Low Reynolds number flows in a corner

Project Report

Submitted by

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Certificate

This is to certify that the report titled **"Low Reynolds number flows in a corner"**, submitted by **H V Vivek**, to the Indian Institute of Technology, Madras, for the award of the degree of **Bachelor of Technology**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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1 Introduction

Assuming the continuum hypothesis, for a fluid with no sources or sinks, the conservation of mass can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

where ρ is the mass density of the fluid and u is the fluid velocity.
Making an assumption of incompressible flow (i.e. $\frac{D\rho}{Dt}$), the conservation of mass equation reduces to:

$$\nabla.\mathbf{u} = 0$$

Similarly, The conservation of momentum equations can be written as:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.(\rho \mathbf{u}\mathbf{u}) + \mathbf{b} = 0$$

Both the laws can be combined to give the well known Navier Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \frac{\mathbf{b}}{\rho}$$

The Navier Stokes equation can be converted to its non dimensional form by introducing the following dimensionless variables:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad t^* = t\frac{U}{L} \quad p^* = \frac{p - p_\infty}{\mu \frac{U}{L}}$$

This gives us the Reynolds number as

$$Re = \frac{\rho U}{L}$$

Now the Navier Stokes equations can be written as:

$$Re\frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* p^* + \nabla^{*2} \mathbf{u}^*$$

2 Low Reynolds Number Flows

The assumption of low Reynolds number implies we assume that

$$Re \approx 0$$

This means that the Navier Stokes equations can be reduced to:

$$\nabla^* p^* = \nabla^{*2} \mathbf{u}^*$$

Rewriting it in dimensional form we get

$$\nabla p = \mu \nabla^2 \mathbf{u}$$

Stream Function
For an incompressible fluid, we have

$$\nabla.\mathbf{u} = 0$$

If divergence of a vector is zero, then the vector quantity can be represented as the curl of a vector potential. In case of a fluid this is known as the Stream function, ψ, written in vector form ψ (0,0,ψ)

$$\mathbf{u} = \nabla \times \psi$$

Vorticity
Vorticity,ω can be defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

In a 2-D case this would give

$$\boldsymbol{\omega} = \frac{1}{2}[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}]\hat{\boldsymbol{e_z}}$$

where u and v are the velocity components in x and y directions. We know that the velocities can be expressed in terms of the Stream function ψ as

$$v = -\frac{\partial \psi}{\partial x} \quad u = \frac{\partial \psi}{\partial y}$$

Therefore we can write

$$\frac{\partial v}{\partial x} = -\frac{\partial^2 \psi}{\partial x^2} \quad \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial y^2}$$

Hence we can write the vorticity in terms of the Stream function as:

$$\boldsymbol{\omega} = -\frac{1}{2}[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}]\hat{\boldsymbol{e_z}}$$

Or

$$\boldsymbol{\omega} = -\frac{1}{2}\nabla^2 \psi \hat{\boldsymbol{e_z}}$$

Taking curl of the previous equation we obtain:

$$\nabla \times [\nabla p = \mu \nabla^2 \mathbf{u}]$$

$$0 = \mu \nabla \times \nabla^2 \mathbf{u}$$

Using the identity $\nabla^2(\nabla \times A) = \nabla \times \nabla^2 A$, we can write

$$\mu \nabla \times (\nabla^2 \mathbf{u}) = \mu \nabla^2 (\nabla \times \mathbf{u})$$

$$2\mu \nabla^2 \boldsymbol{\omega} = 0$$

Now using the relation between the stream function and the vorticity, i.e.

$$\boldsymbol{\omega} = -\frac{1}{2}\nabla^2 \psi \hat{\boldsymbol{e_z}}$$

it can be written as

$$\mu\nabla^2\nabla^2\psi\hat{\boldsymbol{e_z}} = 0$$

Or simply,

$$\nabla^4\psi = 0$$

This is known as the biharmonic equation. This equation can be used to solve for flows associated with a incompressible flow at very low Reynolds number.

3 General Solution to the Biharmonic Equation

A general solution to the biharmonic equation can be constructed to study the various possibilities of flows which occur in a corner. We can assume a solution to be of the form of

$$\psi = r^\lambda f_\lambda(\theta)$$

and search for a variable separable solution to the biharmonic equation.

Using the expression above for ψ, and substituting it into the biharmonic equation leads to the following:

$$\nabla^4\psi = \nabla^4 r^\lambda f_\lambda(\theta)$$

$$= \nabla^2\nabla^2 r^\lambda f_\lambda(\theta)$$

$$= (\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2})r^\lambda f_\lambda(\theta) = 0$$

$$\nabla^2(r^\lambda f_\lambda) = r^{\lambda-2}(\lambda(\lambda-1)f_\lambda + \lambda f_\lambda + f_\lambda'') = r^{\lambda-2}(\lambda^2 f_\lambda + f_\lambda'') = r^k f_k$$

$$\nabla^4(r^\lambda f_\lambda) = \nabla^2\nabla^2(r^\lambda f_\lambda) = \nabla^2 r^k f_k$$

$$\nabla^4(r^\lambda f_\lambda) = \nabla^2 r^k f_k = r^{k-2}(k^2 f_k + f_k'')$$

$$\nabla^4(r^\lambda f_\lambda) = r^{\lambda-4}((\lambda-2)^2(\lambda^2 f_\lambda + f_\lambda'') + \lambda^2 f_\lambda'' + f_\lambda'''' = 0$$

Hence

$$(\lambda-2)^2\lambda^2 f_\lambda + ((\lambda-2)^2 + \lambda^2)f_\lambda'' + f_\lambda'''' = 0$$

$$(\lambda^2 + \frac{\partial}{\partial\theta^2})((\lambda-2)^2 + \frac{\partial}{\partial\theta^2})f = 0$$

is the governing equation for $f = f_\lambda(\theta)$

The general form of the function f_λ can be deduced by taking

$$f_\lambda(\theta) = Xe^{i\beta\theta}$$

and solving for β where X is an arbitrary constant. Substituting this into the governing equation we get

$$(\lambda-2)^2\lambda^2 - ((\lambda-2)^2 + \lambda^2)\beta^2 + \beta^4 = 0$$

Solving the biquadratic equation we get

$$(\beta^2 - \lambda^2)(\beta^2 - (\lambda-2)^2) = 0$$

Therefore the possible values of β are $\lambda, -\lambda, \lambda-2, -(\lambda-2)$ The general form of $f_\lambda(\theta)$ can therefore be written as

$$f_\lambda(\theta) = Ae^{i\lambda\theta} + Be^{-i\lambda\theta} + Ce^{i(\lambda-2)\theta} + De^{-i(\lambda-2)\theta} = 0$$

Since the constants A,B,C,D can be complex, we can reduce the general form to represent it in trigonometric functions giving us

$$f_\lambda(\theta) = Acos(\lambda\theta) + Bsin(\lambda\theta) + Ccos((\lambda-2)\theta) + Dsin((\lambda-2)\theta)$$

where A,B,C,D are arbitrary constants. The values of β takes repeated roots for three values of λ. These values are $\lambda = 0, 1, 2$. For these values the function f_λ is different from the one given above. The f_λ for these special cases are given below:

$$f_0 = A\theta + B\theta^2 + C\theta^3$$

$$f_1 = Acos\theta + Bsin\theta + C\theta cos(\theta) + D\theta sin(\theta)$$

$$f_2 = Acos2\theta + Bsin2\theta + C\theta + D$$

This solution can be confirmed by comparing with the Michell's Solution which offers a solution for the Airy's Stress function which also follows the biharmonic equation. The general solution given by Michell is as follows:

$$\begin{aligned} \psi = A_0 r^2 + B_0 r^2 ln(r) + C_0 ln(r) + D_0 \theta \\ +(A_1 r + B_1 r^{-1} + B_1' r\theta + C_1 r^3 + D_1 r ln(r))cos\theta \\ +(E_1 r + F_1 r^{-1} + F_1' r\theta + G_1 r^3 + H_1 r ln(r))sin\theta \\ +\sum_{n=2}^{\infty}(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-(n+2)})cos(n\theta) \\ +\sum_{n=2}^{\infty}(E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-(n+2)})sin(n\theta) \end{aligned}$$

