$*$-GROUP IDENTITIES ON UNITS OF DIVISION RINGS

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Abstract. Let D be a division ring of characteristic different from 2 with infinite center F and involution $*$ of the first kind. In this paper, among other results, we show that if $\mathcal{U}(D)$ satisfies a $*$-group identity, then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type. This leads to some results. For instance, let N be a $*$-invariant normal subgroup of $\mathcal{U}(D)$ such that all symmetric elements of N is central (this is the case when, for example, each symmetric element is bounded Engel). Then either N is central or $\dim_F D = 4$ and $*$ is of the symplectic type.

1. Introduction

Let R be a ring with center $Z(R)$. An *involution* $*$ of R is a map $* : R \to R, x \mapsto x^*$, satisfying the conditions $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for every $x, y \in R$. The involution is of the *first kind* if $x^* = x$ for every $x \in Z(R)$ (otherwise, $*$ is of the second kind). *In this paper, involutions we consider are of the first kind.* A subset S of R is called $*$-*invariant* if $x^* \in S$ for every $x \in S$. It is obvious that if S is $*$-invariant and a subring of R, then $*$ is also an involution of S. The set of *symmetric elements* of S is denoted by S^+, so, $S^+ = \{x \in S \mid x^* = x\}$.

Let F be a field and n a positive integer ≥ 2. The matrix ring $M_n(F)$ of degree n over F always has an involution. For a matrix $A \in M_n(F)$, denote by A^t the transpose of A. Clearly, t is an involution of $M_n(F)$. Assume that $n = 2m$ is even. For $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_{2m}(F)$ where $A_{11}, A_{12}, A_{21}, A_{22} \in M_m(F)$, set $A^s = \begin{bmatrix} A_{22}^t & -A_{12}^t \\ -A_{21}^t & A_{11}^t \end{bmatrix}$. One can show that s is also an involution of $M_{2m}(F)$. We say that t is the *transpose involution* of $M_n(F)$ and s the *symplectic involution* of $M_{2m}(F)$. Moreover,

Lemma 1.1. [22, Proposition 2.1.4] *Let D be a division ring of characteristic different from 2 and n a positive integer. Let $*$ be any involution on $M_n(D)$. Then, up to an automorphism θ of $M_n(D)$ satisfying $\theta(A^*) = (\theta(A))^*$ for all $A \in M_n(D)$, one of the following conditions occurs:*

1. *there exist an involution $*'$ of D and an invertible diagonal matrix $C = \mathrm{diag}(c_1, c_2, \cdots, c_n)$ such that $c_i^{*'} = c_i$ for all i, and if $A = (a_{ij}) \in M_n(D)$, then $A^* = C^{-1}(a_{ij}^{*'})^t C$ (in this case, we say that $*$ is of the transpose type).*
2. *D is a field, n is even, and $*$ is the symplectic involution (we say that $*$ is of the symplectic type).*

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Let D be a division ring which is finite dimensional over its center F. Assume that $*$ is an involution of D. Let n be the degree of D over F, that is, $\dim_F D = n^2$ and let K be a field containing F. Since $D \otimes_F K \cong M_n(K)$, via the homomorphism $a \mapsto a \otimes 1$, one has that D is a subring of $M_n(K)$. It is easy to see that the map $* \otimes 1 : D \otimes_F K \to D \otimes_F K$, defined by $(a \otimes k)^{*\otimes 1} = a^* \otimes k$ for every $a \otimes k \in D \otimes_F K$, is an involution of $D \otimes_F K$.

Lemma 1.2. [25, Proposition 4.1] *Let D be a division ring of degree n over its center F, that is, $\dim_F D = n^2$. Denote by $\overline{F}$ the algebraic closure of F. Assume that $\psi : D \otimes_F \overline{F} \to M_n(\overline{F})$ is an isomorphism and $*$ is an involution of D. Then, the map*

$$\overline{*} = \psi(* \otimes 1)\psi^{-1} : M_n(\overline{F}) \to M_n(\overline{F})$$

is an involution of $M_n(\overline{F})$ of the first kind. The type of $\overline{*}$ is independent of the choice of isomorphisms ψ.

We say that $*$ is of the symplectic type (resp., the transpose type) if $\overline{*}$ is of the symplectic type (resp., the transpose type).

Let G be a group with center $Z(G)$ and $X = \{x_1, x_2, \dots\}$ be a set of countably non-commuting indeterminates. The free group generated by X and the free product of $\langle X \rangle$ and G over $Z(G)$ is denoted by $\langle X \rangle$ and $\langle X \rangle *_{Z(G)} G$, respectively. An element of $\langle X \rangle$ and of $\langle X \rangle *_{Z(G)} G$ is called a *group monomial* and a *generalized group monomial* over G, respectively. Assume that

$$w(x_1, x_2, \dots, x_n) = a_1 x_{i_1}^{n_1} a_2 x_{i_2}^{n_2} \dots a_t x_{i_t}^{n_t} a_{t+1}$$

is a non-identity element of $\langle X \rangle *_{Z(G)} G$. If

$$w(g_1, g_2, \dots, g_n) = 1$$

for every $g_1, g_2, \dots, g_n \in G$, then we say that w is a *generalized group identity* of G or G *satisfies the generalized group identity* w. If, particularly, $w(x_1, x_2, \dots, x_n)$ is a group monomial (i.e., if $a_1 = a_2 \cdots = a_{t+1} = 1$), then we call w a *group identity* of G.

