SOLUTIONS MANUAL FOR

Instructor's Guide for Exploring Geometry, Second Edition

by

Michael Hvidsten
Gustavus Adolphus College

CRC Press
Taylor & Francis Group

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CHAPTER 1

Geometry and the Axiomatic Method

The development of the axiomatic method of reasoning was one of the most profound events in the history of mathematics, and yet many math majors can finish four years of college-level mathematics without ever really delving into the subject. In this chapter we strive to rectify this situation by exploring axiomatic systems and their properties.

One strand running through the chapter is the search for the "ideal". The golden ratio is the ideal in concrete form, realized through natural and man-made constructions. Deductive reasoning from a base set of axioms is the ideal in abstract form, realized in the crafting of clear, concise, and functional definitions, and in the reasoning employed in well-constructed proofs.

Another strand in the chapter, and which runs through the entire text, is that of the interplay between the concrete and the abstract. Students are encouraged to "play" with concrete lab projects, such as the Golden Ratio, but should also be encouraged to experiment when doing proofs and more abstract thinking. The experimentation in the latter is of the mind, but it can utilize many of the same principles of exploration as students naturally use in the lab. When trying to come up with a proof they should consider lots of examples and they should ask "What if ...?" questions. Most importantly, they should *interact* with the ideas, just as they interact with the software environment.

Student interaction with ideas and discovery of concepts was a primary organizing principle for the text. Interaction is encouraged in three ways. First, topics are introduced and developed in the text. Next, lab projects reinforce concepts, or introduce related ideas. Lastly, project results are discussed, and conclusions drawn, in written lab reports. Students read about concepts and hear them discussed in class; they conduct "experiments" to make the ideas concrete and then conceptualize the ideas by re-telling them in project reports.

Thus, the work done in the lab and in group projects is a critical component of the course. The projects designed to be done in groups have an additional pedagogical advantage in that students speak to each other using mathematical terms and concepts, and by doing so, better internalize such concepts and make them less abstract.

NOTES ON LAB PROJECTS

The lab on the golden ratio should take most students less than one class period to finish. The main difficulty with this first lab will be in learning the functionality of the dynamic software program used to carry out the lab exploration.

Students will have many questions on the format of lab reports. In appendix A of this guide, there is a sample lab report for a "fake" lab on the Pythagorean Theorem. This could be handed out as an example before students complete the first lab project.

SOLUTIONS TO EXERCISES IN CHAPTER 1

1.3 Project 1 - The Ratio Made of Gold

1.3.1 Since $AB = 2$, we have that $\frac{2}{x} = \frac{1+\sqrt{5}}{2}$. Solve for x and clear the denominator of radicals.

1.3.2 Since we are assuming $EB = 2$, then by right triangle BEG we can show that $BG = \frac{1+\sqrt{5}}{2}$. Since $BG = GH$, and $GE = 1$, the result follows.

1.3.3 This exercise can be done right in the lab. The only thing to be careful about is that students do not get carried away with this idea and spend the whole class period on it!

1.3.3 This would be a good exercise to discuss in class the day after the lab. A good discussion question is "Why is the golden ratio so ubiquitous?"

1.4 The Rise of the Axiomatic Method

In this section we focus on *reasoning* in mathematics. The problems in this section may seem quite distant from geometry to the students, but the goal is to have students reason from the definitions and properties that an axiomatic system posits and then argue using just those basic ideas and relationships. This is good mental training. It is all too easy to argue from diagrams when trying to justify geometric statements.

1.4.2 There are two cases to consider. In case I, one pile has $n > 0$ coins and the other has 0. The first player then just picks all n coins and wins. In case II, both piles are of non-zero height. The first player picks enough coins from the larger pile to make the piles even. The other player, on his turn, has to make the piles uneven again. The first player then just continues the strategy of always making the piles even. Eventually the other player must create the situation in Case I, and the first player wins.

1.4.3 Let a set of two different flavors be called a pairing. Suppose there were m children and $n > m$ pairings. By Axiom 2 every pairing is associated to a unique child. Thus, for some two pairings, P_1,P_2 there is a child C associated to both. But this contradicts Axiom 3. Likewise, if $m > n$, then by Axiom 3 some two children would have the same pairing. This contradicts Axiom 2. So, $m = n$ and, since the number of pairings is $4 + 3 + 2 + 1 = 10$, there are 10 children.

1.4.4 Suppose C_1 and C_2 both liked flavors F_1 and F_2. This would violate Axiom 2.

1.4.5 There are exactly four pairings possible of a given flavor with the others. By Axiom 2, there is exactly one child assoicated to each of these four pairings.

1.4.6 If $xz = yz$ then by Axiom 4 we can find z^{-1} such that $(xz)z^{-1} = (yz)z^{-1}$. By associativity we can re-arrange to get $x(zz^{-1}) = y(zz^{-1})$, or $xe = ye$. By Axiom 3 we get $x = y$.

1.4.7 If $xyz = e$, then by Axiom 4, $x^{-1}xyz = x^{-1}$. By Axiom 4 and Axiom 3, $x^{-1}xyz = eyz = yz$ and thus $yz = x^{-1}$. Then, $yzx = x^{-1}x = e$

1.4.8 Suppose that e and f are two identities. Then, for all x in the group we have $x = xe = ex$ and $x = xf = fx$. So, $ex = fx$ and by exercise 1.4.6 we get $e = f$.

1.4.9 First we show that $1 \in M$. By Axiom 4 we know 1 is not the successor of any natural number. In particular, it cannot be a successor of itself. Thus, $1' \neq 1$ and $1 \in M$. Now, suppose $x \in M$. That is, $x' \neq x$. By Axiom 3 we have that $(x')' \neq x'$, and so $x' \in M$. Both conditions of Axiom 6 are satisfied and thus $M = N$.

1.4.10 1 is not the successor of any number by Axiom 4. Let $M = \{1\} \cup \{x | x \neq 1 \textit{ and } \exists u \textit{ with } u' = x\}$. First, $1 \in M$ is obvious. Suppose

$x \in M$. Then, there is u with $u' = x$. So, $(u')' = x'$ and by definition $x' \in M$.

1.4.11 Given x, let $M = \{y|x + y\ is\ defined\}$. Then, by definition $1 \in M$. Suppose $y \in M$. Then, $x + y' = (x + y)'$ is defined and $y' \in M$. So, $M = N$ by Axiom 6. Now, since x was chosen arbitrarily, addition is defined for all x and y.

1.4.12 Case I: $x1 = x$. Case II: $xy' = x + xy$.

1.4.13 This is a good discussion question. Have the students write short essays and then conduct an in-class discussion on the role of abstraction versus application in mathematics. The future high-school teachers should be encouraged to think about how abstraction and application cross-fertilize one another. This viewpoint is one which they should strive to convey to their own students.

1.4.14 This question naturally follows from the previous exercise and again would form the basis of a good discussion vehicle for class. There is virtually no concern for proof and abstraction in current high school mathematics. Have the students really grapple with this. Is this desirable?

1.5 Properties of Axiomatic Systems

This is a "meta" section. We are studying properties of axiomatic systems themselves, considering such systems as mathematical objects in comparison to other systems. This will be foreign territory to almost everyone in the class, but is incredibly fascinating from a philosophical viewpoint. Students so often think of mathematics as an ancient subject, but in this section we can bring in the amazing results of the twentieth century mathematician Kurt Godel. Possible side trips include discussions of information theory and computability in computer science. A good reference here is Gregory Chaitin's book *The Limits of Mathematics* (Springer, 1998.) Additionally, much more could be investigated as to the various philosophies of mathematics, in particular the debates between platonists and constructionists, or between intuitionists and formalists. A good reference here is Edna E. Kramer's *The Nature and Growth of Modern Mathematics* (Princeton, 1981), in particular Chapter 29 on Logic and Foundations.

1.5.1 Let S be the set of all sets which are not elements of themselves. Let P be the proposition that "S is an element of itself". And consider the two propositions P and the negation of P, which we denote as $\neg P$. Assume P is true. Then, S is an element of itself. So, S

is a set which by definition is not an element of itself. So, $\neg P$ is true. Likewise, if $\neg P$ is true then P is true. In any event we get P and $\neg P$ both true, and the system cannot be consistent.

1.5.2 This is another good classroom discussion starter. In particular, if students are assigned a final project, one possible project idea is to elaborate on these philosophical viewpoints.

1.5.3 This is another good final project idea.

1.5.4 For every pair of points we have exactly one line. Thus, there are $3 + 2 + 1 = 6$ lines. The argument for this is exactly the same as we gave in the ice cream flavor example in the previous section. This is a good place to point out that the objects in question are not what is critical – what is important is how objects are related to one another.

1.5.5 Let P be a point. Each pairing of a point with P is associated to a unique line. There are exactly three such pairings.

1.5.6 Here is a simple model. Lines are segments and points are Euclidean points.

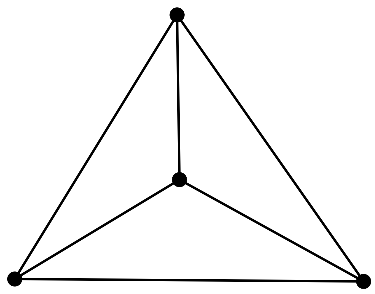


FIGURE 1.1:

1.5.7 Yes. The lines and points satisfy all of the axioms.

1.5.8 Since $(2, 1), (3, 2), (3, 1)$ are not in P then the first axiom is satisfied. Also, the only case where the second axiom holds is for $(1, 2)$ and $(2, 3)$. Since $(1, 3)$ is in P, then the second axiom holds, and this is a model.

1.5.9 If (x, y) is in P, then $x < y$. Clearly, $y < x$ is impossible and the first axiom is satisfied. Also, inequality is transitive on numbers so the second axiom holds and this is a model.

1.5.10 Add the following axiom to the first model: There are exactly 4 elements in S. The example in exercise 1.5.8 is still a model and the new axiom cannot be proved from the original two, and is thus independent. However, this new axiom clearly fails in the example of exercise 1.5.9.

1.5.11 A quick listing of all points and lines and incidence relations shows that Axioms 1 and 2 are satisfied. For Axiom 3, points A, B, C, and D have the property that no subset of three of the points are collinear. For Axiom 4, line $\overleftrightarrow{AB}$ suffices.

1.5.12 By Axiom 3, there are at least 4 points, call them A, B, C, and D, no three of which are collinear. By Axiom 1, there are four lines l_1, l_2, l_3, l_4 incident on pairs of points (A, B),(B, C),(C, D), and (A, D). Suppose one subset of three of these lines were concurrent, say l_1, l_2, and l_3. Since l_1 and l_2 meet at B and l_2 and l_3 meet at C, Then, $B = C$ and the three points A,B, and C are collinear, which is impossible.

1.5.13 The dual to Axiom 1 is "Given two distinct lines, there is exactly one point incident with them both."

Proof: Suppose there were two points A and B incident on both lines. This would contradict Axiom A1.

1.5.14 The dual to Axiom 2 is "Given two distinct points, there is at least one line that is incident with both points."

Proof: By Axiom 1 there is exactly one line incident with both pints, so there is also at least one.

1.5.15 By Axiom 4 there is a line l with $n + 1$ points, say $P_1, \ldots,$ P_{n+1}. By Axiom 3, there must be a point Q that is not on l. Let l_1 be the line incident on P_1 and Q, l_2 be the line incident on P_2 and Q, etc. The $n + 1$ lines $l_1, \ldots, l_{n+1}$ through Q satisfy the dual statement of Axiom 4.

1.6 Euclid's Axiomatic Geometry

In this section we take a careful look at Euclid's original axiomatic system. We observe some of its inadequacies in light of our modern "meta" understanding of such systems, and discuss the one axiom that has been the creative source of much of modern geometry – the Parallel Postulate.

1.6.2 Create a segment $\overline{AB}$ of length b and extend it by an additional segment $\overline{BC}$ of length c. At A construct a perpendicular to $\overleftrightarrow{AB}$ and mark off a segment $\overline{AD}$ of length a. Then, the rectangle on a and $b + c$ is equivalent to the rectangle on a and b together with the rectangle on a and c.

1.6.3 An explanation should be given along with a figure like the following:

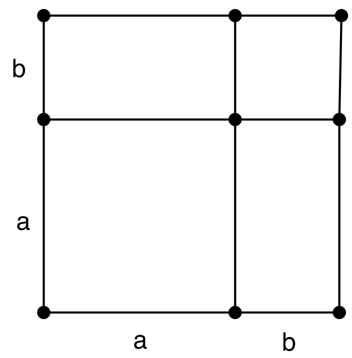


FIGURE 1.2:

1.6.5

$$\begin{aligned} 123 &= 3 \cdot 36 + 15 \\ 36 &= 2 \cdot 15 + 6 \\ 15 &= 2 \cdot 6 + 3 \\ 6 &= 2 \cdot 3 + 0 \end{aligned}$$

Thus, $gcd(123, 36) = 3$.

1.6.6 Let r_k be the last non-zero remainder in the process. That is, $r_{k-1} = q_{k+1}r_k + 0$ and $r_{k-2} = q_k r_{k-1} + r_k$. Then, r_k divides r_{k-1} and also divides r_{k-2}. Since $r_{k-3} = q_{k-1}r_{k-2} + r_{k-1}$, then r_k divides r_{k-3}. Continuing in this way (following the set of equations back up from the last one) we get that r_k divides all remainders and also divides a and b. Suppose d was another divisor of a and b. Then, following the equations down from the first, we get that d divides r_1, r_2, etc, and thus divides r_k. So, $d < r_k$ and r_k is the greatest common divisor.

1.6.7 This exercise is a good starting off point for discussing the importance of definitions in mathematics. Students could propose their definitions of a circle, for example, and then the pluses and minuses of the various definitions could be debated in class. One possible definition for a circle is:

Definition 1.1. A circle with center O and radius length r is the set of points P on the sphere such that the distance along the great circle from O to P is r.

Note that this definition is itself not entirely well-defined, as we have not specified what we mean by distance. Here, again, is a good opportunity to wrestle with the "best" definition of distance. For circles of any radius to exist, distance must be defined so that it grows without bound. Thus, one workable definition is for distance to be net *cumulative* arc length along a great circle as we move from a point O to a point P.

An angle ABC can be most easily defined as the Euclidean angle made by the tangent lines at B to the circles defining $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$.

Then, Postulate 1 is satisfied *most* of the time, as we can construct a unique great circle passing through two points on the sphere, *if* the points are not antipodal. We simply intersect the sphere with the plane through the two points and the center of the sphere.

Postulate 2 is satisfied as we can always extend an arc of a great circle, although we may retrace the existing arc.

Postulate 3 is satisfied if we use the cumulative distance definition as discussed above.

Postulate 4 is automatically satisfied as angles are Euclidean angles.

Postulate 5 is not satisfied, as *every* pair of lines intersects. An easy proof of this is to observe that every line is uniquely defined by a plane through the origin. Two different planes will intersect in a line, and this line must intersect the sphere at two points.

1.6.8 This is false. An easy counterexample is the triangle that is defined in the first octant. It has three right angles.

1.6.9 This is true. Given a plane through the origin, we can always find an orthogonal plane. The angle these planes make will equal the angle of the curves they define on the sphere, as the spherical angles are defined by tangent lines to the sphere, and thus lie in the planes.

1.6.10 This is false. An easy counterexample is constructed by starting with the triangle ABC that is defined in the first octant, as shown. It has three right angles. Let D be the midpoint of $\overline{AB}$. Then, $ACBD$ is a four-sided figure with three right angles. If they complain that this figure is not really four-sided, just move A and B a bit towards C, creating A' and B', and let D' be the intersections of the perpendiculars to $\overline{AC}$ and $\overline{BC}$ at A' and B'. By continuity, the angle at D' must be greater than a right angle.

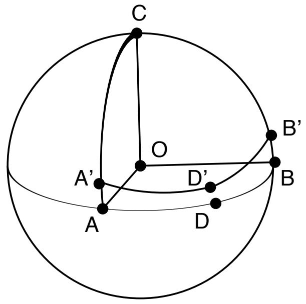


FIGURE 1.3:

1.6.11 Yes, the example given in the answer to exercise 1.6.8.

1.7 Project 2 - A Concrete Axiomatic System

After the last few sections dealing with abstract axiomatic systems, this lab is designed so that students can explore another geometric system through concrete manipulation of the points, lines, etc of that system. The idea here is to have students explore the environment first, then have them make some conjectures about what is similar and what is different in this system as compared to standard Euclidean geometry.

1.7.1 Students should report the results of their experiments here. They will not have the tools to prove these results, but should provide evidence that they have fully explored each idea.

For example, they should report that they tried to construct a rectangle, but were unsuccessful in doing so. In fact, they may discover that if they they construct a four-sided figure with three right angles, the fourth angle is always acute.

The sum of the angles in a triangle will be less than 180 degrees.

Euclid’s construction of an equilateral triangle is valid in hyperbolic geometry. Again, they should provide experimental evidence for this.

Finally, the perpendicular to a line through a point not on the line is a valid construction. Here, it is enough for students to convince themselves that the built-in perpendicular construction creates a new line that *always* stays perpendicular to the original line.

CHAPTER 2

Euclidean Geometry

In this chapter we start off with a very brief review of basic properties of angles, lines, and parallels. When presenting such material, one has to make a choice. One can present the basic results of plane geometry from first principles, starting with an axiomatic system, such as Hilbert's Axioms, carefully laying out such concepts as betweenness, incidence, congruence, and continuity. This approach has several virtues. Students see, perhaps for the first time in their mathematical careers, a logical system built entirely from first principles. They also can clearly determine what theorems, definitions, and axioms are fair game to use in their own proofs of results. On the other hand, a thorough and *complete* development of Hilbert's axioms would necessarily take a substantial portion of a semester-long course in geometry, leaving little time for other, equally important topics such as non-euclidean geometry and transformational geometry.

A second approach is to review, in summary form, some of the most important logical problems of classical Euclidean geometry that axiom writers such as Hilbert attempted to fix, and then to move on to more substantial results in plane geometry. This is the approach taken in Chapter 2. It has the advantage of exposing students to the logical issues facing mathematicians over the last several hundred years and, at the same time, covering significant geometric ideas such as the definition of area, cevians, and circle inversion. One disadvantage of this approach is that students may feel unsure of what they can assume and not assume when working on proofs. In each section of Chapter 2 the author tried to carefully describe what results and assumptions were made in that section. For example, in section 2.1, students are instructed to use the notion of *betweenness* in the way one's tuition would dictate, while at the same time pointing out that this is one of those geometric properties that needs an axiomatic base.

If the more rigorous approach to Euclidean Geometry is desired, the complete foundational development can be found in on-line chapters at the author's website: http://www.gac.edu/˜hvidsten/geom-text.

SOLUTIONS TO EXERCISES IN CHAPTER 2

2.1 Angles, Lines, and Parallels

This section is perhaps the least satisfying section in the chapter for students, since many theorems are referenced without proof. It may be helpful to remind students that these results were no doubt covered in great detail in their high school geometry course, and that a full development of such results would entail a "filling in" of many days (weeks/months) of foundational work based on Hilbert's axioms.

A significant number of the exercises deal with parallel lines. This is for two reasons. First of all, historically there was a great effort to prove Euclid's fifth Postulate by converting it into a logically equivalent statement that was hoped to be easier to prove. Thus, many of the exercises nicely echo this history. Secondly, parallels and the parallel postulate are at the heart of one of the greatest revolutions in math—the discovery of non-Euclidean geometry. This section foreshadows that development, which is covered in Chapters 7 and 8.

2.1.1 It has already been shown that $\angle FBG \cong \angle DAB$. Also, by the vertical angle theorem (Theorem 2.3) we have $\angle FBG \cong \angle EBA$ and thus, $\angle DAB \cong \angle EBA$.

Now, $\angle DAB$ and $\angle CAB$ are supplementary, thus add to two right angles. Also, $\angle CAB$ and $\angle ABF$ are congruent by the first part of this exercise, as these angles are alternate interior angles. Thus, $\angle DAB$ and $\angle ABF$ add to two right angles.

2.1.2 Let ΔABC be a triangle, and consider the sum of the angles at A and B. Extend the angle at A to create an exterior angle. Then, the sum of this exterior angle and the angle at A is 180 degrees, as they make up a line. However, by the Exterior Angle Theorem we know that the exterior angle is greater than the angle at B. Thus, the sum of the angles at A and B is less than the sum of the angle at A and its exterior angle, which is 180.

2.1.3 a. False, right angles are defined solely in terms of congruent angles.

b. False, an angle is defined as *just* the two rays plus the vertex.

c. True. This is part of the definition.

d. False. The term "line" is undefined.

2.1.4 a. A point M is the midpoint of segment $\overline{AB}$ if M is between A and B and $\overline{AM} \cong \overline{MB}$.

b. The perpendicular bisector of segment $\overline{AB}$ is a line through the midpoint M of $\overline{AB}$ that is perpendicular to $\overleftrightarrow{AB}$.

c. The triangle defined by three non-collinear points A, B, C is the union of the line segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$.

d. An equilateral triangle is a triangle whose sides are congruent.

2.1.5 Proposition I-23 states that angles can be copied. Let A and B be points on l and n respectively and let m be the line through A and B. If $t = m$ we are done. Otherwise, let D be a point on t that is on the same side of n as l. (Assuming the standard properties of betweenness) Then, $\angle BAD$ is smaller than the angle at A formed by m and n. By Theorem 2.9 we know that the interior angles at B and A sum to two right angles, so $\angle CBA$ and $\angle BAD$ sum to less than two right angles. By Euclid's fifth postulate t and l must meet.

2.1.6 Given the assumptions stated in the exercise, if we copy $\angle CBA$ to A, creating line n, then by Theorem 2.8 n and l will be parallel. Also, the sum of the interior angles at B and A for lines l and n will sum to two right angles. Thus, lines t and n cannot be coincident. Thus, by Playfair line t cannot be parallel to l.

To show that t and l intersect on the same side of m as D and C, we assume that they intersect on the other side, at some point E. Let F be a point on n that is on the other side of m from C, and let G a point on t on this same side. Then, $\angle BAF$ is less than $\angle BAG$ in measure, and since $\angle BAF \cong \angle CBA$ by Theorem 2.9, we have that the exterior angle $\angle CBA$ to ΔBAE is smaller than an opposite interior angle ($\angle BAG$), which contradicts the Exterior Angle Theorem.

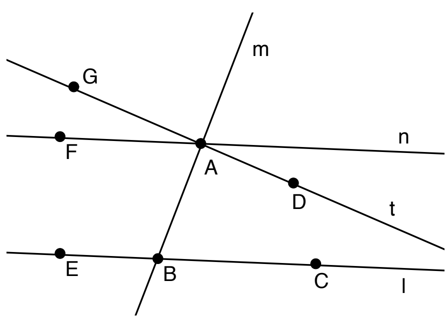


FIGURE 2.1:

2.1.7 First, assume Playfair's Postulate, and let lines l and m be parallel, with line t perpendicular to l at point A. If t does not intersect m then, t and l are both parallel to m, which contradicts Playfair. Thus, t intersects m and by Theorem 2.9 t is perpendicular at this intersection.

Now, assume that whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. Let l be a line and P a point not on l. Suppose that m and n are both parallel to l at P. Let t be a perpendicular from P to l. Then, t is perpendicular to m and n at P. By Theorem 2.4 it must be that m and n are coincident.

2.1.8 To create a parallel to $\overleftrightarrow{BC}$ at A we could just copy $\angle CBA$ to A and use Theorem 2.8. The three angles defined by the triangle at A sum to two right angles, and by Theorem 2.9 we have that the sum of these angles equals the sum of the angles in the triangle.

2.1.9 Assume Playfair and let lines m and n be parallel to line l. If $m \neq n$ and m and n intersect at P, then we would have two different lines parallel to l through P, contradicting Playfair. Thus, either m and n are parallel, or are the same line.

Conversely, assume that two lines parallel to the same line are equal or themselves parallel. Let l be a line and suppose m and n are parallel to l at a point P not on l. Then, n and m must be equal, as they intersect at P.

2.1.10 Assume Playfair and let line t intersect one of the parallel lines m and n, say it intersects m at P. If m did not intersect n, then t and m would be two different lines both parallel to n at P, which contradicts Playfair.

Conversely, assume that if a line intersects one of two parallel lines, it must intersect the other. Let l be a line and suppose m and n are parallel to l at a point P not on l, with $m \neq n$. Then, m intersects n, which is parallel to l. By assumption, m must intersect l, and thus cannot be parallel to l.

2.2 Congruent Triangles and Pasch's Axiom

This section introduces many results concerning triangles and also discusses several axiomatic issues that arose from Euclid's treatment of triangles.

This may be a good point to review Euclid's "proof" of SAS congruence. An interesting discussion point would be to have students voice their opinion as to whether the proof was valid or not.

Also, the history of axiom systems would be a good supplemental activity at this point. Hilbert's axioms did not arise overnight. He took the best of those who came before him, including Pasch, and molded these separate strands into a complete system.

2.2.1 Yes, it could pass through points A and B of ΔABC. It does not contradict Pasch's axiom, as the axiom stipulates that the line cannot pass through A, B, or C.

2.2.2 Construct the diagonal $\overline{AC}$ of the rectangle $ABCD$. Then, a line passing through a side of the rectangle will be a line passing through a side of one of the two triangles defined by the diagonal and the original sides of the rectangle. By Pasch's axiom, this line will either pass through one of the other sides of the triangle, which include the rectangle sides and the diagonal. If it passes through a side of the rectangle, we are done. If it passes through the diagonal, then using Pasch's axiom a second time, we get that it must pass through one of the other two sides of the other triangle, and thus through a side of the rectangle.

The same argument can be used repeatedly to show that a line passing through a side (but not a vertex) of an arbitrary n-gon (and not just a regular n-gon) will intersect a side. Just pick a vertex and construct interior triangles by taking all diagonals from this vertex.

2.2.3 No. Here is a counter-example.

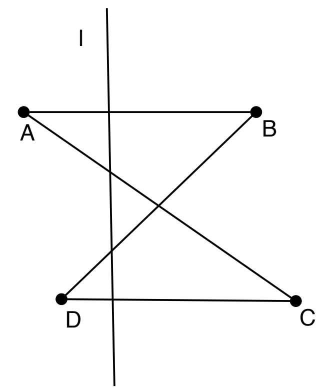


FIGURE 2.2:

2.2.4 If $A = B$, or $B = C$, or $A = C$ the result follows immediately. Otherwise, we can assume all three points are distinct.

If they all lie on a line, then, one is between the other two. In every case, we get that $\overline{AC}$ cannot intersect l.

Assume the points are non-collinear. and that A and C are on opposite sides of l. Then, l intersects $\overline{AC}$ and does not pass through A or C. By Pasch's axiom, it must intersect $\overline{AB}$ or $\overline{BC}$ at a point other than A, B, or C. Say it intersects $\overline{AB}$. This contradicts the assumption that A and B are on the same side. Likewise, if it intersects $\overline{BC}$ we get a contradiction. Thus, A and C are on the same side of l.

2.2.5 If $A = C$ we are done. If A, B, and C are collinear, then B cannot be between A and C, for then we would have two points of intersection for two lines. If A is between B and C, then l cannot intersect $\overline{AC}$. Likewise, C cannot be between A and B.

If the points are not collinear, suppose A and C are on opposite sides. Then l would intersect all three sides of ΔABC, contradicting Pasch's axiom.

2.2.6 A point P is in the interior of a ΔABC if P is in the interior of $\angle ABC$ and in the interior of $\angle BCA$ and in the interior of $\angle CAB$.

2.2.7 Let $\angle ABC \cong \angle ACB$ in ΔABC. Let $\overrightarrow{AD}$ be the angle bisector of $\angle BAC$ meeting side $\overline{BC}$ at D. Then, by AAS, ΔDBA and ΔDCA are congruent and $\overline{AB} \cong \overline{AC}$.

2.2.8 Referring to Fig. 2.1, we can use the SSS triangle congruence theorem on ΔADE and ΔABE to show that $\angle EAB \cong \angle BAE$.

2.2.9 Suppose that two sides of a triangle are not congruent. Then, the angles opposite those sides cannot be congruent, as if they were, then by the previous exercise, the triangle would be isosceles.

