THE NORMALITY OF A MONOMIAL IDEAL AND THE INTEGER ROUNDING PROPERTIES

DO HOANG VIET

The integer rounding properties of matrices are subjects of integer programming and convex geometry. In 1981, S. Baum and Jr. L. E. Trotter found the equivalence between integer rounding properties and the integral decomposition property of polyhedra, see [BL81]. Afterwards, Professor Ngo Viet Trung has studied the relationship between the normality of monomial ideals of polynomial rings and the integer rounding properties. The report is a presentation of results that I studied under his guidance.

Contents

The report consists of two modules having four consecutive sections each. This first module presents insights related to integer linear programming and convex geometry, the integer rounding and the integral decomposition properties. In the second module, we combine subjects of the first with algebraic concepts, the monomial ideals.

In the first section of each module (section one and five), we show the basic concepts of the mudule. The next sections (section two and six) present initial results that will be developed in last sections (section four and eight). Section three and seven are lists of tools we use to prove the main results in two last sections.

Throughout the report, we write points (vectors in vector spaces) and their coordinates (real or rational numbers) in a bold and an usual way respectively, such as an illustration $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$. Additionally, we denote the zero vector and vector whose coordinates are all 1's by $\mathbf{0}$ and $\mathbf{1}$, respectively. If we write $\mathbf{x} < \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$), it means every coordinate of $\mathbf{x}$ is lower than (or equal to) the respective coordinate of $\mathbf{y}$. It is routine to verify that space $\mathbb{R}^n$ with the order is a partially ordered set. We will write $\mathbf{x} \cdot \mathbf{y}$ as the inner product $\mathbf{x}\mathbf{y}^T$, the sum of products of respective coordinates of $\mathbf{x}$ and $\mathbf{y}$. For every number $a \in \mathbb{R}$, we use the symbol $\lfloor a \rfloor$ ($\lceil a \rceil$) as the greatest (lowest) integers that are not larger (lower) than a.

1. The motivation

Initially, we mention some definitions of integer programming and convex geometry, which will be used for the rest of the report.

Let M be an non-zero $n \times m$ matrix with non-negative rational entries. For each vector $\mathbf{w} \in \mathbb{N}^n$, we define the following values:

$$\nu^*(\mathbf{w}, M) = \max\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \leq \mathbf{w}^T\},$$

$$\nu(\mathbf{w}, M) = \max\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{N}^m, M\mathbf{y}^T \leq \mathbf{w}^T\},$$

$$\tau^*(\mathbf{w}, M) = \min\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^m_{\geq 0}, M\mathbf{y}^T \geq \mathbf{w}^T\},$$

$$\tau(\mathbf{w}, M) = \min\{\mathbf{1} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbb{N}^m, M\mathbf{y}^T \geq \mathbf{w}^T\}.$$

It is routine to verify that $\nu^*(\mathbf{w}, M) \geq \nu(\mathbf{w}, M)$ and $\tau^*(\mathbf{w}, M) \leq \tau(\mathbf{w}, M)$. From the perspective of integer programming, we pay attention to when $\nu(\mathbf{w}, M)$ and $\nu^*(\mathbf{w}, M)$ (similarly $\tau(\mathbf{w}, M)$ and $\tau^*(\mathbf{w}, M)$) are most approximate. With this motivation, we come into the following definition.

Definition 1. *An $n \times m$-matrix M has the integer round-down property (integer round-up property) if for every $\boldsymbol{w} \in \mathbb{N}^n$, $\lfloor \nu^*(\boldsymbol{w}, M) \rfloor = \nu(\boldsymbol{w}, M)$ (respectively $\lceil \tau^*(\boldsymbol{w}, M) \rceil = \tau(\boldsymbol{w}, M)$).*

Now, we come into geometric concepts that have a close relationship with the integer rounding properties that we can see in Proposition 1.

Definition 2. *Let P be a polyhedron in $\mathbb{R}^n_{\geq 0}$.*

- (i) *We call P upper comprehensive (lower comprehensive) if for every $\boldsymbol{y} \geq \boldsymbol{x} \in P$ ($\boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{x} \in P$), $\boldsymbol{y}$ must belong to P.*
- (ii) *P is said to have the integral decomposition property if every integral vector of $kP = \{k\boldsymbol{a} \mid \boldsymbol{a} \in P\}$ is a sum of k integral vectors of P for all integers $k \geq 1$.*

In 1981, S. Baum and Jr. L. E. Trotter shown the relationship between above concepts that motivated Professor Ngo Viet Trung and I to investigate the additional relationships between them and even algebra in the second module of this report.

Proposition 1. *Let $P \subsetneq \mathbb{R}^n_+$ be a nonempty polyhedron with nonempty interior satisfying its extreme points are all integral.*

- (i) *If P is upper comprehensive, then for the matrix M whose columns are precisely minimal integral points of P, P has the integral decomposition property if and only if M has the integer round-down property.*
- (ii) *If P is bounded and lower comprehensive, then for the matrix M whose columns are exactly maximal integral points of P, P has the integral decomposition property if and only if M has the integer round-up property.*

Proof. See [BL81, Theorem 1]. □

By considering Proposition 1, we can realize that both integer rounding properties are strictly associated to the integral decomposition property of upper and lower comprehensive polyhedra. This poses a question straightforwardly that whether the integer rounding properties are equivalent or not. In terms of geometry, it is routine to realize that there are certainly not polyhedra being both upper and lower comprehensive. Therefore, if we want to build a bridge between two rounding properties, then we must construct pairs of "dual" polyhedra, an upper and a lower comprehensive ones. We can see the construction in next section and realize that two integer rounding properties are simply different versions of an algebraic condition, the normality of monomial ideals, in the last section.

2. Newton polyhedra

Let $I = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{N}^n$ be a set of finitely many integral points such that $\mathbf{a}_i \not\leq \mathbf{a}_j$ for all $i \neq j$. We are going to use the symbols for the rest of the report. Note that we will not consider the case $I = \{\mathbf{0}\}$ due to its inconvenience. By writing $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{N}^n$ for $1 \leq i \leq m$, we put $(a_{max})_j := \max\{a_{ij} \mid 1 \leq i \leq m\}$, the maximal j-coordinate among that of the points, and $\mathbf{a}_{max} := ((a_{max})_1, \ldots, (a_{max})_n)$. We can see that $\mathbf{a}_{max} \geq \mathbf{a}_i$ for every $1 \leq i \leq m$, so we can define $\mathbf{a}_i^* := \mathbf{a}_{max} - \mathbf{a}_i \in \mathbb{N}^n$.

With the set I above, the *Newton polyhedron* associated to I is defined to be

$$N(I) := \operatorname{conv}\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}.$$

With the aim in the first section, we define a "dual" polytope that is bounded as follows:

$$N^*(I) := \operatorname{conv}\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}.$$

We define M and M^* as the $n \times m$ matrices whose columns are $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ and $\mathbf{a}_1^{*T}, \ldots, \mathbf{a}_m^{*T}$, respectively. Indeed, $N(I)$ and $N^*(I)$ will play roles as polyhedra in Proposition 1. In order to ensure they satisfy the conditions in Proposition 1, we, initially, need to prove their upper and lower comprehensiveness. The process belongs to this section and the next where more complex technical properties will be obtained. Now, we show some useful basic properties of $N(I)$ and $N^*(I)$.

