ANALYSIS • MATH6105 • HOMEWORK #1
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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Question 1. Let $\{\omega_n\}$ be a sequence of non-negative real numbers. For a subset $E \subset \mathbb{N}$, set

$$\mu_\omega(E) := \sum_{n \in E} \omega_n.$$

Show that μ_ω is a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. (Recall that $\mathcal{P}(\mathbb{N}) := \{E : E \subset \mathbb{N}\}$.) Now, let ν be an arbitrary measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Show that ν coincides with the measure μ_ω where $\omega_n := \nu(\{n\})$.

Question 2 (Characterisation of the Euclidean topology on $\mathbb{R}$. Let E be an open subset of $\mathbb{R}$. For $x, y \in E$, denote by $I_{x,y}$ the closed interval between x and y ($I_{x,x} \equiv \{x\}$). Define a relation $\sim$ on E such that $x \sim y$ if and only if $I_{x,y} \subset E$.

(1) Show that $\sim$ is an equivalence relation on E. What are the equivalence classes?
(2) Show that E is the disjoint union of at most countably many open intervals.

Question 3. Using the above question, deduce that if $\mathcal{F}$ is a σ-algebra on $\mathbb{R}$ containing all the intervals of the form $]a, \infty[$ for $a \in \mathbb{R}$, then $\mathcal{F}$ contains the Borel σ-algebra on $\mathbb{R}$.

Question 4 (Pushforward measures). Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f : X \to Y$ be a function between sets. Define

$$f_\#(\mathcal{F}) := \left\{B \subset Y : f^{-1}(B) \in \mathcal{F}\right\}, \qquad [f_\#\mu](B) := \mu\left(f^{-1}(B)\right) \text{ for } B \in f_\#(\mathcal{F}).$$

(1) Show that $(Y, f_\#(\mathcal{F}), f_\#\mu)$ is a measure space.
(2) Now let $Y = \mathbb{R}$ and $(X, \mathcal{F}, \mu)$ be $\left(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mathcal{L}^1\right)$, the Borel σ-algebra and the Lebesgue measure. Determine $(\mathbb{R}, f_\#(\mathcal{F}), f_\#\mu)$ when
   - $f(x) = \tan x$ if $\cos x \neq 0$, and $f(x) = 0$ if $\cos x = 0$;
   - $f(x) = \arctan x$.

Question 5 (The First Borel–Cantelli Lemma). Let $\{E_j\}_{j=1}^\infty$ be a countable family of measurable subsets of $(X, \mathcal{F}, \mu)$. Suppose that $\sum_{j=1}^\infty \mu(E_j) < \infty$, and define

$$\limsup_{j \to \infty} E_j := \{x \in X : x \in E_k \text{ for infinitely many } k\}.$$

Show that $\limsup_{j \to \infty} E_j$ is measurable and μ-null.

Question 6. Give the details of the proof for Carathéodory's theorem:

Theorem 0.1. *Let μ^* be an outer measure on a nonempty set X. Set*

$$\mathcal{M}^* := \left\{A \subset X : \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ for all } E \subset X\right\}.$$

Then $\mathcal{M}^*$ is a σ-algebra on X, and μ^* restricted to $\mathcal{M}^*$ is a complete measure.

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