Suppose in ΔABC that $\overline{AC}$ is greater than $\overline{AB}$. On $\overline{AC}$ we can find a point D between A and C such that $\overline{AD} \cong \overline{AB}$. Then, $\angle ADB$ is an exterior angle to ΔBDC and is thus greater than $\angle DCB$. But, ΔABD is isosceles and so $\angle ADB \cong \angle ABD$, and $\angle ABD$ is greater than $\angle DCB = \angle ACB$.

2.2.10 In the figure accompanying Theorem 2.11, suppose that $\overline{BC}$ was greater than $\overline{YZ}$. Then, we could find a point D between B and C such that $\overline{BD} \cong \overline{YZ}$, and by SAS ΔABD would be congruent to ΔXYZ. This implies that $\angle BAD \cong \angle YXZ$. But, we are given that $\angle BAC \cong \angle YXZ$, and so $\angle BAD \cong \angle BAC$. This implies that D lies on $\overleftrightarrow{AC}$, and that A, B, C are collinear, which is impossible.

2.2.11 Let ΔABC and ΔXYZ be two right triangles with right angles at A and X, and suppose $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$. Suppose $\overline{AB}$ is greater than $\overline{XY}$. Then, we can find a point D between A and B such that $\overline{AD} \cong \overline{XY}$. By SAS $\Delta ADC \cong \Delta XYZ$. Now, $\angle BDC$ is exterior to ΔADC and thus must be greater than 90 degrees. But, ΔCDB is isosceles, and thus $\angle DBC$ must also be greater than 90 degrees. This is impossible, as then ΔCDB would have angle sum greater than 180 degrees.

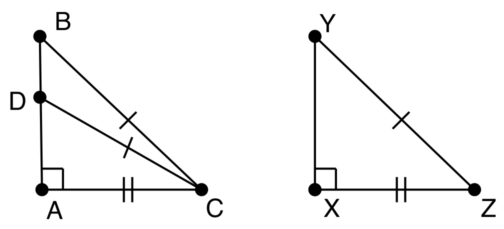


FIGURE 2.3:

2.2.12 Show that ΔADB and ΔBCA are congruent, and then show that ΔADC and ΔBDC are congruent.

2.2.13 We use AAS to show that $\Delta BFH \cong \Delta AFG$ and $\Delta CEI \cong \Delta AEG$. Thus $\overline{BH} \cong \overline{AG} \cong \overline{CI}$ and $BHIC$ is Saccheri. Also, by adding congruent angles in the left case we get that the sum of the angles in the triangle is the same as the sum of the summit angles. In the right case, we need to re-arrange congruent angles.

2.2.14 Assume Playfair and let $ABCD$ be a Saccheri Quadrilateral with base $\overline{AB}$. By Theorem 2.8 we know that $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$. By Theorem 2.9 the summit angles must add to 180 degrees. This, each angle must be 90 degrees.

Conversely, assume that the summit angles of a Saccheri quadrilateral are always right angles. Let ΔABC be a triangle. By the previous exercise, we know that we can construct a Saccheri quadrilateral based on the triangle whose summit angles add to the angle sum of the triangle. Thus, the sum of the angles in a triangle is always 180 degrees. By Exercise 2.1.8, this implies Playfair's axiom is true.

2.2.15 Given quadrilaterals $ABCD$ and $WXYZ$ we say the two quadrilaterals are congruent if there is some way to match vertices so that corresponding sides are congruent and corresponding angles are congruent.

SASAS Theorem: If $\overline{AB} \cong \overline{WX}$, $\angle ABC \cong \angle WXY$, $\overline{BC} \cong \overline{XY}$, $\angle BCD \cong \angle XYZ$, and $\overline{CD} \cong \overline{YZ}$, then quadrilateral $ABCD$ is congruent to quadrilateral $WXYZ$.

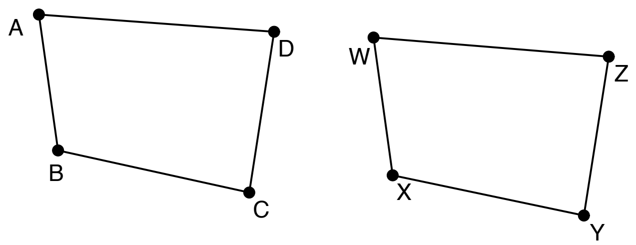


FIGURE 2.4:

Proof: ΔABC and ΔWXY are congruent by SAS. This implies that ΔACD and ΔWYZ are congruent. This shows that sides are correspondingly congruent, and two sets of angles are congruent ($\angle ABC \cong \angle WXY$ and $\angle CDA \cong \angle YZW$). Since $\angle BAC \cong \angle XWY$ and $\angle CAD \cong \angle YWZ$, then by angle addition $\angle BAD \cong \angle XWZ$. Similarly, $\angle BCD \cong \angle XYZ$. $\square$

2.3 Project 3 - Special Points of a Triangle

Students should be encouraged to explore and experiment in this lab project. Ask them if there are any other sets of intersecting lines that one could construct. Or, are there interesting properties of constructed intersecting lines in other polygons?

Some of the students in the class will be future secondary math teachers. This project is one that could be easily transferred to the high school setting. Students could have as an extra credit exercise the task of preparing a similar project for a high school class.

2.3.1 ΔDGB and ΔDGA are congruent by SAS, as are ΔEGB and ΔEGC. Thus, $\overline{AG} \cong \overline{BG} \cong \overline{CG}$. By SSS $\Delta AFG \cong \Delta CFG$ and since the angles at F must add to 180 degrees, the angles at F must be congruent right angles.

2.3.2 We proved in the previous exercise that, if G is the circumcenter, then $\overline{AG} \cong \overline{BG} \cong \overline{CG}$. Thus, the circle centered at G with radius $\overline{AG}$ must pass through the other two vertices.

2.3.3 The angle pairs in question are all pairs of an exterior angle and an interior angle on the same side for a line falling on two parallel lines. These are congruent by Theorem 2.9.

Since $\angle DAB$, $\angle BAC$, and $\angle CAE$ sum to 180 degrees, and $\angle BDA$, $\angle BAD$, and $\angle ABD$ sum to 180 then, using the congruences shown in the diagram, we get that $\angle DBA \cong \angle BAC$. Likewise, $\angle BAD \cong \angle ABC$. By ASA we get that $\Delta ABC \cong \Delta BAD$. Similarly, $\Delta ABC \cong \Delta CEA$ and $\Delta ABC \cong \Delta FCB$.

2.3.4 An altitude of ΔABC will meet a side at right angles. Thus, the altitude meets the side of the associated triangle at right angles, as this side is parallel to the side of ΔABC. Also, by the triangle congruence result of the previous exercise, the altitude bisects the side of the associated triangle.

2.3.5 Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ define an angle and let $\overrightarrow{AD}$ be the bisector. Drop perpendiculars from D to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and assume these intersect at B and C. Then, by AAS, ΔABD and ΔACD are congruent, and $\overline{BD} \cong \overline{CD}$.

Conversely, suppose D is interior to $\angle BAC$ with $\overline{BD}$ perpendicular to $\overrightarrow{AB}$ and $\overline{CD}$ perpendicular to $\overrightarrow{AC}$. Also, suppose that $\overline{BD} \cong \overline{CD}$. Then, by the Pythagorean Theorem $AB^2 + BD^2 = AD^2$ and $AC^2 + CD^2 = AD^2$. Thus, $\overline{AB} \cong \overline{AC}$ and by SSS $\Delta ABD \cong \Delta ACD$. This implies that $\angle BAD \cong \angle CAD$.

2.4.1 Mini-Project:Area in Euclidean Geometry

This section includes the first "mini-project" for the course. These projects are designed to be done in the classroom, in groups of three or four students. Each group should elect a Recorder. The Recorder's sole job is to outline the group's solutions to exercises. The summary should not be a formal write-up of the project, but should give enough a brief synopsis of the group's reasoning process.

The main goal for the mini-projects is to have students discuss geometric ideas with one another. Through the group process, students clarify their own understanding of concepts, and help each other better grasp abstract ways of thinking. There is no better way to conceptualize an idea than to have to explain it to another person.

In this mini-project, students are asked to grapple with the notion of "area". You may want to precede the project by a general discussion of how to best define area. Students will quickly find that it is not such an obvious notion as they once thought. For example, what does it mean for two figures to have the same area?

2.4.1 Construct a diagonal and use the fact that alternate interior angles of a line falling on parallel lines are congruent to generate an ASA congruence for the two sub-triangles created in the parallelogram.

2.4.2 We know that $\overline{AD} \cong \overline{EF}$. Thus, $\overline{AE} \cong \overline{DF}$, as the length of either of these differ by the length of $\overline{DE}$. Also, $\angle AEB \cong \angle DFC$, by Theorem 2.9. By SAS we get that $\Delta AEB \cong \Delta DFC$. Thus, parallelogram $ABCD$ can be split into ΔAEB and $EBCD$ and parallelogram $EBCF$ can be split into ΔDFC and $EBCD$. Clearly, these are two pairs of congruent polygons.

2.4.3 We have that $\overline{AE} \cong \overline{DF}$. Theorem 2.9 says that $\angle AEB \cong \angle DFC$. So, by SAS $\Delta AEB \cong \Delta DFC$. Let G be the point where $\overline{CD}$ intersects $\overline{BE}$. (Such a point exists by Pasch's axiom applied to ΔAEB) Now, parallelogram $ABCD$ can be split into ΔAEB plus ΔBGC minus ΔDGE. Also, parallelogram $EBCF$ can be split into ΔDFC plus ΔBGC minus ΔDGE.

2.4.4 Use Theorem 2.8 and Exercise 2.4.1.

2.4.5 By Theorem 2.9 we know that $\angle BAE$ and $\angle FBA$ are right angles, and thus $ABFE$ is a rectangle. By Theorem 2.9 we have that $\angle DAB \cong \angle CBG$, where G is a point on $\overrightarrow{AB}$ to the right of B. Subtracting the right angles, we get $\angle DAE \cong \angle CBF$. By SAS, $\Delta DAE \cong \Delta CBF$. Then rectangle $AEFB$ can be split into $AECB$ and ΔCBF and parallelogram $DABC$ can be split into $AECB$ and ΔDAE and the figures are equivalent.

Hidden Assumptions? One hidden assumption is the notion that areas are additive. That is, if we have two figures that are not overlapping, then the area of the union is the sum of the separate areas.

2.4.2 Cevians and Area

2.4.6 Since a median is a cevian to a midpoint, then the fractions in the ratio product of Theorem 2.24 are all equal to 1.

2.4.7 Let the triangle and medians be labeled as in Theorem 2.24. The area of ΔAYB will be equal to AYh, where h is the length of a perpendicular dropped from B to $\overleftrightarrow{AC}$. The area of ΔCYB will be equal to CYh, Since $\overline{AY} \cong \overline{CY}$, these areas will be the same and ΔABC will balance along $\overleftrightarrow{BY}$. A similar argument shows that ΔABC balances along each median, and thus the centroid is a balance point for the triangle.

2.4.8 Refer to the figure below. By the previous exercise we know that $1+2+3 = 4+5+6$ (in terms of areas). Also, since 1 and 2 share the same base and height we have $3 = 4$. Similarly, $1 = 2$ and $5 = 6$. Thus, $1 = 6$.

Similarly, $2+3+4 = 1+5+6$ will yield $4 = 5$, and $3+4+5 = 1+5+5$ yields $2 = 3$. Thus, all 6 have the same area.

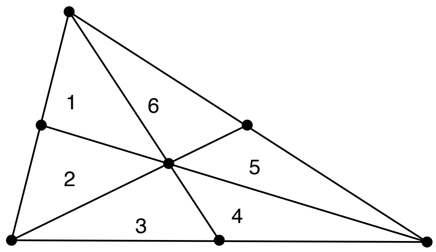


FIGURE 2.5:

2.4.9 Consider median $\overline{BD}$ in ΔABC, with E the centroid. Let $a = BE$ and $b = DE$. Then, the area of ΔDBC is $\frac{(a+b)h}{2}$ where h is the height of the triangle. This is 3 times the area of ΔDEC by the previous exercise. Thus, $\frac{(a+b)h}{2} = 3\frac{bh}{2}$, or $a = 2b$.

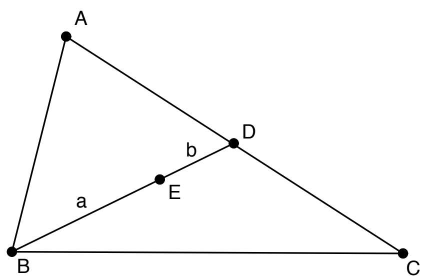


FIGURE 2.6:

2.5 Similar Triangles

As stated in the text, similarity is one of the most useful tools in the geometer's toolkit. Several of the exercises could be jumping off points for further discussion—the definition of the trigonometric functions, the Pythagorean Theorem (this could be a place to review some of the myriad of ways that this theorem has been proved), and Pascal's Theorem and its use in Hilbert's development of arithmetic.

2.5.1 Since $\overleftrightarrow{DE}$ cuts two sides of triangle at the midpoints, then by Theorem 2.27, this line must be parallel to the third side $\overline{BC}$. Thus $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Since the angle at A is congruent to itself, we have by AAA that ΔABC and ΔADE are similar, with proportionality constant of $\frac{1}{2}$.

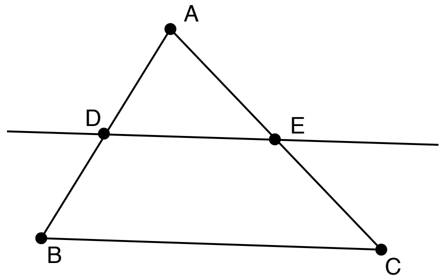


FIGURE 2.7:

2.5.2 Since $\overline{AG} \cong \overline{DE}$ then $\frac{AB}{AG} = \frac{AB}{DE}$ and so $\frac{AB}{AG} = \frac{AC}{DF}$. Since $\overline{AH} \cong \overline{DF}$ we get $\frac{AB}{AG} = \frac{AC}{AH}$. By Theorem 2.27, $\overleftrightarrow{GH}$ and $\overleftrightarrow{BC}$ are parallel.

2.5.3 Let ΔABC and ΔDEF have the desired SSS similarity property. That is sides $\overline{AB}$ and $\overline{DE}$, sides $\overline{AC}$ and $\overline{DF}$, and sides $\overline{BC}$ and $\overline{EF}$ are proportional. We can assume that $\overline{AB}$ is at least as large as $\overline{DE}$. Let G be a point on $\overline{AB}$ such that $\overline{AG} \cong \overline{DE}$. Let $\overleftrightarrow{GH}$ be the parallel to $\overleftrightarrow{BC}$ through G. Then, $\overleftrightarrow{GH}$ must intersect $\overleftrightarrow{AC}$, as otherwise $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ would be parallel. By the properties of parallels, $\angle AGH \cong \angle ABC$ and $\angle AHG \cong \angle ACB$. Thus, ΔAGH and ΔABC are similar.

Therefore, $\frac{AB}{AG} = \frac{AC}{AH}$. Equivalently, $\frac{AB}{DE} = \frac{AC}{AH}$. We are given that $\frac{AB}{DE} = \frac{AC}{DF}$. Thus, $\overline{AH} \cong \overline{DF}$.

Also, $\frac{AB}{AG} = \frac{BC}{GH}$ and $\frac{AB}{AG} = \frac{AB}{DE} = \frac{BC}{EF}$. Thus, $GH \cong EF$.

By SSS ΔAGH and ΔDEF are congruent, and thus ΔABC and ΔDEF are similar.

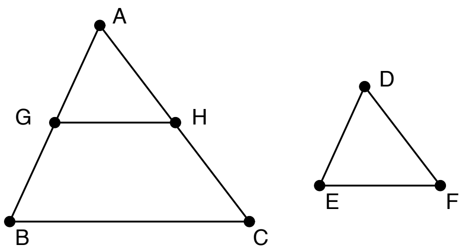


FIGURE 2.8:

2.5.4 Since $\angle ACD$ and $\angle DCB$ have measures summing to 90 degrees, and since $\angle DAC$ and $\angle ACD$ sum to 90, then $\angle DCB \cong \angle CAD (\cong \angle CAB)$. Likewise, $\angle CBD(\cong \angle ABC) \cong \angle ACD$. By AAA ΔDCB and ΔCAB are similar, as are ΔACD and ΔABC. Thus, $\frac{y}{a} = \frac{a}{c}$ and $\frac{x}{b} = \frac{b}{c}$. Or, $y = \frac{a^2}{c}$ and $x = \frac{b^2}{c}$. Thus, $c = x + y = \frac{a^2+b^2}{c}$. The result follows immediately.

2.5.5 Any right triangle constructed so that one angle is congruent to $\angle A$ must have congruent third angles, and thus the constructed triangle must be similar to ΔABC. Since *sin* and *cos* are defined in terms of ratios of sides, then proportional sides will have the same ratio, and thus it does not matter what triangle one uses for the definition.

2.5.6 Drop a perpendicular from C to $\overleftrightarrow{AB}$ intersecting at D. There are two cases. If D is not between A and B, then it is to to the left of A or to the right of B. We can assume it is to the left of B. Then, the angle at A must be obtuse, as $\angle BAC$ is exterior to right triangle ΔACD. If D is between A and B then we can assume the angles at A and B are acute, again by an exterior angle argument.

In the first case, $sin(\angle A) = \frac{CD}{b}$ and $sin(\angle B) = \frac{CD}{a}$. Then, $\frac{sin(\angle A)}{sin(\angle B)} = \frac{a}{b}$.

In the second case, $sin(\angle A) = sin(\angle DAC)$. An exactly analogous argument to the first case finishes the proof.

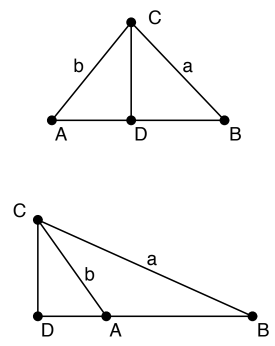


FIGURE 2.9:

2.5.7 If the parallel to $\overleftrightarrow{AC}$ does not intersect $\overleftrightarrow{RP}$, then it would be parallel to this line, and since it is already parallel to $\overleftrightarrow{AC}$, then by Exercise 2.1.9 $\overleftrightarrow{RP}$ and $\overleftrightarrow{AC}$ would be parallel, which is impossible.

By the properties of parallels, $\angle RAP \cong \angle RBS$ and $\angle RPA \cong \angle RSB$. Thus, by AAA ΔRBS and ΔRAP are similar. ΔPCQ and

ΔSBQ are similar by AAA using an analogous argument for two of the angles and the vertical angles at Q.

Thus, $\frac{CP}{BS} = \frac{CQ}{BQ} = \frac{PQ}{QS}$, and $\frac{AP}{BS} = \frac{AR}{BR} = \frac{PR}{SR}$. So, $\frac{CP}{AP}\frac{BQ}{QC} = \frac{CP}{AP}\frac{BS}{CP} = \frac{BS}{AP}$ And, $\frac{CP}{AP}\frac{BQ}{QC}\frac{AR}{RB} = \frac{BS}{AP}\frac{AR}{RB} = \frac{BS}{AP}\frac{AP}{BS} = 1$.

2.5.8 By Theorem 2.25 applied to ΔADE and ΔABC we get $\frac{AD}{AB} = \frac{AE}{AC}$. Again, using Theorem 2.25 on ΔAFE and ΔABG we get $\frac{AF}{AB} = \frac{AE}{AG}$. Thus, $\frac{AD}{AF} = \frac{AG}{AC}$.

2.5.1 Mini-Project: Finding Heights

This mini-project is a good example of an activity future high school geometry teachers could incorporate into their courses. It is a very practical application of the notion of similarity. The mathematics in the first example for finding height is extremely easy, but the interesting part is the data collection. Students need to determine how to get the most accurate measurements using the materials they have on hand.

The second method of finding height is again a simple calculation using two similar triangles, but the students may not see this at first. The interesting part of the project is having them see the connection between the mirror reflection and the calculation they made in part I.

Again, have the students work in small groups with a Recorder, but make sure the Recorder position gets shifted around from project to project.

2.6 Circle Geometry

This section introduces students to the basic geometry of the circle. The properties of inscribed angles and tangents are the most important properties to focus on in this section.

2.6.1 Case 2: A is on the diameter through $\overline{OP}$. Let $\alpha = m\angle PBO$ and $\beta = m\angle POB$. Then, $\beta = 180 - 2\alpha$. Also, $m\angle AOB = 180 - \beta = 2\alpha$.

Case 3: A and B are on the same side of $\overleftrightarrow{PO}$. We can assume that $m\angle OPB > m\angle OPA$. Let $m\angle OPB = \alpha$ and $m\angle OPA = \beta$. Then, we can argue in a similar fashion to the proof of the Theorem using $\alpha - \beta$ instead of $\alpha + \beta$.

2.6.2 Let quadrilateral $ABCD$ be inscribed in the circle, with center O. Then, by Theorem 2.31 $a = \angle OAB = \frac{1}{2}\angle EOB$, where E is the point of intersection of $\overrightarrow{AO}$ with the circle. Also, $b = \angle OAD = \frac{1}{2}\angle EOD$.

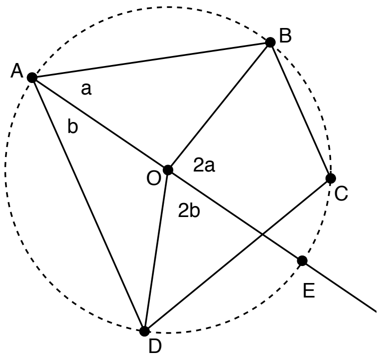


FIGURE 2.10:

Likewise, we would have this relationship for angles generated by $\overrightarrow{CO}$.

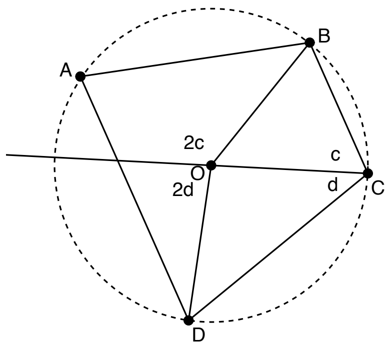


FIGURE 2.11:

Clearly, the sum of the angles at O $(2a + 2b + 2c + 2d)$ is 360, and so the sum of the angles at A and C is 180.

2.6.3 Consider $\angle AQP$. This must be a right angle by Corollary 2.33. Similarly, $\angle BQP$ must be a right angle. Thus, A, Q, and B are collinear.

2.6.4 By Theorem 2.31 we know that $m\angle AOP = 2m\angle ABP$ and $m\angle POC = 2m\angle PBC$. But, $\overrightarrow{BP}$ bisects $\angle ABC$ and so $\angle AOP \cong \angle POC$. Let Q be the point of intersection of $\overline{OP}$ and $\overline{AC}$. Then, $\Delta OQA \cong \Delta OQC$ by SAS. The result follows.

2.6.5 Let $\overline{AB}$ be the chord, O the center, and M the midpoint of $\overline{AB}$. Then $\Delta AOM \cong \Delta BOM$ by SSS and the result follows.

2.6.6 $\angle BAD \cong \angle BCD$ by Corollary 2.32. Likewise, $\angle CBA \cong \angle CDA$. Thus, ΔABP and ΔCDP are similar. The result follows immediately.

2.6.7 Consider a triangle on the diagonal of the rectangle. This has a right angle, and thus we can construct the circle on this angle. Since the other triangle in the rectangle also has a right angle on the same side (the diameter of the circle) then it is also inscribed in the same circle.

2.6.8 First, $m\angle BDA + m\angle CAD = 180 - m\angle DPA = m\angle CPD$. Then, by Theorem 2.31, we have $m\angle BDA + m\angle CAD = \frac{1}{2}(m\angle BOA + m\angle COD)$. Since $m\angle CPD = m\angle BPA$ (Vertical angles), the result follows.

2.6.9 If point P is inside the circle c, then Theorem 2.41 applies. But, this theorem says that $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$, where C and D are the other points of intersections of $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ with the circle. If P is inside c, then C and D are different points. The assumption of Theorem 2.42 says that $m\angle BPA = \frac{1}{2}m\angle BOA$. But, $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$ would then imply that $m\angle COD = 0$, which is impossible as C and D are not collinear with O.

2.6.10 By Theorem 2.36 $m\angle PTA = 90 - m\angle ATO$. Since triangle OAT is isosceles ($\overline{OA}$ and $\overline{OT}$ are radii of c), then $\angle ATO \cong \angle OAT$. Since $m\angle TOA = 180 - (m\angle ATO + m\angle OAT = 180 - 2m\angle ATO$, then $m\angle ATO = \frac{1}{2}(180 - m\angle TOA = 90 - \frac{1}{2}m\angle TOA)$.

So, $m\angle PTA = 90 - m\angle ATO = 90 - (90 - \frac{1}{2}m\angle TOA) = \frac{1}{2}m\angle TOA$.

2.6.11 The angle made by $\overline{BT}$ and l must be a right angle by Theorem 2.36. Likewise, the angle made by $\overline{AT}$ and l is a right angle. Thus, A, T, and B are collinear.

2.6.12 Suppose they intersected at another point P. Then, ΔTBP and ΔTAP are both isosceles triangles. But, this would imply, by the previous exercise, that there is a triangle with two angles greater than a right angle, which is impossible.

2.6.13 Suppose one of the circles had points A and B on opposite sides of the tangent line l. Then $\overline{AB}$ would intersect l at some point P which is interior to the circle. But, then l would pass through an interior point of the circle and by continuity must intersect the circle in two points which is impossible. Thus, either all points of one circle are on opposite sides of l from the other circle or are on the same side.

2.6.14 Let P and Q be points on the tangent, as shown. Then, $\angle BDT \cong \angle BTP$, as both are inscribed angles on the same arc. Likewise, $\angle ACT \cong \angle ATQ$. Since, $\angle BTP \cong \angle ATQ$ (vertical angles), then $\angle BDT \cong \angle ACT$ and the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel.

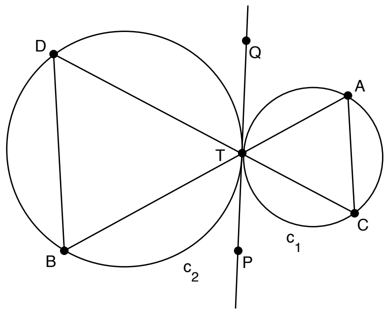


FIGURE 2.12:

2.6.15 By Theorem 2.36, we have that $\angle OAP$ is a right angle, as is $\angle OBP$. Since the hypotenuse ($\overline{OP}$) and leg ($\overline{OA}$) of right triangle ΔOAP are congruent to the hypotenuse ($\overline{OP}$) and leg ($\overline{OB}$) of right triangle ΔOBP, then by Exercise 2.2.10 the two triangles are congruent. Thus $\angle OPA \cong \angle OPB$.

2.6.16 Suppose that the bisector did not pass through the center. Then, construct a segment from the center to the outside point. By the previous theorem, the line continued from this segment must bisect the angle made by the tangents. But, the bisector is unique, and thus the original bisector must pass through the center.

2.6.17 Let A and B be the centers of the two circles. Construct the two perpendiculars at A and B to $\overleftrightarrow{AB}$ and let C and D be the intersections with the circles on one side of $\overleftrightarrow{AB}$.

If $\overleftrightarrow{CD}$ does not intersect $\overleftrightarrow{AB}$, then these lines are parallel, and the angles made by $\overleftrightarrow{CD}$ and the radii of the circles will be right angles. Thus, this line will be a common tangent.

Otherwise, $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$ at some point P. Let $\overleftrightarrow{PE}$ be a tangent to the circle with center A. Then, since ΔPAC and ΔPBD are similar, we have $\frac{AP}{BP} = \frac{AC}{BD}$. Let $\overleftrightarrow{BF}$ be parallel to $\overleftrightarrow{AE}$ with F the intersection of the parallel with the circle centered at B. Then, $\frac{AC}{BD} = \frac{AE}{BF}$. So, $\frac{AP}{BP} = \frac{AE}{BF}$. By SAS similarity, ΔPAE and ΔPBF are similar, and so F is on $\overleftrightarrow{PE}$ and $\angle PFB$ is a right angle. Thus, $\overleftrightarrow{PE}$ is a tangent to the circle centered at B.