Lemma 1. *If $\boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i^* \geq \boldsymbol{a}\}$, then $\boldsymbol{a}_{max} - \boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \leq \boldsymbol{a}\}$. Consequently, if $\boldsymbol{u} \in N^*(I)$, then $\boldsymbol{a}_{max} - \boldsymbol{u} \in N(I)$.*

Proof. The first statement is obvious. For any $\mathbf{u} \in N^*(I)$, we can express $\mathbf{u} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Therefore, $\mathbf{a}_{max} - \mathbf{u} = \sum_{i=1}^t \lambda_i (\mathbf{a}_{max} - \mathbf{u}_i) \in N(I)$ because $\mathbf{a}_{max} - \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ by the previous statement. $\square$

From the above lemma, we can pose a natural question that if $\mathbf{u} \in N(I)$ and $\mathbf{u} \leq \mathbf{a}_{max}$, then we can whether reach a conclusion that $\mathbf{a}_{max} - \mathbf{u} \in N^*(I)$ or not? We can mimic the proof of Lemma 1 for the case as follows: if $\mathbf{u} \in N(I)$, then $\mathbf{u}$ can be express as a convex combination $\mathbf{u} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. However, we cannot obtain an equality $\mathbf{a}_{max} - \mathbf{u} = \sum_{i=1}^t \lambda_i (\mathbf{a}_{max} - \mathbf{u}_i)$. This is because we do not know whether $\mathbf{u}_i \leq \mathbf{a}_{max}$ or not! Indeed, we can also obtain the result later (Lemma 7) when we construct some technical properties that ensure that we can choose the $\mathbf{u}_i$'s satisfying the inequality (Lemma 6). Now we come into another property of Newton polyhedra.

Lemma 2. *The Newton polyhedron $N(I)$ is upper comprehensive, so are $kN(I)$ for every $k \geq 1$.*

Proof. For any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ satisfying $\mathbf{y} \geq \mathbf{x} \in N(I)$, we need to prove $\mathbf{y} \in N(I)$. Initially, we show the statement holds for integral points $\mathbf{x}$ and $\mathbf{y}$. Indeed, according to the definition of $N(I)$, we can express $\mathbf{x} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ for $1 \leq i \leq t$. This results in $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ since $\mathbf{y} \geq \mathbf{x}$. Thus, the equality $\mathbf{y} = \sum_{i=1}^t \lambda_i (\mathbf{y} - \mathbf{x} + \mathbf{u}_i)$ leads $\mathbf{y}$ to be an element of $N(I)$.

Afterwards, we prove the statement also holds for a real point $\mathbf{y}$ and an integral point $\mathbf{x}$. By setting $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$, we obtain that $(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor) \geq \mathbf{x}$ because $\mathbf{x}$ is integral. Therefore, by the previous statement, it must satisfy that $(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor)$ belongs to $N(I)$, so does every vertex of the cube $[\lfloor y_1 \rfloor, \lfloor y_1 \rfloor + 1] \times \cdots \times [\lfloor y_n \rfloor, \lfloor y_n \rfloor + 1]$. Because $\mathbf{y}$ is an element of the cube, $\mathbf{y}$ is a convex combination of these vertices, which leads $\mathbf{y}$ to belong to $\operatorname{conv}\{N(I)\}$, or equivalently $\mathbf{y} \in N(I)$.

Next, we show the conclusion of this lemma holds for real points $\mathbf{x}$ and $\mathbf{y}$. Because $\mathbf{x} \in N(I)$, we can write $\mathbf{x} = \sum_{i=1}^t \lambda_i \mathbf{u}_i$, where $\mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$ and $\lambda_i \geq 0$ and $\sum_{i=1}^t \lambda_i = 1$. We have $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \geq \mathbf{u}_i \in \mathbb{N}^n \cap N(I)$, so $\mathbf{y} - \mathbf{x} + \mathbf{u}_i \in N(I)$ by the last statement. Therefore, the equality $\mathbf{y} = \sum_{i=1}^t \lambda_i (\mathbf{y} - \mathbf{x} + \mathbf{u}_i)$ results in $\mathbf{y} \in \operatorname{conv}\{N(I)\} = N(I)$. Finally, the similar property for $kN(I)$ is obtained straightforwardly. $\square$

Similarly, we will also have $N^*(I)$ is lower comprehensive (Lemma 8). But at that time, our tools are not effective enough to prove that. Now, we indicate a property of $N^*(I)$ that will help us to show its lower comprehensiveness in the next section.

Lemma 3. *$N^*(I)$ is closed in terms of Euclidean norm in $\mathbb{R}^n$.*

Proof. It is routine to verify that the number of elements of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$ is finite, and we put m as the number. Fix an (arbitrary) order $\mathbf{b}_i$'s, $1 \leq i \leq m$, of elements of the set and let A be the matrix $\begin{bmatrix}\mathbf{b}_1^T & \cdots & \mathbf{b}_m^T\end{bmatrix}$. Now, for an arbitrary point $\mathbf{y} \in \mathbb{R}^n$ satisfying that there exists a sequence of vectors $\{\mathbf{y}_i\}_{i=1}^{\infty}$ of $N^*(I)$ converging to $\mathbf{y}$, we claim that $\mathbf{y} \in N^*(I)$. Indeed, since each $\mathbf{y}_i$ is a convex combination of vectors belonging to $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$, we have a vector $\mathbf{z}_i := (y_{i1}, y_{i2}, \ldots, y_{im}) \in [0,1]^m$ such that $\mathbf{1} \cdot \mathbf{z}_i = \sum_{j=1}^m y_{ij} = 1$ and $A\mathbf{z}_i^T = \sum_{j=1}^m y_{ij}\mathbf{b}_j^T = \mathbf{y}_i^T$, for each i. Due to the compactness of $[0,1]^m$, we obtain a subsequence $\{\mathbf{z}_{i'}\}$ of $\{\mathbf{z}_i\}$ that converges to a vector $\mathbf{z} = (y_1, \ldots, y_m)$ belonging to $[0,1]^m$. Since the linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by A is continuous, the fact that $\mathbf{y}_{i'}^T = A\mathbf{z}_{i'}^T$ tends to $\mathbf{y}^T$ allows $\mathbf{y}^T$ to be $A\mathbf{z}^T$. The obvious property $\mathbf{1} \cdot \mathbf{z} = 1$ leads $\mathbf{y}^T$ to be a convex combination of columns of A, which is equivalent to $\mathbf{y} \in N^*(I)$. $\square$

3. Technical properties

The properties in the previous section were elegant results of polyhedra, but results in this section are only technical ones that allow us to prove latter propositions rather than helping understand the geometric structure of polyhedra.

Lemma 4. *If a rational point $\boldsymbol{u}$ is a convex combination of rational points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t \in \mathbb{Q}^n_{\geq 0}$, which means that $\boldsymbol{u} = \sum_{i=1}^t \lambda_i \boldsymbol{u}_i$, where $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=1}^t \lambda_i = 1$, then there exists a sequence of* ***non-negative rational*** *vectors $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \in \mathbb{Q}^t_{\geq 0}$, $j \geq 1$, such that $\sum_{i=1}^t \lambda_i^{(j)} = 1$ and $\boldsymbol{u} = \sum_{i=1}^t \lambda_i^{(j)} \boldsymbol{u}_i$ for every j, and $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \to (\lambda_1, \ldots, \lambda_t)$ as $j \to \infty$.*

Proof. Initially, let S be the subset of $\{1, \ldots, t\}$ such that $s \in S$ if and only if $\lambda_s = 0$. In cases $S \neq \emptyset$, we will write $S = \{s_1, \ldots, s_{|S|}\}$.

