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Intersection graphs of general linear groups

Mai Hoang Bien* and Do Hoang Viet†

Faculty of Mathematics and Computer Science
University of Science, VNU-HCMC, 227 Nguyen Van Cu Street
District 5, Ho Chi Minh City, Vietnam
**mhbien@hcmus.edu.vn*
†vietdohoang10@gmail.com

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Let F be a field and $\mathrm{GL}_n(F)$ the general linear group of degree $n > 1$ over F. The intersection graph $\Gamma(\mathrm{GL}_n(F))$ of $\mathrm{GL}_n(F)$ is a simple undirected graph whose vertex set includes all nontrivial proper subgroups of $\mathrm{GL}_n(F)$. Two vertices A and B of $\Gamma(\mathrm{GL}_n(F))$ are adjacent if $A \neq B$ and $A \cap B \neq \{I_n\}$. In this paper, we show that if F is a finite field containing at least three elements, then the diameter $\delta(\Gamma(\mathrm{GL}_n(F)))$ is 2 or 3. We also classify $\mathrm{GL}_n(F)$ according to $\delta(\Gamma(\mathrm{GL}_n(F)))$. In case F is infinite, we prove that $\Gamma(\mathrm{GL}_n(F))$ is one-ended of diameter 2 and its unique end is thick.

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1. Introduction

The relationship of graph and group theories is interesting and has a long history (over 140 years). Cayley is maybe the first person who associated groups to graphs in 1878 [9]. The main idea of associations is to describe the algebraic properties of a certain group according to its associated graph. To do this, we make a "good association" from a certain group to a graph. We then investigate this associated graph and, finally, apply the results to the base group. The "good association" depends on algebraic properties of groups to study and its applications. For example, see [5, 19, 20] for the definition and applications of Cayley graphs of groups; we refer to [4, 6, 8, 15] for the definition of the commuting graph of groups and its related issues; [1, 23, 24, 28] presented the non-commutating graph of a group, defined by Neumann in 1976; for a good survey and some applications of directed power graphs (see [10, 11, 17]). We refer to [13] for an overview of graphs associated with subgroups of groups.

*Corresponding author.

In this paper, we study the intersection graph of a group. Recall that *an undirected simple graph without loop* Γ is a pair of sets $(\mathcal{V}, \mathcal{E})$ in which $\mathcal{E}$ includes some elements $\{a, b\}$, where a, b are distinct elements in $\mathcal{V}$. The sets $\mathcal{V}$ and $\mathcal{E}$ are called the *vertex set* and *edge set* of Γ, respectively. Every element of $\mathcal{V}$ is called a *vertex* of Γ and every element $\{a, b\} \in \mathcal{E}$ is called an *edge* of Γ. For convenience, we write $a - b$ if $\{a, b\}$ is an edge of Γ and say that a and b are adjacent. If the vertex set $\mathcal{V}$ is finite (obviously, in this case, the edge set $\mathcal{E}$ is also finite), then Γ is called a *finite graph*. In this paper, our graphs are understood to be undirected simple without loop.

Let G be a group. The intersection graph $\Gamma(G) = (\mathcal{V}_G, \mathcal{E}_G)$ of G is a graph defined as follows: the vertex set includes all nontrivial proper subgroup of G. Two vertices A and B of $\Gamma(G)$ are adjacent if $A \neq B$ and $A \cap B \neq \{1\}$. Observe that our assumption allows the case when G is infinite, so the intersection graph $\Gamma(G)$ maybe contains infinite many vertices. Clearly, if G is finite, then the graph $\Gamma(G)$ is finite.

The intersection graph of a group was first defined by Csakany and Pollak in 1969 [12] with inspiration from [7]. There are a lot of interesting results on intersection graphs. In 2010, Shen classified all finite groups whose intersection graphs are disconnected [27]. In 1975, Zelinka investigated the intersection graphs of finite abelian groups. In 2016, Ma evaluated the diameter of intersection graphs of finite non-abelian simple groups [21]. The planarity of intersection graph of finite groups was mentioned in [2, 18]. In 2015, the girth of intersection graph of a group was considered in [3]. Some induced subgraphs of intersection graphs of groups have been also studied. For example, see [14] for the intersection graph of abelian subgroups and [26] the intersection graph of cyclic subgroups of a finite group.

The main purpose of this paper is to investigate the intersection graphs of general linear groups $\mathrm{GL}_n(F)$ of degree $n > 1$ over an arbitrary field F. For convenience, we recall some notion in an arbitrary simple undirected graph without loop. Let Γ be a simple undirected graph without loop, let u and v be two distinct vertices of Γ. A path between u and v is defined as a sequence of distinct vertices $u = v_0 - v_1 - \cdots - v_n = v$ in which the first and last vertices are u and v, respectively, and two consecutive vertices are adjacent. The number n in the sequence is called the *length* of the path. The distance between u and v is defined as the smallest length of all paths between u and v and is denoted by $d(u, v)$. If there is no path between u and v, we write $d(u, v) = \infty$. The graph Γ is called *connected* if $d(u, v) < \infty$ for every two distinct vertices $u, v \in \Gamma$. Additionally, if $d(u, v)$ is bounded when u, v range over $\mathcal{V}$, then the number

$$\delta(\Gamma) = \max\{d(u, v) \mid u, v \in \Gamma, u \neq v\}$$

is called *the diameter* of Γ, otherwise, we denote by $\delta(\Gamma) = \infty$. Remark that for a connected graph Γ, if the vertex set $\mathcal{V}$ is finite, then $\delta(\Gamma)$ is finite. In case Γ is infinite, $\delta(\Gamma)$ is unnecessarily finite. For example, if Γ is a graph which is an infinite path

$$\Gamma = v_0 - v_1 - \cdots - v_n - \cdots,$$

then $\delta(\Gamma) = \infty$. The first aim of this paper is to show that if F is a finite field containing at least three elements and $n > 1$, then $2 \le \delta(\Gamma(\mathrm{GL}_n(F))) \le 3$ (see Sec. 2). In Sec. 3, we classify $\mathrm{GL}_n(F)$ according to the diameter $\delta(\Gamma(\mathrm{GL}_n(F)))$. We spend Sec. 4 for the case when F is infinite. We show that in this case, the diameter $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$. Moreover, since the vertex set of $\Gamma(\mathrm{GL}_n(F))$ is infinite, we have an opportunity to study the set of rays of $\Gamma(\mathrm{GL}_n(F))$.

