Lecture Notes of Analysis

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Content

1 Measure Theory

1.1 Measurable Space

Motivation: Consider $f(x) := \chi_{[0,1]\cap\mathbb{Q}}$, it is not Riemann integrable.But intuitively, $\int_0^1 f\mathrm{d}x = 0$ because there are a lot more points in $[0,1]\setminus\mathbb{Q}$ than $[0,1]\cap\mathbb{Q}$.

Henri Lebesgue: partition the range instead of the domain. $f(x) := \chi_{[0,1]\cap\mathbb{Q}}$ takes only 2 values $\{0,1\}$, $\{f=0\} = \{x\in\mathbb{R} : f(x)=0\} = f^{-1}\{0\} = \mathbb{R}\setminus([0,1]\cap\mathbb{Q})$, $\{f=1\} = \{x\in\mathbb{R} : f(x)=1\} = f^{-1}\{1\} = \mathbb{R}[0,1]\cap\mathbb{Q}$.A reasonable sense of integration should yield

$$\begin{aligned}\int_{\mathbb{R}} f\mathrm{d}x =& \text{Volume}(\{f=0\})\cdot 0 + \text{Volume}(\{f=1\})\cdot 1\\ =&\text{Volume}(\{f=1\}) = \text{Volume}([0,1]\cap\mathbb{Q}) = 0.\end{aligned} \tag{1.1}$$

How to define "Volume"? This is done axiomatically

Definition 1.1 *Let X be a set, $\mathcal{F}\subset\mathcal{P}(X)$*[1] *is said to be a* $\boldsymbol{\sigma-}$ ***algebra*** *if the following held:*

(i) $\mathcal{X}\subset\mathcal{F}$;

(ii) If $A\in\mathcal{F}$, then $X\setminus A\in\mathcal{F}$($\mathcal{F}$ is closed under complement);

(iii) If $A_i\in\mathcal{F}$ for $i=1,2,3,\cdots$, then $\bigcup_{i=1}^{\infty} A_i\in\mathcal{F}$($\mathcal{F}$ is closed under ***countable*** *union);*

Definition 1.2 *If X is a set and $\mathcal{F}\subset\mathcal{P}(X)$ is a $\sigma-$ algebra, then $(X,\mathcal{F})$ is said to be a* ***measurable space****. Each $E\subset\mathcal{F}$ is said to be a* ***measurable set****.*

Definition 1.3 *Let $(X,\mathcal{F}),(Y,\mathcal{G})$ be measurable space. A function $f:X\to Y$ is said to be $(\mathcal{F}/\mathcal{G})-$**measurable** if $\forall B\in\mathcal{G}$, we have $f^{-1}(B)\in\mathcal{F}$.*

Example Let X be a set. Let $(Y,\mathcal{G})$ be a measurable space, let $f:X\to Y$ be a function. Then

$$f^{\#}(\mathcal{G}) = \{f^{-1}(B) : B\in\mathcal{G}\} \tag{1.2}$$

is a $\sigma-$ algebra. It is the smallest $\sigma-$ algebra such that f is measurable.

Remark By definition of $\sigma-$ algebra $\mathcal{F}$,

(1) $\varnothing\in\mathcal{F}$(since $\varnothing = X\setminus X$).

[1] Power set $\mathcal{P}(X) := \{\text{subset of } X\}$

(2) $\mathcal{F}$ is closed under countable/ finite intersection.

(3) $A \setminus B = A \cap (X \setminus B) \subset \mathcal{F}$.

Definition 1.4 *Let X be a set, $\mathcal{T} \subset \mathcal{P}(X)$ is a **topology** if and only if*

(i) $\varnothing \in \mathcal{T}, \mathrm{X} \in \mathcal{T}$;

(ii) $\mathcal{T}$ *is closed under finite intersection;*

(iii) $\mathcal{T}$ *is closed under arbitrary unions.*

$(X, \mathcal{T})$ is called a topological space. Any set $E \in \mathcal{T}$ is called open.

Example (0) Euclidean $\mathbb{R}^n$. Topology consist of $\{B_\delta(x) : x \in \mathbb{R}^n, \delta > 0\}$. Open ball $B_\delta(x) = \{y \in \mathbb{R}^n : |x - y| < \delta\}$

(1) Metric space (X, d), $d : (X, X) \to [0, +\infty[$ such that

(i) Symmetric:$d(x, y) = d(y, x)$;

(ii) Nondegenerate:$d(x, y) = 0$ if and only if $x = y$;

(iii) Triangle Inequality:$d(x, y) + d(y, z) \geqslant d(x, z)$.

Any metric space is a topological space, with topology generated by open metric balls $B_\delta(x) = \{y \in X : d(x, y) < \delta\}$.

Definition 1.5 *Let X be a topological space. The smallest $\sigma-$ algebra cotaining the topology of X is called the **Borel $\sigma-$ algebra**.*

Proposition 1.1 *Let X be a topological spaces, $X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Z}$ be functions, g be continuous. Then*

(1) *If X is a topological space and f is continuous, then $g \circ f$ is continuous.*

(2) *If X is a measurable space and f is measurable, then $g \circ f$ is measurable.*

Proposition 1.2 *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces, $f : X \to Y$ is continuous if and only if $\forall \ O \in \mathcal{T}_Y, f^{-1}(O) \in \mathcal{T}_X$.*

Proof (1) To see that $g \circ f$ is continuous, we need to prove that if $O \in \mathrm{Z}$ is open, then $(g \circ f)^{-1}(O) \subset X$ is open. But

$$(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)) \tag{1.3}$$

Since g is continuous, $g^{-1}(O) \subset Y$ is open. Also, since f is continuous, $f^{-1}(g^{-1}(O))$ is open.

(2) change all "open" to measurable (sets), "continuous" to measurable (functions) in

the above proof. □

CONVENTION Unless specified, otherwise $\sigma-$algebra on topological spaces are Borel $\sigma-$ algebra.

Proposition 1.3 *Let $(X,\mathcal{F}),(Y,\mathcal{G})$ are measurable spaces. Let $u_1,\cdots,u_n : X \to \mathbb{R}$ be (Borel) measurable. Let $\Phi : \mathbb{R}^n \to (Y,\mathcal{G})$ be (Borel) measurable, then $h : X \to Y$*

$$x \mapsto \Phi(u_1(x),\cdots,u_n(x)) \tag{1.4}$$

is measurable.

PROOF By the previous result, it suffices to prove $u_1 \oplus \cdots \oplus u_n : X \to \mathbb{R}^n$

$$x \mapsto (u_1(x),\cdots,u_n(x)) \tag{1.5}$$

is measurable. □

CLAIM Every Borel measurable set in $\mathbb{R}^n$ is countable union of closed rectangles

$$R = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n]. \tag{1.6}$$

Assume the claim, for any $\bar{R} \subset \mathbb{R}^n$,

$$(u_1 \oplus \cdots \oplus u_n)^{-1}(\bar{R}) = \bigcap_{i=1}^{n} u_i^{-1}([a_i,b_i]) \tag{1.7}$$

Lemma 1.1 (Whitney Covering Lemma) *Let $\varnothing \neq \Omega \subset \mathbb{R}^n$ be open. $\exists \{Q_j\}_1^\infty$ (finite) closed cubes in $\mathbb{R}^n$ s.t.*

(a) $\bigcup_{i=1}^{\infty} Q_j = \Omega$;

(b) $\mathring{Q}_j \cap \mathring{Q}_k = \varnothing,\ \forall j \neq k$.

(c) $\sqrt{n}\mathrm{length}(Q) \leqslant \mathrm{dist}(Q_j,\Omega^c) \leqslant 4\sqrt{n}\mathrm{length}(Q)$;

(d) $\frac{1}{4} \leqslant \frac{\mathrm{length}(Q_j)}{\mathrm{length}(Q_k)} \leqslant 4$ *if* $Q_j \cap Q_k \neq \varnothing$;

(e) $\forall j, Q_j$ *intersects with* $12^n Q_k$.

Proposition 1.4 *Let $(X, \mathcal{F}), (Y, \mathcal{G}), (Z, \mathcal{H})$ be measurable spaces. Let $f : X \to Y$ be $\mathcal{F}/\mathcal{G}-$measurable, and let $g : Y \to Z$ be $\mathcal{G}/\mathcal{H}-$measurable. Then $g \circ f : X \to Z$ is $\mathcal{F}/\mathcal{H}-$measurable,*

Proof Consider $E \in \mathcal{H}$ and show that $(g \circ f)^{-1}(E) \in \mathcal{F}$. □

Remark (1) We write $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ to be "measurable" if there is no confusion about $\mathcal{F}$ and $\mathcal{G}$.

(2) If X is a topological space, then we always take the Borel $\sigma-$algebra when we regard X as a measurable space.

(3) $(X, \mathcal{F}) \xrightarrow{f} (Y, \mathcal{G}) \xrightarrow{g} (Z, \mathcal{H})$,$Y, Z$ are topological space.If g is continuous, f is measurable, then $(g \circ f)$ is measurable.

$\mathbb{C} = \mathbb{R}^2$ as a topological space. Can consider the Borel $\sigma-$algebra on $\mathbb{C}$.

Corollary (a) If $u, v : (X, \mathcal{F}) \to \mathbb{R}$ are measurable then so is $f(x) = u(x) + \mathrm{i}v(x), f : (X, \mathcal{F}) \to \mathbb{C}$;

(b) If $f : (X, \mathcal{F}) \to \mathbb{C}$, then so are $\mathrm{Re}(f), \mathrm{Im}(f), |f| : (X, \mathcal{F}) \to \mathbb{R}$;

(c) If $f, g : (X, \mathcal{F}) \to \mathbb{C}$ are measurable, then so are fg and $f + g$;

(d) If $f : (X, \mathcal{F}) \to \mathbb{C}$ is measurable, then $\exists$ measurable $\alpha : (X, \mathcal{F}) \to \mathbb{C}$, s.t. $|\alpha| = 1, f = \alpha\,|f|$.

Proof of (d): Let $Z = \{x \in X : f(x) = 0\}$. Set

$$\alpha(x) = \frac{f(x) + \chi_Z(x)}{|f(x) + \chi_Z(x)|} = \begin{cases} \dfrac{f(x)}{|f(x)|} & \text{for } f(x) \neq 0; \\ 1 & \text{for } f(x) = 0; \end{cases} \tag{1.8}$$

Clearly, $Z = f^{-1}(\{0\})$ is measurable, also, $z \mapsto \dfrac{z}{|z|}$ is measurable on X. By previous proposition 1.4, α is measurable on X. It's easy to check that $|\alpha| = 1$ and $f(x) = \alpha\,|f(x)|$.

Definition 1.6 *Let X be an arbitrary set. Let $\mathcal{E} \subset \mathcal{P}(X)$ be an arbitrary collection of subsets of X, $\exists!$ smallest $\sigma-$algebra on X that contains $\mathcal{E}$.*

Proof (Non constructive) Consider $\Sigma := \{\mathcal{G} \subset \mathcal{P}(X) : \text{G is a } \sigma-\text{algebra}; \mathcal{G} \supset \mathcal{E}\}.\Sigma \neq \varnothing$, since $\mathcal{P}(X) \in \Sigma$. We take $\mathcal{F} := \bigcap \Sigma = \bigcap\limits_{\mathcal{G} \in \Sigma} \mathcal{G}$. Clearly $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}$ is the minimal element in Σ(w.r.t[2] the ordering of inclusion of sets). It remains to check that Σ is a $\sigma-$algebra.

[2] w.r.t is the abbreviation of "with respect to".

Indeed,

(1) $X \in \mathcal{F}$ since $X \in \mathcal{G}, \forall \mathcal{G} \subset \Sigma$.

(2) (Closure under complement) Take $A \in \mathcal{F} := \bigcap \Sigma$.Then $A \in \mathcal{G}$ for all $\mathcal{G} \in$ Z. Since each such $\mathcal{G}$ is a $\sigma-$ algebra, $A^c = X \setminus A \in \mathcal{G} \Rightarrow A^c \in \bigcap \Sigma =: \mathcal{F}$.

(3) (Closure under countable union) Let $A_1, A_2, \cdots, \in \mathcal{F}$. By the definition of $\mathcal{F}$, $A_j \in \mathcal{G}, \forall \mathcal{G} \in \Sigma$ and $j \in \mathbb{N}$. Then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{G}, \forall \mathcal{G} \in \Sigma$. So $\sum_{j=1}^{\infty} A_j \in \bigcap \Sigma = \mathcal{F}$. □

Example If $(X, \mathcal{T})$ is a topological space, $\sigma(\mathcal{T})$ is the Borel $\sigma-$algebra.

Proposition 1.5 *Let $(X, \mathcal{F})$ be a measurable space. Let $Y = [-\infty, +\infty]$(extended real line) and let $f : X \to Y$ be a function.*

(i) If $f^{-1}(]\alpha, +\infty]) \in \mathcal{F}, \forall \alpha \in \mathbb{R}$, then f is measurable.

(ii) If $f^{-1}([q, +\infty]) \in \mathcal{F}, \forall q \in \mathbb{Q}$, then f is measurable.

Remark For (i) use the following:

Claim Any open set in Y is a countable union of "open interval".

For (ii), $\forall \alpha \in \mathbb{R}, \exists \{q_j\} \subset \mathbb{Q}, q_j \searrow \alpha (q_j \gneqq \alpha)$. Then it follows from (i) as $\bigcup_{j=1}^{\infty}[q_j, +\infty] =]\alpha, +\infty]$.

Theorem 1.1 *If $f_n : (X, \mathcal{F}) \to [-\infty, +\infty]$ are Borel measurable, then so are $g := \sup_{n \geqslant 1} f_n, h := \limsup_{n\to\infty} f_n$.*

Remark $g(x) = \left(\sup_{n\geqslant 1} f_n\right)(x) := \sup_{n\geqslant 1} f_n(x)$.

Proof : By proposition 1.5, it suffices to prove that $g^{-1}(]\alpha, +\infty]) \in \mathcal{F}$: Note that

$$\begin{aligned} x \in g^{-1}(]\alpha, +\infty]) &\Leftrightarrow \sup f_n(x) = g(x) > \alpha \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, f_N(x) > \alpha - \varepsilon \\ \Leftrightarrow x \in \bigcup_{N\in\mathbb{N}} f_N^{-1}(]\alpha - \varepsilon, +\infty]) &\Leftrightarrow x \in \bigcap_{\varepsilon\in[0,\frac{1}{100}]\cap\mathbb{Q}} \bigcup_{N\in\mathbb{N}} f_N^{-1}(]\alpha - \varepsilon, +\infty]) \in \mathcal{F}. \end{aligned} \tag{1.9}$$

Recall for $\{a_n\} \subset \mathbb{R}$,

$$\limsup_{n\to\infty} a_n := \inf_{n\in\mathbb{N}} \left(\sup_{k\geqslant n} a_k\right) = \sup\{\lim_{j\to\infty} a_{n_j} : \{a_{n_j}\}\text{is a subsequence of } \{a_n\}^{\#}\} \tag{1.10}$$

$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n = L$.

Corollary If $f, g : (\mathrm{Z}, \mathcal{F}) \to [-\infty, +\infty]$ are measurable, then so are $f \vee g = \max\{f, g\}, f \wedge$

$g = \min\{f, g\}, f^+ = f \vee 0, f^- = -(f \wedge 0)$.

Remark $f = f^+ - f^-$.

Definition 1.7 $f : (X, \mathcal{F}) \to \mathbb{C}$ *is a simple function if and only if* $f(X)$ *is finite.*

Proposition 1.6 *Let* $f : X \to [0, +\infty]$ *be measurable. There exists* $\{\varphi_n\} : X \to [0, +\infty]$ *s.t.*

$$\begin{cases} 0 \leqslant \varphi_1(x) \leqslant \varphi_2(x) \leqslant f(x), & \forall x \in X, \\ \varphi_n(x) \to f(x) & \text{as } n \to \infty. \end{cases} \tag{1.11}$$

PROOF Consider dyadic rationals: for $n \in \mathbb{N}$, set $\mathcal{D}_n = \left\{ \frac{j}{2^n} : j \in \mathbb{N}_0 \right\}$. For each $y \in [0, +\infty[$, $\exists!\, j = j(n, y) \in \mathbb{N}$, s.t. $\frac{j}{2^n} \leqslant j < \frac{j+1}{2^n}$.

Define

$$\varphi_n(x) := \begin{cases} \dfrac{j(f(x), n)}{2^n} & \text{if } 0 \leqslant f(x) < n; \\ n & \text{if } n \leqslant f(x) < +\infty; \end{cases} \tag{1.12}$$

Check φ_n (1) simple; (2)$\varphi_n \leqslant \varphi_{n+1}$;(3) $\phi_n \to f$, $|\phi_n(x) - f(x)| \leqslant \frac{1}{2^n} \to 0$.

1.2 Measure Space

MEASURE: THE VOLUME OF A SET IN A MEASURABLE SPACE

Definition 1.8 *A (Non-Negative) measure is a function* $\mu : \mathcal{F} \to [0, \infty]$ *where* $(X, \mathcal{F})$ *is a measurable space s.t. for disjoint* $\{A_j\}_1^\infty \subset \mathcal{F}$*, we have* $\mu\Big(\coprod_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j)$*. Call* $(X, \mathcal{F}, \mu)$ *a measure space.*

Remark (1)A family of sets $\mathcal{G}$ is disjoint if and only if $\forall A, B \in \mathcal{G}, A \cap B = \varnothing$;

(2) $\coprod$ (square cup) is for disjoint union.

Definition 1.9 *A real/signed measure* $\mu : \mathcal{F} \to [-\infty, +\infty]$ *is given by* $\mu = \mu^+ - \mu^-$ *for measure* μ^+, μ^-*(No"$\infty - \infty$"). A complex measure* $\mu : \mathcal{F} \to \mathbb{C}$ *is given by* $\mu = \lambda + \mathrm{i}\eta$ *for real measure* $\lambda, \eta : \mathcal{F} \to]-\infty, +\infty[$ *.*

Remark For any measure μ on $(X, \mathcal{F})$, always require $\exists A \in \mathcal{F}, A \neq \varnothing$ s.t. $\mu(A) < \infty$.

Property 1.1 *Let $(X,\mathcal{F},\mu)$ be measure space, then (1) $\mu(\varnothing)=0$; (2) Monotonicity: $A\subset B\Rightarrow \mu(A)\leqslant\mu(B)$; (3) If $A_1\subset A_2\subset A_3\subset\cdots$ in $\mathcal{F}, A=\bigcap\limits_{n=1}^{\infty}A_n$, then $\mu(A_n)\nearrow\mu(A)$ as $n\to\infty$.(4) Let $A_1\supset A_2\supset A_3\supset\cdots$ in $\mathcal{F}, A=\bigcap_{n=1}^{\infty}A_n$, and $\mu(A_1)<\infty$, then $\mu(A_n)\searrow\mu(A)$.*

Proof (1)Let $A\in\mathcal{F}$ satisfies $\mu(A)<\infty$, so $\mu(A)=\mu(A\cup\varnothing)=\mu(A)+\mu(\varnothing)\Rightarrow\mu(\varnothing)=0$;

(2) $\mu(B)=\mu(A\cup(B\setminus A))=\mu(A)+\mu(B\setminus A)$.

(3)Define $B_1=A_1,B_2=A_2\setminus A_1,B_3=A_3\setminus A_2$. Then $\{B_j\}_{j=1}^{n}$ is disjoint, and $\coprod\limits_{j=1}^{\infty}B_j=A_n$, thus,

$$\mu(A_n)=\mu\Big(\coprod_{j=1}^{n}B_j\Big)=\sum_{j=1}^{n}\mu(B_j)\nearrow\sum_{j=1}^{\infty}\mu(B_j)=\mu\Big(\coprod_{j=1}^{\infty}B_j\Big)=\mu(A). \tag{1.13}$$

(4) Let $C_i=A_1\setminus A_i$, so $\varnothing=C_1\subset C_2\subset\cdots\subset A_1$. Then $\mu(A_n)=\mu(A_1)-\mu(C_n)$. By (3), $\mu(C_n)\nearrow\mu\Big(\bigcup\limits_{k=1}^{\infty}C_k\Big)=\mu\Big(\bigcup\limits_{k=1}^{\infty}[A_1\setminus A_k]\Big)=\mu\Big(A_1\setminus\bigcap\limits_{k=1}^{\infty}A_k\Big)$. Since $\mu(A_1)<\infty$, we get $\mu(A_n)\searrow\mu\Big(\bigcap_{k=1}^{\infty}A_k\Big)$. □

Remark Consider $A_n=\{n,n+1,n+2,\cdots\},\mu(E)=\#$ elements in $E,\mu(A_n)=\omega.\forall n\in\mathbb{N}$, but $\bigcap\limits_{n=1}^{\infty}A_n=\varnothing$.

Examples (0) (Counting measure) X =set, $\mathcal{F}=\mathcal{P}(X)$,

$$\mathcal{H}^0(E)=\begin{cases}\#\text{of elements in } E\ , & \text{if } E \text{ finite},\\ \infty, & \text{if } E \text{ infinite}.\end{cases} \tag{1.14}$$

(1)Lebesgue measure $\mathcal{L}^1$ on $\mathbb{R}^n$. If $n=1$ and $\mathcal{L}^1$ on $[0,n]$, then $\sigma-$algebra $\mathcal{F}=\sigma(B([0,\infty[)\cup\{\infty\})$;

(2)Dirac Let X be a set $\mathcal{F}=\mathcal{P}(X)$. Let $X_0\in X$, delta measure $\delta_{x_0}:\mathcal{P}(X)\to[0,\infty]$,

$$E\mapsto\begin{cases}1, & \text{if } x_0\in E,\\ 0, & \text{if } x_0\notin E.\end{cases} \tag{1.15}$$

(3)A finite measure μ on $(X,\mathcal{F})$ with $\mu(X)=1$ is a probability measure;

(4)Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $A \in \mathcal{F}$, then $\mathcal{F}\Big|_A := \{A \cap E : E \subset \mathcal{F}\}$ is $\sigma-$ algebra.

(5)Let $(X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)$ be measure spaces. There is a measure space $(\mathcal{X} \times \mathcal{Y}, \Sigma, \lambda).\Sigma = \sigma(\{F \times G \subset X \times Y : F \in \mathcal{F}, G \in \mathcal{G}\}), \lambda(F \times G) = \mu(F) \times \nu(G)$. Product measure space $\Sigma = \mathcal{F} \otimes \mathcal{G}, \lambda = \mu \otimes \nu$.

1.3 Integration

Goal: Define $\int_X f \mathrm{d}\mu$ for $f : (X, \mathcal{F}, \mu) \to \mathbb{C}$. Idea : partition the range of $f(\chi_{[0,1] \cap \mathbb{Q}})$. Nice functions : f that takes only finitely many values. By approximation, can integrate measurable functions.

Definition 1.10 *Let $\mathcal{P} : (X, \mathcal{F}, \mu) \to [0, \infty[$ be a measurable simple functions, i.e. $p(x) = \sum_N^{i=1} a_i \chi_{E_i}(x)$ for $a_i \in [0, \infty[, E_i \in \mathcal{F}, N \in \mathbb{N}$. Define $\int_X p \mathrm{d}\mu := \sum_{i=1}^N a_i \mu(E_i)$.*

Idea:

$$\int_X \phi \mathrm{d}\mu = \int_X \Big(\sum_{i=1}^N a_i \chi_{E_i} \Big) \mathrm{d}\mu \overset{\text{should}}{=\!=} \sum_N^{i=1} a_i \int_X \chi_{E_i} \mathrm{d}\mu = \sum_N^{i=1} a_i \int_{E_i} 1 \mathrm{d}\mu = \sum_{i=1}^N a_i \mu(E_i) \tag{1.16}$$

Definition 1.11 *Let $f : (X, \mathcal{F}, \mu) \to [0, \infty]$ be measurable. Define $\int_X f \mathrm{d}\mu := \sup \Big\{ \int_X \varphi \mathrm{d}\mu : \varphi \text{ simple}, 0 \leqslant \varphi \Big\} (\text{pointwise})$.*

Idea: $\exists$ simple $\{\varphi_j\} \nearrow f$.

Definition 1.12 *Let $f : (X, \mathcal{F}, \mu) \to [0, \infty]$ be measurable. Define*

$$\int_X f \mathrm{d}\mu := \int_X f^+ \mathrm{d}\mu - \int_X f^- \mathrm{d}\mu \tag{1.17}$$

where $f = f^+ - f^-$ for $f^\pm : (X, \mathcal{F}, \mu)$ measurable.

Definition 1.13 *Let $f : (X, \mathcal{F}, \mu) \to [0, \infty]$ be measurable. Then*

$$\int_X f \mathrm{d}\mu = \Big(\int_X \mathrm{Re}(f) \mathrm{d}\mu \Big) + \mathrm{i} \Big(\int_X \mathrm{Im}(f) \mathrm{d}\mu \Big). \tag{1.18}$$

Properties: Assume every function/set below is measurable.

(a) $f \leqslant g \Rightarrow \int_E f \mathrm{d}\mu \leqslant \int_E g \mathrm{d}\mu$;

(b) $0 \leqslant f, A \subset B \ \Rightarrow \int_A f \mathrm{d}\mu \leqslant \int_B f \mathrm{d}\mu, \int_E f \mathrm{d}\mu \equiv \int_X f \chi_E \mathrm{d}\mu$;

(c) For c =constant, $\int_E cf \mathrm{d}\mu = c \int_E f \mathrm{d}\mu$;

(d) If $f = 0$ on E, then $\int_E f \mathrm{d}\mu = 0$, even if $\mu(E) = \infty$;

(e) If $\mu(E) = 0$, then $\int_E f \mathrm{d}\mu = 0$, even if $f = \infty$.

1.4 Convergence Theorem

Problem: $f_n \to f$ (pointwise). $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu \stackrel{?}{=} \int_X f \mathrm{d}\mu \ \ \circledast$. Sufficient conditions for $\circledast$ holds true.

Theorem 1.2 (Monotone Convergence Theorem/MCT) *Let $f_n : (X, \mathcal{F}, \mu) \to [0, \infty]$ be measurable functions, if $f_1 \leqslant f_2 \leqslant f_3 \leqslant \cdots \forall x \in X$, and if $f_n(x) \to f(x)$ pointwise, then f is measurable, and $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$.*

Proof Clearly f is measurable. Also, by monotonicity of integrals we have $\int_X f_n \mathrm{d}\mu \leqslant \int_X f_{n+1} \mathrm{d}\mu$. So $\alpha := \lim_{n\to\infty} \int_X f_n \mathrm{d}\mu$ exists. Also, $\int_X f_n \mathrm{d}\mu \leqslant \int_X f \mathrm{d}\mu$, so $\alpha \leqslant \int_X f \mathrm{d}\mu$.

To show that $\alpha \geqslant \int_X f \mathrm{d}\mu$. By definition of Lebesgue integral, it suffices to prove that

$$\alpha \geqslant (1-\varepsilon) \int_X \varphi \mathrm{d}\mu, \quad \forall \varepsilon \in]0,1[, \quad \forall \text{simple } \ 0 \leqslant \varphi \leqslant f. \tag{1.19}$$

Then the theorem follows by taking $\varepsilon \searrow 0$ and supremising over φ. Fix any such φ and ε. Consider

$$E_n := \{x \in X : f_n(x) \geqslant (1-\varepsilon)\varphi(x)\} \tag{1.20}$$

By the monotonicity of $\{f_n\}$, we have $E_1 \subset E_2 \subset E_3 \subset \cdots$ since $f_n \to f, 0 \leqslant \varphi \leqslant f$, $\varepsilon \in]0,1[$, $X = \bigcup_{n=1}^{\infty}$. Note that

$$\int_X f_n \mathrm{d}\mu \geqslant \int_{E_n} f_n \mathrm{d}\mu \geqslant \int_{E_n} (1-\varepsilon)\varphi \mathrm{d}\mu = (1-\varepsilon)\int_{E_n} \varphi \mathrm{d}\mu. \tag{1.21}$$

Thus, taking $n \searrow \infty$ we obtain

$$\alpha = \lim_{n\to\infty} \int_X f_n \mathrm{d}\mu \geqslant (1-\varepsilon) \lim_{n\to\infty} \int_{E_n} \varphi \mathrm{d}\mu \xlongequal{E_n \searrow \bigcup E_j = X} (1-\varepsilon) \int_X \varphi \mathrm{d}\mu. \tag{1.22}$$

Then inequality (1.19) holds. □

Lemma 1.2 (Fatou's Lemma) *Let $f_n : (X, \mathcal{F}, \mu) \to [0, \infty]$ be measurable, then*

$$\int_X \liminf_{n\to\infty} f_n \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu. \tag{1.23}$$

PROOF Recall that $\liminf\limits_{n\to\infty} f_n = \sup\limits_{k\geqslant 1} g_k$ where $g_k = \inf_{n\geqslant k}$. So

$$\int_X g_k \mathrm{d}\mu \leqslant \int_X f_k \mathrm{d}\mu. \tag{1.24}$$

By MCT, $\lim\limits_{k\to\infty} \int_X g_k \mathrm{d}\mu$ exists and is equal to

$$\int_X \lim_{k\to\infty} g_k \mathrm{d}\mu = \int_X \liminf_{n\to\infty} f_n \mathrm{d}\mu. \tag{1.25}$$

On the RHS, pass to any convergent subsequence,

$$\int_X \liminf_{n\to\infty} f_n \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu. \tag{1.26}$$

□

Example. $X =]0,1], \mu =$ Lebesgue measure, $f_n = n\chi_{]0,\frac{1}{n}]}$. Then $f_n \to f \equiv 0$, but $\int_X f_n \mathrm{d}\mu \equiv 1, \ \forall n$.

Theorem 1.3 (Dominated Convergence Theorem/DCT) *Let $f_n : (X, \mathcal{F}, \mu) \to \mathbb{C}$ be* *measurable functions. Let $f_n \to f$ on X as $n \nearrow \infty$. Assume that there exists a function $g \in L^1(X, \mathcal{F}, \mu, \mathbb{R}_+)$, s.t. $|f_n| \leqslant g$ on X. Then $\liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$.*

Proof Let $f = \lim_{n\to\infty} f_n$. Since $|f_n| \leqslant g \in L^1(\mu)$. So $f \in L^1(\mu)$. Apply Fatou to $2g - |f_n - f|$, then

$$\begin{aligned}\int_X 2g\mathrm{d}\mu &= \int_X \lim_{n\to\infty} \Big(2g - |f_n - f|\Big)\mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X \Big(2g - |f_n - f|\Big)\mathrm{d}\mu \\ &\leqslant \int_X 2g\mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu.\end{aligned} \tag{1.27}$$

So $\limsup_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu \leqslant 0$.
Note that

$$\left|\int_X f_n \mathrm{d}\mu - \int_X f\mathrm{d}\mu\right| = \left|\int_X (f_n - f)\mathrm{d}\mu\right| \leqslant \int_X |f_n - f| \,\mathrm{d}\mu \to 0. \tag{1.28}$$

Then $\liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f\mathrm{d}\mu$. □

Remark (1) The space $L^1(X, \mathcal{F}, \mu, \mathbb{C}) := \{f : (X, \mathcal{F}, \mu) \to \mathbb{C}\text{measurable} \int_X |f| \,\mathrm{d}\mu < \infty\}$ is the space of integrable functions (summable function). Notation: $L^1(X, \mathcal{F}, \mu) := L^1(\mu)$.

Set $\|f\|_{L^1} := \int_X |f| \,\mathrm{d}\mu$. ($\|\cdot\|_{L^1} : L^1(\mu) \to \mathbb{R}_+$). Then $L^1(X, \mathcal{F}, \mu, \mathbb{C})$ is a normed vector space;

(2) Let $(X, \mathcal{F}, \mu)$ be a measure space, $E \in \mathcal{F}$ is null (w.r.t. μ) if and only if $\mu(E) = 0$. A property holds $\mu-$almost everywhere if it holds outside a $\mu-$null set. ($\mu-$ a.e.);

(3) For DCT: we can actually conclude a stronger result:

$$\int_X |f_n - f| \,\mathrm{d}\mu \to 0. \tag{1.29}$$

Proposition 1.7 (Linearity of Lebesgue Integral) *Let $f, g \in L^1(X, \mathcal{F}, \mu, \mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g \in L^1$, with $\int_X (\alpha f + \beta g)\mathrm{d}\mu = \alpha \int_X f\mathrm{d}\mu + \beta \int_X g\mathrm{d}\mu$.*

Proof Case 1: $f, g : X \to [0, \infty], \alpha, \beta \in [0, +\infty[$. Want to show:

$$\circledast \begin{cases} \int_X (f+g)\mathrm{d}\mu = \int_X f\mathrm{d}\mu + \int_X g\mathrm{d}\mu; \\ \int_X \alpha f\mathrm{d}\mu = \alpha \int_X f\mathrm{d}\mu. \end{cases} \tag{1.30}$$

Now we have already proved $\circledast$ for simple functions, then

$$\begin{aligned} \int_X \alpha f\mathrm{d}\mu &:= \sup\Big\{ \int_X \tilde{\varphi}\mathrm{d}\mu : 0 \leqslant \tilde{\varphi} \leqslant \alpha f, \tilde{\varphi}\text{simple}\Big\} \\ &\xlongequal{\varphi=\tilde{\varphi}/\alpha(\alpha\neq 0)} \sup\Big\{ \int_X \alpha\tilde{\varphi}\mathrm{d}\mu : 0 \leqslant \varphi \leqslant f, \varphi\text{simple}\Big\} = \alpha \int_X f\mathrm{d}\mu. \end{aligned} \tag{1.31}$$

For addition, let $\{\varphi_n\}, \{\psi_n\}$ be simple functions s.t. $\varphi_n \nearrow f, \psi_n =\nearrow g$. Then,

$$\begin{aligned} \int_X (f+g)\mathrm{d}\mu &\xlongequal{\text{MCT}} \lim_{n\to\infty} \int_X (\varphi_n + \psi_n)\mathrm{d}\mu \xlongequal{\circledast\text{for simple functions}} \lim_{n\to\infty} \Big\{\Big(\int_X \varphi_n\mathrm{d}\mu\Big) + \Big(\int_X \psi_n\mathrm{d}\mu\Big)\Big\} \\ &= \lim_{n\to\infty} \int_X \varphi_n\mathrm{d}\mu + \lim_{n\to\infty} \int_X \psi_n\mathrm{d}\mu \xlongequal{\text{MCT}} \lim_{n\to\infty} \int_X f\mathrm{d}\mu + \lim_{n\to\infty} \int_X g\mathrm{d}\mu. \end{aligned} \tag{1.32}$$

For f, g simple, $f \in \sum_{i=1}^N \alpha_i \chi_{E_i}, g \in \sum_{j=1}^M \beta_j \chi_{F_j}$, where $\{E_i\}, \{F_j\}$ are disjoint.
Label: $E_i \cap F_j = \Omega_{ij}$, then $\{\Omega_{ij}, E_i \setminus \Omega_{ij}, F_j \setminus \Omega_{ij}\}$ disjoint, we have

$$f + g = \sum_{1\leqslant i\leqslant N, 1\leqslant j\leqslant M} \alpha_i \chi_{E_i\setminus\Omega_{ij}} + \beta_j \chi_{F_j\setminus\Omega_{ij}} + (\alpha_i + \beta_j)\chi_{\Omega_{ij}}. \tag{1.33}$$

Thus, by definition of $\int$ for simple functions,

$$\begin{aligned} \int_X (f+g)\mathrm{d}\mu &= \sum_{i,j} \alpha_i \mu(E_i \setminus \Omega_{ij}) + \beta_j \mu(E_j \setminus \Omega_{ij}) + (\alpha_i + \beta_j)\mu(\Omega_{ij}) \\ &= \sum_{i,j} \alpha_i[\mu(E_i \setminus \Omega_{ij}) + \mu(\Omega_{ij})] + \beta_j[\mu(F_j \setminus \Omega_{ij}) + \mu(\Omega_{ij})] \\ &= \sum_{i=1}^{\infty} \alpha_i(E_i) + \sum_{j=1}^{\infty} \beta_j \mu(F_j) = \int_X f\mathrm{d}\mu + \int_X g\mathrm{d}\mu. \end{aligned} \tag{1.34}$$

Exercise $\alpha \int_X f \mathrm{d}\mu = \int_X \alpha f \mathrm{d}\mu$ for simple f.
Case 2: f, g real-valued. Let $f = f^+ - f^-, g = g^+ - g^-$, then

$$\begin{aligned} f + g = f^+ + g^+ - (f^- + g^-) &= (f+g)^+ - (f+g)^- \\ \Rightarrow (f+g)^+ + f^- + g^- &= (f+g)^- + f^+ + g^+ \\ \Rightarrow \int (f+g)^+ + \int f^- + \int g^- &= \int (f+g)^- + \int f^+ + \int g^+ \end{aligned} \tag{1.35}$$

Thus

$$\int_X (f+g)\mathrm{d}\mu := \int_X [(f+g)^+ - (f+g)^-]\mathrm{d}\mu = \int f^+ - \int f^- + \int g^+ - \int g^- =: \int f + \int g. \tag{1.36}$$

Case 3: f, g complex-valued. Left as **exercise**. □

Proposition 1.8 *Let* $f_n \xrightarrow{\text{a.e.}} f, 0 \leqslant |f_n| \leqslant g_n, g_n \xrightarrow{\text{a.e.}} g$, *and* $\int g_n \to \int g (g_n, g \in L^1)$, *then* $\int f_n \to \int f$.

Proof To prove the proposition, consider

$$h_n := g_n + g - |f_n - f| \geqslant 0, \text{ a.e.} \tag{1.37}$$

Then use Fatou's Lemma:

$$\int \liminf_{n\to\infty} h_n = 2\int g \leqslant \liminf_{n\to\infty} \int h_n = 2\int g - \limsup_{n\to\infty} \int |f_n - f|. \tag{1.38}$$

□

Corollary (1) If $f_n : (X, \mathcal{F}, \mu) \to [0, \infty]$ are measurable, then $\int_X \Big(\sum_{n=1}^{\infty} f_n\Big)\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \mathrm{d}\mu$.
$S_M = \sum_{n=1}^{M} f_n$ apply MCT.

(2) If $f_n : (X, \mathcal{F}, \mu) \to \mathbb{C}$ are measurable, and if $\int_X |f_n| \mathrm{d}\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ (pointwise) converges to an L^1 function f, s.t.

$$\int_X f \mathrm{d}\mu = \sum_{n=1}^{+\infty} \int_X f_n \mathrm{d}\mu. \tag{1.39}$$

(3) If $a_{ij} \geqslant 0,\ \forall 1 \leqslant i < j$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \tag{1.40}$$

Example (1) $f : (X, \mathcal{F}, \mu) \to [0, +\infty]$ measurable. Claim $\lim\limits_{n\to\infty}\int_X n\ln\Big(1+\dfrac{f(x)}{n}\Big)\mathrm{d}\mu = \int_X f\mathrm{d}\mu$.

Proof Using MCT and consider $n\ln\Big(1+\dfrac{f(x)}{n}\Big) = \ln\Big[\Big(1+\dfrac{f(x)}{n}\Big)^n\Big] \underset{n\to\infty}{\nearrow} \ln e^{f(x)} = f(x)$.□

(2) $\sum_{n=0}^{\infty}\dfrac{(2n)!}{4^n n!\,(n+1)!} = \sum\limits_{n=0}^{\infty}\dfrac{(2n)!}{4^n (n!\,)^2}\displaystyle\int_0^1 x^n\mathrm{d}x$. By Taylor's expansion, $(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\dfrac{(2n)!}{4^n (n!\,)^2}x^n$. So by MCT,

$$\begin{aligned}\int_0^1 (1-x)^{-\frac{1}{2}}\mathrm{d}x &= \lim_{n\to\infty}\int_0^{1-\frac{1}{n}}(1-x)^{-\frac{1}{2}}\mathrm{d}x \\ &= \lim_{n\to\infty} -2(1-x)^{\frac{1}{2}}\Big|_0^{1-\frac{1}{n}} = \lim_{n\to\infty}(2-2n^{-\frac{1}{2}}) = 2.\end{aligned} \tag{1.41}$$

(3) $\lim\limits_{n\to\infty}\displaystyle\int_0^1 \frac{n^{\frac{3}{2}}xe^x}{1+n^2x^2}\mathrm{d}x$. Let $f_n(x) = \int_0^1 \dfrac{n^{\frac{3}{2}}xe^x}{1+n^2x^2}\chi_{[0,1]}\mathrm{d}x$. Although $0 \leqslant f_n \to 0$ as $n\to\infty$, $\{f_n\}$ is not uniformly bounded:

$$f_n\Big(\frac{1}{n}\Big) = \frac{n^{\frac{3}{2}}n^{-1}e^{n^{-1}}}{1+n^2(n^{-1})^2} = \frac{\sqrt{n}e^{\frac{1}{n}}}{2}. \tag{1.42}$$

So DCT can't be applied in this way.

Consider $g_n(x) = e\dfrac{n^{\frac{3}{2}}x}{1+n^2x^2}\chi_{[0,1]}$, then $0 \leqslant f_n(x) \leqslant g_n(x),\ \forall x\in\mathbb{R}$. Note that

$$\begin{aligned}\int_{\mathbb{R}} g_n\mathrm{d}\mu &= e\int_0^1 \frac{n^{\frac{3}{2}}x}{1+n^2x^2}\mathrm{d}x \\ &\xlongequal{y=1+n^2x^2} \frac{e}{2\sqrt{n}}\int_1^{1+n^2}\frac{\mathrm{d}y}{y} = \frac{e}{2}\frac{\ln(n^2+1)}{\sqrt{n}} \to 0 \text{ as } n\nearrow\infty.\end{aligned} \tag{1.43}$$

Since $\ln(n^2+1)\approx\ln(n^2) = 2\ln n$, $\dfrac{\ln n}{n^\alpha}\to 0$ as $n\to\infty$, $\forall\alpha>0$. Apply proposition 1.8,

$g \equiv f \equiv 0$ zero function to conclude.

(4) Compute $\lim_{n\to\infty} \int_0^{n\pi} \cos\left(\frac{x}{2n}\right) x^2 e^{-x^3} \mathrm{d}x = \frac{1}{3}$.(DCT)

1.5 Completion of Measure Space

"Completeness": Banach spaces are "complete", $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is "complete" in square-integrable functions.

"No holes": $\mathbb{R}$ is complete but $\mathbb{Q}$ is not.

Definition 1.14 *A measure is $(X, \mathcal{F}, \mu)$ is **complete** if and only if every subset of null sets are in $\mathcal{F}$.*

Note If $N \in \mathcal{F}$ s.t. $\mu(N) = 0$, and if $\hat{N} \subset N$ in general we cannot conclude $\hat{N} \in \mathbb{F}$. But if $\hat{N} \in \mathcal{F}$ and $\hat{N} \subset N \Rightarrow \mu(\hat{N}) = 0$.

Definition 1.15 (also as theorem) *Let $(X, \mathcal{F}, \mu)$ be a measure space, set $\overline{\mathcal{F}} := \{E \cup Z : E \in \mathcal{F}$ and $Z \subset N \in \mathcal{F}$ s.t. $\mu(N) = 0\}$, $\overline{\mu}(E \cup Z) := \mu(E)$. Then $(X, \overline{\mathcal{F}}, \overline{\mu})$ is the **completion** of $(X, \mathcal{F}, \mu)$. It satisfies*

(i) $\mathcal{F}$ is the minimal $\sigma-$algebra containing $\mathcal{F}$ and all subsets of $\mu-$null sets;

(ii) $\overline{\mu}$ is a measure on $(X, \overline{\mathcal{F}})$;

(iii) $\overline{\mu}$ is complete w.r.t. $(X, \overline{\mathcal{F}})$.

Example Let $\mathcal{B}_{\mathbb{R}^n}$ is the **Borel $\boldsymbol{\sigma}-$algebra** on $\mathbb{R}^n$. Then $\overline{\mathcal{B}_{\mathbb{R}^n}}$ is called the **Lebesgue $\boldsymbol{\sigma}-$algebra**.

Fact $\mathcal{B}_{\mathbb{R}^n} \subsetneq \overline{\mathcal{B}_{\mathbb{R}^n}}$.

PROOF Step 0. $\overline{\mu}$ is well defined, i.e. if $E_1 \cup Z_1 = E_2 \cup Z_2$ where $E_i \in \mathcal{F}, Z_i \setminus N_i \in \mathcal{F}, \mu(N_i) = 0, i \in \{1, 2\}$, then $\mu(E_1) = \mu(E_2)$. Indeed, $\mu(E_1) = \mu(E_1 \cap E_2) + \mu(E_1 \setminus E_2)$, but $E_1 \setminus E_2 \in \mathcal{F}$ and $E_1 \setminus E_2 \subset (E_1 \cup Z_1) \setminus E_2 \subset Z_2 \subset N_2 \in \mathcal{F}$. By symmetry, the same holds for $E_2 \setminus E_1$. So $\mu(E_1) = \mu(E_2)$.

Step 1. $\overline{\mathcal{F}}$ is a σ.

(1) $X \in \overline{\mathcal{F}}$ since $X \in \mathcal{F}$.

(2) If $A \in \overline{\mathcal{F}}$, then we need to show that $A^c \in \overline{\mathcal{F}}$. By definition, $A = E \cup Z$ for $E \in \mathcal{F}$, $Z \subset N \in \mathcal{F}, \mu(N) = 0$. Then $A^c = (E \cup Z)^c = E^c \cap Z^c = (E^c \cap N^c) \cup (N \setminus Z)$. Here

$E^c \cap N^c = (E \cap N)^c \in \mathcal{F}$, and $N \setminus Z \subset N$, so $A^c \in \overline{\mathcal{F}}$ by definition of $\mathcal{F}$.

(3) If $A_1, A_2, \cdots \in \overline{\mathcal{F}}$, want to show $\bigcup_{i=1}^{\infty} A_i \in \overline{\mathcal{F}}$. Indeed, $\forall i \in \mathbb{N}, A_i = E_i \cup Z_i$ for $Z_i \subset N_i \in \mathcal{F}, \mu(N_i) = 0$. Then

$$\bigcup_{i=1}^{\infty} A_i = \Big(\bigcup_{i=1}^{\infty} E_i\Big) \cup \Big(\bigcup_{i=1}^{\infty} Z_i\Big)^{3} \tag{1.44}$$

since $\mu\Big(\bigcup_{i=1}^{\infty} N_i\Big) \leqslant \sum_{i=1}^{\infty} \mu(N_i) = 0$. Then $\bigcup_{i=1}^{\infty} A_i \in \overline{\mathcal{F}}$.

Step 2.(Minimality) By construction, $\overline{\mathcal{F}} \supset \mathcal{F}_{\min}$, where $\mathcal{F}_{\min} :=$ the minimal $\sigma-$algebra containing $\mathcal{F}$ and all subsets of null sets. $\overline{\mathcal{F}} \subset \mathcal{F}_{\min}$ is also trivial by definition of $\overline{\mathcal{F}}$.

Step 3. $\overline{\mu}$ is a measure. Clearly $\overline{\mu}(\varnothing) = 0$. Now let $\{A_i\}_{i=1}^{\infty} \subset \overline{\mathcal{F}}$ be disjoint. So $A_i = E_i \cup Z_i$ where $E_i \in \mathcal{F}, Z_i \subset N_i \in \mathcal{F}$ for $\mu(N_i) = 0$. Then $\overline{\mu}\Big(\bigsqcup_{i=1}^{\infty} A_i\Big) = \overline{\mu}\Big(\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{i=1}^{\infty} Z_i\Big) :\xlongequal{\text{def of } \overline{\mu}} \mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i) \xlongequal{\text{def of } \overline{\mu}} \sum \overline{\mu}(A_i)$.

Step 4: $\overline{\mu}$ is complete. Assume that $F \in \overline{\mathcal{F}}.\overline{\mu}(F) = 0, E \subset F$. Want to show that $E \in \overline{\mathcal{F}}$.

Indeed, $F = G \cup Z$ where $G \in \mathcal{F}, Z \subset N \in \mathcal{F}$ with $\mu(N) = 0$. Then $E \subset G \cup Z \subset G \cup N \in \mathcal{F}$. □

1.6 Construction of Measure

Question Given $(X, \mathcal{F})$ measurable space, is there (a canonical way to construct) a (nontrivial) measure?

Definition 1.16 *Let X be a subset. An outer measure μ_* is a function $\mu_* : \mathcal{P}(X) \to [0, +\infty]$ s.t.*

(i) $\mu_*(\varnothing) = 0$;

(ii) $E \subset F \Rightarrow \mu_*(E) \leqslant \mu_*(F)$;

---

[3] $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$, $\bigcup_{i=1}^{\infty} Z_i \subset \bigcup_{i=1}^{\infty} N_i \in \mathcal{F}$ and is μ-null.