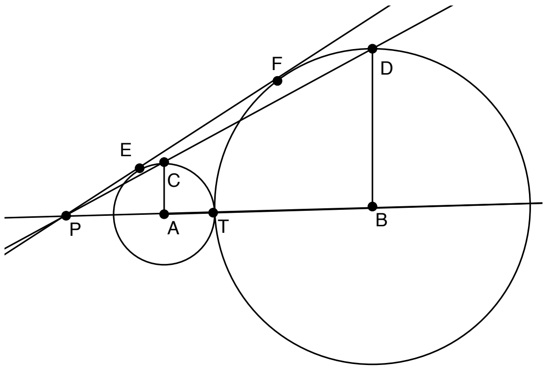


FIGURE 2.13:

2.7 Project 4 - Circle Inversion and Orthogonality

This section is crucial for the later development of the Poincaré model of non-Euclidean (hyperbolic) geometry. It is also has some of the most elegant mathematical results found in the course.

2.7.1 By Theorem 2.31, $\angle Q_2P_1P_2 \cong \angle Q_2Q_1P_2$. Thus, $\angle PP_1Q_2 \cong \angle PQ_1P_2$. Since triangles ΔPP_1Q_2 and ΔPQ_1P_2 share the angle at P, then they are similar. Thus, $\frac{PP_1}{PQ_1} = \frac{PQ_2}{PP_2}$, or $(PP_1)(PP_2) = (PQ_1)(PQ_2)$.

2.7.2 Choose a line from P passing through the center. Then, $PP_1PP_2 = (PO - OP_1)(PO + OP_1) = PO^2 - r^2$.

2.7.3 By similar triangles $\frac{OP}{OT} = \frac{OT}{OP'}$. Since $OT = r$ the result follows.

2.7.4 As P approaches O, the distance OP goes to zero, so the distance OP' must get larger without bound, for the product to remain equal to r^2. Thus, the orthogonal circle radius grows larger without bound as well.

CHAPTER 3

Analytic Geometry

This chapter is a very quick review of analytic geometry. In succeeding chapters, analytic methods will be utilized freely. Thus, if students are not familiar with vectors, basic trigonometry, and complex numbers, then a quick review of this chapter would be necessary. However, for most math majors, the material in this chapter can be safely skipped.

The project on Bézier Curves is quite enjoyable. Even if Chapter 3 is skipped, you may want students to do this project.

Also, the last section in the chapter on Birkhoff's axioms would fit nicely into a course more heavily focused on foundational questions.

SOLUTIONS TO EXERCISES IN CHAPTER 3

3.2 Vector Geometry

3.2.1 If A is on either of the axes, then so is B and the distance result holds by the definition of coordinates. Otherwise, A (and B) are not on either axis. Drop perpendiculars from A and B to the x-axis at P and Q. By SAS similarity, ΔAOP and ΔBOQ are similar, and thus $\angle AOP \cong \angle BOQ$, which means that A and B are on the same line $\overleftrightarrow{AO}$, and the ratio of BO to AO is k.

3.2.2 If $B = (0, 0)$ choose $k = 0$. Otherwise, if A is on the x-axis, then $A = (x, 0)$, with $x \neq 0$, and $B = (x_1, 0)$. Choose $k = \frac{x_1}{x}$. If A is on the y-axis, a similar argument can be used. Otherwise, drop perpendiculars from A and B to the x-axis at P and Q. By AA similarity, ΔAOP and ΔBOQ are similar. The result follows.

3.2.3 The vector from P to Q is in the same direction (or opposite direction) as the vector v. Thus, since the vector from P to Q is $\vec{Q} - \vec{P}$, we have $\vec{Q} - \vec{P} = tv$, for some real number t. In coordinates we have $(x, y) - (a, b) = (tv_1, tv_2)$, or $(x, y) = (a, b) + t(v_1, v_2)$.

3.2.4 By the previous exercise, points $Q = (x, y)$ on the line l can be expressed as $(x, y) = (a, b) + t(v_1, v_2)$. where $P = (a, b)$ is any point on l, and v is a vector in the same direction as $\vec{Q} - \vec{P}$. Then, $x = a + tv_1$ and $y = b + tv_2$.

Now, $-v_2x + v_1y = -v_2a + v_2b$. So, $-v_2x + v_1y + (v_2a - v_1b) = 0$. Choose $A = -v_2$, $B = v_1$, and $C = v_2a - v_1b$.

3.2.5 By Exercise 3.2.3 the line through A and B can be represented by the set of points of the form $\vec{A} + t(\vec{B} - \vec{A})$. Then, $M = \frac{1}{2}(\vec{A} + \vec{B}) = \vec{A} + \frac{1}{2}(\vec{B} - \vec{A})$ is on the line through A and B, and is between A and B. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the distance from A to M is $\sqrt{(\frac{x_1}{2} - \frac{x_2}{2})^2 + (\frac{y_1}{2} - \frac{y_2}{2})^2}$, which is equal to the distance from B to M.

3.2.6 $\vec{W} = \frac{1}{2}(\vec{A}+\vec{B})$, $\vec{X} = \frac{1}{2}(\vec{B}+\vec{C})$, $\vec{Y} = \frac{1}{2}(\vec{C}+\vec{D})$, $\vec{Z} = \frac{1}{2}(\vec{D}+\vec{A})$. Since $\vec{X} - \vec{W} = \frac{1}{2}(\vec{C} - \vec{A})$ and $\vec{Y} - \vec{Z} = \frac{1}{2}(\vec{C} - \vec{A})$, then the two sides ($\overline{WX}$ and $\overline{YZ}$) of $WXYZ$ are parallel. Likewise, the other pair of sides are parallel.

3.3 Project 5 - Bézier Curves

3.3.1 The derivative to $\vec{c}(t)$ is $\vec{c}'(t) = 2\vec{B} - 2\vec{A} + 2t(\vec{C} - 2\vec{B} + \vec{A}$. Then $\vec{c}'(0) = 2\vec{B} - 2\vec{A}$, which is in the direction of $\vec{B} - \vec{A}$ and $\vec{c}'(1) = 2\vec{B} - 2\vec{A} + 2(\vec{C} - 2\vec{B} + \vec{A} = 2\vec{C} - 2\vec{B}$, which is in the direction of $\vec{C} - \vec{B}$.

3.3.2 Basic algebra.

3.3.3 Similar computation to Exercise 3.3.1.

3.4 Angles in Coordinate Geometry

3.4.1 Let $\vec{A} = (\cos(\alpha), \sin(\alpha))$ and $\vec{B} = (\cos(\beta), \sin(\beta))$. Then, from Theorem 3.11 we have $\cos(\alpha - \beta) = \vec{A} \circ \vec{B}$, since $\vec{A}$ and $\vec{B}$ are unit length vectors. The result follows immediately.

3.4.2 Simply replace β by $-\beta$ in the previous exercise and use the fact that $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

3.4.3 By Exercise 3.4.1,

$$\begin{aligned}\cos(\frac{\pi}{2} - (\alpha + \beta)) &= \cos(\frac{pi}{2})\cos(\alpha + \beta) + \sin(\frac{pi}{2}90)\sin(\alpha + \beta)) \\ &= \sin(\alpha + \beta).\end{aligned}$$

Then, use the formula from Exercise 3.4.2 with the term inside cos being $(\frac{pi}{2} - \alpha) + (-\beta)$.

3.4.4 If A and D are on opposite sides of $\overleftrightarrow{BC}$, then $ABDC$ is an inscribed convex quadrilateral, and thus opposite angles are supplementary. So, the measure of angle A is 180 minus the measure of angle D. But, the sine function was defined so that supplementary angles have the same sine, and thus $sin(\angle A) = sin(\angle D)$.

3.5 The Complex Plane

3.5.1

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos(\theta) + i\ \sin(\theta))(\cos(\phi) + i\ \sin(\phi)) \\ &= (\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) + i(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)) \\ &= \cos(\theta + \phi) + i\ \sin(\theta + \phi) \\ &= e^{i(\theta+\phi)} \end{aligned}$$

3.5.2 We have $e^{i\pi} = \cos(\pi) + i\ \sin(\pi) = -1$.

3.5.3 Let $z = e^{i\theta}$ and $w = e^{i\phi}$ and use Exercise 3.4.1.

3.5.4 Let z be a non-zero complex number. Then, $z = re^{i\theta}$, with $r \neq 0$. Let $w = \frac{1}{r}e^{-i\theta}$. Then, $zw = 1$.

3.5.5 The rationalized complex numbers have the form $i\frac{-1}{2}$, i, and $\frac{1}{10} - i\frac{1}{5}$.

3.6 Birkhoff's Axiomatic System for Analytic Geometry

3.6.1 First, if A is associated to $x_A = t_A\sqrt{dx^2 + dy^2}$, where $A = (x, y) = (x_0, y_0) + t_A(dx, dy)$, and B is associated to x_B in a similar fashion, then $|x_A - x_B| = |t_A - t_B|\sqrt{dx^2 + dy^2}$. On the other hand,

$$d(A, B) = \sqrt{(t_A dx - t_B dx)^2 + (t_A dy - t_B dy)^2} = \sqrt{dx^2 + dy^2}|t_A - t_B|$$

3.6.2 Let ΔABC be a triangle with side lengths $a = BC$, $b = AC$, and $c = AB$. Let α, β, and γ be the angles at A, B, and C respectively. Then,

$$\begin{aligned} c^2 &= ||\vec{B} - \vec{A}||^2 = ||(\vec{B} - \vec{C}) - (\vec{A} - \vec{C})||^2 \\ &= ((\vec{B} - \vec{C}) - (\vec{A} - \vec{C})) \circ ((\vec{B} - \vec{C}) - (\vec{A} - \vec{C})) \\ &= ((\vec{B} - \vec{C}) \circ (\vec{B} - \vec{C})) + ((\vec{A} - \vec{C}) \circ (\vec{A} - \vec{C})) - 2((\vec{B} - \vec{C}) \circ (\vec{A} - \vec{C})) \\ &= a^2 + b^2 - 2\ a\ b\ \cos(\theta) \end{aligned}$$

3.6.3 Given a point O as the vertex of the angle, set O as the origin of the coordinate system. Then, identify a ray $\overrightarrow{OA}$ associated to the angle θ, with $A = (x, y)$. Let $a = ||\vec{A}|| = \sqrt{x^2 + y^2}$. Then, $\sin^2(\theta) + \cos^2(\theta) = (\frac{x}{a})^2 + (\frac{y}{a})^2 = \frac{x^2+y^2}{a^2} = 1$.

3.6.4 Discussion question.

3.6.5 Discussion question. One idea is that analytic geometry allows one to study geometric figures by the equations that define them. Thus, geometry can be reduced to the arithmetic (algebra) of equations.

CHAPTER 4

Constructions

In this chapter we cover some of the basic Euclidean constructions and also have a lot of fun with lab projects. The origami project should be especially interesting for students, as it is an axiomatic system with which they can *physically* interact and explore.

The third section on constructibility may be a bit heavy for students, but the relationship between geometric constructibility and algebra is a fascinating one, especially if your students have had some exposure to abstract algebra. Also, all students should know what the three classical construction problems are, and how the pursuit of solutions to these problems has had a profound influence on the development of modern mathematics.

SOLUTIONS TO EXERCISES IN CHAPTER 4

4.1 Euclidean Constructions

4.1.1 Use SSS triangle congruence on ΔABF and ΔDGH.

4.1.2 Let M be the midpoint of $\overline{AB}$. Then $\Delta AMD \cong \Delta BMD$ by SSS, and so $\angle BMD$ is a right angle. A similar argument shows that $\angle CMB$ is a right angle. Thus, M lies on $\overleftrightarrow{CD}$ and the result follows.

4.1.3 Use the SSS triangle congruence theorem on ΔADE and ΔABE to show that $\angle EAB \cong \angle BAE$.

4.1.4 If l is tangent to the circle through B, the result follows from the fact that a radius segment is perpendicular to a tangent. Otherwise, l will intersect the circle again at C. Let $\overrightarrow{AE}$ be the bisector of $\angle BAC$. This ray will intersect $\overline{BC}$ at some point, which we may assume to be E. Then, use SAS congruence on ΔABE and ΔACE.

4.1.5 Use the fact that both circles have the same radius.

4.1.6 It is enough to prove the result for B outside c because if B were inside or on c, we could find another point B' outside of c, and at a distance greater than r from B, and transfer c to B' and then to B.

It is clear, then, that the intersections described in the construction exist. By SSS we have that $\Delta CGO \cong \Delta ECB$, since $\overline{CG}$ and $\overline{CE}$ are common radii of a circle, $\overline{CO}$ and $\overline{CB}$ are common radii, and $\overline{GO}$ and $\overline{EB}$ are common radii.

Now, ΔCBO is equilateral (proved in Exercise 4.1.5). Thus, $\angle OBC \cong \angle COB$. By the triangle congruence just shown, $\angle EBC \cong \angle COG$. Subtracting, we get $\angle OBE \cong \angle GOB$. By SAS, we have $\Delta OBE \cong \Delta GOB$, and thus $\overline{BG} \cong \overline{EO}$ and the result follows.

4.1.7 Let the given line be l and let P be the point not on l. Construct the perpendicular m to l through P. At a point Q on m, but not on l, construct the perpendicular n to m. Theorem 2.7 implies that l and n are parallel.

4.1.8 Construct the midpoint M of $\overline{AB}$. Construct the circle at M of radius AB. Construct the perpendicular to $\overleftrightarrow{AB}$ at M. Let C be an intersection of the perpendicular and the circle. Then, ΔABC is isosceles, as can seen by a SAS argument on ΔAMC and ΔBMC.

4.1.9 On $\overrightarrow{BA}$ construct A' such that $BA' = a$. On $\overrightarrow{BC}$ construct C' such that $BC' = b$. Then, SAS congruence gives $\Delta AB'C'$ congruent to any other triangle with the specified data.

4.1.10 Let $\overrightarrow{AC}$ be a ray not on $\overleftrightarrow{AB}$. Let $\overline{DE}$ be some segment. On $\overrightarrow{AC}$ we can construct a point P_1 such that $\overline{AP_1} \cong \overline{DE}$. Continuing in the same direction as $\overrightarrow{AC}$, we can construct P_2 such that $\overline{P_1P_2} \cong \overline{DE}$. Continuing in this way, we can construct n points $P_1, P_2, \ldots, P_n$ with $\overline{AP_1}$ of length DE, $\overline{P_1P_2}$ of length DE, and so on. Then, construct $\overline{P_nB}$, and construct $P_{n-1}B_{n-1}$ parallel to $\overleftrightarrow{P_nB}$ through P_{n-1}. This must intersect $\overline{AB}$ (Pasch's axiom), so we can assume B_{n-1} is on $\overline{AB}$.

It is easy to show that ΔP_nAB is similar to $\Delta P_{n-1}AB_{n-1}$ (AAA) and so the ratio of P_nA to $P_{n-1}A$, which is $\frac{n}{n-1}$ is the same as the ratio of BA to $B_{n-1}A$.

It is clear that if we again find a parallel to $\overleftrightarrow{P_{n-1}B_{n-1}}$ through P_{n-2} intersecting $\overline{AB}$ at B_{n-2}, creating another pair of similar triangles. Continuing in this way, we get a sequence of point $B_{n-1}, B_{n-2}, \ldots, B_1$, and a sequence of similar triangles. At the last step, we would have that the ratio of P_2A to P_1A, which is 2 to 1, is the same as the ratio of B_2A to B_1A. In other words, $\overline{B_1A} \cong \overline{B_2B_1}$. Unraveling the other similarity ratios yields a sequence of congruent sub-segments of $\overline{AB}$.

4.2 Project 6 - Euclidean Eggs

The contrast between the method of drawing curves used in this project and the method of Bézier Curves covered in Chapter 3 is a good discussion point.

4.2.1 The tangent to one of the circles will meet $\overleftrightarrow{AB}$ at C at right angles by Theorem 2.36. The tangent to the other circle will also meet $\overleftrightarrow{AB}$ at C at a right angle. Since the perpendicular to $\overleftrightarrow{AB}$ at C is unique, the tangents coincide.

4.2.2 This is just a matter of verifying that all appropriate pairs of circles have lines through their centers meeting the point where the circles are tangent. For example, arcs AEC and CI_3D meet smoothly at C, since $\overline{BC}$ passes through the center of the circle through C, I_3, and D.

4.2.3 The construction steps are implied by the figure.

4.3 Constructibility

4.3.1 Just compute the formula for the intersection.

4.3.2 Use the trig identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ with $2\theta = 45$. Since $cos(45) = \frac{\sqrt{2}}{2}$, which is constructible, and since solving for $\sin(\theta)$ involves only simple algebra and square roots, then $\sin(2.5)$ is constructible.

4.3.3 Reverse the roles of the product construction.

4.3.4 It is straightforward to show ΔABD is similar to ΔDBC. Thus, $\frac{BD}{BC} = \frac{AB}{BD}$, or $\frac{BD}{a} = \frac{a}{BD}$. So, $a^2 = BD$.

4.3.5 For $\sqrt{3}$, use a right triangle with hypotenuse 2 and one side 1. For $\sqrt{5}$, use a right triangle with sides of length 1 and 2.

4.3.6 We know that π is not constructible. Then, for all rational numbers r, $r\pi$ is not constructible.

4.3.7 Consider $\frac{a}{\pi}$. This is less than a.

4.3.8 If the points where the circle crosses the x-axis are not constructible, we get 2. Otherwise, by the previous exercise, we have at least one in $[0, a)$. This point generates 4 others on the circle by symmetry.

4.3.9 If a circle of radius r and center (x, y) has x not constructible, then $(x, y + r)$ and $(x, y - r)$ are non-constructible on the circle. We can use the same reasoning if y is not constructible. If the center is constructible, then the previous exercise gives at least two non-constructible points for a circle of radius r whose center is at the origin.

Add (x, y) to these two points to get two non-constructible points on the original circle.

4.3.10

4.4 Mini-Project: Origami Construction

For this project, one will need a good supply of square paper. Commercial origami paper is quite expensive. Equally as good paper can be made by taking notepads and cutting them into squares using a paper-cutter. (Cutting works best a few sheets at a time)

4.3.11 Given $\overline{AB}$, we can fold A onto B by axiom O2. Let l be the fold line of reflection created, and let l intersect AB at C. Then, since the fold preserves length, we have that $AC = CB$, and $\angle ACE \cong \angle ECB$, as show in Fig. 4.1. The result follows.

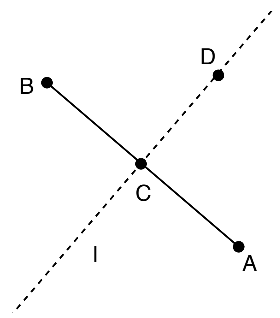


FIGURE 4.1:

4.3.12 Use Axiom O4 twice.

4.3.13 Since the reflection fold across t preserves length, we have $PR = P'R$. Also, the distance from a point to a line is measured along the perpendicular from the point to the line. Thus, the distance from R to l is equal to $P'R$. Thus, the distance from R to P equals the distance from R to l and R is on the parabola with focus P and directrix l.

An interesting extra credit problem for this construction would be to show that t is *tangent* to the parabola at R. One proof is as follows:

Suppose t intersected at another point R' on the parabola. Then, by definition, R' must have been constructed in the same way that R was, so there must be a folding (reflection) across t taking P to some point P'' on l such that $\overleftrightarrow{P''R'}$ is perpendicular to l at P'', and intersects t at R'. Then, by a triangle argument, we can show that $\overleftrightarrow{PP'}$ and $\overleftrightarrow{PP''}$

must both be perpendicular to t at R and R'. Since perpendiculars are unique, we must have that $R = R'$.

(To show, for example, that $\overleftrightarrow{PP'}$ is perpendicular to t at R, we can easily show that $\Delta PQR \cong \Delta P'QR$ by using the angle- and distance-preserving properties of reflections, and then use a second congruent triangle argument to show that $\overleftrightarrow{PP'}$ crosses t at right angles.)

4.3.14 Since a single fold takes P to P' and Q to Q', we know that $PQ = P'Q'$. Also, since l_1 is equidistant with l_2 and l_b, then R is the image of the midpoint of PQ, and $P'R = Q'R$, and l_3 must be a perpendicular bisector of $P'Q'$. Thus, by SAS, $\Delta PQ'R \cong PP'R$.

Also, since $\overleftrightarrow{SP'}$ is perpendicular to l_b, then $SP' = \frac{1}{2}PQ = P'R$. Then, $\Delta PP'S$ and $\Delta PP'R$ are two right triangles with two sides congruent. It follows from the Pythagorean Theorem that the third sides are congruent, and thus SSS implies that $\Delta PP'S \cong \Delta PP'R$.

CHAPTER 5

Transformational Geometry

In this chapter we make great use of functional notation and somewhat abstract notions such as $1-1$ and onto, inverses, composition, etc. Students may wonder how such computations are related to geometry, but that is the very essence of the chapter—that we can understand and investigate geometric ideas with more than one set of foundational lenses.

With that in mind, we will make use of synthetic geometric techniques where they are most elegant and can aid intuition, and at other times we will rely on analytical techniques.

SOLUTIONS TO EXERCISES IN CHAPTER 5

5.1 Euclidean Isometries

5.1.1 Define the function f^{-1} by $f^{-1}(y) = x$ if and only if $f(x) = y$. Then, f^{-1} is well-defined, as suppose $f(x_1) = f(x_2) = y$. Then, since f is $1-1$ we have that $x_1 = x_2$. Since f is onto, we have that for every y in S there is an x such that $f(x) = y$. Thus, f^{-1} is defined on all of S. Finally, $f^{-1}(f(x)) = f^{-1}(y) = x$ and $f(f^{-1}(y)) = f(x) = y$. So, $f \circ f^{-1} = f^{-1} \circ f = id_S$.

Suppose g was another function on S such that $f \circ g = g \circ f = id_S$. Then, $g \circ f \circ f^{-1} = f^{-1}$, or $g = f^{-1}$.

5.1.2 Let f be an isometry. We need to show $f^{-1}(A)f^{-1}(B) = AB$. But, since f is an isometry, we have $f^{-1}(A)f^{-1}(B) = f(f^{-1}(A))f(f^{-1}(B)) = AB$.

5.1.3 Since $g^{-1} \circ f^{-1} \circ f \circ g = g^{-1} \circ g = id$ and $f \circ g \circ g^{-1} \circ f^{-1} = f \circ f^{-1} = id$, then $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

5.1.4 $f(g(A))f(g(B)) = g(A)g(B) = AB$.

5.1.5 Let f be an isometry and let c be a circle centered at O of radius $r = OA$. Let $O' = f(O)$ and $A' = f(A)$. Let P be any point on c. Then, $O'f(P) = f(O)f(P) = OP = r$. Thus, the image of c under f is contained in the circle centered at O' of radius r. Let P' be any other point on the circle centered at O' of radius r. Then, $Of^{-1}(P') = f^{-1}(O')f^{-1}(P') = O'P' = r$. Thus, $f^{-1}(P')$ is a point on c and every such point P' is the image of a point on c, under the map T.

5.1.6 Use SSS congruence.

5.1.7 Label the vertices of the triangle A, B, and C. Then, consider vertex A. Under an isometry, consider the actual position of A in the plane. After applying the isometry, A might remain or be replaced by one of the other two vertices. Thus, there are three possibilities for the position occupied by A. Once that vertex has been identified, consider position B. There are now just two remaining vertices to be placed in this position. Thus, there are a maximum of 6 isometries. We can find 6 by considering the three basic rotations by 0, 120, and 240 degrees, and the three reflections about perpendicular bisectors of the sides.

5.1.8 Suppose f and g agree on three non-collinear points A, B, and C. Then, $f(A) = g(A)$ implies that $g^{-1}(f(A)) = A$. So, the isometry $g^{-1} \circ f$ fixes the three non-collinear points A,B, and C. By Theorem 5.3, $g^{-1} \circ g = id$, or $g = f$.

5.1.9 First, we show that f is a transformation. To show it is $1-1$, suppose $f(x, y) = f(x', y')$. Then, $kx+a = kx'+a$ and $ky+b = ky'+b$. So, $x = x'$ and $y = y'$.

To show it is onto, let (x', y') be a point. Then, $f(\frac{x'-a}{k}, \frac{y'-b}{k}) = (x', y')$.

f is not, in general, an isometry, since if $A = (x, y)$ and $B = (x', y')$ then $f(A)f(B) = kAB$.

5.1.10 Let S be a similarity and ABC a triangle. Let $A' = S(A)$, $B' = S(B)$, and $C' = S(C)$. Then, $A'B'C'$ is again a triangle, as S preserves the betweenness property. The two triangles are then similar, by AAA similarity, since S preserves angle measure.

5.1.11 Let ABC be a triangle and let $A'B'C'$ be its image under f. By the previous exercise, these two triangles are similar. Thus, there is a $k > 0$ such that $A'B' = kAB$, $B'C' = kBC$, and $A'C' = kAC$.

Let D be any other point not on $\overleftrightarrow{AB}$. Then, using triangles ABD and $A'B'D'$ we get that $A'D' = kAD$.

Now, let $\overline{DE}$ be any segment with D not on $\overleftrightarrow{AB}$. Then, using triangles ADE and $A'D'E'$ we get $D'E' = kDE$, since we know that $A'D' = kAD$.

Finally, let $\overline{EF}$ be a segment entirely on $\overleftrightarrow{AB}$, and let D be a point off $\overleftrightarrow{AB}$. Then, using triangles DEF and $D'E'F'$ we get $E'F' = kEF$, since we know that $D'E' = kDE$.

Thus, in all cases, we get that $f(A)f(B) = kAB$.

5.1.12 We need to show that f preserves betweenness and angle measure. Let $\overline{AB}$ be a segment, with $\vec{A} = (x_1, y_1)$ and $\vec{A} = (x_2, y_2)$. We can express $\overline{AB}$ as the set of points $(x, y) = \vec{A}+t(\vec{B}-\vec{A})$, where $0 \leq t \leq 1$. Now, $f(A) = (kx_1, ky_1) = k\vec{A}$ and $f(B) = (kx_2, ky_2) = k\vec{B}$. Then, the segment $\overline{f(A)f(B)}$ is the set of points $(x', y') = k\vec{A} + t(k\vec{B} - k\vec{A})$, where $0 \leq t \leq 1$. But, $k\vec{A} + t(k\vec{B} - k\vec{A}) = k(\vec{A} + t(\vec{B} - \vec{A})) = k(x, y)$. So, the points between $f(A)$ and $f(B)$ are precisely the image of the points between A and B. Thus, f preserves betweenness.

Let points A, O, and B define an angle. Let $v = \vec{A} - \vec{O}$ and $w = \vec{B} - \vec{O}$. Then, the measure of the angle can be derived from the dot product formula $cos(\theta) = \frac{v \circ w}{||v||||w||}$. If we replace v be kv and w by kw we see that the fraction on the right side will remain unchanged.

5.2.1 Mini-Project:Isometries Through Reflection

In this mini-project, students are led through a guided discovery of the amazing fact that, given any two congruent triangles, one can find a sequence of at most three reflections taking one triangle to the other.

5.2.1 First of all, suppose that C and R are on the same side of $\overleftrightarrow{AB}$. Then, since there is a unique angle with side $\overline{AB}$ and measure equal to the measure of $\angle BAC$, then R must lie on $\overrightarrow{AC}$. Likewise, R must lie on $\overrightarrow{BC}$. But, the only point common to these two rays is C. Thus, $R = C$.

If C and R are on different sides of $\overleftrightarrow{AB}$, then drop a perpendicular from C to $\overleftrightarrow{AB}$, intersecting at P. By SAS, ΔPAC and ΔPAR are congruent, and thus $\angle APR$ must be a right angle, and R is the reflection of C across $\overleftrightarrow{AB}$.

5.2.2 Suppose Q, A, and B are collinear. Then, l_1 is the perpendicular bisector of $\overline{QB}$ and the result follows.