Now assume that $n = 2m$ and $*$ is an involution of G (i.e., a map $* : G \to G, g \mapsto g^*$, such that $(gh)^* = h^* g^*$ and $(g^*)^*$ for every $g, h \in G$). If

$$w(g_1, g_2, \dots, g_m, g_1^*, g_2^*, \dots, g_m^*) = 1$$

for every $g_1, g_2, \dots, g_m \in G$, then w is called a $*$-*generalized group identity* of G or we say that G satisfies a $*$-*generalized group identity* w. Again, if particularly $a_1 = a_2 \cdots = a_{t+1} = 1$, then we call w a $*$-*group identity* of G. For convenience, we write $w(x_1, x_1^*, x_2, x_2^*, \dots, x_m, x_m^*)$ instead of $w(x_1, x_2, \dots, x_{2m})$ and naturally we call $w(x_1, x_1^*, x_2, x_2^*, \dots, x_m, x_m^*)$ a $*$-*(generalized) group monomial.* Clearly, each (generalized) group identity is a $*$-(generalized) group identity (where no $*$'s appear). Moreover since xx^* is a symmetric element, a (generalized) group identity on symmetric elements of a group G yields a $*$-(generalized) group identity of G.

Let F be an infinite field of characteristic different from 2 and G be a torsion group. Assume $*$ be any involution on G, and extend it linearly to FG, the group algebra of G over F. In [20], Jespers-Ruiz Marin found the necessary and sufficient conditions for $(FG)^+$ to be commutative. Giambruno-Polcino Milies-Sehgal in [13] found the conditions under which $\mathcal{U}(FG)^+$ satisfies a group identity. The study of the notion of $*$-group identities has been begun in [12], where the authors showed that $\mathcal{U}(FG)$ satisfies a $*$-group identity if and only if $\mathcal{U}(FG)^+$ satisfies a group identity.

Let D be a division ring with center F. By a well known result of Amitsur, if F is infinite and $\mathcal{U}(D)$ satisfies a group identity, then D is commutative [1, Theorem 19]. This result has been extended in several ways (e.g., see [5, 7, 9, 15]). Now, let $*$ be an involution of D. In this paper, we show an involution version of Amitsur's result. More precisely, we show that if $\mathcal{U}(D)$ satisfies a $*$-group identity, then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type (Theorem 2.2). As a result, let R be a semisimple algebra over an infinite field F of characteristic $\neq 2$ with involution $*$. If $\mathcal{U}(R)$ satisfies a $*$-group identity, then each Wedderburn component of R is of dimension at most 4 over its center, and R^+ is central in R (Corollary 2.3). Also we show that is $\langle \mathcal{U}(D)^+ \rangle$ is nilpotent, then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type (Theorem 2.6). Finally, let N be a $*$-invariant normal subgroup of $\mathcal{U}(D)$ such that $N^+ \subseteq F$ (this is the case when, for example, each element of N^+ is bounded Engel). We will see that either N is central or $\dim_F D = 4$ and $*$ is of the symplectic type (Theorem 2.7 and Corollary 2.8).

2. Main Results

For the remainder of this paper D will be a division ring of characteristic different from 2 with center F and involution $*$ of the first kind. Also, $\overline{*}$ and $\overline{F}$ are as in Lemma 1.2. The group of units of a ring R is denoted by $\mathcal{U}(R)$.

Lemma 2.1. *Assume* $\dim_F D = d^2$ *and* $w(x_1, x_1^*, \ldots, x_m, x_m^*)$ *is a $*$-generalized group identity of* $\mathcal{U}(D)$. *Then,* $w(x_1, x_1^{\overline{*}}, \ldots, x_m, x_m^{\overline{*}})$ *is also a $\overline{*}$-generalized group identity of* $\mathrm{GL}_d(\overline{F})$.