Recall that in an infinite graph $\Gamma = (\mathcal{V}, \mathcal{E})$, a *ray* of Γ is a one-way infinite path in Γ, that is, a ray is a countable sequence of distinct vertices of $\mathcal{V}$

$$v_0 - v_1 - \cdots - v_n - \cdots$$

in which two consecutive vertices are adjacent. Let

$$(v_i)_{i\ge 0} = v_0 - v_1 - \cdots - v_n - \cdots \quad \text{and} \quad (u_i)_{i\ge 0} = u_1 - u_2 - \cdots - u_n - \cdots$$

be two rays of Γ. We say that $(u_i)_{i\ge 0}$ and $(v_i)_{i\ge 0}$ are *end-equivalent*, and denote by $(u_i)_{i\ge 0} \equiv (v_i)_{i\ge 0}$, if there exists a ray $(w_i)_{i\ge 0}$ such that $\{w_i \mid i \in \mathbb{N}\} \cap \{u_i \mid i \in \mathbb{N}\}$ and $\{w_i \mid i \in \mathbb{N}\} \cap \{v_i \mid i \in \mathbb{N}\}$ are infinite [16]. It is shown that $\equiv$ is an equivalent relation. For a ray $(u_i)_{i\ge 0}$ of Γ, we denote by $[(u_i)_{i\ge 0}]$ the class of all rays of Γ which are end-equivalent to $(u_i)_{i\ge 0}$. The class $[(u_i)_{i\ge 0}]$ is called an *end* of Γ. If $[(u_i)_{i\ge 0}]$ contains infinitely many disjoint rays, then we say that $[(u_i)_{i\ge 0}]$ is *thick*, otherwise, it is *thin*. We call Γ *one-ended* if Γ has exactly one end. In this paper, we prove that if F is an infinite field, then $\Gamma(\mathrm{GL}_n(F))$ is one-ended and its unique end is thick.

Our notations in this paper are standard. In particular, $\mathrm{GL}_n(F)$ is the general linear group of degree $n \ge 1$ over a field F, $\mathrm{SL}_n(F)$ is the special linear group of degree n over F. It is well know that the center $Z(\mathrm{GL}_n(F))$ is $\{\alpha I_n \mid \alpha \in F^*\}$, so for convenience, αI_n is written shortly by α for every $\alpha \in F^*$. For every $1 \le i, j \le n$, we denote by e_{ij} the matrix in $M_n(F)$ in which the (i,j)-entry is 1 and the other entries are 0. For a group G, the order of G is denoted by $|G|$.

2. The Diameter of $\Gamma(\mathrm{GL}_n(F))$ Over Finite Fields F

Our field F in this paper is assumed to contain at least three elements since we want the intersection graph to be connected. The following example shows that in the case $F = \mathbb{Z}/2\mathbb{Z}$, the intersection graph of $\mathrm{GL}_n(F)$ is unnecessarily connected. Let $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ be the general linear group of degree 2 over the field $\mathbb{Z}/2\mathbb{Z}$. We may check that

$$\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, I_2, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

The general linear group $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ has 6 subgroups: $A = \{I_2\}$, $B = \left\{I_2, \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right\}$, $C = \left\{I_2, \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]\right\}$, $D = \left\{I_2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right\}$, $E = \left\{I_2, \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right\}$ and $G = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$. Hence, the intersection graph $\Gamma(\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}))$ has four vertices B, C, D and E, and no edge. There is no much information of this graph.

For convenience, if A is a vertex in $\Gamma(\mathrm{GL}_n(F))$, then the *order* of A is understood to be the order of A as a subgroup in $\mathrm{GL}_n(F)$. We first borrow a lemma from [12] (or see [27, Lemma 1]).

Lemma 2.1 ([12]). *For a finite group G, if $\Gamma(G)$ is connected, then*

$$\delta(\Gamma(G)) = \max\{d(A,B) \mid A, B \in \mathcal{V}_G \text{ of prime order and } A \neq B\}.$$

Observe that in [27], it was shown that if G is not simple and the intersection graph G is connected, then the diameter $\delta(\Gamma(G))$ of $\Gamma(G)$ is less than or equal to 4 (see [27, Lemma 5]). If F is a field containing at least three elements, then the general linear group $\mathrm{GL}_n(F)$ is not simple for every $n \geq 2$ since $\{I_n\} \neq \mathrm{SL}_n(F) \neq \mathrm{GL}_n(F)$. Hence, if $\Gamma(\mathrm{GL}_n(F))$ is connected, then, by [27, Lemma 5], the diameter of $\Gamma(\mathrm{GL}_n(F))$ is less than or equal to 4. The following result, we evaluate the diameter of $\Gamma(\mathrm{GL}_n(F))$ better in case it is connected.