If Q, A, and B are not collinear, then consider ΔQAB. The line

l_1 will intersect $\overline{QB}$ at some point, say I. Then, $\Delta BAI \cong \Delta QAI$ by SAS. This means that l_1 is the perpendicular bisector of $\overline{QB}$ and that Q is the reflection of B across l_1.

We conclude that if two triangles (ΔABC and ΔPQR) share a point in common ($A = P$), then there is a reflection r that fixes the common point and moves B to Q. Then, either the new triangles are the same, or there is a second reflection that will take $r(C)$ to $r(R)$.

5.2.3 If two triangles (ΔABC and ΔPQR) share no point in common, then by Theorem 5.6 there is a reflection mapping A to P, and by the previous exercise, we would need at most two more reflections to map $\Delta Pr(B)r(C)$ to ΔPQR.

5.2.4 The two triangles are right triangles. Verify they are congruent using the Pythagorean Theorem.

The sequence of reflections can be carried out by mapping A to D, etc, as was outlined above, but this gets messy. Suggest that they look for simpler reflections to carry out the task.

For example, let r_1 be reflection across $y = -1$. That is $r_1(x, y) = (x, -y-2)$. Then, $r_1(A) = (-3, -4)$, $r_1(B) = (-3, -8)$, and $r_1(C) = (-6, -4)$. Let r_2 be reflection across $x = -1$. That is, $r_2(x, y) = (-x-2, y)$. Then, $r_2(r_1(A)) = (1, -4)$, $r_1(B) = (1, -8)$, and $r_1(C) = (4, -4)$.

5.2.2 Reflections

5.2.5

5.2.6 Let $ABCDE$ be a regular pentagon. Let $\overrightarrow{DF}$ be the angle bisector of angle D. Let G be the midpoint of $\overline{AB}$. Then ΔAED and ΔBCD are congruent by SAS, and so ΔADG and ΔBGD are congruent by SSS. Thus, G must lie on $\overleftrightarrow{DF}$ and $\overleftrightarrow{DF}$ is a perpendicular bisector of $\overline{AB}$. Thus, reflection across $\overleftrightarrow{DF}$ takes A to B.

We can assume F is on $\overline{EC}$. Then by SAS, $\Delta DEF \cong \Delta DCF$ which means $\overleftrightarrow{DF}$ is a perpendicular bisector of $\overline{EC}$ and reflection across $\overleftrightarrow{DF}$ takes E to C.

This type of argument could be used with general n-gons, for n odd.

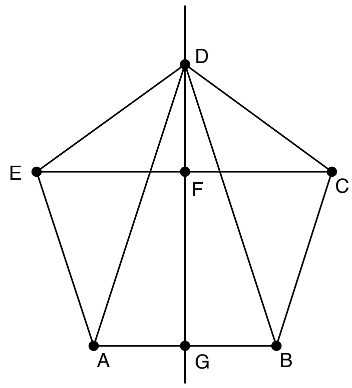


FIGURE 5.1:

5.2.7 Let G be the midpoint of $\overline{AB}$. Then $\Delta AED \cong \Delta BCD$ by SAS and $\Delta AGD \cong \Delta BGD$ by SSS. Thus, $\overleftrightarrow{DG}$ is the perpendicular bisector of $\overline{AB}$, and reflection across $\overleftrightarrow{DG}$ takes A to B. Also, $\overleftrightarrow{DG}$ must bisect the angle at D and by the previous exercise the bisector is a line of reflection. This proof would be easily extendable to regular n-gons, for n odd, by using repeated triangle congruences to show the perpendicular bisector is the angle bisector of the opposite vertex.

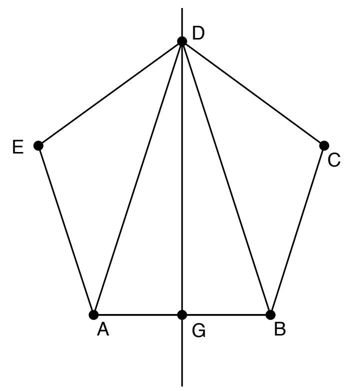


FIGURE 5.2:

5.2.8 In parallelogram $ABCD$ if diagonal $\overline{AC}$ is a line of symmetry then reflection across this line must take B to D. Thus, $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$. Since opposite sides of a parallelogram are congruent, then all sides are congruent.

5.2.9 Suppose that a line of symmetry l for parallelogram $ABCD$

is parallel to side $\overline{AB}$. Then, clearly reflection across l cannot map A to B, as this would imply that l is the perpendicular bisector of $\overline{AB}$.

If reflection mapped A to C, then l would be the perpendicular bisector of a diagonal of the parallelogram. But, since l is parallel to $\overline{AB}$, this would imply that the diagonal must be perpendicular to $\overline{AB}$ as well. A similar argument can be used to show that the other diagonal ($\overline{BD}$) must also be perpendicular to $\overline{AB}$. If this were the case, one of the triangles formed by the diagonals would have angle sum greater than 180 degrees, which is impossible.

Thus, reflection across l must map A to D, and l must be the perpendicular bisector of $\overline{AD}$. Clearly, using the property of parallels, we get that the angles at A and D in the parallelogram are right angles.

5.2.10 Suppose D is on $\overleftrightarrow{AB}$. If $D = Q$, then by definition $CD = r(C)r(D)$. If $D \neq Q$, then by SAS, $\Delta CQD \cong \Delta r(C)QD$, which implies that $CD = r(C)D = r(C)r(D)$.

Suppose D is on the other side of $\overleftrightarrow{AB}$ as C. Again, there are two cases. If D is on $\overleftrightarrow{CQ}$, then the result follows by splitting up $\overline{CD}$ into $\overline{CQ}$ and $\overline{QD}$. On the other hand, if D is not on $\overleftrightarrow{CQ}$, we can look at C and $r(D)$ and argue as we did in the proof of the theorem.

5.2.11 Let r be a reflection across $\overleftrightarrow{AB}$ and let C be a point not on $\overleftrightarrow{AB}$. Then, $r(C)$ is the unique point on the perpendicular dropped to $\overleftrightarrow{AB}$ at a point P on this line such that $CP = r(C)P$, with $r(C) \neq C$. Now, $r(r(C))$ is the unique point on this same perpendicular such that $r(C)P = r(r(C))P$, with $r(r(C)) \neq r(C)$. But since $r(C)P = CP$ and $C \neq r(C)$, then $r(r(C)) = C$. But, then $r \circ r$ fixes three non-collinear points A, B, and C, and so must be the identity.

5.2.12 Let l be a line invariant under a reflection r_m, with m the line of symmetry for r_m. Clearly, $l = m$ is possible, as r_m fixes m. Suppose $l \neq m$ is an invariant line. Let A be a point on l. Then, $r_m(A)$ is again a point on l. For some A on l we have $r_m(A) \neq A$, for is this were not the case, then all points on l would be fixed by r_m, and thus $l = m$. We can assume that $r_m(A) \neq A$. Then, m must be the perpendicular bisector of $\overline{Ar_m(A)}$, which implies that l and m must be perpendicular to one another.

5.2.13 Let A and B be distinct points on l. Then, $r_m \circ r_l \circ r_m(r_m(A)) = r_m(r_l(A)) = r_m(A)$ and likewise, $r_m \circ r_l \circ r_m(r_m(B)) = r_m(B)$. Thus, the line l' through $r_m(A)$ and $r_m(B)$ is fixed by $r_m \circ r_l \circ r_m$ and this triple composition must be equivalent to reflection across l'.

5.2.14 Drop a perpendicular from O to the line intersecting at Q.

By SAS we get the length from O to P is the same as the length from O' to P. Thus, to minimize the total length to V we just minimize the length from O' to P to V. But, the shortest path will be a straight line, so P must be located so that it is on the line through O' and V. Using congruent triangles and vertical angles, we see that the shortest path occurs when the two angles made at P are congruent.

5.3 Translations

5.3.1

5.3.2

5.3.3 Since $(r_2 \circ r_1) \circ (r_1 \circ r_2) = id$, and $(r_1 \circ r_2) \circ (r_2 \circ r_1) = id$, then $r_2 \circ r_1$ is the inverse of $r_1 \circ r_2$. Also, if T has translation vector v, then $T(x, y) = (x, y) + v$. Let S be the translation defined by $S(x, y) = (x, y) - v$. Then, $S \circ T(x, y) = ((x, y) + v) - v = (x, y)$ and $T \circ S((x, y) - v) + v = (x, y)$. Thus, S is the inverse to T.

5.3.4 Let T_1 have translation vector v_1 and T_2 have translation vector v_2. Then, $T_1 \circ T_2(x, y) = T_1((x, y) + v_2) = (x, y) + (v_2 + v_1)$, and $T_1 \circ T_2$ is a translation with translation vector $v_1 + v_2$.

5.3.5 As in the previous exercise, represent T_1 and T_2 in coordinate form.

5.3.6 Choose a coordinate system where l is the x-axis. Then, reflection across l is given by $r_l(x, y) = (x, -y)$. If T is in the same direction as l, then its translation vector can be represented as $v = (v_1, 0)$. Then, $T \circ r_l(x, y) = (x + v_1, -y)$, and $r_l \circ T(x, y) = (x + v_1, -y)$.

If T is not in the direction of l then $v = (v_1, v_2)$ with $v_2 \neq 0$. Then, $T \circ r_l(x, y) = (x + v_1, -y + v_2)$, and $r_l \circ T(x, y) = (x + v_1, -y - v_2)$, so the result will not hold.

5.3.7 Let (x, K) be a point on the line $y = K$. If T is a translation with translation vector $v = (0, -K)$, then, by Exercise 5.3.3, T^{-1} has translation vector of $-v = (0, K)$. Thus, $T^{-1} \circ r_x \circ T(x, K) = T^{-1} \circ r_x(x, 0) = T^{-1}(x, 0) = (x, K)$. So, $T^{-1} \circ r_x \circ T$ fixes the line $y = K$ and so must be the reflection across this line. The coordinate equation for r is given by $T^{-1} \circ r_x \circ T(x, y) = T^{-1} \circ r_x(x, y - K) = T^{-1}(x, -y + K) = (x, -y + 2K)$. So, $r(x, y) = (x, -y + 2K)$.

5.3.8 Let A be a point on one invariant line for T and B a point on another invariant line. Then, the figure $ABT(B)T(A)$ is a parallelogram, and the segments $AT(A)$ and $BT(B)$ are on the two invariant lines. Thus, these lines are parallel. Also, the vector from A to $T(A)$ will be equivalent to the translation vector, and thus all lines are parallel to the displacement vector (considered as a segment).

5.3.9 Let T be a translation with (non-zero) translation vector parallel to a line l. Let m be perpendicular to l at point P. Let n be the perpendicular bisector of $\overline{PT(P)}$, intersecting $\overline{PT(P)}$ at point Q. Then, r_n, reflection about n maps P to $T(P)$. Consider $r_n \circ T$. We have $r_n \circ T(P) = P$. Let $R \neq P$ be another point on m. Then, $PRT(R)T(P)$ is a parallelogram, and thus $\angle PRT(R)$ and $\angle RT(R)T(P)$ are right angles. Let S be the point where n intersects $\overline{RT(R)}$. Then, $\angle RSQ$ is also a right angle. Also, by a congruent triangle argument, we have $\overline{RS} \cong \overline{ST(R)}$, and so n is the perpendicular bisector of $\overline{RT(R)}$ and $r_n \circ T(R) = R$. Since $r_n \circ T$ fixes two points on m we have $r_n \circ T = r_m$, or $T = r_n \circ r_m$.

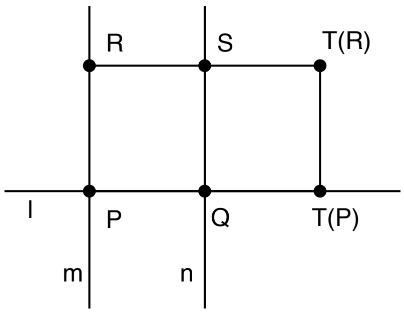


FIGURE 5.3:

5.4 Rotations

5.4.1 First,

$$\begin{aligned} T^{-1} \circ Rot_\phi \circ T(C) &= T^{-1} \circ Rot_\phi \circ T(x, y) \\ &= T^{-1} \circ Rot_\phi(0, 0) \\ &= T^{-1}(0, 0) \\ &= (x, y) \\ &= C \end{aligned}$$

Suppose $T^{-1} \circ Rot_\phi \circ T$ fixed another point P. Then, $Rot_\phi \circ T(P) = T(P)$, which implies that $T(P) = (0, 0)$, or $P = T^{-1}(0, 0) = (x, y) = C$. Thus, $T^{-1} \circ Rot_\phi \circ T$ must be a rotation. What is the angle for this rotation? Consider a line l through C that is parallel to the x-axis. Then, T will map l to the $x - axis$ and Rot_ϕ will map the x-axis to a line m making an angle of ϕ with the x-axis. Then, T^{-1} will preserve this angle, mapping m to a line making an angle of ϕ with l. Thus, the rotation angle for $T^{-1} \circ Rot_\phi \circ T$ is ϕ.

5.4.2 Let P be a point on l. Let T be the translation mapping P to O and let ϕ be the angle measured counter-clockwise from the x-axis to $T(l)$. Then, the map $T^{-1} \circ r_x \circ Rot_{-\phi} \circ T$, where r_x is reflection across the x-axis and $Rot_{-\phi}$ is rotation about the origin of $-\phi$, will result in reflection across l.

5.4.3

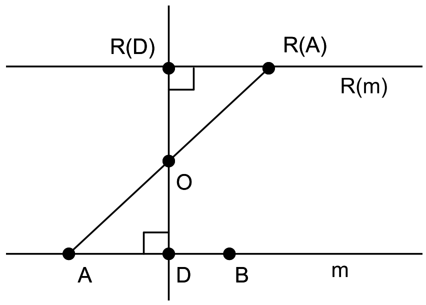
5.4.4 Drop a perpendicular from O to l at A. Then, since $\theta \neq 0$ we have that $R_O(A) \neq A$. If the angle of rotation is less than 180 degrees, then $AOR_O(A)$ forms a triangle, actually an isosceles triangle. If l was invariant, then $R_O(A)$ would lie on l, which implies that both base angles of the triangle would be right angles, which is impossible. If the angle of rotation is 180 degrees, then A and $R_O(A)$ would lie on a line through the origin, and if both points are on l, then l would also pass through the origin.

5.4.5 By the preceding exercise, the invariant line must pass through the center of rotation. Let A be a point on the invariant line. Then, $R_O(A)$ lies on $\overleftrightarrow{OA}$ and $\overline{OA} \cong \overline{OR_O(A)}$. Either A and $R_O(A)$ are on the same side of O or are on opposite sides. If they are on the same side, then $A = R_O(A)$, and the rotation is the identity, which is ruled out. If they are on opposite sides, then the rotation is 180 degrees. If the rotation is 180 degrees, then for every point $A \neq O$ we have that A, O, and $R_O(A)$ are collinear, which means that the line $\overleftrightarrow{OA}$ is invariant.

5.4.6 First, if $R(m)||m$, then m cannot pass through O. Drop a perpendicular from O to m at A, and suppose the rotation angle is less than 180. Then, A, O, and $R(A)$ form a triangle. But, if $R(m)||m$, then one of the angles made by the transversal $AR(A)$ will be greater than or equal to 90, which contradicts the existence of the triangle. Thus, the angle of rotation must be 180.

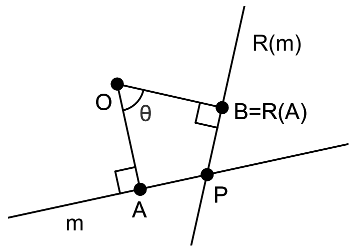
5.4.7 Suppose $R(m)$ is parallel to m. Let O be the center of rotation. If m passes through O, then $R(m)$ also passes through O, which contradicts the lines being parallel. So, we assume m does not pass through O. Let A, B be two points on m and construct $\overline{AR(A)}$. If the angle $\angle BAO$ is a right angle, then $\angle R(B)R(A)R(O)$ is also a right angle, and the rotation angle must be 180 degrees, which is impossible.

So, WLOG we assume that $\angle BAO$ is less than a right angle. Drop a perpendicular from O to m at D. Since R is an isometry , then $\Delta DAO \cong \Delta R(D)R(A)O$ and the vertical angles at O are congruent, which means A, O, and $R(A)$ are collinear. This would mean the rotation angle is 180 degrees, which is impossible.



So, m and $R(m)$ intersect at a point P.

Drop perpendiculars from O to m and $R(m)$ at A and B. Note that $A \neq P$, as if $A = P$ then m and $R(m)$ would coincide. Thus, $OBPA$ is a quadrilateral. Let θ be the rotation angle. Since the sum of the angles in a quadrilateral is 360 degrees, then $\angle BPA$ must have measure $180 - \theta$. But, then the vertical angle at P has measure $180 - m\angle BPA = \theta$.



5.4.8 Let $R = r_l \circ r_m$ be a rotation about the point P where l and m intersect. Then, since $(r_l \circ r_m) \circ (r_m \circ r_l) = id$ and $(r_m \circ r_l) \circ (r_l \circ r_m) = id$, then $R^{-1} = r_m \circ r_l$, and the angle of rotation is the same, but in reverse direction, as the angle is twice the angle between the lines of reflection.

5.4.9 Let R_1 and R_2 be two rotations about P. Let l and m be the lines of reflection for R_1. By Theorem 5.15 we can choose m as a defining line of reflection for R_2 and there is a unique line n such that $R_2 = r_n \circ r_m$. Then, $R_2 \circ R_1 = r_n \circ r_m \circ r_m \circ r_l = r_n \circ r_l$, which is again a rotation about P.

5.4.10 Consider $R^{-1} \circ R'$. This map fixes O and A and thus fixes $\overleftrightarrow{OA}$. So, either $R^{-1} \circ R'$ is a reflection or it is the identity. Since the composition of two rotations about a common point is again a rotation (by the preceding exercise), then $R^{-1} \circ R' = id$ and the result follows.

5.4.11 If $r_1 \circ R = r_2$, then $R = r_1 \circ r_2$. Thus, the lines for r_1 and r_2 must intersect. If they intersected at a point other than the center of rotation for R, then R would fix more than one point, which is impossible.

5.4.12 H is clearly a rotation, by the definition of rotations. The angle of rotation is twice the angle made by the lines of reflection, or twice a right angle, or 180.

5.4.13 Let $m = \overleftrightarrow{AB}$. Then, we can choose lines l and n such that $H_A = r_m \circ r_l$ and $H_B = r_n \circ r_m$. Note that l and n are both perpendicular to m and thus parallel. Then, $H_B \circ H_A = r_n \circ r_l$, which is a translation parallel to l.

5.4.14 Note that $f \circ H_A \circ f^{-1}$ maps $f(A)$ back to itself. If this map fixes any other point P, then $H_A \circ f^{-1}(P) = f^{-1}(P)$, and so $f^{-1}(P) = A$ or $P = f(A)$. Thus, $f \circ H_A \circ f^{-1}$ is a rotation about $f(A)$. Any line through $f(A)$ will get mapped to a line through A by f^{-1}. Then H_A will map this new line to itself, and f will map this half-turned line back to the original line. Thus, by Exercise 5.4.5, $f \circ H_A \circ f^{-1}$ is a half-turn about $f(A)$.

5.5 Project 7 -Quilts and Transformations

This project is another great opportunity for the future teachers in the class to develop similar projects for use in their own teaching. One idea to incorporate into a high school version of the project is to bring into the class the cultural and historical aspects of quilting.

5.5.1

5.5.2

5.5.3 For bilateral symmetry, any reflection line must pass through the center of the quilt pattern. The only patterns which have such symmetry are: 25-Patch Star (horizontal, vertical, 45 degree, and −45 degree lines of symmetry) and Flower Basket (45 degree line of symmetry).

Star Puzzle, Dutch Man's Puzzle, and 25-Patch Star all have rotational symmetry of 90 (and thus 180 and 270) degrees.

Thus, 25-Patch Star is the only pattern with both rotational and bilateral symmetry.

5.5.4 The isometry composed of the two reflections is again a symmetry of the quilt, and will be a rotation, since the lines meet at a point. Since the lines are perpendicular, the rotation angle will be twice 90, or 180 degrees.

5.6 Glide Reflections

5.6.1

5.6.2 Let P'' be the reflection of P across $\overleftrightarrow{QS}$. Then, triangles ΔPQS and $\Delta P''QS$ are congruent, as one is the image of the other under reflection across $\overleftrightarrow{QS}$. Then, the measure of $\angle P'QP''$ is 180 minus twice the measure of $\angle P''QS$, which is equal to 180 minus twice the measure of $\angle PQS$, which is equal to the measure of $\angle QPS$. Thus, by SAS triangles $\Delta P'QP''$ and ΔQPS are congruent. Thus, $\angle PQS \cong \angle QP'P''$ and lines $\overleftrightarrow{P'P''}$ and $\overleftrightarrow{QS}$ are parallel. This implies that $T_{SQ}(P'') = P'$, and so $G(P) = T_{SQ}(r_{SQ}(P)) = T_{SQ}(P'') = P'$.

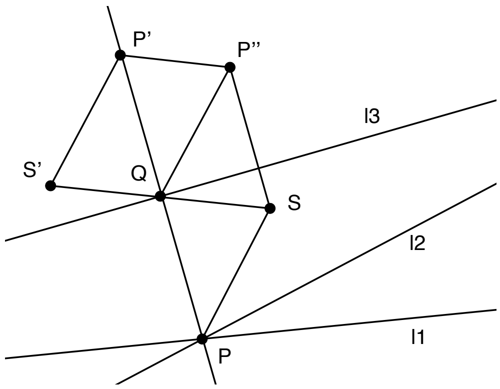


FIGURE 5.4:

5.6.3 Suppose m is invariant. Then, the glide reflection can be written as $G = T_{AB} \circ r_l = r_l \circ T_{AB}$. If $G(G(m)) = m$, then $(T_{AB} \circ r_l) \circ (r_l \circ T_{AB})(m) = T_{2AB}(m) = m$. So, m must be parallel or equal to l, if it is invariant under T_{2AB}. Suppose m is parallel to l. Then, $T_{AB}(m) = m$. So, $G(m) = r_l \circ T_{AB}(m) = r_l(m)$. But, reflection of a line m that is parallel to l cannot be equal to m. Thus, the only line invariant under the glide reflection is l itself.

5.6.4 Choose the coordinate system so that the line of reflection l is the x -axis. Then, if $G = T_v \circ r_l$, we have $G(x, y) = (x, -y) + (v_1, 0)$. If $G(x, y) = (x, y)$ then $x + v_1 = x$, or $v_1 = 0$. Thus, $T_v = id$ and $G = r_l$.

5.6.5 The glide reflection can be written as $G = T_{AB} \circ r_l = r_l \circ T_{AB}$. So, $G \circ G = (T_{AB} \circ r_l) \circ (r_l \circ T_{AB}) = T_{2AB}$.

5.6.6 The set of glide reflections does not form a group, as by the previous exercise a glide reflection squared is a pure translation.

5.6.7 The set does not include the identity element.

5.6.8 By Exercise 5.4.12 the composition of two half-turns about different points is a translation.

5.6.9 The identity (rotation angle of 0) is in the set. The composition of two rotations about the same point is again a rotation by Exercise 5.4.8. The inverse to a rotation is another rotation about the same point by Exercise 5.4.7. Since rotations are functions, associativity is automatic.

5.6.10 The set of translations contains the identity (translation vector $= (0, 0)$). The composition of two translations is again a translation by Exercise 5.3.4. The inverse of a translation is again a translation by Exercise 5.3.3. Since translations are functions, associativity is automatic.

5.6.11 A discussion and diagram would suffice for this exercise.

5.6.12 Every isometry can be expressed as the product of 1, 2, or 3 reflections. In the chapter we have shown that rotations and translations cannot be equivalent to glides and reflections. (Just consider the fixed points) Thus, a product of two reflections cannot be expressible as a product of 1 reflection or 3 reflections (see Theorem 5.19) So an isometry cannot be both direct and opposite.

5.6.13 Rotations and translations are defined as the product of two reflections. The identity can also be written as the product of a reflection with itself. These are then both direct and even. Glides and reflections can be written as the product of three or one reflections. These are then odd and opposite.

5.7 Structure and Representation of Isometries

This section is a somewhat abstract digression into ways of representing transformations and of understanding their structure as algebraic elements of a group. An important theme of the section is the usefulness of the matrix form of an isometry, both from a theoretical viewpoint (classification), as well as a practical viewpoint (animation in computer graphics).

A very interesting digression could be made here into the ubiquity of matrix methods (and thus transformations) in the field of computer animation. There are many excellent textbooks in computer graphics that one could use as reference for this purpose. For example, the book by F.S. Hill listed in the bibliography of the text is a very accessible introduction to the subject.

5.7.1 Let $G_1 = T_{v_1} \circ r_{l_1}$ and $G_2 = T_{v_2} \circ r_{l_2}$ be two glide reflections. If $G_1 \circ G_2$ is a translation, say T_v, then, since $G_1 \circ G_2 = T_v = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2})$, then $T_{v-v_1-v_2} = r_{l_1} \circ r_{l_2}$ and thus $l_1 || l_2$.

On the other hand, if the lines are parallel, then $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ T_v \circ T_{v_2}$, for some vector v.

If the lines intersect, then the composition of r_{l_1} with r_{l_2} will be a rotation, say R, and $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ R \circ T_{v_2}$. This last composition yields a rotation, by Theorem 5.20

5.7.2 $f \circ H_O \circ f^{-1}$ fixes $f(O)$, and by the table we know that this composition must be a rotation or a translation. Thus, it must be a rotation. Also, if l is an invariant line for H_O (i.e., any line through O), then $f \circ H_O \circ f^{-1}$ will fix $f(l)$, and thus $f \circ H_O \circ f^{-1}$ must be a half-turn.

5.7.3 First, $f \circ r_m \circ f^{-1}(f(m)) = f(m)$, so $f(m)$ is a fixed line for $f \circ r_m \circ f^{-1}$. Also, $(f \circ r_m \circ f^{-1})^2 = f \circ r_m \circ f^{-1} \circ f \circ r_m \circ f^{-1} = id$. Thus, $f \circ r_m \circ f^{-1}$, which must be a reflection or glide reflection from looking at Table 5.3, is a reflection. Since it fixes $f(m)$ it must be reflection across $f(m)$.

5.7.4 Let $T_{AB} = r_m \circ r_l$ with l and m parallel. Then, the previous exercise implies $f \circ T_{AB} \circ f^{-1} = f \circ r_m \circ f^{-1} \circ f \circ r_l \circ f^{-1} = r_{f(m)} \circ r_{f(l)}$. Since f preserves parallels, we know that $f(l)$ and $f(m)$ parallel, and so $f \circ T_{AB} \circ f^{-1}$ will be a translation defined by the vector between these lines. But, since f preserves distance and angle, then this vector is $\overrightarrow{f(A)f(B)}$.

5.7.5 Using the previous exercises we have $f \circ r_m \circ T_{AB} \circ f^{-1} = f \circ r_m \circ f^{-1} \circ f \circ T_{AB} \circ f^{-1} = r_{f(m)} \circ T_{f(A)f(B)}$.

5.7.6 First, $f \circ R_{A,\alpha} \circ f^{-1}(f(A)) = f \circ R_{A,\alpha}(A) = f(A)$. Thus, $f(A)$ is a fixed point of $f \circ R_{A,\alpha} \circ f^{-1}$. Since conjugation of a rotation must yield a translation or a rotation, and since translations have no fixed points (unless they are the identity), we can assume that $f \circ R_{A,\alpha} \circ f^{-1}$ is a rotation about $f(A)$ of some angle β.

Let $R_{A,\alpha} = r_l \circ r_m$. Then, $f \circ R_{A,\alpha} \circ f^{-1} = f \circ r_l \circ f^{-1} \circ f \circ r_m \circ f^{-1}$. By Exercise 5.7.3, we then have $f \circ R_{A,\alpha} \circ f^{-1} = r_{f(l)} \circ r_{f(m)}$. Since f preserves angles we have that the rotation angle defined by $f(l)$ and $f(m)$ is the same as that defined by l and m, up to orientation. Thus, if f is a rotation or translation, we have the new rotation angle is α (this can also be seen by looking at Table 5.3). If f is a reflection or glide reflection, the angle orientation will switch, and the new angle will be $-\alpha$.