Proof. This is just a corollary of [25, Theorem 6.4]. ∎

Theorem 2.2. *Suppose that F is infinite and $\mathcal{U}(D)$ satisfies a $*$-group identity. Then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type.*

Proof. Assume that D is non-commutative. We must show $\dim_F D = 4$ and $*$ is of the symplectic type. Let w be a $*$-group identity of $\mathcal{U}(D)$. According to [12, Lemma 1], we may assume w is of the form $w(x, x^*) = x^{n_1}(x^*)^{m_2} \ldots x^{n_t}(x^*)^{m_t}$ where $n_1, m_t \in \{0, \pm 1\}$, and all the other exponents lie in $\{\pm 1, \pm 2\}$. We first claim that D is centrally finite. Assume that D is of infinite dimensional over F. If $w_1(x, y) = w(x, y) = x^{n_1} y^{m_2} \ldots x^{n_t} y^{m_t}$, then, since F is infinite, D satisfies the group identity $w_1(x, y) = 1$ by [10, Theorem, p. 191], which contradicts [1, Theorem 19]. The claim is shown, that is, D is centrally finite. Let $n^2 = \dim_F D$. According to Lemma 2.1, $\mathrm{GL}_n(\overline{F})$ satisfies the $\overline{*}$-group identity $w(x, x^{\overline{*}})$. We next claim that $\overline{*}$ is of the symplectic type. Suppose that $\overline{*}$ is of the transpose type. Via the embedding $\mathrm{GL}_2(\overline{F}) \to \mathrm{GL}_n(\overline{F}), A \mapsto \begin{bmatrix} A & 0 \\ 0 & I_{n-2} \end{bmatrix}$, $\mathrm{GL}_2(\overline{F})$ is $\overline{*}$-invariant, so $\mathrm{GL}_2(\overline{F})$ also satisfies the $\overline{*}$-group identity $w(x, x^{\overline{*}})$. Moreover, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\overline{F})$, then by Lemmas 1.1 and 1.2, $A^{\overline{*}} = \begin{bmatrix} a & cc_1c_2^{-1} \\ bc_1^{-1}c_2 & d \end{bmatrix}$, where $c_1, c_2 \in \overline{F}$. We consider two cases:

Case 1. char$(D) = 0$. Let $K = \mathbb{Q}(c_1^{-1}c_2)$ be the subfield of $\overline{F}$ generated by $c_1^{-1}c_2$ over $\mathbb{Q}$. We may assume that K is a subfield of the complex numbers $\mathbb{C}$. Again, $\mathrm{GL}_2(\overline{F})$ is $\overline{*}$-invariant, so $\mathrm{GL}_2(\overline{F})$ satisfies the $\overline{*}$-group identity $w(x, x^{\overline{*}})$. Choose $r \in \mathbb{Q}$ such that $|c_1^{-1}c_2r^2| \geq 2$, $|c_1^{-1}c_2r^2 - 2| \geq 2$ and $|c_1^{-1}c_2r^2 + 2| \geq 2$. Now consider two matrices $A = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ rc_1^{-1}c_2 & 1 \end{bmatrix}$ in $M_n(K)$. According to [8], $\langle A, B\rangle$ is a free group. On the other hand, $A^{\overline{*}} = B$, so $w(A, B) = w(A, A^{\overline{*}}) = 1$; a contradiction.

Case 2. char$(D) = p > 0$. If $\overline{F}$ is algebraic over the prime subfield P of $\overline{F}$, then so is F. Hence, D is also algebraic over the finite field P, which implies by Jacobson's Theorem (see [21, Theorem 13.11]) that D is commutative. Therefore, $\overline{F}$ is not algebraic over P. Consider $L = P[c_1^{-1}c_2]$ the subring of $\overline{F}$ generated by $c_1^{-1}c_2$ over P. If $c_1^{-1}c_2$ is not algebraic over P, then $P[c_1^{-1}c_2]$ is isomorphic to the polynomial ring $P[t]$. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then, it is easy to check that $A^2 = B^2 = 0$, and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not nilpotent. According to [11, Lemma 3.2], $\langle 1 + (c_1^{-1}c_2)A, 1 + (c_1^{-1}c_2)^2BAB\rangle$ is isomorphic to the free product

$$\langle 1 + (c_1^{-1}c_2)A\rangle * \langle 1 + (c_1^{-1}c_2)^2BAB\rangle.$$

Observe that

$$(1 + (c_1^{-1}c_2)A)^{\overline{*}} = \begin{bmatrix} 1 & c_1^{-1}c_2 \\ 0 & 1 \end{bmatrix}^{\overline{*}} = \begin{bmatrix} 1 & 0 \\ (c_1^{-1}c_2)^2 & 1 \end{bmatrix} = 1 + (c_1^{-1}c_2)^2BAB.$$

So

$$w(1 + (c_1^{-1}c_2)A, 1 + (c_1^{-1}c_2)^2BAB) = w(1 + (c_1^{-1}c_2)A, (1 + (c_1^{-1}c_2)A)^{\overline{*}}) = 1,$$

which is a contradiction. Now if $c_1^{-1}c_2$ is algebraic over P, then $P[c_1^{-1}c_2] = P(c_1^{-1}c_2)$ is a subfield of $\overline{F}$. Because $\overline{F}$ is not algebraic over P, neither is $\overline{F}$ over $P(c_1^{-1}c_2)$. Let $t \in \overline{F}$ be transcendental over $P(c_1^{-1}c_2)$ and consider the polynomial ring $P(c_1^{-1}c_2)[t]$. In this subcase, again by [11, Lemma 3.2], $\langle 1 + tA, 1 + tc_1^{-1}c_2BAB\rangle$ is isomorphic to the free product $\langle 1 + tA\rangle * \langle 1 + tc_1^{-1}c_2BAB\rangle$. Similarly, one has $(1 + tA)^{\overline{*}} = 1 + tBAB$ which is also a contradiction.

