

Discontinuous Galerkin Methods

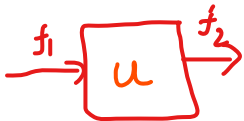
Vishal Hariharan¹ Dr. Aditya Konduri²

¹Indian Institute of Technology Madras

²Indian Institute of Science

Introduction

Consider the one-dimensional scalar conservation law



$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = S(x), \quad x \in \Omega, \quad (1)$$

subject to an appropriate set of initial and boundary conditions

- We will need to decide how we solve this system, and we will need to discretize in space and time
- How does DG compare with already established methods such as FEM, FVM ?

Weak Form

- The weak form of the PDE is the integral (over the global domain Ω) of our solution times some test function ϕ :

$$\int_{\Omega} \frac{\partial u}{\partial t} \phi \, dx + \int_{\Omega} \frac{\partial f(u)}{\partial x} \phi \, dx = 0 \quad (2)$$

- In DG the global domain is split into K elements, where the local solution is defined for a particular element
- The global solution is then the direct sum of each of these local polynomials, a piece-wise polynomial

Domain Decomposition

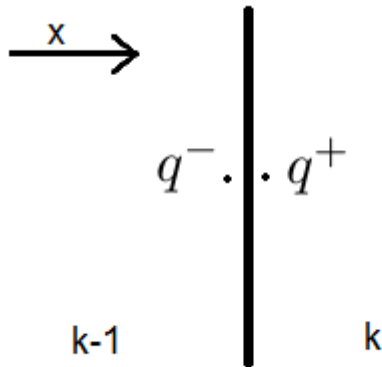
- Integration by-parts to reduce smoothness requirements

$$\int_k \frac{\partial u}{\partial t} \phi \, dx + [f(u)\phi] \Big|_{x_L}^{x_R} - \int_k f(u) \frac{d\phi}{dx} \, dx = 0 \quad (3)$$

- The integration domain is now over the element instead of the whole domain such that $\{x|x \in k, x_L \leq x \leq x_R\}$
- This is similar to a FEM approach, however continuity is not enforced across elements

Flux function

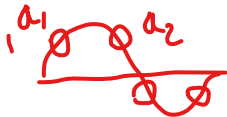
- Many choices: Lax-Friedrichs, Richtmyer, Godunov, Osher, etc
- In the linear case $f(q) = cq$, all the above numerical fluxes coincide with the so-called *upwind* flux
- Flux is very important in wave dominated problems



Upwind flux

Soln approximation

- We need a suitable basis set for our solution approximation
- To get a flexible method, we'd like to be able to generate an arbitrary order solution approximation. How do we do this in code?
- We can think in terms of nodal or modal

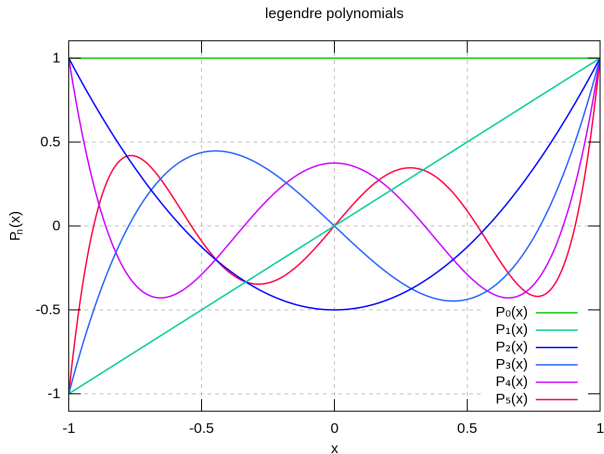


$$u(x) = \sum_{i=0}^M a_i \psi_i(x)$$



(4)

Legendre Polynomials



Basis functions

Let us begin by introducing the affine mapping

$$x \in \mathbf{D}^k : x(r) = x_l^k + \frac{1+r}{2}h^k, \quad h^k = x_r^k - x_l^k \quad (5)$$

with the reference variable $r \in \mathbf{I} = [-1, 1]$. We consider local polynomial approximations of the form

$$x \in \mathbf{D}^k : u_h^k(x(r), t) = \sum_{n=1}^{N_p} \hat{u}_n^k \psi_n(r) = \sum_{n=1}^{N_p} u_h^k(x_i, t) l_i^k(r) \quad (6)$$

Basis functions (contd...)

$$\psi_n(r) = \tilde{P}_{n-1}(r) = \frac{P_{n-1}(r)}{\gamma_{n-1}}$$

where $P_n(r)$ are the classic Legendre polynomials of order n and

$$\gamma_n = \frac{2}{2n+1}$$

$$\int_{-1}^1 P_n P_j = \delta_{nj} w_j$$

is the normalization.