5.7.7 Rotation of (x, y) by an angle ϕ yields $(x\, cos(\phi)\, -$

$y\ sin(\phi), x\ sin(\phi) + y\ cos(\phi))$. Multiplying $x + iy$ by $cos(\phi) + i\ sin(\phi)$ yields the same point. Translation by $v = (v_1, v_2)$ yields $(x+v_1, y+v_2)$. Adding v_1+iv_2 to $x+iy$ yields the same result. Finally, reflection across x is given by $r_x(x, y) = (x, -y)$. Complex conjugation sends $x + iy$ to $x - iy$. Clearly, this has the same effect.

5.7.8 $R_{a,\alpha}(z) = a + (e^{i\alpha}(z - a))$. Then, to find the fixed point, we set $R_{a,\alpha} \circ R_\beta(z) = z$, or $a + (e^{i\alpha}((e^{i\beta} z - a)) = z$, and solve for z.

5.7.9 $T_v \circ R_\beta(z) = (e^{i\beta} z) + v$. To find the fixed point set $(e^{i\beta} z) + v = z$ and solve for z.

5.7.10 Let $A = (2, 3)$ and $B = (-4, 0)$. Then the motions we want are T_A, $R_{A,45} = T_A \circ R_4 5 \circ T_{-A}$ and T_B. T_A is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

T_{-A} is given by

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

R_{45} is given by

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $R_{A,45} = T_A \circ R_{45} \circ T_{-A}$ is

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{-2\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + 2 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{-2\sqrt{2}}{2} + \frac{-3\sqrt{2}}{2} + 3 \\ 0 & 0 & 1 \end{bmatrix}$$

T_B is given by

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These would be placed in the pipeline in the following order: T_A, then $R_{A,45}$, then T_B.

5.8 Project 8 - Constructing Compositions

The purpose of this lab is to make concrete the somewhat abstract notion of composition of isometries. In particular, by carrying out the constructions of the lab, students see how the conditions on compositions of rotations found in Table 5.3 arise naturally.

If the students have difficulty getting started with the first proof, help them out by discussing in general how one can write a rotation as the composition of two reflections through the center of rotation, and point out that the choice of reflection lines is not important – one can choose *any* two lines as long as they make the right angle, namely half the desired rotation angle.

5.8.1 A rotation can be expressed as the composition of two reflections about lines through the center of rotation, as long as the reflection lines make an angle of half the reflection angle. Since m and n are bisectors of the rotation angles, then, $R_{A,\angle EAB} = r_{\overleftrightarrow{AB}} \circ r_n$ and $R_{B,\angle ABE} = r_m \circ r_{\overleftrightarrow{AB}}$, taking into the account the orientation of the rotation angles.

5.8.2 From the previous exercise we know that

$$\begin{aligned} R_{B,\angle ABE} \circ R_{A,\angle EAB} &= r_m \circ r_{\overleftrightarrow{AB}} \circ r_{\overleftrightarrow{AB}} \circ r_n \\ &= r_m \circ r_n \end{aligned}$$

Thus, since $R_{B,\angle ABE} \circ R_{A,\angle EAB} = r_m \circ r_n$ we have that this composition is a rotation about O.

5.8.3 Theorem 5.20

5.8.4 As in the first exercise of the lab, we can write $R_{A,\angle CAB} = r_{AL} \circ r_{AB}$. By the parallel property of lines, we know that $\angle MBN \cong \angle BAL$. Thus, the angle made by the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AL}$ is the same as the angle made by the lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{BN}$. This implies that $R_{B,\angle BAC} = r_{AB} \circ r_{BN}$, taking into the account the orientation of the rotation angles.

Thus,

$$\begin{aligned} R_{A,\angle CAB} \circ R_{B,\angle BAC} &= r_{AL} \circ r_{AB} \circ r_{AB} \circ r_{BN} \\ &= r_{AL} \circ r_{BN} \end{aligned}$$

Since $\overleftrightarrow{AL}$ is parallel to $\overleftrightarrow{BN}$, then the composition of the rotations is a translation by twice the vector between the two reflection lines.

CHAPTER 6

Symmetry

This chapter is quite algebraic in nature—focusing on the different discrete symmetry groups that arise for frieze patterns and wallpaper patterns. The material is, while interesting, not critical to the subsequent chapters, and may be skipped if so desired.

SOLUTIONS TO EXERCISES IN CHAPTER 6

6.1 Finite Plane Symmetry Groups

6.1.1

6.1.2 The symmetry group is the dihedral group of order 4. (4 rotations generated by a rotation of 90 degrees, and reflections generated by a reflection across a perpendicular bisector of a side) This gives 8 symmetries. There are no more, since if we label the vertices and fix a position for a vertex to occupy, we have 4 choices for the vertex to be placed in that position and only two choices for the rest of the vertices. Thus, a maximum of eight symmetries possible.

6.1.3 The dihedral group of order 5. This has order 10, and there can be at most 10 symmetries. (Use an argument like that used in the preceding exercise)

6.1.4 Symmetry group has a maximum of $2n$ symmetries, using an argument like that used in Exercise 6.1.2.

6.1.5 By the previous exercise there are at most $2n$ symmetries. Also, by the work done in Section 5.4 we know there are n rotations, generated by a rotation of $\frac{360}{n}$, that will be symmetries. Let r be a reflection across a perpendicular bisector of a side. This will be a reflection, as will all n compositions of this reflection with the n rotations. This gives $2n$ different symmetries

6.1.6 A dihedral group of order n has n rotations. Let $R_{a,\alpha}$ be the rotation of smallest non-zero angle. Then, $R_{a,\alpha}$ generates all rotations. Let r_l be one of the reflections in the group. Then, by Theorem 6.7 the group has exactly n reflections, all of which can be represented as $r_l \circ R_{a,i\alpha}$. Let $r_m = r_l \circ R_{a,\alpha}$. Then, $r_l \circ r_m = R_{a,\alpha}$, and the two reflections r_l and r_m generate all elements of the group.

6.1.7 The number of symmetries is $2n$. The only symmetries that fix a side are the identity and a reflection across the perpendicular bisector of that side. The side can move to n different sides. Thus, the stated product is $2n$ as claimed.

6.1.6 There would be $2n$ symmetries fixing a face (regular n-gon). If there are f faces, then a total of $2nf$ symmetries. A tetrahedron has 4 faces, all triangles, and thus has $2 \cdot 3 \cdot 4 = 24$ symmetries. Similar reasoning gives 48 symmetries for a cube.

6.2 Frieze Groups

6.2.1 Since $\gamma^2 = \tau$, then $< \tau, \gamma, H >$ is contained in $< \gamma, H >$. Also, it is clear that $< \gamma, H >$ is contained in $< \tau, \gamma, H >$. Thus, $< \tau, \gamma, H >=< \gamma, H >$.

6.2.2 Choose r_u so that it intersects m at the center of H. Then, $r_u \circ H$ will fix this intersection point, and thus cannot be a glide reflection. It must then be a reflection. Since H can be represented as the composition of r_u and r_m, then, $r_u \circ H = r_u \circ r_u \circ r_m = r_m$ and we have $< \tau, r_m, H >$ is contained in $< \tau, r_u, H >$. Since $H^{-1} = H$ we have $r_u = r_m \circ H$ and $< \tau, r_u, H >$ is contained in $< \tau, r_m, H >$. The result follows.

6.2.3 Let r_u and $r_{u'}$ be two reflections across lines perpendicular to m. Then, the composition $r_u \circ r_{u'}$ must be a translation, as these lines will be parallel. Thus, $r_u \circ r_{u'} = T^k$ for some k, and $r_{u'} = r_u \circ T^k$.

6.2.4 Let g and g' be two glide reflections. Then, both are in the direction of the minimal translation vector v. Thus, if m is the midline, then $g^{-1} \circ g' = T_{-jv} \circ r_m \circ r_m \circ T_{kv} = T_{(k-j)v}$, and $g' = g \circ T_{(k-j)v}$.

Also, $g^2 = T_{2jv}$. But, any translation is some multiple of the minimal one, say T^k. Then, j is one-half of k and either this is an integer, or half of an integer.

6.2.5 Consider g^2. This must be a translation, so $g^2 = T_{kv}$ for some k where T_v is the fundamental translation. Then, $g = T_{\frac{k}{2}v} \circ r_m$, where m is the midline. Suppose $\frac{k}{2}$ is an integer, say $\frac{k}{2} = j$. Then, since $T_{(v-jv)}$

is in the group, we have $T_{(v-jv)} \circ g = T_{(v-jv)} \circ T_{\frac{k}{2}v} \circ r_m = T_v \circ r_m$ is in the group.

Otherwise, $\frac{k}{2} = j + \frac{1}{2}$ for some integer j. We can find T_{-jv} in the group such that $T_{-jv} \circ g = T_{\frac{v}{2}} \circ r_m$ is in the group.

6.2.6 From Table 5.3 we know that $H_B \circ H_A$ will be a translation, and if $H_B \circ H_A(A) = H_B(A) = C$, then the translation vector must be $\vec{AC}$. But, the length of $\overline{AC}$ is twice that of $\overline{AB}$. So, we get that $2\vec{AB} = k'v$ for some k'. Now, either k' is even or it is odd. The result follows.

6.2.7 The composition $r_v \circ r_u$ must be a translation. Also, if $r_v \circ r_u(A) = r_v(A) = C$, then the translation vector must be $\vec{AC}$. But, the length of $\overline{AC}$ is twice that of $\overline{AB}$. So, we get that $2\vec{AB} = k'v$ for some k'. Now, either k' is even or it is odd. The result follows.

6.2.8 From Table 5.3 we know that either $\tau \circ r_u$ or $r_u \circ \tau$ must be a reflection, and since neither fixes points on m, these compositions must yield reflections perpendicular to m. Thus, any possible composition of products of τ and r_u can be reduced to a simple product of a translation (τ^k) with zero or more reflections perpendicular to m. Reducing again yields either a simple translation (all reflections cancel out) or a single reflection perpendicular to m. Thus, the subgroup generated by τ and r_u cannot contain r_m or H or γ and none of $< \tau, r_m >$ or $< \tau, H >$ or $< \tau, r_m >$ can be subgroups of $< \tau, r_u >$.

6.2.9 From Table 5.3 we know that $\tau \circ H$ or $H \circ \tau$ is either a translation or a rotation, so it must be either τ^k for some k or H_A for A on m. Thus, any composition of products of τ and H can be reduced ultimately to a simple translation or half-turn, or to some $\tau^j \circ H_B$ or $H_B \circ \tau_j$, which are both half-turns. Thus, the subgroup generated by τ and H cannot contain r_m or r_u or γ and none of $< \tau, r_m >$ or $< \tau, r_u >$ or $< \tau, r_m >$ can be subgroups of $< \tau, H >$.

6.2.10 From Table 5.3 we know that $\tau \circ \gamma$ or $\gamma \circ \tau$ is either a reflection or a glide reflection. Since the glide translation vector is in the same direction as the translation, then we see by the table that either composition must be a glide. Arguing as we did in the previous two exercises, it is clear that any element of $< \tau, \gamma >$ must be either a translation or a glide reflection, and the subgroup result follows.

6.2.11 The compositions $\tau^k \circ r_m$ or $r_m \circ \tau^k$ generate glide reflections with glide vectors kv. The composition of τ with such glide reflections generates other glide reflections with glide vectors $(k+j)v$. The composition of r_m with a glide in the direction of m will generate a translation. Thus, compositions of the three types of symmetries—glides, r_m, and

τ^k—will only generate symmetries within those types. Thus, $< \tau, \gamma >$ cannot be a subgroup of $< \tau, r_m >$, since γ has translation vector of $\frac{v}{2}$ which cannot be generated in $< \tau, r_m >$. Also, neither $< \tau, r_u >$ nor $< \tau, H >$ can be subgroups of $< \tau, r_m >$.

6.2.12 By the previous exercises, the only subgroup possible for each of the 2-generator groups is $< \tau >$. Also, a group that does not include γ as a generator cannot have a subgroup containing γ. (See the explanation for the last exercise for why.) Thus, $< \tau, \gamma, H >$ cannot be a subgroup of $< \tau, H, r_m >$.

Also, in the group $< \tau, H, r_m >$, we can generate a glide by $\tau \circ r_m$ and this glide (plus all others in the group) will have translation vector of kv. But, in $< \tau, \gamma, H >$, all glides will have translation vector of $kv + \frac{v}{2}$. Thus, $< \tau, H, r_m >$ cannot be a subgroup of $< \tau, \gamma, H >$, which essentially means that r_m cannot be generated in $< \tau, \gamma, H >$.

We conclude that the subgroup structure must look like the following diagram:

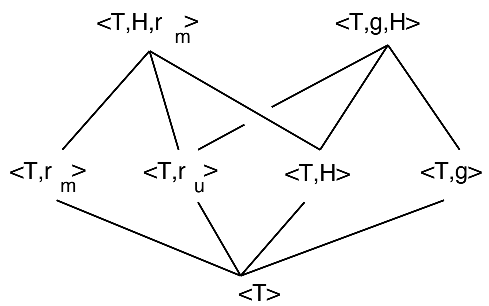


FIGURE 6.1:

6.2.13 First Row: $< \tau >$, $< \tau, \gamma >$. Second Row: $< \tau, \gamma, H >$, $< \tau, r_u >$. Third Row: $< \tau, r_m, H >$, $< \tau, H >$. Last Row: $< \tau, r_m >$.

6.2.14

6.3 Wallpaper Groups

6.3.1 The first is rectangular, the second rhomboidal, and the third is square.

6.3.2 The transformation f is the composition of a translation $(T(x, y) = (x, y) + (0, 1))$ with a reflection $r(x, y) = (-x, y)$ across the y-axis. Since the translation is parallel to the y-axis, the composition is a glide reflection. Thus, f^2 will be a translation by the vector $(0, 2)$. The second translation described in the exercise has translation vector $(1, 1)$. If we consider the triangle defined by these two vectors, we can choose $v = (1, 1)$ and $w = (-1, 1)$ and we get a square lattice on these vectors.

6.3.3 The translation determined by f^2 will be in the same direction as T, so we do not find two independent directions of translation.

6.3.4 H cannot contain glide reflections or translations with nontrivial translation vectors. Thus, the only possible symmetries in H are rotations and reflections. If H has a rotation, and it must be of order 2, then the rotation must be a half-turn. So, H can contain at most one rotation, and so must contain at least two different reflections fixing the given lattice point. Then, the composition of these reflections must be a rotation, and thus a half-turn. Thus, $H = \{id, H, r_l, r_m\}$ where l and m are perpendicular.

6.3.5 The lattice for G will be invariant under rotations about points of the lattice by a fixed angle. By the previous problem, these rotations must be half-turns. By Theorem 6.18 the lattice must be Rectangular, Centered Rectangular, or Square.

6.3.6 By the previous exercise, the lattice for $p2mm$ must be Rectangular, Centered Rectangular, or Square. It cannot be Centered Rectangular. If it was Square, then it would allow 90 degree rotations, which is not possible.

6.3.7 Let C be the midpoint of the vector $v = \vec{AB}$, where v is one of the translation vectors for G. Let m_1 be a line perpendicular to $\overleftrightarrow{AB}$ at A. Then, $T_v = r_{m_1} \circ r_{m_1'}$ where m_1' is a line perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$. But since r_{m_1} is in G, then $r_{m_1} \circ T_v = r_{m_1'}$ is in G. Likewise, we could find a line m_2' perpendicular to the other translation vector $w = \vec{AC}$ at its midpoint, yielding another reflection $r_{m_2'}$. The formulas for these two reflections are $r_{m_1'} = r_{m_1} \circ T_v$ and $r_{m_2'} = r_{m_2} \circ T_w$.

6.3.8 Since the translation vectors for $p2m'm'$ and $p2mm$ are assumed the same, we need only worry about the reflections and half-turns. Also, since the two reflections will generate half-turns, we need only be concerned about the reflections.Using the same notation as the previous exercise, since $r_{m_1'} = r_{m_1} \circ T_v$ and $r_{m_2'} = r_{m_2} \circ T_w$, then $p2m'm' \subseteq p2mm$. Since $r_{m_1} = r_{m_1'} \circ T_v$ and $r_{m_2} = r_{m_2'} \circ T_w$, then $p2mm \subseteq p2m'm'$.

6.3.9 In the preceding exercise, we saw that the group of symmetries can be generated from reflections half-way along the translation vectors. Thus, if we reflect the shaded region, we must get another part of the pattern. Thus, three reflections of the shaded area will fill up the rectangle determined by v and w and the rest of the pattern will be generated by translation.

6.3.10 The point A can be represented as $sv+tw$. Then, $T_{-sv-tw} \circ R_{A,\alpha} \circ T_{sv+tw}$ is again an element of G. But, by Exercise 5.7.6, this composition is the isometry $R_{T_{-sv-tw}(A),\alpha} = R_{O,\alpha}$. Since $R_{O,\alpha}(v)$ must be a lattice point, we have $R_{O,\alpha}(v) = a_{11}v + a_{12}w$. Also, $R_{O,\alpha}(w) = a_{21}v + a_{22}w$. Thus, all a_{ij} must be integers. If we write any vector as a combination of v and w, say $u = cv + dw$, then $R_{O,\alpha}(u) = cR_{O,\alpha}(v) + dR_{O,\alpha}(w) = (a_{11}c + a_{21}d)v + (a_{12}cw + a_{22})dw$. Thus, the matrix for $R_{O,\alpha}$ with respect to the basis spanned by v and w is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If the trace of $R_{O,\alpha}$ is invariant under a change of basis, then the trace of $R_{O,\alpha}$ must be an integer. Thus, $2cos(\frac{360}{n})$ must be an integer. So, $|cos(\frac{360}{n})| = 1$ or $|cos(\frac{360}{n})| = \frac{1}{2}$. The result follows immediately.

6.3.11 If $A = lv + mw$ and $B = sv + tw$, then $0 \leq s, t \leq 1$. The length between A and B is the length of the vector $\vec{B - A} = (s-l)v + (t-m)w$. This length squared is the dot product of $\vec{B - A}$ with itself, i.e., $(s-l)^2(v \bullet v) + 2(s-l)(t-m)(v \bullet w) + (t-m)^2(w \bullet w)$. If $v \bullet w > 0$, then this will be maximal when both $(s-l)$ and $(t-m)$ are maximal. This occurs when $(s-l) = 1$ and $(t-m) = 1$, which holds only if $s = 1 = t$ and $l = m = 0$. If $v \bullet w < 0$, we need $(s-l)$ to be as negative as possible, and $(t-m)$ to be as positive as possible (or vice-versa). In either case, we get values of 0 or 1 for s, t, l, and m.

6.3.12 The angle between $-v$ and the double rotation of v by $r_{a,72}$ will be $180 - 144 = 56 < 60$, and this angle is disallowed by Theorem 6.19.

6.3.13

6.5 Project 9 - Constructing Tessellations

This project on tiling can be an ideal project for future high school teachers. Supplementary material on planar tiling, like that found in *Tilings and Patterns*, by Grunbaum and Shephard, can be introduced during class for those students who are especially interesting in the mathematics of planar tiling.

Additionally, supplementary material on the art of tiling can be introduced by an expanded look at the work of M.C. Escher. A good resource for this would be Doris Schattschnieder's book *M. C. Escher, Visions of Symmetry*.

6.5.1 The symmetry group is $p4g$.

6.5.1

CHAPTER 7

Hyperbolic Geometry

The discovery of non-Euclidean geometry is one of the most important events in the history of mathematics. Much more time could be spent on telling this story, and, in particular, the history of the colorful figures who co-discovered this geometry. The book by Boyer and Merzbach and the University of St. Andrews web site, both listed in the bibliography of the text, are excellent references for a deeper look at this history. Greenberg's book is also an excellent reference.

SOLUTIONS TO EXERCISES IN CHAPTER 7

In section 7.2 we see for the first time the relevance of our earlier discussion of models in Chapter 1. The change of axioms in Chapter 7 (replacing Euclid's fifth postulate with the hyperbolic parallel postulate) requires a change of models. This is a good place to reinforce the idea that in an axiomatic system, it is not important what the terms actually mean; the only thing that matters is the relationships between the terms.

We introduce two different models at this point to help students recognize the abstraction that lies behind the concrete expression of points and lines in theses models.

7.2.2 Mini-Project: The Klein Model

Students should be encouraged to construct on paper the lines, etc, of the Klein model.

7.2.1 Use the properties of Euclidean segments.

7.2.2 Students should present an argument similar to that for circles in the Poincaré Model.

7.2.3 The special case is where the lines intersect at a boundary point of the Klein disk. Otherwise, use the line connecting the poles of the two parallels to construct a common perpendicular.

7.2.4 If $m\angle QPT = 90$,then $\overleftrightarrow{QP}$ will be a common perpendicular to $\overline{AB}$ and $\overline{BC}$. But this implies that $\overleftrightarrow{QP}$ passes through the pole of $\overline{BC}$. We know that the line through the pole of $\overline{AB}$ and B also passes through the pole of $\overline{BC}$. We would then have two different (Euclidean) lines passing between the two poles, which is impossible.

7.3 Basic Results in Hyperbolic Geometry

A danger for students in this section is not seeing the distinction between points at infinity and regular points. Omega triangles share some properties of regular triangles, like congruence theorems and Pasch-like properties, but are not regular triangles—thus necessitating the theorems found in this section.

7.3.1 Use the interpretation of limiting parallels in the Klein model.

7.3.2 Use the interpretation of limiting parallels in the Klein model.

7.3.3 First, if m is a limiting parallel to l through a point P, then $r_l(m)$ cannot intersect l, as if it did, then $r_l^2(m) = m$ would also intersect l. Now, drop a perpendicular from $r_l(P)$ to l at Q, and consider the angle made by Q, $r_l(P)$, and the omega point of $r_l(m)$. If there were another limiting parallel (n) to l through $r_l(P)$ that lies within this angle, then by reflecting back by r_l we would get a limiting parallel $r_l(n)$ that lies within the angle made by Q, P and the omega point of l, which is impossible. Thus, $r_l(m)$ must be limiting parallel to l and reflection maps omega points to omega points, as r_l maps limiting parallels to l to other limiting parallels. Also, it must fix the omega point, as it maps limiting parallels on one side of the perpendicular dropped to l to limiting parallels on that same side.

7.3.4 If r_l fixes an omega point for a line $l' \neq l$, then by the properties of reflection in Chapter 5 we know that l' must be perpendicular to l at some point Q. Let P be a point on l that is not on l'. Then, the angle made by Q, P and the omega point of l' determine the left- and right-limiting parallels for l' through P. But, reflection across l switches left- and right-limiting parallels, as it takes points on one side of l to points on the other side. Thus, r_l does not fix the omega points of l'.

7.3.5 Let P be the center of rotation and let l be a line through P with the given omega point Ω. (Such a line must exist as Ω must correspond to a limiting parallel line m, and there is always a limiting parallel to m through a given point P) Then, we can write $R = r_n \circ r_l$ for another line n passing through P. But, since r_l fixes Ω, and R does as well, then, r_n must fix Ω. But, if n and l are not coincident, then n is not limiting parallel to l and thus cannot have the same omega points as l. By the previous exercise, r_n could not fix Ω. Thus, it must be the case that n and l are coincident and r is the identity.

7.3.6 Use the Exterior Angle Theorem for Omega Triangles.

7.3.7 Let $PQ\Omega$ be an omega triangle and let R be a point interior to the triangle. Drop a perpendicular from Q to $\overleftrightarrow{P\Omega}$ at S. Then, either R is interior to triangle QPS, or it is on $\overleftrightarrow{QS}$, or it is interior to $\angle QS\Omega$. If it is interior to ΔQPS it intersects $\overleftrightarrow{P\Omega}$ by Pasch's axiom for triangles. If it is on $\overleftrightarrow{QS}$ it obviously intersects $\overleftrightarrow{P\Omega}$. If it is interior to $\angle QS\Omega$, it intersects $\overleftrightarrow{P\Omega}$ by the definition of limiting parallels.

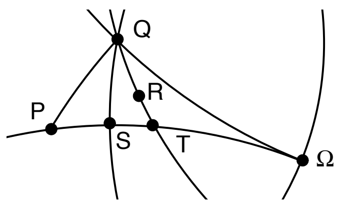


FIGURE 7.1:

7.3.8 Let l be a line passing through R, but not coincident with $\overleftrightarrow{P\Omega}$. If l passes through ΔPQR, but does not intersect P or Q, then by Paschs's axiom for triangles, it must intersect $\overline{PQ}$ or $\overline{PR}$. It cannot intersect $\overline{PR}$ (l and $\overleftrightarrow{P\Omega}$ do not coincide), and so intersects $\overline{PQ}$. If l does not pass through ΔPQR, it must enter Omega triangle $QR\Omega$ and by Theorem 7.5 it must intersect $\overline{Q\Omega}$.

7.3.9 Let l be the line passing through R. Then, either l passes within Omega triangle $PR\Omega$ or it passes within $QR\Omega$. In either case, we know by Theorem 7.5 that l must intersect the opposite side, i.e. it must intersect $\overline{P\Omega}$ or $\overline{Q\Omega}$.

7.3.10 Use the hint to contradict Exercise 7.3.6

7.3.11 Suppose we had another segment $\overline{P'Q'}$ with $\overline{P'Q'} \cong \overline{PQ}$ and let l' be a perpendicular to $\overline{P'Q'}$ at Q'. Let $\overleftrightarrow{P'R'}$ be a limiting parallel to l' at P'. Then, by Theorem 7.8, we know that $\angle QPR \cong \angle Q'P'R'$ and thus, the definition of this angle only depends on h, the length of $\overline{PQ}$.

7.3.12 Let $\overline{PQ}$ have length h' and let R be a point on $\overline{PQ}$ with length $h < h'$. Then, in Omega triangle $QR\Omega$, the angle $\angle QR\Omega = a(h)$ is greater than $\angle QP\Omega = a(h')$ by the Exterior Angle Theorem for Omega Triangles.

7.3.13 Suppose $a(h) = a(h')$ with $h \neq h'$. We can assume that $h < h'$. But, then the previous exercise would imply that $a(h) > a(h')$. Thus, if $a(h) = a(h')$ then $h = h'$.

7.3.14 Given any segment we can construct the angle $a(h)$ as in the previous problem. This angle has a $1-1$ correspondence with the length of the segment. Thus, since angles are absolutely determined, then length must also be absolute, i.e. for a segment, there can be only one length corresponding to the angle constructed from that segment.

7.4 Project 10 - The Saccheri Quadrilateral

Students may experience a flip of orientation for their construction of a Saccheri quadrilateral when moving the quad about the screen. The construction depends on the orientation of the intersections of circles and these may switch as the quad is moved. A construction of the Saccheri quad that dose not have this unfortunate behavior was searched for unsuccessfully by the author. A nice challenge problem for best students in the class would be to ask if they can come up with a better construction. If they can, the author would love to hear about it!

7.4.1 Show that ΔADB and ΔBCA are congruent, and then show that ΔADC and ΔBDC are congruent.

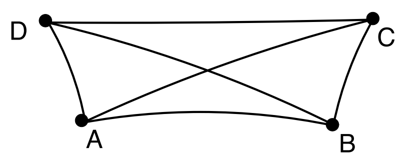


FIGURE 7.2:

7.4.2

- Why? (1): Theorem 7.4.
- Why? (2): Show that the line joining the midpoint of the base and summit segments is perpendicular to both. (The proof is actually in the paragraph preceding Theorem 7.9)

- Why? (3): Since $\overline{AD} \cong \overline{CB}$ and the angles at A and B are right angles, then by Theorem 7.8 the omega triangles are congruent.
- Why? (4): Exterior Angle Theorem for Omega Triangles.
- Why? (5): $m\angle ADC = m\angle AD\Omega + m\angle CD\Omega < m\angle BC\Omega + m\angle EC\Omega = m\angle BCE$. Also, $m\angle BCE + m\angle DCB = 180 > m\angle ADC + m\angle DCB = 2m\angle ADC$.

