

# INTERSECTION GRAPHS OF GENERAL LINEAR GROUPS

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ABSTRACT. Let  $F$  be a field and  $\mathrm{GL}_n(F)$  the general linear group of degree  $n > 1$  over  $F$ . The intersection graph  $\Gamma(\mathrm{GL}_n(F))$  of  $\mathrm{GL}_n(F)$  is a simple undirected graph whose vertex set includes all non-trivial proper subgroups of  $\mathrm{GL}_n(F)$ . Two vertices  $A$  and  $B$  of  $\Gamma(\mathrm{GL}_n(F))$  are adjacent if  $A \neq B$  and  $A \cap B \neq \{I_n\}$ . In this paper, we show that, if  $F$  is a finite field containing at least 3 elements, then the diameter  $\delta(\Gamma(\mathrm{GL}_n(F)))$  is 2 or 3. We also classify  $\mathrm{GL}_n(F)$  according to  $\delta(\Gamma(\mathrm{GL}_n(F)))$ . In case  $F$  is infinite, we prove that  $\Gamma(\mathrm{GL}_n(F))$  is one-ended of diameter 2 and its unique end is thick.

## 1. INTRODUCTION

The relationship of graph and group theories is interesting and has a long history (over 140 years). A. Cayley is maybe the first person who associated groups to graphs in 1878 [7]. The main idea of associations is to describe the algebraic properties of a certain group according to its associated graph. To do this, we make a “good association” from a certain group to a graph. We then investigate this associated graph and, finally, apply the results to the base group. The “good association” depends on algebraic properties of groups to study and its applications. For example, see [3, 16, 17] for the definition and applications of Cayley graphs of groups; we refer to [2, 4, 6, 11] for the definition of the commuting graph of groups and its related issues; [1, 20, 21, 26] presented the non-commutating graph of a group, defined by Neumann in 1976; for a good survey and some applications of directed power graphs, see [8, 9, 13].

In this paper, we study the intersection graph of a group. Recall that an *undirected simple graph without loop*  $\Gamma$  is a pair of sets  $(\mathcal{V}, \mathcal{E})$  in which  $\mathcal{E}$  includes some elements  $\{a, b\}$ , where  $a, b$  are distinct elements in  $\mathcal{V}$ . The sets  $\mathcal{V}$  and  $\mathcal{E}$  are called the *vertex set* and *edge set* of  $\Gamma$  respectively. Every element of  $\mathcal{V}$  is called a *vertex* of  $\Gamma$  and every element  $\{a, b\} \in \mathcal{E}$  is called an *edge* of  $\Gamma$ . For convenience, we write  $a - b$  if  $\{a, b\}$  is an edge of  $\Gamma$  and say that  $a$  and  $b$  are adjacent. If the vertex set  $\mathcal{V}$  is finite (obviously, in this case, the edge set  $\mathcal{E}$  is also finite), then  $\Gamma$  is called a *finite graph*. In this paper, our graphs are understood to be undirected simple without loop.

Let  $G$  be a group. The intersection graph  $\Gamma(G) = (\mathcal{V}_G, \mathcal{E}_G)$  of  $G$  is a graph defined as follows: the vertex set includes all non-trivial proper subgroup of  $G$ . Two vertices  $A$  and  $B$  of  $\Gamma(G)$  are adjacent if  $A \neq B$  and  $A \cap B \neq \{1\}$ . Observe that our assumption allows the case when  $G$  is infinite, so the intersection graph  $\Gamma(G)$  maybe contains infinite many vertices. Clearly, if  $G$  is finite, then the graph  $\Gamma(G)$  is finite.

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*Key words and phrases.* intersection graph; linear group; diameter; ray  
2010 *Mathematics Subject Classification.* 05C12, 05C50, 20C05, 20E99.

The intersection graph of a group was firstly defined by B. Csakany and G. Pollak in 1969 [10] with inspiration from [5]. There are a lot of interesting results on intersection graphs and its subgraphs. In 2010, R. Shen classified all finite groups whose intersection graphs are disconnected [25]. In 1975, B. Zelinka investigated the intersection graphs of finite abelian groups. In 2016, X. Ma evaluated the diameter of intersection graphs of finite non-abelian simple groups [18]. In [23], the induced subgraph of intersection graphs of groups generated by all vertices which are proper cyclic subgroups was studied. The planarity of intersection graph of finite groups was mentioned in [14] and [24].

The main purpose of this paper is to investigate the intersection graphs of general linear groups  $GL_n(F)$  of degree  $n > 1$  over an arbitrary field  $F$ . For convenience, we recall some notion in an arbitrary simple undirected graph without loop. Let  $\Gamma$  be a simple undirected graph without loop, let  $u$  and  $v$  be two distinct vertices of  $\Gamma$ . A path between  $u$  and  $v$  is defined as a sequence of distinct vertices  $u = v_0 - v_1 - \cdots - v_n = v$  in which the first and last vertices are  $u$  and  $v$  respectively and two consecutive vertices are adjacent. The number  $n$  in the sequence is called the *length* of the path. The distance between  $u$  and  $v$  is defined as the smallest length of all paths between  $u$  and  $v$  and is denoted by  $d(u, v)$ . If there is no path between  $u$  and  $v$ , we write  $d(u, v) = \infty$ . The graph  $\Gamma$  is called *connected* if  $d(u, v) < \infty$  for every two distinct vertexes  $u, v \in \Gamma$ . Additionally, if  $d(u, v)$  is bounded when  $u, v$  range over  $\mathcal{V}$ , then the number

$$\delta(\Gamma) = \max\{d(u, v) \mid u, v \in \Gamma, u \neq v\}$$

is called *the diameter* of  $\Gamma$ , otherwise, we denote by  $\delta(\Gamma) = \infty$ . Remark that for a connected graph  $\Gamma$ , if the vertex set  $\mathcal{V}$  is finite, then  $\delta(\Gamma)$  is finite. In case  $\Gamma$  is infinite,  $\delta(\Gamma)$  is unnecessarily finite. For example, if  $\Gamma$  is a graph which is an infinite path

$$\Gamma = v_0 - v_1 - \cdots - v_n - \cdots,$$

then  $\delta(\Gamma) = \infty$ . The first aim of this paper is to show that if  $F$  is a finite field containing at least 3 elements and  $n > 1$ , then  $2 \leq \delta(\Gamma(GL_n(F))) \leq 3$  (see Section 2). In Section 3, we classify  $GL_n(F)$  according to the diameter  $\delta(\Gamma(GL_n(F)))$ . We spend Section 4 for the case when  $F$  is infinite. We show that in this case, the diameter  $\delta(\Gamma(GL_n(F))) = 2$ . Moreover, since the vertex set of  $\Gamma(GL_n(F))$  is infinite, we have an opportunity to study the set of rays of  $\Gamma(GL_n(F))$ .