(*iii*) ($\sigma-$*subadditivity*) $\forall E_1, E_2, \dots \subset X$,

$$\mu_*\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leqslant \sum_{i=1}^{\infty} \mu_*(E_i). \tag{1.45}$$

Given an outer measure, the following "separation criterion" due to CARATHÉODORY gives a natural $\sigma-$ algebra $\mathcal{M}_{\mu_*} \subset \mathcal{P}(X)$ s.t. $\mu_*|_{\mathcal{M}_{\mu_*}}$ is a complete measure.

Definition 1.17 $\mathcal{M}_{\mu_*} := \{E \subset X : \forall A \in X, \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \setminus E)\}$.

Remark (1) If μ_* is a measure, then this condition holds.

(2)$\mu(A) \leqslant \mu_*(A \cap E) + \mu_*(A \setminus E)$ is trivial. So $\mathcal{M}_{\mu_*} := \{E \subset X : \forall A \in X, \mu_*(A) \geqslant \mu_*(A \cap E) + \mu_*(A \setminus E)\}$.

(3) If $N \subset X$ satisfies $\mu_*(N) = 0$, then $N \in \mathcal{M}_{\mu^*}$. Indeed, for any $A \subset X$, $\mu_*(A \cap N) \leqslant \mu_*(N) = 0$. Also, $\mu_*(A) \leqslant \mu_*(A \setminus N) + \mu_*(N) \leqslant \mu_*(A \cap N^c) \leqslant \mu_*(A)$. So $\mu_*(A) \leqslant \mu_*(A \setminus N) + \mu_*(A \cap N)$.

(4) $\mathcal{M}_{\mu_*}$ is closed under complement.

Theorem 1.4 $(X, \mathcal{M}_{\mu_*}, \mu_*|_{\mathcal{M}_{\mu_*}})$ *is a complete measure space.*

PROOF By the previous remarks, it remains to prove:

(1)$\mathcal{M}_{\mu_*}$ is a $\sigma-$ algebra;

(2)μ_* is countably addition over $\mathcal{M}_{\mu_*}$.

Step 1: $\mathcal{M}_{\mu_*}$ is closed under finite unions, and μ_* is finitely additive.

Let $E_1, E_2 \in \mathcal{M}_{\mu_*}$ and let $A \subset X$ be arbitrary. We need to check that $\mu_*(A) \geqslant \mu_*((E_1 \cup E_2) \setminus A)$. Indeed, [4]

$$\begin{aligned} \mu_*(A) \geqslant & \mu_*(A \cap E_2) + \mu_*(A \cap E_2^c) \\ \geqslant & \mu_*(E_1 \cap A \cap E_2) + \mu_*(E_1^c \cap A \cap E_2) + \mu_*(E_1 \cap A \cap E_2^c) + \mu_*(E_1^c \cap A \cap E_2^c) \\ \geqslant & \mu_*(A \cap (E_1 \cup E_2)) + \mu_*(A \setminus (E_1 \cup E_2)). \end{aligned} \tag{1.46}$$

So $E_1 \cup E_2 \in \mathcal{M}_{\mu_*}$. If furthermore $E_1 \cap E_2 = \varnothing$, then

$$\mu_*(E_1 \cup E_2) = \mu_*(E_1 \cap (E_1 \cup E_2)) + \mu_*((E_1 \cup E_2) \setminus E_1) = \mu_*(E_1) + \mu_*(E_2). \tag{1.47}$$

[4] $E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

Step 2: $\mathcal{M}_{\mu_*}$ is closed under countable unions of disjoint subsets, and $\mu_*|_{\mathcal{M}_{\mu_*}}$ is $\sigma-$additive. Let $\{E_1, E_2, \cdots\} \subset \mathcal{M}_{\mu_*}$ be disjoint, set $G_n := \bigcup_{i=1}^{n} G_i$. By step 1, $G_n \in \mathcal{M}_{\mu_*}, \forall n \in \mathbb{N}$. Note that for $A \subset X$:

$$\begin{aligned}\mu_*(G_n \cap A) =& \mu_*(E_n \cap G_n \cap A) + \mu_*(E_n^c \cap G_n \cap A)\\ =& \mu_*(E_n \cap A) + \mu_*(G_{n-1} \cap A) = \sum_{j=1}^{n} \mu_*(E_j \cap A).\text{(by induction)}\end{aligned} \tag{1.48}$$

So $\forall A \subset X$,

$$\mu_*(A) = \mu_*(G_n \cap A) + \mu_*(G_n^c \cap A) = \sum_{j=1}^{n} \mu_*(E_j \cap A) + \mu_*(G_n^c \cap A) \tag{1.49}$$

for $n \in \mathbb{N}$. Sending $n \to \infty$, we get

$$\mu_*(A) \geqslant \mu_*(G_\infty \cap A) + \mu_*(G_\infty^c \cap A). \tag{1.50}$$

Thus $G_\infty \in \mathcal{M}_{\mu_*}$. Finally, take $A = G_\infty$ in (1.49), we get

$$\mu_*(G_\infty) \geqslant \mu_*(G_\infty \cap E_j) + \mu_*(G_\infty \cap G_\infty^c). \tag{1.51}$$

$$\mu_*(A \cap G_\infty) = \mu_*\Big(A \cap \Big(\bigcup_{j=1}^{\infty} E_j\Big)\Big) = \mu_*\Big(\bigcup_{j=1}^{\infty} (A \cap E_j)\Big) \leqslant \sum_{j=1}^{\infty} \mu_*(A \cap E_j). \tag{1.52}$$

So $\mu_*(G_\infty) = \sum_{j=1}^{\infty} \mu_*(E_j)$. $\sigma-$ additive. □

Definition 1.18 *Let X be a set $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra on X, if $\mathcal{A}(\neq \varnothing)$ is closed under complement, finite union and intersection. $\mu_0 : \mathcal{A} \to [0, +\infty]$ is a* ***pre-measure*** *if:*

(i) $\mu_0(\varnothing) = 0$;

(ii) If $\{E_j\}_1^\infty \subset \mathcal{A}$ is a countable disjoint collection, and if $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, then $\mu_0\Big(\bigsqcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \mu_0(E_j)$.

Theorem 1.5 *Let $(X, \mathcal{A}, \mu_0) = $ (set,algebra,pre-measure). Then $\mu_* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure, where*

$$\mu_*(E) := \inf\Big\{\sum_{j=1}^{\infty} \mu_0(E_j) : E \subset \bigcup_{j=1}^{\infty} E_j, E_j \in \mathcal{A}\Big\}. \tag{1.53}$$

Moreover, $\mu_*|_{\mathcal{A}} = \mu_0$ and $\mathcal{A} \subset \mathcal{M}_{\mu_*}$

Proof Step 1: μ_* defines an outer measure.(**exercise**).

Step 2: $\mu_*|_{\mathcal{A}} = \mu_0$. Indeed, $\mu_*(E) \leqslant \mu_0(E), \forall E \subset A$, since E covers itself. For "$\geqslant$", let $E \subset \bigcup_{j=1}^{\infty} E_j$ for $E_j \in \mathcal{A}$. Then $E = \bigsqcup_{j=1}^{\infty} \Big(E \cap \Big(E_j \setminus \bigcup_{l=1}^{j-1} E_l\Big)\Big)$. By definition, $\tilde{E}_j \in \mathcal{A}, \forall j$, and

$$\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(\tilde{E}_j) \leqslant \sum_{j=1}^{\infty} \mu_0(E \cap E_j) \leqslant \sum_{j=1}^{\infty} \mu_0(E_j). \tag{1.54}$$

Take inf over all such covers $\{E_j\}$, we get $\mu_0(E) \leqslant \mu_*(E)$.

Step 3: $\mathcal{A} \subset \mathcal{M}_{\mu_*}$. Take any $E \in \mathcal{A}, A \in \mathcal{P}(X)$, and $\varepsilon > 0$. We show that $\mu_*(A) \geqslant \mu_*(A \cap E) + \mu_*(A \setminus E) - \varepsilon$. Indeed, by definition of μ_*, $\exists \{E_j\} \subset \mathcal{A}$, s.t. $A \subset \bigcup_{j=1}^{\infty} E_j$ with

$$\mu_*(A) \geqslant \sum_{j=1}^{\infty} \mu_0(E_j) - \varepsilon = \sum_{j=1}^{\infty} [\mu_0(E \cap E_j) + \mu_0(E^c \cap E_j)] - \varepsilon \tag{1.55}$$

But $\{E \cap E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ covers $A \cap E$, $\{E^c \cap E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ covers $A \cap E^c$. So by definition of μ_*,

$$\mu_*(A) \geqslant \mu_*(E \cap A) + \mu_*(E^c \cap A) - \varepsilon. \tag{1.56}$$

Here $\varepsilon > 0$ is arbitrary, so $E \in \mathcal{M}_{\mu_*}$. □

Recall The complete measure $\mu = \mu_*|_{\mathcal{M}_{\mu_*}}$ extends the pre-measure μ_0.

Question Is μ the unique complete measure on $\mathcal{M}_{\mu_*}$ extending μ_0?

Claim μ on $(X, \sigma(A))$ is the unique measure extending μ_0 provided that μ is $\sigma-$ finite.

Definition 1.19 *A measure ν on $(X, \mathcal{F})$ is* ***$\sigma-$finite*** *if and only if $X = \bigcup_{i=1}^{\infty} X_i$ for $X_i \in \mathcal{F}$ s.t. $\nu(X_i) < \infty$, $\forall i$.*

In summary, we have proved HAHN-KOLMOGOROV Extension Theorem :

Theorem 1.6 (Hahn-Kolmogorov Extension Theorem) *Let $(X, \mathcal{A}, \mu_0)$ be (set,algebra,pre-measure). There is a canonical extension into a measure space $(X, \sigma(\mathcal{A}))$ in the following sense: $\exists$ outer measure μ_* over X extending μ_0 s.t. its "measurable sub-measure space"*

$$(X, \mathcal{M}_{\mu_*}, \overline{\mu} = \mu_*|_{\mathcal{M}_{\mu_*}}) \supset (X, \sigma(\mathcal{A}), \mu), \tag{1.57}$$

such $(X, \sigma(\mathcal{A}), \mu)$ is unique if μ is $\sigma-$ finite. $\overline{\mu}$ is the completion of μ.

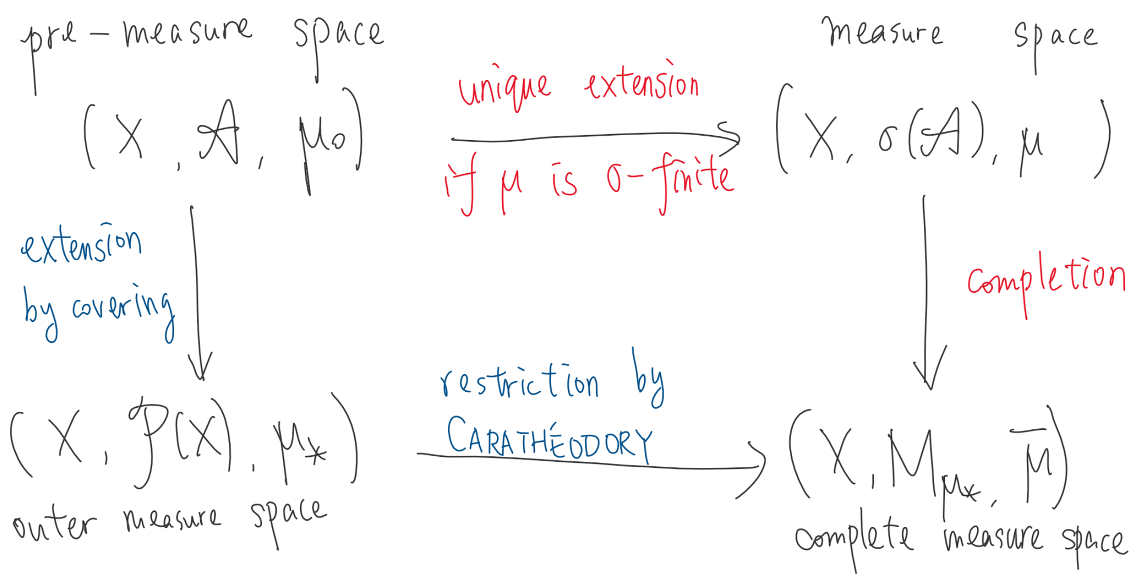

图 1: The Process of Completing Measure Space

Back to $\mathbb{R}^d$.

Convention: $Q \subset \mathbb{R}^d$ is a (closed) cube if and only if $Q = x_0 + [-l, l]^d$ for some $x_0 \in \mathbb{R}^d, l > 0$. "Define" $|Q| = (2l)^d$. Define an outer measure via covering by cubes, for $E \subset \mathbb{R}^d$ any subset, define

$$\mu_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j|, Q_j = \text{cubes} \right\}. \tag{1.58}$$

For this μ_*, $\sigma(A)$ = Borel $\sigma-$algebra of $\mathbb{R}^d$. $\mathcal{M}_{\mu_*}$ = Lebesgue $\sigma-$ algebra. $\bar{\mu}$ = Lebesgue measure.

Example Borel measures on metric spaces. (X, d) = metric space.

Definition 1.20 *An outer measure is "good" if and only if*

$$\mu_*(A \sqcup B) = \mu_*(A) + \mu_*(B), \quad \forall A, B \subset (X, d),\ d(A, B) > 0, \tag{1.59}$$

where $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

Proposition 1.9 *The* CARATHÉODORY *measurable sets* $\mathcal{M}_{\mu_*}$ *contain the Borel* $\sigma-$*algebra, provided that* μ_* *is good.*

PROOF Let us prove that any closed set $K \subset (X, d)$ satisfies $\mu_*(A) \geqslant \mu_*(A \cap K) + \mu_*(A \setminus K), \forall A \subset X$. We only need to show for $\mu_*(A) < \infty$.

Foliate $A \cap K^c$ by closed sets:

$$A \cap K^c = \bigcup_{j=1}^{\infty} C_j,\ C_j := \{x \in A \cap K^c : d(x, K) \geqslant j^{-1}\} \tag{1.60}$$

Since μ_* is good,

$$\mu_*(A) \geqslant \mu_*((K \cap A) \sqcup C_j) = \mu_*(K \cap A) + \mu_*(C_j). \tag{1.61}$$

Claim $\mu_*(C_j) \to \mu_*(A \cap K^c)$ as $j \to \infty$, then $K \in \mathcal{M}_{\mu_*}$. PROOF of the claim: Note that $C_1 \subset C_2 \subset C_3 \subset \cdots$. For any j, $\mu_*(C_j) \geqslant \mu_*(C_j) \geqslant \mu_*(C_j \setminus C_{j-1} \sqcup C_{j-2})$. Since $d(C_j \setminus C_{j-1}, C_{j-2}) > 0$, by "good"ness of outer measure,

$$\begin{aligned} +\infty \geqslant \mu_*(C_j) &\geqslant \mu_*(C_j \setminus C_{j-1}) + \mu_*(C_{j-2}) \\ &\geqslant \cdots \\ &\geqslant \mu_*(C_j \setminus C_{j-1}) + \mu_*(C_{j-2} \setminus C_{j-3}) + \mu_*(C_{j-4} \setminus C_{j-5}) + \cdots \end{aligned} \tag{1.62}$$

Then

$$\cdots + \mu_*(C_{j+2} \setminus C_{j+1}) + \mu_*(C_j \setminus C_{j-1}) + \mu_*(C_{j-2} \setminus C_{j-3}) + \cdots \leqslant \infty, \tag{1.63}$$

$$\cdots + \mu_*(C_{j+1} \setminus C_j) + \mu_*(C_{j-1} \setminus C_{j-2}) + \mu_*(C_{j-3} \setminus C_{j-4}) + \cdots \leqslant \infty. \tag{1.64}$$

That indicates as $j \to \infty$,

$$\mu_*(C_{j+2} \setminus C_{j+1}) + \mu_*(C_{j+4} \setminus C_{j+3}) + \cdots \to 0,\ \mu_*(C_{j+1} \setminus C_j) + \mu_*(C_{j+3} \setminus C_{j+2}) + \cdots \to 0. \tag{1.65}$$

While

$$\mu_*(C_j) \leqslant \mu_*(A \cap K^c) \leqslant \mu_*(C_j) + \mu_*(C_{j+1} \setminus C_j) + \mu_*(C_{j+2} \setminus C_{j+1}) \cdots$$
$$\leqslant \mu_*(C_j) + \mu_*(C_{j+1} \setminus C_j) + \mu_*(C_{j+3} \setminus C_{j+2}) + \cdots \quad (1.66)$$
$$+ \mu_*(C_{j+2} \setminus C_{j+1}) + \mu_*(C_{j+4} \setminus C_{j+3}) + \cdots$$

Sending $j \nearrow \infty$ we get $\mu_*(C_j) \to \mu_*(A \cap K^c)$ □

Borel measures on metric spaces are "regular".

Definition 1.21 *Let μ be a Borel measure on (X, d). It is "locally finite" if and only if for any bounded set $E \subset X$, $\mu(E) < \infty$.*

Remark $E \subset (X, d)$ is bounded if and only if $\exists N$ s.t. $E \subset B(0, N) = \{x \in X : d(x, 0) < N$ if and only if $\mathrm{diam}(E) := \sup\{d(x, y) : x, y \in E\} < \infty$

Theorem 1.7 *Let μ be a locally finite Borel measure on (X, d) and $\mathcal{B}_X$ is Borel $\sigma-$ algebra, then it is*

(i)(Outer Regular)$\forall \varepsilon > 0, \forall E \subset \mathcal{B}_X$, $\exists$ open set $O \supset E$ s.t. $\mu(O \setminus E) < \varepsilon$.

(ii)(Inner Regular)$\forall \varepsilon > 0, \forall E \subset \mathcal{B}_X$, $\exists$ closed set $K \subset E$ s.t. $\mu(E \setminus K) < \varepsilon$.

Proof Consider $\mathcal{F} := \{E \in \mathcal{B}_X : \forall \varepsilon > 0, \exists$ open $O \supset E \supset K$closed s.t. $\mu(E \setminus K) < \varepsilon$ and $\mu(O \setminus E) < \varepsilon\}$. We show that $\mathcal{F}$ is a $\sigma-$ algebra containing the topology, then $\mathcal{F} \supset \mathcal{B}_X$.

(1) $\varnothing \in \mathcal{F}$;(obvious)

(2) $\mathcal{F}$ is closed under complement. If $E \subset \mathcal{F}$, then $\exists$ open $O \supset E \supset K$ closed s.t. $\mu(O \setminus E) < \varepsilon, \mu(E \setminus K) < \varepsilon$. But then closed $O^c \subset E^c \subset K^c$ open and $\mu(E^c \setminus O^c) = \mu(O \setminus E) < \varepsilon$ and $\mu(K^c \setminus E^c) = \mu(E \setminus K) < \varepsilon$.

(3) Closed under countable union. Let $\{E_j\}_1^\infty \subset \mathcal{F}$. Then $\forall j$, $\exists$ open $O_j \supset E_j \supset K_j$ closed s.t. $\mu(O_j \setminus E_j) < \dfrac{\varepsilon}{2^j}.\mu(E_j \setminus K_j) < \dfrac{\varepsilon}{2^j}$. Then $O := \bigcup_{j=1}^\infty O_j$ is open, and

$$\mu(O \setminus \bigcup_j E_j) \leqslant \mu(\bigcup_j (O_j \setminus E_j)) \leqslant \sum_j \frac{\varepsilon}{2^j} < \varepsilon. \quad (1.67)$$

As for the union of inner closed set, we prove

Claim Let $K \subset (X,d)$ be a countable union of closed sets. Then $\exists \tilde{K} \subset X$ closed such that $\tilde{K} \subset \bigcup_{j=1}^{\infty} K_j$ and $\mu(\bigcup_{j=1}^{\infty} K_j \setminus \tilde{K}) < \varepsilon$. Here μ is a Borel measure on (X,d) and it is locally finite.

Proof of the claim W.L.O.G[5] we can assume that $\{K_j\}$ is increasing (for example, consider $K_1, K_1 \cup K_2, K_1 \cup K_2 \cup K_3, \cdots$). Set $K := \bigcup_{j=1}^{\infty} K_j$. Consider $B_n := \{x \in X : d(x, x_0) < n\}$ for some fixed $x_0 \in X$. Then $K_j \cap (\overline{B}_n \setminus B_{n-1}) \nearrow K \cap (\overline{B}_n \setminus B_{n-1})$. Since μ is locally finite, $\forall \varepsilon > 0$, $\exists I(n) \in \mathbb{N}$, s.t.

$$\mu((K \setminus K_{I_{(n)}}) \cap (\overline{B}_n \setminus B_{n-1})) < \frac{\varepsilon}{2^n} \tag{1.68}$$

Then, set

$$\tilde{K} := \bigcup_{n=1}^{\infty} K_{I_{(n)}} \cap (\overline{B_n} \setminus B_{n-1}) \tag{1.69}$$

It holds that

$$\begin{aligned} \mu(K \setminus \tilde{K}) =& \mu\Big(\bigcup_{n=1}^{\infty} [K \cap (\overline{B_n} \setminus B_{n-1})] \setminus \bigcup_{n=1}^{\infty} [K_{I_{(n)}} \cap (\overline{B_n} \setminus B_{n-1})] \Big) \\ \leqslant& \sum_{n=1}^{\infty} \mu\Big(K \setminus K_{I_{(n)}} \cap (\overline{B}_n \setminus B_{n-1}) \Big) \leqslant \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \end{aligned} \tag{1.70}$$

The following claim is applied to prove set $\tilde{K}$ is a closed set.

Claim A set $F \subset (X,d)$ is closed $\Leftrightarrow$ $F \cap \overline{B}_n$ is closed $\forall n \in \mathbb{N}$.

Proof of the claim "$\Rightarrow$" (trivial);

"$\Leftarrow$" Take a sequence $\{x_j\} \subset F$ such that $d(x_j, x_\infty) \to 0$ as $j \to \infty$. By assumption $F \cap \overline{B}_n$ is closed, so the limit $x_\infty \in \overline{B}_n \cap F$. In particular, $x_\infty \in F$, so F is closed.

Given the definition of $\tilde{K}$ and we have

$$\tilde{K} \cap \overline{B}_n = \bigcup_{k=1}^{n} K_{I_{(k)}} \cap (\overline{B_k} \setminus B_{k-1}) \tag{1.71}$$

while $K_{I_{(k)}} \cap (\overline{B_k} \setminus B_{k-1})$ is closed, so $\tilde{K} \cap \overline{B}_n$ is also closed. Then we apply the above claim

[5] W.L.O.G is the abbreviation of "without loss of generality", 不失一般性

to prove that $\tilde{K}$ is closed.[6] $\square$

Remark The assumption X = metric space can be relaxed. The theorem holds for $\sigma-$compact, locally compact, Hausdorff spaces. See Theorem 2.17 in *Rudin*.

1.7 Lebesgue Measure on $\mathbb{R}^d$

Definition 1.22 *Let $E \subset \mathbb{R}^d$ be any set. Define*

$$\mathcal{L}_*(E) := \inf\{\sum_{j=1}^{\infty} |Q_j| : Q_j = \text{ closed cubes}, E \subset \bigcup_i Q_j\}. \tag{1.72}$$

Remark Can replace cubes by rectangles/balls.

Examples (0)$\mathcal{L}_*(\{\text{point}\}) = 0$;

(1) Q = closed cubes $\mathcal{L}_*(Q) = |Q|$.

"$\leqslant$" is clear, since Q covers itself.

"$\geqslant$": Let $Q \subset \bigcup_{j=1}^{\infty} Q_j$ where Q_j are closed cubes. "Fatten Q_j a little" $\exists$ open cubes $U_j \supset Q_j$ where $|U_j| \leqslant (1+\varepsilon)Q_j$. Since Q is compact, so $Q \subset U_1 \cup \cdots \cup U_N$ (after relabelling) . Then by elementary Euclidean geometry (**exercise**),

$$|Q| \leqslant \sum_{j=1}^{N} |U_j| \leqslant (1+\varepsilon) \sum_{j=1}^{N} |Q_j| \leqslant (1+\varepsilon) \sum_{j=1}^{\infty} |Q_j| \tag{1.73}$$

Here $\varepsilon > 0$ is arbitrary. Infimising over $\{Q_j\}$ to get $|Q| \leqslant \mathcal{L}_*(Q)$.

(2) $\mathring{Q}$ = open cube; $\mathcal{L}_*(\mathring{Q}) = \left|\mathring{Q}\right|$;

(3) R = closed rectangle = $\prod_{i=1}^{d}[a_i, b_i] \Rightarrow \mathcal{L}_*(R) = \prod_{i=1}^{d}(b_i - a_i)$.

Proof $|R| \leqslant \mathcal{L}_*(R)$ as in (1). (Left as **exercise**). "$\geqslant$": Divide the rectangle by cubes of length $\delta > 0$. Denote $Q_{\text{IN}} = \{$ cubes lying entirely in $R\}$, $Q_{\text{BDRY}} = \{$ cubes $\cap \partial R \neq \varnothing\}$, then $R = \bigcup_{Q \in Q_{\text{IN}} \sqcup Q_{\text{BDRY}}} Q$. It is easy to see

$$\sum_{Q \in Q_{\text{IN}}} |Q| \leqslant |R|\,. \tag{1.74}$$

[6] 此处证明的完善应当感谢严菁同学做出的贡献.

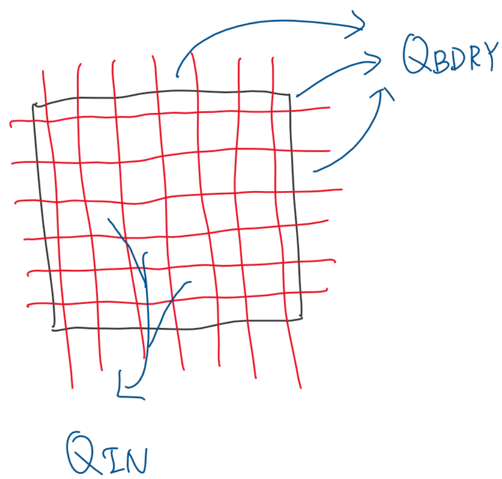

图 2: The division by cubes of length $\delta > 0$

Also, as $\#Q_{\text{BDRY}} \approx$ perimeter of $R \approx O(\delta^{1-\delta})$[7], hence

$$\sum_{Q \in Q_{\text{BDRY}}} |Q| \lesssim \delta^d O(\delta^{1-d}) \lesssim O(\delta). \tag{1.75}$$

So $\displaystyle\sum_{Q \in Q_{\text{BDRY}} \cup Q_{\text{IN}}} |Q| \leqslant |R| + O(\delta)$. Sending $\delta \searrow$ and $\mathcal{L}_*(R) \leqslant |R|$. □

Proposition 1.10 $\mathcal{L}_*$ *is indeed an outer measure on* $\mathbb{R}^d$.

Proof (1)$\mathcal{L}_*(\varnothing) = 0$;

(2) Monotonicity $E \subset F \;\Rightarrow\; \mathcal{L}_*(E) \leqslant \mathcal{L}_*(F)$. Any cover of F is a cover of E.

(3) $\sigma-$subadditively. Let $E = \displaystyle\bigcup_{j=1}^{\infty} E_j$. For each E_j , take a cover $\{Q_j^{(k)}\}_k$ of cubes s.t. $E_j \subset \displaystyle\bigcup_{k=1}^{\infty} Q_j^{(k)}, \mathcal{L}_*(E_j) \geqslant \left|Q_j^{(k)}\right| - \frac{\varepsilon}{2^j}$. Then E is covered by $\{Q_j^{(k)}\}_{j,k}$,so

$$\mathcal{L}_*(E) \leqslant \sum_j \sum_k \left|Q_j^{(k)}\right| \leqslant \sum_{j=1}^{\infty} \left(\mathcal{L}_*(E_j) + \frac{\varepsilon}{2^j}\right) \leqslant \sum_{j=1}^{\infty} (\mathcal{L}_*(E_j)) + \varepsilon. \tag{1.76}$$

[7]边长为 δ 的网格在 $d-1$ 维空间的体积为 δ^{d-1}，因此一个 d 维立方体的其中一个 $d-1$ 维表面可被分为 $O(\delta^{1-d})$ 个部分，而每个部分可以与一个 Q_{BDRY} 中的元素形成对应.

Definition 1.23 *$E \subset \mathbb{R}^n$ is* ***Lebesgue Measurable*** *($E \in \mathcal{M}_{\text{Leb}}$) if and only if $\forall \varepsilon, \exists$ open $O \supset E$ s.t. $\mathcal{L}_*(O \setminus E) < \varepsilon$. Defined by outer regularity.*

Remark We'll prove later that this is equivalent to $\mathcal{L}_*(A) \geqslant \mathcal{L}_*(A \cap E) + \mathcal{L}_*(A \setminus E), \forall E \in \mathcal{M}_{\text{Leb}}$.

Goal: Make $(\mathbb{R}^d, \mathcal{M}_{\text{Leb}}, \mathcal{L}_*|_{\mathcal{M}_{\text{Leb}}})$ a complete measure space.

Proposition 1.11 *$\mathcal{M}_{\text{Leb}}$ is a $\sigma-$ algebra containing $\mathcal{B}_{\mathbb{R}^d}$(=Borel $\sigma-$ algebra on $\mathbb{R}^d$).*

Proof (0) Open set $\in \mathcal{M}_{\text{Leb}}$; $\varnothing \in \mathcal{M}_{\text{Leb}}$.

(1) Closed under countable union. Let $\{E_j\} \subset \mathcal{M}_{\text{Leb}}$, then $\exists$ open $O_j \supset E_j . \mathcal{L}_*(O_j \backslash E_j) < \frac{\varepsilon}{2^j}$. So $O = \bigcup_{j=1}^{\infty} O_j$ is open, and $\mathcal{L}_*(O \setminus \bigcup_{j=1}^{\infty} E_j) \leqslant \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} < \varepsilon$.

(2) $\mathcal{L}_*(E) = \inf\{\mathcal{L}_*(O) : O = \text{open} \supset E\}$.

"$\leqslant$" : By monotonicity of $\mathcal{L}_*$;

"$\geqslant$" : Take $E \subset \cap_{j=1}^{\infty} Q_j$ for closed cubes Q_j, $\sum_j |Q_j| \leqslant \mathcal{L}_*(E) + \varepsilon$.

Take a "fattening " of Q_j into open cubes $U_j \supset Q_j$ s.t. $|U_j| \leqslant (1+\delta)\,|Q_j|$. Set $O = \bigcup_{j=1}^{\infty} U_j$. Then $O \supset \bigcup_j E_j$ and

$$\mathcal{L}_*(O) \leqslant \sum_{j=1}^{\infty} (1+\delta)\,|Q_j| \leqslant (1+\delta)[\mathcal{L}_*(E) + \varepsilon]. \tag{1.77}$$

Then send $\varepsilon, \delta \to 0$.

(3) As a corollary, $\mathcal{M}_{\text{Leb}}$ contains all $N \subset \mathbb{R}^d$, s.t. $\mathcal{L}_*(N) = 0$.

(4) If $F \subset \mathbb{R}^d$ is closed, then $F \in \mathcal{M}_{\text{Leb}}$. By considering $F \cap \overline{B}_n (B_n \equiv B_n(0))$, we can assume that F is compact.

Aside Compactness: Heine-Borel: In $\mathbb{R}^d, d < \infty$, closed + bounded $\Leftrightarrow$ compact. Fact: Heine-Borel property holds $\Leftrightarrow$ X[8] is finite-dimensional .

As

$$\mathcal{L}_*(F) = \inf\{\mathcal{L}_*(O) : O \supset F . O \subset \mathbb{R}^d \text{ open.}\} \tag{1.78}$$

[8] X is a Banach space.

$\forall \varepsilon > 0, \exists$ open $O \supset F$ s.t. $\mathcal{L}_*(O \setminus F) < \varepsilon$. As $O \setminus F$ is open, so we can decompose $O \setminus F$ into almost disjoint union of closed cubes. (e.g. by Whitney Decomposition). $O \setminus F = \bigcup\limits_{j=1}^{\infty} Q_j$.

Here $\operatorname{dist}\Big(\bigcup\limits_{j=1}^{N} Q_j, F\Big) \geqslant \varepsilon_0 > 0$.

(Recall the fact : If $K \subset \mathbb{R}^d$ is compact. $F \subset \mathbb{R}^d$ is closed, and if $K \cap F = \varnothing$, then $\operatorname{dist}(K, F) \geqslant \varepsilon_0 > 0)\Rightarrow$

$$\mathcal{L}_*(O) \geqslant \mathcal{L}_*(F) + \mathcal{L}_*\Big(\bigcup_{j=1}^{N} Q_j\Big) \geqslant \mathcal{L}_*(F) + \sum_{j=1}^{N} |Q_j| \,. \tag{1.79}$$

Sending $N \nearrow \infty$, we get

$$\mathcal{L}_*(O \setminus F) \leqslant \sum_{j=1}^{\infty} |Q_j| \leqslant \mathcal{L}_*(O) - \mathcal{L}_*(F) < \varepsilon. \tag{1.80}$$

Then we have $F \in \mathcal{M}_{\text{Leb}}$.

(5) It has been proved that $\mathcal{M}_{\text{Leb}}$ contains all the closed set. Then $E \in \mathcal{M}_{\text{Leb}} \Rightarrow E^c = \mathbb{R}^d \setminus E \in \mathcal{M}_{\text{Leb}}$, which means that $\mathcal{M}_{\text{Leb}}$ is closed under complement. Since $E \in \mathcal{M}_{\text{Leb}}.\forall n \in \mathbb{N}, \exists O_n$ open s.t. $O_n \supset E$, $\mathcal{L}_*(O_n \setminus E) \leqslant \frac{1}{n}$. Consider $F := \bigcup\limits_{n=1}^{\infty} O_n^c \in \mathcal{M}_{\text{Leb}}$ Also $E^c \setminus F = E^c \cap \bigcap\limits_{n=1}^{\infty} O_n \subset O \setminus E.\forall m \in \mathbb{N}$. So $\mathcal{L}_*(E^c \setminus F) \leqslant \mathcal{L}_*(O_m \setminus E) \leqslant \frac{1}{m}.\Rightarrow E^c \setminus F$ is $\mathcal{L}_*-$null, so $E^c \setminus F \in \mathcal{M}_{\text{Leb}}$ by (3). Thus $E^c = (E^c \setminus F) \sqcup F \in \mathcal{M}_{\text{Leb}}$. $\square$**Conclusion** $\mathcal{M}_{\text{Leb}}$ is a $\sigma-$ algebra $\supset \mathcal{B}_{\mathbb{R}^d}$.

Proposition 1.12 *$\mathcal{L} := \mathcal{L}_*|_{\mathcal{M}_{\text{Leb}}}$ is a complete measure.*

Proof In order to prove that $\mathcal{L}$ is a complete measure, we need to verify that (1)$\mathcal{L}$ is a measure; (2)$\mathcal{L}$ satisfies its completeness on $\mathcal{M}_{\text{Leb}}$.

Completeness is already proved and we only need to verify the $\sigma-$ additivity(remind that the $\sigma-$ subadditivity has been proved before). Let $\{E_j\} \subset \mathcal{M}_{\text{Leb}}$ be disjoint and we need to prove that

$$\forall \varepsilon > 0,\ \mathcal{L}\Big(\bigsqcup_{j=1}^{\infty} E_j\Big) \geqslant \sum_{j=1}^{\infty} \mathcal{L}(E_j) - \varepsilon. \tag{1.81}$$

Assume each E_j is bounded. Since $E_j \in \mathcal{M}_{\text{Leb}}$, $\exists$ closed $K_j \subset E_j$ s.t. $\mathcal{L}(E_j \setminus K_j) \leqslant \dfrac{\varepsilon}{2^j}$. These K_j are compact and disjoint. So

$$\mathcal{L}\Big(\bigsqcup_{j=1}^{N} K_j\Big) = \sum_{j=1}^{N} \mathcal{L}(K_j), \quad \forall N < \infty. \tag{1.82}$$

$\Rightarrow \mathcal{L}(E) \geqslant \sum_{j=1}^{N} \mathcal{L}(K_j) \geqslant \sum_{j=1}^{N} \mathcal{L}(E_j) - \varepsilon$. Send $N \nearrow \infty$ to conclude. □

Proposition 1.13 *$E \subset \mathbb{R}^d$ is in $\mathcal{M}_{\text{Leb}}$ $\Leftrightarrow$ $\forall A \subset \mathbb{R}^d, \mathcal{L}_*(A) \geqslant \mathcal{L}_*(A \cap E) + \mathcal{L}_*(A \cap E^c)$.*

Proof "$\Rightarrow$" Let $E \subset \mathcal{M}_{\text{Leb}}$. Take $A \subset \mathbb{R}^d$ s.t. $\mathcal{L}_*(A) < \infty$. Then $\forall n \in \mathbb{N}$, $\exists$ closed cubes $\{Q_k^{(n)}\}_{k=1}^{\infty}$ s.t. $A \subset \bigcup_{k=1}^{\infty} Q_k^{(n)}, \mathcal{L}_*(A) \geqslant \sum_{k=1}^{\infty} \left|Q_k^{(n)}\right| - \dfrac{1}{n}$. Let $G := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} Q_k^{(n)}$. Then $A \subset G, \mathcal{L}(G \setminus A) = 0$ and $G \in \mathcal{M}_{\text{Leb}}$. $\mathcal{L}(A) = \mathcal{L}(G)$, so

$$\mathcal{L}_*(A \cap E) + \mathcal{L}_*(A \setminus E) \leqslant \mathcal{L}_*(G \cap E) + \mathcal{L}_*(G \setminus E). \tag{1.83}$$

Since $\mathcal{L}$ is a measure, $\mathcal{L}_*(G \cap E) + \mathcal{L}_*(G \setminus E) = \mathcal{L}(G) = \mathcal{L}_*(A)$. So $\mathcal{L}$ satisfies the Carathéodory's criterion.

"$\Leftarrow$". □

Definition 1.24 *$G \subset \mathbb{R}^d$ is G_δ if and only if $G =$ countable $\cap$ of open sets; $F \subset \mathbb{R}^d$ is F_σ if and only if $F =$ countable $\cup$ of close sets;*

Theorem 1.8 *$E \subset \mathbb{R}^d$ is Lebesgue measurable*

- *if and only if $\exists$ $G_\delta-$ set $G \supset E$ s.t. $\mathcal{L}(G \setminus E) = 0$;*
- *if and only if $\exists$ $F_\sigma-$ set $F \subset E$ s.t. $\mathcal{L}(E \setminus F) = 0$.*

Proof Use completeness of $\mathcal{L}$. □

Definition 1.25 (Translation Invariant) *If $E \subset \mathbb{R}^d$ is Lebesgue measurable, then $\mathcal{L}(E) = \mathcal{L}(E + X)$, $\forall x \in \mathbb{R}^d$. Here $E + x := \{y + x : y \in E\}$. This property called **translation invariant** of measure $\mathcal{L}$.*

Theorem 1.9 *Let μ be a Borel measure on $\mathbb{R}^d$, which is translation invariant and $\mu(K) < \infty, \forall K \subset \mathbb{R}^d$ compact. Then $\mu(E) = \lambda \cdot \mathcal{L}(E)$ for some $\lambda \geqslant 0$ constant and $\forall E \in \mathcal{M}_{\text{Leb}}$.*

PROOF Recall that $\mathcal{M}_{\text{Leb}}$ is outer and inner regular. Thus it suffices to prove for $E \subset \mathbb{R}^d$ open. We can decomposition E into $E = \bigsqcup_{j=1}^{\infty} \tilde{Q}_j$. $\tilde{Q}_j = \prod_{i=1}^{d} [v_j^{(i)}, v_j^{(i)} + 2^{-j}[$, i.e. $\tilde{Q}_j$ is half-open/ half- closed or $\tilde{Q}_j$ has edge length 2^{-j}.
Example Consider $\Omega_j = \{\text{cubes of length } 2^{-j} \text{ in } \mathbb{R}^d\}$. First take all cubes in Ω_1 that lie in E. Remove all those in Ω_2, Ω_3 and so on...(**exercise**). So if further suffices to prove for $E = \tilde{Q}_j$. But by translation-invariance and subaddtivity.

$$\mu(\tilde{Q}) = \lambda \mathcal{L}(Q), \quad \lambda = \frac{\mu([0,1]^d)}{\mathcal{L}([0,1]^d)}. \tag{1.84}$$

1.8 Cantor Set and Nonmeasurable Set

Questions: In $\mathbb{Q}^d$,

(1) Are there Lebesgue measurable but not Borel measurable sets?

(2) Are there non-Lebesgue measurable sets? Both: Yes. $\mathcal{B}_{\mathbb{R}^d} \subsetneq \mathcal{M}_{\text{Leb}} \subseteq \mathcal{P}(\mathbb{R}^d)$. For $d = 1$, $\mathbb{R}^d$ has a countable topology base, e.g. $\{B_r(x) \subset \mathbb{R}^d : x \in \mathbb{Q}^d, r \in \mathbb{Q} > 0\}$.
Topological fact: $|B_{\mathbb{R}^d}| = 2^{\aleph_0} = \aleph_1$. But $\forall$ uncountable set $\mathcal{C} \subset \mathbb{R}$ such that every subset of $\mathcal{C}$ is $\mathcal{L}^1-$ null. Since the Lebesgue measure $\mathcal{L}^1$ is complete, all null sets are Lebesgue measurable.

$$|\mathcal{M}_{\text{Leb}}| \geqslant 2^{|\mathcal{C}|} = 2^{\aleph_1} > |\mathcal{B}_{\mathbb{R}^d}|. \tag{1.85}$$

Here $\mathcal{C} = \text{Cantor set} = \bigcap_{n=0}^{\infty} C_n$.

$$C_0 = [0,1]$$
$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$
$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$
$$\cdots$$

C_{j+1} is obtained from C_j by removing "middle-$\frac{1}{3}$" of each connected component of C_j.

Equivalently, $\mathcal{C} = \{\sum_{n=0}^{\infty} \frac{a_n}{3^n} : a_n \in \{0,2\}\} = \{0.a_1a_2a_3 \cdots : a_n \in \{0,2\}\}$(Tertiary expansion)

Properties: (0) $\mathcal{C}$ is closed;

(1) $\mathcal{C}$ is uncountable: "Cantor diagonalisation".

PROOF Suppose that $\mathcal{C}$ were countable. Then we can enumerate $\mathcal{C} = \{c_1, c_2, c_3, \cdots\}$

$$\begin{aligned} c_1 &= 0.d_1^{(1)}d_1^{(2)}d_1^{(3)}\cdots \\ c_2 &= 0.d_2^{(1)}d_2^{(2)}d_2^{(3)}\cdots \\ c_3 &= 0.d_3^{(1)}d_3^{(2)}d_3^{(3)}\cdots \\ &\cdots \end{aligned} \tag{1.86}$$

Consider

$$\delta_j = \begin{cases} 0, & \text{if } d_j^{(j)} = 2; \\ 2, & \text{if } d_j^{(j)} = 0. \end{cases} \tag{1.87}$$

Then $0.\delta_1\delta_2\delta_3 \cdots \in \mathcal{C}$ but it is not in the list. $\to |$. $\square$

(2) $\forall x \neq y \in \mathcal{C}, \exists z \in [0,1] \setminus \mathcal{C}$ in between of x, y. "Totally disconnected"

(3) $\forall x \in \mathcal{C}, \forall \delta > 0,\ B_\delta(x) \cap \mathcal{C} \setminus \{x\} \neq \varnothing$.

(4) $\mathcal{L}^1(\mathcal{C}) = 0$.

Recall $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$, where $C_1 \supset C_2 \supset C_3 \supset \cdots$, $\mathcal{L}^1(\mathcal{C}) \leqslant \mathcal{L}^1(C_n) = \left(\frac{2}{3}\right)^n \to 0$.(By induction)

Remark "$\xi-$ Cantor set" instead of "$\frac{1}{3}-$ Cantor set", $\xi \in]0,1[$.

In fact, we can construct an explicit surjection for $\mathcal{C}$ to $[0,1]$:

$$\begin{aligned} &F : \mathcal{C} \to [0,1] \\ &\sum_k \frac{a_k}{3^k} \mapsto \sum_k \frac{a_k/2}{2^k}. \end{aligned} \tag{1.88}$$

If $]a, b[$ is a connected component of $[0,1] \setminus \mathcal{C}$, then

$$\begin{aligned} a &= 0.a_1 a_2 \cdots a_n 022222 \cdots, \\ b &= 0.a_1 a_2 \cdots a_n 200000 \cdots, \\ F(a) &= \sum_{n}^{i=1} \frac{a_i/2}{2^i} + \sum_{j=n+1}^{\infty} \frac{1}{2^j}, \\ F(b) &= \sum_{i=1}^{n} \frac{a_i/2}{2^i} + \frac{1}{2^{n+1}} + 0. \end{aligned} \tag{1.89}$$

So we can extend F to a continuous function $F : [0,1] \to [0,1]$. It is increasing, F exists and equals to 0 a.e..

Overview $\displaystyle\int_0^1 F' \mathrm{d}x = 0 \neq 1 = F(1) - F(0)$. F violates the fundamental theorem of Calculus.

Definition 1.26 $F : [0,1] \to [0,1]$ *is the* ***Cantor-Lebesgue function*** *or "devil's staircase".*

Now, consider $\Phi(x) = F(x) + x : [0,1] \to [0,2]$.

- increasing;
- continuous;
- $\Phi([0,1] \setminus \mathcal{C}) = \sqcup\{$ affine segments of slope $1\} \Rightarrow \mathcal{L}^1(\Phi([0,1] \setminus \mathcal{C})) \leqslant 1$. But Φ is surjection onto $[0,2]$, so $\mathcal{L}^1(\Phi(\mathcal{C})) \geqslant 1$.

Conclusion Image of $\mathcal{L}^1-$null sets under continuous functions may not be null.

Theorem 1.10 (Vitali) *For $A \subset \mathbb{R}$, if every subset of A is Lebesgue measurable, then $\mathcal{L}^1(\mathcal{A}) = 0$.*

Assume this theorem, then in the previous example, $\exists$ non-Lebesgue measurable $N \subset \Phi(\mathcal{C})$. Consider $\Phi^{-1}(N)$. Since $\Phi^{-1}(N) \subset \mathcal{C}, \mathcal{L}^1(\mathcal{C}) = 0$. So $\Phi^{-1}(N) \in \mathcal{M}_{\text{Leb}}$ and $\mathcal{L}^{-1}(\Phi^{-1}(N)) = 0$.

Consider $\exists$ $\mathcal{L}^1-$measurable set (in fact,$\mathcal{L}^1-$null) and $\exists$ continuous function s.t. the image is not Lebesgue measurable.

Proof of Vitali: Assume $\exists A \subset \mathbb{R}$ s.t. all subsets of A are $\mathcal{L}^1-$ measurable. Consider the relation on $\mathbb{R}$

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q} \tag{1.90}$$

It is an equivalence relation. So $\mathbb{R}$ is partitioned into disjoint subsets $[i] \subset \mathbb{R}$ where $[i] = i+\mathbb{Q}$. That is $\mathbb{R}/\!\sim = \{[i]\}$, each $[i]$ is an equivalence class. Consider $\mathcal{E} :=$ the set obtained by choosing precisely 1 representative from each equivalence class in $\mathbb{R}/\!\sim . \mathbb{R} = \bigsqcup_{r\in\mathbb{Q}}(r+\mathcal{E})$.

Axiom of Choice Let Ω be a set. Let $\{E_\alpha : \alpha \in I\} \subset \mathcal{P}(\Omega)$ be non-empty. There exists a function $F : I \to \Omega$ such that $F(\alpha) \in E_\alpha$ for each $\alpha \in I$.

Definition 1.27 $\{F(\alpha) : \alpha \in I\} \subset \Omega$ *is a choice set.*

Remark Zermelo-Fraenkel Axioms are independent of ”Axiom of Choice”.
Proof of Vitali. Assume $\forall B \subset A, B \in \mathcal{M}_{\text{Leb}}$. Consider an equivalence relation on $\mathbb{R}$

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}. \tag{1.91}$$

By Axiom of Choice, let $\mathcal{E}$ be the choice set of $\mathbb{R}/\!\sim$; that is, $\mathcal{E}$ consists of precisely one representative from each equivalence class in $\mathbb{R}/\!\sim$.
[A general element of $\mathbb{R}/\!\sim$ is $x + \mathbb{Q} = \{x + q : q \in \mathbb{Q}\} \subset \mathbb{R}$].
Claim $\mathbb{R} = \bigsqcup_{r\in\mathbb{Q}}(\mathcal{E} + r)$.
Proof of the claim:

(1) Disjoint. If $x \in (r+\mathcal{E}) \cap (s+\mathcal{E})$, then $e_1, e_2 \in \mathcal{E}$, s.t. $x = r + e_1 = s + e_2$. So $e_1 - e_2 = s - r \in \mathbb{Q}$. Then $[e_1] = [e_2] \in \mathbb{R}/\!\sim$. By definition of $\mathcal{E}$, $e_1 = e_2$, so $s = r \geqslant r + \mathcal{E} = s + \mathcal{E}$.

