Q1) Given: Columns of X are linearly independent
To prove: $X^T X$ is invertible.

Proof: By rank-nullity theorem, we have

$$\text{rank}(X) + \dim(\text{Null}(X)) = n$$

where X is a $m \times n$ matrix.

Now, since X has n linearly independent rows, $\text{rank}(X) = n$.

$X^T X$ is a $n \times n$ matrix. By rank-nullity theorem, we have

$$\text{rank}(X^T X) + \dim(\text{null}(X^T X)) = n$$
$$= \text{rank}(X)$$

Now, null $(X^T X)$ is the vector space containing those vectors u (apart from the zero vector) such that

$$X^T X u = 0$$

However, the equation $Xu = 0$ has no solution apart from the trivial one. If ~~$X^T X v = 0$~~ is a solution for $X^T X u = 0$, we have

$$X^T X v = 0$$
$$\Rightarrow V^T X^T X V = 0 \quad (V^T \text{ is non-zero})$$
$$\Rightarrow (XV)^T (XV) = 0 \text{ or } Xv = 0 \text{ for some non-zero } v$$

$\Rightarrow$ This contradicts our claim.

Hence, there is no such v.

$\Rightarrow \text{rank}(X^T X) = \text{rank}(X) = n$

$\Rightarrow X^T X$ is square and full-rank.

$\Rightarrow |X^T X|$ is non-zero.

$\Rightarrow X^T X$ is invertible

(Q3) $X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$

After ~~adding~~ appending $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to X

$\Rightarrow X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1+1+1+1+1 & 1+2+3+3+4 \\ 1+2+3+3+4 & 1+4+9+9+16 \end{bmatrix}$

$= \begin{bmatrix} 5 & 13 \\ 13 & 39 \end{bmatrix}$

$\Rightarrow \theta = (X^T X)^{-1} X^T y$

$= \begin{bmatrix} 5 & 13 \\ 13 & 39 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$

$= \frac{1}{26} \begin{bmatrix} 39 & -13 \\ -13 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 27 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 5/26 \end{bmatrix} \begin{bmatrix} 10 \\ 27 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/26 \end{bmatrix}$