ANALYSIS • MATH6105 • HOMEWORK #3
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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Question 1. The Luzin theorem says that every bounded, measurable function on $\mathbb{R}^d$ is nearly continuous. Here "nearly" is understood in the Luzin sense — *i.e.*, outside a subset of arbitrarily small measure. The aim of this question is to show that "nearly" cannot be understood as "a.e.".

(1) Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ which equals $\chi_{[0,1]}$ a.e..
(2) Consider the following variant of the standard Cantor set — let $C_0 = [0,1]$, and each time remove the middle ℓ_j of every of the remaining intervals in C_j to form C_{j+1}. By suitably choosing $\{\ell_j\}_{j=0}^{\infty}$, show that for any $\xi \in]0,1[$ one may construct a Cantor-like set $\mathcal{C}_\xi$ which is perfect, totally disconnected, uncountable, and has Lebesgue measure ξ.
(3) By suitably iterating the above construction, construct a measurable subset $E \subset [0,1]$ such that for any open interval $I \subset]0,1[$, both $E \cap I$ and $E^c \cap I$ have positive measure.
(4) Deduce that there exists a measurable function $g : \mathbb{R} \to \mathbb{R}$ satisfying the following property: let $h : \mathbb{R} \to \mathbb{R}$ be any function that equals g *a.e.* on $\mathbb{R}$; then h cannot be continuous at *any* point in $[0,1]$.

Question 2. By considering the Cantor–Lebesgue function (the devil's staircase), or otherwise, construct Lebesgue-measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is *not* Lebesgue-measurable.

Question 3. Prove the following result stated in the lectures:

Theorem 0.1. *Let $f : [a,b] \to \mathbb{R}$ be Borel measurable. We can find sequences $\{f_n\}$ converging to f in the Lebesgue measure such that*

(1) f_n are bounded and Borel measurable;
(2) f_n are simple functions;
(3) f_n are step functions;
(4) f_n are continuous functions.

Question 4. Let $\{f_n\}$ be a sequence of integrable functions on $(X, \mathcal{F}, \mu)$ that converges to f in L^1, namely that $\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0$. Can we conclude that $f_n \to f$ in measure μ?

Question 5. Can you find BV functions on $[0,1]$ that are not absolutely continuous? How about *continuous* BV functions on $[0,1]$ that are not absolutely continuous?

Question 6. Prove that absolutely continuous functions on $[0,1]$ form a closed subspace of BV-functions on $[0,1]$.

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