Recall that in an infinite graph  $\Gamma = (\mathcal{V}, \mathcal{R})$ , a *ray* of  $\Gamma$  is a 1-way infinite path in  $\Gamma$ , that is, a ray is a countable sequence of distinct vertices of  $\mathcal{V}$

$$v_0 - v_1 - \cdots - v_n - \cdots$$

in which two consecutive vertices are adjacent. Let

$$(v_i)_{i \geq 0} = v_0 - v_1 - \cdots - v_n - \cdots \text{ and } (u_i)_{i \geq 0} = u_1 - u_2 - \cdots - u_n - \cdots$$

be two rays of  $\Gamma$ . We say that  $(u_i)_{i \geq 0}$  and  $(v_i)_{i \geq 0}$  are *end-equivalent*, and denote by  $(u_i)_{i \geq 0} \equiv (v_i)_{i \geq 0}$ , if there exists a ray  $(w_i)_{i \geq 0}$  such that  $\{w_i \mid i \in \mathbb{N}\} \cap \{u_i \mid i \in \mathbb{N}\}$  and  $\{w_i \mid i \in \mathbb{N}\} \cap \{v_i \mid i \in \mathbb{N}\}$  are infinite [12]. It is shown that  $\equiv$  is an equivalent relation and if  $\Gamma$  is connected, then  $\Gamma$  contains a ray. For a ray  $(u_i)_{i \geq 0}$  of  $\Gamma$ , we denote by  $[(u_i)_{i \geq 0}]$  the class of all rays of  $\Gamma$  which are end-equivalent to  $(u_i)_{i \geq 0}$ . The class  $[(u_i)_{i \geq 0}]$  is called an *end* of  $\Gamma$ . If  $[(u_i)_{i \geq 0}]$  contains infinitely many disjoint rays, then we say that  $[(u_i)_{i \geq 0}]$  is *thick*, otherwise, it is *thin*. We call  $\Gamma$  *one-ended*

if  $\Gamma$  has exactly one end. In this paper, we prove that if  $F$  is an infinite field, then the intersection  $\Gamma(\mathrm{GL}_n(F))$  is one-ended and its unique end is thick.

Our notations in this paper are standard. In particular,  $\mathrm{GL}_n(F)$  is the general linear group of degree  $n \geq 1$  over a field  $F$ ,  $\mathrm{SL}_n(F)$  is the special linear group of degree  $n$  over  $F$ . It is well known that the center  $Z(\mathrm{GL}_n(F))$  is  $\{\alpha I_n \mid \alpha \in F^*\}$ , so for convenience,  $\alpha I_n$  is written shortly by  $\alpha$  for every  $\alpha \in F^*$ . For every  $1 \leq i, j \leq n$ , we denote by  $e_{ij}$  the matrix in  $M_n(F)$  in which the  $(i, j)$ -entry is 1 and the other entries are 0. For a group  $G$ , the order of  $G$  is denoted by  $|G|$ .

## 2. THE DIAMETER OF $\Gamma(\mathrm{GL}_n(F))$ OVER FINITE FIELDS $F$

Our field  $F$  in this paper is assumed to contain at least 3 elements since we want the intersection graph to be connected. The following example shows that in the case  $F = \mathbb{Z}/2\mathbb{Z}$ , the intersection graph of  $\mathrm{GL}_n(F)$  is unnecessarily connected. Let  $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$  be the general linear group of degree 2 over the field  $\mathbb{Z}/2\mathbb{Z}$ . We may check that

$$\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, I_2, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

The general linear group  $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$  has 6 subgroups:  $A = \{I_2\}$ ,  $B = \left\{ I_2, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ ,  $C = \left\{ I_2, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ ,  $D = \left\{ I_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ ,  $E = \left\{ I_2, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$  and  $G = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ . Hence, the intersection graph  $\Gamma(\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}))$  has 4 vertices  $B, C, D$  and  $E$ , and no edge. There is no much information of this graph.

For convenience, if  $A$  is a vertex in  $\Gamma(\mathrm{GL}_n(F))$ , then the *order* of  $A$  is understood to be the order of  $A$  as a subgroup in  $\mathrm{GL}_n(F)$ . We first borrow a lemma from [10] (or see [25, Lemma 1]).

**Lemma 2.1.** [10] *For a finite group  $G$ , if the intersection graph  $\Gamma(G) = (\mathcal{V}_G, \mathcal{E}_G)$  of  $G$  is connected, then the diameter*

$$\delta(\Gamma(G)) = \max \{d(A, B) \mid A, B \in \mathcal{V}_G \text{ of prime order and } A \neq B\}.$$

Observe that in [25], it was shown that if  $G$  is not simple and the intersection graph  $G$  is connected, then the diameter  $\delta(\Gamma(G))$  of  $\Gamma(G)$  is less than or equal to 4 (see [25, Lemma 5]). If  $F$  is a field containing at least 3 elements, then the general linear group  $\mathrm{GL}_n(F)$  is not simple for every  $n \geq 2$  since  $\{I_n\} \neq \mathrm{SL}_n(F) \neq \mathrm{GL}_n(F)$ . Hence, if  $\Gamma(\mathrm{GL}_n(F))$  is connected, then, by [25, Lemma 5], the diameter of  $\Gamma(\mathrm{GL}_n(F))$  is less than or equal to 4. The following result, we evaluate the diameter of  $\Gamma(\mathrm{GL}_n(F))$  better in case it is connected.

**Lemma 2.2.** *Let  $F$  be a finite field containing at least 3 elements and  $n > 1$ . If the intersection graph  $\Gamma(\mathrm{GL}_n(F))$  is connected, then the diameter  $\delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$ .*

**PROOF.** By Lemma 2.1, it suffices to show that  $d(A, B) \leq 3$  for every two distinct vertices  $A$  and  $B$  in  $\Gamma(\mathrm{GL}_n(F))$  of prime order. Assume that  $A = \langle a \rangle$  and  $B = \langle b \rangle$  are two subgroups of  $\mathrm{GL}_n(F)$  of prime orders  $p$  and  $q$  respectively. Let  $\alpha$  be a primitive element of  $F^*$ , that is,  $F^* = \langle \alpha \rangle$ . Put  $A_1 = \langle a, \alpha \rangle$  and  $B_1 = \langle b, \alpha \rangle$ . Since  $\alpha$  is central,  $A_1$  and  $B_1$  is abelian, so  $A_1$  and  $B_1$  are non-trivial proper subgroups of  $\mathrm{GL}_n(F)$ . It implies that  $A_1$  and  $B_1$  are vertices of  $\Gamma(\mathrm{GL}_n(F))$  and they are adjacent since  $F^* \leq A_1 \cap B_1$  and  $F^*$  contains at least two elements.