7.5 Lambert Quadrilaterals and Triangles

7.5.1 Referring to Figure 7.6, we know ΔACB and ΔACE are congruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FCA$, and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBCD and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{FE}$ and the angle at E is a right angle.

7.5.2 Divide the quadrilateral into two triangles and use Theorem 7.14.

7.5.3 Create two Lambert quadrilaterals from the Saccheri quadrilateral, and then use Theorem 7.13.

7.5.4 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ are as shown. If $HE > AD$, then we can find points I on $\overline{HE}$ and J on $\overline{GF}$ such that $HI > AD$ and $GJ > BC$. A triangle congruence argument shows that $IJGH$ is a Saccheri quadrilateral congruent to $ABCD$ and thus $EFJI$ is a rectangle.

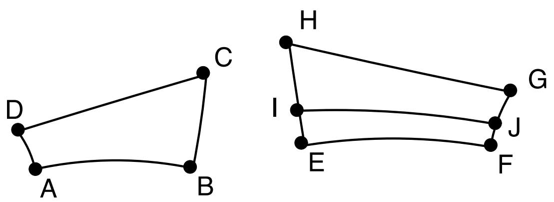


FIGURE 7.3:

7.5.5 Since the angle at O is acute, then OAA' and OBB' are triangles. Also, since $OA < OB$, then A is between O and B, and likewise A' is between O and B'. Thus, the perpendicular n at A to $\overleftrightarrow{AA'}$ will enter $\Delta OBB'$. By Pasch's axiom it must intersect $\overline{OB'}$ or $\overline{BB'}$. It cannot intersect $\overline{OB'}$ as n and $\overleftrightarrow{OB'}$ must be parallel. Thus, n

intersects $\overline{BB'}$ at C. Then, $A'ACB'$ is a Lambert Quadrilateral and $B'C > A'A$. Since C is between B and B' we have $B'B > A'A$.

7.5.6 Suppose l and m are two parallels with the property that for points A,B, and C on l the distance to m (measured by dropping perpendiculars from these points to m), is the same. Let A' and B' be the points where the perpendiculars from A and B intersect m. Then, $A'ABB'$ is a Saccheri Quadrilateral We can assume that C is the midpoint of $\overline{AB}$. Let C' be the midpoint of $\overline{A'B'}$. Then, $C'CBB'$ is a Lambert Quadrilateral, and CC' will be the distance from C to m, which must equal BB'. But, by Theorem 7.13 we know that $CC' < BB'$. Thus, not all of the points on l can be equal distance from m.

7.5.7 Let m be right limiting parallel to l at P and let P' be a point on m to the right of P (i.e. in the direction of the omega point). Let Q and Q' be the points on l where the perpendiculars from P and P' to l intersect l.

We claim that $m\angle QPP' < m\angle Q'P'R$ where R is a point on m to the right of P'. If these angles were equal we would have $\overline{PQ} \cong \overline{P'Q'}$ by Exercise 7.3.11, and thus $QPP'Q'$ would be a Saccheri quadrilateral, which would imply that $\angle Q'P'R$ is a right angle, which is impossible. If $m\angle QPP' > m\angle Q'P'R$, then $PQ < P'Q'$ by Exercise 7.3.12, which would imply that we could find a point S on $\overline{P'Q'}$ with $PQ = Q'S$, yielding Saccheri quadrilateral $PQQ'S$. Then, $\angle PSQ'$ must be acute, which contradicts the Exterior angle theorem for $\Delta PSP'$.

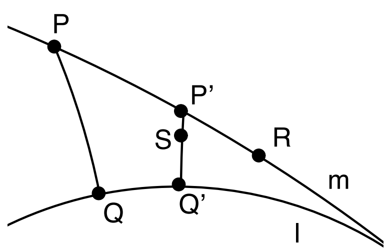


FIGURE 7.4:

Thus, $m\angle QPP' < m\angle Q'P'R$, and the result follows from Exercise 7.3.12.

7.5.8 Suppose the two lines were right limiting parallel. Suppose they had a common perpendicular, say $\overleftrightarrow{PQ}$, with P on l and Q on m. Let R be a point to the right of P on l and drop a perpendicular from

R to m at S. Then, $PQRS$ is a Lambert Quadrilateral and by Theorem 7.13 we have have $RS > PQ$. But, this contradicts the previous exercise.

7.5.9 If they had more than one common perpendicular, then we would have a rectangle.

7.5.10 Let l and m be the two parallel (but not limiting parallel) lines. By use of hyperbolic transformations we can transform l and m so that we can assume m to be the x-axis. Let Ω and Ω' be the omega points for l. Since these will not be the omega points for m, there will be two hyperbolic lines through Ω and Ω' that will be orthogonal to m at B and A respectively. (Construct the Euclidean circle through Ω, and its Euclidean reflection across the x-axis, that is orthogonal to the boundary).

Now, consider all perpendicular (hyperbolic) lines to m between A and B. One such is shown at P. This must intersect l at some point Q (By Pasch's axiom for Omega triangles). Consider $\angle PQR$. Clearly, this will very from a limiting value of 0 as Q approaches Ω' to 360 as Q approaches Ω. Thus, by continuity, there must be a point on l where this angle is a right angle.

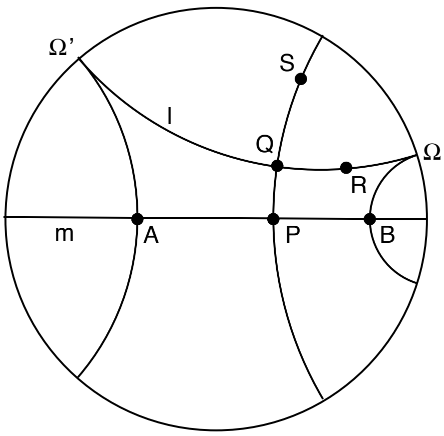


FIGURE 7.5:

7.5.11 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ have $\overline{AB} \cong \overline{EF}$ and $\angle ADC \cong \angle EHG$. If $EH > AD$ then we can find I on $\overline{EH}$ and J on $\overline{FG}$ such that $\overline{EI} \cong \overline{FJ} \cong \overline{AD}$. Then, by repeated application of SAS on sub-triangles of $ABCD$ and $EIJF$ we can show that these two Saccheri Quadrilaterals are congruent. But, this implies that the angles at H and I in quadrilateral $IHGJ$ are supplementary, as are the angles at G and J, which means that we can construct a quadrilateral with angles sum of 360. This contradicts Theorem 7.15, by considering triangles created by a diagonal of $IHGJ$.

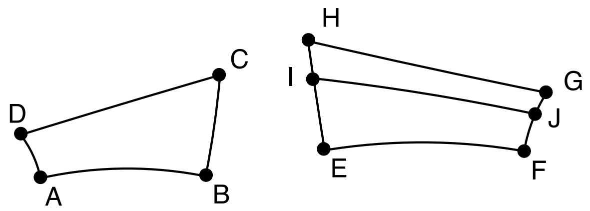


FIGURE 7.6:

7.5.12 The proof is essentially the same as the proof of Theorem 7.17.

7.5.13 No. To construct a scale model, we are really constructing a figure similar to the original. That is, a figure with corresponding angles congruent, and length measurements proportional by a non-unit scale factor. But, AAA congruence implies that any such scale model must have lengths preserved.

7.6 Area in Hyperbolic Geometry

In this section we can refer back to the mini-project we did on area in Chapter 2. That discussion depended on rectangles as the basis for a definition of area. In hyperbolic geometry, no rectangles exist, so the next best shape to base area on is the triangle. This explains the nature of the theorems in this section.

7.6.1 Let J be the midpoint of $\overline{A''B}$ and suppose that $\overleftrightarrow{EF}$ cuts $\overline{A''B}$ at some point $K \neq J$. Then, on $\overleftrightarrow{E''K}$ we can construct a second Saccheri Quadrilateral by the method of dropping perpendiculars from B and C to $\overleftrightarrow{E''K}$. Now, $\overline{BC}$ is the summit of the original Saccheri Quadrilateral $BCIH$ and the new Saccheri Quadrilateral. Thus, if n is the perpendicular bisector of $\overline{BC}$, then n meets $\overleftrightarrow{E''F}$ and $\overleftrightarrow{E''K}$ at right angles. Since E'' is common to both curves, we get a triangle having two right angles, which is impossible.

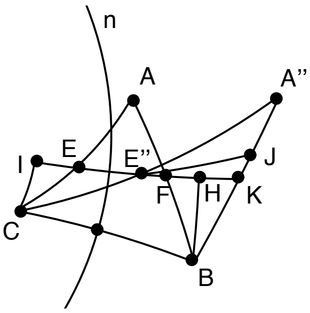


FIGURE 7.7:

7.6.2 Given ΔABC, we know that any two rays making up two sides of an angle of the triangle are not coincident on the same line. Thus, they create a non-zero angle, and the angle sum of the triangle must be non-zero. Thus, the defect can never be 180.

7.6.3 This question can be argued both ways. If we could make incredibly precise measurements of a triangle, then we could possibly measure the angle sum to be less than 180. However, since the universe is so vast, we would have to have an incredibly large triangle to measure, or incredibly good instruments. Also, we could never be sure of errors in the measurement overwhelming the actual differential between the angle sum and 180.

7.7 Project 11 - Tiling the Hyperbolic Plane

A nice supplement to this section would be to bring in several of M. C. Escher's Circle Limit figures and discuss what (n, k) tilings they are based upon.

7.7.1 Reasoning as we did in Section 6.4, we see that if we have k regular n-gons meeting at a common vertex, then

$$180n < 360 + 2n\alpha$$

where $\alpha = \frac{360}{2k}$. Then,

$$\frac{360}{k} > 180 - \frac{360}{n}$$

and dividing by 360 and re-arranging gives

$$\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$$

Thus, since $\frac{1}{3} + \frac{1}{3} > \frac{1}{2}$ we have that a $(3, 3)$ tiling is possible.

7.7.2 From the equation

$$\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$$

We see that there are an infinite number of regular tilings possible in hyperbolic (2-dimensional) geometry.

7.7.3 In a $(6, 5)$ tiling we have regular hexagons meeting 5 at a vertex. The interior angles of the hexagons must be $\frac{360}{5} = 72$. Triangulating such a hexagon by triangles to the center, we see that the central angle must be 60 degrees and the base angles of the isosceles triangles must be 36 degrees (half the interior angle).

Thus, to build the tiling we start with a triangle of angles 60, 36, and 36 and continue the construction just as we did in the lab.

CHAPTER 8

Elliptic Geometry

Hyperbolic and Elliptic geometry are the fundamental examples of non-Euclidean geometry. The connection between the three geometries —Euclidean, Hyperbolic, and Elliptic —and the three possible parallel properties —1, > 1, or 0 parallels through a point to a given line —is one of the most interesting and thought-provoking ideas in geometry. The relationship between geometries and the shape of the universe would be a nice side-trip after covering this chapter.

SOLUTIONS TO EXERCISES IN CHAPTER 8

8.2 Perpendiculars and Poles in Elliptic Geometry

8.2.1 Referring to Figure 8.1, we know ΔACB and ΔACE are congruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FCA$, and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBCD and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{FE}$ and the angle at E is a right angle.

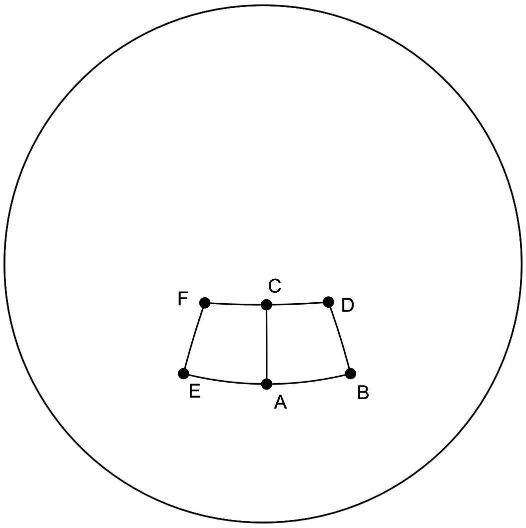


FIGURE 8.1:

8.2.2 Referring to Fig. 8.1 we know ΔACB and ΔACE are congruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FCA$, and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBCD and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{FE}$ and the angle at E is a right angle.

8.2.3 By Exercise 8.2.2 we can create a Saccheri quadrilateral from the Lambert quadrilateral. Since the summit angles in the Saccheri quadrilateral will equal the fourth (non-right) angle in the Lambert quadrilateral, the result follows.

8.2.4 Given a Lambert quadrilateral $ABDC$ with right angles at A, B, and C, suppose that $DB > AC$ Then there is a point E on the line through B, D with D between B and E such that $\overline{BE} \cong \overline{AC}$. It follows that $ABEC$ is a Saccheri quadrilateral, and $\angle ACE \cong \angle BEC$, and both angles are obtuse.

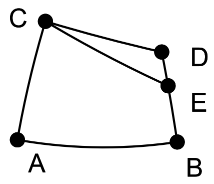


FIGURE 8.2:

However, since E lies within $\angle ACD$ then $\angle ACD$ contains $\angle ACE$. So, $\angle ACE$ must be less than a right angle. This contradicts the fact that $\angle ACE$ must be obtuse.

Thus, $DB \leq AC$. If $DB = AC$, then we would have a Saccheri quadrilateral $ABDC$ and the summit angles would be congruent, which is impossible. We must have, then, that $DB < AC$. A similar argument shows $CD < AB$.

8.2.5 Since there are two perpendiculars from n that meet at O, then by the work in this section, all perpendiculars to n meet at O and O is the polar point.

8.3 Project 12 - Models of Elliptic Geometry

8.3.1 If the lune is defined by an angle of $\frac{2\pi}{n}$, where n is an integer, then, it is clear that n such lunes will fill the entire the sphere, which has area 4π. Thus, the area of one lune must be $\frac{4\pi}{n}$, which is twice the angle swept out. A similar argument can be used if n is a rational number, and then we can claim it true for all n by continuity.

8.3.2 If the lune is defined by an angle of $\frac{2\pi}{n}$ then the area will be $\frac{4\pi R^2}{n}$.

8.3.3 It is possible to make a triangle with all three right angles. Consider the sphere model. With one segment of the triangle on the equator, we can make perpendiculars from the endpoints to get a triangle with two right angles. These will intersect at the top of the sphere. By adjusting the length of the equator segment we can make a triangle with three right angles.

In the circle model for elliptic geometry, as the three vertices approach the boundary circle, the interior angles get closer and closer to 180 degrees. The upper bound would be 180 degrees.

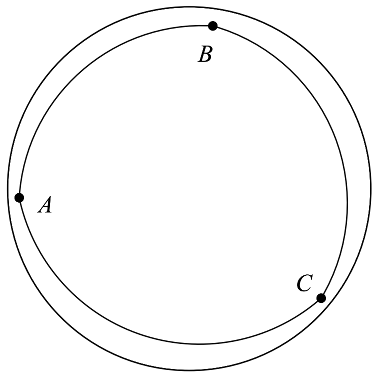


FIGURE 8.3:

8.3.4 The side length should be equal to the polar length. In the sphere model, consider one triangle segment to be on the equator. Then, the other two sides will meet at the pole, as they are perpendiculars to the given side.

8.4 Basic Results in Elliptic Geometry

8.4.1 Consider the line $\overline{OR}$. This line must intersect l at some point Q. Also, it is perpendicular to l at Q, for if it were not perpendicular, then at Q we could construct the perpendicular which must pass through the polar point O. But, then we would have two lines through O and Q. Since $\overline{OR}$ is perpendicular to l at Q, then the distance from Q to O is the polar distance q. There is a unique point along $\overline{OR}$ in the direction of R that is at this distance. Thus, R must be equal to Q and $R$is on l.

8.4.2 Let m be a line passing through O and let P be a point on m that is a distance q away from O. (The point P must exist as the length of a line is $2q$.) Then, by Exercise 8.4.1 P is on l, the polar of O. The perpendicular to l at P intersects O, so the perpendicular must be m, as there is a unique line through P and O.

Conversely, a perpendicular to l passes through O by Theorem 8.1.

8.4.3 Suppose the polar points were the same. Then, by Exercise 8.4.1 points that are a distance of q away from this common polar would have to be on l and also on m. Thus, the two lines must be identical. The line through the two polar points will be perpendicular to both lines by Exercise 8.4.2.

8.4.4 This follows almost immediately from Exercise 8.4.1.

8.4.5 The *exterior* of a circle is the set of points whose distance to the center is larger than the radius. If the radius is equal to the polar distance then the circle is a line and the center of the circle is the polar point of the circle/line. The interior is defined, but would be all of the elliptic plane, as the distance from a point to another point is at most the polar distance (the length of an entire line is $2q$, so the length of any subset (segment) is at most q).

8.4.6 Pick three equally spread out points on a line.

8.4.7 The rotation R is made of two reflections r_m and r_m where m and n are lines passing through O. If P is a point on l, then for $P' = r_m(P)$, we have that the length of $\overline{OP}(= q)$ is equal to the length of $\overline{OP'}$, as r_m fixes O and preserves length. Thus, by Exercise 8.4.1 we have that P' is on l. A similar argument shows that $r_n(P')$ is also on l. Thus, $R(P) = r_n(r_m(P))$ is on l.

8.4.8 By the previous exercise, we know that H fixes l, but not necessarily every point of l. Let P be a point on l. We can assume that $H = r_n \circ r_m$ where r_m and r_n are reflections across lines m and n that pass through O. Also, we know that m and n make right angles. WLOG we can assume that m passes through P. Thus, $H(P) = r_n(r_m(P)) = r_n(P)$. Now, r_n will map line m back to itself, preserving lengths. So, since P is on m, then $r_n(P)$ will be another point on m that is equidistant from O as P. That is, $r_n(P)$ will be a distance q from O, where q is the polar distance. This means that $r_n(P)$ must be on l. We know it is on m. Since these two lines only intersect at P, then $H(P) = r_n(P) = P$.

8.4.9 We can assume that $R_1 = r_n \circ r_p$ and $R_2 = r_p \circ r_m$ are the two rotations. (This is similar to the work done in Chapter 5 on compositions of transformations). Then, $R_1 \circ R_2 = r_n \circ r_m$. If $n = m$ then the composition is the identity, which is a rotation of 0 degrees. If $n \neq m$, then we know that m and m must intersect at a point O. Then, $R_1 \circ R_2 = r_n \circ r_m$ is a composition of reflections with a fixed point and thus is a rotation.

8.4.10 By Exercise 8.4.8 we can write any of the three reflections as a rotation. Since the composition of rotations is again a rotation (Exercise 8.4.9), the result follows.

8.5 Triangles and Area in Elliptic Geometry

8.5.1 Divide the quadrilateral into two triangles and use Theorem 8.9.

8.5.2 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ are as shown. If $HE > AD$, then we can find points I on $\overline{HE}$ and J on $\overline{GF}$ such that $HI = AD$ and $GJ = BC$. A triangle congruence argument shows that $IJGH$ is a Saccheri quadrilateral congruent to $ABCD$ and thus $EFJI$ is a rectangle.

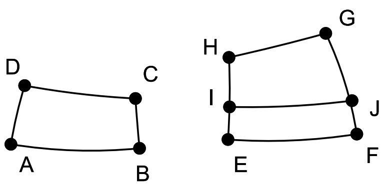


FIGURE 8.4:

8.5.3 If they had more than one common perpendicular, then we would have a rectangle.

8.5.4 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ have $\overline{AB} \cong \overline{EF}$ and $\angle ADC \cong \angle EHG$. If $EH > AD$ then we can find I on $\overline{EH}$ and J on $\overline{FG}$ such that $\overline{EI} \cong \overline{FJ} \cong \overline{AD}$. Then, by repeated application of SAS on sub-triangles of $ABCD$ and $EIJF$ we can show that these two Saccheri Quadrilaterals are congruent. But, this implies that the angles at H and I in quadrilateral $IHGJ$ are supplementary, as are the angles at G and J, which means that we can construct a quadrilateral with angle sum of 360. This contradicts Corollary 8.10

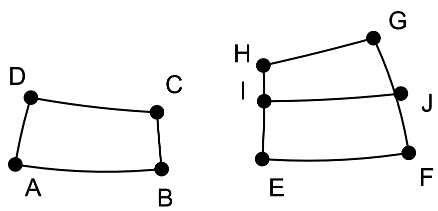


FIGURE 8.5:

8.5.5 Suppose that the intersection of $\overleftrightarrow{EF}$ with $\overline{A\hat{\ }B}$ is not the midpoint. Connect G to the midpoint K of $\overline{A\hat{\ }B}$ and construct a second Saccheri quadrilateral B. Let M be the midpoint of $\overline{BC}$. We know that the line joining the midpoints of a Saccheri quadrilateral is perpendicular to the base and summit. (Exercise 8.2.1) Thus, the perpendicular n of $\overline{BC}$ at M will meet both bases of the quadrilaterals at right angles – it will be a common perpendicular. Thus, point G must be the polar point of n (Exercise 8.2.5). But, this is impossible, as the intersection of n with the bases of the Saccheri quadrilaterals happens at the midpoints of the bases and these points cannot be a distance q from G (Q being the polar distance).

8.5.6 AAA precludes scale models.

8.6 Project 13 - Elliptic Tiling

8.6.1 The tetrahedron will generate the $(3, 3)$ tiling. The octahedron will generate the $(3, 4)$ tiling. The icosahedron will generate the $(3, 5)$ tiling. The cube will generate the $(4, 3)$ tiling. The dodecahedron will generate the $(5, 3)$ tiling.

8.6.2

8.6.3 In a $(3, 5)$ tiling we have equilateral triangles meeting five at a vertex. The interior angles of the triangle must be $\frac{360}{5} = 72$. Open an Elliptic Geometry window in *Geometry Explorer* and create a point at the origin - **Point At Origin** (**Misc** menu). Select the point at the origin and choose **Base Pt of Triangle with Angles...**(**Misc** menu). Enter 72 for all angles. In the Canvas, a new point will be created that corresponds to point B in the equilateral triangle we are trying to construct. To construct the third point C of the triangle, rotate point B about point A by an angle of 72 degrees. This is the basic tile of our tiling. Color in this tile and then use rotations and reflections to generate the entire tiling.

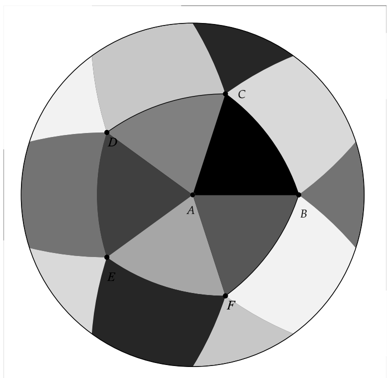


FIGURE 8.6:

CHAPTER 9

Projective Geometry

Projective geometry is the natural culmination of a study of geometry that proceeds from Euclidean to non-Euclidean geometries. While much of the material in this chapter is much more opaque than in previous chapters, the beauty and elegance realized from abstraction should be emphasized. Also, the connection with perspective drawing, and the Elliptic disk model can be used to give students a more concrete basis for understanding concepts such as harmonic sets.

SOLUTIONS TO EXERCISES IN CHAPTER 9

9.2 Project 14 - Perspective and Projection

9.2.1

9.2.2 Clearly, since the two triangles lie in planes Π and Π' then any lines generated from these triangles must also lie in these planes. An immediate conclusion is that the three points D, E, and F must be collinear, as they lie on the intersection of the two planes.

9.2.3 If $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel, then there is no Euclidean point of intersection. However, if we allow the lines to intersect at *the point at infinity*, then the two intersection points of the other sides will form a line that is parallel to $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$. Thus, it makes sense to say that the point at infinity is collinear with the other two points (D and E). A general discussion of these ideas is what the students should report for this question.

9.2.4 Line $\overleftrightarrow{OC}$ crosses the two parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$. Thus, angles $\angle DCB$ and $\angle DC'B'$ are congruent. (Fig. 9.1). Also, Line $\overleftrightarrow{OC}$ crosses the two parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$. Thus, angles $\angle DCA$ and

$\angle DC'A'$ are congruent. Thus, angles $\angle ACB$ and $\angle A'B'C'$ are congruent.

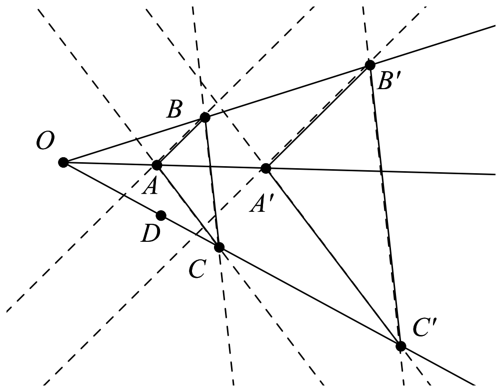


FIGURE 9.1:

By similar reasoning, one can show ΔOBC is similar to $\Delta OB'C''$. Thus, $\overline{BC}$ and $\overline{B'C'}$ are in proportion to $\overline{OC}$ and $\overline{OC'}$. Also, ΔOAC is similar to $\Delta OA'C''$. Thus, $\overline{OC}$ and $\overline{OC'}$ are in proportion to $\overline{AC}$ and $\overline{A'C'}$. So, $\overline{BC}$ and $\overline{B'C'}$ are in proportion to $\overline{AC}$ and $\overline{A'C'}$.

Thus, by SAS similarity, ΔABC is similar to $\Delta A'B'C''$ and we have $\angle ABC$ is congruent to $\angle A'B'C'$. Since, we know that $\angle OBC \cong \angle OB'C'$, then we have that $\angle OBA \cong \angle OB'A'$. By Theorem 2.8, lines $\overline{AB}$ and $\overline{A'B'}$ are parallel.

9.3 Foundations of Projective Geometry

9.3.1 Suppose the lines are l and m. If they both pass through points P and Q, then by axiom A1 the two lines must be the same line.

9.3.2 Use the construction in the hint. If l and m were parallel, then $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$ would be parallel, but that is impossible, as both lines share point Q. Let S be the intersection of l and m. Since l is parallel to $\overleftrightarrow{QR}$ and m is parallel to $\overleftrightarrow{PQ}$, then S cannot be any of the points P, Q, or R.

9.3.3 By the previous exercise, we know that an Affine geometry must have at least four distinct points, say P, Q, R, and S. For each pair of points, we have a unique line, by axiom A1. A quick check of every pairing of one of the six lines with a point not on that line shows that axiom A2 is satisifed. Thus, for these four points and six lines, all three axioms are satisfied.

9.3.4 Let P be a point. By axiom P3 there must be two other points

Q and R such that P, Q, and R are non-collinear. Then, $l = \overleftrightarrow{PQ}$ and $m = \overleftrightarrow{PR}$ are two distinct lines. Now, Q and R define a line n by axiom P1, and this line is different than l or m. Since P, Q, and R are non-collinear, then, if the three lines have a common intersection point, then that point (S) must be different than P, Q, or R. But, then lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ have two distinct points in common, which contradicts Theorem 9.2.

9.3.5 Let P be a point. By axiom P3 there must be two other points Q and R such that P, Q, and R are non-collinear. Then, $l = \overleftrightarrow{PQ}$ and $m = \overleftrightarrow{PR}$ are two distinct lines. Now, Q and R define a line n by axiom P1, and this line is different than l or m. By axiom P4, n must have a third point, say S. Then, $\overleftrightarrow{PS}$ exists and is distinct from both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$.

9.3.6 Let l be a line in A and let P be a point not on l. In the projective plane, we know that l and u intersect at a unique point, call it U. Then, the line $\overleftrightarrow{PU}$ will be a line in A that will not intersect l at a point in A as U is not in A. Thus, $\overleftrightarrow{PU}$ will be parallel to l through P. To show the parallel is unique, suppose m is another parallel to l through P. We know that m and l intersect in the projective plane at some point, call it V. That must mean that V is on u. But, that would imply that m and $\overleftrightarrow{PU}$ intersect at two points, so $V = U$.

9.3.7 It is clear that the example satisfies P3 and P4. Checking all pairs of points and lines will suffice for P1 and P2.