Two subcases lead us a contradiction. The claim is shown, that is, $\overline{*}$ is of the symplectic type. We finally claim that $n = 2$. Assume that $n > 2$. Then, $n = 2\ell$ with $\ell > 1$ (by Lemma 1.1). Via the embedding $\mathrm{GL}_\ell(\overline{F}) \to \mathrm{GL}_{2\ell}(\overline{F})$, $A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, $\mathrm{GL}_\ell(\overline{F})$ is $\overline{*}$-invariant, so $\mathrm{GL}_\ell(\overline{F})$ also satisfies the $\overline{*}$-group identity $w(x, x^{\overline{*}})$. Moreover, the involution $\overline{*}$ of $\mathrm{GL}_\ell(\overline{F})$ is of the transpose type. Using the first path, we have a contradiction. Thus, $n = 2$ and the proof is completed. $\square$

Our next result extends [12, Lemma 5] with similar argument. We include its proof for the sake of completeness.

Corollary 2.3. *Let R be a semisimple algebra over an infinite field F of characteristic $\neq 2$ with involution $*$. If $\mathcal{U}(R)$ satisfies a $*$-group identity, then each Wedderburn component of R is of dimension at most 4 over its center, and R^+ is central in R.*

Proof. Let $R = Re_1 \oplus \cdots \oplus Re_k$, where e_i's are primitive central idempotents. For convinced, let $e = e_i$, for some $1 \leq i \leq k$, and assume $Re \cong M_n(D)$, where D is a division ring and n is a natural number.

First, assume that $e \in R^+$. Then Re is a $*$-invariant subset of R. Therefore, $\mathrm{GL}_n(D) \subseteq \mathcal{U}(R)$ satisfies a $*$-group identity. Suppose that the involution $*$ is of transpose type. Denote the matrix with a one in the i, j position and every other entry zero by E_{ij}. In this case, E_{11} is a symmetric idempotent, hence E_{11} is central by [12, Lemma 3]. Thus, $n = 1$, i.e., $\mathcal{U}(D)$ satisfies a $*$-group identity. So, by Theorem 2.2, D is commutative. Now suppose that the involution $*$ is of the symplectic type. Then by Lemma 1.1, D is a field. Assume $n > 2$, and let $a = E_{1n} + E_{1n}^* = E_{1n} - E_{\frac{n}{2}(\frac{n}{2}+1)}$. Then a is symmetric and square-zero. Let $c = E_{11}, d = E_{nn}$. Then $cd = 0$, but $cad = E_{1n} \neq 0$, which [12, Lemma 3] gives us a contradiction. If $n = 2$, then the symmetric elements are the scalar multiples of the identity matrix, hence, central, and we are done.

Now, suppose that e is not symmetric. Then e^* is also a primitive central idempotent. Let $w(x, x^*)$ be the $*$-group identity on $\mathcal{U}(R)$, as [12, Lemma 1]. Assume $\alpha e, \beta e \in \mathcal{U}(R)$. Then $u = \alpha e + \beta^* e^* + (1 - e - e^*) \in \mathcal{U}(R)$ and $u^* = \beta e + a^* e^* + (1 - e - e^*)$. Since, $w(u, u^*) = 1$, looking only at the first component, we get $w(\alpha e, \beta e) = e$. That is, $w(x_1, x_2)$ is a group identity for $\mathrm{GL}_n(D)$. Now, $n = 1$ by [14, Corollary 3] and D is a field by [1, Theorem 19]. ■

Lemma 2.4. *Assume that F is infinite. Then, $aa^* \in F$ for every $a \in D$ if and only if either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type.*

Proof. Assume that $aa^* \in F$ for every $a \in D$. Then, $\mathcal{U}(D)$ satisfies a $*$-group identity $(xx^*)(yy^*)(xx^*)^{-1}(yy^*)^{-1} = 1$. Therefore, by Theorem 2.2, either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type. Conversely, assume that $\dim_F D = 4$ and $*$ is of the symplectic type. Then, by Lemma 1.2, D can be considered as a subring of $M_2(\overline{F})$ and the involution $\overline{*}$ of $M_2(\overline{F})$ is of the symplectic type. Thus each symmetric element is central, hence $aa^{\overline{*}} \in \overline{F}$ for every $a \in M_2(\overline{F})$. In particular, $aa^* \in F$ for every $a \in D$. ■

Let D be a division ring which has not necessarily equipped with an involution. In [17], Herstein conjectured that if for every $x, y \in \mathcal{U}(D)$, there exists a positive integer $n(x, y)$ such that $(x^{-1}y^{-1}xy)^{n(x,y)} \in F$, then D is commutative. This conjecture holds in case D is centrally finite [17, Theorem 2] or F is uncountable [16]. In general, the conjecture is still open. In our next result, we will find the involution version of [17, Theorem 2].