After that, when $S \neq \emptyset$, we put S' to be the $|S| \times t$ matrix such that the (i, s_i)-entry of S' is 1 for every $1 \leq i \leq |S|$ and the remaining entries are all zero. Otherwise, we regard S' as the "0×0" matrix. Now we put

$$A := \left[\begin{array}{ccccc} 1 & 1 & \ldots & 1 & 1 \\ \mathbf{u}_1^T & \mathbf{u}_2^T & \ldots & \mathbf{u}_{t-1}^T & \mathbf{u}_t^T \\ \hline & & & & \\ & & S' & & \\ & & & & \end{array}\right] \text{ and } B := \begin{bmatrix} 1 \\ \mathbf{u}^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which are a $(n+1+|S|) \times t$ matrix and a $(n+1+|S|) \times 1$ matrix, respectively.

We can observe that the linear system $A\mathbf{x}^T = B$ has the solution $\mathbf{x}_0 = (\lambda_1, \ldots, \lambda_t) \in \mathbb{R}^t_{\geq 0}$. If $\operatorname{rank}(A) = t$, then the linear system has the unique solution $(\lambda_1, \ldots, \lambda_t)$. Moreover, $\mathbf{x}_0$ must be rational because we can solve the linear system by Gaussian elimination and all of entries of both A and B are rational. Therefore, the conclusion holds in the case $\operatorname{rank}(A) = t$, provided that we choose $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) = (\lambda_1, \ldots, \lambda_t)$ for all j. Otherwise, if $\operatorname{rank}(A) = r < t$, then there are exactly $t - r$ free variables among t variables of the solution $\mathbf{x} = (x_1, \ldots, x_t)$. Without loss of generality, we assume that $x_1, \ldots, x_r$ are precisely free variables of the solutions. By Linear Algebra, the dependent variable x_h is a linear combination $f_h(1, x_1, \ldots, x_r)$ of 1 and the free variables with rational coefficients for every $r+1 \leq h \leq t$ due to the rationality of entries of A and B.

A remark that should be noticed is $S \cap \{1, \ldots, r\} = \emptyset$. This is because for each $s \in S$, the variable x_s of the solution $\mathbf{x} = (x_1, \ldots, x_t)$ is usually constant zero, so it is not a free variable. This allows us to choose x_i to be positive for $1 \leq i \leq r$. Now we choose positive rational numbers $\lambda_i^{(j)}$, $j \geq 1$ and $1 \leq i \leq r$, such that $\lambda_i^{(j)} \to \lambda_i$ as $j \to \infty$. If we choose free variable x_i to be $\lambda_i^{(j)}$ for $1 \leq i \leq r$, then the dependent variable $\lambda_h^{(j)} := x_h = f_h(1, x_1, \ldots, x_r)$ tends to $f_h(1, \lambda_1, \ldots, \lambda_r) = \lambda_h$ as $j \to \infty$ for every $r+1 \leq h \leq t$. Moreover, for $h \in S \subseteq \{r+1, \ldots, t\}$, $\lambda_h^{(j)} = f_h(1, \lambda_1, \ldots, \lambda_r) = 0$ for all j. To sum up, we obtain that $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \to (\lambda_1, \ldots, \lambda_t)$ as $j \to \infty$ and $\lambda_s^{(j)} = 0$ if $\lambda_s = 0$ for all j. Therefore, the condition that $(\lambda_1, \ldots, \lambda_t) \in \mathbb{R}_+^t$ allows us to obtain $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)}) \geq \mathbf{0}$, for sufficiently large j. By excluding small indices, we can assume this property holds for every $j \geq 1$. Since each $(\lambda_1^{(j)}, \ldots, \lambda_t^{(j)})$ is a solution of the linear system, it satisfies all of the requirements in the conclusion. The proof is completed. □

The above lemma allows us to change the background of numbers from $\mathbb{R}$ to $\mathbb{Q}$. In latter propositions, we can realize the important role of rational vectors in the report. The following lemma helps us to "replace" "free" vectors with vectors "lower than" $\mathbf{a}_{max}$ in Lemma 6, the main technique of the section.

Lemma 5. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a point of $\mathbb{N}^n$ and k be a positive integer. If there exist elements $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k$ of $\mathbb{N}^n$ such that $\boldsymbol{x} \leq \sum_{i=1}^k \boldsymbol{y}_i$, then there exist elements $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k$ of $\mathbb{N}^n$ satisfying $\boldsymbol{x} = \sum_{i=1}^k \boldsymbol{z}_i$ and $\boldsymbol{z}_i \leq \boldsymbol{y}_i$.*

Proof. We set y_{ij} to be the j-th coordinate of $\mathbf{y}_i$. We only prove this lemma in cases that $\mathbf{x} < \sum_{i=1}^k \mathbf{y}_i$. Let Ω be the subset of $\{1, \ldots, n\}$ such that $x_j < \sum_{i=1}^k y_{ij}$ if and only if $j \in \Omega$. By the assumption, Ω is a non-empty set. These conditions allow us to define a map

$$\omega : \Omega \to \{0, \ldots, n-1\}$$

as follows: For each $t \in \Omega$,

- if $x_t < y_{1t}$, then $\omega(t) = 0$;
- otherwise, $\omega(t)$ is chosen to satisfy $\sum_{i=1}^{\omega(t)} y_{it} \leq x_t < \sum_{i=1}^{\omega(t)+1} y_{it}$.

Afterwards, we set $\mathbf{z}_i$'s. For $j \notin \Omega$, we set $z_{ij} := b_{ij}$. For each $t \in \Omega$,

- put $z_{it} := y_{it}$ if $i \leq \omega(t)$,
- $z_{\omega(t)+1,t} := x_t - \sum_{i=1}^{\omega(t)} y_{it}$,
- $z_{it} := 0$ if $i > \omega(t) + 1$ (we regard $\sum_{i=1}^{0} y_{it}$ as 0).

After setting the z_{ij}'s, we set points $\mathbf{z}_i = (z_{i1}, \ldots, z_{in})$ for each $1 \leq i \leq k$. It is routine to verify that the points $\mathbf{z}_i$'s satisfy the conclusion. □

Now, we come into the main technique of this section that allows us to prove the remain questions in the previous section.