If  $A = A_1$  and  $B = B_1$ , then since  $F^* \leq A, F^* \leq B$  and  $A, B$  are of prime order,  $F^* = A$  and  $F^* = B$ , which contradicts the hypothesis. Hence, either  $A \not\leq A_1$  or  $B \not\leq B_1$ . Without loss of generality, we suppose that  $A \not\leq A_1$ . In this case, we have two cases. If  $B \not\leq B_1$ , then we have the path  $A - A_1 - B_1 - B$ . Hence,  $d(A, B) \leq 3$ . If  $B = B_1$ , then  $B \cap A_1$  contains  $F^*$  and since  $F^*$  contains at least 2 elements, one has the path  $A - A_1 - B$ . In this case,  $d(A, B) \leq 2$ . All our cases lead  $d(A, B) \leq 3$ . The proof is complete. ■

Now we are ready to show our main result in this section.

**Theorem 2.3.** *Let  $F$  be a finite field containing at least 3 elements, let  $n$  be a integer greater than 1 and let  $\Gamma(\mathrm{GL}_n(F))$  be the intersection graph of the general linear group  $\mathrm{GL}_n(F)$ . The following assertions hold:*

- (1) *If  $A$  is a vertex of  $\Gamma(\mathrm{GL}_n(F))$ , then  $d(A, \mathrm{SL}_n(F)) \leq 2$ .*
- (2) *The intersection graph  $\Gamma(\mathrm{GL}_n(F))$  is connected and  $2 \leq \delta(\Gamma(\mathrm{GL}_n(F))) \leq 3$ .*

PROOF. Assume that  $p = \mathrm{Char}(F)$  and  $|F| = p^m$  for some positive integer  $m$ .

1. Since  $F$  contains at least 3 elements,  $\mathrm{SL}_n(F)$  is a non-trivial proper subgroup of  $\mathrm{GL}_n(F)$ , which implies that  $\mathrm{SL}_n(F)$  is a vertex of  $\Gamma(\mathrm{GL}_n(F))$ . Now let  $A$  be a vertex of  $\Gamma(\mathrm{GL}_n(F))$  different from  $\mathrm{SL}_n(F)$ , that is,  $A$  is a non-trivial proper subgroup of  $\mathrm{GL}_n(F)$  and  $A \neq \mathrm{SL}_n(F)$ . We must show that  $d(A, \mathrm{SL}_n(F)) \leq 2$ . Assume that  $d(A, \mathrm{SL}_n(F)) > 2$ . We seek a contradiction. Since  $d(A, \mathrm{SL}_n(F)) > 2$ , the intersection  $A \cap \mathrm{SL}_n(F) = \{I_n\}$ . In particular, if  $[A, A] = \langle aba^{-1}b^{-1} \mid a, b \in A \rangle$  is the derived subgroup of  $A$ , then  $[A, A] \subseteq A \cap \mathrm{SL}_n(F) = \{I_n\}$ , so  $A$  is abelian. Let  $a \in A \setminus \{I_n\}$  and  $b \in \mathrm{SL}_n(F) \setminus \{I_n\}$ . We claim that  $\mathrm{GL}_n(F) = \langle a, b \rangle$ . If  $\langle a, b \rangle \neq \mathrm{GL}_n(F)$ , then we have the path  $A - \langle a, b \rangle - \mathrm{SL}_n(F)$  since  $a \in A \cap \langle a, b \rangle$  and  $b \in \mathrm{SL}_n(F) \cap \langle a, b \rangle$ . Therefore,  $d(A, \mathrm{SL}_n(F)) = 2$ , a contradiction. Hence,  $\mathrm{GL}_n(F) = \langle a, b \rangle$ . The claim is shown. If  $a$  is central, then  $\langle a, b \rangle$  is abelian, so is  $\mathrm{GL}_n(F)$ , a contradiction. Thus,  $a$  is non-central. Now as  $\mathrm{GL}_n(F)/\mathrm{SL}_n(F) \cong F^*$  via the determinant morphism  $\det : \mathrm{GL}_n(F) \rightarrow F^*, a \mapsto \det(a)$ , one has that  $\det(a)$  is a primitive element of  $F^*$ , that is,  $F^* = \langle \det(a) \rangle$ . In particular, the order of  $\det(a)$  is  $p^m - 1$ . We separate our proof into two cases.

*Case 1.* The case  $\mathrm{Char}(F) = p > 2$ . Then, the cardinality of  $F^*$  is  $p^m - 1$ , an even number. If  $a^2 \neq I_n$ , since  $F^* = \langle \det(a) \rangle$  holds when  $a$  ranges over  $A \setminus \{I_n\}$ , then  $F^* = \langle \det(a^2) \rangle$ . Hence, the order of  $\det(a)^2$  is also  $p^m - 1$ . Therefore, the order of  $\det(a)$  is less than or equal to  $\frac{p^m - 1}{2}$ , which contradicts the fact that  $p^m - 1$  is the order of  $\det(a)$ . It implies that  $a^2 = I_n$ . Consequently,  $\det(a)^2 = 1$ . As a result,  $\det(a) = -1$  since  $a \notin \mathrm{SL}_n(F)$ . In particular, the cardinality of  $F^*$  is 2, which implies that  $F \cong \mathbb{Z}/3\mathbb{Z}$ . Moreover, since  $a$  ranges over  $A$ , one has every non-identity element of  $A$  is of order 2 and determinant  $-1$ . Assume that  $a, a_1 \in A$ . Then,  $\det(aa_1) = \det(a)\det(a_1) = (-1)(-1) = 1$ , which implies that  $aa_1 \in \mathrm{SL}_n(F)$ . We conclude that  $aa_1 = I_n$ , that is,  $a_1 = a$  since  $a^2 = I_n$ . Thus  $A = \{I_n, a\}$ . Now put  $B = \{I_n, a, -I_n, -a\}$ . As  $a$  is non-central,  $a \neq -I_n$  and hence  $A \not\leq B$ . We claim that  $d(B, \mathrm{SL}_n(F)) = 1$ . Indeed, if  $n$  is even, then  $\det(-I_n) = 1$ , otherwise, if  $n$  is odd, then  $\det(-a) = (-1)^n \det a = (-1)(-1) = 1$ . As a result,  $B \cap \mathrm{SL}_n(F)$  is non-trivial since either  $-I_n \in \mathrm{SL}_n(F)$  or  $-a \in \mathrm{SL}_n(F)$ , so one has the path  $A - B - \mathrm{SL}_n(F)$ . It concludes that  $d(A, \mathrm{SL}_n(F)) \leq 2$ , a contradiction.