Let G be a group with center $Z(G)$ and $w(x_1, x_2, \ldots, x_n)$ a non-identity group monomial in $x_1, x_2, \ldots, x_n$. Recall that w is said to be a *power central group identity* if for every $g_1, g_2, \ldots, g_n \in G$, there exists natural number $p = p(g_1, g_2, \ldots, g_n)$ such that $w(g_1, g_2, \ldots, g_n)^p \in Z(G)$. Such identity in division rings was studied in several papers (e.g., see [23]). Similarly, we will define notion of $*$-power central group identity. Assume that $*$ is an involution of G and let $w(x_1, x_1^*, \ldots, x_m, x_m^*)$ be a (non-identity) $*$-group monomial. We say that w is a *$*$-power central group identity* if for every $g_1, g_2, \ldots, g_m \in G$, there exists a positive integer $p = p(g_1, g_2, \ldots, g_m)$ such that $w(g_1, g_1^*, \ldots, g_m, g_m^*)^p \in Z(G)$. We say that G satisfies a *$*$-power central group identity* if there exists some $*$-power central group identity w of G.

Theorem 2.5. *Let D be a locally finite dimensional division ring over F. If $\mathcal{U}(D)$ satisfies a $*$-power central group identity, then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type.*

Proof. Assume D is non-commutative. We must show that $\dim_F D = 4$ and $*$ is of the symplectic type. By Jacobson's Theorem, F is infinite. Let $a \in D$ be any element. By Lemma 2.4, it suffices to show $aa^* \in F$. To do it, assume $b \in D$ and let Δ be the subdivision ring of D generated by $S = \{a, a^*, b, b^*\}$ over F. Then $\dim_F \Delta < \infty$. Let D_1 be the subdivision ring of Δ generated by S over the prime subfield P. Then, D_1 is $*$-invariant whose center F_1 is finitely generated over P [4, Lemma 2.6]. Now let $w(x_1, x_1^*, \dots, x_m, x_m^*)$ be a $*$-power central group identity of $\mathcal{U}(D)$, that is, for every $g_1, g_2, \dots, g_m \in \mathcal{U}(D)$, there exists a positive integer $p = p(g_1, g_2, \dots, g_m)$ such that $w(g_1, g_1^*, \dots, g_m, g_m^*)^p \in F$. Then, $w(x_1, x_1^*, \dots, x_m, x_m^*)$ is also a $*$-power central group identity of $\mathcal{U}(D_1)$. As [2, Lemma 2.2], there exists an integer $\ell \geq 1$ such that $w(g_1, g_1^*, \dots, g_m, g_m^*)^\ell \in F_1$. It implies that

$$w(x_1, x_1^*, \dots, x_m, x_m^*)^{-\ell} y^{-1} w(x_1, x_1^*, \dots, x_m, x_m^*)^\ell y = 1$$

is a $*$-group identity of $\mathcal{U}(D_1)$. Since $\dim_{F_1} D_1 < \infty$, we may assume F_1 is infinite and D_1 is non-commutative. Hence, by Theorem 2.2, $\dim_{F_1} D_1 = 4$ and $*$ (as an involution on D_1) is of the symplectic type. Now, Lemma 2.4 implies that $aa^* \in F_1$; consequently, $(aa^*)b = b(aa^*)$. As b ranges over D, $aa^* \in F$, as desired. ∎

It is known that if D is a division ring such that $\mathcal{U}(D)$ is a nilpotent group, then D is commutative. In the following result, we instead assume that the set of symmetric elements of $\mathcal{U}(D)$, denoted by $\mathcal{U}(D)^+$, is nilpotent (see [22, Lemma 4.1.2]).

Theorem 2.6. *Let n be a positive integer such that $[s_1, s_2, \dots, s_n] = 1$ for every $s_1, s_2, \dots, s_n \in \mathcal{U}(D)^+$. Then either D is commutative or $\dim_F D = 4$ and $*$ is of the symplectic type.*

Proof. Assume that D is non-commutative. We must show that $\dim_F D = 4$ and $*$ is of the symplectic type. Let $w(x_1, x_2, \dots, x_n) = [x_1, x_2, \dots, x_n]$. Then, $\mathcal{U}(D)^+$ satisfies the group identity w. Hence, if we let

$$w_1(x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots, x_n^*) = w(x_1x_1^*, x_2x_2^*, \dots, x_na_n^*),$$

then $\mathcal{U}(D)$ satisfies the $*$-group identity w_1. Let F be the center of D. By Theorem 2.2, it suffices to show that F is infinite. Assume that F is finite. We seek a contradiction. As D is non-commutative and F is finite, $\dim_F D = \infty$. Now, by [18, Theorem 2.1.6], $D = F(D^+)$.