Lemma 6. *We have the following results that allow us to "replace" vector $\boldsymbol{u}$ with a vector $\boldsymbol{v} \leq \boldsymbol{a}_{max}$.*

(a) *If $\boldsymbol{u} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \leq \boldsymbol{a}\}$, then there exists $\boldsymbol{v} \leq \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \leq \boldsymbol{a}\}$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$.*

(b) *If $\boldsymbol{u} \in \mathbb{N}^n \cap N(I)$, then there exists $\boldsymbol{v} \leq \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \mathbb{N}^n \cap N(I)$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$.*

(b') *If $\boldsymbol{u} \in \mathbb{Q}^n \cap N(I)$, then there exists $\boldsymbol{v} \leq \boldsymbol{u}$ satisfying $\boldsymbol{v} \in \mathbb{Q}^n \cap N(I)$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$. Moreover, if k is a positive integer satisfying $k\boldsymbol{u} \in \mathbb{N}^n$, then $k\boldsymbol{v} \in \mathbb{N}^n$.*

(c) *If $\boldsymbol{v} \in \mathbb{Q}^n \cap N(I)$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$, then $\boldsymbol{v}$ is a convex combination of vectors belonging to $\{\boldsymbol{a} \in \mathbb{N}^n \mid \exists \boldsymbol{a}_i \leq \boldsymbol{a} \leq \boldsymbol{a}_{max}\}$ with positive **rational** coefficients.*

Proof. For parts (a), (b) and (b'), we set $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$. It is sufficient to prove the lemma in cases that $\mathbf{u} \not\leq \mathbf{a}_{max}$. Let J be the subset of $\{1, \ldots, n\}$ such that $u_i > (a_{max})_i$ if and only if $i \in J$. By the assumption, it must satisfy $J \neq \emptyset$. For each $j \in J$, we put $v_j := (a_{max})_j$; otherwise, we put $v_i := u_i$ if $i \notin J$. It is routine to verify that $\mathbf{v} := (v_1, \ldots, v_n) \leq \mathbf{u}$ and $\mathbf{v} \leq \mathbf{a}_{max}$.

(a) The rest of the proof is to prove that $\mathbf{v} \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. Indeed, because $\mathbf{u} \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, we can assume $\mathbf{u} \geq \mathbf{a}_1 = (a_{11}, \cdots, a_{in})$ without loss of generality. This results in $u_i \geq \mathbf{a}_{1i}$ for every $i \notin J$. Moreover, because $(a_{max})_j = \max\{a_{ij} \mid 1 \leq i \leq m\}$, we have $(a_{max})_j \geq a_{1j}$ for $j \in J$. Combining the inequalities, we obtain that $\mathbf{v} \geq \mathbf{a}_1$.

(b) The rest of the proof is to prove that $\mathbf{v} \in \mathbb{N}^n \cap N(I)$. Because $\mathbf{u} \in N(I)$, we can write $\mathbf{u} = \sum_{i=1}^t \lambda_i \mathbf{u}'_i$, where $\lambda_i > 0$ and $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{u}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. We write $\mathbf{u}'_i = (u'_{i1}, \ldots, u'_{in})$ for $1 \leq i \leq t$. For each $1 \leq i \leq t$, we set $v'_{ij} := (a_{max})_j$ if $j \in J$; otherwise, we put $v'_{ij} := u'_{ij}$ if $j \notin J$. Similarly to (a), we obtain $\mathbf{v}'_i := (v'_{i1}, \ldots, v'_{in}) \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. Therefore, it is routine to verify that $\mathbf{v} = \sum_{i=1}^t \lambda_i \mathbf{v}'_i \in N(I)$.

(b') The proof of the first statement is similar to that of (b). The way of putting $\mathbf{v}$ we used allows us to obtain the second statement straightforwardly.

(c) Because $\mathbf{v} \in N(I)$, we have a convex combination $\mathbf{v} = \sum_{i=1}^t \lambda_i \mathbf{d}_i$, where $\lambda_i \in \mathbb{R}_+$ and $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By Lemma 4, we can choose $\lambda_i \in \mathbb{Q}_+$ for every $1 \leq i \leq t$. Therefore, there exists $k \in \mathbb{N}$ satisfying $k\lambda_i \in \mathbb{N}$ for all i, which implies $k\mathbf{v}$ is a sum of k integral vectors $\mathbf{d}_i$'s. We rewrite $k\mathbf{v} = \sum_{i=1}^k \mathbf{d}_i \in \mathbb{N}^n$, where $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By (a) of this lemma, there exists $\mathbf{d}'_i \leq \mathbf{d}_i$ for each i such that $\mathbf{d}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. As a result,

$$\mathbf{0} \leq k\mathbf{a}_{max} - k\mathbf{v} = \sum_{i=1}^k (\mathbf{a}_{max} - \mathbf{d}_i) \leq \sum_{i=1}^k (\mathbf{a}_{max} - \mathbf{d}'_i).$$

By Lemma 5, there exist $\mathbf{c}'_1, \ldots, \mathbf{c}'_k$ satisfying $k\mathbf{a}_{max} - k\mathbf{v} = \sum_{i=1}^k \mathbf{c}'_i$ and $\mathbf{0} \leq \mathbf{c}'_i \leq \mathbf{a}_{max} - \mathbf{d}'_i$. Thus, $k\mathbf{v} = \sum_{i=1}^k (\mathbf{a}_{max} - \mathbf{c}'_i)$. Because $\mathbf{a}_{max} - \mathbf{c}'_i \geq \mathbf{d}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, we have $\mathbf{c}_i := \mathbf{a}_{max} - \mathbf{c}'_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Finally, we obtain that $\mathbf{v} = \sum_{i=1}^k \frac{1}{k} \mathbf{c}_i$. We have just completed the proof.

□

Part (c) of Lemma 6 allows us to complete the remain questions in the previous section.

Lemma 7. *If $\boldsymbol{v} \in N(I) \cap \mathbb{Q}^n$ and $\boldsymbol{v} \leq \boldsymbol{a}_{max}$, then $\boldsymbol{a}_{max} - \boldsymbol{v} \in N^*(I)$.*

Proof. By (c) of Lemma 6, $\mathbf{v} = \sum_{i=1}^t \lambda_i \mathbf{d}_i$, where $\sum_{i=1}^t \lambda_i = 1$ and $\mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Thus, $\mathbf{a}_{max} - \mathbf{v} = \sum_{i=1}^t \lambda_i (\mathbf{a}_{max} - \mathbf{d}_i) \in N^*(I)$ since $\mathbf{a}_{max} - \mathbf{d}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}^*_i \geq \mathbf{a}\}$. □

The above lemma and Lemma 1 help us to see that the rational points of $N(I)$ and $N^*(I)$ are symmetric through the point $\frac{1}{2}\mathbf{a}_{max}$. At that time, we can pose a question how about real points of the polyhedra. Indeed, we also have the similar result for real points, but it is not necessary for our progress. We can utilize the convergent sequences to prove the results about real points. At the end of the section, we show the lower comprehensiveness of $N^*(I)$ by the analysis-related argument.

Lemma 8. *$N^*(I)$ is lower comprehensive.*

Proof. For any $\mathbf{y} < \mathbf{x} \in N^*(I)$, we have $\mathbf{a}_{max} - \mathbf{y} \geq \mathbf{a}_{max} - \mathbf{x} \in N(I)$ by Lemma 1, and so $\mathbf{a}_{max} - \mathbf{y} \in N(I)$ by Lemma 2. If $\mathbf{y} \in \mathbb{Q}^n$ (or equivalently $\mathbf{a}_{max} - \mathbf{y} \in \mathbb{Q}^n$), then $\mathbf{y} \in N^*(I)$ by Lemma 7. Otherwise, when $\mathbf{y} \in \mathbb{R}^n$ generally, there is a sequence of rational vectors between $\mathbf{x}$ and $\mathbf{y}$ and the sequence converges to $\mathbf{y}$. Thanks to the previous argument, all of the vectors of the sequence belong to $N^*(I)$, so does $\mathbf{y}$ due to the closeness of $N^*(I)$ (Lemma 3). □

4. The integral decomposition property

By results in the previous sections, we can see that $N(I)$ and $N^*(I)$ are quite related. This motivates us to find a common property of both them. And fortunately, the integral decomposition property, the most important characteristic mentioned in the first section, is an one that both $N(I)$ and $N^*(I)$ have or do not have in common.