*Case 2.* The case when  $\mathrm{Char}(F) = 2$ . Then, the cardinality  $|F|$  is  $2^m$  and  $m > 1$  by hypothesis that  $|F| \geq 3$ . Let  $s$  be a prime factor of  $2^m - 1$ . Consider

the element  $a^s$ . If  $a^s \neq I_n$ , then using the same arguments as in Case 1, one has  $\langle \det(a^s) \rangle = F^*$ . Since  $\det(a^s) = \det(a)^s$ , the order of  $\det(a)$  is less than or equal to  $\frac{2^p-1}{s}$  which contradicts the fact that  $\langle \det(a) \rangle = F^*$ . As a result,  $a^s = I_n$ . If  $2^m - 1$  has two distinct prime factors  $s$  and  $t$ , then  $a^s = I_n$  and  $a^t = I_n$ . It implies that  $s = \text{ord}(a) = t$ , a contradiction. Hence,  $2^m - 1 = s^k$  for some prime number  $s$  and natural number  $k$ . By Catalan's Theorem (for example, see [19]),  $k = 1$ , that is,  $s = 2^m - 1$  is prime. Let  $a, a_1 \in A$ . Then,  $\langle a \rangle$  and  $\langle a_1 \rangle$  are groups of prime cardinality  $s$ . Since  $\langle \det(a) \rangle = F^*$ , there exists a positive integer  $\ell$  such that  $\det(a_1) = \det(a)^\ell = \det(a)^\ell$ . Therefore,  $\det(a_1^{-1}a^\ell) = 1$ , so  $a_1^{-1}a^\ell \in \text{SL}_n(F)$ , which implies that  $a_1^{-1}a^\ell = I_n$  by the assumption that  $A \cap \text{SL}_n(F) = \{I_n\}$ . Consequently,  $a_1 = a^\ell \in \langle a \rangle$ . Thus,  $\langle a \rangle = \langle a_1 \rangle$ . It implies that  $A = \{I_n, a, \dots, a^{s-1}\}$ . Now let  $\alpha$  be a primitive element of  $F^*$ , that is,  $F^* = \langle \alpha \rangle$ . In particular, there exists  $1 \leq i \leq s-1$  such that  $\det(a) = \alpha^i$ . Put  $B = \langle \alpha, a \rangle$ . If  $A = B$ , then  $A$  contains  $F^*$ , so  $A = F^*$  since the order  $s$  of  $A$  is prime. It contradicts the fact that  $a$  is non-central. Hence,  $A \not\subseteq B$ . By definition,  $A$  and  $B$  are adjacent. If  $s$  is a divisor of  $n$ , that is,  $n = st$ , then  $\det(\alpha I_n) = \alpha^n = \alpha^{st} = (\alpha^s)^t = 1$ . Otherwise, if  $s$  is not a factor of  $n$ , then there exist integers  $\mu$  and  $\nu$  such that  $\mu n + \nu s = 1$ . It implies that

$$\det(\alpha^{-\mu i} a) = \alpha^{-\mu i} \det(a) = \alpha^{-\mu i} \alpha^i = \alpha^{-\mu i + i} = \alpha^{i\nu s} = (\alpha^s)^{i\nu} = 1.$$

These two cases lead us to conclusion that either  $\alpha I_n \in \text{SL}_n(F)$  or  $\alpha^{-\mu i} a \in \text{SL}_n(F)$ . Observe that  $\alpha I_n$  and  $\alpha^{-\mu i} a$  are non-identity since  $a$  is non-central and  $s$  is not a factor of  $\mu i$ , so the intersection  $\text{SL}_n(F) \cap B$  is non-trivial. Consequently, we have the path  $A - B - \text{SL}_n(F)$ , so  $d(A, \text{SL}_n(F)) \leq 2$ , a contradiction.

Two cases lead to a contradiction. Thus,  $d(A, \text{SL}_n(F)) \leq 2$  for every vertex  $A$  of  $\Gamma(\text{GL}_n(F))$  different from  $\text{SL}_n(F)$ .

2. By (1),  $\Gamma(\text{SL}_n(F))$  is connected. By Lemma 2.2,  $\delta(\Gamma(\text{GL}_n(F))) \leq 3$ . Now we show  $\delta(\Gamma(\text{GL}_n(F))) \geq 2$ . Let  $c = I_n + e_{12}$  and  $d = I_n + e_{21}$  be two elements in  $\text{GL}_n(F)$ . If  $p = \text{Char}(F)$ , then  $c^p = I_n + pe_{12} = I_n$  and  $d^p = I_n + pe_{21} = I_n$ , so  $C = \langle c \rangle$  and  $D = \langle d \rangle$  are subgroups of  $\text{GL}_n(F)$  of order  $p$ . Since  $c \notin \langle d \rangle$  and  $d \notin \langle c \rangle$ , two subgroups  $C$  and  $D$  are distinct. Hence,  $d(C, D) \geq 2$ . Consequently,  $\delta(\Gamma(\text{GL}_n(F))) \geq 2$ . The proof now is complete. ■

The following example shows that 3 is the best upper bound of the diameter of  $\Gamma(\text{GL}_n(F))$ .

**Proposition 2.4.** *The diameter  $\delta(\Gamma(\text{GL}_2(\mathbb{Z}/3\mathbb{Z}))$  of  $\Gamma(\text{GL}_2(\mathbb{Z}/3\mathbb{Z}))$  is 3.*

PROOF. If  $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$  and  $b = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ , then  $a^2 = b^3 = I_2$  and

$$\langle a, b \rangle = \text{GL}_2(\mathbb{Z}/3\mathbb{Z}).$$

It is clear that two vertices  $\langle a \rangle$  and  $\langle b \rangle$  are not adjacent. Hence, by Theorem 2.3,  $2 \leq d(\langle a \rangle, \langle b \rangle) \leq 3$ . If  $d(\langle a \rangle, \langle b \rangle) = 2$ , then there exists a vertex  $C$  of  $\Gamma(\text{GL}_2(\mathbb{Z}/3\mathbb{Z}))$  such that

$$\langle a \rangle - C - \langle b \rangle.$$

Therefore,  $\langle a \rangle \cap C$  and  $\langle b \rangle \cap C$  are non-trivial, which implies that  $a, b \in C$  since the orders of  $a, b$  are prime numbers. Consequently,  $C = \langle a, b \rangle = \text{GL}_2(\mathbb{Z}/3\mathbb{Z})$ , which contradicts the hypothesis. Thus,  $d(\langle a \rangle, \langle b \rangle) = 3$ . By Lemma 2.1,  $\delta(\Gamma(\text{GL}_2(\mathbb{Z}/3\mathbb{Z}))) = 3$ . The proof is complete. ■

### 3. CLASSIFYING OF $\text{GL}_n(F)$ ACCORDING TO THE DIAMETER $\delta(\Gamma(\text{GL}_n(F)))$

In the previous section, we show that if  $F$  is a finite field containing at least 3 elements and  $n > 1$ , then  $2 \leq \delta(\Gamma(\text{GL}_n(F))) \leq 3$ . In this section, we classify  $\text{GL}_n(F)$  according to the diameter  $\delta(\Gamma(\text{GL}_n(F)))$ . It is shown that the general linear group  $\text{GL}_n(F)$  of degree  $n > 1$  over a finite field is generated by two elements [28]. The following result gives us a relationship between the orders of generators and the diameter  $\delta(\Gamma(\text{GL}_n(F)))$ .

