experimental degree distribution, estimate the probability for a randomly picked node to have a degree value between 10 and 20



By reading the plot, the probability for a random node to have a degree between 10 and 20 is the area of the red surface on the picture. Hence,

$$\begin{aligned} p\left(d_i \in [10;20]\right) &= 3 \times 2.10^{-2} + 7 \times 10^{-3} \\ &= 0.067 \end{aligned}$$

4 Deriving the degree distribution for the BA-model

4. a)

$$\Pi_i = \frac{k_i}{\sum_{j=1}^{N} k_j}$$

We can see that

$$\sum_{j=1}^{N} k_j = 2mN$$

If we consider that $N_0 \approx 0$, and because every new vertex added has degree m and every other vertex linked to the new node has its degree increased by one.
Then, we have:

$$\Pi\left(k\right) = N p_{k,N} \times \Pi_i = \frac{N \times k_i \times p_{k,N}}{2mN} = \frac{k_i p_{k,N}}{2m}$$

4. b)

The number of degree k nodes that acquire a new link and turn into $(k+1)$ degree nodes is:

$$n_k^- = \frac{k}{2} p_{k,N}$$

The number of degree $(k-1)$ nodes that acquire a new link, increasing their degree to k is:

$$n_k^+ = \frac{k-1}{2} p_{k-1,N}$$

Thus, for all $k > m$, we get:

$$\begin{aligned} \left(N+1\right) p_{k,N+1} - N p_{k,N} &= n_k^+ - n_k^- \\ &= \frac{k-1}{2} p_{k-1,N} - \frac{k}{2} p_{k,N} \end{aligned}$$

and for $k = m$, we now have:

$$\begin{aligned} \left(N+1\right) p_{m,N+1} - N p_{m,N} &= n_m^+ - n_m^- \\ &= 1 - \frac{m}{2} p_{m,N} \end{aligned}$$

as $n_m^+ = 1$. (Only one node has degree m, the one that is added to the network).

4. c)

We can now let the network grows towards the infinite network size limit. Then let's consider stationary solutions of the two equations:

$$\left(N+1\right) p_{k,N+1} - N p_{k,N} \rightarrow N p_k + p_k - N p_k = p_k$$

$$\left(N+1\right) p_{m,N+1} - N p_{m,N} \rightarrow p_m$$

Thus,

$$p_k = \frac{k-1}{k+2} p_{k-1} \quad k > m$$

$$p_m = \frac{2}{m+2}$$

4. d)

We have:

$$\begin{aligned}
p_{m+1} &= \frac{m}{m+3}p_m = \frac{2m}{(m+2)\,(m+3)} \\
p_{m+2} &= \frac{m}{m+4}p_{m+1} = \frac{m}{m+4}\,\frac{2m}{(m+2)\,(m+3)} = \frac{2m\,(m+1)}{(m+2)\,(m+3)\,(m+4)} \\
p_{m+3} &= \frac{m+2}{m+5}p_{m+2} = \frac{\cancel{m+2}}{m+5}\,\frac{2m\,(m+1)}{\cancel{(m+2)}\,(m+3)\,(m+4)} = \frac{2m\,(m+1)}{(m+3)\,(m+4)\,(m+5)}
\end{aligned}$$

We can see that there is a recursive pattern that will happen at this point. We can replace the denomerator $m+3$ with k. This gives us the equation we were looking for:

$$p_k = \frac{2m\,(m+1)}{k\,(k+1)\,(k+2)}$$