Lemma 2.2. *Let F be a finite field containing at least three elements and $n > 1$. If $\Gamma(\mathrm{GL}_n(F))$ is connected, then $\delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$.*

Proof. By Lemma 2.1, it suffices to show that $d(A,B) \leq 3$ for every two distinct vertices A and B in $\Gamma(\mathrm{GL}_n(F))$ of prime order. Assume that $A = \langle a \rangle$ and $B = \langle b \rangle$ are two subgroups of $\mathrm{GL}_n(F)$ of prime orders p and q, respectively. Let α be a primitive element of F^*, that is, $F^* = \langle \alpha \rangle$. Put $A_1 = \langle a, \alpha \rangle$ and $B_1 = \langle b, \alpha \rangle$. Since α is central, A_1 and B_1 is abelian, so A_1 and B_1 are nontrivial proper subgroups of $\mathrm{GL}_n(F)$. It implies that A_1 and B_1 are vertices of $\Gamma(\mathrm{GL}_n(F))$ and they are adjacent since $F^* \leq A_1 \cap B_1$ and F^* contains at least two elements. If $A = A_1$ and $B = B_1$, then since $F^* \leq A, F^* \leq B$ and A, B are of prime order, $F^* = A$ and $F^* = B$, which contradicts the hypothesis. Hence, either $A \subsetneqq A_1$ or $B \subsetneqq B_1$. Without loss of generality, we suppose that $A \subsetneqq A_1$. In this case, we have two cases. If $B \subsetneqq B_1$, then we have the path $A - A_1 - B_1 - B$. Hence, $d(A,B) \leq 3$. If $B = B_1$, then $B \cap A_1$ contains F^* and since F^* contains at least two elements, one has the path $A - A_1 - B$. In this case, $d(A,B) \leq 2$. All our cases lead $d(A,B) \leq 3$. The proof is complete. □

Now we are ready to show our main result in this section.

Theorem 2.3. *Let F be a finite field containing at least three elements and let n be a integer greater than 1. The following assertions hold*:

(1) *If A is a vertex of $\Gamma(\mathrm{GL}_n(F))$, then $d(A, \mathrm{SL}_n(F)) \leq 2$.*
(2) *$\Gamma(\mathrm{GL}_n(F))$ is connected and $2 \leq \delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$.*

Proof. Assume that $p = \mathrm{Char}(F)$ and $|F| = p^m$ for some positive integer m.

(1) Since F contains at least three elements, $\mathrm{SL}_n(F)$ is a nontrivial proper subgroup of $\mathrm{GL}_n(F)$, which implies that $\mathrm{SL}_n(F)$ is a vertex of $\Gamma(\mathrm{GL}_n(F))$. Now let A be a vertex of $\Gamma(\mathrm{GL}_n(F))$ different from $\mathrm{SL}_n(F)$, that is, A is a nontrivial proper

subgroup of $\mathrm{GL}_n(F)$ and $A \neq \mathrm{SL}_n(F)$. We must show that $d(A, \mathrm{SL}_n(F)) \leq 2$. Assume that $d(A, \mathrm{SL}_n(F)) > 2$. We seek a contradiction. Since $d(A, \mathrm{SL}_n(F)) > 2$, the intersection $A \cap \mathrm{SL}_n(F) = \{I_n\}$. In particular, if $[A, A] = \langle aba^{-1}b^{-1} \mid a, b \in A\rangle$ is the derived subgroup of A, then $[A, A] \subseteq A \cap \mathrm{SL}_n(F) = \{I_n\}$, so A is abelian. Let $a \in A\backslash\{I_n\}$ and $b \in \mathrm{SL}_n(F)\backslash\{I_n\}$. We claim that $\mathrm{GL}_n(F) = \langle a, b\rangle$. If $\langle a, b\rangle \neq \mathrm{GL}_n(F)$, then we have the path $A - \langle a, b\rangle - \mathrm{SL}_n(F)$ since $a \in A \cap \langle a, b\rangle$ and $b \in \mathrm{SL}_n(F) \cap \langle a, b\rangle$. Therefore, $d(A, \mathrm{SL}_n(F)) = 2$, a contradiction. Hence, $\mathrm{GL}_n(F) = \langle a, b\rangle$. The claim is shown. If a is central, then $\langle a, b\rangle$ is abelian, so is $\mathrm{GL}_n(F)$, a contradiction. Thus, a is non-central. Now as $\mathrm{GL}_n(F)/\mathrm{SL}_n(F) \cong F^*$ via the determinant morphism $\det : \mathrm{GL}_n(F) \to F^*, a \mapsto \det(a)$, one has that $\det(a)$ is a primitive element of F^*, that is, $F^* = \langle\det(a)\rangle$. In particular, the order of $\det(a)$ is $p^m - 1$. We separate our proof into two cases.

Case 1. The case $\mathrm{Char}(F) = p > 2$. Then, the cardinality of F^* is $p^m - 1$, an even number. If $a^2 \neq I_n$, since $F^* = \langle\det(a)\rangle$ holds when a ranges over $A\backslash\{I_n\}$, then $F^* = \langle\det(a^2)\rangle$. Hence, the order of $\det(a)^2$ is also $p^m - 1$. Therefore, the order of $\det(a)$ is less than or equal to $\frac{p^m-1}{2}$, which contradicts the fact that $p^m - 1$ is the order of $\det(a)$. It implies that $a^2 = I_n$. Consequently, $\det(a)^2 = 1$. As a result, $\det(a) = -1$ since $a \notin \mathrm{SL}_n(F)$. In particular, the cardinality of F^* is 2, which implies that $F \cong \mathbb{Z}/3\mathbb{Z}$. Moreover, since a ranges over A, one has every non-identity element of A is of order 2 and determinant -1. Assume that $a, a_1 \in A$. Then, $\det(aa_1) = \det(a)\det(a_1) = (-1)(-1) = 1$, which implies that $aa_1 \in \mathrm{SL}_n(F)$. We conclude that $aa_1 = I_n$, that is, $a_1 = a$ since $a^2 = I_n$. Thus $A = \{I_n, a\}$. Now put $B = \{I_n, a, -I_n, -a\}$. As a is non-central, $a \neq -I_n$ and hence $A \subsetneqq B$. We claim that $d(B, \mathrm{SL}_n(F)) = 1$. Indeed, if n is even, then $\det(-I_n) = 1$, otherwise, if n is odd, then $\det(-a) = (-)^n \det a = (-1)(-1) = 1$. As a result, $B \cap \mathrm{SL}_n(F)$ is nontrivial since either $-I_n \in \mathrm{SL}_n(F)$ or $-a \in \mathrm{SL}_n(F)$, so one has the path $A - B - \mathrm{SL}_n(F)$. It concludes that $d(A, \mathrm{SL}_n(F)) \leq 2$, a contradiction.