What are the benefits of using Legendre functions?

Initial approximation

Equating inner products of each of the test functions with our approximation, and with that of initial condition $u_0(x)$,

(Eq 2.2) Cockburn, Bernardo and Shu, Chi-Wang

Runge-Kutta Discontinuous Galerkin Methods for Convection-Dominated Problems

In : *Journal of Scientific Computing* 16/3 (2001), 173-261.

$$\int f \phi = \int u \phi$$

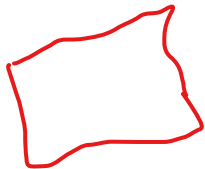
$$\hat{u}_n \simeq \sum_{i=1}^{N_p} u(r_i) \tilde{P}_{n-1}(r) w_i \quad (7)$$

where r_i are the quadrature points and w_i are the weights.

Lagrange interpolation

The nodal approach is convenient to generalize problems to higher dimensions. Therefore, if

$$u(r) \simeq u_h(r) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(r),$$



is an interpolant (i.e., $u(\varepsilon_i) = u_h(\varepsilon_i)$) at the grid points, ε_i ,

$$u_h(r) = \sum_{n=1}^{N_p} u(\varepsilon_i) l_i(r) \quad \text{where} \quad l_i(r) = \prod_{n=1}^{N_p} \frac{r - \varepsilon_j}{\varepsilon_i - \varepsilon_j}$$

1D advection-diffusion eqn

- We have done what seems like quite a bit of tangential work, but it now pays off. Let us illustrate through an example

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \text{in } (0, 1) \times (0, T) \quad (8)$$

$$u(x, 0) = \sin(2\pi x), \quad u(0, t) = u(1, t) \quad \& \quad u_x(0, t) = u_x(1, t)$$

- We will now substitute the expression for our solution approximation and arrive at linear systems describing evolution of the variable $u(x, t)$

Semi-discrete system

Let us revisit the weak form

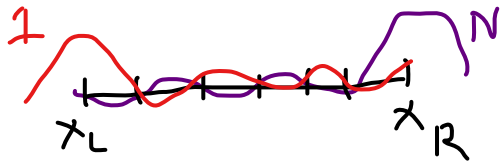
$$\int_k \frac{\partial u}{\partial t} \phi \, dx + [(cu - \alpha u_x) \phi] \Big|_{x_L}^{x_R} - \int_k (cu - \alpha u_x) \frac{d\phi}{dx} \, dx = 0 \quad (9)$$

Plugging in $u(x, t) = \sum_{i=1}^N a_i(t) \psi_i(x)$,

$$\int_k \frac{\partial u}{\partial t} \phi_j \, dx = \sum_{i=1}^N \frac{da_i}{dt} \int_k \psi_i(x) \phi_j(x) \, dx = \sum_{i=1}^N \frac{da_i}{dt} \int_I l_i(r) l_j(r) \frac{h^k}{2} \, dr$$

Semi-discrete system (contd...)

$$\int_k \sum_{i=1}^N \left[ca_i \psi_i(x) \phi'_j(x) - \alpha a_i \psi'_i(x) \phi'_j(x) \right] dx = \sum_{i=1}^N ca_i \int_I l_i(r) l'_j(r) dr - \sum_{i=1}^N \alpha a_i \int_I l'_i(r) l'_j(r) dr$$



$$\text{Upwind : } [u\phi] \Big|_{x_L}^{x_R} = u_K(x_R)\phi_j(x_R) - u_{K-1}(x_R)\phi_j(x_L)$$

$$\text{Downwind : } [u_x\phi] \Big|_{x_L}^{x_R} = \{u_x\}_{K+1}(x_L)\phi_j(x_R) - \{u_x\}_K(x_L)\phi_j(x_L)$$

Semi-discrete system (contd...)

$$\text{Mass matrix } \mathcal{M}^k \doteq \frac{h^k}{2} \int_I l_i(r) l_j(r) dr$$

$$\text{Stiffness matrix } \mathcal{S}^k \doteq c \int_I l_i(r) l'_j(r) dr - \alpha \int_I l'_i(r) l'_j(r) dr$$