For the case that we are considering, that is for flows in a corner, we can reduce the above general solutions by using a few intuitive characteristics. For example, the need for the velocity to be finite as we approach the corner (a valid physical assumption) means that all the coefficients of the inverse functions of r $(B_n, F_n, E_n, H_n, D_n)$ need to be zero. For the same reason, the coefficients of the logarithmic functions in r also need to be zero. A direct dependence on θ can also be ruled out because a streamline at the corner has the same constant value for $\theta = 0$ and $\theta = \alpha$. Hence the solution reduces to the same as the one solved above

$$\psi = r^\lambda (Acos(\lambda\theta) + Bsin(\lambda\theta) + Ccos((\lambda - 2)\theta) + Dsin((\lambda - 2)\theta))$$

The Michell's solution to the Airy's stress function assumes integral solutions. In the general case that is derived above we do not make any such assumptions. The above equation though can be used in cases where λ is a integer.

4 Taylor's Problem

The particular problem considered by G.I. Taylor (1960) is given by two semi-infinite long plates, one at the reference angle of 0 and the other at an angle α. The plate along 0^0 is dragged at a constant velocity U through the corner point. It is important to note that the solution can be a function of θ because there is a discontinuity in the velocity at the corner of the plates and the stream function on both the plates need not be the same.

4.1 Finding the exponent of the problem

In this special case, we shall start by assuming a variable separable solution to the stream function in r and θ:

$$\psi = r^{\lambda} f_{\lambda}(\theta)$$

The radial component of the velocity is given by:

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

The angular component of the velocity is given by:

$$v = -\frac{\partial \psi}{\partial r}$$

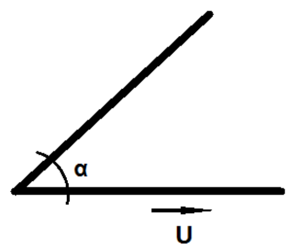


Figure 1: Taylor's Problem

Therefore, for the particular assumed solution, we can write these components as:

$$u = r^{\lambda-1} f'_{\lambda}(\theta)$$
$$v = (\lambda) r^{\lambda-1} f_{\lambda}(\theta)$$

What is interesting to note is that in this problem, the velocity u along the plate at angle 0^0 is a constant $U\hat{e_r}$. This implies that it is not a function of r or θ. Its independence of r can only be possible if $\lambda - 1 = 0$. Or the exponent of this problem, $\lambda = 1$. Since the exponent of the problem is an integer, we can start from the Michell's solution to the biharmonic solution.

4.2 Adapting Michell's Solution

Michell's solution to the biharmonic solution is given by

$$\begin{aligned}
\psi &= A_0 r^2 + B_0 r^2 ln(r) + C_0 ln(r) + D_0\theta \\
&+(A_1 r + B_1 r^{-1} + B'_1 r\theta + C_1 r^3 + D_1 r ln(r)) cos\theta \\
&+(E_1 r + F_1 r^{-1} + F'_1 r\theta + G_1 r^3 + H_1 r ln(r)) sin\theta \\
&+\sum_{n=2}^{\infty}(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-(n+2)}) cos(n\theta) \\
&+\sum_{n=2}^{\infty}(E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-(n+2)}) sin(n\theta)
\end{aligned}$$

Arguments similar to the last time, we can rule out the terms which have a logarithmic function of r and terms wit negative exponents of r. And since we already know that $\lambda = 1$, the stream function can therefore be expressed as:

$$\psi = Pr(Acos\theta + B\theta cos\theta + Csin\theta + D\theta sin\theta)$$

where P is an arbitrary constant.

4.3 Applying Boundary Conditions

$$\psi = Pr(Acos\theta + B\theta cos\theta + Csin\theta + D\theta sin\theta)$$
$$u = P(-Asin\theta + Bcos\theta - B\theta sin\theta + Ccos\theta + Dsin\theta + D\theta cos\theta)$$
$$v = -P(Acos\theta + B\theta cos\theta + Csin\theta + D\theta sin\theta)$$

The boundary conditions for this case are:
No slip Boundary conditions:
$u = U$ for $\theta = 0$
$u = 0$ for $\theta = \alpha$
No normal flow boundary conditions:
$v = 0$ for $\theta = 0$
$v = 0$ for $\theta = \alpha$
Applying these boundary conditions we get:

$$U = P(B + C) \tag{1}$$
$$0 = -P(A) \tag{2}$$
$$0 = P(-Asin\alpha + Bcos\alpha - B\alpha sin\alpha + Ccos\alpha + Dsin\alpha + D\alpha cos\alpha) \tag{3}$$
$$0 = -P(Acos\alpha + B\alpha cos\alpha + Csin\alpha + D\alpha sin\alpha) \tag{4}$$

From (1), since all are arbitrary constants, we can assume that P = U which gives us $B+C = 1$. From (2), since P = U, this implies that A = 0. From (1) and (2), (3) can be reduced to:

$$0 = (Bcos\alpha - B\alpha sin\alpha + Ccos\alpha + Dsin\alpha + D\alpha cos\alpha)$$
$$0 = cos\alpha - B\alpha sin\alpha + Dsin\alpha + D\alpha cos\alpha \tag{5}$$

And (4) can be reduced to

$$0 = B\alpha cos\alpha + (1 - B)sin\alpha + D\alpha sin\alpha \tag{6}$$

Solving the equations, we get:

$$P = U$$
$$A = 0$$
$$B = \frac{sin^2\alpha}{sin^2\alpha - \alpha^2}$$
$$C = \frac{-\alpha^2}{sin^2\alpha - \alpha^2}$$
$$D = \frac{\alpha - cos\alpha sin\alpha}{sin^2\alpha - \alpha^2}$$

4.4 Results

From the results of the previous section, the stream function becomes:

$$\psi = Ur(sin^2\alpha - \alpha^2)^{-1}(\theta sin\alpha sin(\alpha - \theta) + \alpha(\theta sin\theta - \alpha sin\theta)) \tag{7}$$
$$u = U(sin^2\alpha - \alpha^2)^{-1}(sin\alpha(sin(\alpha - \theta) - \theta cos(\alpha - \theta)) + \alpha(sin\theta + \theta cos\theta - \alpha cos\theta)) \tag{8}$$
$$v = -U(sin^2\alpha - \alpha^2)^{-1}(\theta sin\alpha sin(\alpha - \theta) + \alpha(\theta sin\theta - \alpha sin\theta)) \tag{9}$$

As we can see, the velocity components are only a function of the angle θ and are independent of the radial distance. The streamline plots for the Taylor's solution for angles of 30^0,90^0 and 120^0 are given in Figure 2.

