ANALYSIS • MATH6105 • HOMEWORK #3
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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One important theorem we shall use frequently throughout the course is the Fubini(–Tonelli) theorem, which allows us to interchange the order of iterated integrals under the assumption that the integrand is either nonnegative or integrable. However, unfortunately, we won't go over the proof in lectures due to the limited time. Please refer to the texts (*e.g.*, [Folland, §2.5]) to understand the statement and proof of the following.

Theorem 0.1 (Fubini–Tonelli). *Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ-finite measure spaces.*

- *If $f : (X \times Y, \mathcal{F} \otimes \mathcal{G}, \mu \otimes \nu) \to \mathbb{R}$ is a non-negative measurable function, then $g(x) := \int_Y f(x,y)\,\mathrm{d}\nu(y)$ and $h(y) := \int_X f(x,y)\,\mathrm{d}\mu(x)$ are both measurable, and*
$$\iint_{X\times Y} f\,\mathrm{d}(\mu\otimes\nu) = \int_X g(x)\,\mathrm{d}\mu(x) = \int_Y h(y)\,\mathrm{d}\nu(y). \tag{0.1}$$
- *If $f \in L^1(X \times Y, \mathcal{F} \otimes \mathcal{G}, \mu \otimes \nu)$, then for μ-a.e. $x \in X$ the function $y \mapsto f(x,y)$ is in $L^1(Y, \mathcal{G}, \nu)$, and for ν-a.e. $y \in Y$ the function $x \mapsto f(x,y)$ is in $L^1(X, \mathcal{F}, \mu)$. Moreover, the a.e.-defined functions g and h as above are in $L^1(X, \mathcal{F}, \mu)$ and $L^1(Y, \mathcal{G}, \nu)$, respectively, and Equation* (0.1) *holds.*

Then, work out the following problem ([Folland, p.77, Q55]): Let $Q = [0,1]^2$. Investigate the existence and equality of
$$\int_Q f\,\mathrm{d}\mathcal{L}^2, \qquad \int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \text{and } \int_0^1\int_0^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
for the following f:

- $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$;
- $f(x,y) = (1-xy)^{-\alpha}$ for some $\alpha > 0$;
- $f(x,y) = (x-\frac{1}{2})^{-3}\mathbb{1}_{\{0<y<|x-\frac{1}{2}|\}}$

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