(2) Union. $\forall y \in \mathbb{R}$, choose $z \in \mathbb{R}$ s.t. $y \in [z] \in \mathbb{R}/\!\sim$. By definition of $\sim$, $y - z \in \mathbb{Q}$. Say $y - z = q \in \mathbb{Q}$. Note that $\exists \tilde{z} \in \mathcal{E} \cap [z]$, say $\tilde{z} = z + \tilde{q}$ for $\tilde{q} \in \mathbb{Q} \Rightarrow y = \tilde{z} - \tilde{q} + q \in (q - \tilde{q}) + \mathcal{E}$.

Cut $A \subset \mathbb{R}$ by translation of $\mathcal{E}$ for $r \in \mathbb{Q}$, set $A_r := A \cap (r+\mathcal{E})$. By assumption, A_r is $\mathcal{L}^1-$ measurable $\forall r \in \mathbb{Q}$. Consider A_0 first. Take $K \subset A_0$ compact and set $\mathcal{H} = \bigcup_{q_1\in\mathbb{Q}\cap[0,1]}(K + q)$.

Observe: (1) $\mathcal{L}^1(\mathcal{H}) < \infty$; (2) $K \subset \mathcal{E} \Leftrightarrow \{K + q\}_{q\in\mathbb{Q}\cap[0,1]}$ is disjoint. Then

$$\infty > \mathcal{L}^1(\mathcal{H}) = \mathcal{L}^1\Big(\bigsqcup_{\mathbb{Q}\cap[0,1]}(K + q)\Big) = \sum_{q\in\mathbb{Q}\cap[0,1]} \mathcal{L}^1(K + q). \tag{1.92}$$

$\mathcal{L}^1$ is translated-invarianced $= |\mathbb{Q} \cap [0,1]|\, \mathcal{L}^1(K) \Rightarrow \mathcal{L}^1(K) = 0, \forall K \subset A$ compacet $\Rightarrow \mathcal{L}^1(A_0) = 0$ by inner regularity.

Same argument holds for A_r in place of A_0. But $A = \bigsqcup_{r\in\mathbb{Q}} A_r$. So $\mathcal{L}^1(A) = 0$. $\square$

Remark "If" can be replaced by "if and only if".

1.9 Banach-Tarski Paradox

Recall: G is a group if and only if it is a set with a binary operation $\cdot$:

$$\begin{aligned} \cdot:\ & G \times G \to G \\ & (g,h) \mapsto g \cdot h \end{aligned} \tag{1.93}$$

s.t. (*i*) $\exists$ *identity* $e \in G, e \cdot g = g \cdot e = g, \forall g \in G$;

(*ii*) $\exists$ *inverse.* $\forall g \in G$, $\exists\,!\ h \in G$, *s.t.* $g \cdot h = h \cdot g = e$. $(h \equiv g^{-1})$;

(*iii*) *Associativity.* $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

Example $(\mathbb{Z}, +)$. $\mathbb{GL}(n, \mathbb{R}) = \{M = n \times n \text{ matrix } : \det(M) \neq 0\}$. *Group action: Let X be a set G acting (on the left) on X* $\Leftrightarrow$

$$\begin{aligned} \alpha:\ & G \times X \to X \\ & \alpha(g,x) \mapsto g \cdot x \end{aligned} \tag{1.94}$$

(*i*) $e \cdot x = x, \forall x \in X$;

(*ii*) $g \cdot (h \cdot x) = (g \cdot h) \cdot x, \forall g, h \in G, x \in X$.

Definition 1.28 *$G \circlearrowright X, \forall x \in X, G \cdot x = \{g \cdot x : g \in G\}$ is called the orbit of x.*

Example $X = \overline{\mathbb{B}^3} = \{x \in \mathbb{R}^3 : |x| \leqslant 1\}$.$\mathbb{SO}(3) \circlearrowright X$.*(group of (+) rotations). Given $x \in X$,* $\mathrm{SO}(3) \cdot x = \{y \in \overline{\mathbb{B}^3} : |y| = |x|\}$.

Note that $x \in Gy \overset{def}{\Leftrightarrow} x \sim y$ in an equivalence relation. So X is partitioned into orbits.

Definition 1.29 *If $G \circlearrowright X$ has only 1 orbit, then the action is transitive.*

Definition 1.30 *$G \circlearrowright X$, $x \in X$. This stabiliser/isotropy group of X is $G_x = \mathrm{Stab}_G(x) := \{g \in G : gx = x\}$.*

Definition 1.31 *If $G_x = \{e\}, \forall x \in X$, then say $G \circlearrowright X$ is free.*

Definition 1.32 *$G \circlearrowright X$.$E \subset X$ is **G-paradoxical** if and only if $\exists$ disjoint subsets in E*

$$\{A_1, \cdots, A_m; B_1, \cdots, B_m\} \text{ for } m, n \in 1, 2, \cdots, \infty. \tag{1.95}$$

and $\exists$ *group elements* $\{g_1, g_2, \cdots, g_m; h_1, \cdots, h_n\}$ *such that* $E = \bigcup_{i=1}^{m} g_i(A_i) = \bigcup_{j=1}^{n} h_j(B_j)$.

Proposition 1.14 $\mathbb{S}^1$ *is* $\mathrm{SO}(2)-$ *paradoxical.*

Proof *Let* $G = \{g \in \mathrm{SO}(2) : g = \text{ rotation by } 2\pi\alpha,\ \alpha \in \mathbb{Q}\}$. *Let* $\mathcal{A}$ *be a choice set of* $\mathrm{SO}(2)/G$, *let* $M = \{\sigma \cdot 1 : \sigma \in \mathcal{A}\}$ *(*$\sigma =$ *rotation,* $1 = (1,0)^{\mathrm{T}}$*).*

Enumerate G *by* $\{\rho_i\}_{i=1}^{\infty}$. *Set* $M_i = \rho_i(M) \Rightarrow \mathbb{S}^1 = \bigcup_{i=1}^{m} M_i$ *and all* M_i *are congruent.*

Easy to see: $\mathbb{S}^1 = \bigsqcup_{j=1}^{\infty} \tilde{\rho}_{2j}(M_{2j}) = \bigsqcup_{k=1}^{\infty} \tilde{\rho}_{2k-1}(M_{2k-1})$ *for* $\tilde{\rho}_i \in \mathrm{SO}(2)$. $\square$

$$\mathbb{S}^1 = \tilde{\rho}_1(M_1) \sqcup \tilde{\rho}_3(M_3) \sqcup \tilde{\rho}_5(M_5) \sqcup \cdots = \tilde{\rho}_2(M_2) \sqcup \tilde{\rho}_4(M_4) \sqcup \tilde{\rho}_6(M_6) \sqcup \cdots. \tag{1.96}$$

If $\exists$ $\sigma-$ *additive, rotation invariant probability measure* μ *on* $\mathbb{S}^1$,

$$1 = \mu(\mathbb{S}^1) = \cdots = \mu(\mathbb{S}^1) + (\mathbb{S}^1) = 2 \tag{1.97}$$

$\rightarrow|$.

Definition 1.33 $A, B \subset X, G \circlearrowright X$. *A and B are "G− equidecomposable" if and only if* $\exists n \in N$, $A = \bigsqcup_{i=1}^{n} A_i, B = \bigsqcup_{i=1}^{n} B_i, A_i = g_i(B_i)$ *for* $g_i \in G$.

Theorem 1.11 $\mathbb{S}^1 \setminus \{point\}$ *is* $\mathrm{SO}(2)-$ *decomposable to* $\mathbb{S}^1$.

Proof *W.L.O.G. take point* $= 1$. *Let* $A = \{e^{\mathrm{i}n} : n = 1, 2, 3, \cdots\}, B = \mathbb{S}^1$. *Rotate* A *by* $\rho \in \mathrm{SO}(2), \rho(z) = e^{-\mathrm{i}z}$. *So* $\rho(A) \sqcup B = \mathbb{S}^1, A \sqcup B = \mathbb{S}^1 \setminus \{1\}$. $\square$

Definition 1.34 (Free Group) $\mathbb{F}_2 :=< \sigma, \tau >$: *elements words (e.g.* $\sigma\tau\sigma^{-1}\tau\sigma^{-1}\tau^{-1}\sigma$, *but no* $\sigma\sigma^{-1}$, $\sigma^{-1}\sigma, \tau^{-1}\tau$, $\tau\tau^{-1}$*).*

Group operation: concatenation.

$$(\sigma\tau) \cdot (\sigma\tau^{-1}) = \sigma\tau\sigma\tau^{-1}, \quad (\sigma\tau) \cdot (\tau^{-1}) = \sigma. \tag{1.98}$$

"{ }" = empty word = "1" is the identity.

Proposition 1.15 *If* G *is* $G-$ *paradoxical s.t.* $G \circlearrowright X$ *freely, then* X *is* $G-$ *paradoxical.*

Corollary *If $H \leqslant G$. H is $H-$ paradoxical, then G is $G-$ paradoxical.*

$H \circlearrowleft H$, H act on itself freely.

Proposition 1.16 *$G \circlearrowleft X$, X is $G-$ paradoxical, then so is G.*

Proposition 1.17 *$G \circlearrowleft X$. If G is paradoxical, and $G \circlearrowleft X$ freely (has nontrivial fixed point), then X is paradoxical.*

PROOF *Consider $G := \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j)$ for $\{A_i, B_j\}$ disjoint subsets of G; $\{g_i, h_i\}$ elements of G.*

Recall $G \circlearrowleft X \rightsquigarrow x \sim y \Leftrightarrow \exists g \in G$ s.t. $x = gy$, $X/\sim =$ the $G-$ orbit space. Let M be a choice set of the $G-$ orbit space. Recall Axiom of Choice, $\exists$ function

$$\begin{aligned} f : X/\sim &\to X \\ \textit{orbit} &\mapsto \textit{element in this orbit} \end{aligned} \tag{1.99}$$

Then $\{g(M) : g \in G\}$ partitions X. [In Vitali Theorem, $M = \mathcal{E}, g(M) = r + \mathcal{E}, \coprod_{r \in \mathbb{Q}}(r + \mathcal{E}) = \mathbb{R}$]. Indeed, if $g_1(M) \cap g_2(M) \neq \varnothing$, then $\exists x \in X, \alpha, \beta \in M$ s.t. $x = g_1\alpha = g_2\beta$. Then $\alpha = g_1^{-1}g_2\alpha = (g_1^{-1}g_2)\beta$. But M is a choice set, so $\alpha = \beta$. In this case, $g_1 = g_2$, so $g_1(M) = g_2(M)$.

Exercise *$X \subset \bigcup_{g \in G} g(M)$.*

Let $\bigcup_{g \in A_i} g(M) = \tilde{A}_i$, $\bigcup_{h \in B_j} \tilde{B}_j$, they are pairwise disjoint, and

$$X = \bigcup_i g_i(\tilde{A}_i) = \bigcup_i h_j(\tilde{B}_j) \tag{1.100}$$

□

Proposition 1.18 *$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is $\mathrm{SO}(3)$- paradoxical. $\mathrm{SO}(3) = \{$positive 3D rotation $\} = \{A : \mathbb{M}_{3\times 3}(\mathbb{R}) : A^{\mathrm{T}}A = I_3, \det(A) = 1\}$.*

Fact *$\exists$ subgroup $\mathcal{F} \leqslant \mathrm{SO}(3)$ s.t. $\mathcal{F} \cong \mathbb{F}_2$.*

PROOF *Let $\mathcal{D}$ be the set $\{x \in \mathbb{S} : x$ is fixed by some $g \in \mathbb{F} \setminus \{0\}\}$.$\mathbb{D}$ is (at most) countable. Then $\mathcal{F} \circlearrowleft (\mathbb{S}^2 \setminus \mathcal{D})$ without fixed points. But $\mathcal{F} \cong \mathbb{F}_2$ is $\mathcal{F}-$ paradoxical. By a previous proposition, $\mathbb{S}^2 \setminus \mathcal{D}$ is $\mathcal{F}-$ paradoxical, hence $\mathrm{SO}(3)-$ paradoxical.* □

Now, we prove that $\mathbb{S}^2 \setminus \mathcal{D}$ and $\mathbb{S}^2$ are $\mathrm{SO}(3)$ are equidecompsable. Indeed, $\exists$ axis Ξ such that *rotations about Ξ does not fix a point in $\mathcal{D}$.*

Consider

$$\mathcal{A} = \{\theta \in [0, 2\pi[: \text{ For } \rho_\theta = \text{ rotation about } \Xi \text{ of angle } \theta, \exists n \in \mathbb{N}, \rho_\theta^n \text{ sends a point in } \mathcal{D} \text{ to another point in } \mathcal{D}\}. \tag{1.101}$$

Pick $\theta_{\mathcal{A}} \in [0, 2\pi[\backslash \mathcal{A}$. Then $\Lambda = \{\rho^n_{\theta_{\mathcal{A}}}(\mathcal{D}) : n \in \mathbb{N} \geqslant 1\}$ is a disjoint subset of $\mathbb{S}^2$. Therefore,

$$\begin{aligned} \mathbb{S}^2 \setminus \mathcal{D} = \Lambda \sqcup (\mathbb{S}^2 \setminus \Lambda \setminus \mathcal{D}) &\overset{\rho^{-1}_{\theta_{\mathcal{A}}}}{\sim} \rho^{-1}_{\theta_{\mathcal{A}}}(\Lambda) \sqcup (\mathbb{S}^2 \setminus \Lambda \setminus \mathcal{D}) \\ &= (\Lambda \sqcup \mathcal{D}) \sqcup (\mathbb{S}^2 \setminus \Lambda \setminus \mathbb{D}) = \mathbb{S}^2 \end{aligned} \tag{1.102}$$

Theorem 1.12 (Banach-Tarski) *$\overline{\mathbb{B}^3}$ is $\mathrm{Isom}(\mathbb{R}^3)-$paradoxical.*

$\mathrm{Isom}(\mathbb{R}^3) = \{x \mapsto Ax + b, b \in \mathbb{R}, A \in \mathrm{SO}(3)\}$ *rigid motions.*

PROOF *Step 1. $\mathbb{S}^2$ is $\mathrm{SO}(3)-$ paradoxical $\Rightarrow \overline{\mathbb{B}}^3$- paradoxical.*

Step 2. Let Γ be a copy of $\mathbb{S}^1$ passing through $O \in \mathbb{R}^3$, such that $\Gamma^\times = \Gamma \setminus \{0\} \in \overline{\mathbb{B}}^3 \setminus \{0\}$.

*We 've proved $\Gamma \sim \Gamma^\times$.(**Exercise** in the sense of $\mathrm{Isom}(\mathbb{R}^3)-$ equidecomposable)*

$\overline{\mathbb{B}}^3 \setminus \{0\} = (\overline{\mathbb{B}}^3 \setminus \{0\} \setminus \Gamma^\times) \sqcup \Gamma^\times \sim (\overline{\mathbb{B}}^3 \setminus \Gamma) \sqcup \Gamma = \overline{\mathbb{B}}^3$. *So*

$$\begin{aligned} \overline{\mathbb{B}}^3 &= (\overline{\mathbb{B}}^3 \setminus \{0\}) \sqcup \{0\} \sim (\overline{\mathbb{B}}^3 \setminus \{0\}) \cup (\overline{\mathbb{B}}^3 \setminus \{0\}) \cup \{0\} \\ &(\overline{\mathbb{B}}^3 \setminus \{0\} \textit{is paradoxical}) \\ &= \overline{\mathbb{B}}^3 \cup (\overline{\mathbb{B}}^3 \setminus \{0\}) \sim \overline{\mathbb{B}}^3 \cup \overline{\mathbb{B}}^3. \\ &(\overline{\mathbb{B}}^3 \setminus \{0\} \textit{is equidecomposable for } \mathbb{B}^3). \end{aligned} \tag{1.103}$$

□

Corollary *$\exists$ no nonzero, finite measure on $\overline{\mathbb{B}}^3$, which is $\mathrm{Isom}(\mathbb{R}^3)$-invariant such that all subsets of $\overline{\mathbb{B}}^3$ are measurable.*

Claim *$\exists \mathbb{F}_2 \cong \mathcal{F} \leqslant \mathrm{SO}(3)$ subgroup.*

PROOF *Let φ, ρ be rotations about $z-$ and $x-$ axes of angle $\theta = \arccos\left(\frac{3}{5}\right)$.*

$$\Rightarrow \ \varphi^{\pm 1} = \begin{pmatrix} \frac{3}{5} & \mp\frac{4}{5} & 0 \\ \pm\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \rho^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \mp\frac{4}{5} \\ 0 & \pm\frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{1.104}$$

To see $\langle\varphi,\rho\rangle \cong \mathbb{F}_2$, we assume for contradiction $\exists$ nontrivial "reduced word" $w \in \langle\varphi,\rho\rangle$, we prove that $\forall w \in \langle\varphi,\rho\rangle \leqslant \mathrm{SO}(3)$,

$$w \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} /5^k \quad \text{s.t. } 5 \nmid b. \tag{1.105}$$

by induction on $|w|$. □

1.10 Littlewood 3 Principles

Littlewood 3 Principles:

(A)Every set is nearly a finite union of intervals;

(B)Every function is nearly continuous;

(C)Every convergent sequence (of functions) is nearly uniformly convergent.

Theorem 1.13 (Egoroff Theorem) *(C) Let $E \subset \mathbb{R}^d$ be Lebesgue measurable $\mathcal{L}^d(E) < \infty$. Let $\{f_j\}_1^\infty : E \to \mathbb{R}$ be Lebesgue measurable. Assume $f_j \to f$ a.e. on E. Then $\forall\varepsilon > 0$, $\exists A_\varepsilon$ closed $\subset E$, s.t. $\mathcal{L}^d(E \setminus A_\varepsilon) < \varepsilon, f_j \to f$ uniformly on A_ε.*

Example $f_n(x) = x^n$ on $[0,1]$. $f_n \to f \equiv 0$ pointwise on $[0,1[$ but not uniformly. $\forall\varepsilon > 0, \forall n \in \mathbb{N}, \exists$ small $\delta > 0$ s.t. $f(1-\delta) > \varepsilon$. But $\forall\delta' > 0$ on the closed interval $A_{\delta'} = [0, 1-\delta']$, $f_n \to f$ uniformly.

Proof WLOG, can assume $f_n \to f$ everywhere on E. We quantify the positive convergence by defining

$$E_j^n := \{x \in E : |f_j(x) - f(x)| < \frac{1}{n}\}, E_{(k)}^j := \bigcap_{j \geqslant k} E_j^n. \tag{1.106}$$

Then $\tilde{E}_{(k)}^n \subset \tilde{E}_{(k+1)}^n \nearrow E$ as $k \nearrow \infty$ for each $n \in \mathbb{N}$. So $\mathcal{L}^d(E) = \lim_{k\to\infty} \mathcal{L}^d(\tilde{E}_k^n)$. Pick a subsequence $\{E_{(k_\nu)}^n\}_{\nu\in\mathbb{N}}$ s.t. $\mathcal{L}^d(E \setminus \tilde{E}_{(k_\nu)}^n) \leqslant \dfrac{1}{2^n}$ for $\nu \geqslant \nu_0(n)$.

Consider the "tail event"

$$\tilde{A}_\varepsilon = \bigcap_{k \geqslant N_0} \tilde{E}_{(k_{\nu_0})}^n = \bigcap_{n \geqslant n_0} \bigcap_{j \geqslant k} E_j^n \tag{1.107}$$

for each $N_0 = N_0(\varepsilon)$ to be determined.

Observe that on $\tilde{A}_\varepsilon$, $|f_j - f| < \dfrac{1}{n}$, $\forall n \geqslant N_0$, $\forall j \geqslant k_{\nu_0(n)}$. Then $\forall \nu > 0$, by choosing $N_0 \geqslant \dfrac{1}{\nu}$ we get $|f_j - f| < \varepsilon$ on $\tilde{A}_\varepsilon$ $\forall j \geqslant k_{\nu_0(n)}, \forall n \geqslant N_0 \geqslant f_j$ converges uniformly to f on $\tilde{A}_\varepsilon$. Also, $\mathcal{L}^d(E \setminus \tilde{A}_\varepsilon) = \mathcal{L}^d\Big(\bigcup_{n=N_0}^{\infty}(E \setminus \tilde{E}^{(n)}_{k_{\nu_0(n)}})\Big) \leqslant \sum_{n=N_0}^{\infty} \dfrac{1}{2^n} < \dfrac{\varepsilon}{2}$ for suitable $N_0 = N_0(\varepsilon)$.

Finally, $\tilde{A}_\varepsilon$ is a $G_\delta-$ set, hence $\mathcal{L}^d-$ measurable. By the inner regularity of $\mathcal{L}^d$, $\exists$ closed set $A_\varepsilon \setminus \tilde{A}_\varepsilon$ with $\mathcal{L}^d(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \dfrac{\varepsilon}{2}$. This A_ε is the desired set. $\square$

Theorem 1.14 (Lusin's Theorem) *Let $E \subset \mathbb{R}^d$ be Lebesgue measurable with $\mathcal{L}^d(E) < \infty$. Let $f : E \to \mathbb{R}$ be Lebesgue measurable, $|f(x)| < \infty, \forall x \in E$. Then $\forall \varepsilon > 0$, $\exists$ closed $K_\varepsilon \setminus E$ s.t. $\mathcal{L}^d(E \setminus K_\varepsilon) \leqslant \varepsilon$ and $f|_{K_\varepsilon}$ is continuous.*

Example $f = \chi_{\mathbb{Q}\cap[0,1]}$. We can choose an open set $\mathcal{O} \subset \mathbb{R}$ s.t. $\mathcal{O} \setminus (\mathbb{Q} \cap [0,1])$ and $\mathcal{L}^1(\mathcal{O}) < \varepsilon$. Then $f|_{[0,1]\setminus\mathcal{O}}$ is continuous.
Remark "$f|_{K_\varepsilon}$ is continuous" means that "$f|_{K_\varepsilon}$: $K_\varepsilon \to R$ is continuous".
Convention: $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ Borel measurable if and only if f is $(B_{\mathbb{R}^{d_1}}/B_{\mathbb{R}^{d_2}})-$ measurable.
Recall $\mathbb{B}_{\mathbb{R}^{d_1}} \subsetneq \mathbb{M}_{\mathrm{Leb}(\mathbb{R}^{d_1})}$.
Fact Compositions of Lebesgue measurable functions are not necessarily Lebesgue measurable. Consider $\Phi(x) = F(x) + x : [0,1] \to [0,2]$.
Proof of Lusin's Theorem. First, consider $f = \chi_F$ for $F \in \mathcal{M}_{\mathrm{Leb}}, F \subset E$. By inner regularity, $\exists K_\varepsilon \subset F$ closed with $\mathcal{L}^d(F \setminus K_\varepsilon) < \varepsilon$. Then $f|_{K_\varepsilon} = \chi_{K_\varepsilon} \equiv 1$ on K_ε.

Now let φ be a simple function on E, i.e. $\varphi = \sum_{j=1}^N \alpha_j \chi_{E_j}$ for $\alpha_j \in \mathbb{R}$. $E_j \in \mathcal{M}_{\mathrm{Leb}}, N \in \mathbb{N}$. By considering refinements, we may assume $\{E_j\}^N$ are disjoint.

Apply the previous argument, to get disjoint compact subsets $K_j \subset E_j, 1 \leqslant j \leqslant N$ s.t. $\mathcal{L}^d(E_j \setminus K_j) < \dfrac{\varepsilon}{2^N}$. $\forall j = 1, \cdots, N$. Consider $\tilde{\varphi} = \sum_{i=1}^N \alpha_j \chi_{K_j}$.

$\tilde{\varphi}$ is continuous in $\bigsqcup_{j=1}^N K_j$, since

$$\mathrm{dist}(K_i, K_j) \geqslant c_0 > 0, \quad \forall i \neq j. \tag{1.108}$$

That is, $\forall$ simple function $\varphi : E \to \mathbb{R}$, $\exists K_\varepsilon \subset E$ compact, $\exists \tilde{\varphi} \in C^0(K_\varepsilon)$ s.t. $\mathcal{L}^d(E \setminus K_\varepsilon) < \varepsilon, \tilde{\varphi} = \varphi$ on K_ε.

Finally, let $f : E \to \mathbb{R}$ be Lebesgue measurable. Take simple functions $\varphi \to \mathbb{R}$ s.t.

$\varphi_j \nearrow f$ pointwise. Let $K_j \setminus E$ be compact s.t. $\mathcal{L}^d(E \setminus K_j) \leqslant \frac{\varepsilon}{2^k}$ and $\varphi_j|_{K_j}$ is continuous.

By Egoroff, $\exists$ closed set $F_\varepsilon \subset E$ s.t.

$$\mathcal{L}^d(E \setminus F_\varepsilon) < \frac{\varepsilon}{2} \text{ and } \varphi_j \to f \text{ uniformly on } F_\varepsilon \tag{1.109}$$

The proof is complete by choosing $K_\varepsilon = F_\varepsilon \cap \bigcap_{j=1}^{\infty} K_j$. □

Lusin's theorem cannot be strengthed to "a.e.". For example, for $f = \chi_{[0,\infty[}$, $\nexists\, g \in C^0(\mathbb{R})$ s.t. $g = f$ $\mathcal{L}^1-$a.e. In fact, $\exists \mathcal{L}^1-$ measurable $E \subset [0,1]$ such that $\forall g =$ (Lebesgue measurable) function that equals $\mathcal{L}^1-$ a.e. χ_E, g must be continuous at every point in $[0,1]$.

Generalization:

Theorem 1.15 (Alberti's Lusin Theorem(1991)) *Let $\mathbb{R}^d$ be an open set. Let $\boldsymbol{v}$ be a Borel vector field on Ω. Then $\forall \varepsilon > 0$, $\exists$ closed set $K_\varepsilon \subset \Omega$ s.t. $\mathcal{L}^d(\Omega \setminus K_\varepsilon) < \varepsilon_1$ and $\boldsymbol{v} = \nabla\varphi$ for some $\varphi \in C_0^1(K_\varepsilon)$ on K_ε.*

Theorem 1.16 *Let $(X, \mathcal{F}, \mu)$ be a locally compact Hausdorff topological space. Let $f : X \to \mathbb{C}$ be measurable. Let $A \subset X$ be $\mu(A) < \infty$ and $f \equiv 0$ on A^c. Then $\forall \varepsilon > 0, \exists g \in C_c^0(X)$ s.t. $\mu(\{f \neq g\}) < \varepsilon$. (In addition, can require $\|g\|_{\sup(X)} \leqslant \|f\|_{\sup(X)}$)*

All the arguments for the $\mathbb{R}^d$ can pass through, except that we don't know if $\tilde{\varphi} = \sum_j \alpha_j \chi_{K_j}$ (where $\{K_j\}_1^N$ are disjoint compact subsets of X) is continuous.

No metric space structure here.

To overcome this difficulty, we use Hausdorff. For $K_i \cap K_j = \varnothing$ compace set, $\exists$ open set $U_i \supset K_i, U_j \supset K_j, U_i \cap U_j = \varnothing$.

Tool:(C^0-Urysohn Lemma) For X = locally compact Hausdorff topological space $V \subset X$ open, $K \subset V$ compact. $\exists \psi \in C^0(X)$, s.t. $\psi \equiv 1$ on K, $\psi \equiv 0$ on V^c, $0 \leqslant \psi \leqslant 1$ on X.

Notation: $K \prec \psi \prec V$ means that $\psi \equiv 1$ on K, $\psi \equiv 0$ on V^c. $\text{supp}(\psi) = \overline{\{\psi \neq 0\}}$.

Proof Enumerate $\mathcal{Q} \cap [0,1] = \{r_i\}_{i=1}^\infty$, $r_1 = 0, r_2 = 1$. Choose $K \subset V_1 \Subset V_0 \Subset V$[9]. If

[9] $U \Subset V \Leftrightarrow U \subset \overline{U} \subset V$

$V_{r_1}, V_{r_2}, \cdots, V_{r_n}$ has been chosen such that $V_{r_i} \subset V_{r_j}$ whenever $r_i > r_j$, then choose $V_{r_{n+1}}$ as follows: note that (by the choice of r_1,r_2),$\exists \alpha, \beta \in \{1, 2, \cdots, n\}$ s.t. $r_\alpha :=$ the largest of $\{r_i\}_{i=1}^n$ that is less than r_{n+1}, $r_\beta :=$ the smallest of $\{r_i\}_{i=1}^n$ that is greater than r_{n+1}. Choose $V_{r_{n+1}}$ s.t. $V_{r_\beta} \Subset V_{r_{n+1}} \Subset V_{r_\alpha}$ (nested sequence of open subsets). In this way we obtained open sets $\{V_r : r \in \mathbb{Q} \cap [0, 1]\}$ s.t. $V_r \Subset V_t$ whenever $r > t$.

Define for $r, s \in \mathbb{Q} \cap [0, 1]$

$$f_r(x) = \begin{cases} r, & \text{if } x \in V_r, \\ 0, & \text{if } x \notin V_r, \end{cases} \quad g_r(x) = \begin{cases} 1, & \text{if } x \in \overline{V_s}, \\ s, & \text{if } x \notin \overline{V_s}. \end{cases} \tag{1.110}$$

where

$$f := \sup_{r \in \mathbb{Q} \cap [0,1]} f_r, \quad g := \sup_{s \in \mathbb{Q} \cap [0,1]} g_s \tag{1.111}$$

Aside semicontinuity:

Definition 1.35 *$f : X =$ topological space $\to \mathbb{R}$(or $\overline{\mathbb{R}}$) is*

- *LSC (lower semicontinuous) : $\{f > \alpha\}$ is open $\forall \alpha \in \mathbb{R}$;*
- *USC (upper semiconitnuous) : $\{f < \alpha\}$ is open $\forall \alpha \in \mathbb{R}$.*

By definition, continuous $\Leftrightarrow$ LSC + USC.

Back to Urysohn.f_r is LSC:

$$\{f_r > \alpha\} = \begin{cases} X, & \text{if } \alpha < 0, \\ V_r, & \text{if } 0 \leqslant \alpha < r, \\ \varnothing, & \text{if } \alpha \geqslant r. \end{cases} \tag{1.112}$$

Then $f := \sup_{r \in \mathbb{Q} \cap [0,1]} f_r$ is LSC, because $\{f > \alpha\} = \bigcup_{r \in \mathbb{Q} \cap [0,1]} \{f_r > \alpha\}$. Similarly, $g = \inf_s g_s$ is USC.

Also, $0 \leqslant f \leqslant 1$, $f \equiv 1$ on K, $f \equiv 0$ on V^c. So it remains to prove $f = g$.

- $f \leqslant g$: If $f_r(x) > g_s(x)$, then $r > s$ but $x \in V_r \setminus \overline{V_s}$;
- $f \geqslant g$: If $f(x) < g(x)$, then $\exists r, s \in \mathbb{Q} \cap [0, 1]$, s.t.

$$\sup_{\tilde{r}} f_{\tilde{r}} = f(x) < r < s < g(x) = \inf_{\tilde{s}} g_{\tilde{s}} \tag{1.113}$$

$\Rightarrow x \notin V_r, x \in \overline{V_s}$. So $x \in \overline{V_s} \setminus V_r$, but then $s < r$. $\rightarrow|$. □

Corollary (Partition of Unity) $X =$ locally compact Hausdorff topological space, $K \subset X$ compact, $K \subset V_1 \cup V_2 \cup \cdots V_n$. Then $\exists \rho_1, \cdots, \rho_n \in C^0(X)$, $\rho_j \prec V_j$ and $\sum_{j=1}^{n} \rho_j(X) = 1, \ \forall K$.

1.11 Riesz Representation Theorem

Definition 1.36 *$C_c^0(X) = \{f : X \to \mathbb{R}, f \text{ is continuous}, \operatorname{supp}(f) \subset X \text{ compact}\}$. Λ is a positive linear functional on X if and only if $\Lambda \in [C_c^0(X)]^*$ s.t. $\Lambda f \leqslant 0, \forall f \leqslant 0 \in C_c^0(X)$.*

Definition 1.37 *Let $X =$ Hausdorff topological space. A measure μ on Borel $\sigma-$ algebra $\mathcal{B}(X)$ is* Radon *($\mu \in \mathcal{M}(X)$) if and only if it is*

(i) locally finite: $\mu(K) < \infty, \forall K$ compact;

(ii) outer regular: $\forall B \in \mathcal{B}(X)$, $\mu(B) = \inf\{\mu(O) : O \supset B \text{ open}\}$.

(iii) inner regular on open sets: O open, $\mu(O) = \sup\{\mu(K) : K \subset O \text{ compact}\}$.

Example(1) $\mathcal{L}|_{\mathcal{B}_{\mathbb{R}^d}}$ on $\mathbb{R}^d$;

(2) Dirac delta: $\delta_{x_0} := 1$ if $x_0 \in E$ and 0 on elsewhere;

(3) Gaussian measure on $\mathbb{R}^d$;

(4) Probability measure on Polish space (homeomorphic to separable complete metric space).

Non-example Sorgenfrey line $X = ([0, 1[)$, Sorgenfrey topology $\mathcal{T}_{sor}$ is the topology generated by $\{[\alpha, \beta[: 0 \leqslant \alpha < \beta \leqslant 1\}$.

Claim $\mathcal{L}^1$ on X is not inner regular.

Proof This is because all the compact sets are at most countable. Indeed, let $K \subset X$ be compact. Fix $x \in K$, cover K by $\{[x, 1[, \ [0, x - \frac{1}{n}[\}_{n \in \mathbb{N}} \subset \mathcal{T}_{sor}$. Since it has a finite subcover, $\exists q = q(x) \in \mathbb{Q} \cap [0, 1[$ such that $]q(x), x] \cap K = \{x\}$, so $\{]q(x), x] : x \in K\}$ is pairwise disjoint. The map $K \hookrightarrow \mathbb{Q}$ is injective, $x \mapsto a$ rational in $\{]q(x), x]\}$, so $|K| \leqslant |\mathbb{Q}| =$ countable. □

Theorem 1.17 (Riesz) *Let $X =$ local compact Hausdorff topological space. For all $\Lambda \in [C_c^0(X)]^*$ positive linear functional, $\exists$ nice measure μ on some $\sigma-$ algebra $\mathcal{F} \supset \mathcal{B}$ such that $\Lambda f = \int_X f \mathrm{d}\mu, \ \ \forall f \in C_c^0(X)$. "nice"= Complete + Radon.*

Proof Step 1:(Uniqueness) Let $\mu, \tilde{\mu}$ be Radon measures representing Λ. By regularity of $\mu, \tilde{\mu}$, it suffices to check that $\mu(K) = \mu(\tilde{K}), \forall K \subset X$ compact. Pick open $O \supset K$ s.t.

$\mu(O \setminus K) < \varepsilon$. By Urysohn, $\exists K \prec f \prec O$, which means $f \equiv 1$ on K, $f \in C_c^0(O)$. Then $\tilde{\mu}(K) = \int_X \chi_K \mathrm{d}\tilde{\mu} = \int_X f \mathrm{d}\tilde{\mu} = \Lambda \tilde{f} = \int_X$, and $\mu(K) + \varepsilon \geqslant \mu(0) = \int_X \chi_O \mathrm{d}\mu \geqslant \int_X f \mathrm{d}\mu$. Here ε is arbitrary, so $\tilde{\mu}(K) \leqslant \mu(K)$. By symmetry, $\tilde{\mu}(K) \geqslant \mu(K)$.

Step 2: Construct $\sigma-$ algebra $\mathcal{F}$ and measure μ. For $V \subset X$ open, set

$$\mu(V) := \sup\{\Lambda f : f \prec V\}. \tag{1.114}$$

By construction, μ is incrasing, thus we can extend the definition of μ to all subsets of X by setting

$$\hat{\mu}(E) := \inf\{\mu(O) : O \subset X \text{ open}, O \supset E\}. \tag{1.115}$$

In the sequel, write $\hat{\mu} = \mu$.

Define $\mathcal{F} := \{E \subset X : \mu(E) < \infty, \mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$. $\mathcal{F}_{\text{loc}} := \{E \subset X : E \cap K \in \mathcal{F}, \forall K \text{ compact}\}$.

Step 3: Note that μ is monotone, then if $\mu(E) = 0$, then $E \subset \mathcal{F}$.

Remark Λ is also monotone. If $f \leqslant$, then $\Lambda g = \Lambda f + \Lambda(g - f) \geqslant \Lambda f$.

Step 4: μ is $\sigma-$ subadditive on $\mathcal{P}(X)$. If $V_1, V_2 \subset X$ are open, by Urysohn, $\exists C^0$ p.o.u.[10], $\psi_1 \prec V_1$, $\psi_2 \prec V_2$, $\psi_1 + \psi_2 \equiv 1$ on some compact $K \subset V_1 \cap V_2$. Take $g \prec V_1 \cup V_2$, s.t.supp$(g) \subset K$, $\Lambda g = \Lambda(\psi_1 g) + \Lambda(\psi_2 g) \leqslant \mu(V_1) + \mu(V_2)$, where $\psi_1 g \prec V_1, \psi_2 g \prec V_2$. So $\mu(V_1 \cup V_2) \leqslant \mu(V_1) + \mu(V_2)$ by supremising over all $g \prec V_1 \cup V_2$. Finite subaddtivity can be extended to σ, which has been left as **exercise**.

Step 5: If K is compact, then $K \in \mathcal{F}$, $\mu(K) = \inf\{\Lambda f : K \prec f\}$.

"$\leqslant$": For any $f \succ K$, consider $V_{f>\alpha}, \forall \alpha \in]0,1[$. Then $\forall g \prec V_\alpha$, $\alpha g \leqslant f$ and $K \subset V_\alpha$. Then $\mu(K) \leqslant \mu(V_\alpha) = \sup\{\Lambda g : g \prec V_\alpha\} \leqslant \alpha^{-1} \Lambda f$ by monotonicity of Λ. Send $\alpha \nearrow 1$.

"$\geqslant$": We show that $\forall \varepsilon > 0$, $\exists f \succ K$, s.t. $\Lambda f \leqslant \mu(K) + \varepsilon$. Indeed, by the definition of μ, $\exists$ open $V \supset K$ s.t. $\mu(V) \leqslant \mu(K) + \varepsilon$. By Urysohn, take ψ with $K \prec \psi \prec V \;\Rightarrow \Lambda \psi \leqslant \mu(V) \leqslant \mu(K) + \varepsilon$.

Step 6: Every open set U satisfies $\mu(U) = \sup\{\mu(K) : K \text{ compact } \subset U\}$. Left as **exercise**. (Hint: $\Lambda f \leqslant \mu(\mathrm{supp}(f))$ for $f \in C_c^0(X)$).

Step 7: (μ is $\sigma-$ additive on $\mathcal{F}$). Again, we prove finite additivity. Then $\sigma-$ additivity follows by a routine argument.

Let K_1, K_2 be compact, $K_1 \cap K_2 = \varnothing$. By Hausdorff and Urysohn, $\exists f \in C_c^0(X)$ s.t.

[10] p.o.u. is the abbreviation of "partition of unity"

$0 \leqslant f \leqslant 1, f \equiv 0$ on K_1, $f \equiv 1$ on K_2. By step 5, $\mu(K_1 \cup K_2) = \inf\{\Lambda g : g \succ K_1 \cup K_2\}$, so $\exists g \succ K_1 \cup K_2$, $\Lambda g - \varepsilon < \mu(K_1 \cup K_2)$. Then $fg \succ K_2, (1-f)g \succ K_1$, thus

$$\mu(K_1 \cup K_2) \geqslant \Lambda(fg) + \Lambda((1-f)g) - \varepsilon \geqslant \mu(K_2) + \mu(K_1) - \varepsilon. \tag{1.116}$$

Step 8: $\forall E \in \mathcal{F}, \varepsilon > 0$, $\exists$ compact $K \subset E \subset O$ open with $\mu(O \setminus K) < \varepsilon$. Indeed $\exists K \subset E \subset O, \mu(O) < \mu(E) + \frac{\varepsilon}{2}$, $\mu(E) < \mu(K) + \frac{\varepsilon}{2}$. But $O \setminus K$ is open. By step 6, $O \setminus K \in \mathcal{F}$. Apply finite additivity of μ on $\mathcal{F}$ (step 7) $\Rightarrow$ $\mu(O \setminus K) + \mu(E) - \frac{\varepsilon}{2} \leqslant \mu(O) = \mu(K) + \mu(O \setminus K) \leqslant \mu(K) + \frac{\varepsilon}{2}$.

Step 9: $\mathcal{F}$ is closed under $\cup, \cap, \setminus$.(**exercise**).

Step 10: $\mathcal{F}_{\text{loc}}$ is $\sigma-$ algebra; $\mathcal{F} \supset \mathcal{B}(X)$.

Step 11: $\mathcal{F} = \{E \in \mathcal{F}_{\text{loc}} | \mu(E) < \infty\}$;

Step 12: Prove that $\Lambda f = \int_X f \mathrm{d}\mu, \forall f \in C_c^0(X)$. Call $\mathrm{supp}(f) = K$, $f(X) \subset [a,b]$. Note that it is enough to prove "$\leqslant$".

Discretize f: choose $\{y_0 < a < y_1 < y_2 < \cdots < y_n = b\}$ such that

$$y_{j+1} - y_j < \varepsilon. \tag{1.117}$$

Let $E_j = K \cap \{y_{j-1} \leqslant f < y_j\}$. Cover E_j by open O_j s.t.

$$\mu(E_j) + \delta > \mu(O_j), \tag{1.118}$$

$$f|_{O_j} < y_j + \eta. \tag{1.119}$$

$\varepsilon, \delta, \eta$ in (1.117),(1.118),(1.119) are to be determined. Let $\{\psi_j\}_{j=1}^n$ be a p.o.u. suborbinating to $\{O_j\}$, $\sum \psi_j \equiv 1$ on K. So

$$\begin{aligned}\Lambda f &= \sum_{j=1}^n \Lambda(\psi_j f) \leqslant \sum_{j=1}^n (y_j + \eta)\Lambda\psi_j \leqslant \sum_{j=1}^n (y_j + \eta)\mu(O_j) \\ &\leqslant \sum_{j=1}^n (y_j + \eta)(\mu(E_j) + \delta) = \sum_{j=1}^n [(y_j - \varepsilon) + \varepsilon + \eta](\mu(E_j) + \delta).\end{aligned} \tag{1.120}$$

But $f|_{E_j} \geqslant y_j - \varepsilon$, $K = \bigsqcup_{j=1}^{n} E_j$, so

$$\sum_{j=1}^{n}(y_j - \varepsilon)\mu(E_j) = (y_j - \varepsilon)\int_X \chi_{E_j} \mathrm{d}\mu \leqslant \int_X f \mathrm{d}\mu. \tag{1.121}$$

Then

$$\Lambda f \leqslant \int_K f + \delta \sum_{j=1}^{n}(y_j - \varepsilon) + (\varepsilon + \eta)\sum_{j=1}^{n}(\mu(E_j) + \delta). \tag{1.122}$$

Choosing $\eta = \varepsilon$, $\delta = \dfrac{\varepsilon}{100n(|b| \vee 1)}$ to get

$$\Lambda f \leqslant \int_X f \mathrm{d}\mu + C_0 \varepsilon, \tag{1.123}$$

where C_0 depends only on $|b|$ and $\mu(K)$. Sending $\varepsilon \searrow 0$ and we get $\Lambda f \leqslant \int_X f \mathrm{d}\mu$. $\square$

Convention A measure μ on $(X, \mathcal{F})$ is called positive/non-negative measure to emphasize that μ takes values in $[0, \infty]$.

Definition 1.38 *A complex measure is a function $\mu : \mathcal{F} \to \mathbb{C}$, s.t. $\forall E \in \mathcal{F}$ and $\forall$ partition $\{E_j\}_{n=1}^{\infty} \subset \mathcal{F}$ with $E = \bigsqcup_{j=1}^{n} E_j, \mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$.*

Definition 1.39 *A real/signal measure is a $\mathbb{R}-$valued complex measure.*

Definition 1.40 *Let $\mu : \mathcal{F} \to \mathbb{C}$ be a complex measure on $(X, \mathcal{F})$. The total variation measure of μ is the function $|\mu| : \mathcal{F} \to [0, \infty]$ given by*

$$|\mu|(E) := \sup\{\sum_{j=1}^{\infty} |\mu(E_j)| : E = \bigsqcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{F}\} \tag{1.124}$$

Theorem 1.18 *Let μ be a complex measure on $(X, \mathcal{F})$.*

(1) *μ defines a positive measure on $(X, \mathcal{F})$*

(2) *$|\mu|$ dominates μ in the sense that $|\mu|(E) \geqslant |\mu(E)|, \forall E \in \mathcal{F}$. In addition, $|\mu|$ is the smallest positive measure dominating μ.*

(3) $\|\mu\| := |\mu|(X)$. *For X = locally compact Hausdorff topological space, $\mathcal{M}^{\mathbb{C}}(X) := \{$complex regular Borel measure$\}$*[11]*, equipped with $\|\cdot\|$ is a Banach space.*

Recall $(\mathcal{B}, \|\cdot\|)$ is a Banach space if and only if it is a complete[12] normed vector space.
Remark μ is regular if and only if $|\mu|$ is regular.

Proof (1) Let $E \in \mathcal{F}, E = \bigsqcup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{F}$ be arbitrary.

$|\mu|(E) \geqslant \sum_{i=1}^{\infty} |\mu|(E_i)$. Take any $t_i \in \mathbb{R}$ with $t_i \leqslant |\mu|(E_i)$. Then $\exists \{A_{ij}\}_{j=1}^{\infty} \subset \mathcal{F}$ s.t. $E_i = \bigsqcup_{j=1}^{\infty} A_{ij}$ for each i and $\sum_{j=1}^{\infty} |\mu(A_{ij})| \geqslant t_i$. But $E = \bigsqcup_{i=1}^{\infty}\bigsqcup_{j=1}^{\infty} A_{ij}$, so $|\mu|(E) \geqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\mu(A_{ij})| \geqslant \sum_{i=1}^{\infty} t_i$.

$|\mu|(E) \leqslant \sum_{i=1}^{\infty} |\mu|(E_i)$. By definition of $|\mu|(E)$, we need to prove that $\forall$ partition $E = \bigsqcup_{k=1}^{\infty} B_k$, we have $|\mu(B_k)| \leqslant \sum_{i=1}^{\infty} |\mu|(E_i)$. Note that $\{B_k \cap E_i\}$ partitions B_k for each fixed i and E_i for each fixed k. So

$$\sum_{k=1}^{\infty} |\mu(B_k)| = \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(B_k \cap E_i) \right| \leqslant \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |\mu(B_k \cap E_i)| \leqslant \sum_{i=1}^{\infty} |\mu|(E_i). \tag{1.125}$$

(2) Dominating is clear: since $E \cup \varnothing = E$ is a partition. Minimality: if a positive measure λ satisfies $\lambda(E) \geqslant |\mu(E)|, \forall E \in \mathcal{F}$, then for any partition $\{E_i\}_{i=1}^{\infty} \subset \mathcal{F}$ for E, $\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geqslant \sum_{i=1}^{\infty} |\mu(E_i)|$. Taking sup over all partitions, we get $\lambda(E) \geqslant |\mu|(E)$.

(3) $\|\mu\| := |\mu|(X)$ is called the "total variation norm" of μ. □

Theorem 1.19 (Riesz Representation) *Let μ be a complex measure on $(X, \mathcal{F})$ = locally*

[11] Indeed a vector space. Inner and outer regularity holds.

[12] If $\|x_n - x_l\| \to 0$ as $n, l \to \infty$, then $\exists x \in \mathcal{B}$ s.t. $\|x_n - x\| \to 0$ as $n \to \infty$.

compact Hausdorff space. Then $\exists$ *isomprphism (of vector spaces)* $C_0^0(X)$

$$\begin{gathered}[C_0^0(X)]^* \simeq \mathcal{M}^{\mathbb{C}}(X) \\ \Phi_\mu \hookleftarrow \mu \\ \Phi_\mu(f) := \int_X f \mathrm{d}\mu. \end{gathered} \tag{1.126}$$

Moreover, this isomprphism is an isometry $\|\Phi_\mu\| = \|\mu\| := |\mu|(X)$.

Definition 1.41 $C_0^0(X) := \{f \in C^0(X) : \forall \varepsilon > 0,\ \exists$ *compact* $K \subset X$ *s.t.* $\sup_{x \in X \setminus K} |f(x)| < \varepsilon\}$.
The "natural" norm on $C^0(X)$ *is the supremum norm.* $\Phi_\mu \in [C_0^0(X)]^* = \{$*bounded linear functions* $X \to \mathbb{R}\}$ *has the "operator" norm:* $\|\Phi_\mu\| := \sup\{|\Phi_\mu(f)| : f \in C_0^0, \|f\| \leqslant 1\}$.