9.3.8 Let *abcd* be a complete quadrilateral. Let $E = a \cdot b$ and $F = c \cdot d$, $G = a \cdot c$ and $H = b \cdot d$, and $I = a \cdot d$ and $J = b \cdot c$ be the pairs of opposite vertices of the quadrilateral.

Consider $EFGH$. Suppose three of the points E,F,G,or$H$ are collinear. Suppose E, F, G are collinear. Then, $a = b$, which is impossible. Likewise, we can show that no subset of three of E, F, G, H are collinear. Then, $EFGH$ is a complete quadrangle. The pairs of opposite sides are $e = EF$ and $f = GH$, $a = EG$ and $d = FH$, and $b = EH$ and $c = FG$.

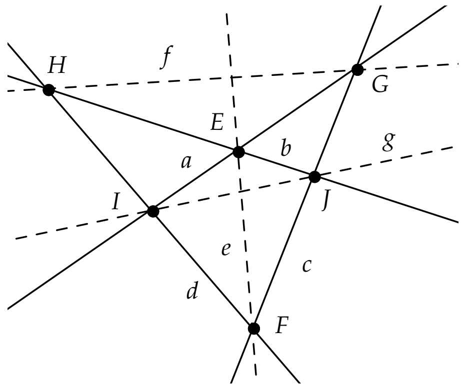


FIGURE 9.2:

Now, the diagonal sides of *abcd* are the lines e, f, and $g = IJ$. Suppose that these three lines were concurrent. That means the intersection point K of e and f must be on $\overleftrightarrow{IJ}$. Thus, I, J, K are collinear. Now, the diagonal vertices of $EFGH$ are $K = e \cdot f$, $I = a \cdot d$, and $J = b \cdot c$. If I, J, K are collinear, this contradicts the fact that $EFGH$ is a complete quadrangle. So, the diagonal sides of *abcd* are not concurrent.

9.3.9 Since Axioms P(n)1 and P(n)2 exactly match P1 and P2, then P1 and P2 are satisfied. P(n)3 guarantees the existence of three non-collinear points, so Axiom P3 is satisfied.

Let P, Q, R, and S be the four points guaranteed by Axiom P(n)3. Then, $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$,$\overleftrightarrow{PS}$, $\overleftrightarrow{QR}$, $\overleftrightarrow{QS}$ and $\overleftrightarrow{RS}$ must be distinct lines. Suppose there were a line m with only two points A and B. Since m intersects $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$, then either one of A or B is P, or A is on one of the lines and B is on the other.

Suppose $A = P$. Now m intersects $\overleftrightarrow{QR}$ and $\overleftrightarrow{QS}$ at a point other than P, so $B = Q$. Then, $m = \overleftrightarrow{PQ}$. But, we know that $\overleftrightarrow{PQ}$ must have a third point as it must intersect $\overleftrightarrow{RS}$ at a point other than P or Q. So, $A \neq P$.

Suppose A is on $\overleftrightarrow{PQ}$ and B is on $\overleftrightarrow{PR}$. Since m must intersect $\overleftrightarrow{QR}$ at some point, then A is on $\overleftrightarrow{QR}$ or B is on $\overleftrightarrow{QR}$. Then, $A = Q$ and $B = R$. Then, $m = \overleftrightarrow{QR}$. But, we know that $\overleftrightarrow{QR}$ must have a third point as it must intersect $\overleftrightarrow{PS}$ at a point other than Q or R. So, it cannot be the case that A is on $\overleftrightarrow{PQ}$ and B is on $\overleftrightarrow{PR}$.

We conclude that there cannot be a line with only two points. A similar proof (simpler) shows that there cannot be a line with only one point.

9.3.10 The verification is straight-forward.

9.3.11 Since Moulton lines are built from pieces of Euclidean lines, then P3 and P4 follow immediately.

For P2, any pair of Moulton lines that are standard Euclidean lines will either intersect in the regular Euclidean plane or will intersect at a point at infinity. For other Moulton lines, if the two lines are defined by different slopes m_1 and m_2, then consider the regular Euclidean lines of those slopes. Either those lines will intersect in the region $x \leq 0$ or $x > 0$. Since the Moulton "pieces" defining the line have the same slope, then the Moulton lines with slopes m_1 and m_2 will also intersect where the Euclidean lines did. The case left is the case where the Moulton lines have the same slope. In this case, they intersect at the point at infinity. So, P2 holds.

For P1, let A and B be two point in the plane. If A and B are both in the left half-plane ($x \leq 0$) we just find the line $y = m'x + b$ joining them and define $y = 2m'x + b$ in the right half-plane ($x > 0$) to get the Moulton line incident on A and B. A similar construction works if A and B are both in the right half-plane ($x > 0$).

If A is in the left half-plane and B in the right, but A is above or at the same level as B, then the slope of the line joining the two points is negative or zero, and the Moulton line is just the Euclidean line incident on the points.

The only case left is where A is in the left half-plane and B in the right, and A is below B. It is clear from the hint that the fraction $\dfrac{slope(\overleftrightarrow{BY})}{slope(\overleftrightarrow{AY})}$ range from 0 to infinity as Y ranges from Y_1 to Y_0. Thus, by continuity there is a Y value where this fraction is equal to 2. Let $m = slope(\overleftrightarrow{BY})$. This will generate the unique Moulton line from A to B.

9.3.12 All of the lines defined by the triangles are regular Euclidean lines, so the Euclidean points D and E would be the intersections one would expect by Desargues Theorem. However, the line defined by D and E takes a "jog" at the intersection with the y-axis and the *Moulton* line does not go through E.

9.4 Transformations in Projective Geometry and Pappus's Theorem

9.4.1 Closure: Let π_1 and π_2 be projectivities. Then, each is constructed from a composition of perspectivities. The composition of π_1 with π_2 will again be a composition of perspectivities and thus is a projectivity.

Associativity: This is automatic (inherited) from function composition.

Identity: A perspectivity from a pencil of points back to itself (or pencil of lines back to itself) is allowed, and thus the identity is a projectivity.

Inverses: Given any basic perspectivity from, say a pencil of points on l to a pencil of points on l', will have its inverse be the revers mapping from the pencil at l' to the pencil at l. Since a projectivity is the composition of perspectivities, then the inverse will be the composition of inverse perspectivities (in reverse order).

9.4.2 By Exercise 9.3.4 we know there is a line $l' \neq l$. Let X,Y, and Z be three distinct points on l'. By Theorem 9.6 there is a perspectivity π_1 mapping A to X, B to Y, and C to Z. Again, by this theorem, there is a perspectivity π_2 mapping X to A', Y to B', and Z to C'. Then, π_2 composed with π_1 is the desired projectivity.

9.4.3 Dual to Theorem 9.6: Let a,b,c be three distinct lines incident on point P and a', b', and c' be three distinct lines incident on point P', with $P \neq P'$. Then, there is a projectivity taking a,b,c to a',b',c'.

Dual to Corollary 9.7: Let a,b,c and a', b', and c' be two sets of distinct lines incident on point P. Then, there is a projectivity taking a,b,c to a',b',c'.

9.4.4 The case for Pappus's Theorem assumes that point A is the intersection of l and l' as shown. Let $X = AB' \cdot A'B$ and $Y = AC' \cdot A'C$ and $Z = BC' \cdot B'C$. Then, since $A = X = Y$, it is clear that X,Y, and Z are collinear.

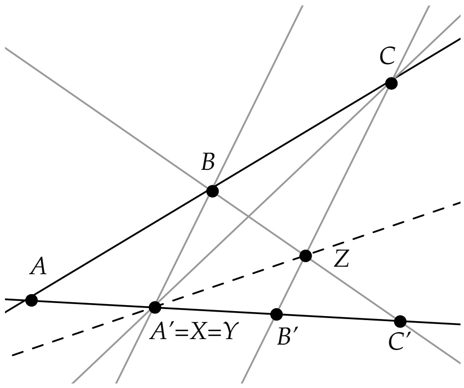


FIGURE 9.3:

9.4.5 Let π_1 and π_2 be projectivities having the same value on points A, B, and C on l. Then $\pi_1 \circ \pi_2^{-1}$ will leave the points A, B, and C on l invariant. By P7, $\pi_1 \circ \pi_2^{-1} = id$, or $\pi_1 = \pi_2$.

9.4.6 Using the dual to axiom P7 (Theorem 9.13), the argument is precisely the dual argument to that given in Exercise 9.4.6.

9.4.7 Consider a projectivity of pencils of points from line l to line l'. Theorem 9.6 gives us a construction for a projectivity as the composition of two perspectivities if $l \neq l'$. If $l = l'$, then let A, B, and C be projectively related to A', B', and C' on l'. Let m be a line not identical to l and P a point not on m. Let π be the perspectivity defined by center P and line m. Let X, Y, and Z be the points on m that are perspective from P to A', B', and C'. Then, by Theorem 9.6 we can find a sequence of two perspectivities taking A, B, and C to X, Y, and Z. Then, π maps X, Y, Z to A', B', C'.

9.4.8 First Why? T, U, and P are collinear by Pappus's Theorem. **Second Why?** U, V, and Q are collinear again by Pappus's Theorem. **Third Why?** Since $R = CA \cdot A'C'$ and since V is on $A'C'$ and T is on AC, then $R = CT \cdot A'V$. Since S is on BC then Q is on CS. SInce Q is alos on UV be part 2, then $Q = CS \cdot UV$. A similar argument shows $P = A'S \cdot TU$. **Fourth Why?** R, P, and Q are collinear by Pappus's Theorem.

9.4.9 WLOG assume $A = A'$. Then, $P = AB \cdot A'B' = A$ and $R = AC \cdot A'C' = A$. Clearly, there is a line defined on P, Q, and R as $P = R$.

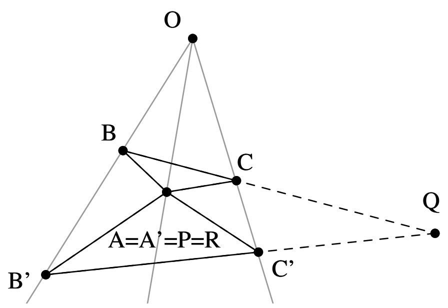


FIGURE 9.4:

9.4.10 The condition $S_{A'B'} = A'B' \cdot BC = C$ implies that A',B' and C are collinear while $S_{A,C} = B'$ implies that A, C and B' are collinear. Thus, A, A', C and B' are collinear, which is not possible for a Desargues configuration. (No two points on a side of a triangle are collinear with the center of perspectivity)

Note that there is no ordering on the points A',B', and C' given in the problem. So, the problem is really assuming that one of the lines in one of the triangles intersects one of the "non-corresponding" lines at a point other than a vertex. This gives six possible intersections to pick from.

9.5 Models of Projective Geometry

9.5.1 P1: Given two "regular" points $(x_1, y_1, 1)$ and $(x_2, y_2, 1)$, the Euclidean line defined by these points will be the only line so defined in the plane $z = 1$, and thus will be the unique projective line on these points. Given $(x, y, 1)$ and a point at infinity $(x', y', 0)$. there is a unique line through $(x, y, 1)$ with slope $\frac{y'}{x'}$. This is the projective line through the points. Given two points at infinity, the line at infinity is the only line through these points.

P2: For two regular lines in the plane $z = 1$, there is only one point of intersection. For a regular line l and the line at infinity, let $\frac{y}{x}$ be the slope of l, Then, the point $(x, y, 0)$ is the point at infinity that is on l.

P3: $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ are not collinear.

P4: Every regular line clearly has at least three points. The line at infinity contains an infinite number of points of the form $(x, y, 0)$.

9.5.2 The projectivity can be represented by the 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let $P = (\alpha_1, \beta_1)$, $Q = (\alpha_2, \beta_2)$, and $R = (\alpha_3, \beta_3)$ (in parametric coordinates on the line). If these points are invariant, then $\lambda_1 P = AP$, $\lambda_2 Q = AQ$, and $\lambda_3 R = AR$. That is, these three points (as vectors) are eigenvectors of A. But, if a 2x2 matrix is not a multiple of the identity, then the matrix has at most two different eigenvectors. So, $A = \lambda I$. On projective points, A then acts as the identity.

9.5.3 All points at infinity must lie on the line at infinity. Points at infinity are of the form $(x, y, 0)$ where both x and y cannot be simultaneously zero. Let $[u, v, w]$ be the coordinate for the line at infinity. Then $ux + vy + 0w = 0$ for all choices of x and y. Clearly, $u = v = 0$. Since the coordinate vector for a line cannot be the zero vector, then $w \neq 0$. Thus, $[0, 0, 1]$ must be the homogeneous coordinates for the line at infinity.

9.5.4 From $X = \alpha' R + \beta' S$ we substitute $R = \lambda_1 P + \lambda_2 Q$ and $S = \mu_1 P + \mu_2 Q$ to get $X = \alpha'(\lambda_1 P + \lambda_2 Q) + \beta'(\mu_1 P + \mu_2 Q)$. So, $X = \alpha'\lambda_1 P + \alpha'\lambda_2 Q) + \beta'\mu_1 P + \beta'\mu_2 Q$. Since $X = \alpha P + \beta Q$ we get $\alpha = \lambda_1 \alpha' + \mu_1 \beta'$ and $\beta = \lambda_2 \alpha' + \mu_2 \beta'$.

9.5.5 Let $P = (x_1, y_1, 1)$ and $Q = (x_2, y_2, 1)$. Let $x = x_2 - x_1$ and $y = y_2 - y_1$. Then, the point $(x, y, 0)$ will be the point at infinity, as it points (as a vector) in the direction of the slope of the line.

9.5.6 Check all four subgroups of three of the points. In each case, put the three into a 3x3 matrix. In all four cases, the matrix is nonsingular. Thus, by Theorem 9.16 no group of three will be collinear.

9.5.7 Let P, Q, R, and S be the points of the first complete quadrangle. Let $X_1 = (1, 0, 0)$, $X_2 = (0, 1, 0)$, $X_3 = (0, 0, 1)$ and $X_4 = (1, 1, 1)$. By Theorem 9.24 there is a unique collineation π_1 taking P to X_1, Q to X_2, R to X_3 and S to X_4. Let P', Q', R', and S' be the points of the second complete quadrangle. Then, there is a unique collineation π_2 taking P' to X_1, Q' to X_2, R' to X_3 and S' to X_4. Then, $\pi_2^{-1} \circ \pi_1$ will take P, Q, R, and S to P', Q', R', and S'.

Suppose there was another collineation π' taking P, Q, R, and S to P', Q', R', and S'. Then, $\pi_2 \circ \pi'$ takes P to X_1, Q to X_2, R to X_3 and S to X_4. Uniqueness implies that $\pi_1 = \pi_2 \circ \pi'$ or $\pi' = \pi_2^{-1} \circ \pi_1$.

9.5.8 The identity map leaves the four points invariant. By Theorem 9.25 such a map is unique.

9.5.9 Let A be the collineation and let P and Q be distinct points on l. Let l' be the line that is the image of l under A. Let P' and Q' be distinct points of l'. Given X on l, we have $X = \alpha P + \beta Q$. Also,

$AX = \alpha' P' + \beta' Q'$. Now,

$$\begin{aligned} AX &= A(\alpha P + \beta Q) \\ &= \alpha AP + \beta AQ \\ &= \alpha(\lambda_1 P' + \lambda_2 Q') + \beta(\mu_1 P' + \mu_2 Q') \\ &= (\alpha\lambda_1 + \beta\mu_1)P' + (\alpha\lambda_2 + \beta\mu_2)Q' \end{aligned}$$

Thus, $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

9.5.10 Suppose the collineation is affine. An ideal point $[l]$ represents the set of lines that are parallel to a given line l. Suppose the collineation maps l to a point l'. Let m be a line parallel to l. Then, the collineation maps m to a line m' that is parallel to l', and thus in the set of parallels to l', i.e. to the ideal point represent by l'. Thus, the collineation maps the ideal point represented by l to the ideal point represented by l'.

Suppose a collineation maps ideal point [l] to ideal point [l']. Then, the collineation must map all lines m in $[l]$ (defined as the parallels to l) to lines m' in $[l']$ (defined as the parallels to l'). That is, it maps lines parallel to l to lines parallel to l'. Thus, it maps parallel lines to parallel lines.

9.5.11 Let $X = (x, y, 0)$ be the homogeneous coordinates for an ideal point. A collineation can be represented by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Then, $AX = (\alpha, \beta, gx + hy + 0)$. This must again be an ideal point by Exercise 9.5.11. So, $gx + hy = 0$ for all choices of x and y. Then, $x = 0 = y$. For A to be non-singular $i \neq 0$. Then, $\frac{1}{i}A$ will be equivalent to A as a collineation and will have third row equivalent to [0 0 1].

9.6 Project 15 - Ratios and Harmonics

9.6.1 $\frac{\text{dist}(A,C)}{\text{dist}(A,B)} = \frac{||(P+t_1v)-(P+t_3v)||}{||(P+t_1v)-(P+t_2v)||} = \frac{|t_1-t_3|||v||}{|t_1-t_2|||v||} = \frac{|t_3-t_1|}{|t_2-t_1|}$.

9.6.2 $\frac{\text{dist}(D,F)}{\text{dist}(D,E)} = \frac{||M(C-A)||}{||M(B-A)||} = \frac{||M(P+t_3v)-(P+t_1v)||}{||M((P+t_2v)-(P+t_1v))||} = \frac{||M(t_3v-t_1v)||}{||M(t_2v-t_1v)||} = \frac{|t_3-t_1|||M(v)||}{|t_2-t_1|||M(v)||} = \frac{|t_3-t_1|}{|t_2-t_1|}$. This last term equals $\frac{\text{dist}(A,C)}{\text{dist}(A,B)}$ by Exercise 9.6.1.

9.6.3 This is just a matter of showing that the ratios are preserved. For example, if

$$r = R(A, B; C, D) = \frac{AC\ \ BD}{AD\ \ BC},$$

then

$$R(B, A; D, C) = \frac{BD\ \ AC}{BC\ \ AD} = r.$$

9.6.4 If

$$r = R(A, B; C, D) = \frac{AC\ \ BD}{AD\ \ BC},$$

then

$$R(B, A; C, D) = \frac{BC\ \ AD}{BD\ \ AC} = \frac{1}{r}.$$

A similar argument can be used to show the other results stated in the exercise.

9.6.5

$$R(C1, E; G, C2) = \frac{C1G\ \ EC2}{C1C2\ \ EG} = \frac{\frac{2}{3}C1C2\ \ \frac{1}{5}C1C2}{C1C2\ \ (-(\frac{1}{3} - \frac{1}{5})C1C2)} = \frac{\frac{2}{15}}{-\frac{2}{15}} = -1.$$

9.7 Harmonic Sets

9.7.1 For the quadrangle, $ABCD$ let $E = AB \cdot CD$, $F = AC \cdot BD$, and $G = AD \cdot BC$.

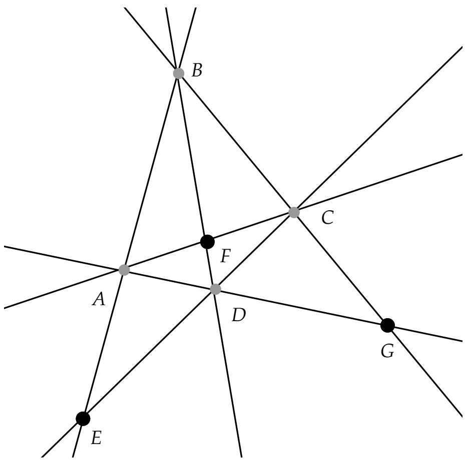


FIGURE 9.5:

Consider a line through two of the diagonal points, say the line EF. Suppose A was on EF. Then, F is on $AE = AB$. But, this implies that D is on AB, which contradicts the fact that A, B, and D must be non-collinear. A similar argument will show that EF cannot intersect any of the other three points B, C, or D. Similarly, the result holds for EG and FG.

9.7.2 The proof is exactly the dual to the proof of Exercise 9.7.1. Students should be encouraged to think of the proof this way.

9.7.3 Let O be the intersection of FF' and HH'. Then, the perspectivity from O maps EF to $E'F'$, thus maps l to l'. (See Fig. 9.6) The perspectivity maps E,F, and H to E', F', and H'. Let I'' be th image of I under this perspectivity. Then, by Corollary 9.31 we have that $H(E', F'; H', I'')$. But, by Theorem 9.29 we know that the fourth point in a harmonic set is uniquely defined, based on the first three points. Thus, $I' = I''$.

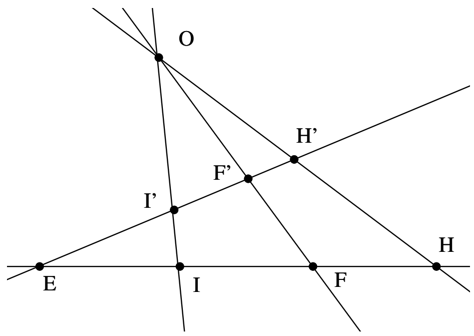


FIGURE 9.6:

9.7.4 $\frac{\mathrm{dd}(A,C)}{\mathrm{dd}(A,D)} = \frac{||(\alpha P+Q)-(\gamma P+Q)||}{||(\alpha P+Q)-(\delta P+Q)||} = \frac{|\alpha-\gamma|||P||}{|\alpha-\delta|||P||} = \frac{|\alpha-\gamma|}{|\alpha-\delta|} = \frac{|\gamma-\alpha|}{|\delta-\alpha|}$. Likewise, $\frac{\mathrm{dd}(B,D)}{\mathrm{dd}(B,C)} = \frac{|\delta-\beta|}{|\gamma-\beta|}$.

9.7.5 Let A,B,C, and D have parametric homogeneous coordinates (α_1, α_2), (β_1, β_2), (γ_1, γ_2) and (δ_1, δ_2). We can assume that the homogeneous parameters for the four points are equivalently $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$ and $(\delta, 1)$, where $\alpha = \frac{\alpha_1}{\alpha_2}$, $\beta = \frac{\beta_1}{\beta_2}$, $\gamma = \frac{\gamma_1}{\gamma_2}$, and $\delta = \frac{\delta_1}{\delta_2}$. From Theorem 9.33 we have

$$R(A, B; C, D) = \frac{dd(A, C)\, dd(B, D)}{dd(B, C)\, dd(A, D)}$$
$$= \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)}$$

If we do a replacement using $\alpha = \frac{\alpha_1}{\alpha_2}$, $\beta = \frac{\beta_1}{\beta_2}$, $\gamma = \frac{\gamma_1}{\gamma_2}$, and $\delta = \frac{\delta_1}{\delta_2}$. we get

$$\begin{aligned}
R(A,B;C,D) &= \frac{(\frac{\gamma_1}{\gamma_2} - \frac{\alpha_1}{\alpha_2})(\frac{\delta_1}{\delta_2} - \frac{\beta_1}{\beta_2})}{(\frac{\gamma_1}{\gamma_2} - \frac{\beta_1}{\beta_2})(\frac{\delta_1}{\delta_2} - \frac{\alpha_1}{\alpha_2})} \\
&= \frac{\frac{1}{\gamma_2\alpha_2}(\gamma_1\alpha_2 - \gamma_2\alpha_1)\frac{1}{\delta_2\beta_2}(\delta_1\beta_2 - \delta_2\beta_1)}{\frac{1}{\gamma_2\beta_2}(\gamma_1\beta_2 - \gamma_2\beta_1)\frac{1}{\delta_2\alpha_2}(\delta_1\alpha_2 - \delta_2\alpha_1)} \\
&= \frac{(\gamma_1\alpha_2 - \gamma_2\alpha_1)(\delta_1\beta_2 - \delta_2\beta_1)}{(\gamma_1\beta_2 - \gamma_2\beta_1)(\delta_1\alpha_2 - \delta_2\alpha_1)}
\end{aligned}$$

9.7.6 Let a,b,c, and d have parametric homogeneous coordinates (α_1, α_2), (β_1, β_2), (γ_1, γ_2) and (δ_1, δ_2) with respect to lines m_1 and m_2 passing through O. Then, by Theorem 9.26, we have that A, B, C, and D can be represented by these same coordinates for points P and Q on l. The result then follows from Corollary 9.34 and Definition 9.27.

9.7.7 Use the coordinates for A, B, C, and D as set up in Theorem 9.33. Let D' have coordinates $(\delta', 1)$. Then,

$$\begin{aligned}
R(A,B;C,D) &= \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)} \\
&= R(A,B;C,D') \\
&= \frac{(\gamma - \alpha)(\delta' - \beta)}{(\gamma - \beta)(\delta' - \alpha)}
\end{aligned}$$

Thus, $\dfrac{\delta' - \beta}{\delta' - \alpha} = \dfrac{\delta - \beta}{\delta - \alpha}$. Or, $(\delta' - \beta)(\delta - \alpha) = (\delta - \beta)(\delta' - \alpha)$. So, $-\delta'\alpha - \delta\beta = -\delta'\beta - \delta\alpha$. Thus, $\delta'(\alpha - \beta) = \delta(\alpha - \beta)$ and $\delta' = \delta$.

9.7.8 For the parametric form of the cross-ratio we have

$$R(A,B;C,D) = \frac{(\gamma - \alpha)(\delta - \beta)}{(\gamma - \beta)(\delta - \alpha)}$$

If this ratio were equal to 0 or infinity then one of the terms in the fraction (e.g. $\gamma - \delta$) would have to be zero. But, this means that two of our points were the same point.

If the cross-ratio is equal to 1, then by Exercise 9.6.4 we have that $R(A,D;C,B) = \frac{1}{1-0}$ which has already been ruled out.

9.7.9 Let the parametric coordinates of A, B, C, D, and E be

$(\alpha,1)$, $(\beta,1)$, $(\gamma,1)$, $(\delta,1)$, and $(\epsilon,1)$. Then,

$$\begin{aligned} R(A,B;C,D)R(A,B;D,E) &= \frac{(\gamma-\alpha)(\delta-\beta)}{(\gamma-\beta)(\delta-\alpha)}\frac{(\delta-\alpha)(\epsilon-\beta)}{(\delta-\beta)(\epsilon-\alpha)} \\ &= \frac{(\gamma-\alpha)(\epsilon-\beta)}{(\gamma-\beta)(\epsilon-\alpha)} \\ &= R(A,B;C,E) \end{aligned}$$

9.7.10 By Theorem 9.37 we know that $R(A,B;C,D) = -1$. By Exercise 9.6.3 we know that the cross-ratio is preserved if any two pairs of points are inter-changed. Thus $R(C,D;A,B) = -1$. By Theorem 9.37 we then have $H(C,D;A,B)$.

9.7.11 If the coordinates for A and C are α and γ then the coordinates for B would be $\beta = \dfrac{\alpha+\gamma}{2}$. Then, for a fourth point D with coordinate δ we have

$$\begin{aligned} R(A,C;B,D) &= \frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\gamma)(\delta-\alpha)} \\ &= \frac{(\frac{\gamma-\alpha}{2})(\delta-\gamma)}{(\frac{\alpha-\gamma}{2})(\delta-\alpha)} \\ &= -\frac{\delta-\gamma}{\delta-\alpha} \end{aligned}$$

If D has coordinates of the point at infinity, then the cross-ratio will be equal to -1 and the four points will form a harmonic set.

9.8 Conics and Coordinates

9.8.1 According to Theorem 9.12 the projectivity defining the conic is equivalent to the composition of two perspectivities. The proof is by contradiction. Suppose that the line AB corresponded to itself under the projectivity. Interpret Lemma 9.10 as a statement about the pencils of points on l, m, and n. The dual to this lemma would be:

"Given three pencils of lines at P, R, and Q with $P \neq Q$, suppose there is a projectivity taking the pencil at P to the pencil at Q. If $l = PQ$ is invariant under the projectivity, then, then the pencil at P is perspective to the pencil at Q."

In the case of the exercise, the line l would be AB. If AB it is invariant, then by the lemma the projectivity is a perspectivity. This implies that the point conic is singular.

9.8.2 Use the classification from Theorem 9.40 which lists the possible types of singular conics. All the singular conics that arise from non-identity projectivities do not satisfy the condition that given five distinct points on the conic, no three are collinear.