**Theorem 3.1.** *Let  $F$  be a finite field containing at least 3 elements, let  $n \geq 2$  and let  $\text{GL}_n(F)$  be the general linear group of degree  $n$  over  $F$ . Then,  $\delta(\Gamma(\text{GL}_n(F))) = 3$  if and only if  $\text{GL}_n(F)$  is generated by two elements of prime order.*

PROOF. Assume that  $\delta(\Gamma(\text{GL}_n(F))) = 3$ . Then, by Lemma 2.1, there exist two vertices  $A, B$  in  $\Gamma(\text{GL}_n(F))$  of prime order such that  $d(A, B) = 3$ . Since  $A$  and  $B$  are of prime order,  $A = \langle a \rangle$  and  $B = \langle b \rangle$  where  $a$  and  $b$  are of prime order. Moreover,  $A$  and  $B$  are distinct, so  $\langle a, b \rangle$  is different from  $A$  and  $B$ . If  $\langle a, b \rangle \neq \text{GL}_n(F)$ , then one has the path  $A - \langle a, b \rangle - B$ , which implies that  $d(A, B) = 2$ , a contradiction. Hence,  $\text{GL}_n(F) = \langle a, b \rangle$ .

Conversely, assume that  $\text{GL}_n(F) = \langle a, b \rangle$ , where  $a, b$  are elements in  $\text{GL}_n(F)$  of prime order. Let  $A = \langle a \rangle$  and  $B = \langle b \rangle$ . Then, since  $A$  and  $B$  are abelian,  $A$  and  $B$  are proper subgroups of  $\text{GL}_n(F)$ , which implies that  $A$  and  $B$  are vertices in  $\Gamma(\text{GL}_n(F))$  of prime order. As  $A$  and  $B$  are distinct and of prime order, the intersection  $A \cap B = \{I_n\}$ , which implies that  $d(A, B) > 1$ . If  $d(A, B) = 2$ , then there exists a vertex  $C \in \Gamma(\text{GL}_n(F))$  different from  $A$  and  $B$  such that  $A - C - B$  is a path. Therefore,  $A \cap C \neq \{I_n\}$  and  $B \cap C \neq \{I_n\}$ . Again, because  $A$  and  $B$  are of prime order,  $A \leq C$  and  $B \leq C$ . Consequently,  $a, b \in C$ . Thus,  $\text{GL}_n(F) = \langle a, b \rangle \leq C$ , which contradicts the hypothesis. It implies that  $d(A, B) \geq 3$ . By Lemma 2.1 and Theorem 2.3,  $\delta(\Gamma(\text{GL}_n(F))) = 3$ . ■

We next find some conditions on the cardinality of  $F$  such that  $\delta(\Gamma(\text{GL}_n(F))) = 2$ .

**Proposition 3.2.** *Let  $F$  be a finite field contain at least 3 elements, let  $n \geq 2$  and let  $\text{GL}_n(F)$  be the general linear group of degree  $n$  over  $F$ . If the multiplicative group  $F^* \neq \langle \alpha, \beta \rangle$  for every elements  $\alpha, \beta \in F^*$  of prime order, then  $\delta(\Gamma(\text{GL}_n(F))) = 2$ .*

PROOF. By Lemma 2.1 and Theorem 2.3, it suffices to show that  $d(A, B) \leq 2$  for every two distinct vertices  $A, B$  of  $\Gamma(\text{GL}_n(F))$  of prime order. Assume that  $A, B$  are two distinct vertices of  $\text{GL}_n(F)$  of prime order. Since  $A$  and  $B$  are distinct and of prime order as subgroups of  $\text{GL}_n(F)$ ,  $A \cap B = \{I_n\}$ . Suppose that  $A = \langle a \rangle$  and  $B = \langle b \rangle$ , where  $a$  and  $b$  are distinct elements of  $\text{GL}_n(F)$  of prime orders  $p$  and  $q$  respectively. Then,  $\det(a), \det(b) \in F^*$  of orders  $p$  and  $q$  respectively. By hypothesis,  $F^* \neq \langle \det(a), \det(b) \rangle$ , which implies that  $\text{GL}_n(F) \neq \langle a, b \rangle$ . Therefore,  $\langle a, b \rangle$  is a vertex of  $\Gamma(\text{GL}_n(F))$  and one has a path  $A - \langle a, b \rangle - B$ . Thus,  $d(A, B) \leq 2$ . The proof is complete. ■

**Corollary 3.3.** *Let  $F$  be a finite field containing at least 3 elements and  $n \geq 2$ . Then  $\delta(\Gamma(\text{GL}_n(F))) = 2$  if  $|F^*|$  satisfies one of following conditions:*

- (1) *There exist prime numbers  $h, k, \ell > 1$  such that  $h k \ell$  is a divisor of  $|F^*|$ .*
- (2) *The cardinality  $|F^*| = h^2$  for some prime number  $h$ .*

PROOF. 1. Let  $\alpha, \beta \in F^*$  be of prime orders  $h$  and  $k$  respectively, then the order of the subgroup  $\langle \alpha, \beta \rangle$  of  $F^*$  is at most  $hk$ , so since  $|F^*| > hk\ell$ ,  $F^* \neq \langle \alpha, \beta \rangle$ . By Proposition 3.2,  $\delta(\Gamma(\text{GL}_n(F))) = 2$ .

2. Assume that  $\text{Char}(F) = p$ . Then  $|F^*| = p^\mu$  for some positive integer  $\mu$ . Hence,  $p^\mu - 1 = h^2$ , equivalently,  $p^\mu - h^2 = 1$ . According to Catalan's theorem (see [19]),  $\mu = 1$ . If  $h$  is odd, then  $p$  is even and greater than 2, which is a contradiction. Therefore,  $h = 2$ . It implies that  $p = 5$ . Let  $A = \langle a \rangle$  and  $B = \langle b \rangle$  be subgroups of  $\text{GL}_n(F)$  of prime order  $h$  and  $k$  respectively. Put  $\alpha = \det(a)$  and  $\beta = \det(b)$ . One has  $\alpha^h = 1$  and  $\beta^k = 1$  since  $a^h = I_n$  and  $b^k = I_n$ . Hence, the orders of  $\alpha$  and  $\beta$  are either prime numbers or 1. Because  $|F^*| = 4$ , the orders of  $\alpha$  and  $\beta$  are divisors of 2. If either  $a \in \text{SL}_n(F)$  or  $b \in \text{SL}_n(F)$ , then either  $\alpha = 1$  or  $\beta = 1$ . Without loss of generality, we assume that  $\alpha = 1$ . Then  $\langle \alpha, \beta \rangle = \langle \beta \rangle$  of order 1 or 2. Therefore,  $F^* \neq \langle \alpha, \beta \rangle$ . By Proposition 3.2,  $\delta(\Gamma(\text{GL}_n(F))) = 2$ . Now assume that both  $a, b$  are not in  $\text{SL}_n(F)$ , that is,  $\alpha \neq 1$  and  $\beta \neq 1$ . Then,  $\alpha, \beta$  are of order 2. As  $\bar{4}$  is the unique element of order 2 in  $\mathbb{Z}/5\mathbb{Z}$ ,  $\alpha = \beta$ , then  $\langle \alpha, \beta \rangle = \langle \alpha \rangle$  of order 2, which implies that  $F^* \neq \langle \alpha, \beta \rangle$ . Again according to Proposition 3.2,  $\delta(\Gamma(\text{GL}_n(F))) = 2$ . ■

Now we give some examples of general linear groups  $\text{GL}_n(F)$  with diameter  $\delta(\Gamma(\text{GL}_n(F))) = 3$ . To do this, by the previous corollary, we need a field  $F$  such that the order of the multiplicative group  $F^*$  is prime or a product of two distinct prime numbers. The next main result in this section considers fields whose multiplicative groups is of prime order. The class of such fields are “big”. For example, if  $n$  is a positive integer such that  $2^n - 1$  is a prime number<sup>1</sup>, then the field with  $2^n$  elements belongs to this class.