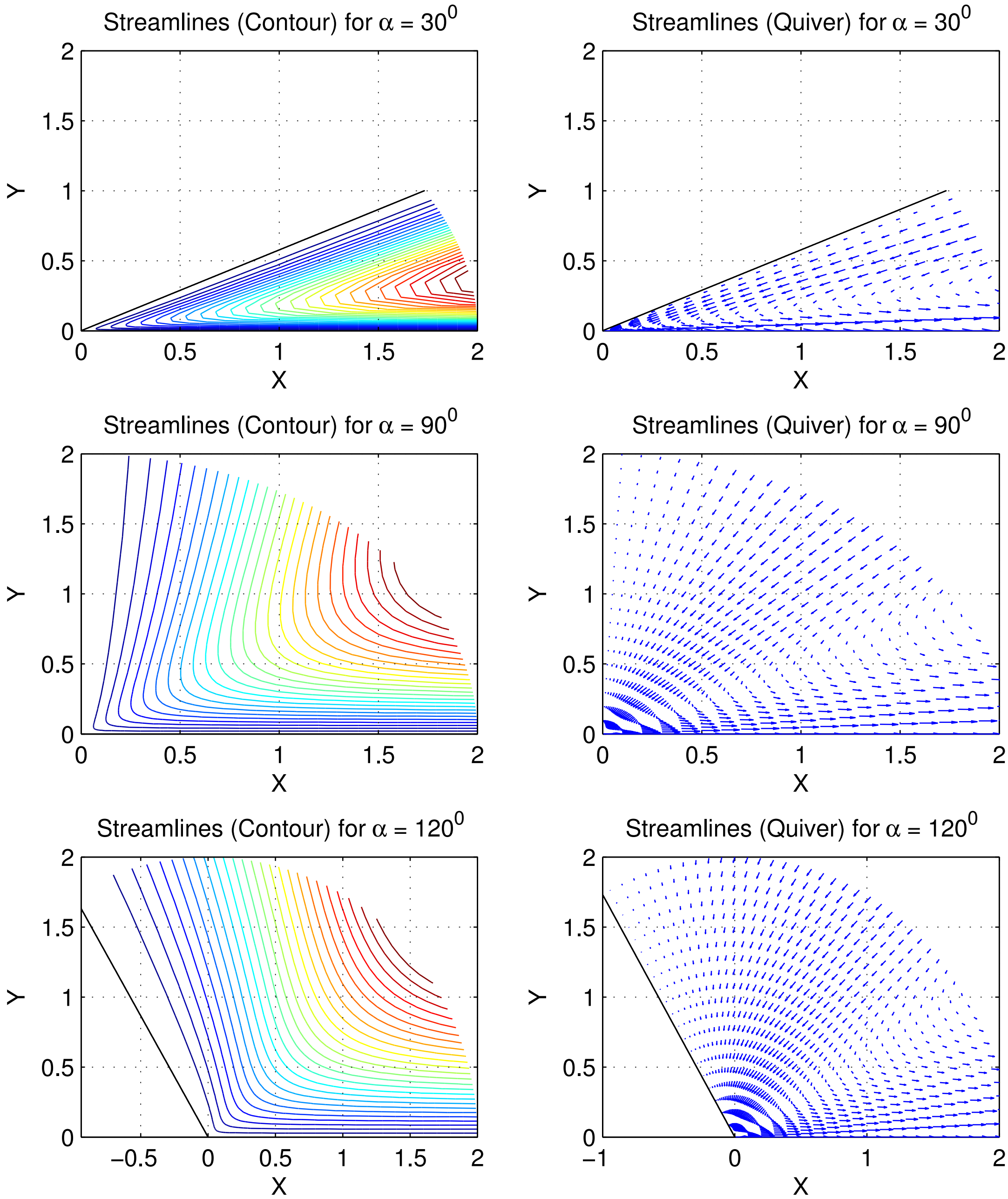


Figure 2: Taylor's Solution for 30^0,90^0 and 120^0

4.5 Linear Stability Analysis

The velocity components in r and θ directions for the Taylor's problem are:

$$U = C(sin^2\alpha - \alpha^2)^{-1}(sin\alpha(sin(\alpha - \theta) - \theta cos(\alpha - \theta)) + \alpha(sin\theta + \theta cos\theta - \alpha cos\theta)) \quad (10)$$

$$V = -C(sin^2\alpha - \alpha^2)^{-1}(\theta sin\alpha sin(\alpha - \theta) + \alpha(\theta sin\theta - \alpha sin\theta)) \quad (11)$$

where C is the velocity of the plate.

4.5.1 Linearized Conservation Equations

Let U and V be the velocity components in the r and θ directions and $\boldsymbol{U}$ be the velocity in vector form such that $\boldsymbol{U} = U\hat{r_r} + V\hat{e_\theta}$ and pressure P. For the flow we are considering we have:

$$U = U(\theta) \;\; V = V(\theta) \quad (12)$$

Then the perturbed flow can be modeled as $\boldsymbol{U+u}$ and $P + p$ such that

$$\boldsymbol{U+u} = (U + u)\hat{e_r} + (V + v)\hat{r_\theta} \quad (13)$$

The continuity and the Navier-Stokes equations for an incompressible unperturbed flow flow can be written as follows:

$$div(\boldsymbol{U}) = 0$$
$$\partial_t \boldsymbol{U} + (\boldsymbol{U}.\nabla)\boldsymbol{U} = -\nabla P + \frac{1}{Re}(\nabla^2 \boldsymbol{U})$$

The above equations are also satisfied by the perturbed flow. These equations can be written as:

$$div(\boldsymbol{U}) + div(\boldsymbol{u}) = 0$$
$$\partial_t \boldsymbol{U} + \partial_t \boldsymbol{u} + (\boldsymbol{U}.\nabla)\boldsymbol{U} + (\boldsymbol{u}.\nabla)\boldsymbol{U} + (\boldsymbol{U}.\nabla)\boldsymbol{u} + (\boldsymbol{u}.\nabla)\boldsymbol{u} = -\nabla P - \nabla p + \frac{1}{Re}(\nabla^2 \boldsymbol{U}) + \frac{1}{Re}(\nabla^2 \boldsymbol{u})$$

Subtracting Eq.4.5.1 from Eq. 4.5.1 and linearizing the equation ($(\boldsymbol{u}.\nabla)\boldsymbol{u}$ can be neglected), we get the linearized perturbation equations:

$$div(\boldsymbol{u}) = 0 \quad (14)$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}.\nabla)\boldsymbol{U} + (\boldsymbol{U}.\nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}(\nabla^2 \boldsymbol{u}) \quad (15)$$

Writing the momentum equations in component form, we have

$$\partial_t u + U\partial_r u + \frac{V}{r}\partial_\theta u + u\partial_r U + \frac{v}{r}\partial_\theta U - 2\frac{Vv}{r} = -\partial_r p + \frac{1}{Re}(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}) \quad (16)$$

$$\partial_t v + U\partial_r v + \frac{V}{r}\partial_\theta v + \frac{Vu}{r} + u\partial_r V + \frac{v}{r}\partial_\theta V + \frac{Uv}{r} = -\frac{1}{r}\partial_\theta p + \frac{1}{Re}(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial u}{\partial \theta}) \quad (17)$$

Since the velocities are invariant in r direction we can take the following normal mode form for perturbations:

$$u = \hat{u}(\theta)e^{i(kr - \omega t)}$$
$$v = \hat{v}(\theta)e^{i(kr - \omega t)}$$

$$p = \hat{p}(\theta)e^{i(kr-\omega t)}$$

Adding the fact that, due to Eq. 12, the above equations can be written as:

$$(-i\omega)\hat{u} + U(ik)\hat{u} + \frac{V}{r}\partial_\theta \hat{u} + \frac{\hat{v}}{r}\partial_\theta U - 2\frac{V\hat{v}}{r} = -(ik)\hat{p} + \frac{1}{Re}((\frac{ik}{r} - k^2 + \frac{\partial_{\theta\theta}}{r^2})\hat{u} - \frac{\hat{u}}{r^2} - \frac{2}{r^2}\frac{\partial \hat{v}}{\partial \theta}) \tag{18}$$

$$(-i\omega)\hat{v} + U(ik)\hat{v} + \frac{V}{r}\partial_\theta \hat{v} + \frac{V\hat{u}}{r} + \frac{\hat{v}}{r}\partial_\theta V + \frac{U\hat{v}}{r} = -\frac{1}{r}\partial_\theta \hat{p} + \frac{1}{Re}((\frac{ik}{r} - k^2 + \frac{\partial_{\theta\theta}}{r^2})\hat{v} - \frac{\hat{v}}{r^2} - \frac{2}{r^2}\frac{\partial \hat{u}}{\partial \theta}) \tag{19}$$