Case 2. The case when $\mathrm{Char}(F) = 2$. Then, the cardinality $|F|$ is 2^m and $m > 1$ by hypothesis that $|F| \geq 3$. Let s be a prime factor of $2^m - 1$. Consider the element a^s. If $a^s \neq I_n$, then using the same arguments as in Case 1, one has $\langle\det(a^s)\rangle = F^*$. Since $\det(a^s) = \det(a)^s$, the order of $\det(a)$ is less than or equal to $\frac{2^p-1}{s}$ which contradicts the fact that $\langle\det(a)\rangle = F^*$. As a result, $a^s = I_n$. If $2^m - 1$ has two distinct prime factors s and t, then $a^s = I_n$ and $a^t = I_n$. It implies that $s = \mathrm{ord}(a) = t$, a contradiction. Hence, $2^m - 1 = s^k$ for some prime number s and natural number k. By Catalan's Theorem (for example, see [22]), $k = 1$, that is, $s = 2^m - 1$ is prime. Let $a, a_1 \in A$. Then, $\langle a\rangle$ and $\langle a_1\rangle$ are groups of prime cardinality s. Since $\langle\det(a)\rangle = F^*$, there exists a positive integer ℓ such that $\det(a_1) = \det(a)^\ell = \det(a^\ell)$. Therefore, $\det(a_1^{-1}a^\ell) = 1$, so $a_1^{-1}a^\ell \in \mathrm{SL}_n(F)$, which implies that $a_1^{-1}a^\ell = I_n$ by the assumption that $A \cap \mathrm{SL}_n(F) = \{I_n\}$. Consequently, $a_1 = a^\ell \in \langle a\rangle$. Thus, $\langle a\rangle = \langle a_1\rangle$. It implies that $A = \{I_n, a, \ldots, a^{s-1}\}$. Now let α be a primitive element of F^*, that is, $F^* = \langle\alpha\rangle$. In particular, there exists $1 \leq i \leq s-1$ such that $\det(a) = \alpha^i$. Put $B = \langle\alpha, a\rangle$. If $A = B$, then A contains F^*, so $A = F^*$

since the order s of A is prime. It contradicts the fact that a is non-central. Hence, $A \subsetneq B$. By definition, A and B are adjacent. If s is a divisor of n, that is, $n = st$, then $\det(\alpha I_n) = \alpha^n = \alpha^{st} = (\alpha^s)^t = 1$. Otherwise, if s is not a factor of n, then there exist integers μ and ν such that $\mu n + \nu s = 1$. It implies that

$$\det(\alpha^{-\mu i} a) = \alpha^{-\mu i} \det(a) = \alpha^{-\mu i}\alpha^i = \alpha^{-\mu i + i} = \alpha^{i\nu s} = (\alpha^s)^{i\nu} = 1.$$

These two cases lead us to conclusion that either $\alpha I_n \in \mathrm{SL}_n(F)$ or $\alpha^{-\mu i} a \in \mathrm{SL}_n(F)$. Observe that αI_n and $\alpha^{-\mu i} a$ are non-identity since a is non-central and s is not a factor of μi, so the intersection $\mathrm{SL}_n(F) \cap B$ is nontrivial. Consequently, we have the path $A - B - \mathrm{SL}_n(F)$, so $d(A, \mathrm{SL}_n(F)) \leq 2$, a contradiction.

Two cases lead to a contradiction. Thus, $d(A, \mathrm{SL}_n(F)) \leq 2$ for every vertex A of $\Gamma(\mathrm{GL}_n(F))$ different from $\mathrm{SL}_n(F)$.

(2) By (1), $\Gamma(\mathrm{SL}_n(F))$ is connected. By Lemma 2.2, $\delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$. Now we show $\delta(\Gamma(\mathrm{GL}_n(F))) \geq 2$. Let $c = I_n + e_{12}$ and $d = I_n + e_{21}$ be two elements in $\mathrm{GL}_n(F)$. If $p = \mathrm{Char}(F)$, then $c^p = I_n + pe_{12} = I_n$ and $d^p = I_n + pe_{21} = I_n$, so $C = \langle c \rangle$ and $D = \langle d \rangle$ are subgroups of $\mathrm{GL}_n(F)$ of order p. Since $c \notin \langle d \rangle$ and $d \notin \langle c \rangle$, two subgroups C and D are distinct. Hence, $d(C, D) \geq 2$. Consequently, $\delta(\Gamma(\mathrm{GL}_n(F))) \geq 2$. The proof now is complete. □

The following example shows that 3 is the best upper bound of the diameter of $\Gamma(\mathrm{GL}_n(F))$.

Proposition 2.4. $\delta(\Gamma(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}))) = 3$.