We firstly need the following lemma whose proof is straightforward (or see [27]).

**Lemma 3.4.** *Let  $F$  be a finite field such that the order of the multiplicative group  $F^*$  is prime. If  $\alpha$  is a primitive element of  $F^*$ , then  $\text{GL}_2(F)$  is generated by  $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ .*

**Proposition 3.5.** *If  $F$  is a finite field whose multiplicative group  $F^*$  is of prime order, then  $\delta(\Gamma(\text{GL}_2(F))) = 3$ .*

PROOF. Let  $p = \text{Char}(F)$ . Then  $|F| = p^n$  for some positive integer  $n$ . If  $p > 2$ , then the order  $F^*$  is prime if and only if  $F = \mathbb{Z}/3\mathbb{Z}$ . Hence,  $\delta(\Gamma(\text{GL}_2(F))) = 3$  by Proposition 2.4. Now, we consider the case when  $p = 2$ . Put  $q = 2^n - 1$ . By hypothesis,  $q$  is prime. According to Lemma 3.4, if  $\alpha$  is a primitive element of  $F^*$ , then  $\text{GL}_2(F)$  is generated by two elements  $a = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$  and  $b = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ .

Observe that  $a^q = \begin{bmatrix} \alpha^q & 0 \\ 0 & 1 \end{bmatrix} = I_2$ , so the order of  $a$  is  $q$  since  $q$  is prime. One may check that  $b^3 = I_2$ , which implies that the order of  $b$  is 3. Hence,  $\text{GL}_2(F)$  is generated by two elements  $a$  and  $b$  in which the orders of  $a$  and  $b$  are prime. Because of Theorem 3.1,  $\delta(\Gamma(\text{GL}_n(F))) = 3$ . ■

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<sup>1</sup>Such prime numbers are called Mersenne prime

## 4. INTERSECTION GRAPHS OF GENERAL LINEAR GROUPS OVER INFINITE FIELDS

We first borrow the following result which is a special case of [22, Theorem 5.4].

**Lemma 4.1.** *The general linear group of degree greater than 1 over an infinite field is infinitely generated.*

**Theorem 4.2.** *The intersection graphs of general linear groups of degree greater than 1 over an infinite field are connected of diameter 2.*

PROOF. Assume that  $F$  is an infinite field and  $n$  is a natural number greater than 1. We must show that the intersection graph  $\Gamma(\text{GL}_n(F))$  of the general linear group  $\text{GL}_n(F)$  is connected of diameter 2. Let  $A$  and  $B$  be two distinct vertices of  $\Gamma(\text{GL}_n(F))$ . If either  $A \subseteq B$  or  $B \subseteq A$ , then  $A \cap B \neq \{1\}$ , so  $d(A, B) = 1$ . Assume that  $A \not\subseteq B$  and  $B \not\subseteq A$ . Let  $a \in A \setminus B$  and  $b \in B \setminus A$ . Put  $C = \langle a, b \rangle$ . Then, by Lemma 4.1,  $C$  is different from  $\text{GL}_n(F)$ , which implies that  $C$  is a vertex of  $\Gamma(\text{GL}_n(F))$ . Moreover,  $a \in C \cap A$  and  $b \in C \cap B$ , so  $C \cap A \neq \{I_n\}$  and  $C \cap B \neq \{I_n\}$ . Hence, we have a path  $A - C - B$ . Therefore,  $d(A, B) \leq 2$ .

Now, we show that  $\delta(\Gamma(\text{GL}_n(F))) = 2$ . Put  $a = I_n + e_{12}$  and  $b = I_n + e_{21}$ . Let  $U = \langle a \rangle$  and  $V = \langle b \rangle$  be the subgroups of  $\text{GL}_n(F)$  generated by  $a$  and  $b$  respectively. Then, it is clear that  $U = \{I_n + me_{12} \mid m \in \mathbb{Z}\}$  and  $V = \{I_n + me_{21} \mid m \in \mathbb{Z}\}$ , so  $U \cap V = \{I_n\}$  which implies that  $d(U, V) > 1$ . Thus,  $\delta(\Gamma(\text{GL}_n(F))) = 2$  ■

Now, we show results on ends of  $\Gamma(\text{GL}_n(F))$  and begin with the following remark.

**Remark 4.3.** Let  $\Gamma$  be an infinite graph. If

$$(v_i)_{i \geq 0} = v_0 - v_1 - \cdots - v_n - \cdots$$

is a ray of  $\Gamma$ , then the ray  $(v_i)_{i \geq k} = v_k - v_{k+1} - \cdots - v_n - \cdots$  is end-equivalent to  $(v_i)_{i \geq 0}$  for every  $k \in \mathbb{N}$ .

Recall that a group  $G$  is called *Artinian* (resp., *Noetherian*) if every descending chain (resp., ascending chain) of subgroups stabilizes after a finite stage, that is, if  $G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$  (resp.,  $G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$ ) is a sequence of subgroups of  $G$ , then there exists a positive integer  $m$  such that  $G_m = G_n$  for every  $n \geq m$ .

**Lemma 4.4.** *Let  $G$  be a non-trivial group with intersection  $\Gamma(G)$ . If  $G$  is non-Artinian or non-Noetherian, then  $\Gamma(G)$  contains infinitely many distinct rays.*

PROOF. We show the lemma for the non-Noetherian case, the non-Artinian one is similar.

