

part 1.

$$1. \sigma''(x) = (\sigma(x)(1-\sigma(x)))'$$

$$= [\sigma(x)(1-\sigma(x))] (1-\sigma(x)) + \sigma(x) [-\sigma(x)(1-\sigma(x))]$$

$$= \sigma(x)(1-\sigma(x)) [1-\sigma(x)-\sigma(x)]$$

$$= \sigma(x)(1-\sigma(x))(1-2\sigma(x)) \quad \blacksquare$$

$$2. \sigma'''(x) = [\sigma(x)(1-\sigma(x))(1-2\sigma(x))']$$

$$= [\sigma(x)(1-\sigma(x))] (1-\sigma(x))(1-2\sigma(x)) + \sigma(x) [-\sigma(x)(1-\sigma(x))](1-2\sigma(x)) + \sigma(x)(1-\sigma(x)) [-2\sigma(x)(1-\sigma(x))]$$

$$= \sigma(x)(1-\sigma(x)) \left[(1-\sigma(x))(1-2\sigma(x)) - \sigma(x)(1-2\sigma(x)) - 2\sigma(x)(1-\sigma(x)) \right]$$

$$= \sigma(x)(1-\sigma(x)) \left[(1-2\sigma(x))^2 - 2\sigma(x) + 2\sigma^2(x) \right]$$

$$= \sigma(x)(1-\sigma(x)) \left[(1-4\sigma(x)+4\sigma^2(x)) - 2\sigma(x) + 2\sigma^2(x) \right]$$

$$= \sigma(x)(1-\sigma(x)) (6\sigma^2(x) - 6\sigma(x) + 1) \quad \blacksquare$$

part 2. Compare $\tanh(x)$ & $\sigma(x)$

$$\begin{cases} \textcircled{1} \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\times \frac{e^x}{e^x} \right) = \frac{e^{2x} - 1}{e^{2x} + 1} \\ \textcircled{2} \sigma(x) = \frac{1}{1 + e^{-x}} \end{cases}$$

$$1^\circ \sigma(2x) = \frac{1}{1 + e^{-2x}} \left(\frac{e^{2x}}{e^{2x}} \right) = \frac{e^{2x}}{e^{2x} + 1}$$

$$\begin{aligned} 2^\circ 2\sigma(2x) - 1 &= \frac{2e^{2x}}{e^{2x} + 1} - 1 = \frac{2e^{2x}}{e^{2x} + 1} - \frac{e^{2x} + 1}{e^{2x} + 1} \\ &= \frac{e^{2x} - 1}{e^{2x} + 1} \end{aligned}$$

$$\text{Therefore, } \tanh(x) = 2\sigma(2x) - 1 \quad \blacksquare$$