Without loss of generality, we assume that n is the smallest integer such that $[s_1, s_2, \dots, s_n] = 1$ for every $s_1, s_2, \dots, s_n \in \mathcal{U}(D)^+$. If $n = 2$, then $\mathcal{U}(D)^+$ is commutative, so is $D = F(D^+)$, which is a contradiction. Hence, we assume that $n \geq 3$. Then, for every $s \in \mathcal{U}(D)^+$, one has $[[s_1, s_2, \dots, s_{n-1}], s] = 1$ for every $s_1, s_2, \dots, s_{n-1} \in \mathcal{U}(D)^+$. As a result, $[s_1, s_2, \dots, s_{n-1}] \in F$ for every $s_1, s_2, \dots, s_{n-1} \in \mathcal{U}(D)^+$. By the minimality of n, there exist $s_1, s_2, \dots, s_{n-1} \in \mathcal{U}(D)^+$ such that $[s_1, s_2, \dots, s_{n-1}] = \alpha \in F \setminus \{1\}$. For convenience, put $a = [s_1, s_2, \dots, s_{n-2}]$ and $b = s_{n-1}$. Then $ab \neq ba$ and $a^{-1}b^{-1}ab = \alpha$.

Let $q = |F|$. Since $b = \alpha a^{-1}ba$, one has

$$b^{q-1} = (\alpha a^{-1}ba)^{q-1} = (\alpha)^{q-1}(a^{-1}ba)^{q-1} = a^{-1}b^{q-1}a,$$

which implies that $b^{q-1}a = ab^{q-1}$. Repeat arguments above, one has $ba^{q-1} = a^{q-1}b$ and $b(a^*)^{q-1} = (a^*)^{q-1}b$. Because $[a, s] \in F$ for every $s \in \mathcal{U}(D)^+$, $[a, aa^*] = \beta \in F$, equivalently, $a(aa^*a^{-1}(a^*)^{-1})a^{-1} = \beta$, which implies that $aa^*a^{-1}(a^*)^{-1} = \beta$. Similarly, $a^{q-1}a^* = a^*a^{q-1}$ and $a(a^*)^{q-1} = (a^*)^{q-1}a$. Let $D_1 = F(a, a^*, b)$, the division subring of D generated by a, a^*, b over F. Then D_1 is $*$-invariant and non-commutative, because $ab \neq ba$. In conclusion, if F_1 is the center of D_1, then we have relations of a, a^*, b as follows:

(1) $ab = \alpha ba$;
(2) $ba^* = \alpha a^*b$;
(3) $aa^* = \beta a^*a$;
(4) $a^{q-1}, (a^*)^{q-1}$ and b^{q-1} belong to F_1,

where $\alpha, \beta \in F \subseteq F_1$. It implies that D_1 is finite dimensional over F_1. Therefore, F_1 is infinite. According to Theorem 2.2, $\dim_{F_1} D_1 = 4$ and $*$ is of the symplectic type of D_1. Now, by [27, Proposition 2.1], $D_1^+ = F_1$. In particular, $b \in F_1$. As a corollary, $ab = ba$ which contradicts the way we choose a and b. The proof is complete. ■

Theorem 2.7. *Suppose that F is infinite and N is a $*$-invariant normal subgroup of $\mathcal{U}(D)$. If $N^+ \subseteq F$, then either N is central or $\dim_F D = 4$ and $*$ is of the symplectic type.*

Proof. Assume that N is non-central. We will show that $\dim_F D = 4$ and $*$ is of the symplectic type. For every $a \in N$, by the fact that N is $*$-invariant, $aa^* \in N^+ \subseteq F$. Since N is normal in D^*, for every $a \in N$ and $b, c \in \mathcal{U}(D)$ we have

$$\begin{aligned} 1 &= ((bab^{-1})(bab^{-1})^*)^{-1}c^{-1}(bab^{-1})(bab^{-1})^*c \\ &= (b^{-1})^*(a^{-1})^*b^*ba^{-1}b^{-1}c^{-1}bab^{-1}(b^{-1})^*a^*b^*c. \end{aligned}$$

Let $a \in N \setminus F$, so D satisfies a $*$-generalized group identity

$$w(x, x^*, y, y^*) = (x^{-1})^*(a^{-1})^*x^*xa^{-1}x^{-1}y^{-1}xax^{-1}(x^{-1})^*a^*x^*y.$$