Assume that  $G$  is non-Noetherian, that is, there exists a sequence of subgroups

$$G_1 \leq G_2 \leq G_3 \leq \cdots \leq G_n \leq \cdots$$

Let  $p_1 = 2 < p_2 = 3 < \cdots < p_n < \cdots$  be a sequence of infinitely many distinct prime numbers. For any  $i \geq 1$ , we consider the sequence

$$G_{p_i} \leq G_{p_i^2} \leq G_{p_i^3} \leq \cdots \leq G_{p_i^m} \leq \cdots$$

Then, in  $\Gamma(G)$ , we have a ray

$$G_{p_i} - G_{p_i^2} - G_{p_i^3} - \cdots - G_{p_i^m} - \cdots$$



Moreover, if  $i \neq j$ , then  $p_i^{n_i} \neq p_j^{m_j}$  for every  $n_i, n_j > 0$ , so  $G_{p_i^{n_i}} \neq G_{p_j^{m_j}}$  for every  $i \neq j$  and  $n_i, n_j > 0$ . Hence,  $\Gamma(G)$  contains infinitely many distinct rays. The proof is complete. ■

**Theorem 4.5.** *Let  $F$  be an infinite field and  $n > 1$ . Then the intersection graph  $\Gamma(\text{GL}_n(F))$  is one-ended and its end is thick.*

PROOF. Since  $\Gamma(\text{GL}_n(F))$  is connected by Theorem 4.2, it contains a ray, namely,  $[(A_i)_{i \geq 0}]$ . We first show that  $[(A_i)_{i \geq 0}]$  is the unique end of  $\Gamma(\text{GL}_n(F))$ . Assume  $(B_i)_{i \geq 0} = B_0 - B_1 - \dots - B_n - \dots$  is an arbitrary ray of  $\Gamma(\text{GL}_n(F))$ . We must find a ray  $(C_i)_{i \geq 0} = C_0 - C_1 - \dots - C_n - \dots$  of  $\Gamma(\text{GL}_n(F))$  such that  $\{C_i \mid i \in \mathbb{N}\} \cap \{A_i \mid i \in \mathbb{N}\}$  and  $\{C_i \mid i \in \mathbb{N}\} \cap \{B_i \mid i \in \mathbb{N}\}$  are infinite. If  $\{A_i \mid i \in \mathbb{N}\}$  contains infinitely many vertices  $B_i$ , then we choose  $(C_i)_{i \geq 0} = (A_i)_{i \geq 0}$ . If  $(B_i)_{i \geq 0}$  contains infinitely many vertices  $A_i$ , then we choose  $(C_i)_{i \geq 0} = (B_i)_{i \geq 0}$ . In two cases, we have  $\{C_i \mid i \in \mathbb{N}\} \cap \{A_i \mid i \in \mathbb{N}\}$  and  $\{C_i \mid i \in \mathbb{N}\} \cap \{B_i \mid i \in \mathbb{N}\}$  are infinite, so  $(A_i)_{i \geq 0} \equiv (B_i)_{i \geq 0}$ . Now, we assume that there exists a natural number  $k$  such that

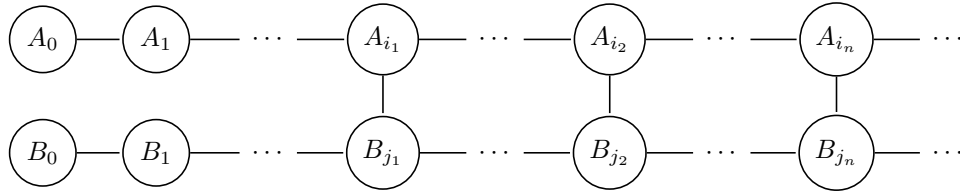
$$\{A_i \mid i \geq k\} \cap \{B_i \mid i \geq k\} = \emptyset.$$

Since,  $(A_i)_{i \geq 0} \equiv (A_i)_{i \geq k}$ , where  $(A_i)_{i \geq k} = A_k - A_{k+1} - \dots - A_n - \dots$  for every  $k \in \mathbb{N}$  (Remark 4.3) and the relation  $\equiv$  is equivalent, without loss of generality, we assume that

$$\{A_i \mid i \in \mathbb{N}\} \cap \{B_i \mid i \in \mathbb{N}\} = \emptyset.$$

We have two cases:

*Case 1.* The case when for every  $k \in \mathbb{N}$ , there exists  $i_k, j_k \geq k$  such that  $A_{i_k}$  and  $B_{j_k}$  are adjacent.

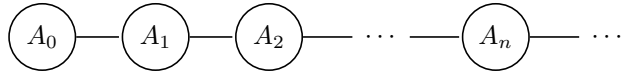


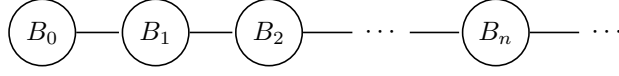
We choose the ray  $(C_i)_{i \geq 0}$  which is the square zigzag one

$$A_1 - \dots - A_{i_1} - B_{j_1} - B_{j_1+1} - \dots - B_{j_2} - A_{i_2} - A_{i_2+1} - \dots - A_{i_3} - B_{j_3} - \dots$$

in the above diagram. One has  $(C_i)_{i \geq 0}$  is a ray since all vertices in  $\{A_i \mid i \in \mathbb{N}\} \cup \{B_i \mid i \in \mathbb{N}\}$  are distinct. Moreover,  $A_{i_k} \in \{C_i \mid i \in \mathbb{N}\} \cap \{A_i \mid i \in \mathbb{N}\}$  and  $B_{j_k} \in \{C_i \mid i \in \mathbb{N}\} \cap \{B_i \mid i \in \mathbb{N}\}$  for every  $k \in \mathbb{N}$ , so  $(C_i)_{i \geq 0}$  is the ray we want to get.

*Case 2.* The case when there exists  $k \in \mathbb{N}$  such that for every  $i_k, j_k \geq k$ ,  $A_{i_k}$  and  $B_{j_k}$  are not adjacent. By using the same arguments as in the previous case, without loss of generality, we assume that  $k = 0$ , that is,  $A_i$  and  $B_j$  are not adjacent for every  $i, j \geq 0$ .





We first construct a sequence of vertices  $\{D_i \mid i \in \mathbb{N}\}$ , where  $D_i$  is generated by at most  $3(i+1)$  elements as subgroups in  $\text{GL}_n(F)$ , in induction on  $i \geq 0$  as follow: Since  $A_0$  and  $B_0$  are not adjacent, let  $a_0 \in A_0 \setminus B_0$ ,  $b_0 \in B_0 \setminus A_0$  and  $c_0 = 1$ . Put  $D_0 = \langle a_0, b_0, c_0 \rangle$ . Assume that

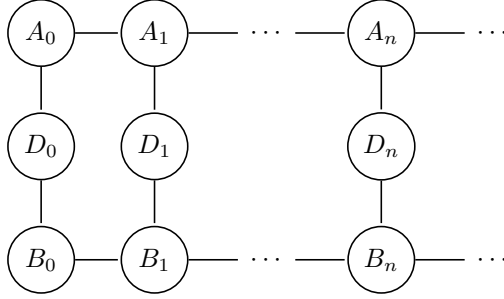
$$D_i = \langle a_0, a_1, \dots, a_i, b_0, b_1, \dots, b_i, c_0, c_1, \dots, c_i \rangle$$

is defined. As  $A_{i+1}$  and  $B_{i+1}$  are not adjacent, there exists  $a_{i+1} \in A_{i+1} \setminus B_{i+1}$  and  $b_{i+1} \in B_{i+1} \setminus A_{i+1}$ . Observe that  $\text{GL}_n(F)$  is infinitely generated, so there exists  $c_{i+1} \in \text{GL}_n(F) \setminus D_i$ . Put

$$D_{i+1} = \langle a_0, a_1, \dots, a_{i+1}, b_0, b_1, \dots, b_{i+1}, c_0, c_1, \dots, c_{i+1} \rangle.$$

Now, we list some properties of  $\{D_i \mid i \in \mathbb{N}\}$  that we use in this theorem.

