Second excercise sheet

Heldert Villegas Jaramillo
Suraj Kuswaha

November 3, 2025

1 Let $f : \{0, \dots, N-1\} \to \mathbb{C}$, then

$$\begin{aligned}
\sum_x |f(x)|^2 &= \sum_x \left| \frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x} \right|, \text{ definition of the inverse Fourier transform} \\
&= \sum_x \left(\frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x} \right) \overline{\left(\frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x} \right)}, \text{ since } z\bar{z} = |z|^2 \\
&= \sum_x \frac{1}{N} \left(\sum_\omega F(\omega) e^{i\omega x} \right) \left(\sum_\omega \overline{F(\omega)} e^{-i\omega x} \right), \text{ properties of complex conjugation}
\end{aligned}$$

Now we will distribute and separate the product in two parts. The first one will correspond to those summands in which the frequency is the same, the second one in which they are different. Take into account also that $|e^{ix}| = 1$ for all $x \in \mathbb{R}$.

$$\begin{aligned}
\sum_x |f(x)|^2 &= \frac{1}{N} \sum_x \left(\sum_\omega |F(\omega)|^2 + \sum_{\omega_0 \neq \omega_1} F(\omega_0) \overline{F(\omega_1)} e^{i(\omega_0 - \omega_1)x} \right) \\
&= \sum_\omega |F(\omega)|^2 + \frac{1}{N} \sum_x \sum_{\omega_0 \neq \omega_1} F(\omega_0) \overline{F(\omega_1)} e^{i(\omega_0 - \omega_1)x}, \text{ distribution of the summation over } x \\
&= \sum_\omega |F(\omega)|^2 + \frac{1}{N} \sum_{\omega_0 \neq \omega_1} F(\omega_0) \overline{F(\omega_1)} \sum_{x=0}^{N-1} e^{i(w_0 - w_1)x}, \text{ change in the order of summation} \\
&= \sum_\omega |F(\omega)|^2 + \frac{1}{N} \sum_{\omega_0 \neq \omega_1} F(\omega_0) \overline{F(\omega_1)} \left(\frac{e^{i(w_0 - w_1)N} - 1}{e^{i(w_0 - w_1)} - 1} \right), \text{ finite geometric summation}
\end{aligned}$$

Since $\omega_0 - \omega_1 = \frac{2\pi k}{N}$ for some $k \in \{0, \dots, N-1\}$ we have that $e^{i(w_0 - w_1)N} - 1 = 1 - 1 = 0$, and therefore the second summand on the previous line is 0, yielding thus Parseval's theorem.