Assume that $\dim_F D = \infty$. Then, since F is infinite, we can apply [10, Theorem, p. 191] to deduce that D satisfies the generalized group identity

$$w(x_1, x_2, y_1, y_2) = x_2^{-1}(a^{-1})^*x_2x_1a^{-1}x_1^{-1}y^{-1}x_1ax_1^{-1}x_2^{-1}a^*x_2y,$$

which contradicts [15, Theorem 2]. Hence, D is centrally finite. Let $\dim_F D = d^2$. Then, $w(x, x^{\overline{*}}, y, y^{\overline{*}})$ is also a $\overline{*}$-generalized group identity of $\mathrm{GL}_d(\overline{F})$ by Lemma 2.1 (here, of course, a denotes under inclusion $D \hookrightarrow M_d(\overline{F})$). Observe that, this identity can write as

$$w(x, x^{\overline{*}}, y, y^{\overline{*}}) = ((xax^{-1})(xax^{-1})^{\overline{*}})^{-1}y^{-1}(xax^{-1})(xax^{-1})^{\overline{*}}y.$$

As a result, $(PaP^{-1})(PaP^{-1})^{\overline{*}} \in \overline{F}$ for every $P \in \mathrm{GL}_d(\overline{F})$.

We claim that $*$ is of the symplectic type. Deny the claim and assume $\overline{*}$ is of the transpose type, that is, there exists $C = \mathrm{diag}(c_1, c_2, \ldots, c_d) \in \mathrm{GL}_d(\overline{F})$ such that $A^{\overline{*}} = CA^tC^{-1}$ for every $A \in \mathrm{GL}_d(\overline{F})$. Since $\overline{F}$ is algebraically closed, by according to the Jordan normal form, there exists $P \in \mathrm{GL}_d(\overline{F})$ such that

$$PaP^{-1} = \begin{bmatrix} \lambda_1 & \lambda_{12} & & & \\ & \lambda_2 & \lambda_{23} & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & \lambda_{(d-1)d} \\ & & & & \lambda_d \end{bmatrix}$$

where $\lambda_i \in \overline{F}$ and $\lambda_{i(i+1)} \in \{0,1\}$. Computing

$$\begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_d \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ \lambda_{12} & \lambda_2 & & & \\ & \lambda_{23} & \lambda_3 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{(d-1)d} & \lambda_d \end{bmatrix} \begin{bmatrix} c_1^{-1} & & & \\ & c_2^{-1} & & \\ & & \ddots & \\ & & & c_d^{-1} \end{bmatrix},$$

we have

$$(PaP^{-1})^{\overline{*}} = \begin{bmatrix} \lambda_1 & & & & \\ c_2c_1^{-1}\lambda_{12} & \lambda_2 & & & \\ & c_3c_2^{-1}\lambda_{23} & \lambda_3 & & \\ & & \ddots & \ddots & \\ & & & c_dc_{d-1}^{-1}\lambda_{(d-1)d} & \lambda_d \end{bmatrix}.$$

Now it is clear that the $(i, i+1)$-entry of the matrix $(PaP^{-1})(PaP^{-1})^{\overline{*}}$ is $\lambda_{i(i+1)}\lambda_{i+1}$ and (i,i)-entry is λ_i^2 for every i. As $(PaP^{-1})(PaP^{-1})^{\overline{*}} \in \overline{F}$ and $\lambda_i \neq 0$ (since a is invertible), one has $\lambda_{i(i+1)} = 0$ for every i and $\lambda_1^2 = \lambda_2^2 = \cdots = \lambda_d^2$. Consequently, $(PaP^{-1})^2 \in \overline{F}$, which implies that $a^2 \in \overline{F}$. Thus $a^2 \in F$ since $a \in D$. We observe that a ranges over $N \setminus F$, so every element of N is radical over F. By [6, Lemma 2.4], N is central, a contradiction. Hence, the claim is shown. that is, $*$ is of the symplectic type. Thus, $d = 2\ell$ and if $A \in M_{2\ell}(\overline{F})$ with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_{2\ell}(\overline{F})$, where $A_{11}, A_{12}, A_{21}, A_{22} \in M_\ell(\overline{F})$, one has $A^{\overline{*}} = \begin{bmatrix} A_{22}^t & -A_{12}^t \\ -A_{21}^t & A_{11}^t \end{bmatrix}$.

It suffices to show $\ell = 1$. Suppose that $\ell > 1$. We claim that a (as an element of $\mathrm{GL}_d(\overline{F})$) is a diagonalizable matrix. Since $\overline{F}$ is algebraically closed, there exists $P \in \mathrm{GL}_{2\ell}(\overline{F})$ such that

$$PaP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} \lambda_1 & \lambda_{12} & & & \\ & \lambda_2 & \lambda_{23} & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & \lambda_{(\ell-1)\ell} \\ & & & & \lambda_\ell \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \lambda_{\ell(\ell+1)} & 0 & \cdots & 0 \end{bmatrix},$$

and

$$A_{22} = \begin{bmatrix} \lambda_{\ell+1} & \lambda_{(\ell+1)(\ell+2)} & & & & \\ & \lambda_{\ell+2} & \lambda_{(\ell+2)(\ell+3)} & & & \\ & & \lambda_{\ell+3} & \ddots & & \\ & & & \ddots & \lambda_{(2\ell-1)2\ell} \\ & & & & \lambda_{2\ell} \end{bmatrix}$$

