Homework Assignment # 1 (due Monday, Sept. 25)

deadline: 2023.9.25, 8:50

1. Write computer programs to approximately compute $\ln 2$ with a series.

   (a) Apply the series
   $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$
   to compute $\ln 2$. Find the minimum number of terms in the series for the computed value to have an absolute error less than 10^{-6}.

   (b) Apply the series
   $$\ln \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + + \frac{x^{2n-1}}{2n-1} + \cdots\right)$$
   with $x = 1/3$ to compute $\ln 2$. Find the minimum number of terms in the series for the computed value to have an absolute error less than 10^{-6}.

2. Write computer programs to numerically compute the integral
   $$I_n = \int_0^1 \frac{x^n}{x+10}\,\mathrm{d}x$$
   for integer $n > 0$.

   (a) Note that for $n = 0$, $I_0 = \ln \frac{11}{10}$. Use the recursion
   $$I_k = \frac{1}{k} - 10 I_{k-1} \quad \text{for } k = 1, 2, 3, \cdots, n$$
   to compute the integrals, I_{10} and I_{20}.

   (b) Note that
   $$\frac{1}{11(n+1)} < I_n < \frac{1}{10(n+1)}.$$
   The integral I_n can be approximated by the average of the lower and upper bounds. For positive integers $m > n$, we have the recursion
   $$I_{k-1} = \frac{1}{10}\left(\frac{1}{k} - I_k\right) \quad \text{for } k = m, m-1, \cdots, n+1.$$
   Let $m = 40$ and approximate the integral at $m = 40$ by $I_m \approx \dfrac{21}{220(m+1)}$. Use the recursive relation above to compute the integrals, I_{10} and I_{20} .

   (c) Do the previous two problems again. But this time please represent real numbers in the single precision in your programs. Compare the results with different precisions.

3. Design your own method for numerically computing the integral $I_n = \int_0^1 x^n e^x\,\mathrm{d}x$. Write a computer program to implement it, make numerical experiments and report on what you observe.

4. Let

$$S(n) = \sum_{k=1}^{n} \frac{1}{k^2}$$

be the partial sum with n be a positive integer. It is known that the limit of the partial sum $S(n)$ as n tends to infinity is $\pi^2/6$. That is,

$$\lim_{n\to\infty} S(n) = \frac{\pi^2}{6}.$$

Write a computer program to compute the partial sum for $n = 10^2, 10^3, 10^4, 10^5, 10^6$. In your program, please use the single precision to represent floating point numbers. Check the convergence of the computed partial sums. Explain on what you observe.

5. Write a computer program to find all zero points of $f(x) = \sin(10x) - x$ with bisection method.

6. Prove that the iterative formulation

$$x_{n+1} = \cos x_n$$

generates a sequence that converges to the solution of the equation $x = \cos x$ for any initial guess x_0.

7. Consider solving the nonlinear equation $f(x) = x^2 + x - 2 = 0$. Let $\varphi(x) = x + f(x)$. The function $\varphi(x)$ has two fixed points $x^* = 1$ and $x^* = -2$.

   (a) Write a computer program to check whether the iteration,

   $$x_{n+1} = \varphi(x_n) \quad \text{for } n = 0, 1, 2, \cdots,$$

   generates a convergent sequence with different initial guess $x_0 \neq 1, -2, 0, -1$.

   (b) Write a computer program to check the convergence of the iteration above accelerated by Aitken's technique, which now reads

   $$x_{n+1} = \frac{y_0 y_2 - y_1^2}{y_0 + y_2 - 2y_1}$$

   with $y_0 = x_n$, $y_1 = \varphi(y_0)$ and $y_2 = \varphi(y_1)$, for $n = 0, 1, 2, \cdots$, until the absolute value of the nonlinear function is sufficiently small, $|f(x_{n+1})| < \varepsilon = 10^{-8}$. Please feel free to choose your own initial guess x_0 for the iteration.

8. Consider solving the nonlinear equation $f(x) = x^3 = 0$. Let $\varphi(x) = x + f(x)$. It is obvious that $x^* = 0$ is the unique fixed point of the function $\varphi(x)$.

   (a) Show that the iteration below

   $$x_{n+1} = \varphi(x_n) \quad \text{for } n = 0, 1, 2, \cdots,$$

   for any non-zero initial guess $x_0 \neq 0$ does not converge.

(b) Write a computer program to numerically verify that the iteration accelerated by Aitken's technique, which reads

$$x_{n+1} = \frac{y_0 y_2 - y_1^2}{y_0 + y_2 - 2y_1}$$

with $y_0 = x_n$, $y_1 = \varphi(y_0)$ and $y_2 = \varphi(y_1)$, for $n = 0, 1, 2, \cdots$, however converges. Choose the tolerance ε to be 10^{-8} for stopping the iteration, which appears in the condition $|f(x_{n+1})| < \varepsilon$. Please print out the computed x_n and $f(x_n)$ during the iteration.