Proof. If $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$, then $a^2 = b^3 = I_2$ and

$$\langle a, b \rangle = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}).$$

It is clear that two vertices $\langle a \rangle$ and $\langle b \rangle$ are not adjacent. Hence, by Theorem 2.3, $2 \leq d(\langle a \rangle, \langle b \rangle) \leq 3$. If $d(\langle a \rangle, \langle b \rangle) = 2$, then there exists a vertex C of $\Gamma(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}))$ such that

$$\langle a \rangle - C - \langle b \rangle.$$

Therefore, $\langle a \rangle \cap C$ and $\langle b \rangle \cap C$ are nontrivial, which implies that $a, b \in C$ since the orders of a, b are prime numbers. Consequently, $C = \langle a, b \rangle = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$, which contradicts the hypothesis. Thus, $d(\langle a \rangle, \langle b \rangle) = 3$. By Lemma 2.1, $\delta(\Gamma(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})))$ $= 3$. The proof is complete. □

3. Classifying of $\mathrm{GL}_n(F)$ According to the Diameter $\delta(\Gamma(\mathrm{GL}_n(F)))$

In Sec. 2, we show that if F is a finite field containing at least three elements and $n > 1$, then $2 \leq \delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$. In this section, we classify $\mathrm{GL}_n(F)$ according to the diameter $\delta(\Gamma(\mathrm{GL}_n(F)))$. It is shown that the general linear group $\mathrm{GL}_n(F)$ of degree $n > 1$ over a finite field is generated by two elements (e.g. see [29]). The following result gives us a relationship between the orders of generators and the diameter $\delta(\Gamma(\mathrm{GL}_n(F)))$.

Theorem 3.1. *Let F be a finite field containing at least three elements and let $n \geq 2$. Then, $\delta(\Gamma(GL_n(F))) = 3$ if and only if $GL_n(F)$ is generated by two elements of prime order.*

Proof. Assume that $\delta(\Gamma(\mathrm{GL}_n(F))) = 3$. Then, by Lemma 2.1, there exist two vertices A, B in $\Gamma(\mathrm{GL}_n(F))$ of prime order such that $d(A, B) = 3$. Since A and B are of prime order, $A = \langle a\rangle$ and $B = \langle b\rangle$ where a and b are of prime order. Moreover, A and B are distinct, so $\langle a, b\rangle$ is different from A and B. If $\langle a, b\rangle \neq \mathrm{GL}_n(F)$, then one has the path $A - \langle a, b\rangle - B$, which implies that $d(A, B) = 2$, a contradiction. Hence, $\mathrm{GL}_n(F) = \langle a, b\rangle$.

Conversely, assume that $\mathrm{GL}_n(F) = \langle a, b\rangle$, where a, b are elements in $\mathrm{GL}_n(F)$ of prime order. Let $A = \langle a\rangle$ and $B = \langle b\rangle$. Then, since A and B are abelian, A and B are proper subgroups of $\mathrm{GL}_n(F)$, which implies that A and B are vertices in $\Gamma(\mathrm{GL}_n(F))$ of prime order. As A and B are distinct and of prime order, the intersection $A \cap B = \{I_n\}$, which implies that $d(A, B) > 1$. If $d(A, B) = 2$, then there exists a vertex $C \in \Gamma(\mathrm{GL}_n(F))$ different from A and B such that $A - C - B$ is a path. Therefore, $A \cap C \neq \{I_n\}$ and $B \cap C \neq \{I_n\}$. Again, because A and B are of prime order, $A \leq C$ and $B \leq C$. Consequently, $a, b \in C$. Thus, $\mathrm{GL}_n(F) = \langle a, b\rangle \leq C$, which contradicts the hypothesis. It implies that $d(A, B) \geq 3$. By Lemma 2.1 and Theorem 2.3, $\delta(\Gamma(\mathrm{GL}_n(F))) = 3$. ☐

We next find some conditions on the order of F such that $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$.

Proposition 3.2. *Let F be a finite field contain at least three elements and let $n \geq 2$. If the multiplicative group $F^* \neq \langle \alpha, \beta\rangle$ for every elements $\alpha, \beta \in F^*$ of prime order, then $\delta(\Gamma(GL_n(F))) = 2$.*

Proof. By Lemma 2.1 and Theorem 2.3, it suffices to show that $d(A, B) \leq 2$ for every two distinct vertices A, B of $\Gamma(\mathrm{GL}_n(F))$ of prime order. Assume that A, B are two distinct vertices of $\mathrm{GL}_n(F)$ of prime order. Since A and B are distinct and of prime order as subgroups of $\mathrm{GL}_n(F)$, $A \cap B = \{I_n\}$. Suppose that $A = \langle a\rangle$ and $B = \langle b\rangle$, where a and b are distinct elements of $\mathrm{GL}_n(F)$ of prime orders p and q, respectively. Then, $\det(a), \det(b) \in F^*$ of orders p and q, respectively. By hypothesis, $F^* \neq \langle \det(a), \det(b)\rangle$, which implies that $\mathrm{GL}_n(F) \neq \langle a, b\rangle$. Therefore, $\langle a, b\rangle$ is a vertex of $\Gamma(\mathrm{GL}_n(F))$ and one has a path $A - \langle a, b\rangle - B$. Thus, $d(A, B) \leq 2$. The proof is complete. ☐

Corollary 3.3. *Let F be a finite field containing at least three elements and $n \geq 2$. Then $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$ if $|F^*|$ satisfies one of the following conditions*:

(1) *There exist prime numbers $h, k, \ell > 1$ such that $hk\ell$ is a divisor of $|F^*|$.*
(2) *The cardinality $|F^*| = h^2$ for some prime number h.*

Proof. (1) Let $\alpha, \beta \in F^*$ be of prime orders h and k, respectively, then the order of the subgroup $\langle \alpha, \beta \rangle$ of F^* is at most hk, so since $|F^*| > hk\ell$, $F^* \neq \langle \alpha, \beta \rangle$. By Proposition 3.2, $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$.