Since the Legendre functions form a complete set, we can represent Lagrange interpolants through Legendre basis to simplify above expressions

(2008) **Making it work in one dimension** In: Nodal Discontinuous Galerkin Methods : Texts in Applied Mathematics, vol 54. Springer, New York

$$[c\hat{u}\phi] \Big|_{x_L}^{x_R} = \begin{bmatrix} -cu_{K-1} & 0 & 0 & \dots & 0 & 0 & cu_K \end{bmatrix}^T$$

$$[\alpha\hat{u}_x\phi] \Big|_{x_L}^{x_R} = \begin{bmatrix} -\alpha\{u_x\}_K & 0 & 0 & \dots & 0 & 0 & \alpha\{u_x\}_{K+1} \end{bmatrix}^T$$

$$\text{Flux vector } \hat{f} = \{c[\hat{u}\phi] \Big|_{x_L}^{x_R} - \alpha[\hat{u}_x\phi] \Big|_{x_L}^{x_R}\}$$

Assembly of system

- We can now combine the mass, stiffness, and numerical flux terms

$$\mathcal{M}^k \frac{\partial \mathbf{u}_h^k}{\partial t} + \hat{f} - \mathcal{S}^T \mathbf{u}_h^k = 0 \quad (10)$$

- Solving for $\frac{\partial \mathbf{u}_h^k}{\partial t}$,

$$\frac{\partial \mathbf{u}_h^k}{\partial t} = \{\mathcal{M}^k\}^{-1} \left(\mathcal{S}^T \mathbf{u}_h^k - \hat{f} \right)$$

Time discretization

We obtain from the semi-discrete equation

$$\frac{\partial \mathbf{u}_h}{\partial t} = \mathcal{L}_h(\mathbf{u}_h, t)$$

RK4 trial steps to determine \mathbf{u}_h , the vector of unknowns at next time step

$$\mathbf{k}^{(1)} = \mathcal{L}_h(\mathbf{u}_h^n, t^n)$$

$$\mathbf{k}^{(2)} = \mathcal{L}_h(\mathbf{u}_h^n + \frac{1}{2}\Delta t \mathbf{k}^{(1)}, t^n + \frac{1}{2}\Delta t)$$

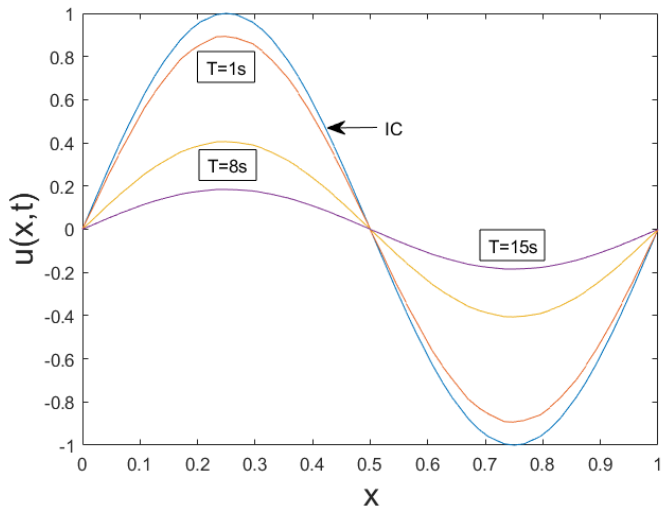
$$\mathbf{k}^{(3)} = \mathcal{L}_h(\mathbf{u}_h^n + \frac{1}{2}\Delta t \mathbf{k}^{(2)}, t^n + \frac{1}{2}\Delta t)$$

$$\mathbf{k}^{(4)} = \mathcal{L}_h(\mathbf{u}_h^n + \Delta t \mathbf{k}^{(3)}, t^n + \Delta t)$$

$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n + \frac{1}{6}\Delta t \left(\mathbf{k}^{(1)} + 2\mathbf{k}^{(2)} + 2\mathbf{k}^{(3)} + \mathbf{k}^{(4)} \right)$$

Soln evolution

DG Solution



MATLAB simulation

Convergence & stability

$$\left| \frac{C \Delta t}{\Delta x} \right| < 1$$

- How does choice of initial parameters affect stability?
- Does the smoothness of the initial soln seem to effect the rate of convergence? (e.g. $\sin(x)$ vs gaussian curve)
- Does h or p refinement seem to be more efficient?