Along with the continuity equation given by

$$(ik)\hat{u} + \frac{\hat{u}}{r} + \frac{1}{r}\partial_\theta \hat{v} = 0 \tag{20}$$

$$ik(U-c)\hat{u} + \frac{V}{r}\partial_\theta \hat{u} + \frac{\hat{v}}{r}(\partial_\theta U - 2V) = -(ik)\hat{p} + \frac{1}{Re}((\frac{ik}{r} - k^2 + \frac{\partial_{\theta\theta}}{r^2} - \frac{1}{r^2})\hat{u} - \frac{2}{r^2}\frac{\partial \hat{v}}{\partial \theta}) \tag{21}$$

$$ik(U-c)\hat{v} + \frac{V}{r}\partial_\theta \hat{v} + \frac{V\hat{u}}{r} + \frac{\hat{v}}{r}(\partial_\theta V + U) = -\frac{1}{r}\partial_\theta \hat{p} + \frac{1}{Re}((\frac{ik}{r} - k^2 + \frac{\partial_{\theta\theta}}{r^2} - \frac{1}{r^2})\hat{v} - \frac{2}{r^2}\frac{\partial \hat{u}}{\partial \theta}) \tag{22}$$

The above equations are inhomogenous and can be only solved using numerical schemes. Such an exercise will be left to future scope at this point of time. But we can notice that at large r values, the above equations can be reduced to a homogenous form:

$$ik(U-c)\hat{u} = -(ik)\hat{p} + \frac{1}{Re}((-k^2)\hat{u}) \tag{23}$$

$$ik(U-c)\hat{v} = \frac{1}{Re}((-k^2)\hat{v}) \tag{24}$$

Cross-differentiating the above equations and using the stream function ψ, the above equations can be combined to give

$$ik(U-c)\nabla\hat{\psi} = -\frac{k^2}{Re}(\nabla\hat{\psi})$$

Or

$$(ik(U-c) + \frac{k^2}{Re})\nabla\hat{\psi} = 0 \tag{25}$$

5 Corner flow with wall rotation

Another problem which can be solved using integral solutions to λ is that of a corner flow with wall rotation. The plates are rotated about the origin at a constant angular velocity ω towards each other as shown below. Though the flow is unsteady, the acceleration terms can be considered negligible near the origin and the conservation equations still reduce to the biharmonic equation.

5.1 Finding the exponent of the problem

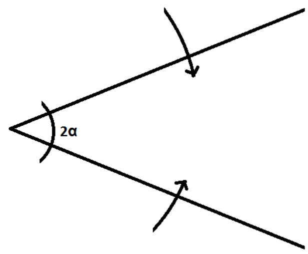


Figure 3: Corner Flow with Rotation

Let us assume a series solution to the stream function. Let

$$\psi = \sum_{\lambda=\lambda_1}^{\lambda_n} A_\lambda r^\lambda f_\lambda(\theta)$$

Now the velocity in the angular direction is given by

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\sum_{\lambda=\lambda_1}^{\lambda_n} \lambda A_\lambda r^{\lambda-1} f_\lambda(\theta)$$

At $\theta = \alpha$, we can see that at any point (r, α), we can write

$$u_\alpha = -(\omega)r$$

Therefore

$$-\sum_{\lambda=\lambda_1}^{\lambda_n} \lambda A_\lambda r^{\lambda-1} f_\lambda(\alpha) = -(\omega)r$$

Equating the powers of r on both sides, we can conclude that

$$A_\lambda = 0$$

for all λ except

$$\lambda - 1 = 1$$

$$\lambda = 2$$

Setting $\lambda = 2$ we can write the stream function as:

$$\psi = A_2 r^2 f_2(\theta)$$

Since

$$-\sum_{\lambda=\lambda_1}^{\lambda_n} \lambda A_\lambda r^{\lambda-1} f_\lambda(\theta) = -(\omega)r$$

$$-2A_2 r f_2(\alpha)(\theta) = -(\omega)r$$

$$A_2 = \frac{\omega}{2f_2(\alpha)}$$

Since the constants in f_2 are arbitrary and are chosen based on boundary conditions, we can set $f_2(\alpha) = \frac{1}{2}$ making $A_2 = \omega$. Therefore the stream function can be written as

$$\psi = \omega r^2 f_2(\theta)$$

where

$$f_2(\theta) = Acos(2\theta) + Bsin(2\theta) + C\theta + D$$

This $f(\theta)$ is a degenerate form of the general solution since there exist equal roots (0) when $\lambda = 2$ and can be obtained from the Michell's solution similar to the previous problem.

5.2 Boundary Conditions

The boundary conditions are again given by the no-slip and no-normal flow conditions. The no-slip boundary conditions imply that the radial velocity at $\theta = \pm\alpha$ is zero.

$$u_r|_{\pm\alpha} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \omega r f_2'(\pm\alpha) = 0$$

for all r. Therefore

$$f_2'(\alpha) = f_2'(-\alpha) = 0$$

The no-normal flow condition implies that the normal flow at the walls is equal to the component of the wall velocity in the normal direction. In polar coordinates, the normal direction at any radial ray is the same as the angular direction, i.e. e_θ. Hence

$$u_\theta(\alpha) = -\frac{\partial\psi}{\partial r} = -2\omega r f_2(\alpha) = -\omega r$$

This implies

$$f_2(\alpha) = \frac{1}{2}$$

Similarly

$$u_\theta(-\alpha) = -\frac{\partial\psi}{\partial r} = -2\omega r f_2(-\alpha) = \omega r$$

$$f_2(\alpha) = -\frac{1}{2}$$

From the boundary conditions, we get the following system of equations:

$$Acos(2\alpha) + Bsin(2\alpha) + C\alpha + D = \frac{1}{2}$$

$$Acos(2\alpha) - Bsin(2\alpha) - C\alpha + D = -\frac{1}{2}$$

$$-2Asin(2\alpha) + 2Bcos(2\alpha) + C = 0$$

$$2Asin(2\alpha) + 2Bcos(2\alpha) + C = 0$$

Solving the above system of equations simultaneously we get:

$$A = 0$$

$$D = 0$$

$$B = \frac{1}{2}(sin2\alpha - 2cos2\alpha)^{-1}$$

$$C = -cos2\alpha(sin2\alpha - 2cos2\alpha)^{-1}$$

5.3 Results

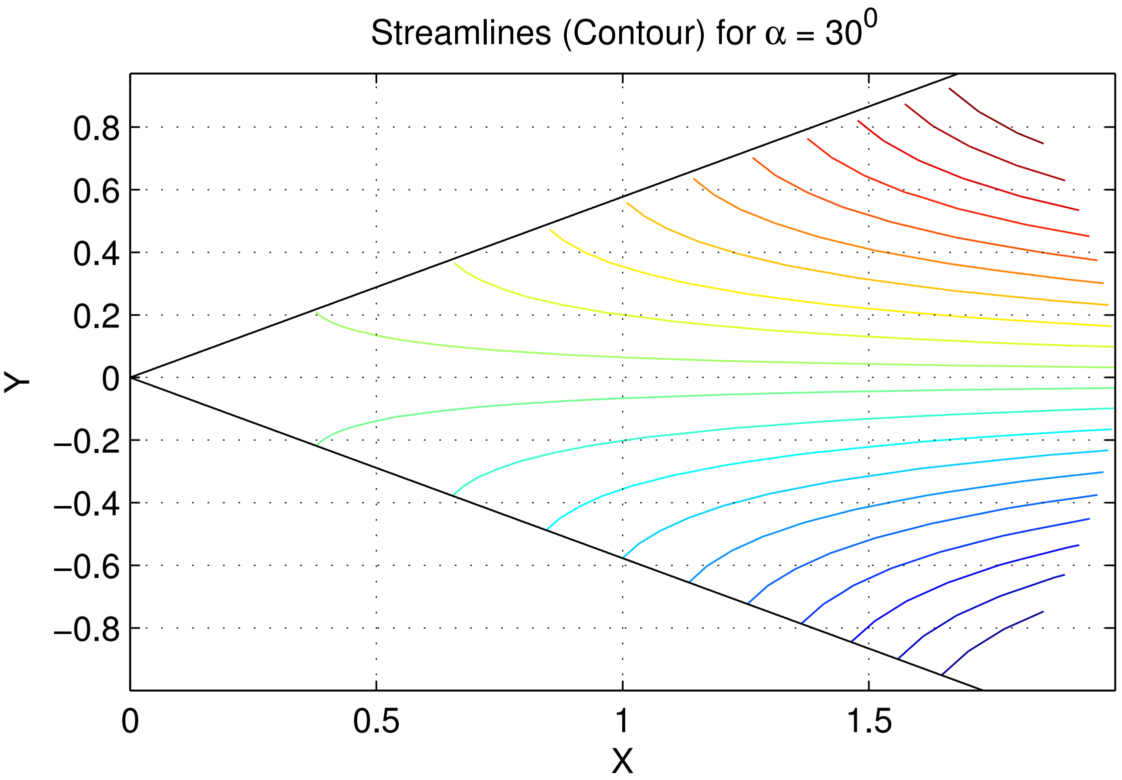
Applying the results from the previous section, the function $f_2(\theta) = \frac{1}{2}(sin2\alpha - 2cos2\alpha)^{-1}(sin2\theta - 2\theta cos2\alpha)$. This gives the stream function and the velocities as:

$$\psi = \frac{1}{2}\omega r^2(sin2\alpha - 2\alpha cos2\alpha)^{-1}(sin2\theta - 2\theta cos2\alpha) \tag{26}$$

$$u_r = \omega r(sin2\alpha - 2\alpha cos2\alpha)^{-1}(cos2\theta - cos2\alpha) \tag{27}$$

$$u_\theta = -\omega r(sin2\alpha - 2\alpha cos2\alpha)^{-1}(sin2\theta - 2\theta cos2\alpha) \tag{28}$$

The streamline plots for this solution for angle of 30^0 are given in Figure 4.



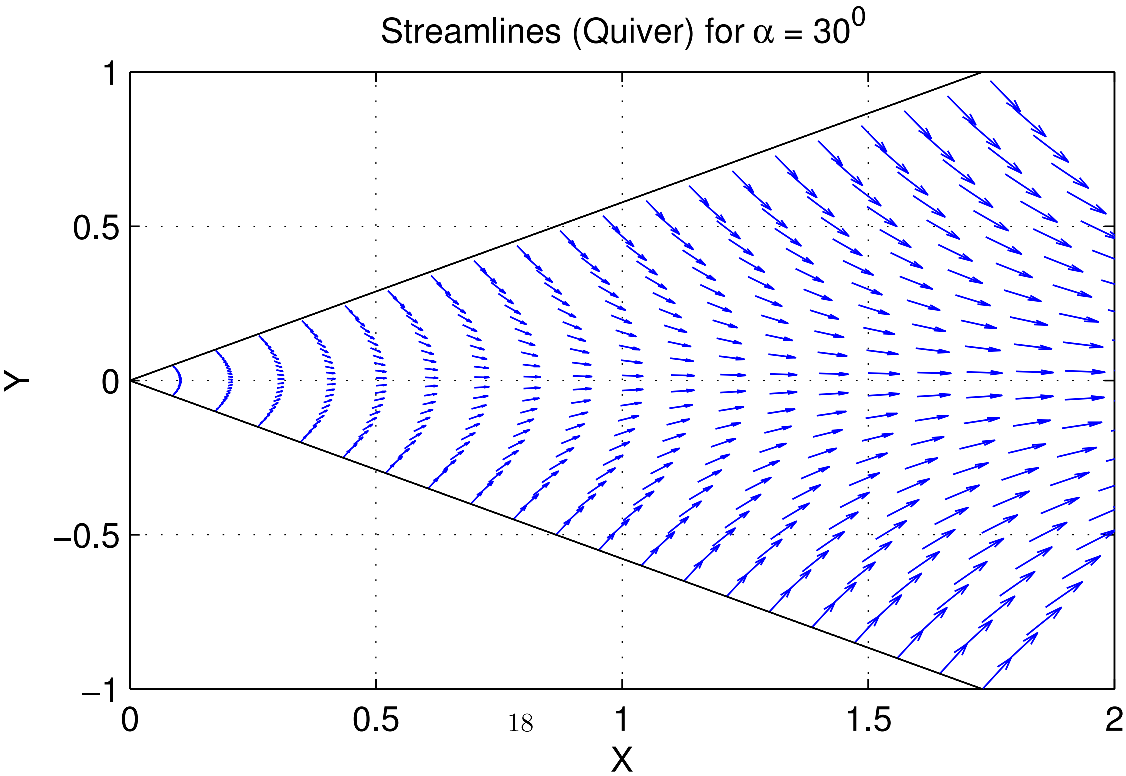


Figure 4: Corner flow with Plate Rotation for 30^0

6 Moffatt Eddies

Another flow which can be represented by the biharmonic equation is the creeping flow in a corner which occurs due to arbitrary disturbance at a distance far away from the corner. Assuming creeping flow conditions near the corner, we can show that the behavior of the flow is asymptotic and it qualitatively does not depend on the external flow. Dean and Montagnon (1946) were the first to solve this problem. But it was H.K. Moffatt (1964) who extended the solution to suggest the existence of eddies in the flow.

6.1 Solution

We can use the stream function expanded in a series form to represent the flow in a corner

$$\psi = \sum_{1}^{\infty} A_n r^{\lambda_n} f_{\lambda_n}(\theta)$$

Let the λ_n be such that

$$1 < Re(\lambda_1) < Re(\lambda_2) < ...$$

But very close to the corner, which is the case we wnt to investigate, the stream function reduces to just the first term

$$\psi \approx A_1 r^{\lambda_1} f_{\lambda_1}(\theta)$$

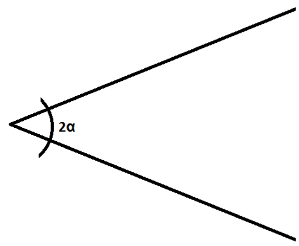


Figure 5: Moffatt Eddies - Problem Setup

provided obviously that $A_1 \neq 0$.

The general flow can be divided into a symmetric and anti-symmetric flows depending on the odd/even nature of $f_{\lambda_1}(\theta)$. The general condition will be a superposition of both these flows and hence we can study them separately to elucidate the qualitative nature of the flow.

6.1.1 Antisymmetric Case

In this case $f_\lambda(\theta)$ is even. This implies

$$f_\lambda(\theta) = Acos\lambda\theta + Ccos(\lambda - 2)\theta$$

The boundary conditions come from the vanishing velocities on both the boundaries (From No slip and No normal flow conditions).