1.12 Decomposition of Measure

Given $(X, \mathcal{F})$. Let μ be a positive measure and let λ_1, λ_2 be complex measure.

Definition 1.42 (1) $\lambda_1 << \mu$(λ_1 *is absolutely continuous w.r.t.* μ) *if and only if* $\mu(E) = 0$ *for* $E \in \mathcal{F} \Rightarrow \lambda_1(E) = 0$.
(2) $\lambda_1 \perp \lambda_2$($\lambda_1, \lambda_2$ *are mutually singular*) *if and only if* λ_i *are concentrated on* $A_i (i = 1, 2)$ *with* $A_1 \cap A_2 = \varnothing$. *Here "*$\lambda$ *is concentrated on* A*"* $\Leftrightarrow \forall B \in \mathcal{F}, B \cap A = \varnothing, \lambda(B) = 0$.

Example Consider λ_1 on $\mathbb{R}$ given by $\lambda_1(E) := \int_E f \mathrm{d}x$ for continuous f. Then $\lambda_1 << \mathcal{L}^1$.

Theorem 1.20 *Let* μ *be a positive,* $\sigma-$ *finite measure on* $(X, \mathcal{F})$ *and let* λ *be a complex measure on* $(X, \mathcal{F})$.

(1)(*Lebesgue Decomposition*) $\exists!\,(\lambda_{\text{ac}}, \lambda_{\text{s}}), \lambda_{\text{ac}} << \mu, \lambda_{\text{s}} \perp \mu$.

(2)(*Radon-Nikodym*) $\exists!\, h \in \mathcal{L}^1(X, \mathcal{F}, \mu; \mathbb{C})$ *s.t.* $\forall E \in \mathcal{F}$, $\int_E \mathrm{d}\lambda_{\text{ac}} = \lambda_{\text{ac}}(E) = \int_E h \mathrm{d}\mu$, *we note* $\mathrm{d}\lambda_{\text{ac}} = h\mathrm{d}\mu$.

Recall $h \in \mathcal{L}^1 \Leftrightarrow \int_X |h|\, \mathrm{d}\mu < \infty$. Notation: $h := \dfrac{\mathrm{d}\lambda_{\text{ac}}}{\mathrm{d}\mu}$ is the Radon-Nikodym deriative.
Example $(\Omega, \mathcal{F}, \text{Prob}) \xrightarrow{Y}$,

$$Y_\# \text{Prob}(E) := \text{Prob}(\{\omega \in \Omega : Y(\omega) \in E\}) = \text{Prob}(Y \in E) \tag{1.127}$$

when $Y_{\#}(\mathrm{Prob})$ is also continuous w.r.t. $\mathcal{L}^1$ on $\mathbb{R}$. Y is called a(n) (absolutely) continuous random variable.

$$f = \frac{\mathrm{d}Y_{\#}(\mathrm{Prob})}{\mathrm{d}\mathcal{L}^1} = \text{p.d.f. of } Y,\ \mathrm{Prob}(E) = \int_E f\mathrm{d}x. \tag{1.128}$$

Proof (Von Neumann)

Step 1: (Uniqueness) If $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}} = \hat{\lambda}_{\mathrm{ac}} + \hat{\lambda}_{\mathrm{s}}$ for $\lambda_{\mathrm{ac}}, \hat{\lambda}_{\mathrm{ac}} \ll \mu$ and $\lambda_{\mathrm{s}} \perp \mu, \hat{\lambda}_{\mathrm{s}} \perp \mu$, then

$$\mu \gg \lambda_{\mathrm{ac}} = \hat{\lambda}_{\mathrm{ac}} = \hat{\lambda}_{\mathrm{s}} \perp \mu. \tag{1.129}$$

then it equals to 0. If $\lambda_{\mathrm{ac}}(E) = \int_E h\mathrm{d}\mu = \int_E \hat{h}\mathrm{d}\mu$ for $h, \hat{h} \in \mathcal{L}^1(\mu)$, then $\int_E (h - \hat{h})\mathrm{d}\mu = 0, \forall E \in \mathcal{F} \Rightarrow h = \hat{h}, \mu-$ a.e..

Step 2: The case that λ is a positive, finite measure.

Claim: $\exists w \in \mathcal{L}^1(\mu)$ s.t. $0 < \omega < 1$ everywhere on X. Since μ is $\sigma-$finite, cut $X = \bigcup_{n=1}^{+\infty} E_n, E_n \in \mathcal{F}$ with $\mu(E_n) < \infty$. Then take

$$w := \frac{\chi_{E_n}}{2^n(\mu(E_n) + 1)}. \tag{1.130}$$

Now, consider a new measure φ s.t.

$$\mathrm{d}\varphi = \mathrm{d}\lambda + w\mathrm{d}\mu. \tag{1.131}$$

Consider functional $\Lambda : \mathcal{L}^2(X, \mathcal{F}, \varnothing; \mathbb{R}), f \mapsto \int_X fg\mathrm{d}\phi$. $|\Lambda f| \leqslant \mathrm{const} \cdot \|f\|_{L^2}, \|f\|_{L^2} = \sqrt{\int_X |f|^2 \mathrm{d}\phi}$. Λ is clearly linear. To see it is bounded,

$$|\Lambda f| = \left|\int_X f\mathrm{d}\lambda\right| \leq \int_X |f| \mathrm{d}\lambda \leq \int_X |f| \mathrm{d}\varphi = \langle |f|, \chi_X\rangle \leqslant \|f\| \|\chi_X\|_2 = \|f\|\sqrt{\varphi(X)}. \tag{1.132}$$

So $\|\Lambda\|_{L^2(\varphi)\to\mathbb{R}} := \sup\{|\lambda f| : \|f\| \leq 1\} \leq \sqrt{\varphi(X)} < \infty$. By Riesz Representation Hilbert spaces[13],

$$\exists!\, g \in L^2(\varphi) \text{ s.t. } \int_X f \mathrm{d}\lambda = \Lambda f = \int_X fg \mathrm{d}\varphi, \forall f \in L^2(\varphi). \tag{1.133}$$

Note that by choosing $f = \chi_E$, we get $\lambda(E) = \int_E g\mathrm{d}\varphi, \forall E \in \mathcal{F}$. Then

$$1 \geqslant \frac{\lambda(E)}{\varphi(E)} = \frac{1}{\varphi(E)} \int_E g\mathrm{d}\varphi \geq 0 \text{ for } \varphi(E) > 0 \tag{1.134}$$

This indicates $0 \leq g \leq 1$, $\varphi-$a.e.. Then applying (1.133) and (1.131) and we have

$$\int_X (1-g) f \mathrm{d}\lambda = \int_X fgw \mathrm{d}\mu. \tag{1.135}$$

Define $\lambda_s := \lambda|_{\{g=1\}}$[14]. Then, if $f = \chi_{\{g=1\}} (\in L^2(\varphi))$, then LHS of (1.135) is 0 and RHS of (1.135) is $\int_{\{g=1\}} w\mathrm{d}\mu (w \not\equiv 0)$, then $\mu(\{g=1\}) = 0$.

But λ_s is concentrated on $\{g = 1\}$ so $\lambda_s \perp \mu$. Then define $\lambda_{ac} := \lambda - \lambda_s = \lambda|_{\{0 \leq g < 1\}}$. Now, choose $f = 1 + g + \cdots + g^n \in L^2(\varphi)$ for some $n \in \mathbb{N}$, then

$$\|f\|_{L^2(\varphi)} \leq (n+1)\|1\|_{L^2(\varphi)} = (n+1)\sqrt{\varphi(X)}. \tag{1.136}$$

We plug f into (1.135) and get

$$\int_X (1 - g^{n+1}) \mathrm{d}\lambda = \int_X (g + g^2 + \cdots + g^{n+1}) w \mathrm{d}\mu. \tag{1.137}$$

By Dominated Convergence Theorem, $\lambda(X \cap \{0 \leq g < 1\}) \equiv \lambda_{ac}(X)$. RHS: integrand is monotone. Say $(g + g^2 + \cdots + g^{n+1})w \nearrow h$ pointwise, by MCT, RHS $\to \int_E h\mathrm{d}\mu$. So $h = \frac{\mathrm{d}\lambda_{ac}}{\mathrm{d}\mu}$ as required. $\square$

For general complex measure λ, write $\lambda = \mathrm{Re}(\lambda) + \mathrm{i}\mathrm{Im}(\lambda)$ (signed (real) measure).

A real measure $\tilde{\lambda} = \tilde{\lambda}^+ - \tilde{\lambda}^-$ where $\lambda^{\pm}$ are positive measure.

An alternative definition for absolute continuity:

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[13]Let $\Psi \in \mathcal{H}^* =$ Bounded linear operator $\mathcal{H} \to \mathbb{R}$, then $\exists!\, v \in \mathcal{H}$ s.t. $\Psi(\omega) = \langle w, v\rangle\ \forall w \in \mathcal{H}$. Recall $\|\Psi\|$(the operator norm) $= \sup\{|\Psi(u)| : u \in \mathcal{H}\}$. In $\mathcal{H}$, let $\langle\cdot,\cdot\rangle$ be the inner product, then $\|v\| = \sqrt{\langle v, v\rangle}$.

[14]$\lambda_s(A) := \lambda(A \cap \{g = 1\})$

Proposition 1.19 *Let μ be a positive measure and λ be complex measure on $(X, \mathcal{F})$. $\lambda \ll \mu \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : |\lambda(E)| < \varepsilon$ whenever $\mu(E) < \delta$ for $E \in \mathcal{F}$.*

Polar Decomposition:

Q: Given a complex measure μ. Can one define $\int_X f \mathrm{d}\mu$?

A: Note that $\mu \ll |\mu|$, so $\exists h \in L^1(|\mu|)$ s.t. $\mathrm{d}\mu = h\mathrm{d}\,|\mu|$, $|h| = 1, |\mu| -$a.e..

Proof of $|h| = 1, |\mu| -$a.e.. $|h| \leq 1, |\mu| -$a.e.:

Suppose $|h| \geqslant 1+\varepsilon$ on some $\Omega \in \mathcal{F}$ with $|mu|\,(\Omega) > 0$. Consider $[h]_\Omega := \dfrac{1}{|\mu|\,(\Omega)} \displaystyle\int_\Omega h\mathrm{d}\,|\mu|$. Then $|[h]_\Omega| \geq 1+\varepsilon$ by monotonicity of $\displaystyle\int \mathrm{d}\,|\mu|$. But by definition of h, $|[h]_\Omega| = \left|\dfrac{\mu(\Omega)}{|\mu|\,(\Omega)}\right| \leqslant 1$.

$|h| \geqslant 1, |\mu| -$a.e.:

Consider $B := \{|h| \leq 1-\delta\}$. Let $\{B_j\}_1^\infty \subset \mathcal{F}$ by any partition of B. Then

$$\sum_j |\mu(B_j)| = \sum_j \left|\int_{B_j} h\mathrm{d}\mu\right| \leqslant (1-\delta) \sum_j |\mu|\,(B_j) = (1-\delta)\,|\mu|\,(B), \tag{1.138}$$

so $|\mu|\,(B) \leqslant (1-\delta)\,|\mu|\,(B) \geq |\mu|\,(B) = 0$.

In summary, $\forall$ complesx measure μ,

$$h := \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|} \text{ satisfies } |h| = 1. \tag{1.139}$$

It is customary to take h s.t. $|h(x)| = 1\ \forall x \in X,\ |\mu| -$a.e..

Now, suppose $\lambda \ll \mu$ = positive measure $\left|\dfrac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right| = 1$. Then by Radon-Nikodym, $\mathrm{d}\lambda = g\mathrm{d}\mu$, $\lambda\mathrm{d}\,|\lambda| = g\mathrm{d}\mu$, $\overline{h}h\mathrm{d}\,|\lambda| = \mathrm{d}\,|\lambda| = \overline{h}g\mathrm{d}\mu$, we have $\mathrm{d}\,|\lambda| = |g|\,\mathrm{d}\mu$.

In summary, $\dfrac{\mathrm{d}|\lambda|}{\mathrm{d}\mu} = \left|\dfrac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right|, \mu-$a.e..

Hahn-Decomposition for Signal Measure

Let μ be a signed measure on $(X, \mathcal{F})$. Anagolous to decomposition of # s/ functions/ matrices:

$$\mu = \mu^+ - \mu^- \text{ for } \mu^\pm = \frac{|\mu| \pm \mu}{2}. \tag{1.140}$$

Theorem 1.21 (Hahn) *Let μ be a signal measure on $(X, \mathcal{F})$. Then* $\exists X^{\pm} \in \mathcal{F}, X = X^+ \sqcup X^-$, $\mu^+ = \mu|_{X^+}, \mu^- = -\mu|_{X^-}$.

Proof Consider the polar decomposition $\mathrm{d}\mu = h\mathrm{d}\,|\mu|$. Since μ is real, $h = \pm 1$. Set $X^{\pm} := \{h = \pm 1\} = h^{-1}\{\pm 1\}$. Then $\forall E \in \mathcal{F}, \mu^+(E) := \dfrac{|\mu| + \mu}{2}(E) = \dfrac{1}{2}\displaystyle\int_E (1+h)\mathrm{d}\mu = \dfrac{1}{2}\displaystyle\int_{X^{\pm}\cap E} (1+1)\mathrm{d}\,|\mu|^2 = |\mu|\,(X^+ \cap E)$. $\square$

2 Functional Analysis

2.1 Basic Notations

k = field = $\mathbb{R}$ or $\mathbb{C}$, E, X = vector space over k.

Definition 2.1 *A **seminorm** on E is a function $\|\cdot\| : E \to \mathbb{R}$ s.t.*

(1) $\|\cdot\| \geqslant 0,\ \forall x \in E$;

(2) $\|\lambda x\| = |\lambda|\,\|x\|,\ \forall \lambda \in k, x \in E$;

(3) $\|x+y\| \leqslant \|x\| + \|y\|,\ \forall x, y \in E$.

Definition 2.2 *If, in addition, $\|x\| = 0 \ \Rightarrow\ x = 0$, then $\|\cdot\|$ is a norm.*

Example(of norm) If E is an inner product space, then E has a norm $\|x\| := \sqrt{\langle x, x\rangle}$.

Proposition 2.1 *If $(E, \|\cdot\|)$ is an NVS*[15]*, and $E_1 \leqslant E$ is a subspace, then $(E_1, \|\cdot\|)$ is an NVS.*

Example (1) k^n with norms ($x = (x_1, \cdots, x_n)$), $\|x\|_1 := \sum_{i=1}^{n} |x_i|$, $\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, $\|x\|_\infty := \max_{1\leqslant i\leqslant n} |x_i|$. They are equivalent.

Definition 2.3 *Two norms $\|\cdot\|, |\cdot|$ on an NVS E are equivalent if and only if $\exists\ 0 < c \leqslant C < \infty$ s.t.*

$$c\|x\| \leqslant |x| \leqslant C\|x\|.\ \forall x \in E. \tag{2.1}$$

Proposition 2.2 *Let $(E, \|\cdot\|)$ be a seminormed VS. Let $E_0 := \{x \in E : \||x|\| = 0\}$ (Note $E_0 \leqslant E$ is a subspace). Then E/E_0 is naturally an NVS, with $\|\cdot\|$,*

$$\|x + E_0\| := \inf\{\||x+y|\| : y \in E_0\} \tag{2.2}$$

Proof (Well def) If $x + E_0 = \tilde{x} + E_0$. We need to check $\|x + E_0\| = \|\tilde{x} + E_0\|$. Indeed, $\|x + E_0\| = \inf\{\||x + y'|\| : y' + E_0\} = \inf\{\||\tilde{x} + \tilde{y} - y + y'|\| : y' + E_0\} = \|\tilde{x} + E_0\|$.

(1)$\|x + E_0\| \geqslant 0$ is clear, with equality if and only if $\inf\{\||x+y|\| : y + E_0\} = 0$. Thus

[15] normed vector space

$\forall \varepsilon > 0, \exists \tilde{x} = x + y, y \in E_0$, s.t. $|||\tilde{x}||| < \varepsilon, 0 \leqslant |||x||| \leqslant |||\tilde{x}||| + |||y||| < \varepsilon$. So $x \in E_0 = 0$ in E/E_0.

(2)$\forall \lambda \in k, x + E_0 \in E/E_0, \|\lambda(x + E_0)\| \equiv \|(\lambda x) + E_0\| := \inf\{|||x + y||| : y \in E_0\} \overset{\tilde{y}=y/\lambda}{=} \inf\{|||\lambda x + \lambda \tilde{y}|||\} = |\lambda| \inf\{|||x + \tilde{y}||| : y \in E_0\} = |\lambda| \, \|x + E_0\|$.

(3)

$$\begin{aligned} &\|(x + E_0) + (y + E_0)\| := \|x + y + E_0\| = \inf\{|||x + y + z||| : z \in E_0\} \\ \leqslant &\inf\{|||x + z_1 + y + z_2||| : z_1 + z_2 \in E_0\} \leqslant \inf\{|||x + z_1||| + |||y + z_2||| : z_1, z_2 \in E_0\} \\ \leqslant &\|x + E_0\| + \|y + E_0\|. \end{aligned} \tag{2.3}$$

□

Example $\forall p \in [1, \infty], (k^n, \|\cdot\|_p)$ is an NVS, norm $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$ $(p < \infty)$.

To see $\triangle-$ineq: $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$,

$$\left(\sum_i |x_i + y_i|^p \right)^{\frac{1}{p}} \leqslant \left(\sum_i |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p \right)^{\frac{1}{p}}. \tag{2.4}$$

Hölder:

$$\sum_i |a_i b_i| \leqslant \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \left(\sum_i |a_i|^q \right)^{\frac{1}{q}}, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{2.5}$$

Proof

$$\left(\sum_i |x_i + y_i|^p \right)^{\frac{1}{p}} = \left(\sum_i |x_i + y_i| \cdot |x_i + y_i|^{p-1} \right)^{\frac{1}{p}} \leqslant \left(\sum_i |x_i| \, |x_i + y_i|^{p-1} + \sum_i |y_i| \, |x_i + y_i|^{p-1} \right)^{\frac{1}{p}}. \tag{2.6}$$

Using Hölder's inequality and we obtain

$$\left(\sum_i |x_i| \, |x_i + y_i|^{p-1} \right)^{\frac{1}{p}} \leqslant \left[\sum_i |x_i|^p \right]^{\frac{1}{p^2}} \left(\sum_i |x_i + y_i|^{q(p-1)} \right)^{\frac{p-1}{p^2}}, \tag{2.7}$$

Similarly for $\left(\sum_i |y_i| \, |x_i + y_i|^{p-1} \right)^{\frac{1}{p}}$. Thus,

$$\left(\sum_i |x_i + y_i|^p \right)^{\frac{1}{p}} \leqslant \left\{ \left(\sum_i |x_i|^p \right)^{\frac{1}{p^2}} + \left(\sum_i |x_i|^p \right)^{\frac{1}{p^2}} \right\} \left(\sum_i |x_i + y_i|^{q(p-1)} \right)^{\frac{p-1}{p^2}} \tag{2.8}$$

Rearrange and we get $\|x+y\|_p \leqslant \|x\|_p + \|y\|_p$. □

Example $\ell^p = \left\{x = \{x_j\}_1^\infty : \left(\sum_{j=1}^{\infty}|x_j|^p\right)^{\frac{1}{p}} < \infty\right\}, \ell^\infty = \{\text{bounded sequences}\}$, $C_{00} = \left\{x = \{x_j\}_1^\infty : \exists N \in \mathbb{N}, \text{s.t. } x_j = 0, \forall j \geqslant N\right\}$,$C_0 = \{x : x_j \to 0 \text{ as } j \to \infty\}$, $C = \{x : x \text{ is a convergent sequence}\}$.$C_{00} \subsetneq C_0 \subsetneq C \subsetneq \ell^\infty$.

Property 2.1 *Let Y, Z be closed subspaces of $X = NVS$. $Y+Z := \{y+z : y \in Y, z \in Z\} \leqslant X$ is not necessarily closed.*

Example $X = \ell^2 \times \ell^1$, $\|(a,b)\|_X = \|a\|_{\ell^2} + \|b\|_{\ell^1}$, $Y = \{(0,y) : y \in \ell^1\}, Z = \{(x,x) : x \in \ell^1\}$ are closed subspaces in X. Here $\ell^1 \subset \ell^2, \forall\{a_j\} \in \ell^1, \sum|a_j| < \infty$, then $\{a_j\} \in \ell^\infty$. So $\|\{a_j\}_{\ell^2}^2\| = \sum|a_j|^2 \leqslant \|\{a_j\}\|_{\ell^\infty}\|\{a_j\}\|_{\ell^1}$[16].

Take $\zeta_n = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, 0, \cdots) \subset C_{00}$. Then $(\zeta_n, 0) = (0, -\zeta_n) + (\zeta_n, \zeta_n) \in Y + Z$. But $\zeta_n \to (1, \frac{1}{2}, \frac{1}{3}, \cdots) \in \ell^2 \setminus \ell^1$. So $(\zeta_n, 0) \to \ell^2 \times \ell^1 \setminus Y + Z$, which indicates $Y + Z$ is not closed.

Definition 2.4 *A sequence $\{x_n\} \subset (X, \|\cdot\|)$ is Cauchy if and only if $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n, l \geqslant N$, then $\|x_n - x_l\| \leqslant \varepsilon$.*

Definition 2.5 *An NVS which is complete is called a Banach space. "Complete" means any Cauchy sequence converges in this space.*

Definition 2.6 *A Hilbert space is a Banach space equipped with an inner product.*

Proposition 2.3 *$(X, \|\cdot\|) =$ Banach space. $Y \leqslant X$ is a subspace. Then $(Y, \|\cdot\|)$ is a Banach space $\Leftrightarrow$ Y is closed.*

Proof "$\Rightarrow$" Take $\{y_n\} \subset Y$, s.t. $\|y_n - x\| \to 0$ as $n \to \infty$ for function $x \in X$. We want to show $x \in Y$. Since $\{y_n\}$ is convergent, $\{y_n\}$ is Cauchy in Y. But $(Y, \|\cdot\|)$ is Banach, $y_n \to \tilde{x}$ for some $\tilde{x} \in Y$, so $x = \tilde{x} \in Y$.

"$\Leftarrow$" Assume $(Y, \|\cdot\|)$ is closed in X. Let $\{y_n\} \subset Y$ be Cauchy. Since $Y \subset X$ and $(X, \|\cdot\|)$ is Banach, then $y_n \to x \in X$. But $x \in Y$ since Y is closed.

[16]这里注意 2 范数是平方求和的形式，平方的其中一项不变，另一项放到 ℓ_∞ 范数即可得到此不等式

Theorem 2.1 *An NVS $(X, \|\cdot\|)$ is Banach if and only if every absolutely convergent series converges.*

Example $\sum_{j=1}^{\infty} \|x_j\| < \infty \Rightarrow$ For $S_n := \sum_{j=1}^{n} x_j$, $\exists y \in X$ s.t. $\|S_n - y\| \to 0$ as $n \to \infty$.

Proof "$\Rightarrow$" If X is Banach, $\sum x_j$ absolutely convergent, then $\left\{S_n : \sum_{j=1}^{n} x_j\right\}$ is Cauchy, hence convergent:

$$\|S_n - S_l\| = \|\sum_{j=l+1}^{n} x_j\| \leqslant \sum_{j=l+1}^{n} \|x_j\| \to 0 \ \text{ as } l, n \nearrow \infty. \tag{2.9}$$

"$\Leftarrow$" Let $\{x_n\} \subset (X, \|\cdot\|)$ be Cauchy. We want to find $\{y_k\} \subset X$ s.t. $\sum \|y_k\| < \infty$ and $\sum y_k \to$ limit of x_n.

Take $n_1 < n_2 < \cdots$ s.t. $\|x_k - x_l\| \leqslant 2^{-j}$, $\forall k, l \geqslant n_j$. Then set $y_1 = x_{n_1}, y_2 = x_{n_2} - x_{n_1}, \cdots$. Here, $\sum y_k$ is absolutely convergent since $\sum \|y_k\| \leqslant 1$. By assumption, $\sum y_k$ converges to $L \in X$. Here $\sum_{k=1}^{M} y_k = x_{n_M}$. That is, every Cauchy sequence $\{x_n\} \subset X$ has a convergent subsequence. □

Lemma 2.1 *Let $(X, \|\cdot\|)$ be an NVS. Let $\{x_n\} \subset X$ be Cauchy, then $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\} \exists$ a convergent subsequence.*

Proposition 2.4 *$(X, \|\cdot\|)$ is a Banach space. $Y \leqslant X$ is closed, then X/Y is Banach with $\|x + Y\|_{\text{quot}} := \inf\{\|x + y\| : y \in Y\}$.(Left as **exercise**).*

Remark If X is Banach, $Y \leqslant X$ is proper dense, then Y is not Banach.
Example For $1 \leqslant p < \infty$, C_{00} is dense in ℓ^p, $C_{00} \subsetneq \ell^p$, but C_{00} is not Banach.
Examples of Banach spaces:
(1) $k^n, k = \mathbb{R}$ or $\mathbb{C}$;
(2) $\ell^p = \left\{\{a_j\}_{j=1}^{\infty} \subset k : \sum_{j=1}^{\infty} |a_j|^p < \infty\right\}, 1 \leqslant p < \infty$;
(3) $X =$ Hausdorff, $\mathrm{BC}(X) = \{f : X \to k : f \text{is bounded continuous}\}, \|f\| := \sup\{|f(x)| : x \in X\}$.$C^0(\mathbb{R})$ and $C_0^0(\mathbb{R})$ is Banach, but $C_c^0(\mathbb{R})$ is not Banach.
(4) $\ell^\infty =$ bounded sequences is Banach, where $\|\{a_j\}\|_{\ell^\infty} = \inf\{M > 0 : |a_j| \leqslant M, \forall j \in$

$N\}.C_0 \subsetneq C \subsetneq \ell^\infty$ is Banach, while $C_{00} \subsetneq C_0$ is not Banach.
(5) $L^p(X, \mathcal{F}, \mu), 1 \leqslant p \leqslant \infty$ is Banach.

More Cautionary Tales

(1) $X =$ Banach, $Y, Z \leqslant X$ closed. $Y \cong Z$, but it may be $X/Y \ncong X/Z$.
Eg. $X = \ell^2$; $Y = \{(0, x_2, x_3, \cdots) \in \ell^\infty\}$, $Z = \{(0, 0, x_3, x_4, \cdots) \in \ell^\infty\}$;
(2) X is NVS; $Y \leqslant X$ is Banach, $(X/Y, \|\cdot\|_{\text{quotient}})$ is Banach $\Rightarrow$ X is Banach.
(3) X is Banach, $Y \leqslant X$ is closed, but it may be $X \ncong Y \oplus X/Y$.

2.2 Linear Operator

Recall X, Y are vector spaces. $T : X \to Y$ is linear if and only if $T(\lambda x + \mu y) = \lambda Tx + \mu Ty, \forall x, y \in X, \lambda, \mu \in k$.

Definition 2.7 *If $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are NVS, then a linear operator $T : X \to Y$ is bounded if and only if $M < \infty$, s.t. $\|Tx\| \leqslant M\|x\|_X, \forall x \in X$.*

Definition 2.8 *The operator norm of T is*

$$\|T\| := \inf\{M > 0 : \|Tx\|_Y \leqslant M\|x\|_X,\ \forall x \in X\}, \tag{2.10}$$

$\|Tx\|_Y \leqslant M\|x\|_X \Leftrightarrow \|T\Big(\dfrac{x}{\|x\|_X}\Big)\|_Y \leqslant M.$

Fact $\|T\| = \sup\{\|Tx\|_Y : \|x\| = 1\} = \sup\{\|Tx\|_Y : \|x\| \leqslant 1\} = \sup\{\|Tx\|_Y : \|x\| < 1\}$.

Proposition 2.5 *Let X, Y be NVS. Let $T : X \to Y$ be linear. The following are equivalent:*
(1) *T is continuous;*
(2) *T is continuous at 0;*
(3) *T is bounded;*
(4) *T is Lipschitz;*
(5) *$T(\mathcal{B}_X)$ is bounded in Y.*

Notation $(X, \|\cdot\|) =$ NVS. $\mathcal{B}_X =$ closed unit ball $= \{x \in X, \|x\| \leqslant 1\}$; $\mathbb{S}_X =$ unit sphere $= \{x \in X, \|x\| = 1\}$.

PROOF (1) $\Leftrightarrow$ (2). "$\Rightarrow$" is trivial.

"$\Leftarrow$" If T is continuous at 0, take any $x \in X, h \in X$. $T(x+h) - Tx = Th - T0$. $\Rightarrow$ so T is continuous at x.

(3)$\Leftrightarrow$ $\exists M$, s.t. $\|Tx\| \leqslant M\|x\|, \forall x \in X$.$\Leftrightarrow$ T is Lipschitz at 0 $\overset{(3)\Leftrightarrow(2)}{\Longleftrightarrow}$ T is Lipschitz.

(2)$\Leftrightarrow$ (3).

"$\Rightarrow$" If $\|Tx\| \leqslant M\|x\|, \forall x \in X$. Take $\delta = \varepsilon/M$, then $\|x\| \leqslant \delta \to \|Tx\| \leqslant M\delta = \varepsilon$.

"$\Leftarrow$" By assumption, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\|Tx\| \leqslant \varepsilon$ wherever $\|x\| \leqslant \delta$. For any $\tilde{x} \in X \setminus \{0\}$, then $\dfrac{\delta \tilde{x}}{2\|\tilde{x}\|} =: x \in \mathcal{B}_\delta(x)$. So $\|Tx\| = \|T\dfrac{\delta\tilde{x}}{2\|\tilde{x}\|}\| = \dfrac{\delta}{2\|\tilde{x}\|}\|T\tilde{x}\| \leqslant \varepsilon$, i.e. $\|T\tilde{x}\| \leqslant \dfrac{2\varepsilon}{\delta}\|\tilde{x}\|, \forall \tilde{x} \in X$. Then fix $\varepsilon > 0$ (hence $\delta > 0$) and set $M = 2\varepsilon/\delta$ and we complete the proof.

(3)$\Rightarrow$ (5): by definition.

(5)$\Rightarrow$ (3): $T(\mathcal{B}_X)$ is bounded if and only if $\exists M > 0$ s.t. $T(\mathcal{B}_X) \subset \mathcal{B}_M(0) \subset Y$, $\Leftrightarrow$ $\|Tz\|_Y \leqslant M, \forall z \in \mathcal{B}_X$. Thus $\forall x \in X \setminus \{0\}$,

$$\|Tx\| = \|x\| \cdot \|T\frac{x}{\|x\|}\| \leqslant \|x\|M. \tag{2.11}$$

(5)$\Rightarrow$ (2): If T is not continuous at 0, then $\exists \varepsilon > 0$, s.t. $\forall \delta > 0, \exists x \in X$, s.t. $\|Tx\| \geqslant \varepsilon$ but $\|x\| \leqslant \delta$. Take $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$, we can get $\{x_n\} \searrow 0$, but $\|T\tilde{x}\| = \dfrac{1}{\sqrt{\|x_n\|}}\|Tx_n\| \geqslant \dfrac{\varepsilon}{\sqrt{\|x_n\|}} \nearrow \infty$. □

Notation Let X,Y be NVS, define $\mathcal{B}(X,Y)$ as set of bounded linear operators, and $\mathcal{B}(X) = \mathcal{B}(X,X)$.

Theorem 2.2 *If $X = NVS$, $Y =$ Banach space, then $(\mathcal{B}(X,Y), \|\cdot\| =$ operator norm is Banach$)$.*

Proposition 2.6 *$X, Y, Z = NVS$. Let $T \in \mathcal{B}(X,Y), S \in \mathcal{B}(Y,Z)$, then $\|S \circ T\| \leqslant \|S\|\|T\|$.*

Proof $\forall x \in X$, $\|STx\|_Z \leqslant \|S\|\|Tx\|_Y \leqslant \|S\|_{\mathcal{B}(Y,Z)}\|T\|_{\mathcal{B}(X,Y)}\|x\|_X$. Take sup over $x \in \mathcal{B}_X$ to get $\|ST\| \leqslant \|S\|\|T\|$. □

Example (1) Shift operators on sequence spaces:

$L : \ell^1 \to \ell^1,\ (x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, x_4, \cdots)$

$R : \ell^1 \to \ell^1,\ (x_1, x_2, x_3, \cdots) \mapsto (0, x_1, x_2, \cdots)$

$\|Rx\|_{\ell^1} = \|(0, x_1, x_2, \cdots)\|_{\ell^1} = \sum_{j=1}^\infty |x_j| = \|x\|_{\ell^1}, \|R\| = 1$; $\|Lx\|_{\ell^1} = \displaystyle\sum_{j=2}^{\infty} |x_j| \leqslant$

$\|x\|_{\ell^1} \Rightarrow \|L\| \leqslant 1$, but $\|B\| = \sup\{\|Lz\|_{\ell^1} : \|z\|_{\ell^1} \leqslant 1\}$. Take $z = (0,1,0,\cdots)$ and $\|Lz\| = \|(0,1,0,\cdots)\|_{\ell^1} \Rightarrow \|L\| = 1$.

(2)$X, Y = C^0([0,1],\mathbb{R}), \|\cdot\|_{C^0} = \|\cdot\|_{\sup}. T : X \to Y, f \to (Tf)(x) = xf(x)$, $\|Tf\|_{C^0} = \|xf(x)\|_{C^0} \leqslant \|f\|_{C^0} \Rightarrow \|T\| \leqslant 1$. Take $f \equiv 1$, and $\|Tf\|_{C^0} = \|x\|_{C^0} = 1 \Rightarrow \|T\| = 1$

(3) $T : L^2([0,1]) \to L^2([0,1]), f \mapsto xf(x)$,

$$\|Tf\|_{L^2} = \Big(\int_0^1 |xf(x)|^2 \,\mathrm{d}x\Big)^{1/2} \leqslant \Big(\int_0^1 |f(x)|^2 \,\mathrm{d}x\Big)^{1/2} = \|f\|_L^2 \tag{2.12}$$

Then $\|T\|_{\mathcal{B}(L^2,L^2)} \leqslant 1$.

Q: Does there exists $f_\star \in L^2, \|f_\star\| \leqslant 1$ s.t. $\|Tf_\star\|_{L^2} = 1$?

Let $f_n(x) = n\chi_{[1-\frac{1}{n},1]}, \|f_n\|_{L^2} = 1$ and $\|Tf_n\|_{L^2}^2 = 1 - \dfrac{1}{n^2} - \dfrac{1}{3n^4} \to 1$.

Remark Sup in $\|T\|$ may fail to be obtained, but sometimes can find maximising sequence.

Q: Why is it so hard to find an unbounded linear operator?

Fact (by Open Mapping Theorem) X, Y = Banach, $T \in \mathcal{B}(X,Y)$ bijective, then T^{-1} is bounded.

Example $\dfrac{\mathrm{d}}{\mathrm{d}x} : C^1(\mathbb{R}) \to C^0(\mathbb{R})$. In C^1 norm, $\|f\|_{C^1} := \|f\|_{\sup} + \|f'\|_{\sup}$, $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is bounded, but in C^0 norm, $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is unbounded.

2.3 Invertibility

$(X, \|\cdot\|)$ = NVS , $T \in \mathcal{B}(X)$.

Example $M_\alpha : \ell^{(x)} \to \ell^{(x)}, \alpha = (\alpha_1, \alpha_2, \cdots), (x_1, x_2, \cdots) \mapsto (\alpha_1 x_1, \alpha_2 x_2, \cdots)$ and $M_\alpha^{-1}(y_1, y_2, \cdots) = \Big(\dfrac{y_1}{\alpha_1}, \dfrac{y_2}{\alpha_2}, \cdots\Big)$, so M_α^{-1} is bounded if and only if $\inf\limits_{j\in\mathbb{N}} |\alpha_j| > 0$.

More generally, for T^{-1} to be bounded, we need

$$\|T^{-1}y\| \leqslant M\|y\|, \quad \forall y, \tag{2.13}$$

Here T is surjection; so (2.13) $\Leftrightarrow$ $\|x\| \leqslant M\|Tx\| \Leftrightarrow \|Tx\| \geqslant \delta\|x\|$ for some $\delta > 0$.

Theorem 2.3 *Let X be Banach, $T \in \mathcal{B}(X)$. If $\exists \delta > 0$ s.t. $\|Tx\|_X \geqslant \delta\|x\|_X, \forall x \in X$ $\circledast$. Then T is injection and $T(X) = \mathrm{ran}T$ is closed. If in addition, $\overline{T(X)} = X$, then T is invertible.*

Proof (1)Injectivity is clear since $\circledast \Rightarrow \ker T = \{0\}$;

(2)Take $\{y_n\} \subset \text{ran}T$ and $y_n \to y$, then $\exists \{x_n\} \in X$ s.t. $y_n = Tx_n$. But

$$\|x_n - x_l\| \leqslant \delta \|Tx_n - Tx_l\| = \delta \|y_n - y_l\| \to 0 \text{ as } n, l \nearrow \infty. \tag{2.14}$$

So $\{x_n\}$ is Cauchy. But X is Banach, so $x_n \to x \in X$, then $y_n = Tx_n \to Tx = y$ for T bounded. Hence, by uniqueness of limits, $y = Tx \in \text{ran}T$. That is, T has closed range.

(3) If $\overline{T(X)} = \overline{\text{ran}(T)} = X$, then for $\text{ran}(T)$ closed , T is surjective, so T is invertible as a linear map. In fact,

$$\|T^{-1}y\| \leqslant^{[17]} \frac{1}{\delta}\|T(T^{-1}y)\| = \frac{1}{\delta}\|y\|, \quad \forall y \in X, \tag{2.15}$$

so $\|T^{-1}\| \leqslant \delta^{-1}$. □

Theorem 2.4 *X = Banach, $T \in \mathcal{B}(X), \|T\| < 1 \Rightarrow 1 - T$ is invertible.*

Proof Guess $(1-T)^{-1} = \sum_{k=0}^{\infty} T^k$, and consider $S_n = \sum_{k=1}^{n} T^k$. Since $\|T\| \leqslant 1$, $\|T^k\| \leqslant \|T\|^k$, so $S_n \in \mathcal{B}(X)$. Also, $\sum T^k$ absolutely converge, and $\mathcal{B}(X)$ is Banach when X is Banach. Thus, $S_n \to S_\infty$ in $\mathcal{B}(X)$.

But $S_\infty(1-T) \leftarrow S_n(1-T) = \sum_{k=0}^{\infty} T^k(1-T) = 1 - T^{n+1} \to 1$ in $\mathcal{B}(X)$, similarly, $(1-T)S_\infty = 1$. So $S_\infty = (1-T)^{-1}$. □

Corollary For $S, T \in \mathcal{B}(X)$, X = Banach, T is invertible, and $\|S\| \leqslant \|T^{-1}\|^{-1}$, then $S + T$ is invertible.

Definition 2.9 *Let $GL(X) = \{T \in \mathcal{B}(X) : T$ is invertible$\}$ be* ***general linear group****, then $GL(X) \subset \mathcal{B}(X)$ is open, since $\forall T \in GL(X), \mathring{\mathbb{B}}(T, \|T^{-1}\|^{-1}) \subset GL(X)$, where $\mathbb{B}(a, \rho) := \{x \in X : \|x - a\| < \rho\}$.*

Definition 2.10 *X = Banach space over k. The* ***spectrum*** *of $T \in \mathcal{B}(X)$ is $\sigma(T) = \text{Spec}(T) := \{\lambda \in k : \lambda 1 - T$ is noninvertible$\}$.*

[17] by $\circledast$.

Theorem 2.5 Spec(T) *is compact.*

Proof (Closed) If $\{\lambda_j\} \subset \mathrm{Spec}(T), \lambda_j \to \lambda$; then $(\lambda_j 1 - T)$ is noninvertible. But $GL(X)$ is open, so $(\lambda_j 1 - T)$ is non-invertible. So $\lambda 1 - T \in \mathrm{Spec}(T)$.

(Bounded) If $\lambda > \|T\|$, then $\|\lambda^{-1}\| < 1 \Rightarrow 1 - \lambda^{-1}T$ is invertible $\Rightarrow \lambda 1 - T$ is invertible $\Rightarrow \lambda \notin \mathrm{Spec}(T)$. Hence $\mathrm{Spec}(T) \subset \mathcal{B}(0, \|T\|)$. $\square$

Theorem 2.6 *Let $k \in \mathbb{R}$ or $\mathbb{C}$. For any nonempty compact $\mathcal{K} \subset k$, $\exists$ Banach space $X, T \in \mathcal{B}(X)$ s.t. $\mathcal{K} = \mathrm{Spec}(T)$.*

Proof Let $\mathcal{K}_0$ be a dense, countable subset of $\mathcal{K}$. $\mathcal{K}_0 = \{\lambda_i\}_1^\infty$. Let $X = \ell^1$ and

$$\begin{aligned} T :& \ell^1 \to \ell^1, \\ &(x_1, x_2, \cdots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \cdots) \end{aligned} \tag{2.16}$$

Thus $\lambda_i \in \mathrm{Spec}(T), \forall i$. Since $(0, \cdots, 0, 1, 0, \cdots) \in \mathrm{Ker}(\lambda_i 1 - T)$, in which 1 is on the i-th place, so $\mathcal{K}_0 \subset \mathrm{Spec}(T) \Rightarrow \overline{\mathcal{K}_0} = \mathcal{K} \subset \overline{\mathrm{Spec}(T)} = \mathrm{Spec}(T)$.

To see $\mathrm{Spec}(T) \subset \mathcal{K}$, take $\lambda \in \mathcal{K}^c$, then for $\mathcal{K}$ compact in $k \in \mathbb{R}$ or $\mathbb{C}$, $\mathrm{dist}(\lambda, \mathcal{K}) \geqslant \delta > 0$. Clearly

$$(\lambda 1 - T)^{-1} = \Big(\frac{1}{\lambda - \lambda_1} x_1, \frac{1}{\lambda - \lambda_2} x_2, \cdots\Big) \tag{2.17}$$

But $\left|\dfrac{1}{\lambda - \lambda_i}\right| \leqslant \delta^{-1} < \infty, \forall i$, so $\|(\lambda 1 - T)^{-1}\| \leqslant \delta^{-1}$. $\square$

Remark If $k \in \mathbb{C}, X =$ Banach space over $\mathbb{C}$, $T \in \mathcal{B}(X)$ by a (deep) theorem, then $\mathrm{Spec}(T) \neq \varnothing$.

For $k \in \mathbb{R}$, possibly $\mathrm{Spec}(T) = \varnothing$.

Example $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} \quad \Rightarrow \det(\lambda 1 - T) = \lambda^2 + 1. \tag{2.18}$$

2.4 Properties of Finite Dimensional Banach Spaces

Lemma 2.2 $\mathbb{S}_1 = \{x \in X : \|x\|_1 = 1\}$ *is compact in* $(X, \|\cdot\|_1)$.

Proof Let $(V, \|\cdot\|_1)$ be a metric space, then compactness is equivalent to sequential compactness.

Let $\{v_j\} \subset \mathbb{S}_1 \subset V$ be any sequence. Let $\{e_1, \cdots, e_n\}$ be the given basis for V. Then $v_j = \sum_{j=1}^{n} v_j^{(i)} e_i$ for each $j = 1, 2, 3, \cdots$. By assumption, $\sum_{j=1}^{n} \left| v_j^{(i)} \right| = 1, \forall j$. Then for each fixed i, $\{v_j^{(i)}\} \subset k$ is bounded. So $\exists$ subsequence $\{v_{j_{k(i)}}^{(i)}\}$ converging to $v_\infty^{(i)}$. By refining the subsequence (not labelled) $v_{j_k}^{(i)} \to v_\infty^{(i)}, \forall i \in \{1, \cdots, n\}$. Clearly, $\sum_{i=1}^{n} \left| v_\infty^{(i)} \right| = 1$, i.e. $v_\infty = \sum_{i=1}^{n} v_\infty^{(i)} e_i \in \mathbb{S}_1$ and $v_{j_k} \to v_\infty$. □

Lemma 2.3 (F.Riesz) *Let X be a NVS. If Y is a proper closed subspace of X, then* $\forall \varepsilon > 0, \exists x \in \mathbb{S}_X = \{z \in X : \|z\| = 1\}$ *s.t.* $\operatorname{dist}(x, Y) \geqslant 1 - \varepsilon$.[18]

Proof Consider X/Y; choose any $[z] \in X/Y$ s.t. $1 \geqslant \|[z]\| > 1 - \varepsilon$. Then, $\forall$ representation $\tilde{z} \in X$ of $[z]$ with $\|\tilde{z}\| \leqslant 1$. Consider $x = \dfrac{\tilde{z}}{\|\tilde{z}\|}$. Then

$$\operatorname{dist}(x, Y) = \operatorname{dist}\Big(\frac{\tilde{z}}{\|\tilde{z}\|}, Y\Big) = \inf\Big\{\|\frac{\tilde{z}}{\|\tilde{z}\|}, y\| : y \in Y\Big\} = \frac{1}{\|\tilde{z}\|}\operatorname{dist}(\tilde{z}, Y). \tag{2.19}$$

□

Definition 2.11 (Heine-Borel Property) *If every closed and bounded set is compact, then we say* ***Heine-Borel property*** *holds.*

Theorem 2.7 (Heine-Borel) *Let X be a Banach space, then Heine-Borel property holds in X* $\Leftrightarrow$ $\dim X < \infty$.

Proof "$\Rightarrow$" If $\dim X = \infty$, let $x_1 \in \mathbb{S}_X$ be arbitrary. And if $\{x_1, \cdots, x_n\} \subset \mathbb{S}_X$ have been chosen, pick $x_{n+1} \in \mathbb{S}_X$ by Riesz Lemma, s.t. $\operatorname{dist}(x_{n+1}, \operatorname{span}\{x_1, \cdots, x_n\}) \geqslant 0.99$. Clearly, $\{x_j\}$ has no convergent subsequence, then Heine-Borel property fails.
"$\Leftarrow$": If $\dim X < \infty$, denote $\dim X = n$. Then for any closed and bounded set A, $\exists a \in \mathbb{R}^*$, s.t. $A \subset [-a, a]^n$. For any open cover $\mathcal{F}$ of A, denote $\tilde{\mathcal{F}} = \mathcal{F} \cup \{X \setminus A\}$, then $\tilde{\mathcal{F}}$ is obviously

[18] F.Riesz 引理给出在无穷维范数空间中的单位球不是紧的. 但是注意这里的紧性是在由范数诱导的强拓扑意义下的，因此这里要和后面的 Banach-Alaoglu 定理相区分：在弱 * 拓扑的意义下共轭空间中闭单位球是弱 * 紧的.

the open cover of $[-a,a]^n$, by using the compactness of $[-a,a]^n$, there exist a finite subcover of $[-a,a]^n$, and we denote it as $\tilde{\mathcal{F}}_f \subset \tilde{\mathcal{F}}$. Then $\tilde{\mathcal{F}}_f \setminus \{X \setminus A\}$ is a finite subcover of A. □

Theorem 2.8 *Let X = finite dimensional vector space. Then any norms on X are equivalent.*

PROOF Let $n = \dim X = \infty$. Let $\{e_1, e_2, \cdots, e_n\} \subset X$ be any basis. Then $\forall x \in X,\ \exists!\,(\lambda_1, \cdots, \lambda_n)$ s.t. $x = \sum_{i=1}^{n} \lambda_i e_i$. Set $\|x\|_1 := \sum_{i=1}^{n} |\lambda_i|$. Let $\|\cdots\|$ be any norm on X.
Claim A: $\|\cdot\| := (X, \|\cdot\|_1) \to \mathbb{R}$ is Lipschitz. Indeed $\forall x, y \in X,\ x = \sum_{i=1}^{n} \lambda_i e_i, y = \sum_{i=1}^{n} \mu_i e_i$,

$$\|x-y\| = \|\sum_{i=1}^{n}(\lambda_i - \mu_i)e_i\| \leqslant \left(\sum_{i=1}^{n} |\lambda_i - \mu_i|\right) \max_{1\leqslant i\leqslant n} \|e_i\| = \|x-y\|_1. \tag{2.20}$$

Claim B: $\mathbb{S}_1 = \{x \in X : \|x\|_1 = 1\}$ is compact in $(X, \|\cdot\|_1)$. Then $0 < c \leqslant \|x\| \leqslant C < \infty$ on $\mathbb{S}_1 \Rightarrow c \leqslant \dfrac{\|x\|}{\|x\|_1} \leqslant C$ on $X \setminus \{0\}$. □
Corollary Any finite-dimensional NVS is Banach.

