Homework Assignment # 2 (due Oct.11)

deadline: 2023.10.11, 23:59

1. Write computer programs to find a solution by iteration within the tolerance $\varepsilon = 10^{-8}$ to the equation
$$f(x) = 10x(1-x)\left(x - \frac{1}{4}\right) - \frac{1}{4} = 0,$$
with the initial guess $x_0 = 0$.

   (a) Solve by the Newton method
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 1, 2, \cdots$$
until $|f(x_{n+1})| < \varepsilon$. Check whether the sequence $\{x_n\}$ converges or not. If it converges, check whether it has linear convergence or quadratic convergence. Explain what you observe.

   (b) Let
$$\varphi(x) = x - \gamma \frac{f(x)}{f'(x)}$$
with $\gamma = 1/2$. Show that the iteration below
$$x_{n+1} = \varphi(x_n) \quad \text{for } n = 0, 1, 2, \cdots$$
has linear convergence if it converges.

   (c) Accelerate the iteration above by Aitken's technique
$$x_{n+1} = \frac{y_0 y_2 - y_1^2}{y_0 + y_2 - 2y_1}$$
with $y_0 = x_n$, $y_1 = \varphi(y_0)$ and $y_2 = \varphi(y_1)$, for $n = 0, 1, 2, \cdots$. Report on your observation and explain.

2. Do problem 1 again for the following equation
$$f(x) = x^3 - x - 3 = 0,$$
with the initial guess $x_0 = 0$.

3. Do problem 1 again for the following equation
$$f(x) = x^4 - 4x^2 + 4 = 0,$$
with the initial guess $x_0 = 1$.

4. Let $n > 0$ be an integer and $\{z_i\}_{i=0}^{n}$ be $(n+1)$ randomly generated distinct points on the interval $[0, 1]$.

(a) Show that the function given by

$$f_n(x) = \sum_{i=0}^{n} \frac{1}{x - z_i}$$

has n roots on the interval $[0, 1]$.

(b) Write a computer program to find all the roots of $f_n(x)$ for $n = 3, 4, 5, 6$.

5. Please compute the LU decomposition by hand for the following matrices.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 4 & -1 & 0 & 5 \\ -4 & 3 & -3 & -5 \\ 2 & 2 & -3 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

6. Let $n > 0$ be an integer, and $\mathbf{A}$ be the dense matrix (every entry is nonzero) given by

$$\mathbf{A} = \frac{1}{n+1} \begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \ddots & 6 & 3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}_{n \times n}$$

Write a computer program to find the inverse of the matrix $\mathbf{A}$. Print out the inverse matrices $\mathbf{A}^-1$ for $n = 4$ and $n = 8$. Report the computer times used by your program to find the inverse for $n = 100, 200, 400, 800$. Find the relation of the computer time with the dimension of the matrix.

7. Write a computer program to solve the linear system below

$$(n+1)^2 \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{n \times n} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}.$$

Here, $f_i = (3x_i + x_i^2)e^{x_i}$ and $x_i = i/(n+1)$ for $i = 1, 2, \cdots, n$. Please print out the numerical solution with six decimal digits for $n = 9$, and make plots to show the the datasets $\{(x_i, u_i)\}_{i=1}^n$ for $n = 99, 199, 399, 799$.

8. Suppose $\mathbf{A}$ is an $n \times n$ invertible real matrix. Show that its QR decomposition in the form as we discussed in the classroom is unique.

9. Make QR decomposition by hand or computer for the matrix below

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

10. Let $\mathbf{A} = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ be such a matrix whose $(i,j)^{th}$ entry equals $1/(i+j-1)$ for $i,j = 1,2,\cdots,n$. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ be, respectively, the unknown and known vectors. Please write a computer program to solve the system $\mathbf{Ax} = \mathbf{b}$ with the Gauss elimination method. Please choose the right hand side $\mathbf{b}$ such that the exact solution reads

$$\mathbf{x} = (x_1, x_2, \cdots, x_n)^T = (1, 1, \cdots, 1)^T.$$

Denote the computed solution by $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^T$. Print out the maximum-norm error

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty := \max_{1\le i\le n} |x_i - \tilde{x}_i|$$

for $n = 4, 5, 6, \cdots$, until the Gauss elimination method fails to work. Report on what you observe and make your own explanation.