$f_\lambda(\pm\alpha) = 0$ and $f'_\lambda(\pm\alpha) = 0$

Applying boundary conditions, we get

$$Acos\lambda\alpha + Ccos(\lambda - 2)\alpha = 0$$

$$A\lambda sin\lambda\alpha + C(\lambda - 2)sin(\lambda - 2)\alpha = 0$$

Solving the equations simultaneously we get

$$\lambda sin\lambda\alpha cos(\lambda - 2)\alpha = (\lambda - 2)cos\lambda\alpha sin(\lambda - 2)\alpha$$

Using the identities

$$sinA + sinB = sin\frac{A + B}{2}cos\frac{A - B}{2}$$

We get,

$$sin2\mu\alpha = -\mu sin2\alpha$$

where $\mu = \lambda - 1$

Assuming λ is complex, we can take

$$\mu = p + iq$$

Substituting it in the equations we get the following two equations by equating the real and imaginary parts on both sides:

$$sin(2\alpha p)\ cosh(2\alpha q) = -psin2\alpha$$

$$cos(2\alpha p)\ sinh(2\alpha q) = -qsin2\alpha$$

Further taking $\epsilon = 2\alpha p$ and $\eta = 2\alpha q$ and $k = sin2\alpha/2\alpha$, the two equations can be written as

$$sin\epsilon\ cosh\eta = -k\epsilon$$

$$cos\epsilon\ sinh\epsilon = -k\eta$$

The equations above can be solved numerically to obtain the values of ϵ and η for different values of α. It was found my Dean and Montagnon (1946), that no complex solution exists for $alpha \leq \alpha_1$ where α_1 is a critical angle. For corner angles above this angle, only real solutions exist and the real solutions increase from 1 at α_1 to infinity at π.

It was Moffatt who in 1964 suggested that the complex solutions for angles below the critical angle could mean the existence of eddies. This can be seen by finding the transverse component of velocity:

$$\psi \approx r^{\lambda_1}[Acos\lambda\theta + Ccos(\lambda - 2)\theta]$$

Multiplying and dividing by $A' r_0^{\lambda_1} sec(\lambda - 2)\alpha$, we get

$$\psi = A' \left(\frac{r}{r_0}\right)^{\lambda_1} [cos\lambda\theta]cos(\lambda - 2)\alpha + \frac{C}{A}cos(\lambda - 2)\theta cos(\lambda - 2)\alpha$$

Using the equation obtained by applying the first boundary condition,

$$Acos\lambda\alpha + Ccos(\lambda - 2)\alpha = 0$$

we can write

$$\frac{C}{A} = -\frac{cos\lambda\alpha}{cos(\lambda - 2)\alpha}$$

Substituting back in the equation, we get

$$\psi = A' \left(\frac{r}{r_0}\right)^{\lambda_1} [cos\lambda\theta cos(\lambda - 2)\alpha - cos(\lambda - 2)\theta cos\lambda\alpha]$$

The transverse component of the velocity along the centerline $\theta = 0$ can be found by

$$v_\theta|_{\theta=0} = -\frac{\partial\psi}{\partial r} = (a + ib)\frac{1}{r}\left(\frac{r}{r_0}\right)^{\lambda_1}$$

Here

$$a + ib = -\lambda_1 A'[cos(\lambda - 2)\alpha - cos\lambda\alpha]$$

$$\left(\frac{r}{r_0}\right)^{\lambda_1} = \left(\frac{r}{r_0}\right)^{1+p+iq} = \left(\frac{r}{r_0}\right)^{1+p} exp(iq_1 ln\left(\frac{r}{r_0}\right)) = \left(\frac{r}{r_0}\right)^{1+p} (cos(q_1 ln\left(\frac{r}{r_0}\right)) + isin(q_1 ln\left(\frac{r}{r_0}\right)))$$

Since only the real part of the velocity component can be understood,

$$v_\theta|_{\theta=0} = Re((a+ib)\frac{1}{r}\left(\frac{r}{r_0}\right)^{\lambda_1})$$

$$= \gamma\frac{1}{r}\left(\frac{r}{r_0}\right)^{1+p_1} sin(q_1 ln\left(\frac{r}{r_0}\right) + \xi)$$

This expression changes sign infinitely as r approaches zero. Infact the value is found to be zero for

$$q_1 ln\left(\frac{r}{r_0}\right) + \xi = -n\pi$$

where n = 1,2,...

Solving for r from the above equation also gives the center of each eddy:

$$r = r_0 e^{(-n\pi-\xi)/q_1}$$

The ratio of the distance between successive eddies is

$$\rho_1 = \frac{r_n}{r_{n+1}} = e^{n\pi/q_1}$$

which is a constant for a given angle α. This ratio is an indication of the fall in the dimensions of successive eddies. Higher this ratio, then the dimensions of the eddies decrease more rapidly. In each eddy, the velocity component along $\theta = 0$, has a local maximum. These maxima occur at

$$r = r_{n+\frac{1}{2}} = r_0 e^{-\epsilon/q_1} e^{-(n+\frac{1}{2})\pi/q_1}$$

The velocity at these points is $v_{n+\frac{1}{2}}$, is

$$v_{n+\frac{1}{2}} = \frac{\gamma}{r_0^{p_1+1}} r^{p_1}_{n+\frac{1}{2}}$$

This velocity can be taken as a measure of intensity. Hence the ratio by which the intensities fall of can be calculated as, ω_1,

$$\omega_1 = \frac{v_{n+\frac{1}{2}}}{v_{n+\frac{3}{2}}} = e^{\pi p_1/q_1}$$

This again is function of only the angle α. These ratios ρ_1 and ω_1 are recorded in Table. 2.

The contour plots for the obtained stream function are shown in Figure 6. The plots clearly show the Moffatt eddies. It can also be seen that the same plots on zooming, shown on the right hand side show a similar pattern, proving the existence of an infinite series of vortices.

6.1.2 Symmetric Case

In this case $f_\lambda(\theta)$ is odd. This implies

$$f_\lambda(\theta) = Bsin\lambda\theta + Dsin(\lambda-2)\theta$$

The boundary conditions come from the vanishing velocities one of boundaries (From No slip and No normal flow conditions) and at the centerline. This can also be taken as a situation when a plate is inserted into a fluid at an angle α. In that case the centerline is the free surface.

$f_\lambda(\pm\alpha) = 0$ and $f'_\lambda(\pm\alpha) = 0$
Applying boundary conditions, we get

$$Bsin\lambda\alpha + Dsin(\lambda - 2)\alpha = 0$$

$$B\lambda cos\lambda\alpha + D(\lambda - 2)cos(\lambda - 2)\alpha = 0$$

Solving the equations simultaneously we get

$$\lambda cos\lambda\alpha sin(\lambda - 2)\alpha = (\lambda - 2)sin\lambda\alpha cos(\lambda - 2)\alpha$$

Using the identities

$$sinA + sinB = sin\frac{A+B}{2}cos\frac{A-B}{2}$$

We get,

$$sin2\mu\alpha = \mu sin2\alpha$$

where $\mu = \lambda - 1$

Similar to last time we can consider $\mu = \epsilon + i\eta$ and calculate the values for ϵ and η. In this case it is found that the critical angle $2\alpha \approx 78^0$ about exactly the half of the critical angle in the antisymmetric case. The dimensions of these eddies fall off less rapidly than the antisymmetric case while the intensity fall off more rapidly. These are indicated in the values of ρ_2 and ω_2 in the Table 1.