2.5 Separability

Definition 2.12 *Let X be a topological space. Then X is separable if and only if $\exists Y$ countable $\subset X$, s.t. $\overline{Y} = X$.*

Example (1)$(C^0[0,1], \|\cdot\|_{C^0} = \|\cdot\|_{\sup})$ is separable, since polynomials are dense (Stone-Weierstrass).

(2)$\ell^p(1 \leqslant p < \infty)$ is separable. C_{00} is dense in ℓ^p. Let $x^{(n)} = (x_1, x_2, \cdots, x_n, 0, \cdots)$, then $x^{(n)} \to x_\infty$ in ℓ^p.

Let $\mathcal{F} := \bigcup_{n=1}^{\infty} \{(x_1, \cdots, x_n, 0, \cdots) : x_j \in \mathbb{Q}\}$, then $\mathcal{F}$ is a countable dense subset of ℓ^p.
Similarly, C and C_0 are separable.

Proposition 2.7 *ℓ^p is nonseparable.*

PROOF Assume $\mathcal{D}$ is a dense subset of ℓ^∞. Try to find an embedding $f := \mathcal{P}(\mathbb{N}) \hookrightarrow$ D dense in ℓ^∞.

Note that $\forall E \subset \mathbb{N}$, $\exists s(E) \in \mathcal{D}$ s.t. $\|s(E) - \chi_E\|_{\ell^\infty} \leqslant \dfrac{1}{4}$. But $\forall E_1 \neq E_2$ in $\mathcal{P}(\mathbb{N})$,

$\|\chi_{E_1} - \chi_{E_2}\|_{\ell^\infty} \geqslant 1$. Hence, $\|s(E_1) - s(E_2)\|_{\ell^\infty} \geqslant \|\chi_{E_1} - \chi_{E_2}\|_{\ell^\infty} - \|s(E_1) - \chi_{E_1}\|_{\ell^\infty} - \|\chi_{E_2} - s(E_2)\|_{\ell^\infty} \geqslant \frac{1}{2}$. Hence $s : \mathcal{P}(\mathbb{N}) \hookrightarrow \mathcal{B}$ is injective. □

Proposition 2.8 *$\mathcal{B}(\ell^2) = \mathcal{B}(\ell^2, \ell^2)$ is nonseparable.*

Proof Look for injection $\iota : \ell^\infty \hookrightarrow \mathcal{B}(\ell^2)$. Define for $a = \{a_i\} \in \ell^\infty, \iota(a) \in \mathcal{B}(\ell^2)$ as follows: $\forall b \in \ell^2$, $(\iota(a))(b) = \{a_i b_i\} \in \ell^2$.

$\{a_i b_i\} \in \ell^2$ since

$$\sum |a_i b_i|^2 \leqslant \|a\|_{\ell^\infty}^2 \sum |b_i|^2 = \|a\|_{\ell^\infty}^2 \|b\|_{\ell^2}^2 \tag{2.21}$$

$\Rightarrow \|\iota(a)\|_{\mathcal{B}(\ell^2)} \leqslant \|a\|_{\ell^\infty} \Rightarrow \|\ell\|_{\mathcal{B}(\ell^\infty, \mathcal{B}(\ell^2))} \leqslant 1$.
Claim : $\|\iota\| = 1$.

$\forall \varepsilon > 0$, for $a = \{a_i\} \in \mathbb{S}_{\ell^\infty}$, choose $n_0 \in \mathbb{N}$ s.t. $|a_{n_0}| > 1 - \varepsilon$. Then $(\iota(a))(e_{n_0})$ satisfies $\|\iota(a)(e_{n_0})\|_{\ell^2} = |a_{n_0}| > 1 - \varepsilon$, so $\|\eta\| \geqslant 1$.

Now $\exists \iota : \ell^\infty \to \mathcal{B}(\ell^2)$ is an isometry, then $\mathcal{B}(\ell^2)$ is nonseparable. □

Definition 2.13 *Let $(X, \mathcal{F}, \mu)$ be measurable space,*

$$\|f\|_{L^p(\mu)} := \Big(\int_X |f(x)|^p \, \mathrm{d}\mu(x) \Big)^{\frac{1}{p}} \tag{2.22}$$

$L^p(\mu) := \{f : (X, \mathcal{F}, \mu) \to k : \|f\|_{L^p(\mu)} < \infty\} \backslash$ "μ-null".

Proposition 2.9 *$L^p(\mathbb{R}^n)$ is separable.*

Since $C_0^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ by modification. Alternatively, use piecewise linear functions to approximate any given $f \in L^p$.

Definition 2.14 *Let X be a Banach space. $S \subset X$ is an Hamel basis if and only if* $\operatorname{span}(S) = X$, *where* $\operatorname{span}(S) = \Big\{ \sum_{i=1}^{n} \lambda_i s_i | \lambda_i \in k, s_i \in S, n < \infty \Big\}$.

Fact S is either finite or uncountable.

2.6 L^p Space

Given $f : (X, \mathcal{F}, \mu) \to \mathbb{R}$ or $\mathbb{C}$, define

$$\|f\|_p := \Big\{ \int_X |f|^p \, \mathrm{d}\mu \Big\}^{\frac{1}{p}}, 1 \leqslant p < infty \tag{2.23}$$

Then $\|\cdot\|_p$ satisfies

(a) $\|\lambda f\|_p = |\lambda| \, \|f\|_p, \forall \lambda \in \mathbb{R}$ or $\mathbb{C}$;

(b) $\|f\|_p = 0$ if and only if $f = 0,\ \mu-$a.e.;

(c) $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.

Remark (c) $\forall \lambda \in [0, 1]$, $\|\lambda f + (1 - \lambda)g\|_p \leqslant \lambda\|f\|_p + (1 - \lambda)\|g\|_p$, so triangle inequality $\Leftrightarrow$ convexity of $\|\cdot\|_p$.

Consider $\mathcal{L}^p(X, \mathcal{F}, \mu) := \{f : (X, \mathcal{F}, \mu) \to \mathbb{R}$ or $\mathbb{C}$ measurable $: \|f\|_p < \infty\}$. There is a natural equivalence relation $\sim$. $f \sim g \Leftrightarrow f = g\ \mu-$a.e..

Definition 2.15 *$L^p(X, \mathcal{F}, \mu) = L^p(X) = L^p(\mu) := \mathcal{L}^p(X, \mathcal{F}, \mu)/\sim$. Can view $L^p(X, \mathcal{F}, \mu)$ as a vector space.*

Convention: Say "L^p-functions".

For $p = \infty$,

$$\begin{aligned} \|f\|_\infty = \|f\|_{L^\infty(\mu)} &= \text{ess sup}\,|f| \\ = &\inf\{M > 0 : |f(x)| \leqslant M \text{ for } \mu - a.e.x \in X\} \\ = &\inf\{M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\} \end{aligned} \tag{2.24}$$

$L^\infty(X, \mathcal{F}, \mu) = \mathcal{L}^\infty(X, \mathcal{F}, \mu)/\sim, \mathcal{L}^\infty(X, \mathcal{F}, \mu) := \{f : (X, \mathcal{F}, \mu) \to \mathbb{R} \text{ or } \mathbb{C} : \|f\|_\infty < \infty\}$.

Definition 2.16 *A set K is convex if and only if $x, y \in K \ \Rightarrow [x, y] \subset K, \forall x, y \in K$.*

Definition 2.17 *A function φ is convex in Ω if and only if $\varphi(\lambda x + (1-\lambda)y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y), \forall 0 < \lambda < 1$ if and only if "epi-graph" is convex.* $\text{epi}(f) := \{(x, y) : y \geqslant f(x)\}$.

Theorem 2.9 (Jensen's Inequality) *Let $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ be convex. Let $(X, \mathcal{F}, \mu)$ be a finite measure space ($\mu(X) < \infty$). Let $f : (X, \mathcal{F}, \mu) \to \mathbb{R}$ be L^1, then $\langle \mathcal{J} \circ f\rangle \geqq \mathcal{J}(\langle f\rangle)$, where $\langle\varphi\rangle := \dfrac{1}{\mu(X)} \displaystyle\int_X \varphi \mathrm{d}\mu$.*

Proof $\mathcal{J}$ is convex, each point on the graph of $\mathcal{J}$ has a supporting line: $\forall s \in \mathbb{R}, \exists V \in \mathbb{R}$, s.t. $\mathcal{J}(t) \geqslant \mathcal{J}(s) + V(t - s)$.

Take $s = \langle f\rangle \Rightarrow\ \exists V \in \mathbb{R}, \forall t \in \mathbb{R}$,

$$\mathcal{J}(t) \geqslant \mathcal{J}(\langle f\rangle) + V(t - \langle f\rangle) \tag{2.25}$$

Then take $t = f(x)$ and integrate over X,

$$\int_X \mathcal{J} \circ f(x) \mathrm{d}\mu(x) \geqslant \int_X \mathcal{J}\langle f\rangle \mathrm{d}\mu(x) + V\left[\int_X f(x)\mathrm{d}\mu(x) - \int_X \langle f\rangle \mathrm{d}\mu\right] = \int_X \mathcal{J}\langle f\rangle \mathrm{d}\mu(x). \quad (2.26)$$

Example $\mathcal{J}(x) = x^2$, then

$$\int_X \mathcal{J} \circ f(x)\mathrm{d}\mu(x) - \int_X \mathcal{J}\langle f\rangle \mathrm{d}\mu(x) \geqslant \int_X |f(x)|^2 \mathrm{d}\mu(x) - \left(\int_X |f(x)| \mathrm{d}\mu(x)\right)^2 =: \mathrm{Var}(f) \geqslant 0. \quad (2.27)$$

□

Theorem 2.10 (Hölder's Inequality) *Let* $1 \leqslant p \leqslant \infty$, $\dfrac{1}{p}+\dfrac{1}{p'} = 1$ *(p' is the Hölderconjugate of p). Let* $f \in L^p(X, \mathcal{F}, \mu), g \in L^p$, *then* $fg \in L^1$. *Also,* $\|fg\|_{L^1} \leqslant \|f\|_{L^p} \cdot \|g\|_{L^{p'}}$.

$$\int_X |fg| \mathrm{d}\mu \leqslant \left(\int_X |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |g|^{p'} \mathrm{d}\mu\right)^{\frac{1}{p'}} \quad (2.28)$$

Proof W.L.O.G., assume $f \geqslant 0, g \geqslant 0$. Also, if $p = \infty$, clearly $\displaystyle\int |fg| \mathrm{d}\mu \leqslant \|f\|_{L^\infty(\mu)} \int |g| \mathrm{d}\mu$. Similarly, $p = 1$ is clear. So, W.L.O.G., we can assume $1 < p < \infty$. We can also assume that $g > 0$ a.e.. Then, set $\varphi = fg^{-\frac{p'}{p}}, \mathrm{d}\nu = g^{p'}\mathrm{d}\mu$.

Apply Jensen's inequality to φ, ν and $\mathcal{J}(t) = t^p$: $\langle \mathcal{J} \circ \varphi\rangle \geqq \mathcal{J}(\langle\varphi\rangle)$

$$\langle J \circ \varphi\rangle = \frac{1}{\nu(X)} \int_X f^p g^{-p'} \mathrm{d}\nu = \frac{\int_X f^p \mathrm{d}\mu}{\int_X g^{p'}\mathrm{d}\mu}. \quad (2.29)$$

Also,

$$RHS = \frac{\int_X f^p \mathrm{d}\mu}{\int_X g^{p'}\mathrm{d}\mu} = (\langle\varphi\rangle)^p = \left(\frac{\int_X \varphi \mathrm{d}\nu}{\nu(X)}\right)^p = \left(\frac{\int_X fg^{-\frac{p'}{p}} g^{p'} \mathrm{d}\nu}{\int_X g^{p'}\mathrm{d}\mu}\right)^p = \frac{\left(\int_X fg\mathrm{d}\mu\right)^p}{\left(\int_X g^{p'}\mathrm{d}\mu\right)^p} \quad (2.30)$$

Then

$$\frac{\left(\int_X f^p \mathrm{d}\mu\right)^{\frac{1}{p}}}{\left(\int_X g^{p'} \mathrm{d}\mu\right)^{\frac{1}{p}}} \geqslant \frac{\int_X fg\mathrm{d}\mu}{\int_X g^{p'}\mathrm{d}\mu} \tag{2.31}$$

which indicates

$$\int fg\mathrm{d}\mu \leqslant \Big(\int_X f^p\mathrm{d}\mu\Big)^{\frac{1}{p}}\Big(\int_X g^{p'}\mathrm{d}\mu\Big)^{\frac{1}{p'}} \tag{2.32}$$

Theorem 2.11 (Minkovski's Inequality) *Let $(X,\mathcal{F},\mu)$ and $(Y,\mathcal{G},\nu)$ be a $\sigma-$finite measure spaces. Let $f=f(x,y):(X\times Y,\mathcal{F}\otimes\mathcal{G},\mu\otimes\nu)$ be $\mu\otimes\nu-$ measurable. Let $1\leqslant p<\infty$. Then*

$$\int_Y\left[\int_X f(x,y)^p\mathrm{d}\mu(x)\right]^{\frac{1}{p}}\mathrm{d}\nu(y)\geqslant\left\{\int_X\left[\int_Y f(x,y)\mathrm{d}\nu(y)\right]^p\mathrm{d}\mu(x)\right\}^{\frac{1}{p}}. \tag{2.33}$$

Proof Note that $\int_X f(x,y)^p\mathrm{d}\mu(x)$ and $\Phi(x)=\int_Y f(x,y)\mathrm{d}\nu(y)$ are measurable (Fubini). Assume that

(1) $f>0$ on a set of positive $\mu\otimes\nu-$measure;

(2) RHS of (2.33)$<\infty$.

Now,

$$\begin{aligned}
&\int_X \Phi^p\mathrm{d}\mu(x)=\int_X\Phi^{p-1}\Phi\mathrm{d}\mu(x)=\int_X\Big[\int_Y f(x,y)\mathrm{d}\nu(y)\Big]\Phi^{p-1}(x)\mathrm{d}\mu(x)\\
=&\int_Y\Big[\int_X f(x,y)\Phi^{p-1}\mathrm{d}\mu(x)\Big]\mathrm{d}\nu(y)\\
\leqslant&\int_Y\Big[\int_X f(x,y)^p\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\Big[\int_X\Phi^{(p-1)p'}(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p'}}\mathrm{d}\nu(y)\\
=&\Big[\int_X\Phi^p(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p'}}\left\{\int_Y\Big[\int_X f(x,y)^p\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\mathrm{d}\nu(y)\right\},
\end{aligned} \tag{2.34}$$

Hence

$$\left\{\int_X\Phi^p\mathrm{d}\mu(x)\right\}^{1-\frac{1}{p'}}\equiv\left\{\int_X\left[\int_Y f(x,y)\mathrm{d}\nu(y)\right]^p\mathrm{d}\mu(x)\right\}^{\frac{1}{p}}\leqslant\left\{\int_Y\Big[\int_X f(x,y)^p\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\mathrm{d}\nu(y)\right\}. \tag{2.35}$$

□

Corollary Let $f(x,y)=\varphi(x)1_{y_1}+\psi(x)1_{y_2}$, let $\nu=\delta_{y_1}+\delta_{y_2}$, RHS= $\Big(\int_X[\varphi(x)+\psi(x)]^p\mathrm{d}\mu(x)\Big)^{\frac{1}{p}}\leqslant$ LHS= $\|\varphi\|_{L^p}+\|\psi\|_{L^p}\Rightarrow\triangle-$inequation for L^p.

Proposition 2.10 (Interpolation) *Let $1\leqslant p<q<r<\infty$. Then $L^p\cap L^r\subset L^q$.*

Proof

$$\int|f|^q\,\mathrm{d}\mu=\int|f|^{\alpha q}|f|^{(1-\alpha)q}\,\mathrm{d}\mu\leqslant\Big(\int|f|^{\alpha qs}\,\mathrm{d}\mu\Big)^{\frac{1}{s}}\Big(\int|f|^{(1-\alpha)qs}\,\mathrm{d}\mu\Big)^{\frac{1}{s}}\tag{2.36}$$

Take α s.t. $\alpha qs=p,(1-\alpha)qs'=r$

$$\int|f|^q\,\mathrm{d}\mu\leqslant\Big(\int|f|^p\,\mathrm{d}\mu\Big)^{\frac{\alpha q}{p}}\Big(\int|f|^r\,\mathrm{d}\mu\Big)^{\frac{(1-\alpha)q}{p}}\tag{2.37}$$

Therefore, $\|f\|_{L^q}\leqslant\|f\|_{L^p}^{\alpha}\|f\|_{L^r}^{1-\alpha}$ provided that $\dfrac{\alpha}{p}+\dfrac{1-\alpha}{r}=\dfrac{1}{q}$. □

Proposition 2.11 *If $\mu(X)<\infty$, $1\leqslant p<q\leqslant\infty$, then $L^q(X)\subset L^p(X)$.*

Proof Assume $f\in L^q$,

$$\int_X|f|^p\,\mathrm{d}\mu=\int_X|f|^p\,\chi_X\mathrm{d}\mu\leqslant\Big(\int_X|f|^{p\cdot\frac{q}{p}}\,\mathrm{d}\mu\Big)^{\frac{p}{q}}\Big(\int_X\chi_X^{\frac{q}{q-p}}\mathrm{d}\mu\Big)^{\frac{q-p}{q}}\tag{2.38}$$

then $\|f\|_{L^p}^p\leqslant\|f\|_{L^q}^p(\mu(X))^{\frac{q-p}{p}}$, and $\|f\|_{L^p}\leqslant\|f\|_{L^q}(\mu(X))^{\frac{1}{p}-\frac{1}{q}}$. □

Definition 2.18 (Convexity of L^p) *Convexity of L^p is defined by the convexity of its unit ball: $\forall x,y\in X$ s.t. $\|x\|=1=\|y\|$, then the midpoint of $[x,y]=\dfrac{x+y}{2}$ lies in $\mathcal{B}_X$.*

Definition 2.19 (Uniformly Convexity of L^p) *Uniformly Convexity of L^p is defined by the uniformly convexity of its unit ball: $\forall x,y\in X,0<\varepsilon\leqslant 2$ s.t. $\|x\|=1=\|y\|$, $\|x-y\|\geqslant\varepsilon$ then the midpoint of $[x,y]=\dfrac{x+y}{2}$ lies **deeply** in $\mathcal{B}_X$, i.e. $\exists\delta(\varepsilon)>0$ s.t. $1-\|\dfrac{x+y}{2}\|\geqslant\delta(\varepsilon)>0$.*

Theorem 2.12 *$\forall 1 < p < \infty$, $L^p(X, \mathcal{F}, \mu)$ is uniformly convex.*

In fact, we have the following precise description:

Theorem 2.13 (Olof Hanner 1956) *For $f, g \in L^p$, we have*

$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \geqslant \Big(\|f\|_{L^p} + \|g\|_{L^p}\Big)^p + \Big(\|f\|_{L^p} + \|g\|_{L^p}\Big)^p + \Big|\|f\|_{L^p} - \|g\|_{L^p}\Big|^p \quad \text{if } 1 \leqslant p \leqslant 2 \tag{2.39}$$

and

$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \leqslant \Big(\|f\|_{L^p} + \|g\|_{L^p}\Big)^p + \Big(\|f\|_{L^p} + \|g\|_{L^p}\Big)^p + \Big|\|f\|_{L^p} - \|g\|_{L^p}\Big|^p \quad \text{if } p \geqslant 2. \tag{2.40}$$

If we take $\tilde{f} = \dfrac{f+g}{2}$, $\tilde{g} = \dfrac{f-g}{2}$, then we obtain

$$2^p\Big(\|\tilde{f}\|_{L^p}^p + \|\tilde{g}\|_{L^p}^p\Big) \geqslant \Big(\|\tilde{f}+\tilde{g}\|_{L^p} + \|\tilde{f}-\tilde{g}\|_{L^p}\Big)^p + \Big|\|\tilde{f}+\tilde{g}\|_{L^p} - \|\tilde{f}-\tilde{g}\|_{L^p}\Big|^p \quad \text{if } 1 \leqslant p \leqslant 2 \tag{2.41}$$

the right hand side of (2.41) $\geqslant 2\|\tilde{f}+\tilde{g}\|_{L^p}^p + p(p-1)\|\tilde{f}+\tilde{g}\|_{L^p}^{p-2}\|\tilde{f}-\tilde{g}\|_{L^p}^2$[19], so we have

$$\|\tilde{f}+\tilde{g}\|_{L^p} \leqslant \|\tilde{f}\|_{L^p} + \|\tilde{g}\|_{L^p}. \tag{2.42}$$

which indicates that Hanner's inequation is stronger than $\triangle-$inequation. It also claim that $\triangle-$inequation can be deduced without Minkovski inequation.

Now we take any $f, g \in L^p, \varepsilon > 0$, $\|f\|_L^p = \|g\|_L^p = 1$, and $\|f-g\|_{L^p} \geqslant \varepsilon$.

Case 1: $p \geqslant 2$. By Hanner's inequation,

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p + \left\|\frac{f-g}{2}\right\|_{L^p}^p \leqslant 1, \tag{2.43}$$

So $\left\|\dfrac{f+g}{2}\right\|_{L^p}^p \leqslant 1 - \left(\dfrac{\varepsilon}{2}\right)^p$, then $1 - \left\|\dfrac{f+g}{2}\right\|_{L^p}^p \geqslant 1 - \left[1 - \left(\dfrac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}} = \delta(\varepsilon)$.

Case 2: $1 \leqslant p \leqslant 2$. Consider $\tilde{f}, \tilde{g}$ and $\xi(a,b) = (a+b)^p + |a-b|^p$. $\xi(a,b)$ is strictly increasing in both a and b. Then by (2.41), $2 \geqslant \xi(\|\tilde{f}\|_p, \|\tilde{g}\|_p)$ since $\|f\|_{L^p} = \|g\|_{L^p} = 1$. But $\|f-g\| \geqslant \varepsilon$, so

$$\xi\Big(1, \frac{\varepsilon}{2}\Big) > \xi(1,0) = 2 \geqslant \xi(\|\tilde{f}\|_p, \|\tilde{g}\|_p) \geqslant \xi\Big(\|\tilde{f}\|_p, \frac{\varepsilon}{2}\Big) \geqslant \xi\Big(0, \frac{\varepsilon}{2}\Big). \tag{2.44}$$

[19] This deduction is followed by: if $a \geqslant b$,then $(a+b)^p + (a-b)^p = b^p\left[\left(\frac{a}{b}+1\right)^p + \left(\frac{a}{b}-1\right)^p\right]$

By continuity, $\exists$ unique $\delta = \delta(\varepsilon)$ s.t. $\xi\Big(1-\delta, \dfrac{\varepsilon}{2}\Big) = 2$. But $\xi\Big(\|\tilde{f}\|_{L^p}, \dfrac{\varepsilon}{2}\Big) \leqslant 2$ for $1 \leqslant p \leqslant 2$, hence $\|\tilde{f}\|_{L^p} = \|\dfrac{f+g}{2}\|_{L^p} \leqslant 1-\delta(\varepsilon)$.

Exercise $L^p(1 \leqslant p \leqslant \infty)$ is a Banach space.

Remark $p = 2$ case: $\|f+g\|_{L^2}^2 + \|f-g\|_{L^2}^2 = 2\|f\|_{L^2}^2 + 2\|g\|_{L^2}^2$, which is known as **parallelogram law**.

Let $(L^p)^* = \mathcal{B}(L^p, \mathrm{R} \text{ or } \mathrm{C})$.

Theorem 2.14 (Duality) *$[L^p(X, \mathcal{F}, \mu)]^* \cong L^{p'}(X, \mathcal{F}, \mu)$ for $1 \leqslant p < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.*

Proof We shall prove that

$$\begin{aligned} \Phi :& L^{p'} \to (L^p)^* \\ & f \mapsto \Phi f : [g \mapsto \int_X fg \mathrm{d}\mu] \end{aligned} \tag{2.45}$$

have these following properties:

(1)$\forall f \in L^{p'}, \Phi f$ is a bounded linear operator on L^p. Linearity is obvious. Boundness:

$$\|\Phi f\|_{(L^p)^*} := \sup\{(\Phi f)(g) : g \in L^p, \|g\|_{L^p} \leqslant 1\} \tag{2.46}$$

$$(\Phi f)(g) = \int fg \mathrm{d}\mu \leqslant \|f\|_{L^{p'}} \|g\|_{L^p} \tag{2.47}$$

then $\|\Phi f\|_{(L^p)^*} \leqslant \|f\|_{L^{p'}}$.

(2) Φ is a bounded linear operator.

$$\text{Linearity: } \Phi(\alpha f_1 + \beta f_2) = \alpha\Phi(f_1) + \beta\Phi(f_2) \tag{2.48}$$

$$\text{Linearity: } \|\Phi\|_{\mathcal{B}(L^{p'},(L^p)^*)} := \sup\{\|\Phi\|_{(L^p)^*} : f \in L^p \leqslant 1\} \leqslant 1. \tag{2.49}$$

(3) Φ is a linear isometry, i.e.

$$\|\Phi\|_{\mathcal{B}(L^{p'},(L^p)^*)} = 1 \tag{2.50}$$

It suffices to prove that $\forall f \in L^{p'} \setminus \{0\}$, $\exists g \in L^p$, $\|g\|_{L^p} = 1$ s.t. $(\Phi f)(g) = \|f\|_{L^{p'}}$.

Let $g = \dfrac{f\,|f|^{p'-2}}{\|f\|^{p'-1}}$, then

$$(\Phi f)(g) = \int_X fg\mathrm{d}\mu = \int_X \frac{f^2\,|f|^{p'-2}}{\|f\|^{p'-1}}\mathrm{d}\mu = \int_X \frac{|f|^{p'}}{\|f\|^{p'-1}}\mathrm{d}\mu = \frac{\|f\|^{p'}}{\|f\|^{p'-1}} = \|f\|_{L^{p'}}. \tag{2.51}$$

Also,

$$\|g\|_{L^p}^p = \frac{\int_X \left|\bar{f}\,|f|^{p'-2}\right|\mathrm{d}\mu}{\|f\|_{L^{p'}}^{(p'-1)p}} = \frac{\int_X \left|f^{(p'-1)p}\right|\mathrm{d}\mu}{\|f\|_{L^{p'}}^{(p'-1)p}} = \frac{\int_X |f|^{p'}\,\mathrm{d}\mu}{\|f\|_{L^{p'}}} = 1. \tag{2.52}$$

(4) Φ is injective. If $f \in \mathrm{Ker}\Phi$, then $\forall g \in L^p, \int_X fg\mathrm{d}\mu = 0$. This is possible if and only if $f = 0$ $\mu-$a.e. $\Leftrightarrow f = 0$ in $L^{p'}$.

As Φ is linear, $\mathrm{Ker}\Phi = \{0\} \Leftrightarrow \Phi$ is injective.

(5) Φ is surjective, i.e. given any $T \in (L^p)^*$, $\exists f \in L^{p'}$ s.t. $\forall g \in L^p$, $Tg = \displaystyle\int_X fg\mathrm{d}\mu$.

Assume for the moment $\mu(X) < \infty$. Consider

$$\begin{aligned} \nu_T :&\mathcal{F} \to [0, \infty[,\\ &A \mapsto T(\chi_A) \end{aligned} \tag{2.53}$$

If $\mu(A) = 0$, then $\nu_T(A) := T(\chi_A) = T(0) = 0$, $\chi_A = 0$ $\mu-$a.e.. Thus $\nu_T \ll \mu$. Then by Radon-Nikodym derivative $f = \dfrac{\mathrm{d}\nu_T}{\mathrm{d}\mu} \in L^1(X, \mu)$, so that $\nu_T(A) \equiv T(\chi_A) = \displaystyle\int_A f\mathrm{d}\mu = \int_X f\chi_A\mathrm{d}\mu$.

Since $T \in (L^p)^*$, so by linearity,

$$Tg = \int_X fg\mathrm{d}\mu, \quad \forall g \in L^p \cap \{\text{simple functions}\}. \tag{2.54}$$

Claim We can further extend (2.54) to $g \in L^\infty(\mu)$.
To see this: note that

$$\{\text{simple functions}\} \cap L^p = \{\text{simple functions}\} \cap L^\infty \subset L^\infty(\mu) \subset L^p(\mu) \text{ as } \mu(X) < \infty. \tag{2.55}$$

$Y = \{\text{simple functions}\} \cap L^p$ is dense in $L^p(\mu)$, which indicates Y is dense in L^∞.
Claim If $Z = NVS$, $X \leqslant Z$ and $\overline{X} = Z$, and if Y is a Banach space, then $\forall T \in \mathcal{B}(X, Y)$, $\exists$

unique $\tilde{T} \in \mathcal{B}(Z,Y)$, s.t. $\tilde{T}|_X = T$, $\|\tilde{T}\| = \|T\|$.

Claim $f \in L^{p'}$.

For $p = 1$, $|f| \leqslant \|T\|\ \mu - a.e.\ \Rightarrow\ f \in L^\infty$, then $f \in L^{p'}$.

For $p > 1$, we want to estimate

$$\int_X |f|^{p'} \mathrm{d}\mu = \sup\left\{ \int_X h\mathrm{d}\mu : 0 \leqslant h \leqslant |f'|^{p'}, h \text{ simple} \right\} \tag{2.56}$$

Fix any such h, consider $g = \mathrm{sgn}(h)h^{\frac{1}{p}}$[20]. Here $g \in L^\infty(\mu)$ and

$$|Tg| = \left|\int_X fg\mathrm{d}\mu\right| = \int_X |f|\, h^{\frac{1}{p}}\mathrm{d}\mu \geqslant \int_X h^{\frac{1}{p}} h^{\frac{1}{p}}\mathrm{d}\mu = \int_X h\mathrm{d}\mu. \tag{2.57}$$

So $\int_X h\mathrm{d}\mu \leqslant \|T\| \cdot \|g\|_{L^p}$.

But $\|g\|_{L^p} = \left\{ \int_X h\mathrm{d}\mu \right\}^{\frac{1}{p}}$,

$$\int_X h\mathrm{d}\mu \leqslant \|T\|^{p'}, \forall h \text{ simple }, 0 \leqslant h \leqslant |f|^{p'}. \tag{2.58}$$

Thus $\|f\|_{L^{p'}} \leqslant \|T\|$. In particular, $f \in L^{p'}$.

Now we remove the restriction of finite space ($\mu(X) < \infty$), for general $(X, \mathcal{F}, \mu)$, and:

For $p > 1$, Let f be in $[L^p(X)]^*$. For $A \subset X$, we have $\|T - T|_A\| = (\|T\|^{p'} - \|T|_A\|^{p'})^{1/p'}$ for $p > 1$. Besides, $\forall p \geq 1$, $\|T\| = \sup\{\|T|_A\| : A \in \mathcal{F}, \mu(A) < \infty\}$. Then we can find a set sequence $\{A_n\} \in \mathcal{F}, \mu(A_n) < \infty, \|T - T|_{A_n}\| \to 0$ as $n \to \infty$, then Radon-Nikodym derivative $f_n := \dfrac{\mathrm{d}\nu_{T|_{A_n}}}{\mathrm{d}\mu}$ is a Cauchy sequence in $L^{p'}$.

Let $f = \lim\limits_{n\to\infty} f_n$ in $L^{p'}$. Then

$$Tf \overset{\text{by limit}}{=\!=\!=\!=} \int_X fg\mathrm{d}\mu, \tag{2.59}$$

hence proves surjectivity.

For $p = 1$, we furthermore need $\sigma-$finiteness to approximate T by cutoff. $\square$

Corollary Let $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $(L^p)^{**} = (L^{p'})^* \cong L^p$. Moreover by the proof of

[20]Function $\mathrm{sgn}(f)$ figures out the sign of f, i.e. $\mathrm{sgn}(f) = \begin{cases} |f|/f, & \text{if } f \neq 0, \\ 0, & \text{if } f = 0. \end{cases}$

duality theorem, the isometric isomprphism is given by

$$\begin{aligned} \mathcal{J} :& L^p \to (L^p)^{**} = [(L^p)^*]^*, \\ & f \mapsto \mathcal{J}(f,) \end{aligned} \tag{2.60}$$

where $\mathcal{J}$ defines as follows: $\forall T \in (L^p)^*$, $[\mathcal{J}(f)](T) := T(f)$.

Definition 2.20 *A Banach space is **reflexive** if $\mathcal{J} : X \to X^{**}, [\mathcal{J}(x)](T) = Tx, \forall x \in X, T \in X^*$ is an isometric isomorphism.*

Then we can obtain from the duality theorem: if $1 < p < \infty$, then L^p is reflexive.
Fact $\exists X$ non-reflexive but $X \cong X^{**}$ isometric isomorphism. (R.D.James. 1921)

2.7 Hahn-Banach Theorem

Definition 2.21 *Let X be a vector space. $p : X \to \mathbb{R}$ is a **positive homogeneous subadditivity functional** if and only if*

(1) $p(\lambda x) = \lambda p(x),\ \forall x, y \in X$;

(2) $p(x + y) \leqslant p(x) + p(y),\ \forall x, y \in X$.

Lemma 2.4 (Zorn) *Let $(\Sigma, \preccurlyeq)$ be a partially ordered set. If every totally ordered subset $\Sigma_1 \subset \Sigma$ has an upper bound, then Σ has a maximal element.*

Remark The upper bound is not necessarily in this totally ordered subset.

Theorem 2.15 (Hahn-Banach) *Let X be a vector space over $\mathbb{R}$. Let $Y \leqslant X$ be a subspace of X. Let p be a positive homogeneous subadditivity functional. If $f \in Y^*$(linear functional) satisfies $f \leqslant p$ on Y, then $\exists F \in X^*$, s.t. $F|_Y = f$, $F \leqslant p$ on X.*

Proof Consider

$$\Sigma := \{(Z, \varphi) : Y \leqslant Z \leqslant X, \varphi \in Z^* \text{ s.t. } \varphi \leqslant p \text{ on } Z, \varphi|_Y = f\}. \tag{2.61}$$

It has a natural partial order:

$$(Z_1, \varphi_1) \preccurlyeq (Z_2, \varphi_2) \Leftrightarrow Z_1 \subset Z_2,\ \varphi_2|_{Z_1} = \varphi_1. \tag{2.62}$$

Note that if $\{(Z_\alpha, \varphi_\alpha) : \alpha \in I\} \subset \Sigma$ is totally ordered, then (Z, φ) s.t. $Z = \bigcup_{\alpha \in I} Z_\alpha, \varphi(x) := \varphi_\alpha(x)$ if $x \in Z_\alpha$ is a maximal element.

By Zorn's lemma, Σ has a maximal element, $\sup(Z, \varphi) \in \Sigma$. If we can show that $Z = X$, then we are done.($\varphi = F$ is the desired extension of f).

Assume for contradiction that $Z \neq X$. Pick $x_1 \in X \setminus Z$. Set $Z^{(1)} = \operatorname{span}(Z \cup \{x_1\}) = Z \oplus \operatorname{span}\{x_1\}$. We look for a one-step extension $\varphi^{(1)} \in [Z^{(1)}]^*$, s.t.

$$\varphi^{(1)}(z + tx_1) = f(z) + t\beta \text{ for } \beta \in \mathbb{R}. \tag{2.63}$$

Such β needs to satisfy

$$f(z) + t\beta \leqslant p(z + tx_1) \forall z \in Z, \forall t \in \mathbb{R}. \tag{2.64}$$

It suffices to check for $t = \pm 1$. Indeed, if $f(z) = \pm\beta \leqslant \beta \leqslant p(z \pm x_1), \forall z \in Z$, then

$$\varphi^{(1)}(z + t\beta) = \varphi^{(1)}\Big[t\Big(\frac{z}{t} + \beta\Big)\Big] = t\varphi^{(1)}\Big(\frac{z}{t} + \beta\Big) \leqslant tp\Big(\frac{z}{t} + x_1\Big) = p(z + tx_1) \tag{2.65}$$

for $t > 0$. Similarly for $t < 0$.

Now we look for $\beta \in \mathbb{R}$, s.t.

$$\varphi^{(1)}(z \pm x_1) \leqslant p(z \pm x_1), \forall z \in Z \tag{2.66}$$

Here $\varphi^{(1)}(z + tx_1) = \varphi(z) + t\varphi^{(1)}(x_1)$.

Observe that (2.66) is equivalent to

$$\begin{cases} \beta \leqslant p(z + x_1) - \varphi(z), \\ -\beta \leqslant p(z' - x_1) - \varphi(z'), \end{cases} \forall z, z' \in Z. \tag{2.67}$$

such $\beta \in \mathbb{R}$ exists if and only if $\varphi(z + z') \leqslant p(z + x_1) + p(z' - x_1), \forall z, z' \in Z$.Since $\varphi \leqslant p$ on Z and p is subadditive, this holds automatically.

we have $(Z \oplus \operatorname{span}\{x_1\}, \varphi^{(1)}) \in \Sigma$. This contradicts the maximality of (Z, φ). $\square$

Remark Hahn-Banach $\Leftrightarrow$ Zorn $\Leftrightarrow$ Axiom of choice.

Theorem 2.16 (Hahn-Banach norm-preserving extension) *Let X be an NVS, $Y \leqslant X, f \in Y^*$, then $\exists F \in X^*, F|_Y = f, \|F\|_{X^*} = \|f\|_{Y^*}$.*

Proof Define for $x \in X$, $p(x) := \|f\|_{Y^*}\|x\|_X$. Then $|f| \leqslant p$ on Y and p is a positive homogeneous subadditive functional. By Hahn-Banach, $\exists F \in X^*$ s.t. $F|_Y = f, F \leqslant p$, here $F \leqslant p \quad \Leftrightarrow \quad F(x) \leqslant \|f\|_{Y^*}\|x\|_X, \Rightarrow \quad \|F\|_{X^*} \leqslant \|f\|_{Y^*}$. But F is an extension of f. So $\|F\|_{X^*} \geqslant \|f\|_{Y^*}$. □

Remark Also holds for $\mathbb{C}-$NVS.

Corollary If X is an NVS, then $\forall x \in X \setminus \{0\}, \exists f \in X^*$, s.t. $\|f\|_{X^*} \leqslant 1$ and $f(x) = \|x\|$.

Proof Define $f_0 : \text{span}\{x\} \to \mathbb{R}$ by $f_0(tx) := |t|\,\|x\|$. Clearly $f_0 \in (\text{span}\{x\})^*, \|f_0\| = 1$. Extend f_0 to $f \in X^*$ by Hahn-Banach s.t. $\|f\|_{X^*} = 1$. □

Corollary $\forall x \in X$,

$$\|x\| = \sup\{|f(x)| : f \in \mathcal{B}_{X^*}, f \in X^*, \|f\| = 1\} \tag{2.68}$$

Remark $\forall f \in X^*$, by definition,

$$\|f\|_{X^*} = \sup\{|f(x)| : x \in \mathcal{B}_X\}. \tag{2.69}$$

Proof "$\geqslant$" $\forall f \in \mathcal{B}_{X^*}$, $\|f(x)\| \leqslant \|f\|_{X^*}\|x\|_X \leqslant \|x\|_X$;

"$\leqslant$" sup is attained at the norming functional. □

Theorem 2.17 *Let X be Banach and let $Y \leqslant X$ be closed subspace. Then $\forall x \in X \setminus Y$, $\exists f \in X^*$ s.t. $\|f\|_{X^*} = 1, f(y) = 0\ \forall y \in Y$, and $f(x) = \text{dist}(x, Y)$.*

Proof Consider $f_0 = \text{span}(Y \cup \{x_0\})$, $f_0 = (y + tx_0) = t\delta, \forall y \in Y, t \in \mathbb{R}$ where $\delta := \text{dist}(x_0, Y)$.

Clearly, f_0 is linear and $f_0|_Y = 0$. Also, $\forall z \in Y \oplus \text{span}\{x_0\}$, say $z = y + tx_0 (t \in \mathbb{R}, y \in Y)$, we have $|f_0(x)| = t\delta = \dfrac{t\|z\|\delta}{\|y + tx_0\|} = \dfrac{\|z\|\delta}{\|\frac{y}{t} + x_0\|} \leqslant \|z\|$, since $\|\frac{y}{t} + x_0\| \geqslant \text{dist}(x_0, Y) = \delta$ for $t > 0$, where $\delta := \text{dist}(x_0, Y)$. Similarly for $t < 0$. So $\|f_0\|_{(Y \oplus \text{span}\{x_0\})^*} \leqslant 1$.

But clearly $\|f_0\| \geqslant 1$. Thus we can extend f_0 to $f \in X^*$ by Hahn-Banach. □

Proposition 2.12 *Let X be a Banach space, then X^* separable $\Rightarrow$ X separable.*

Proof Let $\{f_n\}_{n=1}^{\infty}$ is a dense subset of $\mathbb{S}_{X^*}$. Take $\{x_n\} \subset \mathbb{S}_X$ s.t. $f_n(x_n) > \dfrac{1}{2}$.

Let $Y = \overline{\text{span}\{x_n\}}$. If $Y \neq X$, by Hahn-Banach, $\exists g \in X^*$,s.t. $\|g\| = 1, g|_Y \equiv 0$. Then

$\exists f_n$ s.t. $\|f_n - g\|_{X^*} \leqslant \frac{1}{4}$. But then $|g_{x_n}| \geqslant \Big| |f_n(x_n)| - |(f_n - g)(x_n)| \Big| \geqslant \frac{1}{4}$, which contradicts to $g|_Y = 0$. Then $Y = X$, $\mathrm{span}\{x_n\}$ is dense in X. Then $\bigcup_{N=1}^{\infty} \Big\{ \sum_{j=1}^{N} \lambda_j x_j : \lambda_j \in \mathbb{Q} \Big\}$ is a countable subset of X. □

Definition 2.22 *Let X be a NVS, $C \subset X$ is convex, $x \in C$. Then $f \in X^*$ is said to be a supporting functional of C at x if and only if $\|f\| = 1$, $f(x) = \sup\{f(y) : y \in C\}$.*

Remark If $C = \mathcal{B}_X$, then $\sup\{f(y) : y \in C\} = \|f\|_{X^*}$.

Definition 2.23 *Let X be a Banach space and $C \subset X$. Let*

$$\begin{aligned} \mu_C : X &\to \mathbb{R} \\ x &\mapsto \inf\{\lambda > 0 : x \in \lambda C\}. \end{aligned} \tag{2.70}$$

*μ_C is named as **Minkovski functional** or **gauge**.*

Lemma 2.5 *If X be a Banach space, $C \subset X$ be convex and $0 \in \mathrm{int}(C)$.[21] Then μ_C is a positive homogeneous sublinear functional on X. Also, $\{\mu_C < 1\} \subset C \subset \{\mu_C \leqslant 1\}$.*

Proof If $\mu_C(x) < 1$, then $0 < \lambda < 1$. But then $x \in C$, because $x = \lambda\Big(\frac{x}{\lambda}\Big) + (1-\lambda)\cdot 0$.

The same argument shows that if $0 < \tilde{t} < t, x \in \tilde{t}C$, then $x \in tC$. Take any $t > \mu_C(x), s > \mu_C(y)$, then $x \in tC, y \in sC, x + y \in tC + sC = (t+s)\Big[\frac{t}{t+s}C + \frac{s}{t+s}C\Big]$, then $\mu_C(x+y) \leqslant t + s$. □

Theorem 2.18 *Let X be a Banach space and $C \subset X$ be convex and closed. If $x_0 \notin C$, then $\exists f \in X^*$, s.t. $f(x_0) > \sup\{f(x) : x \in C\}$.*

Proof W.L.O.G. $0 \in C$. Write $\delta = \mathrm{dist}(x_0, C) > 0$. Let $D = \{x \in X \mathrm{dist}(x, C) \leqslant \frac{\delta}{2}\}$, then $0 \in \mathrm{int}(D)$, D is closed and convex. Then μ_D is a positive homogeneous subadditive functional.

[21]这里 $\mathrm{int}(C)$ 表示集合 C 的内点集.

If $\mu_D(x_0) < 1$, then $\forall \varepsilon > 0, \exists \lambda \leqslant 1 + \varepsilon$ s.t. $x_0 \in \lambda D$. But D is closed, so $x_0 \in D$, which meets contradiction. So $\mu_D(x_0) > 1$.

Now define f on $\text{span}\{x_0\}$ by $f(tx_0) = t\mu_D(x_0)$, then $f \leqslant \mu_D$ on $\text{span}\{x_0\}$. Then by Hahn-Banach, we extend to $f \in X^*$. Then $f \leqslant \mu_D$ on X. It remains to check that $f(x_0) > \sup\{f(x) : x \in C\}$. Indeed, RHS$\leqslant 1$, since $\forall x \in C \subset D, f(x) \leqslant \mu_D(x) \leqslant 1$, but LHS$=f(x_0) = \mu_D(x_0) > 1$. □

Corollary Let X be a Banach space and $A, B \subset X$ be disjoint convex sets, and A is open. Then $\exists f \in X^*, \lambda \in \mathbb{R}$, s.t. $f(a) < \lambda, \forall a \in A, f(b) \geqslant \lambda, \forall b \in B$.

Proof Consider $A - B$, it is a convex open set. Also, $A \cap B\varnothing \;\Rightarrow\; 0 \notin A - B$. By Hahn-Banach, $\exists g \in X^*$, $0 = g(0) > g(z), \forall z \in A - B \;\Leftrightarrow\; g(a) < g(b), \forall a \in A$ and $b \in B$. Take $\lambda := \inf_B g$, then $g \geqslant \lambda$ on B, and $g \leqslant \lambda$ on A.

To check strict inequality on A, suppose $\exists \tilde{x} \in A$, s.t. $g(\tilde{x}) = \lambda$. Since A is open, $\exists \delta > 0$ s.t. $\eta \in \mathring{B}_\delta(\tilde{x})$ s.t. $g(\eta) \neq 0$. If $g(\eta) \gtrless$, then $g(\tilde{x} \pm \eta) > \lambda$, which meets contradiction. □

2.8 Baire Category Theorem

Definition 2.24 *Let S be a topological space, then $E \subset S$ is* ***nowhere dense*** *if* $\text{int}(\bar{E}) = \varnothing$. *$E \subset S$ is said to be of the* ***first category*** *or "****meagre****" if and only if E is a countable union of nowhere dense sets. $E \subset S$ is said to be of the* ***second category*** *or "****non-meagre****" if and only if E is not the first category.*

2.8.1 Baire Category Theorem

Theorem 2.19 (Baire Category Theorem) *If S is a complete measure space or a locally compact Hausdorff space, then the intersection of countably many dense open subsets is still dense.*

Proof Let X be a complete metric space. Let $V_1, V_2. \cdots$ be dense open subsets of S. Let $O \subset S$ be any open set. Want to prove $O \cap \bigcap_{i=1}^{\infty} V_i \neq \varnothing$.

Define $O_0 = O$. If for $n \geqslant 1$, open set $O_{n-1} \neq \varnothing$ has been chosen, pick $O_n \neq \varnothing$, $\overline{O_n} \subset \overline{V_n} \subset \overline{O_{n-1}}$. In the case of complete measure, can take $O_n = \mathring{\mathcal{B}}_X(x_n, r_n)$ where $r_n < \dfrac{1}{n}$.

Set $K := \bigcap_{n=1}^{\infty} \overline{O_n}$. Note that $K \subset \bigcap_{n=1}^{\infty} (V_n \cap O)$. $\{\overline{O_n}\}$ is a sequence of closed nested balls, hence the centers x_n are Cauchy. Since X is complete, so $x_n \to x \in X$. Thus $x \in K$, and $\varnothing \neq K \subset \left(\bigcap_{n=1}^{\infty}\right) \cap O$. □

Corollary Complete metric space and locally compact Hausdorff spaces are non-meagre.

PROOF Let $\{E_i\}_{i=1}^{\infty}$ be any nowhere dense sets. Then $(\overline{E_i})^c$ are open and dense. By Baire category theorem, $\bigcap_{i=1}^{\infty}(\overline{E_i})^c$ is dense. In particular, $\left(\bigcup_{i=1}^{\infty}\right)^c = \bigcap_{i=1}^{\infty}(\overline{E_i})^c \neq S$. □

Recall By definition 2.14, *Let X be a Banach space. $S \subset X$ is an Hamel basis if and only if* $\operatorname{span}(S) = X$, *where* $\operatorname{span}(S) = \left\{\sum_{i=1}^{n} \lambda_i s_i | \lambda_i \in k, s_i \in S, n < \infty\right\}$. S is a maximal linearly independent in X. Equivalently, every $x \in X$ can be represented as a unique (finite) linear combination of elements in S.