- (1) All vertices  $\{D_i \mid i \in \mathbb{N}\}$  are distinct since  $c_{i+1} \notin D_i$  for every  $i \geq 0$  distinct.
- (2) For every  $i \in \mathbb{N}$ , there is a path  $A_i - D_i - B_i$  because of the fact that  $a_i \in D_i \cap A_i$  and  $b_i \in D_i \cap B_i$ .



- (3) For every  $i \in \mathbb{N}$ ,  $D_i \notin \{A_i \mid i \in \mathbb{N}\} \cup \{B_i \mid i \in \mathbb{N}\}$ . To see this, assume that  $D_i = A_i$  for some  $i \in \mathbb{N}$ . Then  $A_i$  and  $B_i$  are adjacent, which is a contradiction. Hence,  $D_i \notin \{A_i \mid i \in \mathbb{N}\}$  for every  $i \geq 0$ . Similarly,  $D_i \notin \{B_i \mid i \in \mathbb{N}\}$  for every  $i \geq 0$ .

By choosing the ray which is the square zigzag path

$$(C_i)_{i \geq 0} = A_0 - D_0 - B_0 - B_1 - D_1 - A_1 - A_2 - D_2 - \dots,$$

we see that  $(C_i)_{i \geq 0}$  meets  $(A_i)_{i \geq 0}$  and  $(B_i)_{i \geq 0}$  at infinitely many vertices.

Now, since  $\text{GL}_n(F)$  is infinitely generated,  $\text{GL}_n(F)$  is non-Artinian. By Lemma 4.4, the ray  $[(A_i)_{i \geq 0}]$  is thick. ■

## REFERENCES

- [1] A. Abdollahi, S. Akbari, and H.R. Maimani, *Non-commuting graph of a group*, J. Algebra **298** (2006), 468–492.
- [2] C. Ambrozie, J. Bracic, B. Kuzma, and V. Muller, *The commuting graph of bounded linear operators on a Hilbert space*, J. Funct. Anal. **264** (2013), 1068–1087.
- [3] V. Bafna and P. A. Pevzner, *Genome rearrangements and sorting by reversals*, SIAM J. Comput. **25** (1996), 272–289.
- [4] E. A. Bertram, *Some applications of graph theory to finite groups*, Discrete Math. **44** (1983), 31–43.
- [5] J. Bosak, *The graphs of semigroups. Theory of Graphs and Its Applications*, Proc. Sym- pos., Smolenice 1963, Publ. House Czechoslovak Acad. Sci., Praha, 1964, 119–125.
- [6] S. Brauer and K.A. Fowler, *On groups of even order*, Ann. Math. **62** (1955), 565–583.

- [7] A. Cayley, *Desiderata and suggestions: No. 2. the theory of groups: Graphical representation*, Amer. J. Math. **1** (1878), 174–176.
- [8] I. Chakrabarty, S. Ghosh, and M.K. Sen, *Undirected power graphs of semigroups*, Semigroup Forum **78** (2009), 410–426.
- [9] M. Chowdhury J. Abawajy, A. Kelarev, *Power graphs: A survey*, Electron. J. Graph Theory and Appl. **1** (2013), 125–147.
- [10] B. Csakeany and G. Pollak, *The graph of subgroups of a finite group*, Czechoslovak Math. J. **19** (1969), 241–247.
- [11] A. Everett, *Commuting involution graphs for 3-dimensional unitary groups*, Electron. J. Combin. **18** (2011), 11 pp.
- [12] R. Halin, *Über unendliche Wege in Graphen*, Mathematische Annalen **157** (1964) 125–137.
- [13] N. Imani, H. Sarbazi-Azad, S.G. Akl, and P. Moinzadeh, *Chromatic sets of power graphs and their application to resource placement in multicomputer networks*, Comput. Math. Appl. **58** (2009), 403–413.
- [14] S. Kayacan and E. Yaraneri, *Finite groups whose intersection graphs are planar*, J. Korean Math. Soc. **52** (2015), 81–96.
- [15] A. V. Kelarev and S. J. Quinn, *A combinatorial property and power graphs of groups*, (English summary) Contrib. Gen. Algebra **12** (2000), 229–235.
- [16] A. Kelarev, J. Ryan, and J. Yearwood, *Cayley graphs as classifiers for data mining: The influence of asymmetries*, Discrete Math. **309** (2009), 5360–5369.
- [17] S. Lakshmivarahan, J. Jwo, and S.K. Dhall, *Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey*, Parallel Comput. **19** (1993), 361–407.
- [18] X. Ma, *On the diameter of the intersection graph of a finite simple group*, Czechoslovak Math. J. **66** (2016), 365–370.
- [19] P. Mihailescu, *Primary Cyclotomic Units and a Proof of Catalan's Conjecture*, J. Reine Angew. Math. **572** (2004), 167–195.
- [20] A.R. Moghaddamfar, W.J. Shi, W. Zhou, and A.R. Zokayi, *On the noncommuting graph associated with a finite group*, Siberian Math. J. **46** (2005), 325–332.
- [21] B.H. Neumann, *A problem of Paul Erdos on group*, J. Aust. Math. Soc. **21** (1976), 467–472.
- [22] N. K. Ngoc, M. H. Bien and B. X. Hai, *Free subgroups in almost subnormal subgroups of general skew linear groups*, Algebra i Analiz **28** (2016), No. 5, 220–235; English translation in St. Petersburg Math. J. **28** (2017), No. 5, 707–717.
- [23] R. Rajkumar and P. Devi, *Intersection graphs of cyclic subgroups of groups*, Electron. Notes Discrete Math. **53** (2016), 15–24.
- [24] H. Ahmadi and B. Taeri, *Planarity of the intersection graph of subgroups of a finite groups*, J. Algebra Appl. **15** (2016) 1650040, 19 pages.
- [25] R. Shen, *Intersection graphs of subgroups of finite groups*, Czechoslovak Math. J. **60** (2010), 945–950.
- [26] A. A. Talebi, *On the non-commuting graphs of group  $D_{2n}$* , Int. J. Algebra **2** (2008), 957–961.
- [27] D. E. Taylor, *Pairs of Generators for Matrix Groups. I*, 2006, <https://www.maths.usyd.edu.au/u/don/papers/genAC.pdf>.
- [28] C. Waterhouse, William, *Two Generators for the General Linear Groups over Finite Fields*, Linear and Multilinear Algebra **24** (1989), 227–230.

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