Proposition 2. *$N(I)$ has the integral decomposition property iff so does $N^*(I)$.*

Proof. Assume $N(I)$ has the integral decomposition property. We claim that $N^*(I)$ also has the property. Let k be a positive integer and $\mathbf{a}$ an integral vector of $kN^*(I)$. We have $\mathbf{a} = k\mathbf{b}$ for some $\mathbf{b} \in N^*(I)$. By Lemma 1, we have $\mathbf{a}_{max} - \mathbf{b} \in N(I)$, or equivalently $k\mathbf{a}_{max} - \mathbf{a} \in kN(I) \cap \mathbb{N}^n$. Because $N(I)$ has the integral decomposition property, there exist integral vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in N(I)$ such that $k\mathbf{a}_{max} - \mathbf{a} = \sum_{i=1}^k \mathbf{v}_i$. By (b) of Lemma 6, we can choose integral elements $\mathbf{v}'_1, \ldots, \mathbf{v}'_k \in N(I)$ such that $\mathbf{v}'_i \leq \mathbf{a}_{max}$ and $\mathbf{v}'_i \leq \mathbf{v}_i$ for every $1 \leq i \leq k$. This implies $k\mathbf{a}_{max} - \mathbf{a} \geq \sum_{i=1}^k \mathbf{v}'_i$, so $\mathbf{a} \leq \sum_{i=1}^k (\mathbf{a}_{max} - \mathbf{v}'_i)$. By Lemma 5, there exist integral vectors $\mathbf{d}_1, \ldots, \mathbf{d}_k$ such that $\mathbf{a} = \sum_{i=1}^k \mathbf{d}_i$ and $\mathbf{0} \leq \mathbf{d}_i \leq \mathbf{a}_{max} - \mathbf{v}'_i$. This leads to $\mathbf{a}_{max} - \mathbf{d}_i \geq \mathbf{v}'_i \in N(I)$. Combining with Lemma 2, we obtain $\mathbf{a}_{max} - \mathbf{d}_i \in N(I)$ for every $1 \leq i \leq k$. By lemma 7, $\mathbf{d}_i \in N^*(I)$. Therefore, $\mathbf{a}$ is a sum of k integral vectors of $N^*(I)$. This means that $N^*(I)$ also has the integral decomposition property.

Conversely, assume $N^*(I)$ has the integral decomposition property. For $k \in \mathbb{N}$ and $\mathbf{a} \in kN(I) \cap \mathbb{N}^n$, we need to show that $\mathbf{a}$ is a sum of k integral vectors of $N(I)$. Indeed, we can express $\mathbf{a} = k\mathbf{b}$, where $\mathbf{b} \in \mathbb{Q}^n \cap N(I)$. By (b') of Lemma 6, there exists $\mathbf{b}' \leq \mathbf{b}$ satisfying $\mathbf{b}' \in \mathbb{Q}^n \cap N(I)$ and $\mathbf{b}' \leq \mathbf{a}_{max}$. The statement (c) of Lemma 6 allows us to have $\mathbf{b}' = \sum_{i=1}^t \lambda_i \mathbf{c}_i$, where $\lambda_i \in \mathbb{Q}_+$, $\sum_{i=1}^t \lambda_i = 1$, $\mathbf{c}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a} \leq \mathbf{a}_{max}\}$. Therefore, we have $\mathbf{a}_{max} - \mathbf{c}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}^*_i \geq \mathbf{a}\}$, so $\mathbf{a}_{max} - \mathbf{b}' = \sum_{i=1}^t \lambda_i(\mathbf{a}_{max} - \mathbf{c}_i) \in N^*(I)$. It is worth noting that $k\mathbf{b}' \in \mathbb{N}^n$ because $k\mathbf{b} = \mathbf{a} \in \mathbb{N}^n$ (see (b') of Lemma 6). This results in $k\mathbf{a}_{max} - k\mathbf{b}' \in \mathbb{N}^n \cap kN^*(I)$. Consequently, we obtain $k\mathbf{a}_{max} - k\mathbf{b}' = \sum_{i=1}^k \mathbf{v}_i$, where $\mathbf{v}_i \in \mathbb{N}^n \cap N^*(I)$ because of the integral decomposition property of $N^*(I)$. By Lemma 1, we have $\mathbf{a}_{max} - \mathbf{v}_i \in N(I)$. As a particular consequence, it must satisfy that $\mathbf{a} - k\mathbf{b}' + \mathbf{a}_{max} - \mathbf{v}_1 \in N(I)$ since $\mathbf{a} \geq k\mathbf{b}'$ (see Lemma 2). To sum up, the equality $\mathbf{a} = (\mathbf{a} - k\mathbf{b}' + \mathbf{a}_{max} - \mathbf{v}_1) + \sum_{i=2}^k (\mathbf{a}_{max} - \mathbf{v}_i)$ shows $\mathbf{a}$ is a sum of k integral vectors of $N(I)$. The proof is completed. □

The above proposition is the end of the first module. From the next section, we will apply results in previous sections to algebraic subjects.

5. Monomial ideals

Let $K[x_1, \ldots, x_n]$ be the polynomial ring over a field K and I a monomial ideal of the ring. In order to be more simplistic, we write $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a}$ is an n-dimensional positive integral vector $(a_1, \ldots, a_n)$, as $x_1^{a_1} \ldots x_n^{a_n}$. As a consequence of Hilbert's Basis Theorem in Commutative Algebra, I is generated by a set of finitely many monomials, namely $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m})$. Without loss of generality, we assume that $\{\mathbf{x}^{\mathbf{a}_i}, 1 \leq i \leq m\}$ is a minimal set of generators of I, which is equivalent to there are no i, j satisfying that $i \neq j$ and $\mathbf{a}_i \leq \mathbf{a}_j$ (this is the condition we mentioned at the beginning of section two). By utilizing the symbols in the previous sections, we define Newton polyhedron and the "dual" one of the monomial ideal I as those associated to the sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $\{\mathbf{a}^*_1, \ldots, \mathbf{a}^*_m\}$, respectively, and we also write them as $N(I)$ and $N^*(I)$.

The first observation is that $\mathbf{x}^{\mathbf{a}} \in I^k$ if and only if $\mathbf{a} \geq \sum_{i=1}^m c_i \mathbf{a}_i$ for some $c_i \in \mathbb{N}$ satisfying that $\sum_{i=1}^m c_i = k$. Because of the upper comprehensiveness of $N(I)$ and the fact that $\frac{\mathbf{a}}{k} \geq \sum_{i=1}^m \frac{c_i}{k} \mathbf{a}_i$, it must satisfy $\mathbf{a} \in kN(I)$. However, if $\mathbf{a} \in kN(I) \cap \mathbb{N}^n$, $\mathbf{x}^{\mathbf{a}}$ does not certainly belong to I^k, so we need to find a wider ideal depending on I that contains integral vectors of $kN(I)$. In order to clarify the issue, we recall the following concept: an element $z \in K[x_1, \ldots, x_n]$ is *integral* over I if z is a root of the following monic polynomial:

$$z^d + c_1 z^{d-1} + \cdots + c_{d-1} z + c_d = 0, \tag{1}$$

for some degree d and $c_i \in I^i$ for every $1 \leq i \leq d$. The set of integral elements over I in $K[x_1 \ldots, x_n]$ is called *integral closure* of I in the polynomial ring, namely $\overline{I}$. Indeed, it is a subring of $K[x_1 \dots, x_n]$. I is *integrally closed* in $K[x_1 \dots, x_n]$ if $\overline{I} = I$. More strictly, we call I *normal* if $\overline{I^k} = I^k$ for every $k \in \mathbb{N}$. Initially, we have the following lemma, it illustrates what closure of powers of I is generated by.