9.8.3 Suppose the conic is defined by pencils of points at A and B. Let C, D, E, and F be four other distinct points on the conic. By Theorem 9.41, we know that no subset of three of A, B, C, D, E, and F are collinear. Thus, by Theorem 9.42, the points P, Q, and R defined in the theorem are collinear. If we switch A with C, B with D, and E with F, then the points P, Q, and R are unchanged as points of intersection. Thus, they remain collinear. So, by Theorem 9.42, the point conic is defined by pencils at C and D.

9.8.4 Dual to Theorem 9.42 Let a, b, c, d, e, and f be distinct lines such that no subset of three of the six lines is concurrent. Let p be the line on $a \cdot e$ with $c \cdot f$, q the line on $a \cdot d$ with $c \cdot b$, and r the line on $b \cdot e$ with $d \cdot f$. Then c, d, e, and f are on a non-singular line conic determined by projectively related pencils of points on a and b if and only if p, q, and r are concurrent.

Here is an illustration:

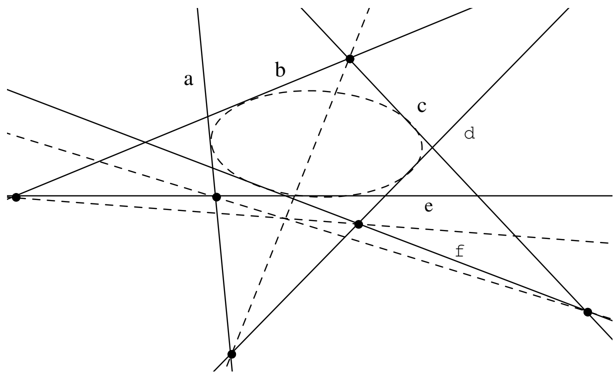


FIGURE 9.7:

9.8.5 If they intersected in five or more points, they would have to be the same conic, by Theorem 9.45.

9.8.6 Re-write the equation as: $2(x^2+4xy+4y^2)+4y^2-4xz+2yz+10z^2=0$. Complete the square: $2(x+2y)^2+4y^2-4xz+2yz+10z^2=0$. Let $x'=x+2y$. Then, $x=x'-2y$. Substituting x' into the equation we get $2x'^2+4y^2-4(x'-2y)z+2yz+10z^2=0$. Simplifying we get $2x'^2+4y^2-4x'z+10yz+10z^2=0$.

9.8.7 Re-write the equations as: $2(x^2 - 2xz + z^2) + 4y^2 + 10yz + 10z^2 = 0$. Let $x' = x - z$. Then, we get $2x'^2 + 4y^2 + 10yz + 10z^2 = 0$.

9.8.8 Re-write the equations as: $2x^2 + 4y^2 + 10(z^2 + yz)$. Complete the square: $2x^2 + 4y^2 + 10((z + \frac{1}{2}y)^2 - \frac{1}{4}y^2) = 0$ Simplifying we get $2x^2 + 4y^2 + 10z'^2 - \frac{5}{2}y^2 = 0$ or $2x^2 + \frac{3}{2}y^2 + 10z'^2 = 0$.

CHAPTER 10

Fractal Geometry

Much of the material in this chapter is at an advanced level, especially the sections on contraction mappings and fractal dimension—Sections 10.5 and 10.6. But this abstraction can be made quite concrete by the computer explorations developed in the chapter. In fact, the computer projects are the *only* way to really understand these geometric objects on an intuitive level, which is probably why no other geometry text covers fractals in great detail.

If one wants to make the chapter more accessible, one could leave out most of section 10.5 and all of 10.6 and cover the other material briefly, focusing most of the student's time on the lab projects.

SOLUTIONS TO EXERCISES IN CHAPTER 10

10.3 Similarity Dimension

The notion of dimension of a fractal is very hard to make precise. In this section we present one simple way to define dimension, but students should know that there are other ways to define dimension as well, each useful for a particular purpose and all agreeing with integer dimension, but not necessarily with each other.

10.3.1 Theorem 2.27 guarantees that the sides of the new triangles are parallel to the original sides. Then, we can use SAS congruence to achieve the result.

10.3.2 The pattern for the total length of each successive stage of the construction is

$$\begin{aligned} l &= 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \dots \\ &= 1 - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k \\ &= 1 - \frac{1}{3}\frac{1}{1-\frac{2}{3}} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

10.3.3 At each successive stage of the construction, 8 new squares are created, each of area $\frac{1}{9}$ the area of the squares at the previous stage. Thus, the pattern for the total area of each successive stage of the construction is

$$\begin{aligned} l &= 1 - \frac{1}{9} - \frac{8}{81} - \frac{64}{9^3} - \dots \\ &= 1 - \frac{1}{9}\sum_{k=0}^{\infty}\left(\frac{8}{9}\right)^k \\ &= 1 - \frac{1}{9}\frac{1}{1-\frac{8}{9}} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Thus, the area of the final figure is 0.

10.3.4 Each sub-square has side-length of $\frac{1}{3}$ the length of a square at the previous stage of construction. Thus, the similarity ratio is $\frac{1}{3}$. Clearly, each of the 8 new sub-sub-squares created at stage n can be scaled up to match the sub-square at stage $n-1$. The result follows from Definition 10.3.

10.3.5 The similarity dimension would be $\frac{\log(4)}{\log(3)}$.

10.3.6 Each sub-cube has length $\frac{1}{3}$ of the original cube, so the similarity ration is $\frac{1}{3}$. Also, at each stage a cube is replaced by 27 sub-cubes, of which 7 are removed. Thus, the similarity ratio is $\frac{\log(20)}{\log(3)}$, which is about 2.727. Thus, the fractal is more "solid-like". For a more "surface-like" fractal, one could remove all eight corner cubes, in addition to the 7 cubes removed in the Menger sponge construction. The similarity dimension of this fractal would be $\frac{\log(12)}{\log(3)}$, which is about 2.262.

10.3.7 Split a cube into 27 sub-cubes, as in the Menger sponge construction, and then remove all cubes except the eight corner cubes and the central cube. Do this recursively. The resulting fractal will have similarity dimension $\frac{\log(9)}{\log(3)}$, which is exactly 2.

10.4 Project 16 - An Endlessly Beautiful Snowflake

Here is another lab that future teachers could modify for a high school setting. If students want a challenge, they could think of other templates based on a simple segment, generalizing the Koch template and the Hat template from exercise 10.4.4

10.4.1 At stage 0 the Koch curve has length 1. At stage 1 it has length $\frac{4}{3}$. At stage 2 it has length $\frac{16}{9} = \frac{4^2}{3^2}$, since each segment is replaced by the template, which is $\frac{4}{3}$ as long as the original segment. Thus, at stage n the length will be $\frac{4^n}{3^n}$, and so the length will go to infinity.

10.4.2 From stage 0 to 1 we add three triangles, each of height and base that is $\frac{1}{3}$ the original triangle. Thus, each new triangle has area $\frac{1}{9}$ the original triangle area and the total area at stage 1 is $1 + \frac{3}{9}$. At stage 2, each of the four segments making up a side of the figure will be replaced by four new segments, thus creating 4 times as many triangles on that side. Each new triangle will have area reduced by a factor of 9. Thus, the total area at stage 2 is $1 + \frac{3}{9} + 4\frac{3}{9^2}$. At stage 3, each side will have 4 times as many triangles again, each of area reduced by a factor of $\frac{1}{9}$. The pattern then is as shown in the exercise.

The sum of the series is

$$\begin{aligned} A &= 1 + \frac{3}{9}\sum_{k=0}^{\infty}\left(\frac{4}{9}\right)^k \\ &= 1 + \frac{3}{9}\frac{1}{1-\frac{4}{9}} \\ &= 1 + \frac{3}{9}\frac{9}{5} \\ &= 1 + \frac{3}{5} \\ &= \frac{8}{5} \end{aligned}$$

10.4.3 The similarity dimension will be that of the template replacement fractal. The similarity ratio is $\frac{1}{3}$ and it takes 4 sub-objects to create the template. Thus, the similarity dimension is $\frac{\log(4)}{\log(3)}$.

10.4.4 The similarity dimension is $\frac{\log(5)}{\log(3)}$.

10.4.5 Students like to design their own fractals.

10.6 Fractal Dimension

Sections 10.5 and 10.6 are quite "thick" mathematically. To get some sense of the Hausdorff metric, have students compute it for some simple pairs of compact sets. For example, two triangles in different positions. Ample practice with examples will help students get a feel for the mini-max approach to the metric and this will also help them be successful with the homework.

10.6.1 A function f is continuous if for each $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ when $0 < |x - y| < \delta$. Let S be a contraction mapping with contraction factor $0 \leq c < 1$. Then, given ϵ, let $\delta = \epsilon$ (if $c = 0$) and $\delta = \frac{\epsilon}{c}$ (if $c > 0$).

If $c = 0$ we have $0 = |S(x) - S(y)| \leq |x - y| < \delta = \epsilon$.

If $c > 0$, we have $|S(x) - S(y)| \leq c|x - y| < c\frac{\epsilon}{c} = \epsilon$.

10.6.2 Let A and B be two disjoint, but congruent, squares, each with a side on the x-axis.

10.6.3 Property (2): Since $d_{\mathcal{H}}(A, A) = d(A, A)$, and since $d(A, A) = max\{d(x, A)|x \in A\}$, then we need to show $d(x, A) = 0$. But, $d(x, A) = min\{d(x, y)|y \in A\}$, and this minimum clearly occurs when $x = y$; that is, when the distance is 0.

Property (3): If $A \neq B$ then we can always find a point x in A that is not in B. Then, $d(x, B) = min\{d(x, y)|y \in B\}$ must be greater than 0. This implies that $d(A, B) = max\{d(x, B)|x \in A\}$ is also greater than 0.

10.6.4 We know that

$$\begin{aligned}
d(A \cup B, C) &= max\{d(x, C)|x \in A \cup B\} \\
&= max\{d(x, C)|x \in A \text{ } or \text{ } x \in B\} \\
&= max\{max\{d(x, C)|x \in A\}, max\{d(x, C)|x \in B\}\} \\
&= max\{d(A, C), d(B, C)\}
\end{aligned}$$

10.6.5 We know that

$$\begin{aligned} d(A, C \cup D) &= max\{d(x, C \cup D)|x \in A\} \\ &= max\{min\{d(x,y)|x \in A \, and \, y \in C \, or \, D\}\} \\ &= max\{min\{min\{d(x,y)|x \in A \, y \in C\}, min\{d(x,y)|x \in A, y \in D\}\}\} \\ &= max\{min\{d(x,C), d(x,D)\}|x \in A\} \end{aligned}$$

The last expression is clearly less than or equal to $max\{d(x, C)|x \in A\} = d(A, C)$ and also less than or equal to $max\{d(x, D)|x \in A\} = d(A, D)$.

10.6.6 Referring to the figures in this section illustrating the construction of Sierpinski's triangle, we see that the triangle is always contained in the squares used in the construction process. These squares can then be used in the computation of $\mathcal{N}_n(S)$, where S is Sierpinski's triangle. Thus, 3 squares of side-length $\frac{1}{2}$ can cover the fractal, as can be seen in the second figure of the process. From the third figure, we see that 9 squares of side-length $\frac{1}{4}$ will cover. Each of these will be split into 3 sub-squares of side-length $\frac{1}{2^3}$ at the next stage of construction, yielding $27 = 3^3$ covering boxes total. Thus, $\mathcal{N}_n(S) = 3^n$ and

$$D_F(S) = \lim_{n \to \infty} \frac{log(3^n)}{log(2^n)} = \frac{log(3)}{log(2)}$$

This value matches the value we calculated for the similarity dimension of Sierpinski's triangle.

10.6.7 There are three contraction mappings which are used to construct Sierpinski's triangle. Each of them has contraction scale factor of $\frac{1}{2}$. Thus, we want $(\frac{1}{2})^D + (\frac{1}{2})^D + (\frac{1}{2})^D = 1$, or $3(\frac{1}{2})^D = 1$, Solving for D we get $D = \frac{log(3)}{log(2)}$.

10.6.8 Start with X being the unit interval. Then, $S_1(X)$ together with $S_2(X)$ is again the unit interval, which means that the unit interval is contained in the attracting set. Now, the only fixed point for S_1 is 0 and the fixed points for S_2 are 0 and 1. Also, both contractions will map points outside of $[0, 1]$ towards this interval. Thus, $[0, 1]$ is the attracting set for the system and the fractal dimension must be 1. However, $S_1([0, 1]) = [0, \frac{1}{2}]$ and $S_2([0, 1]) = [\frac{1}{3}, 1]$. This violates the overlapping criteria of the Theorem.

10.7 Project 17 - IFS Ferns

Students should not worry too much about getting exactly the same numbers for the scaling factor and the rotations that define the fern.

The important idea is that they get the right *types* of transformations (in the correct order of evaluation) needed to build the fern image. For exercise 10.7.5 it may be hopeful for students to copy out one piece of the image and then rotate and move it so it covers the other pieces, thus generating the transformations needed.

10.7.1 The rotation matrix R is given by

$$\begin{bmatrix} \cos(\frac{5\pi}{180}) & \sin(\frac{5\pi}{180}) \\ -\sin(\frac{5\pi}{180}) & \cos(\frac{5\pi}{180}) \end{bmatrix} \approx \begin{bmatrix} 0.996 & 0.087 \\ -0.087 & 0.996 \end{bmatrix}$$

The scaling matrix S is given by

$$\begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$$

If we let T be the translation in the vertical direction by h, then $T_1 = T \circ S \circ R$, which after rounding to the nearest tenth, matches the claimed affine transformation in the text.

10.7.2 The rotation matrix R is given by

$$\begin{bmatrix} \cos(\frac{50\pi}{180}) & -\sin(\frac{50\pi}{180}) \\ \sin(\frac{50\pi}{180}) & \cos(\frac{50\pi}{180}) \end{bmatrix} \approx \begin{bmatrix} 0.643 & -0.766 \\ 0.766 & 0.643 \end{bmatrix}$$

The scaling matrix S is given by

$$\begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}$$

If we let T be the translation in the vertical direction by h, then $T_2 = T \circ S \circ R$, which after rounding to the nearest hundredth, matches the claimed affine transformation in the text.

10.7.3 The rotation matrix R is given by

$$\begin{bmatrix} \cos(\frac{-60\pi}{180}) & -\sin(\frac{-60\pi}{180}) \\ \sin(\frac{-60\pi}{180}) & \cos(\frac{-60\pi}{180}) \end{bmatrix} \approx \begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix}$$

The scaling matrix S is given by

$$\begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}$$

The reflection matrix r is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we let T be the translation in the vertical direction by $\frac{h}{2}$, then $T_3 = T \circ S \circ R \circ r$, which after rounding to the nearest hundredth, matches the claimed affine transformation in the text.

10.7.4 The scale factors are 0.8, 0.3, and 0.3. Thus, $2(0.3)^D + (0.8)^D = 1$. Estimating D (by graphing or using a root-finding algorithm) we find $D \approx 1.58$.

10.7.5 For the lower left portion of the shape, we need to scale the whole figure down by a little less than 0.5, say by 0.48. Also, we need to rotate the figure by 90 degrees and then translate it back by 0.5 in the x-direction to put it in place. Let T_1 be the net transformation accomplishing this. Then $T_1(x, y) = (-0.48y+0.5, 0.48x)$. Let T_2 be the transformation for the upper left portion. Then $T_2(x, y) = (0.5x, 0.5y+0.5)$. Let T_3 be the transformation for the upper right portion. Then $T_3(x, y) = (0.48y + 0.5, -0.48x + 1.0)$. Finally. let T_4 be the transformation for the small inner part. Then $T_4(x, y) = (0.3x+0.3, 0.3y+0.3)$ would work.

10.9 Grammars and Productions

This section will be very different from anything the students have ever done before, except for those students who have had computer science courses. However, the connection between re-writing and axiomatic systems is a deep one. One could view a theorem as essentially a re-writing of various symbols and terms used to initialize a set of axioms. Also, turtle geometry is a very concrete way to view re-writing and so we have a nice concrete realization of an abstract idea.

10.9.1 Repeated use of production rule 1 will result in an expression of the form $a^n S b^n$. Then, using production rule 2, we get $a^n b^n$.

10.9.2 Stage 0 is shown in Fig. 10.1. Stage 1 is shown in Fig. 10.2. The template is shown in Fig. 10.3. The similarity dimension will be $\frac{\ln(8)}{\ln(4)}$.

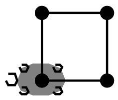


FIGURE 10.1:

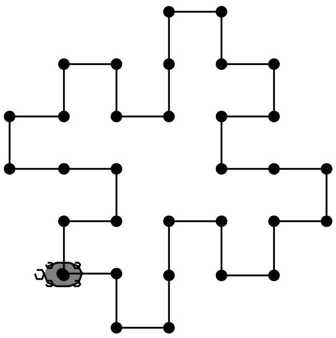


FIGURE 10.2:

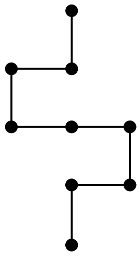


FIGURE 10.3:

10.9.3 The level 1 rewrite is $+RF - LFL - FR+$. This is shown in Fig. 10.4. The level 2 rewrite is $+-LF+RFR+FL-F-+RF-LFL-FR+F+RF-LFL-FR+-F-LF+RFR+FL-+$. This is shown in Fig. 10.5. For the last part of the exercise, the students should recognize that all interior "lattice" points (defined by the length of one segment) are actually visited by the curve. Thus, as the level increases (and we scale the curve back to some standard size) the interior points will cover space, just as the example in section 9.9 did.

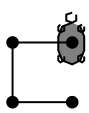


FIGURE 10.4:

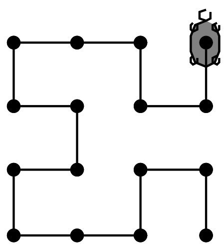


FIGURE 10.5:

10.9.4 If we rotate the left template 180 degrees about Q we will get the right template. An example that works on a 7x7 grid is shown in Fig. 10.6.

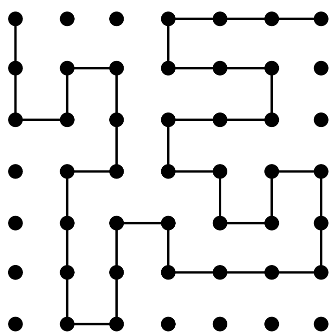


FIGURE 10.6:

10.10 Project 18 - Words Into Plants

Grammars as representations of growth is an idea that can be tied in nicely with the notion of genetics from biology. A grammar is like a blueprint governing the evolution of the form of an object such as a bush, in much the same way that DNA in its expression as proteins governs the biological functioning of an organism.

The material is this section could be adopted for classroom use by future teachers in a much simplified form, or used in its present form for highly gifted high school students wanting more challenging mathematical ideas.

10.10.1 The start symbol was rewritten twice.

10.10.2 The productions are $X- > F[+X][-X]FX$ and $F- > FF$.

10.10.3 Here's one simple example, plus the image generated from rewriting to a level of 3 (Fig. 10.7).

Productions: $X- > F[+X][++X][-X][--X]X$ (Use a small turn angle)

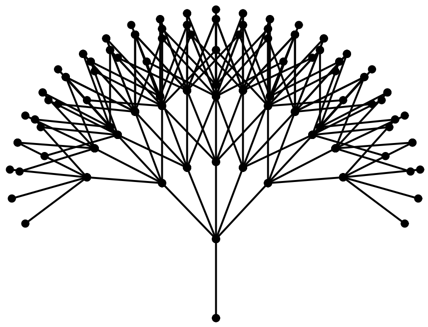


FIGURE 10.7:

APPENDIX A

Sample Lab Report

Pollie Gonn
MCS 303 Project 0
September 12, 2003
The Amazing Pythagorean Theorem

Introduction

The Pythagorean Theorem is perhaps the most famous theorem in geometry, if not in all of mathematics. In this lab, we look at one method of proving the Pythagorean Theorem by constructing a special square. Part I of this report describes the construction used in the proof and Part II gives a detailed explanation of why this construction works, that is why the construction generates a proof of the Pythagorean theorem. Finally, we conclude with some comments on the many proofs of the Pythagorean Theorem.

Part I:

To start out our investigation of the Pythagorean Theorem, we assume that we have a right triangle with legs b and a and hypotenuse c. Our first task construction is that of a segment sub-divided into two parts of lengths a and b. Since a and b are arbitrary, we just create a segment, attach a point, hide the original segment, and draw two new segments as shown.

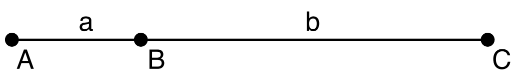


FIGURE A.1:

Then, we construct a square on side a and a square on side b. The purpose of doing this is to create two regions whose total area is a^2+b^2. Clever huh? Constructing the squares involved several rotations, but was otherwise straightforward.

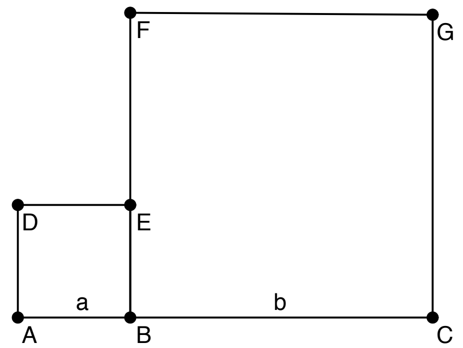


FIGURE A.2:

The next construction was a bit tricky. We define a translation from B to A and translate point C to get point H. Then, we connect H to D and H to G, resulting in two right triangles. In part II, we will prove that both of these right triangles are congruent to the original right triangle.

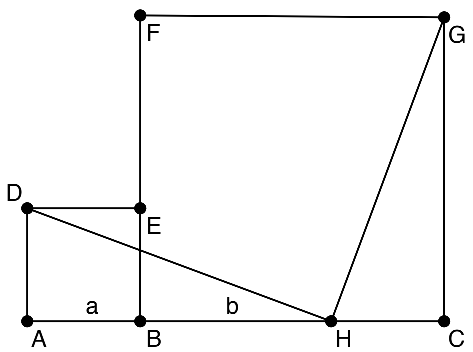


FIGURE A.3:

Next, we hide segment BC and create segments BH and HC. This is so that we have well-defined triangle sides for the next step - rotating right triangle ADH 90 degrees about its top vertex, and right triangle HGC -90 degrees about its top vertex.

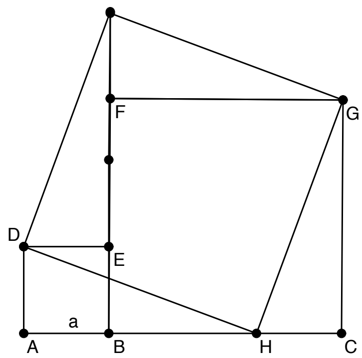


FIGURE A.4:

Part II:

We will now prove that this construction yields a square (on DH) of side length c, and thus, since the area of this square is clearly equal to the sum of the areas of the original two squares, we have $a^2 + b^2 = c^2$, and our proof would be complete. By SAS, triangle HCB must be congruent to the original right triangle, and thus its hypotenuse must be c. Also, by SAS, triangle DAH is also congruent to the original triangle, and so its hypotenuse is also c. Then, angles AHD and CHG(= ADH) must sum to 90 degrees, and the angle DHG is a right angle. Thus, we have shown that the construction yields a square on DH of side length c, and our proof is complete.

Conclusion:

This was a very elegant proof of the Pythagorean Theorem. In researching the topic of proofs of the Pythagorean Theorem, we discovered that over 300 proofs of this theorem have been discovered. Elisha Scott Loomis, a mathematics teacher from Ohio, compiled many of these proofs into a book titled *The Pythagorean Proposition*, published in 1928. This tidbit of historical lore was gleaned from the Ask Dr. Math website (http://mathforum.org/library/drmath/view/62539.html). It seems that people cannot get enough of proofs of the Pythagorean Theorem.

APPENDIX B

Sample Lab Grading Sheet

Sample Grade Sheet for Project 1 - The Ratio Made of Gold

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Solution to Exercise 1.3.1
  - 5 Solution to Exercise 1.3.2
  - 5 Solution to Exercise 1.3.3
  - 5 Solution to Exercise 1.3.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 2 - A Concrete Axiomatic System

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 4 Discussion of Euclid's Five Postulates
  - 4 Construction of Rectangles
  - 4 Sum of Angles in a Triangle
  - 4 Euclid's Equilateral Triangle Construction
  - 4 Perpendicular to a Line through a Point Not on the Line
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 3 - Special Points of a Triangle

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 4 Exercise 2.3.1
  - 4 Exercise 2.3.2
  - 4 Exercise 2.3.3
  - 4 Exercise 2.3.4
  - 4 Exercise 2.3.5
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 4 - Circle Inversion and Orthogonality

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 2.7.1
  - 5 Exercise 2.7.2
  - 5 Exercise 2.7.3
  - 5 Exercise 2.7.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 5 - Bézier Curves

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 3.3.1
  - 5 Exercise 3.3.2
  - 5 Exercise 3.3.3
  - 5 Exercise 3.3.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 6 - Euclidean Eggs

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 15 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 4.2.1
  - 5 Exercise 4.2.2
  - 5 Exercise 4.2.3
- Total Points for Project (out of 25 possible)

Sample Grade Sheet for Project 7 - Quilts and Transformations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 5.5.1
  - 5 Exercise 5.5.2
  - 5 Exercise 5.5.3
  - 5 Exercise 5.5.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 8 - Constructing Compositions

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 5.8.1
  - 5 Exercise 5.8.2
  - 5 Exercise 5.8.3
  - 5 Exercise 5.8.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 9 - Constructing Tesselations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 10 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 6.5.1
  - 5 Exercise 6.5.2
- Total Points for Project (out of 20 possible)

Sample Grade Sheet for Project 10 - The Saccheri Quadrilateral

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 15 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 7.4.1
  - 2 Exercise 7.4.2 part i
  - 2 Exercise 7.4.2 part ii
  - 2 Exercise 7.4.2 part iii
  - 2 Exercise 7.4.2 part iv
  - 2 Exercise 7.4.2 part v
- Total Points for Project (out of 25 possible)

Sample Grade Sheet for Project 11 - Tiling the Hyperbolic Plane

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 15 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 7.7.1
  - 5 Exercise 7.7.2
  - 5 Exercise 7.7.3
- Total Points for Project (out of 25 possible)

Sample Grade Sheet for Project 12 - Models of Elliptic Geometry

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 8.3.1
  - 5 Exercise 8.3.2
  - 5 Exercise 8.3.3
  - 5 Exercise 8.3.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 13 - Elliptic Tiling

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 8.6.1
  - 10 Exercise 8.6.2
  - 5 Exercise 8.6.3
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 14 - Perspective and Projection

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 20 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 9.2.1
  - 5 Exercise 9.2.2
  - 5 Exercise 9.2.3
  - 5 Exercise 9.2.4
- Total Points for Project (out of 30 possible)

Sample Grade Sheet for Project 15 - Ratios and Harmonics

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 25 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 9.6.1
  - 5 Exercise 9.6.2
  - 5 Exercise 9.6.3
  - 5 Exercise 9.6.4
  - 5 Exercise 9.6.5
- Total Points for Project (out of 35 possible)

Sample Grade Sheet for Project 16 - An Endlessly Beautiful Snowflake

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 25 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 10.4.1
  - 5 Exercise 10.4.2
  - 5 Exercise 10.4.3
  - 5 Exercise 10.4.4
  - 5 Exercise 10.4.5
- Total Points for Project (out of 35 possible)

Sample Grade Sheet for Project 17 - IFS Ferns

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 25 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 10.7.1
  - 5 Exercise 10.7.2
  - 5 Exercise 10.7.3
  - 5 Exercise 10.7.4
  - 5 Exercise 10.7.5
- Total Points for Project (out of 35 possible)

Sample Grade Sheet for Project 18 - Words Into Plants

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and a conclusion.
  - 5 Spelling and Grammar
- 15 points - Discussion of Project Work and Solutions to Exercises
  - 5 Exercise 10.10.1
  - 5 Exercise 10.10.2
  - 5 Exercise 10.10.3
- Total Points for Project (out of 25 possible)