(2) Assume that $\mathrm{Char}(F) = p$. Then $|F^*| = p^\mu$ for some positive integer μ. Hence, $p^\mu - 1 = h^2$, equivalently, $p^\mu - h^2 = 1$. According to Catalan's theorem (see [22]), $\mu = 1$. If h is odd, then p is even and greater than 2, which is a contradiction. Therefore, $h = 2$. It implies that $p = 5$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be subgroups of $\mathrm{GL}_n(F)$ of prime order h and k, respectively. Put $\alpha = \det(a)$ and $\beta = \det(b)$. One has $\alpha^h = 1$ and $\beta^k = 1$ since $a^h = I_n$ and $b^k = I_n$. Hence, the orders of α and β are either prime numbers or 1. Because $|F^*| = 4$, the orders of α and β are divisors of 2. If either $a \in \mathrm{SL}_n(F)$ or $b \in \mathrm{SL}_n(F)$, then either $\alpha = 1$ or $\beta = 1$. Without loss of generality, we assume that $\alpha = 1$. Then $\langle \alpha, \beta \rangle = \langle \beta \rangle$ of order 1 or 2. Therefore, $F^* \neq \langle \alpha, \beta \rangle$. By Proposition 3.2, $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$. Now assume that both a, b are not in $\mathrm{SL}_n(F)$, that is, $\alpha \neq 1$ and $\beta \neq 1$. Then, α, β are of order 2. As $\overline{4}$ is the unique element of order 2 in $\mathbb{Z}/5\mathbb{Z}$, $\alpha = \beta$, then $\langle \alpha, \beta \rangle = \langle \alpha \rangle$ of order 2, which implies that $F^* \neq \langle \alpha, \beta \rangle$. Again according to Proposition 3.2, $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$. $\square$

Now we give some examples of general linear groups $\mathrm{GL}_n(F)$ with diameter $\delta(\Gamma(\mathrm{GL}_n(F))) = 3$. To do this, by the previous corollary, we need a field F such that the order of the multiplicative group F^* is prime or a product of two distinct prime numbers. The next main result in this section considers fields whose multiplicative groups are of prime order. The class of such fields is "not small". For example, if n is a positive integer such that $2^n - 1$ is a prime number (such prime numbers are called Mersenne prime), then the field with 2^n elements belongs to this class.

We first need the following lemma.

Lemma 3.4. *Let F be a finite field of characteristic 2 such that the order of the multiplicative group F^* is prime. If λ is a non-identity element of F^*, then $\mathrm{GL}_2(F)$ is generated by $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. By [29], there exists $\alpha, \beta, \gamma \in F^*$ such that $\mathrm{GL}_2(F)$ is generated by $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix}$. Put $a = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then, it suffices to show $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix}$ belong to $\langle a, b \rangle$ for every $\alpha, \beta, \gamma \in F^*$. Since F^* is of prime order, $F^* = \langle \lambda \rangle$. Moreover, $a^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & 1 \end{bmatrix}$ for every $n \in \mathbb{N}$, which implies that $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \in \langle a, b \rangle$ for every

$\alpha \in F^*$. Now we can check that

$$a^{-4}bab^2a^2ba^2b^2a^2bab^2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^4 \end{bmatrix}.$$

Since $(4, 2^n - 1) = 1$, the element λ^4 is non-identity. It implies that $F^* = \langle \lambda^4 \rangle$. Hence, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right] \in \langle a, b \rangle$ for every $\alpha \in F^*$. Therefore, for every $\beta, \gamma \in F^*$,

$$\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix} = \begin{bmatrix} \beta\gamma^{-1} & 0 \\ 0 & 1 \end{bmatrix} b^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \in \langle a, b \rangle.$$

The proof is complete. □

Proposition 3.5. *If F is a finite field whose multiplicative group F^* is of prime order, then $\delta(\Gamma(\mathrm{GL}_2(F))) = 3$.*

Proof. Let $p = \mathrm{Char}(F)$. Then $|F| = p^n$ for some positive integer n. If $p > 2$, then the order F^* is prime if and only if $F = \mathbb{Z}/3\mathbb{Z}$. Hence, $\delta(\Gamma(\mathrm{GL}_2(F))) = 3$ by Proposition 2.4. Now, we consider the case when $p = 2$. Put $q = 2^n - 1$. By hypothesis, q is prime. According to Lemma 3.4, if α is a non-identity element of F^*, then $\mathrm{GL}_2(F)$ is generated by two elements $a = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Observe that $a^q = \left[\begin{smallmatrix} \alpha^q & 0 \\ 0 & 1 \end{smallmatrix}\right] = I_2$, so the order of a is q since q is prime. One may check that $b^3 = I_2$, which implies that the order of b is 3. Hence, $\mathrm{GL}_2(F)$ is generated by two elements a and b in which the orders of a and b are prime. Because of Theorem 3.1, $\delta(\Gamma(\mathrm{GL}_n(F))) = 3$. □

4. Intersection Graphs of General Linear Groups Over Infinite Fields

We first borrow the following result which is a special case of [25, Theorem 5.4].

Lemma 4.1. *The general linear group of degree greater than 1 over an infinite field is infinitely generated.*

Theorem 4.2. *The intersection graphs of general linear groups of degree greater than 1 over an infinite field are connected of diameter 2.*

Proof. Assume that F is an infinite field and n is a natural number greater than 1. We must show that the intersection graph $\Gamma(\mathrm{GL}_n(F))$ of the general linear group $\mathrm{GL}_n(F)$ is connected of diameter 2. Let A and B be two distinct vertices of $\Gamma(\mathrm{GL}_n(F))$. If either $A \subseteq B$ or $B \subseteq A$, then $A \cap B \neq \{1\}$, so $d(A, B) = 1$. Assume that $A \nsubseteq B$ and $B \nsubseteq A$. Let $a \in A \backslash B$ and $b \in B \backslash A$. Put $C = \langle a, b \rangle$. Then, by Lemma 4.1, C is different from $\mathrm{GL}_n(F)$, which implies that C is a vertex of $\Gamma(\mathrm{GL}_n(F))$. Moreover, $a \in C \cap A$ and $b \in C \cap B$, so $C \cap A \neq \{I_n\}$ and $C \cap B \neq \{I_n\}$. Hence, we have a path $A - C - B$. Therefore, $d(A, B) \leq 2$.