Lemma 9. $\overline{I^k} = (\boldsymbol{x}^{\boldsymbol{a}} \mid \boldsymbol{x}^{d\boldsymbol{a}} \in I^{dk}$ *for some* $d)$. *In particular,* I *is integral closed if and only if* $I = (\boldsymbol{x}^{\boldsymbol{a}} \mid \boldsymbol{x}^{d\boldsymbol{a}} \in I^d$ *for some* $d)$.

Proof. By [Vil01, Proposition 7.3.2, page 234], $\overline{I^k}$ is a monomial ideal. Thus, the conclusion is obtained according to [Vil01, Proposition 7.3.3, page 234]. ☐

From the geometric perspective, we have the following "nice" lemma, and it let we know which ideal consists of integral points $\mathbf{a}$ satisfying $\mathbf{a} \in kN(I)$ as what we wanted before.

Lemma 10. $\overline{I^k} = (\boldsymbol{x}^{\boldsymbol{a}} \mid \boldsymbol{a} \in kN(I) \cap \mathbb{N}^n)$.

Proof. By [Vil01, Proposition 7.3.12, page 237], $\overline{I^k} = (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{ conv } \{\mathbf{u} \mid \mathbf{x}^{\mathbf{u}} \in I^k\} \cap \mathbb{N}^n)$. Now, we have the following inclusions:

$$\begin{aligned}
\overline{I^k} &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{ conv}\{\mathbf{u} \mid \mathbf{x}^{\mathbf{u}} \in I^k\} \cap \mathbb{N}^n) \\
&\subseteq (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{ conv}\{\mathbf{u} \mid \mathbf{u} \in kN(I)\} \cap \mathbb{N}^n) \\
&= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \text{ conv}\{kN(I)\} \cap \mathbb{N}^n) \\
&= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in kN(I) \cap \mathbb{N}^n) \text{ (because } N(I) \text{ is convex).}
\end{aligned}$$

Conversely, for $\mathbf{a} \in kN(I) \cap \mathbb{N}^n$, we have $\mathbf{a} = k\mathbf{u}$ for some $\mathbf{u} \in N(I) \cap \mathbb{Q}^n$. By Lemma 4, $\mathbf{u}$ is a convex combination $\sum_{i=1}^t \lambda_i \mathbf{y}_i$, where $\lambda_i \in \mathbb{Q} \cap [0, 1]$ and $\mathbf{y}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. By multiplying both terms by an integer λ so that $\lambda\lambda_i$'s are all integers, we obtain that $\lambda\mathbf{u}$ is a sum of λ vectors of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$, and so, $\lambda\mathbf{a} = \lambda k\mathbf{u}$ is a sum of $k\lambda$ elements of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$. This means that $\mathbf{x}^{\lambda\mathbf{a}}$ is divided by $k\lambda$ monomials of I, or equivalently $(\mathbf{x}^{\mathbf{a}})^\lambda \in (I^k)^\lambda$. Thus, $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ by Lemma 9. The proof is completed. ☐

Now, let M and M^* denote the matrices whose columns are $\mathbf{a}_i$'s and $\mathbf{a}_i^*$'s, respectively, as the section two. We can see the clear relationship between the monomial ideals and the values ν and ν^* mentioned at the beginning of the report.

Proposition 3. *For every* $k \in \mathbb{N}$,

(a) $\boldsymbol{x}^{\boldsymbol{a}} \in I^k$ *if and only if* $\nu(\boldsymbol{a}, M) \geq k$;
(b) $\boldsymbol{x}^{\boldsymbol{a}} \in \overline{I^k}$ *if and only if* $\nu^*(\boldsymbol{a}, M) \geq k$.

Proof. (a) $\mathbf{x}^{\mathbf{a}} \in I^k$ if and only if $\mathbf{a} \geq \sum_{i=1}^m c_i \mathbf{a}_i$ for some $c_i \in \mathbb{N}$ satisfying that $\sum_{i=1}^m c_i = k$. Equivalently, the vector $\mathbf{y} := (c_1, \dots, c_m) \in \mathbb{N}^m$ satisfies $M\mathbf{y}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{y} = k$, which means $\nu(\mathbf{a}, M) \geq k$.

(b) By Lemma 9, $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ if and only if $\mathbf{x}^{d\mathbf{a}} \in I^{dk}$ for some $d \in \mathbb{N}$, which is equivalent to $\nu(d\mathbf{a}, M) \geq dk$ by (a). Thus, the condition that $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ implies $\nu^*(d\mathbf{a}, M) \geq \nu(d\mathbf{a}, M) \geq dk$, so $\nu^*(\mathbf{a}, M) \geq k$. Conversely, we consider the cases that $\nu^*(\mathbf{a}, M) \geq k$. We need to show that $\nu(d\mathbf{a}, M) \geq dk$ for some $d \in \mathbb{N}$, or equivalently there exists a rational vector $\mathbf{z} \in \mathbb{Q}^m_{\geq 0}$ such that $M\mathbf{z}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{z} \geq k$. Indeed, let $\mathbf{y} \in \mathbb{R}^m_+$ be the solution vector for $\nu^*(\mathbf{a}, M)$, i.e., $M\mathbf{y}^T \leq \mathbf{a}^T$ and $\mathbf{1} \cdot \mathbf{y} = \nu^*(\mathbf{a}, M)$. There are some following cases:

(i) When $\mathbf{1} \cdot \mathbf{y} = \nu^*(\mathbf{a}, M) > k$, we choose a sequence of non-negative rational vectors $\{\mathbf{y}_i\}_{i=1}^\infty$ satisfying $\mathbf{y}_i \leq \mathbf{y}$ and $\mathbf{y}_i \to \mathbf{y}$ when $i \to \infty$. It is routine to verify that $M\mathbf{y}_i^T \leq \mathbf{a}^T$ for every i and $\mathbf{1} \cdot \mathbf{y}_j \geq k$ for all sufficiently large j. The conclusion holds in this case provided that we choose $\mathbf{z} = \mathbf{y}_j$.