Proposition 2.13 *Any vector space has a Hamel basis.*

PROOF Let X be a vector space, and define $\mathcal{F}$ is a collection of set which is a non-empty linear independent subset of X. Then we define an order in $\mathcal{F}$ by inclusion. Let $\mathcal{F}_t \subset \mathcal{F}$ be a totally ordered subcollection, then $\tilde{A} := \bigcup_{A \in \mathcal{F}_t} A$ is an upper bound of $\mathcal{F}_t$, by Zorn's Lemma, there exist a maximal in $\mathcal{F}$. □

Corollary A Banach space is either finite-dimensional, or has uncountable Hamel basis.

PROOF Let X be an $\infty-$dimensional Banach space. Assume for contradiction, $\exists S = \{s_i\}_1^{\infty}$ a Hamel bass for X. Let $V_i = \operatorname{span}\{s_1, \cdots, s_n\} \leqslant X$, then $X = \bigcup_{i=1}^{\infty} V_i$.

But each V_i is closed and $\operatorname{int}(V_i) = \infty$, i.e. V_i is nowhere dense. So X is of the first category. But complete metric space is of second category, and then we lead to a contradiction. □

Remark Baire categories describes the scale in topological sense, while measure describes the scale in measure space.

2.8.2 Banach-Steinhaus Theorem

Definition 2.25 *Let X, Y be topological VS. Let $\Gamma \subset \mathcal{L}(X,Y) := \{T : X \to Y \, linear\}$, Γ is equicontinuous if and only if $\forall O_Y \in W \subset Y$, $\exists O_X \in V \subset X$, W, V are open, s.t. $\gamma(V) \subset W$,$\forall \gamma \in \Gamma$.*

Remark If $\Gamma = \{\gamma\}$, Γ is equicontinuous $\Leftrightarrow$ γ is continuous $\Leftrightarrow$ Y is continuous.

Lemma 2.6 *Let X, Y be topological VS. Let $\Gamma \subset \mathcal{L}(X,Y)$ be equicontinuous. Let $E \subset X$ be bounded. Then $\exists F \subset Y$ bounded s.t. $\gamma(E) \subset F, \forall \gamma \in \Gamma$.*

Proof Let $F := \bigcup_{\gamma\in\Gamma} \gamma(E)$. Want to show that F is bounded. Since Γ is equicontinuous, $\forall O_Y \in W \subset Y$, $\exists O_X \in V \subset X$, W, V are open, s.t. $\gamma(V) \subset W$, $\forall \gamma in \Gamma$. As $E \subset X$ is bounded, $\exists t > 0$ s.t. $tV \supset E$. Then $\forall Y \in \Gamma$, $\gamma(E) \subset \gamma(tV) \subset t\gamma(V) \subset tW$. □

Theorem 2.20 (Banach-Steinhaus Theorem) *X, Y are topological VSs. $\Gamma \subset \mathcal{L}(X,Y) \cap C^0(X,Y)$ Let $\mathcal{B} = \{x \in X : \Gamma(x) = \{\gamma(x) : \gamma \in \Gamma\}$ is bounded in $Y\}$. If $\mathcal{B}$ is non-meagre, then $\mathcal{B} = X$ and Γ is equicontinuous.*

Proof Let $W, U \subset Y$ be balanced neighborhoods of O_Y s.t. $\overline{U} + \overline{U} \subset W$. Consider $E = \bigcap_{\gamma\in\Gamma} \gamma^{-1}(\overline{U})$. If $x \in \mathcal{B}$, then $\exists n \in \mathbb{N}$, s.t. $\gamma(x) \in nU, \forall \gamma \in \Gamma \quad \Rightarrow x \in nE$. Then $\mathcal{B} \subset \bigcup_{n=1}^{\infty} nE$.

Since $\mathcal{B}$ is non-meagre, then for at least one n s.t. (nE) is non-meagre. So E has non-empty interior since E is closed. Take $x_0 \in \text{int}(E)$, then $E - x_0$ contains a neighborhood V of O_X ($O_X \in V \subset E - x_0$). So $\forall \gamma \in \Gamma, \gamma(V) \subset \gamma(E - x_0) = \gamma(E) - \gamma(x_0) \subset \overline{U} - \overline{U} \subset W$ for $W \subset Y$ fixed. So Γ is equicontinuous. □

Remark $\mathcal{B}$ is defined by pointwise boundedness $\rightsquigarrow \Gamma$ is uniformly bounded.

Corollary Let X be Banach and Y is a NVS. If $\Gamma \subset \mathcal{B}(X,Y)$ satisfies $\sup_{\|\gamma\|} \|\gamma x\|_Y \leqslant M < \infty$, then $\sup_{\gamma\in\Gamma} \|\gamma\|_{\mathcal{B}(X,Y)} \leqslant M < \infty$, M is independent of X.

Application Let X be Banach space and Y be an NVS. Let $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(X,Y)$. If for each $x \in X$, $\lim_{n\to\infty} (T_n x)$. Clearly $T \in \mathcal{L}(X,Y)$.

Claim $T \in \mathcal{B}(X,Y)$. Indeed $\|T\| \leqslant \limsup_{n\to\infty} \|T_n\|$, which is finite by Banach-Steinhaus.

Application Let X, Y, Z be NVSs. Let $B : X \times Y \to Z$ be a bilinear map. The followings are equivalent:

(i)B is continuous;

(ii)B is continuous at $(0,0)$;

(iii)B is bounded. $\exists C > 0$, s.t. $\forall x \in X, y \in Y, \|B(x,y)\|_Z \leqslant C\|x\|_X \|y\|_Y$.

Claim If either X or Y is Banach, then (i)$\sim$(iii) is equivalent to

(iv) B is separately continuous.

Proof Assume that X is Banach. Set $B_y \in X^*$ for any $y \in Y$, where $B_y(x) = B(x,y)$. Then

$$\sup\Big\{\|B_y(x)\|_Z\colon \|y\|_Y\leqslant 1\Big\} \leqslant M_x \leqslant \infty. \tag{2.71}$$

By Banach-Steinhaus,

$$\sup\Big\{\|B_y\|_{\mathcal{B}(X,Z)}\colon \|y\|_Y\leqslant 1\Big\} \leqslant M \leqslant \infty. \tag{2.72}$$

Therefore,

$$\|B(x,y)\|_Z= \Big\|B\Big(x, \frac{y}{\|y\|_Y}\Big)\Big\|_Z\|y\|_Y= \Big\|B_{\frac{y}{\|y\|}}(x)\Big\|_Z\|y\|_Y\leqslant M\|x\|_X\|y\|_Y \tag{2.73}$$

This proves (iv)⇒(iii). □

2.8.3 Open Mapping Theorem

Definition 2.26 *Let S,T be topological spaces. A function $f : S \to T$ is open if and only if $\forall O \subset S$ open, $f(O) \subset T$ is open.*

Definition 2.27 *$f : S \to T$ is open at $p \in S$ if $\forall$ open neighborhood $p \in U \subset S, f(U) \in T$ contains an open neighborhood of $f(p)$.*

Lemma 2.7 *Let X,Y be topological VS, $T \in \mathcal{L}(X,Y)$. Then T is open if and only if T is open at O_X.*

Lemma 2.8 *Let X be Banach, Y be NVS, $T \in \mathcal{B}(X,Y)$.*

(1) *Suppose that $\exists\ \varepsilon \in]0,1[, M > 0$, s.t. $\forall y \in \mathring{\mathcal{B}}_Y(0,1)$, $\mathrm{dist}(y, T(\mathring{\mathcal{B}}_X(0,M))) < \varepsilon$. Then $\mathring{\mathcal{B}}_Y(0,1) \subset T(\mathring{\mathcal{B}}_X(0,\frac{M}{1-\varepsilon}))$.*

(2) *If $T(\mathring{\mathcal{B}}_X(0,M))$ contains a dense subset of $\mathring{\mathcal{B}}_Y(0,1)$, then $\mathring{\mathcal{B}}_Y(0,1) \subset T(\mathring{\mathcal{B}}_X(0,M))$.*

Proof (1) Let $y = y_1 \in \mathring{\mathcal{B}}_Y(0,1)$. Find $x_1 \in \mathring{\mathcal{B}}_X(0,M)$, s.t. $\|y_1 - Tx_1\|_Y\leqslant \varepsilon$. Let $y_2 := y_1 - Tx_1$, then $\|y_2\|< \varepsilon \Rightarrow \Big\|\dfrac{y_2}{\varepsilon}\Big\| < 1$. Then $\exists z_2 \in \mathring{\mathcal{B}}_X(0,M)$ s.t. $\Big\|Tz_2 - \dfrac{y_2}{\varepsilon}\Big\| < \varepsilon$. Let $x_2 = \varepsilon z_2$, then $\Big\|T\Big(\dfrac{x_2}{\varepsilon}\Big) - \dfrac{y_2}{\varepsilon}\Big\| < \varepsilon$, $\|y_2 - Tx_2\|< \varepsilon^2$.

Suppose $x_1, \cdots, x_n, y_1, \cdots, y_n$ have been chosen s.t.

$$\begin{cases} \|y_k\|_Y < \varepsilon^{k-1}, \ \|x_k\|_Y < M\varepsilon^{k-1}; \\ \|y_k - Tx_k\|_Y < \varepsilon^k, \quad \forall k \in \{1, \cdots, n\}. \end{cases} \tag{2.74}$$

Then, choose $y_{n+1} := y_n - Tx_n$. Indeed, since $\varepsilon^{-n}\|y_{n+1}\| < 1$, $\exists z_{n+1} \in T(\mathring{\mathcal{B}}_X(0, M))$ s.t. $\|\varepsilon^{-n}y_{n+1} - Tz_{n+1}\| < \varepsilon$. Let $x_{n+1} := \varepsilon^n z_{n+1}$. Then $\|x_{n+1}\|_X < M\varepsilon^n$. Also, $\|y_{n+1} - Tx_{n+1}\| < \varepsilon^{n+1}$. So we can choose $\{x_i\}_1^\infty, \{y_i\}_1^\infty$ satisfying (2.74) by induction.

Note that $\displaystyle\sum_{n=1}^{\infty}\|x_n\|_X < M\sum_{n=1}^{\infty}\varepsilon^{n-1} < \infty$, $\varepsilon < 1$. But X is Banach, so $\displaystyle\sum_{n=1}^{\infty} x_n = x$ for some $x \in X$. Also, $y_n = y - Tx_1 - Tx_2 - \cdots - Tx_n$ has norm $\|y_n\| < \varepsilon^{n-1}$. So $y_n \to y = Tx$.

(2)$\forall\varepsilon > 0$, by assumption,

$$\mathring{\mathcal{B}}_Y(0, 1) \subset T(\mathring{\mathcal{B}}_X(0, 1))\mathring{\mathcal{B}}_Y(0, \varepsilon) \tag{2.75}$$

So $\mathring{\mathcal{B}}_Y(0, 1-\varepsilon) \subset T(\mathring{\mathcal{B}}_X(0, 1))$, then

$$\mathring{\mathcal{B}}_Y(0, 1) = \bigcup_{\varepsilon \in]0,1[} \mathring{\mathcal{B}}_Y(0, 1-\varepsilon) \subset T(\mathring{\mathcal{B}}_X(0, 1)) \tag{2.76}$$

□

Theorem 2.21 (Open Mapping Theorem) *Let $T \in \mathcal{B}(X, Y)$ for $X =$ Banach space and $Y =$ NVS. If $T(X)$ is of the second category in Y, then*

(i) T is surjective;

(ii) T is open;

(iii) T is Banach.

Proof (ii)⇒(i): If $T(X) \leqslant Y$ is an open subspace of Y, then $T(X) = Y$.

(ii)⇒(iii): ∃(vector space) isomorphism:

$$X/\mathrm{Ker}T \cong T(X) \stackrel{(i)}{=\!=} T. \tag{2.77}$$

If $Z \leqslant Y$ be an open subspace, then $\mathring{\mathcal{B}}_Y(0, \delta) \subset Z$ for some $\delta > 0$. But $\forall y \in Y \setminus \{0\}$, $\dfrac{\delta}{2}\dfrac{y}{\|y\|} \in \mathring{\mathcal{B}}_Y(0, \delta) \subset Z$. Now, Z is a subspace, so $y \in Z$, $Y = Z$.

Let

$$\begin{aligned} \Phi : X/\mathrm{Ker}T &\to T(X) \\ x + \mathrm{Ker}T &\mapsto Tx \end{aligned} \tag{2.78}$$

Need to see that Φ is also a homeomorphism.

By general topology,

$$\begin{aligned} \pi : X &\to X/\mathrm{Ker}T \\ x &\mapsto x + \mathrm{Ker}T \end{aligned} \tag{2.79}$$

is open and continuous.

Φ is continuous: $\forall$ open $V \subset Y$, $\Phi(V) = \pi(T^{-1}(V))$ =open;

Φ is open: $\forall$ open $U \subset X/\mathrm{Ker}T$, $\Phi(U) = T(\pi^{-1}(U))$ =open.

proof of (ii): T is open. Write $T(X) = \bigcup_{n=1}^{\infty} F_n, F_n = \overline{T(\mathcal{B}_X(0,n))}$. As $T(X)$ is of the second category, for at least one n, $F_n \supset \mathring{\mathcal{B}}_Y(y,\varepsilon)$ for some $y \in Y, \varepsilon > 0$. Note that $F_n \supset \mathring{\mathcal{B}}_Y(-y,\varepsilon)$ by symmetry of F_n. Also, F_n is convex $\Rightarrow$

$$\mathring{\mathcal{B}}_Y(0,\varepsilon) \subset \frac{1}{2}\mathring{\mathcal{B}}_Y(y,\varepsilon) + \frac{1}{2}\mathring{\mathcal{B}}_Y(-y,\varepsilon) \subset F_n. \tag{2.80}$$

By $\mathring{\mathcal{B}}_Y(0,\varepsilon) \subset T(\mathring{\mathcal{B}}_Y(0,n))$, then T is open at O_X by linearity of T. Equivalently, T is open. □

2.8.4 Inverse Function Theorem

Theorem 2.22 (Inverse Function Theorem) *Let X, Y be Banach spaces. Let $T \in \mathcal{B}(X,Y)$. If T is a bijective, then T is a homeomorphic isomorphism.*

Corollary Let X be a vector space. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X s.t. $(X,\mathcal{T}_1), (X,\mathcal{T}_2)$ are Banach spaces and $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

Proof Consider $Id : (X,\mathcal{T}_2) \to (X,\mathcal{T}_1)$. □

2.8.5 Close Graph Theorem

Definition 2.28 *For $f : X \to Y$(not necessary linear),* $\text{graph}_f = \{(x, fx) : x \in X\} \subset X \times Y$.

Theorem 2.23 (Close Graph Theorem) *Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X,Y) \equiv \{\text{linear map } X \to Y\}$. Then $T \in \mathcal{B}(X,Y)$ if and only if* graph_T *is closed in $X \times Y$.*

Proof "$\Rightarrow$": More generally, if X is any topological space, Y is a Hausdorff topological space, and $f : X \to Y$ is continuous, then $\text{graph}_f \subset X \times Y$ is closed.

Consider $\Omega = X \times Y \setminus \text{graph}_f$. Let $(x_0, y_0) \in \Omega$, then $y_0 \neq f(x_0)$. Since Y is Hausdorff, $\exists U \subset Y$ open s.t. $y_n \in U, \exists\ V \subset Y$ open s.t. $f(x_0) \in V$, and $U \cap V = \varnothing$. Then $(x_0, y_0) \in f^{-1}(V) \times U \subset X \times Y$ is open.

"$\Leftarrow$": If graph_T is closed in $X \times Y$, then it is a Banach space. Consider

$$\begin{aligned} pr : \text{graph}_T &\to X \\ (x, Tx) &\mapsto x. \end{aligned} \tag{2.81}$$

Clearly it is a continuous linear bijection. So it is a homeomorphic isomorphism by Inverse Function Theorem. Moreover, $\Theta : \text{graph}_T \to Y$ is continuous. So $T = \Theta \circ pr^{-1}$ is continuous. $\square$

2.9 Weak and Weak* Topologies

Definition 2.29 *Let X be a vector space with (algebraic) dual $X' = \{T_0 X \to \mathbb{R}$ linear$\}$. Let $Y \leqslant X'$. Set $\delta(X, Y) :=$ coarsest topology on X s.t. every element of Y is continuous = the topology generated by basic open neighborhoods $\{x \in X : |f_i(x - x_0)| < \varepsilon, \forall 1 \leqslant i \leqslant n\}$ for some $x_0 \in X, f_1, \cdots f_n \in Y, \varepsilon > 0, n \in \mathbb{N}$.*

Definition 2.30 *Let X is NVS, $\delta = (X, X^*) =$ the weak topology on X. $x_n \rightharpoonup x$ in weak topology on X if and only if $Tx_n \to Tx, \forall T \in X^*$. $\delta(X^*, \mathcal{T}_X(X) \subset X^{**}) =$ the weak * topology on X^*. $T_n \overset{*}{\rightharpoonup}$ if and only if $\forall x \in X$, $T_n x \to Tx$.*

Recall

$$\begin{aligned} \mathcal{J}_X : X &\to X^{**} \\ x &\mapsto \mathcal{J}_X(x) \end{aligned} \tag{2.82}$$

where $\forall T \in X^*, [\mathcal{J}_X](T) = Tx$.

Example By Riesz Representation Theorem, $\Omega \subset \mathbb{R}^n$ compact, $[C^0(\Omega)]^* = \mathcal{M}(\Omega)$. For Banach $X = C^0(\Omega)$, the weak * convergence $\Leftrightarrow$ convergence in integral:

$$\mu_n \overset{*}{\rightharpoonup} \mu \ \Leftrightarrow \ \int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu,\ \forall f \in C^\infty. \tag{2.83}$$

Example $X = \ell^2 = \{a = \{a_n\}_{n=1}^\infty | \sum |a_n|^2 < \infty\}$.

Claim $\{e_n\}_{n=1}^\infty, e_=(0, \cdots, 0, 1, 0, \cdots)$ converges weakly to 0.

Proof $(\ell^2)^* \cong \ell^2$ s.t. $\forall \phi \in (\ell^2)^*, \exists b = \{b_n\} \in \ell^2$, s.t. $\phi(a) = (a_1 b_1, a_2 b_2, \cdots)$. Then $\forall \phi \in (\ell^2)^*$, say $b_\phi = (b_1, b_2, \cdots) \subset \ell^2$, that corresponds to ϕ via $(\ell^2)^* \cong \ell^2$,

$$\phi(e_n) = b \overset{n\to\infty}{\longrightarrow} 0 \text{ as } \sum |b_n|^2 < \infty. \tag{2.84}$$

Hence $e_n \rightharpoonup 0$ but $\|e_n\|_{\ell^2} = 1$. $\square$

Lemma 2.9 *Let E_1, E_2, E_3 be vector spaces. Let $f : E_1 \to E_3, g : E_1 \to E_2$ be linear, then $\exists$ linear $h : E_2 \to E_3$ s.t. $f = h \circ g \ \Leftrightarrow\ \mathrm{Ker} f \supset \mathrm{Ker} g$.*

Proof Clearly "$\Rightarrow$" holds.

"$\Leftarrow$": Define $h_0 : g(E_1) \to E_3$ s.t. $h_0(g(x)) := f(x)$. This is well-defined, since $x, \hat{x} \in E_1$ s.t. $g(x) = g(\hat{x})$, then $x - \hat{x} \in \mathrm{Ker} g$. But $\mathrm{Ker} g \subset \mathrm{Ker} f$ by assumption, so $f(x) = f(\hat{x})$. Then extend h_0 to $h \in \mathcal{L}(E_2, E_3)$ to finish the proof. $\square$

Corollary Let X be a vector space, $f, f_1, \cdots, f_n \in X$. If $\bigcap_{i=1}^n \mathrm{Ker} f_i \subset \mathrm{Ker} f$, then $f \in \mathrm{span}\{f_1, \cdots, f_n\}$.

Proof Let $E_1 = X, E_2 = \mathrm{R}^n, E_3 = \mathbb{R}, g = (f_1, \cdots, f_n)$. So $\exists h \in \mathbb{R}^n \to \mathbb{R}$ s.t. $f = h \circ (f_1, \cdots, f_n)$. That is, $f(x) = \sum_{i=1}^n \lambda_i f_i(x)$ for $h = (\lambda_1, \cdots, \lambda_n) \Leftrightarrow f \in \mathrm{span}\{f_1, \cdots, f_n\}$. $\square$

Property 2.2 (1) *If $\dim X < \infty$, then weak topology is the norm topology on X, weak * topology is the norm topology on X.*

(2) *If $\dim X = \infty$, then weak topology $\subsetneq$ the norm topology on X, weak * topology $\subsetneq$ the norm topology on X.*

(3) *Every weakly open set in X is norm open;*

(4) *If $\dim X = \infty$, then every weakly open set in X is not bounded;*

(5) *Weakly bounded $\Rightarrow$ norm bounded in X; weakly * bounded $\Rightarrow$ norm bounded in X^**

Proof (1) Let $O \subset X^*$ be open in $\|\cdot\|_{X^*}$. We prove that O is weakly * open. Note that

$$\exists f_0 \in X^*,\ \varepsilon > 0 \text{ s.t. } f_0 + \varepsilon \mathcal{B}_{X^*} \subset O \tag{2.85}$$

Let $\{e_1, \cdots, e_n\}$ be a basis for X, and set

$$|||f||| := \max |f(e_1)|\,,\ \forall f \in X^*. \tag{2.86}$$

Since $(X, \|\cdot\|)$ is finite dimensional, $\exists \delta > 0$, s.t. $\|\cdot\|_{X^*} < \varepsilon$ whenever $|||\cdot||| < \delta$, so $\Big\{ f \in X^* : \max\limits_{1 \leqslant i \leqslant n} |f(e_i) - f_0(e_i)| < \delta \Big\}$ is weakly * open, and $\subset \{f \in X^* : \|f - f_0\|_{X^*} < \varepsilon\} \subset O$.

(2) we shall prove a stronger result.

(4) Let $\dim X = \infty, \varnothing \neq O \subset X$. If O is weakly open, then O is unbounded.

Proof of (4)$\Rightarrow$(2): Clearly by definition, any open set in weak topology is in norm topology. Then note that $\mathring{\mathcal{B}}_X$ is in the norm topology but not in the weak topology.

Proof of (4): Let $O \subset X$ be weakly open, W.L.O.G. $O_X \in O$. Then $\exists \varepsilon > 0, f_1, \cdots, f_n \in X^*$ s.t. $O_X \in \bigcap\limits_{i=1}^{n} \mathrm{Ker}(f_i) \subset \{x \in X : |f_i(x)| < \varepsilon, \forall 1 \leqslant i \leqslant n\} \subset O$.

Claim $Z := \bigcap\limits_{i=1}^{n} \mathrm{Ker} f_i \neq \{0\}$. Indeed, if $Z = \{0\}$, then $\forall f \in X^*, Z \subset \mathrm{Ker} f$. By the previous lemma, $f \in \mathrm{span}\{f_1, \cdots, f_n\}$. But X is ∞-dimensional. That makes contradiction.

So $\exists x_* \in Z \subset \{0\}$. But $Z \leqslant X$, so $\mathrm{span}\{x_*\} \subset Z \subset O$, hence O is unbounded.

(3) A basis for the weak topology is

$$\bigcap_{i=1}^{n} \{x \in X : \|f_i(x - x_0) < \varepsilon\|\} \tag{2.87}$$

This is open in the norm topology.

(5) Let $K \subset X$ be weakly bounded. Then $\{|Tx| : T \in X^*\}$ is bounded $\forall x$. So it follows by Banach-Steinhaus. □

Remark (1) For a TVS X, A is bounded if and only if $\forall$ open neighborhood N, $0 \in N$, $A \subset \lambda N$ for some $\lambda > 0$;

(3) $(\mathcal{B}_X, w)$ is metrisable $\Leftrightarrow X^*$ is separable; $(\mathcal{B}_{X^*}, w^*)$ is metrisable $\Leftrightarrow X$ is separable;

(5) If $\{x_n\} \subset X$ weakly converges to x, then $\|x_n\| \leqslant M < \infty$; if $\{T_n\} \subset X^*$ weakly * converges to T, then $\|T_n\| \leqslant M < \infty$.

Theorem 2.24 (Banach-Alaoglu) *If X is a Banach space, then $\mathcal{B}_{X^*}$ is compact in weak * topology.*

Proof Consider $[-1,1]^{\mathcal{B}_X} = \{f : \mathcal{B}_X \to [-1,1]\} = \prod_{a\in\mathcal{B}_X} I_a, I_a = [-1,1]$. This is compact by Tychonoff theorem. Note that $\Omega := \{f|_{\mathcal{B}_X} : f \in \mathcal{B}_{X^*}\} \subset [-1,1]^{\mathcal{B}_X}$.

Also,

$$\begin{aligned}(\mathcal{B}_{X^*}, w^*) &\to (\Omega, \text{pointwise topology}) \\ f &\mapsto f|_{\mathcal{B}_X}\end{aligned} \tag{2.88}$$

is homeomorphism. So it suffices to show that Ω is closed.

Let $\{f_\alpha\}_{\alpha\in\mathcal{I}} \subset \Omega$ be a net, $f_j \to F$ pointwise. Set

$$f(x) := \begin{cases} \|x\| F\Big(\dfrac{x}{\|x\|}\Big) & \text{for } x \in X \setminus \{0\}, \\ 0 & \text{for } x = 0. \end{cases} \tag{2.89}$$

f is linear. In addition, $\|f\|_{X^*} = \sup\{f(x) : x \in \mathcal{B}_X\}$, where $f_\alpha(x) \to f(x)$ for each x, $\|f_\alpha\|_X^* \leqslant 1 \Rightarrow \|f\|_{X^*} \leqslant 1$. Also $f|_{\mathcal{B}_X} = F$, then $F \in \Omega$. □

Theorem 2.25 (Kakutani) *X is a Banach space, then X is reflexive $\Leftrightarrow$ $\mathcal{B}_X$ is weakly compact.*

Proof X is reflexive, then $\mathcal{B}_{\mathcal{J}_X(X)} = \mathcal{B}_{X^{**}}$, which is weak * compact by using Banach-Alaoglu theorem. Note that $\mathcal{J}_X$ is a homeomorphism between $\mathcal{B}_X$ in weak topology and $\mathcal{B}_{X^{**}}$ in weak * topology. In particular, $\mathcal{B}_X$ is weak compact. □

Theorem 2.26 (Hahn-Banach separation theorem for weak * topology) *Let X be a $\mathbb{R}$-Banach space. If $A \subset X$ is weakly * closed convex set, $f \in X^* \setminus A$, then $\exists x \in X$ s.t. $f(x) > \sup_A g$.*

Proof (X^*, w^*) is Hausdorff, so $\exists$ weak * open U, $O_X \subset U \subset X^*$ s.t. $(f+U)\cap A = \varnothing$. We can take $U = \{g \in X^* : |g(x_i)| < \varepsilon, \forall 1 \leqslant i \leqslant n\}$. U is symmetric and convex. By symmetry, $f \notin\in A + U = \bigcup_{a\in A} a + U$ is weak * open, hence open, and also convex.

By Hahn-Banach, $\exists \phi \in X^{**}$ s.t.

$$\phi(f) > \sup_{g \in A+U} \phi(g) \geqslant \sup_{g \in A} \phi(g). \tag{2.90}$$

Claim $\phi \in \mathcal{J}(X)$.

Indeed, fix $g_0 \in A$. Then $\sup\limits_U \phi \leqslant \phi(f) - \phi(g_0) = C_0 < \infty$. If $h \in \bigcap\limits_{i=1}^{n} \mathrm{Ker}\mathcal{J}_X(x_i)$, then $th \in U$ for some $t > 0 \;\Rightarrow \phi(th) \leqslant C$. Then $\phi(h) \leqslant \dfrac{C_0}{t}$. By symmetry, $\phi(-h) \leqslant \dfrac{C_0}{t}$, then $\phi(h) = 0$. So $\mathrm{Ker}\phi = \cap_{i=1}^{n} \mathrm{Ker}(\mathcal{J}_X(x_i))$, $\phi = \sum\limits_{i=1}^{n} \lambda_i \mathcal{J}_X(x_i) = \mathcal{J}_X\Big(\sum\limits_{i=1}^{n} \lambda_i x_i\Big)$, then $\phi \in \mathcal{J}_X(x)$.
□

Theorem 2.27 (Goldstein) *Let X be an NVS, and $\mathcal{B}_{X^{**}}$ is the unit ball in X^{**}. Then $\mathcal{B}_{X^{**}}$ is the weak * closure of $\mathcal{J}(\mathcal{B}_X)$.*

Proof "$\subset$" $\mathcal{J}_X(\mathcal{B}_X) \subset \mathcal{B}_{X^{**}}$. By Banach-Alaoglu, $(\mathcal{B}_{X^{**}}, w^*)$ is compact, then $\overline{\mathcal{J}_X(\mathcal{B}_X)}^{w^*} \subset \mathcal{B}_{X^{**}}$.

"$\supset$" If $W := \overline{\mathcal{J}_X(\mathcal{B}_X)}^{w^*} \subsetneq \mathcal{B}_{X^{**}}$, then $\exists \xi_0 \in \mathcal{B}_{X^{**}} \setminus W$, by Hahn-Banach in weak * version, $\exists f \in X^*$ s.t. $\xi_n(f) > \sup\{\eta(f) : \eta \in W\} =: S$. Clearly $S > 0$. W.L.O.G., take $S = 1$. Thus $\|f\|_{X^*} = \sup\{|f(x)| : x \in \mathcal{B}_X\} = \sup\{\eta(f) : \eta \in \mathcal{J}_X(\mathcal{B}_X)\} \leqslant S = 1$. But $\xi_0(f) > S = 1$, that makes contradiction. □

2.10 Hilbert Spaces

Definition 2.31 *An* ***inner product*** *on a vector space X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ s.t.*

(i) $\forall y \in X, x \mapsto \langle \cdot, \cdot \rangle$ *is linear;*
(ii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ *(conjugate linear);*
(iii) $\langle x, x \rangle \geqslant 0, \forall x \in X;$
(iv) $\langle x, x \rangle = 0 \;\Rightarrow\; x = 0.$

Theorem 2.28 (Cauchy-Schwarz inequality) *Let X be an inner product space, then $|\langle x, y \rangle| \leqslant \|x\| \cdot \|y\|$, where $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on X.*

Proof If $\langle y,y\rangle=0$, then $y=0$. If $\langle y,y\rangle\neq 0$, then

$$\begin{aligned}0\leqslant&\left\langle x-\frac{\langle x,y\rangle}{\langle y,y\rangle}y, x-\frac{\langle x,y\rangle}{\langle y,y\rangle}y\right\rangle\\ =&\left\langle x, x-\frac{\langle x,y\rangle}{\langle y,y\rangle}y\right\rangle-\frac{\langle x,y\rangle}{\langle y,y\rangle}\left\langle y, x-\frac{\langle x,y\rangle}{\langle y,y\rangle}y\right\rangle=\langle x,x\rangle-\frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}\Rightarrow\text{Cauchy-Schwarz inequality}\end{aligned}\tag{2.91}$$

$\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$ is a norm follows from the triangle inequality. Indeed, $\forall x,y\in X$,

$$\begin{aligned}\|x+y\|^2=\langle x+y,x+y\rangle=&\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle\\ =&\|x\|^2+\|y\|^2+2\mathrm{Re}\langle x,y\rangle\leqslant\|x\|^2+\|y\|^2+2\,|\langle x,y\rangle|\\ \leqslant&\|x\|^2+\|y\|^2+2\|x\|\|y\|=(\|x\|+\|y\|)^2.\end{aligned}\tag{2.92}$$

□

Corollary $\langle\cdot,\cdot\rangle:(X,\|\cdot\|)\times(X,\|\cdot\|)\to\mathbb{C}$ is continuous. In particularly, for any fixed $y\in X$, $\langle\cdot,y\rangle\in(X,\|\cdot\|)^*$.

Definition 2.32 *A* ***Hilbert space*** *is a Banach space equipped with inner product.*

Proposition 2.14 *The norm $\|\cdot\|$ on a Hilbert space $\mathcal{H}$ satisfies the* ***parallelogram law****:*

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.\tag{2.93}$$

Proposition 2.15 (Polarisation)

$$\begin{aligned}\langle x,y\rangle=&\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right)\ \textit{over}\ \mathbb{R}\\ =&\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2+\mathrm{i}\|x+\mathrm{i}y\|^2-\mathrm{i}\|x-\mathrm{i}y\|^2\right)\ \textit{over}\ \mathbb{C}\end{aligned}\tag{2.94}$$

Remark If a norm $\|\cdot\|$ on a Banach space X satisfies the parallelogram law, then by defining $\langle\cdot,\cdot\rangle$ via polarisation, then $\langle\cdot,\cdot\rangle$ is an inner product, and $\langle x,x\rangle=\|x\|^2$.

Let $\mathcal{H}$ denote Hilbert space.

Definition 2.33 *Say $x,y\in\mathcal{H}$ are* ***orthogonal*** *to each other, $x\perp y$ if $\langle x,y\rangle=0$,*

Definition 2.34 *For $x \in \mathcal{H}$, $S \subset \mathcal{H}$, say $x \perp S$ if and only if $x \perp y, \forall y \in S$.*

Definition 2.35 *If $\mathcal{F} \leqslant \mathcal{H}$ is a subspace, then $\mathcal{F}^\perp := \{h \in \mathcal{H} : h \perp \mathcal{F}\}$ is called the* ***orthogonal complement*** *of $\mathcal{F}$.*

Proposition 2.16 *$\mathcal{F} \leqslant \mathcal{H} \Rightarrow \mathcal{F}^\perp \leqslant \mathcal{H}$ is a closed subspace.*

PROOF Clearly $\mathcal{F}^\perp \leqslant \mathcal{H}$. Take any $f, h_n \in \mathcal{H}$ s.t. $\langle f, h_n \rangle = 0, \forall n$ and $h_n \to h$ in $\mathcal{H}$. But $\langle f, \cdot \rangle$ is a continuous functional over $\mathcal{H}$. So $\langle f, h \rangle = 0$, i.e. $\mathcal{F}^\perp$ is closed. □

Now, note that for $\mathcal{F} \leqslant \mathcal{H}$, $\mathcal{F} \cap \mathcal{F}^\perp = \{0\}$ (If $H \in \mathcal{F} \cap \mathcal{F}^\perp$, then by definition of $\mathcal{F}^\perp$,$\langle h, h \rangle$=0), so we can form the (algebraic) direct sum $\mathcal{F} \oplus \mathcal{F}^\perp \leqslant \mathcal{H}$.
Let

$$\begin{aligned} \Phi : \mathcal{F} \oplus \mathcal{F}^\perp &\to \mathcal{F} + \mathcal{F}^\perp \leqslant \mathcal{H} \\ (x, y) &\mapsto x + y \end{aligned} \tag{2.95}$$

then Φ is an isomorphism (onto $\mathcal{F} + \mathcal{F}^\perp$).

Theorem 2.29 *If $\mathcal{F} \leqslant \mathcal{H}$ is closed, then Φ is an isomorphism onto $\mathcal{H}$.*

PROOF We only need to check that $\mathcal{H} = \mathcal{F} + \mathcal{F}^\perp$. Indeed, we prove that $\forall x \in \mathcal{H}$, $\exists$ "closest" element $x_1 \in \mathcal{F}$ to x s.t. $x - x_1 \in \mathcal{F}^\perp$.

Let $d := \text{dist}(x, \mathcal{F}) := \inf\{\|x - y\| : y \in \mathcal{F}\}$. Take $\{y_n\} \subset \mathcal{F}$ s.t. $\|x - y_n\|^2 \leqslant d^2 + \dfrac{1}{n}$.
Claim $\{y_n\}$ is Cauchy sequence.

Indeed, by parallelogram law,

$$\|2x - (y_n + y_m)\|^2 + \|y_m - y_n\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 \tag{2.96}$$

So

$$\begin{aligned} &\|y_m - y_n\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\left\|x - \frac{y_n + y_m}{2}\right\| \\ &\leqslant 2\left(d^2 + \frac{1}{n}\right) + 2\left(d^2 + \frac{1}{m}\right) - 4d^2 = \frac{2}{n} + \frac{2}{m} \to 0 \text{ as } n, m \to \infty. \end{aligned} \tag{2.97}$$

Thus $y_n \to x_1$ in $\mathcal{F}$, where $\|x - x_1\| = \lim\limits_{n \to \infty} \|x - y_n\| = d$. But $x_2 = x - x_1$, then $\|x_2\|^2 = d^2$. If $x_2 \notin \mathcal{F}^\perp$, then $\exists z \in \mathcal{F}$, s.t. $\langle x_2, z \rangle > 0$.

Then $\forall\varepsilon > 0$,

$$\begin{aligned} \|x-(x_1+\varepsilon z)\|^2 &= \|x_2-\varepsilon z\|^2 = \|x_2\|^2 - 2\varepsilon\langle x_2, z\rangle + \varepsilon^2\|z\|^2 \\ &= d^2 - \varepsilon\Big[2\langle x_2, z\rangle - \varepsilon\|z\|^2\Big] < d^2 \end{aligned} \tag{2.98}$$

for $\varepsilon > 0$ small enough. This contradicts that $\operatorname{dist}(x, \mathcal{F}) = d$. □

Remark $\mathcal{F} \leqslant \mathcal{H}$ closed $\Rightarrow$ $\mathcal{H} = \mathcal{F} \oplus \mathcal{F}^\perp$. Take $y \in \mathcal{F}, z \in \mathcal{F}^\perp, \langle y, z\rangle = 0$; i.e. $y \in \mathcal{F}^{\perp\perp}$, $\Rightarrow$ $\mathcal{F} \subset \mathcal{F}^{\perp\perp}$.

Conversely, $\forall x \in \mathcal{H}\setminus\mathcal{F}$, then $x = y+z$ for $y \in \mathcal{F}, z \in \mathcal{F}^\perp$. Then $\langle z, x\rangle = \langle z, y\rangle + \|z\|^2 = \|z\|^2 \neq 0$. Thus $x \in \mathcal{F}^{\perp\perp}$, so $\mathcal{F}^{\perp\perp} \subset \mathcal{F}$,

Corollary If $\mathcal{H}$ is a Hilbert space, $\mathcal{F} \leqslant \mathcal{H}$ closed, then $\mathcal{F} = \mathcal{F}^{\perp\perp}$.

Remark Let X be a Banach, $Y \leqslant X$. The analogue for $Y^\perp$ is the annihilator:

$$\operatorname{ann}(Y) = Y^\circ := \{T \in X^* : Ty = 0, \forall y \in Y\} \tag{2.99}$$

Then $Y^{\circ\circ} := (Y^\circ)^\circ = \{\xi \in X^{**} : \xi(T) = 0, \forall T \in Y^\circ\} \Rightarrow \mathcal{J}_X[Y] \subset Y^{\circ\circ} \subset X^{**}$.

Theorem 2.30 (Riesz Representation for Hilbert space) *$\mathcal{H} \cong \mathcal{H}^*$ by a natural isomorphism.*

In fact ,$\forall\Phi \in \mathcal{H}^*$, $\exists!\, y \in \mathcal{H}$ s.t. $\Phi(x) = \langle x, y\rangle$. $\Phi \mapsto y$ is an isomorphism (of Banach space).

But this is false for general Banach space, even if $\dim(X = \text{Banach}) < \infty$.

$$\{\psi : X' \to X \text{ is isomorphism }\} \xleftrightarrow{\text{one-to-one}} GL(X) \tag{2.100}$$

Definition 2.36 *Let $\mathcal{H}$ be Hilbert space, $S \subset \mathcal{H}$ is an orthonormal set if and only if $\forall x, y \in S, \langle x, y\rangle = 0, \forall z \in S,\ \|z\| = 1$.*

Definition 2.37 *An orthonormal basis(ONB) in $\mathcal{H}$ is a maximal orthonormal set.*

Theorem 2.31 *Every separable Hilbert space has an ONB.*

Proof By Zorn's lemma. □

Corollary If $\{e_\mu\} \subset \mathcal{H}$ is an ONB, then $\overline{\operatorname{span}\{e_\mu\}} = \mathcal{H}$.

Proof **Exercise**. □

Corollary Let $\mathcal{H}$ be a separable Hilbert space with $\dim\mathcal{H} = \infty$. $\mathcal{H}$ has an ONB $\{e_i\}_1^\infty$. Also, $\forall$ ONB $\{\tilde{e}_i\}_{i=1}^\infty, \forall x \in \mathcal{H}$,

$$x = \sum_{i=1}^{\infty} \langle x, \tilde{e}_i \rangle \tilde{e}_i \tag{2.101}$$

$\langle x, \tilde{e}_i \rangle$ is called **Fourier Coefficient**.

Example $\mathcal{H} = L^2(\mathbb{T}), \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. $\{e_n(x) := e^{\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ form an ONB for $\mathcal{H}$. Inner product on $\mathcal{H} : \langle f, g \rangle = \int_0^{2\pi} f\overline{g}\mathrm{d}x$. Then $\forall f \in {}^2(\mathrm{T}), f(x) = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n(x)$. That is,

$$f(x) = \sum_{n\in\mathbb{Z}} \left[\int f(y) e^{-\mathrm{i}ny} \mathrm{d}y \right] e^{\mathrm{i}nx} \tag{2.102}$$

in the L^2 sense. Let $\hat{f}(n) = \int f(y)e^{-\mathrm{i}ny}\mathrm{d}y$ be the n-th Fourier coefficient, then

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{\mathrm{i}nx} = \sum_{n\in\mathbb{Z}} \hat{f}(n) \Big[\cos(nx) + \mathrm{i}\sin(nx) \Big] \tag{2.103}$$

which is called Fourier expansion.

Proof By an earlier result, $\mathcal{H}$ have an ONB $\{e_\mu\}_{\mu\in\mathcal{I}}$. Note that

$$\|e_\mu - e_\lambda\|_{\mathcal{H}} = \sqrt{\langle e_\mu - e_\lambda, e_\mu - e_\lambda \rangle} = \sqrt{\|e_\mu\|^2 + \|e_\lambda\|^2} = \sqrt{2}. \tag{2.104}$$

Note that I must be countable. If not, consider dense $\mathcal{D} \subset \mathcal{H}$. Then $\forall \mu \in I$, $\exists d_\mu \in \mathcal{D}$ s.t. $\|e_\mu - e_\lambda\|_{\mathcal{H}} < \frac{\sqrt{2}}{4}$. Then for $\mu \neq \lambda, \|d_\mu - d_\lambda\| \geqslant \frac{\sqrt{2}}{2}$, so $\mathcal{D}$ is uncountable, contradicting that $\mathcal{H}$ is separable.

Conclusion $\mathcal{H}$ has a countable ONB $\{e_1, e_2, \cdots\}$.

Claim $\forall x \in \mathcal{H}$, $n \in N$,

$$\operatorname{dist}\Big(x, \operatorname{span}\{e_1, \cdots, e_n\}\Big) = \Big\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \Big\|. \tag{2.105}$$

Proof To prove the claim, note that

$$\operatorname{dist}\Big(x, \operatorname{span}\{e_1, \cdots, e_n\}\Big) = \inf\Big\{\Big\|x - \sum_{i=1}^{n} \lambda_i e_i\Big\|, \lambda_1, \cdots \lambda_n \in \mathbb{C}\Big\}. \tag{2.106}$$

But

$$\begin{aligned}&\Big\|x - \sum_{i=1}^{n} \lambda_i e_i\Big\|^2 = \Big\langle x - \sum_{i=1}^{n} \lambda_i e_i, x - \sum_{j=1}^{n} \lambda_j e_j\Big\rangle \\ =&\|x\|^2 + \sum_{i=1}^{n} \|\lambda_i\|^2 - 2\operatorname{Re}\Big\langle x, \sum_{i=1}^{n} \lambda_i e_i\Big\rangle = \|x\|^2 + \sum_{i=1}^{n} \|\lambda_i - \langle x, e_i\rangle\|^2 - \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 .\end{aligned} \tag{2.107}$$

since

$$\Big\langle \sum_{i=1}^{n} \lambda_i e_i, \sum_{j=1}^{n} \lambda_j e_j\Big\rangle = \sum_i \sum_j \lambda_i \overline{\lambda_j} \langle e_i, e_j\rangle = \sum_j \lambda_j \overline{\lambda_j} = \sum_j |\lambda_j|^2 . \tag{2.108}$$

So $\operatorname{dist}\Big(x, \operatorname{span}\{e_1, \cdots, e_n\}\Big) = \Big\|x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i\Big\| = \sqrt{\|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i\rangle|^2}$. This proves the claim. $\square$

Then since

$$\overline{\operatorname{span}\{e_i\}_1^\infty} = \mathcal{H}, \text{ for any } \varepsilon > 0, \tag{2.109}$$

$\exists N = N(\varepsilon) \in \mathbb{N}, \operatorname{dist}\Big(x, \operatorname{span}\{e_1, \cdots, e_n\}\Big) < \varepsilon$, $\forall n \geqslant N$, so

$$\Big\|x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i\Big\|_{\mathcal{H}} \xrightarrow{n \nearrow \infty} 0. \tag{2.110}$$

Corollary (Bessel) Let $\{e_i\}_1^\infty$ be an ONB for Hilbert $\mathcal{H}$,

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leqslant \|x\|^2, \ \forall n \in \mathbb{N}, x \in \mathcal{H}. \tag{2.111}$$

Also, $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 = \|x\|^2$ for Parseval's inequality.

Example $f \in L^2(\mathbb{T})$, then

$$\|f\|_{L^2}^2 = \int |f|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2. \tag{2.112}$$

Remark Suppose that Parseval holds. $\forall x \in \mathcal{H}$, where $\{e_i\}$ is only known to be an orthonormal set. Then $\{e_i\}$ must be an ONB.
(Hint: If not, consider $\overline{\text{span}\{e_i\}} \lneqq \mathcal{H}$).

2.11 Spectrum

Let X be an NVS, $T \in \mathcal{B}(X)$. $\text{Spec}(T) = \{\lambda \in \mathbb{C} : \lambda 1 - T \text{ is non-invertible in } \mathcal{B}(X)\}$.
So, $\lambda \in \text{Spec}(T) \Rightarrow$
Either (i) $\lambda 1 - T$ is not injective $\Leftrightarrow \text{Ker}(\lambda 1 - T) \not\subset 0\} \Leftrightarrow \exists v \in X \setminus \{0\}$ s.t. $Tv = \lambda v \Leftrightarrow \lambda$ is an eigenvalue of T;

or (ii) $\lambda 1 - T$ is nor surjective (e.g. $(\lambda 1 - T)(X)$ is not dense in X);

or (iii) $(\lambda 1 - T)^{-1}$(algebraic inverse) is unbounded.

Definition 2.38 *If thete exists $\{x_n\} \subset \mathbb{S}_X$ s.t. $\|\lambda x_n - Tx_n\|_X \to 0$, then λ id in the approximate spectrum of T.*

Fact(Spectral Mapping) Let $p(z)$ be a polynomial over $\mathbb{C}$, then for $X = \text{NVS}$, $T \in \mathcal{B}(X)$,

$$p(\text{Spec}(T)) = \text{Spec}(p(T)). \tag{2.113}$$

3 Homework

3.1 第一次作业

问题 3.1 令 $\{\omega_n\}$ 为非负实数序列. 对于子集 $E \in \mathbb{N}$, 定义

$$\mu_\omega(E) := \sum_{n\in E} \omega_n \tag{3.1}$$

证明 μ_ω 是 $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ 上的测度.(回忆 $\mathcal{P} := \{E : E \subset \mathbb{N}\}$.)

此时, 令 ν 为 $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ 上的任意测度, 证明 ν 与测度 μ_ω 一致, 其中 $\omega_n := \nu(n)$.

证明: 首先说明 $\mathcal{P}(\mathbb{N})$ 构成一个 $\sigma-$ 代数. 因为 $\mathcal{P}(\mathbb{N})$ 包含 X 的所有可能的子集, 因此 $\mathbb{N} \in \mathcal{P}(\mathbb{N})$, 并且 $\mathcal{P}(\mathbb{N})$ 对取余集和取可数并是封闭的.

下面证明 μ_ω 构成 $\sigma-$ 代数 $\mathcal{P}(\mathbb{N})$ 的测度. 即对于 $\{A_i\}$ 为 $\mathcal{P}(\mathbb{N})$ 中互不相交的可数集族, 有

$$\mu_\omega\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{n\in\bigcup_{i=1}^\infty A_i} \omega_n = \sum_{i=1}^\infty \sum_{n\in A_i} \omega_n = \sum_{i=1}^\infty \mu_\omega(A_i). \tag{3.2}$$

因此 μ_ω 构成 $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ 上的测度.

下面证明测度 ν 能表示成 (3.1) 的形式. 事实上, 由测度的性质, 对 $\mathbb{N}$ 的任意子集 E, 有

$$\nu(E) = \nu\Big(\bigcup_{n\in E}\{n\}\Big) = \sum_{n\in E} \nu(\{n\}) = \sum_{n\in E} \nu(n) = \sum_{n\in E} \omega_n. \tag{3.3}$$

因此 ν 与测度 μ_ω 一致.