The contour plots for the obtained stream function are shown in Figure 7. The plots clearly show the Moffatt eddies. It can also be seen that the same plots on zooming, shown on the right hand side show a similar pattern, proving the existence of an infinite series of vortices. The eddies formed are also symmetric about the line $\theta = 0$.

An axisymmetric version of this solution also exists. This was derived by P.N. Shankar (2007) in "Moffatt Eddies in the Cone".

$2\alpha^0$	ϵ_2	η_2	$ln(\rho_2)$	$ln(\eta_2)$
10	7.8518	2.7447	0.1998	8.9872
30	7.8519	2.2185	0.6085	9.1249
50	7.8521	2.6183	1.0471	9.4214
70	7.8524	2.4846	1.5448	9.9289
90	7.8528	2.2923	2.1528	10.7623
110	7.8532	2.0222	2.9825	12.2001
130	7.8535	1.6307	4.3712	15.1304

Table 1: Length and Velocity Scale Factors for Symmetric Moffatt Eddies

$2\alpha^0$	ϵ_1	η_1	$ln(\rho_1)$	$ln(\eta_1)$
10	4.6710	2.2185	0.2471	6.6145
30	4.6735	2.1769	0.7556	6.7445
50	4.6782	2.0912	1.3110	7.0281
70	4.6846	1.9555	1.9627	7.5259
90	4.6919	1.7583	2.8065	8.3829
110	4.6992	1.4752	4.0886	10.0077
130	4.7061	1.0380	6.8669	14.2428