(ii) We consider the cases where $\mathbf{1}\cdot\mathbf{y} = \nu^*(\mathbf{a}, M) = k$. The condition that $M\mathbf{y}^T \le \mathbf{a}^T$ implies that $(a_{1i},\dots,a_{mi})\cdot\mathbf{y} \le a_i$, where a_{ji} and a_i are the i-th coordinate of $\mathbf{a}_j$ and $\mathbf{a}$, respectively, for every $1\le j\le m$ and $1\le i\le n$. Let S be the set of coordinates i's satisfying $(a_{1i},\dots,a_{mi})\cdot\mathbf{y} = a_i$ iff $i\in S$. When $S=\emptyset$, we choose a sequence of rational vector $\mathbf{y}_i$ such that $\mathbf{y}\le\mathbf{y}_i$ and $\mathbf{y}_i\to\mathbf{y}$ as $i\to\infty$. Thus, for every $1\le i\le n$, we have $(a_{1i},\dots,a_{mi})\cdot\mathbf{y}_i \to (a_{1i},\dots,a_{mi})\cdot\mathbf{y} < a_i$. Therefore, for sufficiently large j, it must satisfy $(a_{1i},\dots,a_{mi})\cdot\mathbf{y}_j \le a_i$ for every $1\le i\le n$, or equivalently $M\mathbf{y}_j^T\le\mathbf{a}^T$. Since $\mathbf{y}_i\ge\mathbf{y}$ for all i, we have $\mathbf{1}\cdot\mathbf{y}_j\ge k$. The conclusion holds in cases $S=\emptyset$ if we choose $\mathbf{z}=\mathbf{y}_j$.
Otherwise, when $S\ne\emptyset$, without loss of generality, we assume that $S=\{1,\dots,t\}$. Put $\mathbf{u}_i := (a_{i1},\dots,a_{it})\in\mathbb{N}^t$ for every $1\le i\le m$, and $\mathbf{b}=(a_1,\dots,a_t)$. In the other words, $\mathbf{u}_i$'s and $\mathbf{b}$ are the images of $\mathbf{a}_i$'s and $\mathbf{a}$ of the projection from $\mathbb{R}^n$ onto the vector space $\mathbb{R}^t$ of the first t coordinates, respectively. It is routine to verify that $\frac{\mathbf{b}}{k} = \sum_{i=1}^m \frac{y_i}{k}\mathbf{u}_i$, where y_i is the i-th coordinate of $\mathbf{y}$. By Lemma 2, there exists a sequence of non-negative rational vectors $(y_1^{(p)},\dots,y_m^{(p)})$ such that $\frac{\mathbf{b}}{k}=\sum_{i=1}^m y_i^{(p)}\mathbf{u}_i$ (or equivalently $\frac{a_i}{k} = (a_{1i},\dots,a_{mi})\cdot(y_1^{(p)},\dots,y_m^{(p)})$ for every $1\le i\le t$), $\sum_{i=1}^m y_i^{(p)}=1$ and $(y_1^{(p)},\dots,y_m^{(p)})\to\frac{\mathbf{y}}{k}$ as $p\to\infty$. For each $i\notin S$, the fact that $(a_{1i},\dots,a_{mi})\cdot\mathbf{y}<a_i$ implies that $(a_{1i},\dots,a_{mi})\cdot(y_1^{(p)},\dots,y_m^{(p)})\le\frac{a_i}{k}$ for sufficiently large p. As a result, for sufficiently large p, we have $M(y_1^{(p)},\dots,y_m^{(p)})^T\le\frac{\mathbf{a}^T}{k}$, so $M(ky_1^{(p)},\dots,ky_m^{(p)})^T\le\mathbf{a}^T$. The conclusion holds provided that we choose $\mathbf{z}=(ky_1^{(p)},\dots,ky_m^{(p)})$. The proof is completed.

□

6. The integral closeness

In order to have enough conditions to show the final proposition in the report, we need to show $N(I)$ and $N^*(I)$ have fully the requirements in Proposition 1. This is the maximization and minimization of $\mathbf{a}_i^*$'s and $\mathbf{a}_i$'s in $N^*(I)$ and $N(I)$ respectively.

Proposition 4. *The three following conditions are equivalent:*

(a) $N^*(I)\cap\mathbb{N}^n = \{\boldsymbol{a}\in\mathbb{N}^n \mid \exists\boldsymbol{a}_i^*\ge\boldsymbol{a}\}$;
(b) $N(I)\cap\mathbb{N}^n = \{\boldsymbol{a}\in\mathbb{N}^n \mid \exists\boldsymbol{a}_i\le\boldsymbol{a}\}$;
(c) *I is integrally closed;*
(d) $\{\boldsymbol{a}_i^*, 1\le i\le m\}$ *consists of all maximal integral points of $N^*(I)$. In other words, columns of M^* are precisely maximal integral points of $N^*(I)$;*
(e) $\{\boldsymbol{a}_i, 1\le i\le m\}$ *consists of all minimal integral points of $N(I)$. In other words, columns of M are precisely minimal integral points of $N(I)$.*

Proof. $(a)\Rightarrow(b)$: Assume that $N^*(I)\cap\mathbb{N}^n=\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i^*\ge\mathbf{a}\}$, we claim that $N(I)\cap\mathbb{N}^n=\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i\le\mathbf{a}\}$. Indeed, for each $\mathbf{u}\in N(I)\cap\mathbb{N}^n$, there exists $\mathbf{v}\in N(I)\cap\mathbb{N}^n$ and $\mathbf{v}\le\mathbf{a}_{max}$ and $\mathbf{v}\le\mathbf{u}$ by (b) of Lemma 6. Thus, by Lemma 7,

$$\mathbf{a}_{max}-\mathbf{v}\in N^*(I)\cap\mathbb{N}^n=\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i^*\ge\mathbf{a}\}.$$

This implies that $\mathbf{v}\ge\mathbf{a}_i$ for some i, and so $\mathbf{u}\in\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i\le\mathbf{a}\}$. Thus, $N(I)\cap\mathbb{N}^n\subseteq\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i\le\mathbf{a}\}$. Additionally, the converse inclusion is verified routinely.

$(b)\Rightarrow(a)$: If $N(I)\cap\mathbb{N}^n=\{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i\le\mathbf{a}\}$, then we have

$$\begin{aligned} N^*(I)\cap\mathbb{N}^n &\subseteq \{\mathbf{a}_{max}-\mathbf{a}\mid\mathbf{a}\in N(I)\cap\mathbb{N}^n, \mathbf{a}\le\mathbf{a}_{max}\} \text{(by Lemma 1)}\\ &= \{\mathbf{a}_{max}-\mathbf{a}\mid\exists\mathbf{a}_i\le\mathbf{a}\le\mathbf{a}_{max}\}\\ &= \{\mathbf{a}\in\mathbb{N}^n\mid\exists\mathbf{a}_i^*\ge\mathbf{a}\}.\end{aligned}$$

The converse inclusion is verified routinely, so we obtain the equality.

$(b) \Leftrightarrow (c) \Leftrightarrow (e)$: This is a consequence of Lemma 10.

$(a) \Leftrightarrow (d)$: This is an obvious statement! □

7. Technical results

Afterwards, by using the above proposition, we come into the following result that shows clearly the equivalence between the normality of the monomial ideal I and the integer round-down property of M. Its way of proving is purely technical and its role is also a tool of reaching Proposition 5. However, Proposition 5 is its comprehensive version.

Lemma 11. *I is normal if and only if M has the integer round-down property.*

Proof. Assume that I is normal. For an arbitrary vector $\mathbf{a} \in \mathbb{N}^n$, we need to show that $\nu(\mathbf{a}, M) = \lfloor \nu^*(\mathbf{a}, M) \rfloor$. By setting $k := \lfloor \nu^*(\mathbf{a}, M) \rfloor$, we have $\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$ by (ii) of Proposition 3. Due to the normality of I, $\mathbf{x}^{\mathbf{a}}$ belongs to I^k, and so $\nu(\mathbf{a}, M) \geq k$ by (i) of Proposition 3. Consequently, it must satisfy $\nu(\mathbf{a}, M) = k$.