Now, we show that $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$. Put $a = I_n + e_{12}$ and $b = I_n + e_{21}$. Let $U = \langle a \rangle$ and $V = \langle b \rangle$ be the subgroups of $\mathrm{GL}_n(F)$ generated by a and b, respectively. Then, it is clear that $U = \{I_n + me_{12} \,|\, m \in \mathbb{Z}\}$ and $V = \{I_n + me_{21} \,|\, m \in \mathbb{Z}\}$, so $U \cap V = \{I_n\}$ which implies that $d(U, V) > 1$. Thus, $\delta(\Gamma(\mathrm{GL}_n(F))) = 2$. □

Now, we show results on ends of $\Gamma(\mathrm{GL}_n(F))$ and begin with the following remark.

Remark 4.3. Let Γ be an infinite graph. If

$$(v_i)_{i\geq 0} = v_0 - v_1 - \cdots - v_n - \cdots$$

is a ray of Γ, then the ray $(v_i)_{i\geq k} = v_k - v_{k+1} - \cdots - v_n - \cdots$ is end-equivalent to $(v_i)_{i\geq 0}$ for every $k \in \mathbb{N}$.

Recall that a group G is called *Artinian* (respectively, *Noetherian*) if every descending chain (respectively, ascending chain) of subgroups stabilizes after a finite stage, that is, if $G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$ (respectively, $G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$) is a sequence of subgroups of G, then there exists a positive integer m such that $G_m = G_n$ for every $n \geq m$.

Lemma 4.4. *Let G be a nontrivial group with intersection $\Gamma(G)$. If G is non-Artinian or non-Noetherian, then $\Gamma(G)$ contains infinitely many distinct rays.*

Proof. We show the lemma for the non-Noetherian case, the non-Artinian one is similar.

Assume that G is non-Noetherian, that is, there exists a sequence of subgroups

$$G_1 \lneqq G_2 \lneqq G_3 \lneqq \cdots \lneqq G_n \lneqq \cdots.$$

Let $p_1 = 2 < p_2 = 3 < \cdots < p_n < \cdots$ be a sequence of infinitely many distinct prime numbers. For any $i \geq 1$, we consider the sequence

$$G_{p_i} \lneqq G_{p_i^2} \lneqq G_{p_i^3} \lneqq \cdots \lneqq G_{p_i^m} \lneqq \cdots.$$

Then, in $\Gamma(G)$, we have a ray

$$G_{p_i} - G_{p_i^2} - G_{p_i^3} - \cdots - G_{p_i^m} - \cdots.$$

Moreover, if $i \neq j$, then $p_i^{n_i} \neq p_j^{m_j}$ for every $n_i, n_j > 0$, so $G_{p_i^{n_i}} \neq G_{p_j^{m_j}}$ for every $i \neq j$ and $n_i, n_j > 0$. Hence, $\Gamma(G)$ contains infinitely many distinct rays. The proof is complete. □

Theorem 4.5. *Let F be an infinite field and $n > 1$. Then $\Gamma(\mathrm{GL}_n(F))$ is one-ended and its end is thick.*

Proof. Since Lemmas 4.1 and 4.4, $\Gamma(\mathrm{GL}_n(F))$ contains a ray, namely, $[(A_i)_{i\geq 0}]$. We first show that $[(A_i)_{i\geq 0}]$ is the unique end of $\Gamma(\mathrm{GL}_n(F))$. Assume $(B_i)_{i\geq 0} = B_0 - B_1 - \cdots - B_n - \cdots$ is an arbitrary ray of $\Gamma(\mathrm{GL}_n(F))$. We must find a ray

$(C_i)_{i\geq 0} = C_0 - C_1 - \cdots - C_n - \cdots$ of $\Gamma(\mathrm{GL}_n(F))$ such that $\{C_i \,|\, i \in \mathbb{N}\} \cap \{A_i \,|\, i \in \mathbb{N}\}$ and $\{C_i \,|\, i \in \mathbb{N}\} \cap \{B_i \,|\, i \in \mathbb{N}\}$ are infinite. If $\{A_i \,|\, i \in \mathbb{N}\}$ contains infinitely many vertices B_i, then we choose $(C_i)_{i\geq 0} = (A_i)_{i\geq 0}$. If $(B_i)_{i\geq 0}$ contains infinitely many vertices A_i, then we choose $(C_i)_{i\geq 0} = (B_i)_{i\geq 0}$. In two cases, we have $\{C_i \,|\, i \in \mathbb{N}\} \cap \{A_i \,|\, i \in \mathbb{N}\}$ and $\{C_i \,|\, i \in \mathbb{N}\} \cap \{B_i \,|\, i \in \mathbb{N}\}$ are infinite, so $(A_i)_{i\geq 0} \equiv (B_i)_{i\geq 0}$. Now, we assume that there exists a natural number k such that

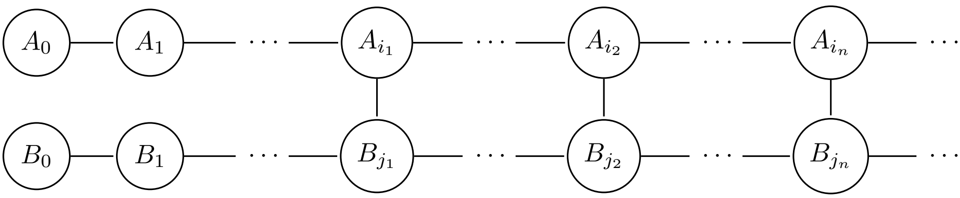
$$\{A_i \,|\, i \geq k\} \cap \{B_i \,|\, i \geq k\} = \emptyset.$$

Since $(A_i)_{i\geq 0} \equiv (A_i)_{i\geq k}$, where $(A_i)_{i\geq k} = A_k - A_{k+1} - \cdots - A_n - \cdots$ for every $k \in \mathbb{N}$ (Remark 4.3) and the relation $\equiv$ is equivalent, without loss of generality, we assume that

$$\{A_i \,|\, i \in \mathbb{N}\} \cap \{B_i \,|\, i \in \mathbb{N}\} = \emptyset.$$

We have two cases.