Table 2: Length and Velocity Scale Factors for Antisymmetric Moffatt Eddies

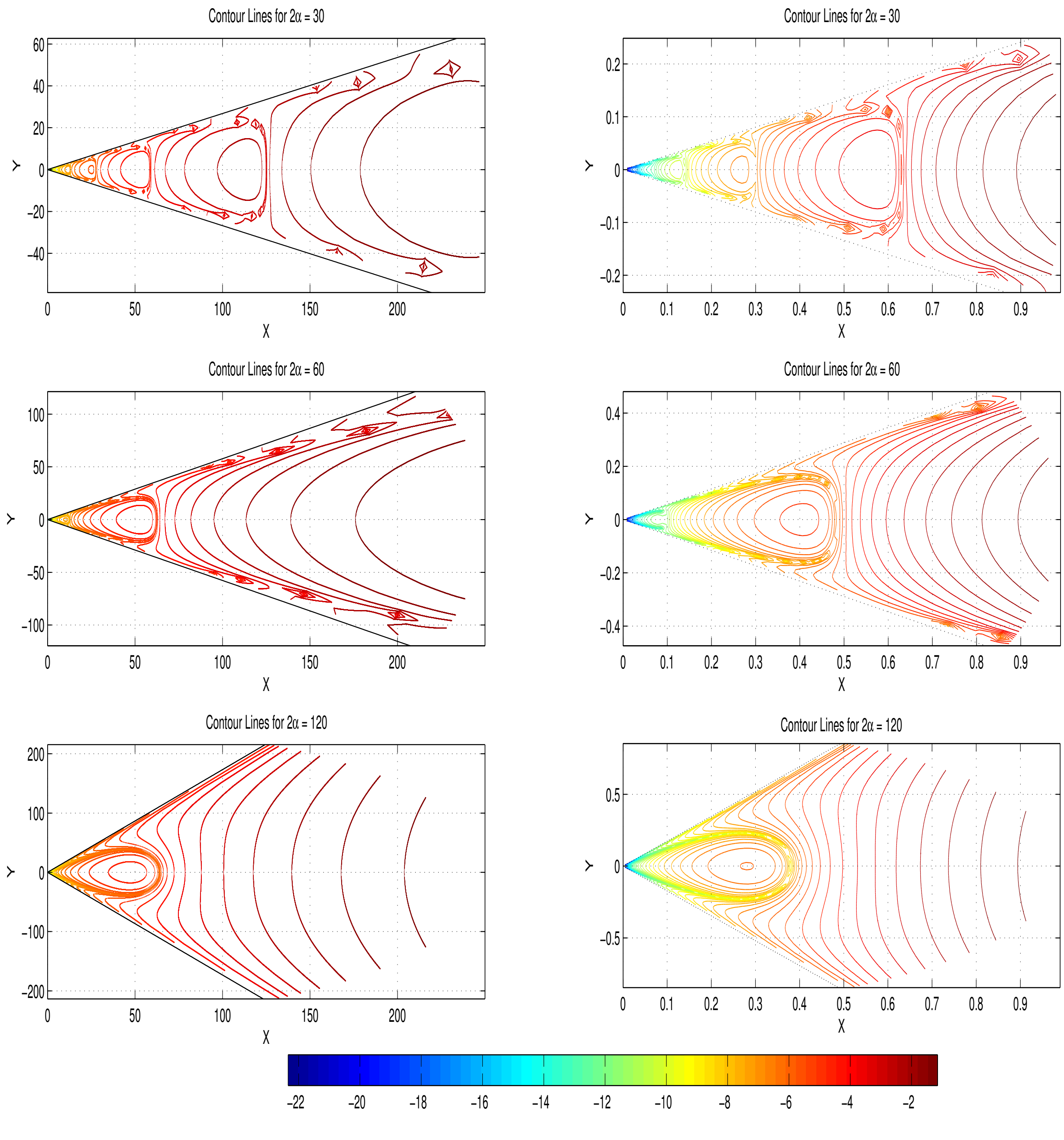


Figure 6: Moffatt's Eddies (Antisymmetric) for 30^0,60^0 and 120^0

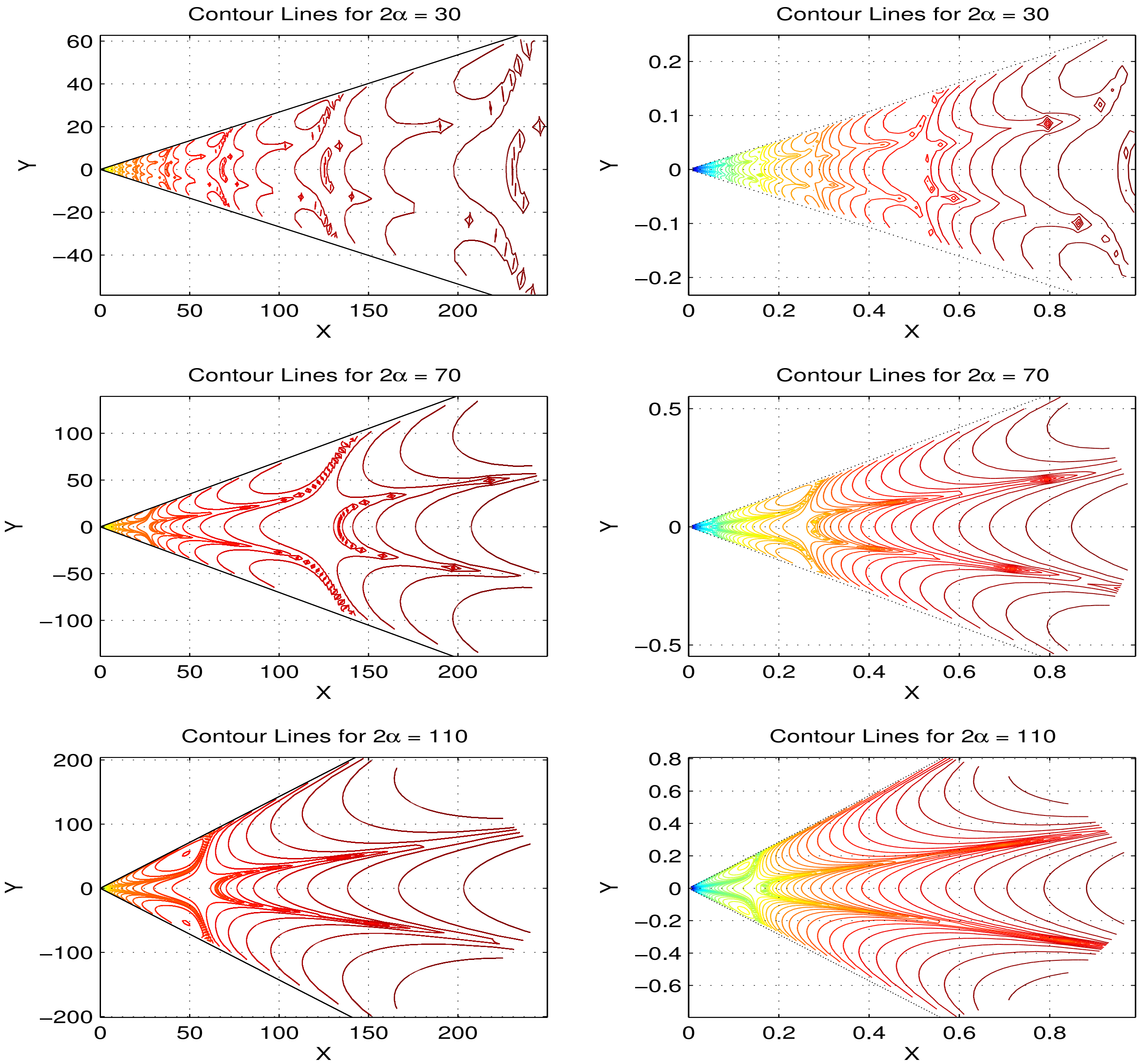


Figure 7: Moffatt's Eddies (Symmetric) for 30^0,70^0 and 110^0

7 Creeping Flow in a Rotating Frame

The aim of this section is to try and model the creeping fluid flow which we might observe in a rotating frame. Here we assume the whole setup where the fluid is located is rotating at a angular velocity Ω. We will try and find how the equations of motion change in such a case.

7.1 Equations of motion in a Rotating Frame

For any vector quantity $\mathbf{P}$, the rate of change of $\mathbf{P}$ can be written as

$$\frac{d\mathbf{P}}{dt} = (\frac{d\mathbf{P}}{dt})_R + \mathbf{\Omega} \times \mathbf{P}$$

where the subscript R represents the quantity in the rotating frame. Hence the momentum equation can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} + \frac{d\mathbf{\Omega}}{dt} \times x + \mathbf{\Omega} \times (\mathbf{\Omega} \times x) = \frac{-\nabla p*}{\rho} + \nu\nabla^2\mathbf{u} + \frac{\mathbf{b}}{\rho}$$

Assuming a time independent $\mathbf{\Omega}$ and working with a reduced pressure where

$$p = p^* - \mathbf{\Omega}^2(r^2 - (\mathbf{\Omega}.x)^2)^2$$

we can rewrite the equation as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = \frac{-\nabla p}{\rho} + \nu\nabla^2\mathbf{u} + \frac{\mathbf{b}}{\rho}$$

Similar to the non rotating frame, we can create a non dimensional equation from the above equation using

$$x^* = \frac{x}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad t^* = t\frac{U}{L} \quad p^* = \frac{p - p_\infty}{\mu\frac{U}{L}} \quad \mathbf{\Omega}^* = \frac{L}{U}\mathbf{\Omega}$$

The resulting equation is

$$Re(\frac{Du^*}{Dt^*} + 2\mathbf{\Omega}^* \times \mathbf{u}^*) = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*$$

7.2 Creeping flow in a Rotating Frame

We use the equation obtained from the previous section and make the creeping flow assumption, i.e

$$Re \approx 0$$

But this would lead to the same equation we obtained in the previous section, implying no difference in the kind of motion observed which would be a trivial pursuit. Hence we also make the following assumption

$$Re\mathbf{\Omega} = O(1)$$

which implies that the $Re\mathbf{\Omega}$ term stays in the equation. This leads to

$$2Re(\mathbf{\Omega}^* \times \mathbf{u}^*) = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^*$$

Bringing it back to dimensional form we have

$$2\mathbf{\Omega} \times \mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u}$$

Using the continuity equation, similar to the derivation of the biharmonic equation for non-rotating frame, we can get a similar equation by using the stream function ψ

$$\nabla \times [2\mathbf{\Omega} \times \mathbf{u} + \nabla p = \mu \nabla^2 \mathbf{u}]$$

$$2\nabla \times \mathbf{\Omega} \times \mathbf{u} = \mu \nabla \times \nabla^2 \mathbf{u}$$

Using the vector triple product identity,

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla.\boldsymbol{b}) - \boldsymbol{b}(\nabla.\boldsymbol{a}) + (\boldsymbol{b}.\nabla)\boldsymbol{a} - (\boldsymbol{a}.\nabla)\boldsymbol{b}$$

we can write

$$2(\mathbf{\Omega}(\nabla.\mathbf{u}) - \mathbf{u}(\nabla.\mathbf{\Omega}) + (\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla)\mathbf{u}) = \mu \nabla \times \nabla^2 \mathbf{u}$$

From the continuity equation, we know that $\nabla.\mathbf{u} = 0$. Hence, the equation reduces to

$$2(-\mathbf{u}(\nabla.\mathbf{\Omega}) + (\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla)\mathbf{u}) = \mu \nabla \times \nabla^2 \mathbf{u}$$

$$2((\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla + \nabla.\mathbf{\Omega})\mathbf{u}) = \mu \nabla \times \nabla^2 \mathbf{u}$$

Using the identity $\nabla^2(\nabla \times \boldsymbol{A}) = \nabla \times \nabla^2 \boldsymbol{A}$, we can write

$$\mu \nabla \times (\nabla^2 \mathbf{u}) = \mu \nabla^2 (\nabla \times \mathbf{u})$$

$$2\mu \nabla^2 \boldsymbol{\omega} = 2((\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla + \nabla.\mathbf{\Omega})\mathbf{u})$$

Assuming a 2D case, we can use the relation between the stream function and the vorticity, i.e.

$$\boldsymbol{\omega} = -\frac{1}{2}\nabla^2 \psi \hat{\boldsymbol{e_z}}$$

it can be written as

$$-\mu \nabla^2 \nabla^2 \psi \hat{\boldsymbol{e_z}} = 2((\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla + \nabla.\mathbf{\Omega})\mathbf{u})$$

$$\nabla^4 \psi \hat{\boldsymbol{e_z}} = -\frac{2}{\mu}((\mathbf{u}.\nabla)\mathbf{\Omega} - (\mathbf{\Omega}.\nabla + \nabla.\mathbf{\Omega})\mathbf{u})$$

If we assume a constant rotation , i.e.$\mathbf{\Omega} = \Omega \hat{e_z}$, and then equating the components on both sides we get:

$$\nabla^4 \psi = 0 \tag{29}$$

which is the same as the case with no rotation. Also if we take a spatially invariant rotation, i.e., $\frac{\partial \Omega_j}{\partial x_i} = 0$ (where $x_{i,j}$ = x,y,z), then we have

$$\nabla^4 \psi \hat{\boldsymbol{e_z}} = -\frac{2}{\mu}(\mathbf{\Omega}.\nabla)\mathbf{u}) \tag{30}$$

The left hand side of the equation is a component in z-direction whereas the right-hand side has components only in x and y directions. This can only mean that our assumption that the flow is 2-dimensional is wrong. This implies that the flow cannot remain 2-dimensional under such a rotation. Amore comprehensive approach will be attempted in the future.

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