Conversely, if $\nu(\mathbf{a}, M) = \lfloor \nu^*(\mathbf{a}, M) \rfloor$ for all $\mathbf{a} \in \mathbb{N}^n$. For every $k \in \mathbb{N}$, we claim that $I^k = \overline{I^k}$. Indeed, we get a monomial $\mathbf{x}^{\mathbf{a}} \in \overline{I^k} \setminus I^k$ if $\overline{I^k} \neq I^k$. By Proposition 3, it must satisfy that $\nu^*(\mathbf{a}, M) \geq k$ and $\nu(\mathbf{a}, M) \leq k - 1$, which contradicts the assumption of the case. The proof is completed. □

After obtaining the relationship between the normality and the integer round-down property, we can pose a question how about the normality and the integer round-up property? Fortunately, we also have the similar result, Proposition 5. But before reaching it, we need to mention some useful properties that not only help us to understand the structure of polyhedra, but allow us also to prove the final proposition.

Lemma 12. *If M^* has the integer round-up property, then I is integrally closed.*

Proof. By Proposition 4, it is sufficient to show that $N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Assume that there exists an integral vector $\mathbf{u} \in N^*(I) \cap \mathbb{N}^n \setminus \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$. Thus, $\tau^*(\mathbf{u}, M^*) \leq 1$ because $\mathbf{u}^T$ is a convex combination of columns of M^*. Meanwhile, $\tau(\mathbf{u}, M^*) > 1$ because $\mathbf{u} \not\leq \mathbf{a}_i^*$ for all i. This contradicts the round-up property of M^*. □

After seven sections, now we have enough equipment that allow us to reach the final result.

8. The normality

Now, we have the following main result, it is a consequence of the previous. Furthermore, it is another presentation of Proposition 1 of S. Baum and Jr. L. E. Trotter in an algebraic language.

Proposition 5. *The following conditions related to the normality of I are equivalent:*

(a) *I is integrally closed and $N^*(I)$ has the integral decomposition property;*
(b) *I is integrally closed and $N(I)$ has the integral decomposition property;*
(c) *M^* has the integer round-up property;*
(d) *M has the integer round-down property.*
(e) *I is normal.*

Proof. $(a) \Leftrightarrow (b)$: This is a consequence of Proposition 2.

$(a) \Rightarrow (c)$: The integral closeness of I allows the columns of M^* to be precisely maximal integral points of $N^*(I)$ by Proposition 4. By (b) of Proposition 1 and Lemma 8, M^* has the integer round-up property.

$(c) \Rightarrow (a)$: If M^* has the integer round-up property, then I is integrally closed by Proposition 4, and so the columns of M^* are precisely maximal integral points of $N^*(I)$ by Proposition 4. By (b) of Proposition 1 and Lemma 8, $N^*(I)$ has the integral decomposition property.

$(b) \Rightarrow (d)$: We can cope with the part in a similar way in the above part. Meanwhile, we show another method of proving the part by Lemma 11 to clarify the equivalence between the normality of I and the integral decomposition property of $N(I)$. By Lemma 11, it is sufficient to show that I is normal, i.e., $I^k = \overline{I^k}$ for every $k \in \mathbb{N}$. However, we can obtain it from the integral decomposition property of $N(I)$ as follows:

$$\begin{aligned}
\overline{I^k} &= (\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in kN(I) \cap \mathbb{N}^n)(\text{ Lemma 10}) \\
&\subseteq \left(\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} = \sum_{i=1}^{k} \mathbf{u}_i, \mathbf{u}_i \in N(I) \cap \mathbb{N}^n\right) (\text{ The integral decomposition property of } N(I)) \\
&= (\mathbf{x}^{\mathbf{u}_1} \cdots \mathbf{x}^{\mathbf{u}_k} \mid \mathbf{u}_i \in N(I) \cap \mathbb{N}^n) \\
&= (\mathbf{x}^{\mathbf{u}_1} \cdots \mathbf{x}^{\mathbf{u}_k} \mid \mathbf{u}_i \in \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}) \\
&= I^k
\end{aligned}$$

The converse inclusion is verified routinely, so we have proved that I is integrally closed.

$(d) \Rightarrow (b)$: By Lemma 11, I is normal. In particular, I is integrally closed. This implies M consists of all minimal integral points of $N(I)$ by Proposition 4. By (a) of Proposition 1 and Lemma 2, $N(I)$ has the integral decomposition property.

$(b) \Rightarrow (e)$: If I is integrally closed and $N(I)$ has the integral decomposition property, then, according to Proposition 5, M has the integer round-down property. I, thus, is normal thanks to Lemma 11.

$(e) \Rightarrow (b)$: When I is normal, I is integrally closed obviously. Moreover, by Lemma 11, M has the integer round-down property, so $N(I)$ has the integral decomposition property according to Proposition 5. The proof is completed. □

In conclusion, after studying the whole report, we can summary the correspondence between algebraic, geometric and integer programming-related subjects as well as their properties in the following table. In the table, two first rows are definitions and properties of subjects respectively. Each of the remaining rows is a list of equivalent conditions of each subject.

$\{\mathbf{a}_i\} \subset \mathbb{N}^n$	$N(I)$	$N^*(I)$	M	M^*	I
Definition	convex hull of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$	convex hull of $\{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$	$\begin{bmatrix}\mathbf{a}_1^T & \cdots & \mathbf{a}_m^T\end{bmatrix}$	$\begin{bmatrix}\mathbf{a}_1^{*T} & \cdots & \mathbf{a}_m^{*\,T}\end{bmatrix}$	$(\mathbf{x}^{\mathbf{a}} \mid \exists \mathbf{a}_i \leq \mathbf{a})$
Geometric property	upper comprehensive Lemma 2	lower comprehensive Lemma 8			
Lemmas 1, 7 If $\mathbf{u} \leq \mathbf{a}_{max}$	$\mathbf{u} \in N(I) \cap \mathbb{N}^n$	$\mathbf{a}_{max} - \mathbf{u} \in N^*(I)$	$\nu^*(\mathbf{u}, M) \geq 1$ Proposition 3	$\tau^*(\mathbf{a}_{max} - \mathbf{u}, M^*) \leq 1$	$\mathbf{x}^{\mathbf{u}} \in \overline{I}$
Lemma 10 Proposition 3	$\mathbf{a} \in kN(I) \cap \mathbb{N}^n$		$\nu^*(\mathbf{a}, M) \geq k$		$\mathbf{x}^{\mathbf{a}} \in \overline{I^k}$
Proposition 4	$N(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i \leq \mathbf{a}\}$	$N^*(I) \cap \mathbb{N}^n = \{\mathbf{a} \in \mathbb{N}^n \mid \exists \mathbf{a}_i^* \geq \mathbf{a}\}$	columns are minimal integral points of $N(I)$	columns are maximal integral points of $N^*(I)$	integrally closed $I = \overline{I}$
Proposition 5 If $I = \overline{I}$	integral decomposition	integral decomposition	integer round-down	integer round-up	normal $I^k = \overline{I^k}$

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