Case 1. The case when for every $k \in \mathbb{N}$, there exists $i_k, j_k \geq k$ such that A_{i_k} and B_{j_k} are adjacent.

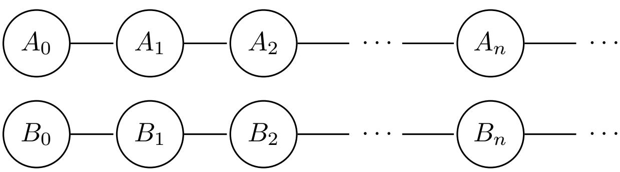


We choose the ray $(C_i)_{i\geq 0}$ which is the square zigzag one

$$A_1 - \cdots - A_{i_1} - B_{j_1} - B_{j_1+1} - \cdots - B_{j_2} - A_{i_2} - A_{i_2+1} - \cdots - A_{i_3} - B_{j_3} - \cdots$$

in the above diagram. One has $(C_i)_{i\geq 0}$ is a ray since all vertices in $\{A_i \,|\, i \in \mathbb{N}\} \cup \{B_i \,|\, i \in \mathbb{N}$ are distinct. Moreover, $A_{i_k} \in \{C_i \,|\, i \in \mathbb{N}\} \cap \{A_i \,|\, i \in \mathbb{N}\}$ and $B_{j_k} \in \{C_i \,|\, i \in \mathbb{N}\} \cap \{B_i \,|\, i \in \mathbb{N}\}$ for every $k \in \mathbb{N}$, so $(C_i)_{i\geq 0}$ is the ray we want to get.

Case 2. The case when there exists $k \in \mathbb{N}$ such that for every $i_k, j_k \geq k$, A_{i_k} and B_{j_k} are not adjacent. By using the same arguments as in the previous case, without loss of generality, we assume that $k = 0$, that is, A_i and B_j are not adjacent for every $i, j \geq 0$.



We first construct a sequence of vertices $\{D_i \,|\, i \in \mathbb{N}\}$, where D_i is generated by at most $3(i+1)$ elements as subgroups in $\mathrm{GL}_n(F)$, in induction on $i \geq 0$ as follow: Since

A_0 and B_0 are not adjacent, let $a_0 \in A_0 \backslash B_0$, $b_0 \in B_0 \backslash A_0$ and $c_0 = 1$. Put $D_0 = \langle a_0, b_0, c_0 \rangle$. Assume that

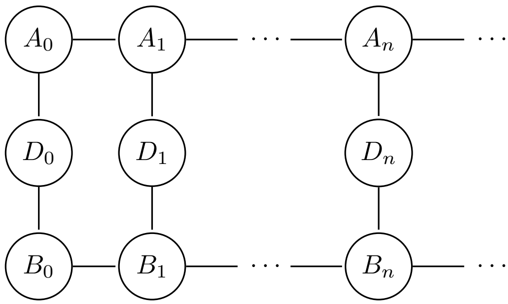
$$D_i = \langle a_0, a_1, \ldots, a_i, b_0, b_1, \ldots, b_i, c_0, c_1, \ldots, c_i \rangle$$

is defined. As A_{i+1} and B_{i+1} are not adjacent, there exists $a_{i+1} \in A_{i+1} \backslash B_{i+1}$ and $b_{i+1} \in B_{i+1} \backslash A_{i+1}$. Observe that $\mathrm{GL}_n(F)$ is infinitely generated, so there exists $c_{i+1} \in \mathrm{GL}_n(F) \backslash D_i$. Put

$$D_{i+1} = \langle a_0, a_1, \ldots, a_{i+1}, b_0, b_1, \ldots, b_{i+1}, c_0, c_1, \ldots, c_{i+1} \rangle.$$

Now, we list some properties of $\{D_i \,|\, i \in \mathbb{N}\}$ that we use in this theorem:

(1) All vertices $\{D_i \,|\, i \in \mathbb{N}\}$ are distinct since $c_{i+1} \notin D_i$ for every $i \geq 0$ distinct.
(2) For every $i \in \mathbb{N}$, there is a path $A_i - D_i - B_i$ because of the fact that $a_i \in D_i \cap A_i$ and $b_i \in D_i \cap B_i$.



(3) For every $i \in \mathbb{N}$, $D_i \notin \{A_i \,|\, i \in \mathbb{N}\} \cup \{B_i \,|\, i \in \mathbb{N}\}$. To see this, assume that $D_i = A_i$ for some $i \in \mathbb{N}$. Then A_i and B_i are adjacent, which is a contradiction. Hence, $D_i \notin \{A_i \,|\, i \in \mathbb{N}\}$ for every $i \geq 0$. Similarly, $D_i \notin \{B_i \,|\, i \in \mathbb{N}\}$ for every $i \geq 0$.

By choosing the ray which is the square zigzag path

$$(C_i)_{i \geq 0} = A_0 - D_0 - B_0 - B_1 - D_1 - A_1 - A_2 - D_2 - \cdots,$$

we see that $(C_i)_{i \geq 0}$ meets $(A_i)_{i \geq 0}$ and $(B_i)_{i \geq 0}$ at infinitely many vertices.

Now, since $\mathrm{GL}_n(F)$ is infinitely generated, $\mathrm{GL}_n(F)$ is non-Artinian. By Lemma 4.4, the ray $[(A_i)_{i \geq 0}]$ is thick. $\square$

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