If a is not diagonalizable, then we can choose P such that $\lambda_{\ell(\ell+1)} \neq 0$. Hence, $(PaP^{-1})^{\overline{*}} = \begin{bmatrix} A_{22}^t & -A_{12}^t \\ 0 & A_{11}^t \end{bmatrix}$. As $(PaP^{-1})(PaP^{-1})^{\overline{*}} \in \overline{F}$, the entry at $(1, 2\ell)$ is 0, which implies that $\lambda_1\lambda_{\ell(\ell+1)} = 0$. Moreover, $\lambda_{\ell(\ell+1)} \neq 0$, so $\lambda_1 = 0$. This contradicts the fact that a is invertible. Thus, the claim is shown, that is, a is diagonalizable. Therefore, $PaP^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ where

$$A_{11} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_\ell \end{bmatrix}, A_{22} = \begin{bmatrix} \lambda_{\ell+1} & & & \\ & \lambda_{\ell+2} & & \\ & & \ddots & \\ & & & \lambda_{2\ell} \end{bmatrix}.$$

Now we claim that $\lambda_1^2 = \lambda_2^2 = \cdots = \lambda_{2\ell}^2$. One has $(PaP^{-1})^{\overline{*}} = \begin{bmatrix} A_{22} & 0 \\ 0 & A_{11} \end{bmatrix}$. By changing two suitable rows of P, without loss of generality, it suffices to show that $\lambda_1^2 = \lambda_{2\ell}^2$. Since $(PaP^{-1})(PaP^{-1})^* \in \overline{F}$, $\lambda_1\lambda_{\ell+1} = \lambda_{2\ell}\lambda_\ell$. Again, by changing the ℓ-th and $(\ell+1)$-th rows of P, one has $\lambda_1\lambda_\ell = \lambda_{2\ell}\lambda_{\ell+1}$. It implies that

$$\lambda_1^2\lambda_\ell\lambda_{\ell+1} = \lambda_{2\ell}^2\lambda_\ell\lambda_{\ell+1}$$

Observe that a is invertible, so each $\lambda_i \neq 0$ which implies that $\lambda_1^2 = \lambda_{2\ell}^2$. The claim is shown. Therefore, $a^2 = Pa^2P^{-1} = (PaP^{-1})^2 = \lambda_1^2 I_{2\ell} \in \overline{F}$. Since $a \in D$, $a^2 \in F$. As a ranges over $N \setminus F$, N is radical over F, which implies that N is central [6, Lemma 2.4]. A contradiction to the hypothesis. Thus, $\ell = 1$, that is, $d = 2$. The proof is complete. ∎

Let G be a group. For x, y in G, define $[x, {}_1y] = [x, y] = x^{-1}y^{-1}xy$, and inductively, $[x, {}_{k+1}y] = [[x, {}_ky], y]$ for each natural number k. An element $a \in G$ is called *left Engel* if for each $g \in G$, there exists a natural number $n = n(g)$, depending on g, such that $[g, {}_na] = 1$. If $n \in \mathbb{N}$ is such that the relation $[g, {}_na] = 1$ holds for each $g \in G$, then a is said to be *left n-Engel*. Denote the set of all the left Engel and the left n-Engel elements of G by $L(G)$ and $L_n(G)$, respectively. The group G is called Engel if $G = L(G)$, and called *bounded Engel* if $G = L_n(G)$ for some $n \in \mathbb{N}$.

Let N be a normal subgroup of $\mathcal{U}(D)$. It is known that if N is a locally nilpotent group, then N is central [19]. Also, if N is a bounded Engel group or if F is uncountable and N is an Engel group, then N is central (see [24, Theorem 1.1] and [3, Corollary 1.2]). We close this paper by a result which provides an involution version of these results.

Corollary 2.8. *Let N be a $*$-invariant normal subgroup of $\mathcal{U}(D)$. Assume one of the following cases occurs:*

(1) D is of locally finite dimensional over F and $N^+ \subseteq L(N)$;
(2) F is uncountable and $N^+ \subseteq L(N)$;

(3) F is infinite and $N^+ \subseteq L_m(N)$ for some natural number m.

Then either N is central or $\dim_F D = 4$ and $*$ is of the symplectic type.

Proof. If D is commutative, there is nothing to do. Assume D is non-commutative, so in either cases F is infinite.

If D is a locally finite dimensional division algebra, then by [26, Corollary 3.5.7], $L(N)$ coincides with the Hirsch-Plotkin radical of N, which is a normal locally nilpotent subgroup of N. This implies that $L(G) \subseteq F$ by [19], thus $N^+ \subseteq F$. If either (2) or (3) occurs, then by [3, Theorem 1.1] and [3, Proposition 1.3], we again deduce that $N^+ \subseteq F$. Now, the result follows from Theorem 2.7. ■

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