1. Consider a forward SDE

$$dx_t = f(x_t, t)\, dt + g(x_t, t) dW_t,$$

show that the corresponding probability flow ODE is written as

$$dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\frac{\partial}{\partial x}\log p(x_t, t) \right] dt.$$

<pf.>

1° We know that $dx_t = f(x_t,t)dt + g(x_t,t)dW_t$, where the density $p(x,t)$, content Fokker-Planck
i.e, $\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p)$ — ①
and we want to find some deterministic ODE:
$dx_t = \nu(x_t,t)dt$, s.t, the desity content $\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\nu p)$ — ②

2° Let $-\frac{\partial}{\partial x}(\nu p) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(g^2 p)\right)$
$\therefore -\frac{\partial}{\partial x}(\nu p) = -\frac{\partial}{\partial x}\left(fp - \frac{1}{2}\partial_x(g^2 p)\right)$
Thus, we have the equation: $\nu p = fp - \frac{1}{2}\partial_x(g^2 p)$
$\therefore \nu = f - \frac{1}{2} \times \frac{\partial_x(g^2 p)}{p}$ — ③
Moreover, we can rewrite $\partial_x(g^2 p) = (\partial_x g^2 p + g^2(\partial_x p)) \Rightarrow \frac{\partial_x(g^2 p)}{p} = \partial_x g^2 + g^2 \times \frac{\partial_x p}{p}$
And, we know that $\frac{\partial_x p}{p} = \partial_x \log p$
③ : $\nu = f - \frac{1}{2}\partial_x g^2 - \frac{g^2}{2}\partial_x \log p$

3° Therefore, $\frac{dx_t}{dt} = \nu = f - \frac{1}{2}\partial_x g^2 - \frac{g^2}{2}\partial_x \log p$, $dx_t = [f - \frac{1}{2}\partial_x g^2 - \frac{g^2}{2}\partial_x \log p]dt$ ▨

3. Unanswered Questions

There are unanswered questions from the lecture, and there are likely more questions we haven't covered.

What happens in the multi-dimensional case if $g(t)$ is a matrix or depends on x?
How does the probability flow ODE change, and how do we handle $\nabla \cdot D$ in general?