问题 3.2 设 E 是 $\mathbb{R}$ 上的一个开子集, 对 $x, y \in E$, 定义 $I_{x,y}$ 为 x 与 y 之间的一个闭区间 (因此 $I_{x,x} = \{x\}$). 定义 E 上的关系 $\sim$ 使得 $x \sim y$ 当且仅当 $I_{x,y} \subset E$.

(1) 证明 $\sim$ 是 E 上的一个等价类. 这些等价类都是什么?

(2) 因此, 或以其他方式, 证明: E 是至多可数多个不交的开区间之并.

证明: (1) 首先, 对 $x \in E$, 有 $\{x\} = I_{x,x} \subset E$. 对任意 x, y, 有 $I_{x,y} = I_{y,x}$, 因此

$$x \sim y \Leftrightarrow I_{x,y} \subset E \Leftrightarrow I_{y,x} \subset E \Leftrightarrow y \sim x. \tag{3.4}$$

对任意 x, y, z, 记 $\tilde{I} = [\min\{x, y, z\}, \max\{x, y, z\}]$. 若 $x \sim y$, $y \sim z$, 则 $I_{x,y} \subset E$, $I_{y,z} \subset E$, 于是 $\tilde{I} \subset E$. 因此 $I_{x,z} \subset \tilde{I} \subset E$, 这给出 $x \sim z$. 综上所述, $\sim$ 构成 E 上的一个等价类.

对每一个 E 中的元素 a，定义 $b=\inf\{x:a\sim x\}$，则 b 或属于集合 $\{x:a\sim x\}$，或为集合 $\{x:a\sim x\}$ 的聚点. 若 b 属于集合 $\{x:a\sim x\}$，则 $I_{a,b}\subset E$，并且由 E 是开集可知存在 $\delta>0$ 使得 $(b-\delta,b+\delta)\in E$，而这给出 $b-\dfrac{\delta}{2}\in\{x:a\sim x\}$，这与 b 的定义矛盾. 因此 b 不属于 a 的等价类，但 b 是集合 $\{x:a\sim x\}$ 的聚点，并且总可以找到足够小的 $\varepsilon>0$，使得 $b+\varepsilon\in\{x:a\sim x\}$，这表明了 $I_{b+\varepsilon,a}\subset E$. 于是 $]b,a[$ 属于 a 的等价类，由于 b 不属于 a 的等价类，因此对任意 $b'\leqslant b$，有 $I_{b',a}\not\subset E$，即 $b'\notin\{x:a\sim x\}$. 同理，若定义 $c=\sup\{x:a\sim x\}$，则同样可推出 $]a,c[$ 属于 a 的等价类，但对任意 $c'\geqslant c$，有 $c'\notin\{x:a\sim x\}$. 于是 a 的等价类为开区间 $]b,c[$，其中 $b=\inf\{x:a\sim x\},c=\sup\{x:a\sim x\}$. 并且对任意两个这样构成的开区间，其必然是不交的，否则这两个开区间之并可合成同一个等价类. 综上所述，E 中的等价类为不交的开区间族.

(2) 由于每一个等价类都构成一个开区间，因此可以任取这个开区间中一个有理数与此开区间对应，并且对任意两个开区间，因为它们是不交的，所以它们所选出的有理数一定是不同的. 这给出了有理数集的子集到此开区间族的一个一一映射，所以这些开区间是可数多个的. 由于 E 中的等价类是不交的开区间族，因此 E 可以写为这些不交的开区间之并，这些开区间是可数多个的，因此 E 是至多可数多个不交的开区间之并.

问题 3.3 *使用上面的问题，推导出：如果 $\mathcal{F}$ 是 $\mathbb{R}$ 上包含所有形如 $]a,+\infty[,a\in\mathbb{R}$ 的区间的 $\sigma-$ 代数，则 $\mathcal{F}$ 包含 $\mathbb{R}$ 上的 Borel $\sigma-$ 代数.*

证明：首先对任意 $a<a'$，有 $]a,+\infty[\in\mathcal{F}$，$]a',+\infty[\in\mathcal{F}$. 于是 $]-\infty,a']=]a',+\infty[^c\in\mathcal{F}$，$]a,a']=]a,+\infty[\cap]-\infty,a']\in\mathcal{F}$. 然后再取 $b>a$，并且 a' 依次取极限为 b 的递增序列 $\{a'_k\}_{k=1}^{\infty}$，此时 $]a,b[=\bigcup\limits_{k=1}^{\infty}]a,a'_k]\in\mathcal{F}$，以上推导表明，对任意的 a,b，开区间 $]a,b[\in\mathcal{F}$. 结合上题结论，$\mathbb{R}$ 上的任意开集都可以表示为不交开区间的可数并，因此 $\mathbb{R}$ 上的任意开集都属于 $\mathcal{F}$.

根据 Borel $\sigma-$ 代数的定义 (即最小的含 $\mathbb{R}$ 上所有开集的 $\sigma-$ 代数)，而 $\sigma-$ 代数 $\mathcal{F}$ 含 $\mathbb{R}$ 上所有的开集，因此 $\mathcal{F}$ 包含 $\mathbb{R}$ 上的 Borel $\sigma-$ 代数.

问题 3.4 *设 $(X,\mathcal{F},\mu)$ 是一个测度空间，并令 $f:X\to Y$ 为集合之间的函数. 定义*

$$f_{\#}(\mathcal{F}):=\{B\subset Y:f^{-1}(B)\in\mathcal{F}\},\qquad [f_{\#}\mu](B):=\mu(f^{-1}(B))\ \text{ 对 }\ B\in f_{\#}(\mathcal{F}).\tag{3.5}$$

(1) 证明 $(Y,f_{\#}(\mathcal{F}),f_{\#}\mu)$ 是一个测度空间；

(2) 现在令 $Y=\mathbb{R}$ 以及 $(X,\mathcal{F},\mu)$ 为 $(\mathbb{R},\mathfrak{B}(\mathbb{R}),\mathcal{L}^1)$，Borel $\sigma-$ 代数和 Lebesgue 测度. 定出 $(\mathbb{R},f_{\#}(\mathcal{F}),f_{\#}\mu)$ 当

- 当 $\cos x \neq 0$ 时 $f(x) = \tan x$，当 $\cos x = 0$ 时 $f(x) = 0$；
- $f(x) = \arctan x$.

证明：(1) 首先证 $f_{\#}\mathcal{F}$ 是一个 $\sigma-$ 代数. 注意到 $\mathcal{F}$ 是一个 $\sigma-$ 代数，因此

(i)$Y = f(X)$，$X \in \mathcal{F}$，因此 $Y \in f_{\#}(\mathcal{F})$；

(ii) 若 $B \subset Y, B \in f_{\#}(\mathcal{F})$，则 $f^{-1}(B) \in \mathcal{F}$，$X \setminus f^{-1}(B) \in \mathcal{F}$. 若要证 $Y \setminus B \in f_{\#}(\mathcal{F})$，只需证明 $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$. 注意到，

$$x \in X \setminus f^{-1}(B) \Leftrightarrow x \notin f^{-1}(B) \Leftrightarrow f(x) \notin B \Leftrightarrow f(x) \in Y \setminus B \Leftrightarrow x \in f^{-1}(Y \setminus B). \tag{3.6}$$

因此 $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$；

(iii) 对 $B_1, B_2, \cdots \in f_{\#}(\mathcal{F})$，下面证明 $\bigcup\limits_{n=1}^{\infty} B_n \in f_{\#}(\mathcal{F})$. 设 $f(A_n) = B_n$，下面证明 $\bigcup\limits_{n=1}^{\infty} B_n = f\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big)$. 注意到，

$$x \in \bigcup_{n=1}^{\infty} B_n \Leftrightarrow \exists N, x \in B_N \Leftrightarrow \exists N, f^{-1}(x) \in A_N \Leftrightarrow f^{-1}(x) \in \bigcup_{n=1}^{\infty} A_n \Leftrightarrow x \in f\Big(\bigcup_{n=1}^{\infty} A_n\Big). \tag{3.7}$$

因此 $\bigcup\limits_{n=1}^{\infty} B_n = f\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big)$. 综上所述，$f_{\#}\mathcal{F}$ 是一个 $\sigma-$ 代数.

再证 $[f_{\#}\mu]$ 为一个测度，此处仅需证明 $[f_{\#}\mu]$ 满足可数可加性. 令 $B_1, B_2, \cdots \in f_{\#}(\mathcal{F})$ 为不交的集合序列，并定义 $f(A_n) = B_n$，此时 $A_1, A_2, \cdots \in f_{\#}(\mathcal{F})$ 也是不交的，若不然，设存在 $n \neq m$ 以及 x，使得 $x \in A_n \cap A_m$，则则 $f(x) \in f(A_n \cap A_m) = B_n \cap B_m = \varnothing$，而这将导出矛盾. 因此

$$[f_{\#}\mu]\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} [f_{\#}\mu](B_n). \tag{3.8}$$

给出 $[f_{\#}\mu]$ 的可数可加性. 综上所述，$(Y, f_{\#}(\mathcal{F}), f_{\#}\mu)$ 是一个测度空间.

(2) 当 $\cos x \neq 0$ 时 $f(x) = \tan x$, 当 $\cos x = 0$ 时 $f(x) = 0$. 若仅在 X 上的 $\Big(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big)$ 中考虑 f^{-1} 的作用，注意到 $\tan x$ 为连续函数，因此在此种限制下 $f^{-1}(B)$ 能落在 $\Big(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big)$ 上的 Borel $\sigma-$ 代数中的集合 B 均属于 $f_{\#}\mathcal{F}$，这样的集合即可组成 $\mathbb{R}$ 上的 Borel $\sigma-$ 代数. 此时再根据由于 Borel $\sigma-$ 代数中包含单点集，并且由于 $\tan x$ 的周期为 π，因此可知 $f_{\#}\mathcal{F}$ 为 $\mathbb{R}$ 上的 Borel $\sigma-$ 代数. 测度 $f_{\#}\mu$ 的表现：对集合 $B \in f_{\#}(\mathcal{F})$，若 $f^{-1}(B)$ 限制在

$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ 上的测度为 0，则 $[f_{\#}\mu](B)=0$；若 $f^{-1}(B)$ 限制在 $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ 上的测度大于 0，则 $[f_{\#}\mu](B)=+\infty$.

当 $f(x)=\arctan x$ 时，由于 $\arctan x$ 是定义在 $\mathbb{R}$ 上，值域为 $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ 的连续函数，故 $f_{\#}\mathcal{F}$ 为 $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ 上的 Borel $\sigma-$ 代数. 此时的 $[f_{\#}\mu](B)$ 即如 $[f_{\#}\mu](B):=\mu(f^{-1}(B))$ 中之所定义.

问题 3.5 设 $f:\mathbb{R}\to\overline{\mathbb{R}}=[-\infty,+\infty]$ 为一个可积函数，并且令 $\lambda>0$. 证明

$$\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|\geqslant\lambda\})\leqslant\frac{1}{\lambda}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x. \tag{3.9}$$

然后推导

(1) $f(x)\in\mathbb{R}a.e.$;

(2) $\displaystyle\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x=0$ 当且仅当 $f(x)=0$ $a.e.$.

证明：

$$\begin{aligned}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x\geqslant&\int_{|f(x)|\geqslant\lambda}|f(x)|\,\mathrm{d}x\geqslant\int_{|f(x)|\geqslant\lambda}(|f(x)|-\lambda)\mathrm{d}x+\lambda\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|\geqslant\lambda\})\\ &\geqslant\lambda\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|\geqslant\lambda\})\end{aligned} \tag{3.10}$$

上式左右同时除以 λ，可得

$$\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|\geqslant\lambda\})\leqslant\frac{1}{\lambda}\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x. \tag{3.11}$$

此时令 $\lambda\to\infty$，可知 (3.11) 右侧 $\to 0$. 由此可知 $\mathcal{L}^1(f(x)\notin\mathbb{R})=0$，这表明 $f(x)\in\mathbb{R}$ a.e..

以 $f(x)=0$ a.e. 知 $\displaystyle\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x=0$ 则为显然. 下面由 $\displaystyle\int_{\mathbb{R}}|f(x)|\,\mathrm{d}x=0$ 给出 $f(x)=0$ a.e.. 注意到

$$\begin{aligned}&\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|>0\})=\mathcal{L}^1\Big(\bigcup_{k=-\infty}^{+\infty}\{x\in\mathbb{R}:2^k\leqslant|f(x)|<2^{k+1}\}\Big)\\ =&\sum_{k=-\infty}^{+\infty}\mathcal{L}^1(\{x\in\mathbb{R}:2^k\leqslant|f(x)|<2^{k+1}\})\leqslant\sum_{k=-\infty}^{+\infty}\mathcal{L}^1(\{x\in\mathbb{R}:|f(x)|\geqslant 2^k\})\end{aligned} \tag{3.12}$$

然而对任意 $k \in \mathbb{Z}$，都有

$$\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| \geqslant 2^k\}) \leqslant \frac{1}{\lambda}\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x = 0. \tag{3.13}$$

因此 $\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| \geqslant 2^k\}) = 0$，$\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| > 0\}) \leqslant \sum_{k=-\infty}^{+\infty} \mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| \geqslant 2^k\}) = 0$. 于是

$$\mathcal{L}^1(\{x \in \mathbb{R} : |f(x)| > 0\}) = 0. \tag{3.14}$$

由此可知 $\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x = 0$ 当且仅当 $f(x) = 0$ a.e..

问题 3.6 设 $f \in L^1(X, \mathcal{F}, \mu)$. 证明对任意 $\varepsilon > 0$，存在 $\delta > 0$ 使得对任意满足 $\mu(E) < \delta$ 的 $E \in \mathcal{F}$，都有 $\int_E |f|\,\mathrm{d}\mu < \varepsilon$ 成立.

证明：由 $f \in L^1(X, \mathcal{F}, \mu)$ 可设存在实数 $M \geqslant 0$ 使得

$$\int_X |f|\,\mathrm{d}\mu = M. \tag{3.15}$$

取 $X_\lambda = \left\{x \in X : |f| \geqslant \lambda\right\}$，$X_\infty = \left\{x \in X : |f| = +\infty\right\}$. 先证存在 $\lambda > 0$ 使得 $\int_{X_\lambda} |f|\,\mathrm{d}\mu < \frac{\varepsilon}{2}$. 倘若不然，设存在趋于 $+\infty$ 的实数序列 $\{\lambda_n\}$ 和单调递减序列 $\{a_n\}$ 满足

$$a_n = \int_{X_{\lambda_n}} |f|\,\mathrm{d}\mu \tag{3.16}$$

知各 a_n 存在下界 $\frac{\varepsilon}{2}$. 由单调收敛原理知 $\{a_n\}$ 存在极限 $a^* \geqslant \frac{\varepsilon}{2}$，这时对 (3.16) 右侧取极限，由控制收敛定理和 (3.15)，有

$$\begin{aligned}\lim_{n\to\infty}\int_{X_{\lambda_n}} |f|\,\mathrm{d}\mu &= \lim_{n\to\infty}\int_X |f|\,\chi_{X_{\lambda_n}}\mathrm{d}\mu = \int_X \lim_{n\to\infty}|f|\,\chi_{X_{\lambda_n}}\mathrm{d}\mu \\ &= \int_X |f|\,\chi_{X_\infty}\mathrm{d}\mu = \int_{X_\infty} |f|\,\mathrm{d}\mu\end{aligned} \tag{3.17}$$

由 $a^* = \int_{X_\infty} |f| \,\mathrm{d}\mu$ 有 $\int_{X_\infty} |f| \,\mathrm{d}\mu \geqslant \frac{\varepsilon}{2}$. 注意到

$$\int_{X_\infty} |f| \,\mathrm{d}\mu = \mu(X_\infty) \cdot (+\infty) \in \{0, +\infty\}, \tag{3.18}$$

(这里用到 $0 \cdot (+\infty) = 0$ 的约定) 则 $\int_{X_\infty} |f| \,\mathrm{d}\mu = +\infty$，这与 (3.15) 矛盾. 因此存在正实数 λ_0 使得 $\int_{X_\lambda} |f| \,\mathrm{d}\mu < \frac{\varepsilon}{2}$.

此时我们可以取 $\delta = \min\left\{\frac{\varepsilon}{\lambda_0}, \mu(X_{\lambda_0})\right\}$，这时对任意 E 满足 $\mu(E) < \delta$，可将其分为 $E \cap X_{\lambda_0}$ 和 $E \setminus X_{\lambda_0}$ 两部分，其中

$$\int_{E\cap X_{\lambda_0}} |f| \,\mathrm{d}\mu \leqslant \int_{X_{\lambda_0}} |f| \,\mathrm{d}\mu \leqslant \frac{\varepsilon}{2} \tag{3.19}$$

$$\int_{E\setminus X_{\lambda_0}} |f| \,\mathrm{d}\mu \leqslant \lambda_0 \mu(E \cap X_{\lambda_0}) \leqslant \lambda_0 \cdot \frac{\varepsilon}{2\lambda_0} = \frac{\varepsilon}{2} \tag{3.20}$$

综上，对如上选取的 δ，有

$$\int_E |f| \,\mathrm{d}\mu = \int_{E\cap X_{\lambda_0}} |f| \,\mathrm{d}\mu + \int_{E\setminus X_{\lambda_0}} |f| \,\mathrm{d}\mu \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.21}$$

问题 3.7 (Borel-Cantelli 引理) 设 $\{E_j\}_{j=1}^{\infty}$ 是可数的 $(X, \mathcal{F}, \mu)$ 的可测子集族. 设 $\sum_{j=1}^{\infty} \mu(E_j) < \infty$，并且定义

$$\limsup_{j\to\infty} E_j := \{x \in X : \text{对无穷多个}k, \text{有}x \in E_k\text{成立}\}. \tag{3.22}$$

证明 $\limsup_{j\to\infty} E_j$ 是可测的并且 $\mu-$ 空的.

证明：首先给出以下表示：

$$\limsup_{j\to\infty} E_j = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k. \tag{3.23}$$

这是因为，存在无穷多个 $x \in E_k \iff x \in \bigcup_{k=j}^{\infty} E_k, \forall j$. 由于 $\sigma-$ 代数 $\mathcal{F}$ 对可数交与可数并封

闭，因此 $\limsup_{j\to\infty} E_j \in \mathcal{F}$，即 $\limsup_{j\to\infty} E_j$ 是可测的. 注意到，

$$\mu\Big(\bigcup_{k=j}^{\infty} E_k\Big) \leqslant \sum_{k=j}^{\infty} \mu(E_k),\quad \mu\Big(\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k\Big) \leqslant \min_j \Big\{\mu\Big(\sum_{k=j}^{\infty} \mu(E_k)\Big)\Big\}, \tag{3.24}$$

而由 $\sum_{j=1}^{\infty} \mu(E_j) < \infty$，知

$$\lim_{j\to\infty} \sum_{k=j}^{\infty} \mu(E_k) = 0. \tag{3.25}$$

因此 $\mu\Big(\limsup_{j\to\infty} E_j\Big) = 0.$

问题 3.8 使用以上的问题，或以其他方式，证明以下命题:

设 $\{f_n\}$ 是 $[0,1]$ 上的可测函数序列，满足 $|f_n(x)| < \infty$ 对 $x \in [0,1]$ a.e. 成立. 则存在一个正整数序列 $\{a_n\}$ 使得 $\dfrac{f_n(x)}{a_n} \to 0$ 对几乎处处的 x 成立.

证明：首先知道

$$\mu(\{x : |f_n(x)| = \infty\}) = 0, \quad \forall n \in \mathbb{N}. \tag{3.26}$$

下面证明，对任意 n，都可找到正实数 $M_n > 0$，使得

$$\mu(\{x : |f_n(x)| > M_n\}) < \frac{1}{2^n} \tag{3.27}$$

若不然，考虑序列 $\{M_{n,k}\}_{k=1}^{\infty}$，令其单调递增并趋于无穷，且满足

$$\mu(\{x : |f_n(x)| > M_{n,k}\}) \geqslant \frac{1}{2^n} \tag{3.28}$$

则必有

$$\mu(\{x : |f_n(x)| = +\infty\}) = \bigcap_{k=1}^{\infty} \mu(\{x : |f_n(x)| > M_{n,k}\}) = \mu\Big(\liminf_{k\to\infty}\{x : |f_n(x)| > M_{n,k}\}\Big) \geqslant \frac{1}{2^n}. \tag{3.29}$$

可得到矛盾，因此这样的 M_n 必然存在. 令 $a_n = nM_n$，则

$$\mu\Big(\Big\{x : \frac{|f_n(x)|}{a_n} > \frac{1}{n}\Big\}\Big) = \mu(\{x : |f_n(x)| > M_n\}) < \frac{1}{2^n}. \tag{3.30}$$

令 $E_n = \left\{x : \dfrac{|f_n(x)|}{a_n} > \dfrac{1}{n}\right\}$，则 $\sum_{n=1}^{\infty} \mu(E_n) < \infty$；由上一问题的结论知 $\mu\Big(\limsup_{n\to\infty} E_n\Big) = 0$；而对任意 $x \notin \limsup_{n\to\infty} E_n$，仅存在有限多个 n 使得 $x \in E_n$ 成立，而这样的点都能保证 $\dfrac{f_n(x)}{a_n} \to 0$ 成立. 因此原问题证毕.

问题 3.9 设 $f_n : (X, \mathcal{F}, \mu) \to [0, \infty]$ 对任意 $n \in \mathbb{N}$ 都是可测的. 假设 $f_1 \geqslant f_2 \geqslant f_3 \geqslant \cdots \geqslant 0$，并且当 $n \to \infty$ 时 f_n 点态收敛到 f. 假设我们还有 $f_1 \in L^1(X, \mathcal{F}, \mu)$.

证明 $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$. 我们是否可以去掉 $f_1 \in L^1(X, \mathcal{F}, \mu)$ 的假设？

证明: 我们可以在条件 $L^1(X, \mathcal{F}, \mu)$ 下证明 $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$. 构造函数序列 $\{f_1 - f_n\}$，由题目条件可知

$$0 \leqslant f_1 - f_2 \leqslant f_1 - f_3 \leqslant \cdots \leqslant f_1 - f_n \leqslant \cdots \leqslant f_1 - f. \tag{3.31}$$

且有 $\lim_{k\to\infty} (f_1 - f_k) = f_1 - f$, a.e.. 此时由 MCT 定理可知

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu. \tag{3.32}$$

在去掉 $f_1 \in L^1(X, \mathcal{F}, \mu)$ 的假设后，我们容易找到反例. 设 $X = \mathbb{R}$，$f(x) \equiv 0$，而

$$f_n(x) = \begin{cases} 0, & |x| \in [0, n], \\ 1, & |x| \in \ ]n, +\infty[. \end{cases} \tag{3.33}$$

此时可知当 $n \to \infty$ 时，f_n 点态逼近于函数 f，但是 $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = +\infty$ 对任何 $n \in \mathbb{N}$ 成立. 但是 $\lim_{n\to\infty} \int_X f \mathrm{d}\mu = 0$，由此给出问题的反例.

问题 3.10 设 $(X, \mathcal{F}, \mu)$ 是一个有限的测度空间. 设 $f_n : (X, \mathcal{F}, \mu) \to \mathbb{C}$ 是一个有界可测函数的序列. 假设 f_n 在 X 上一致逼近于 f.

证明 $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu$. 我们是否可以去掉 $\mu(X) < \infty$ 的假设？

证明: 首先在有限空间内证明问题，由于 f_n 在 X 上一致逼近于 f，因此对任意 $\varepsilon > 0$，存

在 $N\in\mathbb{N}$，使得当 $n>N$ 时，有 $|f_n-f|<\varepsilon$ 在整个 X 上都成立. 此时

$$\int_X f_n\mathrm{d}\mu-\int_X f\mathrm{d}\mu=\int_X (f_n-f)\mathrm{d}\mu\leqslant\int_X |f_n-f|\,\mathrm{d}\mu\leqslant\varepsilon\mu(X). \tag{3.34}$$

由于 $\varepsilon>0$ 可以任取，因此 $\lim\limits_{n\to\infty}\int_X f_n\mathrm{d}\mu=\int_X f\mathrm{d}\mu$ 得到证明.

下面给出去掉 $\mu(X)<\infty$ 以后，等式不成立的一个反例. 设 $X=\mathbb{R}$，$f(x)\equiv 0$，而

$$f_n(x)=\begin{cases}0, & |x|\in[0,n],\\ \dfrac{1}{x}, & |x|\in\ ]n,+\infty[.\end{cases} \tag{3.35}$$

此时我们可以验证 f_n 在 X 上一致逼近于 f，但是 $\lim\limits_{n\to\infty}\int_X f_n\mathrm{d}\mu=+\infty$ 对任何 $n\in\mathbb{N}$ 成立. 但是 $\lim\limits_{n\to\infty}\int_X f\mathrm{d}\mu=0$，由此给出问题的反例.

问题 3.11 设中 $U\subset\mathbb{R}^n$ 是一个开集，$f,g:U\to\mathbb{R}$ 是两个连续函数，假设存在一个 *Lebesgue*-空的集合 $Z\subset U$ 使得 $f(x)=g(x)$ 对所有 $x\in U\setminus Z$ 成立. 证明 $f(x)=g(x)$ 对所有 $x\in U$ 成立.

证明：我们定义函数 $h(x)=f(x)-g(x)$，易知 $h(x)$ 也是一个连续函数，$h(x)=0$ 对所有 $x\in U\setminus Z$ 成立，下面证 $h(x)=0$ 对所有 $x\in U$ 成立.

取任意 $x_0\in Z$，则由于 $Z\subset U$ 是 $\mathbb{R}$ 中的开集，因此存在 r_0，使得 $B(x_0,r_0)\subset U$. 此时构造趋于 0 的单调递减的正实数序列 $r_1\geqslant r_2\geqslant\cdots\geqslant r_n\geqslant\cdots>0$. 于是对任意 $B(x_0,r_k),k\in\mathbb{N}^*$，存在 $x_k\in B(x_0,r_k)\setminus Z$，否则 $B(x_0,r_k)\subset Z$，但由于 $\mu(B(x_0,r_k))>0$，此时与 $\mu(Z)=0$ 产生矛盾，因此这样的序列 $\{x_k\}$ 是存在的，并且都有 $x_k\in U\setminus Z$. 这表明了 $h(x_k)=0$. 注意到 $x_k\to x_0$，而 $h(x)$ 是连续函数，因此

$$h(x_0)=\lim_{k\to\infty}h(x_k)=0. \tag{3.36}$$

因此对每一个 $x\in Z$，均有 $h(x)=0$ 成立. 结合 $h(x)=0$ 对所有 $x\in U\setminus Z$ 可知，$h(x)=0$ 对所有 $x\in U$ 成立.

问题 3.12 证明以下控制收敛定理的变形.

设 $f_n:(X,\mathcal{F},\mu)\to\mathbb{C}$ 是可测函数的序列. 设 f_n 在 μ 的意义下几乎处处收敛于 f. 假设存在可测函数序列 $g_n:(X,\mathcal{F},\mu)\to[0,\infty]$ 和一个可积函数 $G:(X,\mathcal{F},\mu)\to[0,\infty]$ 使得

$0 \leqslant |f_n| \leqslant g_n$ 在 $\mu-$ 意义下几乎处处成立且有

$$\int_X |g_n - G| \,\mathrm{d}\mu \to 0, \quad 当 n \to \infty. \tag{3.37}$$

则 $\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$

证明：首先证明 $f \in L^1(X, \mathcal{F}, \mu)$. 由于 G 是可积函数并且

$$\int_X |g_n - G| \,\mathrm{d}\mu \to 0, \quad 当 n \to \infty. \tag{3.38}$$

并且根据

$$\int_X |g_n - G| \,\mathrm{d}\mu \geqslant \int_X (g_n - G)\mathrm{d}\mu = \int_X g_n \mathrm{d}\mu - \int_X G\mathrm{d}\mu. \tag{3.39}$$

因此存在一个正整数 N，当 $n \geqslant N$ 时，有 $g_n \in L^1(X, \mathcal{F}, \mu)$，并且 $\int_X |g_n| \,\mathrm{d}\mu$ 有上界. 这里我们不妨设 $N = 1$，并且上界为 M. 再根据条件 $0 \leqslant |f_n| \leqslant g_n$ 在 $\mu-$ 意义下几乎处处成立可知 $f_n \in L^1(X, \mathcal{F}, \mu)$. 由 f_n 在 μ 的意义下几乎处处收敛于 f 可知 f 是可测的；由 Fatou 引理，

$$\int_X |f| \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} |f_n| \,\mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X |f_n| \,\mathrm{d}\mu \tag{3.40}$$

可知 $f \in L^1(X, \mathcal{F}, \mu)$.

首先我们先证命题对满足条件的 $f_n : (X, \mathcal{F}, \mu) \to \mathbb{R}$ 成立. 定义 $h_n := G + f_n + |g_n - G|, \tilde{h}_n := G - f_n + |g_n - G|$，注意到 $|g_n - G| \geqslant g_n - G \geqslant \max\{-f_n - G, f_n - G\}$，因此有 $h_n \geqslant 0, \tilde{h}_n \geqslant 0$. 于是结合 Fatou 引理，有

$$\begin{aligned}
\int_X G\mathrm{d}\mu + \int_X f\mathrm{d}\mu &= \int_X \liminf_{n\to\infty} (G + f_n)\mathrm{d}\mu = \int_X \liminf_{n\to\infty} h_n \mathrm{d}\mu \\
&\leqslant \liminf_{n\to\infty} \int_X h_n \mathrm{d}\mu = \liminf_{n\to\infty} \int_X (G + f_n + |g_n - G|)\mathrm{d}\mu \\
&= \int_X G\mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu + \lim_{n\to\infty} \int_X |g_n - G| \,\mathrm{d}\mu \\
&= \int_X G\mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n \mathrm{d}\mu.
\end{aligned} \tag{3.41}$$

$$
\begin{aligned}
\int_X G\mathrm{d}\mu - \int_X f\mathrm{d}\mu = \int_X \liminf_{n\to\infty}(G-f_n)\mathrm{d}\mu = \int_X \liminf_{n\to\infty} \tilde{h}_n \mathrm{d}\mu \\
\leqslant \liminf_{n\to\infty} \int_X \tilde{h}_n\mathrm{d}\mu = \liminf_{n\to\infty}\int_X (G-f_n+|g_n-G|)\mathrm{d}\mu \\
= \int_X G\mathrm{d}\mu + \liminf_{n\to\infty}\int_X(-f_n)\mathrm{d}\mu + \lim_{n\to\infty}\int_X |g_n-G|\,\mathrm{d}\mu \\
= \int_X G\mathrm{d}\mu - \limsup_{n\to\infty}\int_X f_n\mathrm{d}\mu.
\end{aligned}
\tag{3.42}
$$

在 (3.41)(3.42) 两边消去 $\int_X G\mathrm{d}\mu$，可得

$$
\limsup_{n\to\infty}\int_X f_n\mathrm{d}\mu \leqslant \int_X f\mathrm{d}\mu \leqslant \liminf_{n\to\infty}\int_X f_n\mathrm{d}\mu. \tag{3.43}
$$

注意到

$$
\liminf_{n\to\infty}\int_X f_n\mathrm{d}\mu \leqslant \limsup_{n\to\infty}\int_X f_n\mathrm{d}\mu. \tag{3.44}
$$

因此可知 (3.43) 中不等号均应当取等，且 $\limsup$ 和 $\liminf$ 均可换成 $\lim$. 因此

$$
\lim_{n\to\infty}\int_X f_n\mathrm{d}\mu = \int_X f\mathrm{d}\mu. \tag{3.45}
$$

下面我们证明对满足条件的 $f_n : (X,\mathcal{F},\mu)\to\mathbb{C}$ 成立. 此时令 $f_n = \mathrm{Re}(f_n)+\mathrm{i}\mathrm{Im}(f_n)$，由于 $0\leqslant \max\{|\mathrm{Re}(f_n)|,|\mathrm{Im}(f_n)|\}\leqslant |f_n|\leqslant g_n$，因此对 $\mathrm{Re}(f_n)$ 和 $\mathrm{Im}(f_n)$ 也可以使用上述结论，则

$$
\lim_{n\to\infty}\int_X \mathrm{Re}(f_n)\mathrm{d}\mu = \int_X \mathrm{Re}(f)\mathrm{d}\mu,\quad \lim_{n\to\infty}\int_X \mathrm{Im}(f_n)\mathrm{d}\mu = \int_X \mathrm{Im}(f)\mathrm{d}\mu \tag{3.46}
$$

分别成立. 于是

$$
\begin{aligned}
\lim_{n\to\infty}\int_X f_n\mathrm{d}\mu = \lim_{n\to\infty}\int_X(\mathrm{Re}(f_n)+\mathrm{i}\mathrm{Im}(f_n))\mathrm{d}\mu = \lim_{n\to\infty}\int_X \mathrm{Re}(f_n)\mathrm{d}\mu + \mathrm{i}\lim_{n\to\infty}\int_X \mathrm{Im}(f_n)\mathrm{d}\mu \\
= \int_X \mathrm{Re}(f)\mathrm{d}\mu + \int_X \mathrm{i}\mathrm{Im}(f)\mathrm{d}\mu = \int_X(\mathrm{Re}(f)+\mathrm{i}\mathrm{Im}(f))\mathrm{d}\mu = \int_X f\mathrm{d}\mu.
\end{aligned}
\tag{3.47}
$$

问题 3.13 设 $\alpha \in]0,1[$. 证明 *Gamma* 函数

$$\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1} e^{-x} \mathrm{d}x \tag{3.48}$$

作为 *Lebesgue* 积分是良定义的.

证明：将积分区域分为 $]0,1[$ 以及 $]1,+\infty[$，则

$$\int_0^1 x^{\alpha-1} e^{-x} \mathrm{d}x \leqslant \int_0^1 x^{\alpha-1} \mathrm{d}x = \frac{1}{\alpha} x^{\alpha}\Big|_{x=0}^{1} = \frac{1}{\alpha}. \tag{3.49}$$

$$\int_1^{+\infty} x^{\alpha-1} e^{-x} \mathrm{d}x \leqslant \int_1^{+\infty} e^{-x} \mathrm{d}x = e^{-x}\Big|_{x=0}^{1} = e^{-1}. \tag{3.50}$$

因此函数作为 Lebesgue 积分是良定义的.

问题 3.14 考虑 *Shannon* 尺度函数

$$\mathrm{sinc}(x) = \frac{\sin x}{x} \tag{3.51}$$

其是否在 $]0,1[$ 与 $]1,+\infty[$ 上可积？验证你的答案.

解：该函数在 $]0,1[$ 上可积，在 $]1,+\infty[$ 上不可积. 首先由 $|\sin x| < x$ 可知函数的绝对值在 $]0,1[$ 上存在上界 1，因此

$$\int_0^1 |\mathrm{sinc}(x)| \, \mathrm{d}x \leqslant 1. \tag{3.52}$$

可知函数 $\mathrm{sinc}(x)$ 在 $]0,1[$ 上可积. 而在 $]1,+\infty[$ 上，函数 $\mathrm{sinc}(x)$ 的绝对值的积分

$$\int_1^{+\infty} |\mathrm{sinc}(x)| \, \mathrm{d}x = +\infty. \tag{3.53}$$

因此函数 $\mathrm{sinc}(x)$ 在 $]1,+\infty[$ 上不可积.

问题 3.15 求出以下极限

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{nx \sin x}{1 + n^{\alpha} x^{\alpha}} \mathrm{d}x, \quad \text{其中 } \alpha > 1. \tag{3.54}$$

解：注意到

$$\left|\frac{nx\sin x}{1+n^\alpha x^\alpha}\right| \leqslant \frac{nx}{1+n^\alpha x^\alpha} \tag{3.55}$$

取 $t=nx$，定义函数 $\varphi(t)=\dfrac{t}{1+t^\alpha}$，则

$$\varphi'(t)=\frac{1+t^\alpha-\alpha t\cdot t^{\alpha-1}}{(1+t^\alpha)^2}=\frac{1-(\alpha-1)t^\alpha}{(1+t^\alpha)^2} \tag{3.56}$$

当 $t=(\alpha-1)^{1/\alpha}:=t_0$ 时，$\varphi'(t_0)=0$；当 $t>t_0$ 时，$\varphi'(t)<0$，函数单调下降；当 $t<t_0$ 时，$\varphi'(t)>0$，函数单调上升. 因此 $\varphi(t)$ 在 $t=t_0$ 处取到最大值 $\dfrac{(\alpha-1)^{1/\alpha}}{\alpha}$. 此时取函数 $\Phi(x)=\dfrac{(\alpha-1)^{1/\alpha}}{\alpha}$，可作为函数 $\dfrac{nx\sin x}{1+n^\alpha x^\alpha}$ 的控制函数，由控制收敛定理，

$$\lim_{n\to\infty}\int_0^{2\pi}\frac{nx\sin x}{1+n^\alpha x^\alpha}\mathrm{d}x=\int_0^{2\pi}\lim_{n\to\infty}\frac{nx\sin x}{1+n^\alpha x^\alpha}\mathrm{d}x. \tag{3.57}$$

注意到 $\lim\limits_{t\to+\infty}\varphi(t)=0$，因此

$$\lim_{n\to\infty}\frac{nx\sin x}{1+n^\alpha x^\alpha}=0. \tag{3.58}$$

因此

$$\lim_{n\to\infty}\int_0^{2\pi}\frac{nx\sin x}{1+n^\alpha x^\alpha}\mathrm{d}x=\int_0^{2\pi}\lim_{n\to\infty}\frac{nx\sin x}{1+n^\alpha x^\alpha}\mathrm{d}x=0. \tag{3.59}$$

问题 3.16 证明

$$\lim_{n\to\infty}\int_0^{n^2} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\mathrm{d}x=\frac{1}{2}. \tag{3.60}$$

证明:

$$\int_0^{n^2} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\mathrm{d}x=\int_0^{+\infty} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]}\mathrm{d}x. \tag{3.61}$$

注意到

$$n\Big(\sin\frac{x}{n}\Big)\leqslant n\frac{x}{n}=x \tag{3.62}$$

因此有控制函数 xe^{-x^2} 满足

$$n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]}\leqslant xe^{-x^2} \tag{3.63}$$

控制函数 xe^{-x^2} 是可积的，因为

$$\int_0^{+\infty} xe^{-x^2}\mathrm{d}x = \frac{1}{2}\int_0^{+\infty} e^{-x^2}\mathrm{d}(x^2) = \frac{1}{2}e^{-x^2}|_{x=0}^{+\infty} = \frac{1}{2}. \tag{3.64}$$

此时

$$\lim_{n\to\infty}\int_0^{+\infty} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]}\mathrm{d}x = \int_0^{+\infty}\lim_{n\to\infty} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]}\mathrm{d}x. \tag{3.65}$$

而由于

$$\lim_{n\to\infty} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]} = xe^{-x^2} \tag{3.66}$$

因此

$$\int_0^{+\infty}\lim_{n\to\infty} n\Big(\sin\frac{x}{n}\Big)e^{-x^2}\chi_{[0,n^2]}\mathrm{d}x = \int_0^{+\infty} xe^{-x^2}\mathrm{d}x = \frac{1}{2}. \tag{3.67}$$

即为 (3.60) 之结果.

问题 3.17 证明对 $n \geqslant 2$ 或者 $0 \leqslant y \leqslant n$，有

$$\Big(1-\frac{y}{n}\Big)^{-n} \geqslant \Big(1-\frac{y}{n+1}\Big)^{-(n+1)}. \tag{3.68}$$

因此，或以其他方式，证明对 $\alpha > 0$ 和 $\beta > -1$，有

$$\lim_{n\to\infty} n^{\alpha}\int_0^1 x^{\alpha-1}e^{-n\beta x}(1-x)^n\mathrm{d}x = (\beta+1)^{-\alpha}\Gamma(\alpha), \tag{3.69}$$

其中 Γ 是问题 13 中定义的 *Gamma* 函数.

证明:

$$\Big(1-\frac{y}{n}\Big)^{-n} \geqslant \Big(1-\frac{y}{n+1}\Big)^{-(n+1)}. \tag{3.70}$$

等价于

$$n\ln\Big(1-\frac{y}{n}\Big) \leqslant (n+1)\ln\Big(1-\frac{y}{n+1}\Big) \tag{3.71}$$

定义函数 $\varphi(x)=x\ln\left(1-\dfrac{y}{x}\right)$，则

$$\begin{aligned}\varphi'(x)&=\ln\left(1-\frac{y}{x}\right)+x\frac{\frac{y}{x^2}}{1-\frac{y}{x}}=\ln\left(1-\frac{y}{x}\right)+\frac{y}{x-y}\\&=-\ln\left(\frac{x}{x-y}\right)+\frac{y}{x-y}=-\ln\left(1+\frac{y}{x-y}\right)+\frac{y}{x-y}\geqslant 0.\end{aligned}\tag{3.72}$$

因此 $\varphi(x)$ 是单调递增函数，有

$$n\ln\left(1-\frac{y}{n}\right)\leqslant(n+1)\ln\left(1-\frac{y}{n+1}\right)\tag{3.73}$$

成立，则第一问得证.

下面令 $x=\dfrac{y}{n}$，并代入式 (3.69)，可得

$$\textbf{原式}=n^\alpha\int_0^n\left(\frac{y}{n}\right)^{\alpha-1}e^{-n\beta\frac{y}{n}}\left(1-\frac{y}{n}\right)^n\mathrm{d}\left(\frac{y}{n}\right)=\int_0^{+\infty}y^{\alpha-1}e^{-\beta y}\left(1-\frac{y}{n}\right)^n\chi_{[0,n]}(y)\mathrm{d}y\tag{3.74}$$

注意到 $\left(1-\dfrac{y}{n}\right)^n\leqslant\left(1-\dfrac{y}{2}\right)^2$，此时可以 $y^{\alpha-1}e^{-\beta y}\left(1-\dfrac{y}{2}\right)^2$ 为控制函数，由控制收敛定理，有

$$\begin{aligned}\lim_{n\to\infty}\int_0^{+\infty}y^{\alpha-1}e^{-\beta y}\left(1-\frac{y}{n}\right)^n\chi_{[0,n]}(y)\mathrm{d}y&=\int_0^{+\infty}\lim_{n\to\infty}y^{\alpha-1}e^{-\beta y}\left(1-\frac{y}{n}\right)^n\chi_{[0,n]}(y)\mathrm{d}y\\&=\int_0^{+\infty}y^{\alpha-1}e^{-(\beta+1)y}\mathrm{d}y\end{aligned}\tag{3.75}$$

令 $z=(\beta+1)y$，可得

$$\int_0^{+\infty}y^{\alpha-1}e^{-(\beta+1)y}\mathrm{d}y=\int_0^{+\infty}(\beta+1)^{\alpha-1}z^{\alpha-1}e^{-z}(\beta+1)\mathrm{d}z=(\beta+1)^\alpha\int_0^{+\infty}z^{\alpha-1}e^{-z}\mathrm{d}z=(\beta+1)^\alpha\Gamma(\alpha).\tag{3.76}$$

问题 3.18 设 $\alpha>-1$. 证明 $f(x):=x^\alpha\ln x$ 在 $]0,1[$ 上可积，并且

$$\int_0^1 f(x)\mathrm{d}x=-(1+\alpha)^{-2}.\tag{3.77}$$

于是，可以对 $\beta > -1$ 推导出 $g(x) = x^{\beta}(1-x)^{-1}\ln x$ 在 $]0,1[$ 上可积，并且

$$\int_0^1 g(x)\mathrm{d}x = -\sum_{n=1}^{\infty}(n+\beta)^{-2}. \tag{3.78}$$

证明：当 $\alpha > 0$ 时，可知 $\lim\limits_{n\to 0} x^{\alpha}\ln x = 0$，此时积分区间内不含瑕点，函数可积. 下面讨论 $-1 < \alpha \leqslant 0$ 的情形. 此时由于

$$\lim_{n\to\infty}\frac{x^{\alpha}\ln x}{x^{\frac{-1+\alpha}{2}}} = \lim_{n\to\infty} x^{\frac{1+\alpha}{2}}\ln x = 0. \tag{3.79}$$

而 $\displaystyle\int_0^1 x^{\frac{-1+\alpha}{2}}\mathrm{d}x$ 收敛，所以 $f(x) := x^{\alpha}\ln x$ 在 $]0,1[$ 上可积. 此时

$$\begin{aligned}\int_0^1 x^{\alpha}\ln x\mathrm{d}x &= (1+\alpha)^{-1}\int_0^1 \ln x\mathrm{d}(x^{\alpha+1}) = (1+\alpha)^{-1}x^{\alpha+1}\ln x\Big|_{x=0}^{1} - (1+\alpha)^{-1}\int_0^1 x^{\alpha}\mathrm{d}(\ln x)\\ &= -(1+\alpha)^{-1}\int_0^1 x^{\alpha}\mathrm{d}x = -(1+\alpha)^{-2}.\end{aligned} \tag{3.80}$$

另可证 $g(x) = x^{\beta}(1-x)^{-1}\ln x$ 在 $]0,1[$ 上可积，此时仅需讨论 $x=1$ 处的敛散性. 注意到 $\ln x < x-1$，因此 $g(x)$ 在 $x=1$ 处为常义积分，此时 $g(x)$ 在 $]0,1[$ 上可积.

定出函数序列 $g_k(x) := \displaystyle\sum_{j=1}^{k} x^{\beta-1+j}\ln x$, 则 $g(x)$ 为 $g_k(x)$ 之极限函数, 并且 $g(x) \geqslant g_k(x)$. 由逐项积分定理，可得

$$\int_0^1 g(x)\mathrm{d}x = \int_0^1 \sum_{j=1}^{k} x^{\beta-1+j}\ln x\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^1 x^{\beta-1+j}\ln x\mathrm{d}x = -\sum_{n=1}^{\infty}(n+\beta)^{-2}. \tag{3.81}$$

则问题第二个结论亦得证.

3.2 第二次作业

问题 3.19 我们在课堂上讨论了以下的结果：

定理 设 μ 为度量空间 (X,d) 上的一个局部有界的 *Borel* 测度. 则它是

- 外正则的：对任意的 $\varepsilon > 0$ 和任意的 *Borel* 集 $E \subset X$，存在开集 $O \subset X$ 包含 E 使得 $\mu(O \setminus E) < \varepsilon$;

• 内正则的：对任意 $\varepsilon>0$ 和任意 *Borel* 集 $E\subset X$，存在一个闭集 $K\subset X$ 包含于 E 使得 $\mu(E\setminus K)<\varepsilon$.

为了证明这个结果，我们引入了

$$\mathcal{F}=\{Borel\ 集E\subset X:\forall\varepsilon>0,\exists 开集O和闭集K使得O\supset E\supset K,\mu(E\backslash K)<\varepsilon,\mu(O\backslash E)<\varepsilon\},\tag{3.82}$$

然后证明 $\mathcal{F}$ 是一个 $\sigma-$ 代数. 通过仔细地推导出 $\mathcal{F}$ 包含 (X,d) 上的 *Borel* $\sigma-$ 代数来完成这个证明.

证明：已知 $\mathcal{F}$ 是一个 $\sigma-$ 代数，然后 Borel$\sigma-$ 代数是包含所有开集的最小 $\sigma-$ 代数，因此我们只需证明 $\mathcal{F}$ 中包含所有的开集. 显然开集满足外正则性，只需证明开集满足内正则性.

取球心于原点的半径为整数的球序列 $B(n)$，使其分割开集 E 得到序列 $E_n=E\cap(B(n)\setminus B(n-1))$. 我们先证明对 E_n 和任取的 $\varepsilon>0$ 可找到闭集 K_n 使得 $\mu(E_n\setminus K_n)<\varepsilon$. 首先根据局部有界性知 $\mu(B(n))<\infty$，因此 $\mu(E_n)$ 有界. 然后我们考虑集合

$$K_d=\{x\in E_n:\mathrm{dist}(x,(B(n)\setminus B(n-1))\setminus E_n)\geqslant d\}\tag{3.83}$$

此集合有如下性质：

(1)$\mu(K_0)=\mu(E_n)$;

(2) 当 $d\neq 0$ 时，K_d 为闭集；

(3)K_d 在 $d\to 0$ 时单调递增.

下面我们证明 $\lim\limits_{d\to 0}(E_n\setminus K_d)=\varnothing$，为此考虑满足以下条件的点 $x\in\bigcap\limits_{k=1}^{\infty}(E_n\setminus K_{\frac{1}{k}})$，因此 $\mathrm{dist}(x,(B(n)\setminus B(n-1))\setminus E_n)<\dfrac{1}{k},\forall k\in\mathbb{N}^*$，于是 $\mathrm{dist}(x,(B(n)\setminus B(n-1))\setminus E_n)=0$，但 E_n 是开集，因此这样的点不存在. 此时可知 $\lim\limits_{d\to 0}\mu(E_n\setminus K_d)=0$，因此必存在 d_0 使得 $\mu(E_n\setminus K_{d_0})<\varepsilon$，取 $K_n=K_{d_0}$ 可知此时子命题成立. 然后对于变动的 n 分别取 $\varepsilon_n<\dfrac{\varepsilon}{2^n}$，可使每一个 E_n 均被一个内部的闭集所逼近，并且这种逼近的测度之差不会超过 $\sum\limits_n\dfrac{\varepsilon}{2^n}=\varepsilon$.

引用课上证明的结论，存在 E 的一个闭子集 $\tilde{K}$ 使得 $\tilde{K}\subset\bigcup\limits_{j=1}^{\infty}K_j$ 并且 $\mu\Big(\bigcup\limits_{j=1}^{\infty}K_j\backslash\tilde{K}\Big)<\varepsilon$. 此时可得到闭集 $\tilde{K}\subset E$ 并且 $\mu(E\setminus\tilde{K})<2\varepsilon$. 以上可以证明开集 E 的内正则性.

问题 3.20 设 $R\subset\mathbb{R}^d$ 是一个闭长方体，亦即对有限对数 $a_i\leqslant b_i,i\in\{1,2,\cdots,d\},R=\prod\limits_{i=1}^{d}[a_i,b_i]$. 定义 $|R|:=\prod\limits_{i=1}^{d}(b_i-a_i)$. 仔细地证明 $|R|\leqslant\mathcal{L}_*(R)$，$R$ 的 *Lebesgue* 外测度.(另一

个方向在课堂上已经得到证明).

证明：让我们复述一遍 $\mathcal{L}_*$ 的定义：

$$\mathcal{L}_*(E) := \inf\{\sum_{j=1}^{\infty} |Q_j| : Q_j = \text{ 闭集 } F \subset \bigcup_j Q_j\} \tag{3.84}$$

注意到长方体的交集依然是长方体，因此可以计算它的体积. 任意一个符合要求的 $\bigcup_j Q_j$ 应当满足 $F \subset \bigcup_j Q_j$，因此 $|R| \leqslant \sum_j |R \cap Q_j| \leqslant \sum_j |Q_j|$. 因此

$$|R| \leqslant \inf\{\sum_{j=1}^{\infty} |Q_j| : Q_j = \text{ 闭集 } F \subset \bigcup_j Q_j\} = \mathcal{L}_*(E). \tag{3.85}$$

问题 3.21 $E \subset \mathbb{R}^d$ 被称作 *Lebesgue* 可测的当且仅当对于任意的 $\varepsilon > 0$，存在一个开集 $O \supset E$ 使得 $\mathcal{L}_*(O \setminus E) < \varepsilon$. 这个测度被称作 CARATHÉODORY 可测的当且仅当对于任意的 $A \subset \mathbb{R}^d, \mathcal{L}_*(A) \geqslant \mathcal{L}_*(E \cap A) + \mathcal{L}_*(E^c \cap A)$. 在课堂中我们已经仔细地证明 E 是 *Lebesgue* 可测的可以推出它是 CARATHÉODORY 可测的. 证明另一个方向.

证明: 取 $\bigcup_j Q_j$ 覆盖 E 并且 $\mu\Big(\bigcup_j Q_j\Big) \leqslant \mu(E)+\varepsilon$, 然后再将 Q_j 扩展为开方体 $\tilde{Q}_j \supset Q_j$ 使得 $\mu\Big(\bigcup_j \tilde{Q}_j\Big) \leqslant \mu\Big(\bigcup_j Q_j\Big)+\varepsilon$. 此时 $\tilde{Q} := \bigcup_j \tilde{Q}_j$ 为开集, 覆盖 E 并且 $\mathcal{L}_*(\tilde{Q})-\mathcal{L}_*(E) < 2\varepsilon$. 将 $\tilde{Q}$ 代入 $\mathcal{L}_*(A) \geqslant \mathcal{L}_*(E\cap A)+\mathcal{L}_*(E^c\cap A)$ 中的 A 即可证明 $\mathcal{L}_*(E^c\cap Q) \leqslant \mathcal{L}_*(\tilde{Q})-\mathcal{L}_*(E) < 2\varepsilon$, 于是可知 Lebesgue 可测.

问题 3.22 证明 $E \subset \mathbb{R}^d$ 是 *Lebesgue* 可测的当且仅当存在一个 $G_\delta-$ 集 $G \supset E$ 使得 $\mathcal{L}^d(G \setminus E) = 0$，当且仅当存在一个 $F_\sigma-$ 集 $F \subset E$ 使得 $\mathcal{L}^d(E \setminus F) = 0$.

证明：由零测集可测易知当存在一个 $G_\delta-$ 集 $G \supset E$ 使得 $\mathcal{L}^d(G \setminus E) = 0$ 或存在一个 $F_\sigma-$ 集 $F \subset E$ 使得 $\mathcal{L}^d(E \setminus F) = 0$ 时 $E \subset \mathbb{R}^d$ 是 Lebesgue 可测的.

若 $E \subset \mathbb{R}^d$ 是 Lebesgue 可测的，由内外正则性，对任意 $k \in \mathbb{N}^*$ 可取 $O_{\frac{1}{k}} \supset E \supset K_{\frac{1}{k}}$ 使得开集 $O_{\frac{1}{k}}$ 满足 $\mathcal{L}^d(O_{\frac{1}{k}} \setminus E) < \varepsilon$，闭集 $K_{\frac{1}{k}}$ 满足 $\mathcal{L}^d(O_{\frac{1}{k}} \setminus K) < \varepsilon$. 取 $O := \bigcap_k O_{\frac{1}{k}}$ 和 $K := \bigcup_k K_{\frac{1}{k}}$ 即可找到题中要求的集合.

问题 3.23 考虑形式为 $\mathcal{Q}_j := \{x+[0,2^{-j}[^d\colon x \in (2^{-j}\mathbb{Z})^d\}$ 的二进方体. 证明对任意开集 $O \subset \mathbb{R}^d$，它可以被写成集族 $\bigcup_{j=1}^{\infty} \mathcal{Q}_j$ 中集合的不交并. 然后，如课堂中所给出的轮廓，证明任意 $\mathbb{R}^d$ 上的局部有限，平移不变的 *Borel* 测度等于 *Lebesgue* 测度乘以一个缩放因子.(提示：对 j 使用数学归纳法将是有效的).

证明：首先我们考虑集族 $\bigcup_{j=1}^{\infty} \mathcal{Q}_j$ 中所有的二进方体，记 Q_{jk} 为 $[0,2^{-j}[$ 经过平移得到的第 k 个二进方体，并定义指标集 I 使得

$$(j,k) \in I \Leftrightarrow Q_{jk} \subset O \text{ 并且不存在 } (j',k') \text{ s.t. } Q_{jk} \subsetneq Q_{j'k'} \subset O. \tag{3.86}$$

显然这样的定义是良定义的，并且 $\bigcup_{(j,k)\in I} Q_{jk}$ 是不交并. 由定义显然 $\bigcup_{(j,k)\in I} Q_{jk} \subset O$. 下面证明 $\bigcup_{(j,k)\in I} Q_{jk} \supset O$，考虑任意 $x \in O$，由于 O 是开集，则必然存在 x 的球形邻域 $B(x) \subset O$. 此球形邻域必包含一个形如 Q_{jk} 的二进方体，其对应的指标 (j,k) 应当要么属于 I，要么存在 (j',k')，使得 $(j',k') \in I$ 并且 $Q_{jk} \subset Q_{j'k'}$. 由此可知 $x \in \bigcup_{(j,k)\in I} Q_{jk}$.

注意到以上不交并是可数并，因此对任意开集 O，有 $\mu(O) = \sum_{jk} \mu(Q_{jk})$，$\mathcal{L}(O) = \sum_{jk} \mathcal{L}(Q_{jk})$. 如果我们设 $\lambda = \dfrac{\mu([0,1]^d)}{\mathcal{L}([0,1]^d)}$，则 $\lambda = \dfrac{\mu(Q_{jk})}{\mathcal{L}(Q_{jk})}$，因此

$$\mu(O) = \sum_{jk} \mu(Q_{jk}) = \lambda \sum_{jk} \mathcal{L}(Q_{jk}) = \lambda \mathcal{L}(O). \tag{3.87}$$

于是 $\lambda = \dfrac{\mu(O)}{\mathcal{L}(O)}$. 由此可知任意 $\mathbb{R}^d$ 上的局部有限，平移不变的 Borel 测度等于 Lebesgue 测度乘以一个缩放因子.

3.3 第三次作业

问题 3.24 *Lusin* 定理表示每一个 $\mathbb{R}^d$ 上的有界、可测函数都是近乎连续的. 这里的“近乎”理解为 *Lusin* 意义上的——例如，在一个测度任意小的子集以外的部分. 这个问题的目标是为了说明“近乎”不能被理解为“几乎处处”.

(1) 证明不存在一个连续函数 $f : \mathbb{R} \to \mathbb{R}$ 几乎处处等于 $\chi_{[0,1]}$;

(2) 考虑以下基本康托集的变种——令 $C_0=[0,1]$，每一次去掉 C_j 中间的集合 l_j 得到 C_{j+1}，通过适当地选择 $\{l_j\}_{j=0}^{\infty}$，证明对任意 $\xi\in]0,1[$ 我们能够构造一个类康托集 $\mathcal{C}_\xi$ 使得它是完美的、完全不连通的、不可数的，并且具有 *Lebesgue* 测度 ξ；

(3) 通过合适的迭代以上构造，构建一个可测集 $E\subset[0,1]$ 使得它对于任意开区间 $I\subset]0,1[$，$E\cap I$ 和 $E^c\cap I$ 有正测度；

(4) 推导出存在可测函数 $g:\mathbb{R}\to\mathbb{R}$ 满足以下性质：令 $h\to\mathbb{R}\to\mathbb{R}$ 为任意在 $\mathbb{R}$ 上几乎处处等于 g 的函数，h 不可能在 $[0,1]$ 中的任意点处是连续的.

Definition 3.1 一个集合是完美的如果它没有孤立点. 一个集合是完全不连通的如果它没有连通的子集.

证明：(1) 反证法，设存在这样的函数 f. 我们证明存在序列 $x_n\to 0-$ 使得 $f(x_n)=0$. 注意到 f 几乎处处等于 $\chi_{[0,1]}$，因此对任意 $\delta_n>0$，对相应的区间 $]-\delta_n,0[$，存在其中的点 x_n 使得 $f(x_n)=0$. 我们令 δ_n 单调递减趋于 0 即可得到相应的序列 x_n. 同理存在序列 $y_n\to 0+$ 使得 $f(y_n)=1$，由题意 $x=0$ 处函数连续，因此

$$0=\lim_{n\to\infty}f(x_n)=f(0)=\lim_{n\to\infty}f(y_n)=1, \tag{3.88}$$

矛盾. 因此这样的连续函数 f 不存在.

(2) 我们考虑以下的构造：C_j 由有限个区间构成，每次构造在每个区间的中间以区间的中点为中心向两侧各去掉一个子区间，使得子区间的测度小于原区间的测度，并且 C_j 总共去掉的区间总长度等于 $\xi_j>0$. 我们只需取这样的 ξ_j 使得

$$\sum_{j=0}^{\infty}\xi_j=1-\xi \tag{3.89}$$

显然这样构造的集合 $\mathcal{C}_\xi$ 的 Lebesgue 测度为 ξ，显然这样的集合是不可数的. 下面证明这个集合是完美的和完全不连通的.

(i) 完美，即没有孤立点. 任取 $x\in\mathcal{C}_\xi$ 和 $\delta>0$，我们论证存在 $\tilde{x}\in\mathcal{C}_\xi\cap]x,x+\delta[$. 我们使用反证法，若不然，则取定 $x_1\in]x,x+\delta[$，$x_1\notin\mathcal{C}_\xi$，必存在 j 使得 $x_1\in C_{j-1}\setminus C_j$，此时应当存在 $x_1'<x_1$ 使得 $[x_1',x_1]\notin\mathcal{C}_\xi$. 我们不妨设 x_1' 是 C_j 中可以取到的最小的一个，这样可以使得 $[x,x_1']\in\mathcal{C}_\xi$. 然后取 $x_2\in]x,x_1'[$，由假设 $x_2\notin\mathcal{C}_\xi$，于是必存在 k 使得 $x_2\in C_{k-1}\setminus C_k$，此时存在 $x_2'>x_2$ 使得 $[x_2,x_2']\notin\mathcal{C}_\xi$. 我们同样不妨设 x_2' 是 C_k 中可以取到的最大的一个. 于是我们得到 $x<x_2'<x_1'<x+\delta$，并且 $]x_2',x_1'[$ 是构成 C_k 的一个区间，这个区间中必存在一个点 $\tilde{x}\in\mathcal{C}_\xi$，与假设矛盾. 由此可知 $\mathcal{C}_\xi$ 是完美的.

(ii) 完全不连通，即不存在连通的子集. 用反证法，设 $[a,b]\in\mathcal{C}_\xi$，注意到 C_j 的构造方法，知构成 C_j 的区间中测度最大的不超过 $1/2^j$，当 $j\to 0$ 时，$1/2^j\to 0<b-a$，由此可知 $[a,b]$ 不可能包含于 $\mathcal{C}_\xi$，即 $\mathcal{C}_\xi$ 是完全不连通的.

(3) 任取测度为 $0<\xi<1$ 的类康托集合，记为 $\mathcal{C}_1$. 这个集合满足以下两个性质：

(i) 这个集合的补集 $\mathcal{C}_1^c$ 可写成不交区间的可数并；

(ii) 对任何区间 I，若其不含于构成 $\mathcal{C}_1^c$ 的任意一个区间，$I\cap\mathcal{C}_1$ 的测度满足 $0<\mu(I\cap\mathcal{C}_1)<\mu(I)$.

我们将以上性质记为性质 (I)，记构成 $\mathcal{C}_1^c$ 的所有区间测度的上确界为 $s(\mathcal{C}_1)$.

然后我们在构成 $\mathcal{C}_1^c$ 的每个区间上构造类康托集合，此时形成一个新的集合 $\mathcal{C}_2$，我们令这个集合的测度等于 ξ_2，满足 $0<\xi_1<\xi_2$.$\mathcal{C}_2$ 依然满足性质 (I)，并且我们可使构成 $\mathcal{C}_2^c$ 的区间测度的上确界 $s(\mathcal{C}_2)<\dfrac{1}{2}s(\mathcal{C}_1)$.

以此类推，我们可以依次构造集合 $\mathcal{C}_n$，他们都满足性质 (I)，并且 $0<\xi_1<\xi_2<\cdots<\xi_n$，并且 $s(\mathcal{C}_i)<\dfrac{1}{2}s(\mathcal{C}_{i-1})$，然后我们获得集合 $\mathcal{C}=\bigcup\limits_{n=1}^{\infty}\mathcal{C}_n$. 我们证明可以适当地取 ξ_i，使得 $\mathcal{C}$ 满足题目所述之性质.

注意到 $s(\mathcal{C}_j)\to 0$，因此对任意区间 I，必存在 J，使得 I 不包含于 $\mathcal{C}_J$，此时可知 $\mu(I\cap\mathcal{C}_J)>0$，于是 $\mu(I\cap\mathcal{C})>0$. 同时必存在 J'，使得 I 包含构成 $\mathcal{C}_{J'}^c$ 的区间 I'，此时我们只需选取 I' 上适当的构造，使得 $\mu(I'\cap\mathcal{C}^c)>0$ 即可. 我们仅需论证当 $I'=[0,1]$ 时的可行性，而此时只需取 $\lim\limits_{n\to\infty}\xi_n<1$ 即可.

因此存在这样的可测集 $E=\mathcal{C}$ 满足题目所述之性质.

(4) 令 $g=\chi_E$，其中 E 为满足 (3) 中条件的集合. 假设连续函数 h 几乎处处等于 g，此时由于 $\mu(E)>0$，存在 $x_0\in E$，$h(x_0)=1$，由连续函数的性质，存在邻域 $U(x_0,\delta)$ 使得 $\forall x\in U(x_0,\delta)$，有 $h(x)>\dfrac{1}{2}$，但注意 $\mu(U(x_0,\delta))=2\delta>0$，而 h 几乎处处等于 g，因此 $U(x_0,\delta)\subset E$，此时 $]x_0-\delta,x_0+\delta[$ 与 E^c 不交，由此导出矛盾.

问题 3.25 考虑 *Cantor-Lebesgue* 函数或其他函数，构造 *Lebesgue* 可测函数 $f,g:\mathbb{R}\to\mathbb{R}$ 使得 $f\circ g$ 不是 *Lebesgue* 可测的.

解：设 Cantor-Lebesgue 函数 $F(x)$，$\Phi(x)=\dfrac{x+F(x)}{2}$，记 C 是 $[0,1]$ 中的 Cantor 集，W 为 $\Phi(C)$ 中的不可测子集.

取 $f(x)=\chi_{\Phi^{-1}(W)}$，令 $g(x)=\Phi^{-1}(x)$，则 $f(x)=0$，a.e.，$g(x)$ 是严格递增的连续函数，因此 f,g 可测，但是 $f\circ g$ 不是 Lebesgue 可测的.

3.4 第四次作业

问题 3.26 设 X 是局部紧 *Hausdorff* 拓扑空间. 验证 $C_0^0(X) = \overline{C_c^0(X)}$. 这里 $C_0^0(X)$ 是连续函数 f 在无穷远处取为 0 的空间, 例如, 对任意 $\varepsilon > 0$ 都存在紧集 $K \subset X$ 使得 $\|f\|_{C^0(X\setminus K)} < \varepsilon$; $C_c^0(X)$ 是紧支集连续函数空间; 其闭集在 $\|\cdot\|_{C^0}$ 的意义下取得.

证明; 只需证对任意 $f \in C_0^0(X)$, 都存在 $f_\varepsilon \in C_c^0(X)$, 使得 $\|f_\varepsilon - f\| \leqslant \varepsilon$.

首先我们证明 f 取值在实数上的情形. 对 $\varepsilon > 0$, 定义

$$f_\varepsilon = \begin{cases} f - \operatorname{sgn}(f)\varepsilon, & |f| > \varepsilon; \\ 0. & |f| \leqslant \varepsilon \end{cases} \tag{3.90}$$

f_ε 是一个具有紧支集的连续函数, 因此 $f_\varepsilon \in C_c^0(X)$, 并且 $\|f_\varepsilon - f\| \leqslant \varepsilon$.

当 f 取值在复数上时, 分别对 $\mathrm{Re}(f)$ 和 $\mathrm{Im}(f)$ 找以上形式的逼近函数, 分别为 $\tilde{f}_{\mathrm{r}}$ 和 $\tilde{f}_{\mathrm{i}}$, 然后定义 $f_\varepsilon = \tilde{f}_{\mathrm{r}} + \mathrm{i}\tilde{f}_{\mathrm{i}}$, 则 $f_\varepsilon \in C_c^0(X)$, 并且 $\|f_\varepsilon - f\| \leqslant 2\varepsilon$.

问题 3.27 设 $(X, \mathcal{F})$ 是可测空间, μ 是 $(X, \mathcal{F})$ 上的一个正测度, 然后令 $\lambda, \lambda_1, \lambda_2$ 是 $(X, \mathcal{F})$ 上的复测度. 证明以下的基本性质:

(1) 如果 λ 集中在 A 上, 则 $|\lambda|$ 也是;

(2) 如果 $\lambda_1 \perp \lambda_2$, 则 $|\lambda_1| \perp |\lambda_2|$;

(3) 如果 $\lambda_1 \perp \mu$ 及 $\lambda_2 \perp \mu$, 则 $\lambda_1 + \lambda_2 \perp \mu$;

(4) 如果 $\lambda_1 \ll \mu$ 及 $\lambda_2 \ll \mu$, 则 $\lambda_1 + \lambda_2 \ll \mu$;

(5) 如果 $\lambda \ll \mu$, 则 $|\lambda| \ll \mu$;

(6) 如果 $\lambda_1 \ll \mu$ 及 $\lambda_2 \perp \mu$, 则 $\lambda_1 \perp \lambda_2$;

(7) 如果 $\lambda \ll \mu$ 及 $\lambda \perp \mu$, 则 $\lambda = 0$.

证明: (1) 只讨论 λ 为符号测度的情形, 复测度的情形由实部和虚部线性组合而成. 如果 λ 集中在 A 上, 则 $\forall B \in \mathcal{F}, B \cap A = \varnothing, \lambda(B) = 0$. 此时由 Jordan-Hahn 分解可得存在 D 使得 $\lambda^+(B) = \lambda(B \cap D) \geqslant 0, \lambda^-(B) = -\lambda(B \cap D^c) \leqslant 0$. 但是由于 $B \cap D, B \cap D^c \in \mathcal{F}$, 并且这两个集合与 A 都不交, 因此 $\lambda^+(B) = \lambda(B \cap D) = 0, \lambda^-(B) = -\lambda(B \cap D^c) = 0$, 此时有 $|\lambda|(B) = \lambda^+(B) + \lambda^-(B) = 0$, 此时可知 $|\lambda|$ 也集中在 A 上.

(2) 由条件可得 λ_i 集中在 A_i 上 $(i = 1, 2)$, $A_1 \cap A_2 = \varnothing$, 此时由 (1) 的结论, $|\lambda_i|$ 集中在 A_i 上 $(i = 1, 2)$, 因此 $|\lambda_1| \perp |\lambda_2|$.

(3) 由条件可得存在 A_1, A_2, B, 使得 $A_1 \cap B = A_2 \cap B = \varnothing$, 并且 λ_i 集中在 A_i 上 $(i = 1, 2)$, μ 集中在 B 上. 考虑 $\lambda_1 + \lambda_2$ 集中在 $A_1 \cup A_2$ 上, 并且 $(A_1 \cup A_2) \cap B = \varnothing$. 则

$\lambda_1 + \lambda_2 \perp \mu$.

(4) 由条件,对任意集合 $A \in \mathcal{F}$,若 $\mu(A) = 0$,则 $\lambda_1(A) = \lambda_2(A) = 0$,此时 $(\lambda_1+\lambda_2)(A) = 0$,因此 $\lambda_1 + \lambda_2 \ll \mu$.

(5) 由条件,对任意集合 $A \in \mathcal{F}$,若 $\mu(A) = 0$,则 $\lambda(A) = 0$. 此时由 Jordan-Hahn 分解可得存在 D,使得 $\lambda^+(A) = \lambda(A\cap D), \lambda^-(A) = -\lambda(A\cap D^c)$. 注意到 $A\cap D$ 和 $A\cap D^c$ 都是 A 的子集,因此 $\mu(A\cap D) = \mu(A\cap D^c) = 0$,此时有 $\lambda^+(A) = \lambda(A\cap D) = 0, \lambda^-(A) = -\lambda(A\cap D^c) = 0$,于是 $|\lambda|(A) = \lambda^+(A) + \lambda^-(A) = 0$,此时可得 $|\lambda| \ll \mu$.

(6) 由于 $\lambda_2 \perp \mu$,知存在集合 A, B,$A \cap B = \varnothing$ 并且 λ_2 集中于 A,μ 集中于 B. 由 $\lambda_1 \ll \mu$ 知 λ_1 集中于 B. 因此 $\lambda_1 \perp \lambda_2$.

(7) 设 μ 集中于 A,由于 $\lambda \ll \mu$ 知对任意 $B \in A^c$,$\lambda(B) = 0$;由于 $\lambda \perp \mu$ 知对任意 $C \in A$,$\lambda(C) = 0$. 注意到 $B \cup C$ 可以取遍 $\mathcal{F}$,因此 $\lambda = 0$.

问题 3.28 令 $(X, \mathcal{F}, \mu)$ 是测度空间. 集合 $\Sigma \subset L^1(\mu)$ 被称作一致可积的如果对任意的 $\varepsilon > 0$ 我们可以找到 $\delta > 0$ 使得

$$\sup\left\{\left|\int_E f\mathrm{d}\mu\right| : f \in \Sigma, E \in \mathcal{F} \text{ 满足 } \mu(E) < \delta\right\} \leqslant \varepsilon. \tag{3.91}$$

(1) 证明对任意有限子集 $\Sigma \subset L^1(\mu)$ 是一致可积的;

(2) 证明如果 $\mu(X) < \infty$, $\{f_n\}$ 是一致可积的,并且 $f_n \to f$ $\mu-a.e.$,且 $|f| < \infty$ $\mu-a.e.$,则 $f \in L^1(\mu)$,除此以外,$\lim\limits_{n\to\infty}\int_X |f_n - f|\,\mathrm{d}\mu = 0$. 以及,通过考虑 $\mathbb{R}$ 上的 *Lebesgue* 测度,证明假设 $\mu(X) < \infty$ 不能够被省略;

(3) 构造一个 $[0, 1]$ 上的函数序列 $\{f_n\}$ 使得对任意 x 有 $f_n \to 0$,$\int f_n\mathrm{d}\mu \to 0$,但是 $\{f_n\}$ 在 $\mathcal{L}^1$ 的意义下不是一致可积的;

(4) 除此之外,证明如果 $\mu(X) < \infty$,$\{f_n\} \subset L^1(\mu)$,并且 $\lim\limits_{n\to\infty}\int_E f_n\mathrm{d}\mu$ 对每个 $E \in \mathcal{F}$ 都存在,则 $\{f_n\}$ 是一致可积的.

证明:(1) 显然若 Σ 是单元素集,则 Σ 是一致可积的;因此我们可以建立一个映射 $\varphi_f : \varepsilon \mapsto \delta$,使得对 f 满足 (3.91) 中的限制. 此时考虑 $\Sigma = \{f_1, \cdots, f_n\}$ 是有限集合的情形,此时对于任意 $\varepsilon > 0$,取 $\delta = \max\limits_{1\leqslant i\leqslant n}\{f_i\}$,此 δ 满足 (3.91) 中的限制,因此 Σ 是一致可积的.

(2) 取定 $\varepsilon > 0$,将 X 的子集 X' 分割成有限个测度不超过 δ 的不交的集合,集合的数量记为 N;此时可知在这些集合上 f 的积分不超过 $N\varepsilon$. 于是我们可以首先取 X 中的子集 $X_{n+} := \{x : f_n(x) > 0\}$,对 X_{n+} 进行上述论证可知 $\int_{X_{n+}} f\mathrm{d}\mu = \int_X f_n^+\mathrm{d}\mu < N\varepsilon$;

同理可以论证 $\int_X f_n^- \mathrm{d}\mu < N\varepsilon$. 于是我们知道 $\int_X |f_n| \mathrm{d}\mu < 2N\varepsilon$. 由 $f_n \to f$ $\mu-$a.e. 可知 $\int_X |f_n| \mathrm{d}\mu = \int_X |f| \mathrm{d}\mu < 2N\varepsilon$. 这证明了 $f \in L^1(\mu)$. 由于 $f_n \to f$ $\mu-$a.e.，所以 $\Sigma \cup \{f\}$ 也是一致可积的；记集合 $\Sigma' = \{f_n - f\}$，则可知 Σ' 也是一致可积的. 任意取定 $\varepsilon > 0$，由条件存在 $\delta > 0$ 使得对任意 E 满足 $\mu(E) < \delta$，有 $\int_E (f_n - f)^+ \mathrm{d}\mu < \varepsilon, \int_E (f_n - f)^- \mathrm{d}\mu < \varepsilon$，则 $\int_E |f_n - f| \mathrm{d}\mu \leqslant 2\varepsilon$. 考虑集合 $X_{\varepsilon,n} = \{|f_n - f| \leqslant \dfrac{\varepsilon}{\mu(X)}\}$，由 $f_n \to f$ $\mu-$a.e. 知存在 N 使得当 $n > N$ 时，$\mu(X_{\varepsilon,n}^c) < \delta$，此时

$$\int_X |f_n - f| = \int_{X_{\varepsilon,n}} |f_n - f| \mathrm{d}\mu + \int_{X_{\varepsilon,n}^c} |f_n - f| \mathrm{d}\mu \leqslant \mu(X) \cdot \frac{\varepsilon}{\mu(X)} + 2\varepsilon = 3\varepsilon. \tag{3.92}$$

于是可知 $\lim\limits_{n\to\infty} \int_X |f_n - f| \mathrm{d}\mu = 0$.

下面给出一个 $\mu(X) < +\infty$ 不能省略的例子. 设 $X = [0, +\infty[$，$f = 1, f_n = \chi_{[0,n]}$. 此时 $\{f_n\}$ 是一致可积的，并且 $f_n \to f$ $\mu-$a.e.，且 $|f| < \infty$ $\mu-$a.e.，但是 $f \notin L^1(\mu)$.

(3) 取 $f_n (n \geqslant 3)$ 为

$$f_n = \begin{cases} n, \ x \in]0, \dfrac{1}{n} + \dfrac{1}{n^2}], \\ -n, \ x \in [1 - \dfrac{1}{n}, 1[, \\ 0, \ \text{其他位置} \end{cases} \tag{3.93}$$

则 1.f_n 逐点趋于 0；2.$\int f_n \mathrm{d}\mu = \dfrac{1}{n} \to 0$；3. 取 $\varepsilon = 1$，对任意 $0 < \delta < \dfrac{1}{2}$，存在 $\dfrac{1}{n} < \delta$，此时 $\int_{[0,\delta]} f_n \mathrm{d}\mu \leqslant \int_{[0,\frac{1}{n}]} f_n \mathrm{d}\mu = 1 = \varepsilon$. 由此可知 $\{f_n\}$ 满足 (3) 中所要求的条件；

(4) 定义 $\rho(A,B) = \int |\chi_A - \chi_B| \mathrm{d}\mu$，则 $(\mathcal{F}, \rho)$ 是一个完备度量空间 (测度 0 的模集)，且对每个 n, $E \to \int_E f_n \mathrm{d}\mu$ 是连续的. 若 $\varepsilon > 0$, 则存在 E_0, δ, N 使得当 $\rho(E, E_0) < \delta$, $n > N$ 时,

$$\left| \int_E (f_n - f_N) \mathrm{d}\mu \right| < \varepsilon. \tag{3.94}$$

若 $\mu(A) < \delta$，用 $B = E_0 \setminus A$ 和 $C = E_0 \cup A$ 来代替 E，此时 (3.94) 仍然成立，从而用 A 替代 E，2ε 替代 ε，(3.94) 也成立. 现在在 $\{f_1, \cdots, f_N\}$ 上应用 (1) 的结论，存在 $\delta' > 0$ 使

得当 $\mu(A) < \delta'$ 时，有

$$\left|\int_A f_n \mathrm{d}\mu\right| < 3\varepsilon,\ n = 1, 2, 3, \cdots. \tag{3.95}$$

3.5 第五次作业

问题 3.29 证明 $\ell^p(\mathbb{C}) := \left\{a = \{a_j\}_{j=1}^{\infty} \subset \mathbb{C} : \|a\|_p < \infty\right\}$ 对每个 $1 \leqslant p \leqslant \infty$ 是 *Banach* 空间；这里 $\|a\|_p := \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}$. 同时证明 $c_0 := \left\{\{a_j\}_{j=1}^{\infty} \in \ell^\infty : a_j \to 0 \text{ 当 } j \to \infty\right\}$ 是 ℓ^∞ 的一个闭子空间.(因此这是一个相对于范数 $\|\cdot\|_\infty$ 而言的 *Banach* 空间).

证明：首先证明 (1)$p < \infty$ 的情况：对 $\ell^p(\mathbb{C})$ 和范数 $\|a\|_p := \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}$，满足正定性、数乘、三角不等式.

正定性：$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}} = 0 \Leftrightarrow a_j = 0,\ \forall j \Leftrightarrow a = \mathbf{0}$；

数乘：$\forall \lambda \in \mathbb{C}$，$\|\lambda a\|_p = \left(\sum_{j=1}^{\infty} |\lambda a_j|^p\right)^{\frac{1}{p}} = |\lambda| \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}} = |\lambda|\, \|a\|_p$；

三角不等式：由 Minkovski 不等式可得

$$\|a+b\|_p = \left(\sum_{j=1}^{\infty} |a_j + b_j|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty} \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \leqslant \lim_{n\to\infty} \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{\frac{1}{p}} = \|a\|_p + \|b\|_p. \tag{3.96}$$

然后证明 ℓ^p 是完备的. 任取 ℓ^p 中的 Cauchy 列 $\{a^{(n)}\}$，此时对于固定的 j，有 $\{a_j^n\}$ 相对于 n 是 Cauchy 列，因此存在极限 $\lim_{n\to\infty} a_j^{(n)} := a_j^{(\infty)}$. 此时取 $a^{(\infty)} = \{a_j^{(\infty)}\}$，可知 $a^{(\infty)}$ 为 Cauchy 列 $\{a^{(n)}\}$ 收敛之极限，由此知 ℓ^p 是完备的.

(2)$p = \infty$ 的情况：对 $\ell^\infty(\mathbb{C})$ 和范数 $\|a\|_\infty := \sup_n |a_j|$，满足正定性、数乘、三角不等式.

正定性：$\|a\|_\infty = \sup_n |a_j| = 0 \Leftrightarrow a = \mathbf{0}$；

数乘：$\forall \lambda \in \mathbb{C}$，有 $\|\lambda a\|_\infty = \sup_j |\lambda a_j| = |\lambda| \sup_j |a_j| = |\lambda|\, \|a\|_\infty$；

三角不等式：$\|a + b\| = \sup_j |\lambda a_j + b_j| \leqslant \sup_j |\lambda a_j| + \sup_j |\lambda b_j| = \|a\|_\infty + \|b\|_\infty$.

然后证明 ℓ^∞ 是完备的，此处步骤与 $p < \infty$ 的时候是完全一致的.

问题 3.30 令 $(X, \|\cdot\|)$ 是一个 *Banach* 空间，然后设 $Y \leqslant X$ 是一个闭的 *(*向量*)* 子空间. 证

明商空间 X/Y 是赋有商范数

$$\|x+Y\|_{\text{quot}} := \inf\{\|x+y\| : y \in Y\}. \tag{3.97}$$

的 *Banach* 空间.

证明：首先证明 X/Y 是一个赋范线性空间.

(1) 正定性：$\|x+Y\|_{\text{quot}} := \inf\{\|x+y\| : y \in Y\} \geqslant 0$，当等号取到时 $\|[x]\| = 0$，此时要么存在 y 使得 $x+y=0$，要么对 x 存在 $\{y_n\}_{n=1}^{\infty}$，使得 $\|x+y_n\| \to 0$，由 X 是 Banach 空间知 $x+y_n$ 亦存在极限 $x+y$ 满足 $\|x+y\|=0$，此时 $x+y=0$，可知 $[x]=[0]$.

(2) 数乘：$\forall \lambda \in \mathbb{C}$，有

$$\|\lambda x+Y\| = \|\lambda x+\lambda Y\| \leqslant |\lambda|\,\|x+Y\|; \tag{3.98}$$

(3) 三角不等式：$\|x_1+Y+x_2+Y\| = \|x_1+x_2+Y\| = \inf\{\|x_1+x_2+y\|\}$. 对任意 $\varepsilon>0$，可取 y_1, y_2 使得 $\|x_1+y_1\| < \|x_1+Y\|+\varepsilon$，$\|x_2+y_2\| < \|x_2+Y\|+\varepsilon$，于是 $\inf\{\|x_1+x_2+y\|\} < \|x_1+Y\|+\|x_2+Y\|+\varepsilon$，这给出 $\|x_1+x_2+Y\| \leqslant \|x_1+Y\|+\|x_2+Y\|$.

然后证明 X/Y 在商范数意义下构成 Banach 空间. 考虑 X/Y 中的 Cauchy 列 $\{[x_n]\}$，并按以下方法选取元素：取任意 $a \in X$ 满足 $\|[a]\| \neq 0$，映射 $\phi(\lambda) = \lambda a$，易知对每个 $[x_n]$ 都存在唯一的 λ_n 使得 $\phi(\lambda_n) \in [x_n]$. 此时可以商范数和在直线 $\phi(\lambda)$ 上 λ 的差的比值应当是一个固定的常数，因此 $\{\lambda_n\}$ 也是一个 Cauchy 列，此时设 $\{\lambda_n\}$ 的序列极限为 λ，由 X 是 Banach 空间可知 $\lambda a \in X$. 由以上所述距离之间的关系，可知 $[\lambda a]$ 是 Cauchy 列 $[x_n]$ 的极限，我们可知 X/Y 是 Banach 空间.

问题 3.31 $C_0^0(\mathbb{R})$ 和 $C_c^0(\mathbb{R})$ 在上确界范数下是 *Banach* 空间吗？证明你的答案.

证明：$C_0^0(\mathbb{R})$ 是 Banach 空间，$C_c^0(\mathbb{R})$ 不是 Banach 空间.

$C_0^0(\mathbb{R})$ 是 Banach 空间：对任意 $\varepsilon>0$，任意 $n \geqslant 1$ 以及函数 $f_n \in C_0^0(\mathbb{R})$，存在 $X_n > 0$ 使得 $|x| > X_n$ 时，有 $|f_n|(x) < \varepsilon$. 若设 f_n 在上确界范数下有极限函数 f，则对任意 $\varepsilon' > 0$，存在 N，使得 $|f_N - f| < \varepsilon'$. 然后此时取 X_N，则 $|f_N| < \varepsilon$ 在 $|x| > X_N$ 时成立. 这给出 $|f| \leqslant |f_N| + |f_N - f| < \varepsilon + \varepsilon', |x| > X_N$. 由于 ε 和 ε' 可以任取，我们得到 $f \in C_0^0(\mathbb{R})$.

$C_c^0(\mathbb{R})$ 不是 Banach 空间. 取

$$f_n(x) = \begin{cases} e^{-x^2}, |x| < n, \\ \Big(-\operatorname{sgn}(x)x + n + e^{-n^2}\Big) \vee 0, |x| \geqslant n. \end{cases} \tag{3.99}$$

则 $f_n \in C_c^0(\mathbb{R})$，但是 $f_n \to f = e^{-x^2} \notin C_c^0(\mathbb{R})$，这给出 $C_c^0(\mathbb{R})$ 不是 Banach 空间.

问题 3.32 设 X 是赋范向量空间并且 Y 是 *Banach* 空间. 考虑 $\mathcal{B}(X,Y)$，有界线性算子 $T: X \to Y$ 空间. 证明 $\mathcal{B}(X,Y)$ 是一个 *Banach* 空间.

证明：首先可知 $\mathcal{B}(X,Y)$ 按范数 $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$ 构成赋范线性空间 (由课上证明). 下面证明 $\mathcal{B}(X,Y)$ 是完备的.

设 $\{T_n\}_1^\infty$ 为 Cauchy 列，则 $\forall \varepsilon > 0, \exists N = N(\varepsilon)$，使得对 $\forall x \in X$ 有

$$\|T_{n+p}x - T_n x\| \leqslant \varepsilon \|x\|, \ (\forall n > N, \forall p \in \mathbb{N}). \tag{3.100}$$

于是 $T_n x \to y \in Y(n \to \infty)$，记此 $y = Tx$，下面证 $T \in \mathcal{B}(X,Y)$. 不难看出 T 是线性的，下面证 T 有界. 事实上，$\exists n \in \mathbb{N}$ 使得

$$\|Tx\| = \|y\| \leqslant \|T_n x\| + 1 \leqslant (\|T_n\| + 1)\|x\| \ (\forall x \in X, \|x\| = 1). \tag{3.101}$$

即得 $\|T\| \leqslant \|T_n\| + 1$.

问题 3.33 令 $X = C^0([-1,1],\mathbb{R})$ 赋有上确界范数. 定义 $T : X \to \mathbb{R}$ 为

$$T\phi := \int_0^1 \phi - \int_{-1}^0 \phi. \tag{3.102}$$

证明 $T \in X^* = \mathbb{B}(X,\mathbb{R})$ 有算子模 $\|T\| = 2$. 同时证明算子范数不能在 X 的单位球上取到，即，不存在 $\phi \in X$ 满足 $\|\phi\| = 1$ 且 $T(\phi) = 2$.

证明：取函数

$$\phi_n = \begin{cases} 1, \quad x > \dfrac{1}{n} \\ nx, \quad -\dfrac{1}{n} \leqslant x \leqslant \dfrac{1}{n} \\ -1, \quad x < -\dfrac{1}{n} \end{cases} \tag{3.103}$$

此时 $\phi_n \in X$, $\|\phi_n\| = 1$, 并且 $T\phi_n = (1 - \frac{1}{n}) + 2n \times \frac{1}{2n^2} + (1 - \frac{1}{n}) = 2 - \frac{1}{n}$. 这给出 $\|T\| \geqslant 2$; 又因为对任意 $\|\phi\| = 1$，有

$$|T\phi| = \left| \int_0^1 \phi - \int_{-1}^0 \phi \right| \leqslant \int_0^1 |\phi| + \int_{-1}^0 |\phi| \leqslant 2\|\phi\|. \tag{3.104}$$

下面证明不存在 $\phi \in X$ 满足 $\|\phi\| = 1$ 且 $T(\phi) = 2$，若不然，假设存在这样的 ϕ，这说明 ϕ 几乎处处等于 $\mathrm{sgn}(x)$. 由连续函数的介值性知存在 x_0 使得 $\phi(x_0) = \dfrac{1}{2}$，找到 x_0 的一个邻域使得在此邻域中有 $\left|\phi - \dfrac{1}{2}\right| < \dfrac{1}{4}$，但这样的邻域不是零测的，因此导出矛盾.

问题 3.34 设 X 是 ℓ^2 的以下的子空间：

$$X := \Big\{a = \{a_j\} \in \ell^2 : \{ja_j\} \in \ell^1\Big\}. \tag{3.105}$$

定义 $P : X \to X$ 为 $P(\{a_j\}) := \Big(\sum_{j=1}^{\infty} ja_j, 0, 0, 0, \cdots\Big)$. 验证 P 是一个投影算子.X 是 *Banach* 空间吗？

解：P 是投影算子应当验证 (1)P 是线性算子；(2)$P^2 = P$. 线性性显然，然后由于算子 P 对第一项的权重是 1，因此若 a 从第二项开始都为 0，则 $Pa = a$. 于是 $P^2 = P$.

X 不是 Banach 空间. 取序列 a_n 满足

$$a_{nj} = \begin{cases} \dfrac{1}{j}, & j \leqslant n; \\ 0, & j > n. \end{cases} \tag{3.106}$$

易知 a_n 在 ℓ^2 上有极限，但是 $\sum_{j=1}^{\infty} ja_{nj} = \sum_{j=1}^{n} 1 = n \to \infty$，这说明 a_n 的极限不属于 X.

问题 3.35 定义 E 是一个拓扑向量空间. 子集 $U \subset E$ 是平衡的如果对 $x \in U$ 和 $c \in k(= \mathbb{R}\ or\ \mathbb{C})$ 满足 $|c| \leqslant 1$ 可以推出 $cx \in U$. 如果 U 是 $0 \in E$ 的一个凸的平衡邻域，定义

$$|||x||| := \inf\Big\{c \in \mathbb{R} : c > 0, \frac{x}{c} \in U\Big\}. \tag{3.107}$$

证明 $|||\cdot|||$ 是 U 上的一个半模.[这被称作 U 的规范/*Minkovski* 泛函.]

证明：我们需要验证 $|||\cdot|||$ 满足

(1) 正定性，显然 $|||x|||>0$；

(2) 数乘，$|||\lambda x||| = \inf\Big\{c \in \mathbb{R} : c > 0, \dfrac{\lambda x}{c} \in U\Big\} = |\lambda| \inf\Big\{c \in \mathbb{R} : c > 0, \dfrac{x}{c} \in U\Big\} = |\lambda|\, |||x|||$；

(3) 三角不等式，对任意 $\varepsilon>0, x, y \in U$，记 $\lambda_1=|||\lambda x|||+\dfrac{\varepsilon}{2}, \lambda_2=|||\lambda y|||+\dfrac{\varepsilon}{2}$，则

$$\frac{x}{\lambda_1} \in U,\ \frac{y}{\lambda_2} \in U \tag{3.108}$$

由凸集的性质可知

$$\frac{x+y}{\lambda_1+\lambda_2}=\frac{x}{\lambda_1} \frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{y}{\lambda_2} \frac{\lambda_1}{\lambda_1+\lambda_2} \in U, \tag{3.109}$$

这给出 $|||x+y||| \leqslant \lambda_1+\lambda_2$，而令 $\varepsilon \rightarrow 0$ 可得 $|||x+y||| \leqslant |||x|||+|||y|||$.

3.6 第六次作业

问题 3.36 设 X, Y 是赋范线性空间并记 $T \in \mathcal{B}(X, Y)$. 回忆 T 的算子范数由 $\|T\|:=\sup \{\|T x\|_Y: x \in \mathbf{B}_X\}$ 给定，其中 $\mathbf{B}_X:=\{x \in X:\|x\|_X \leqslant 1\}$ 是 X 中的闭单位球. 验证

$$\begin{aligned}
\|T\| & =\inf \{M>0:\|T x\|_Y \leqslant M\|x\|_X, \forall x \in X\}=\sup \left\{\frac{\|T x\|_Y}{\|x\|_X}: x \in X \backslash\{0\}\right\} \\
& =\sup \{\|T x\|_Y: x \in X,\|x\|<1\}=\sup \{\|T x\|_Y: x \in X,\|x\|=1\} .
\end{aligned} \tag{3.110}$$

证明：首先，

$$\begin{aligned}
& \inf \{M>0:\|T x\|_Y \leqslant M\|x\|_X, \forall x \in X\}=\inf \{M>0: \frac{\|T x\|_Y}{\|x\|_X} \leqslant M, \forall x \in X \backslash\{0\}\} \\
= & \sup \left\{\frac{\|T x\|_Y}{\|x\|_X}: \forall x \in X \backslash\{0\}\right\}=\sup \left\{\frac{\|T(\lambda x)\|_Y}{\|(\lambda x)\|_X}: \forall \lambda \neq 0, x \in X,\|x\|=1\right\} \\
= & \sup \left\{\frac{|\lambda|\,\|T x\|_Y}{|\lambda|\,\|x\|_X}: \forall \lambda \neq 0, x \in X,\|x\|=1\right\}=\sup \left\{\|T x\|_Y: x \in X,\|x\|=1\right\}
\end{aligned} \tag{3.111}$$

然后注意到范数数乘的运算性质，并且 $\|x\|=1$ 中的点可由 $\|x\|<1$ 中的点列逼近，因此将 $\sup \left\{\|T x\|_Y: x \in X,\|x\|=1\right\}$ 中的等号换成 $\leqslant$ 或 $<$，此时的取值是不变的.

问题 3.37 设 $(X, \mathcal{F}, \mu)$ 是一个测度空间，并令 $\varphi \in \ell^{\infty}(X, \mu)$，考虑乘法算子 $M_{\varphi} \in \mathcal{B}(\ell^p(X, \mu))$，对 $1 \leqslant p \leqslant \infty$ 给出 $M_{\varphi}(f):=\varphi f($ 点态乘法 $)$. 求出算子范数 $\|M_{\varphi}\|$.

解：考虑函数 f 满足 $\|f\|_p=1$，则 $\|M_{\varphi}(f)\|=\|\varphi f\|_p \leqslant\|\varphi\|_{\infty}\|f\|_p=\|\varphi\|_{\infty}$. 另一方面，对任意 $\varepsilon>0$，取 $M>\|\varphi\|_{\infty}-\varepsilon$，定义 $X_M=\{x:|\varphi(x)|>M\}$ 则 $\mu(X_M)>0$，取其有限正

测子集 $\tilde{X}$，其测度为 δ，则定义函数

$$f = \begin{cases} \mu(\delta)^{-\frac{1}{p}}, \ x \in \tilde{X} \\ 0, \qquad\quad x \notin \tilde{X} \end{cases} \tag{3.112}$$

则 $\|f\|_p = \left(\int_{\tilde{X}} \left(\mu(\delta)^{-\frac{1}{p}}\right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\int_{\tilde{X}} \mu(\delta)^{-1} \mathrm{d}\mu \right)^{\frac{1}{p}} = 1$. 此时可得在 $x \in \tilde{X}$ 上有 $M_\varphi(f) = \varphi\mu(\delta)^{-\frac{1}{p}}$，于是 $|M_\varphi(f)| > M\mu(\delta)^{-\frac{1}{p}}$. 于是

$$\|M_\varphi(f)\|_p > \left(\int_{\tilde{X}} (M\mu(\delta)^{-\frac{1}{p}})^p \right)^{\frac{1}{p}} \geqslant M > \|\varphi\|_\infty - \varepsilon. \tag{3.113}$$

令 $\varepsilon \to 0$ 可得 $\|M_\varphi(f)\| \geqslant \|\varphi\|_\infty$. 综上所述 $\|M_\varphi(f)\| = \|\varphi\|_\infty$.

问题 3.38 令 X, Y 为赋范线性空间并令 $T \in \mathcal{B}(X, Y)$. 证明算子 $T_0 : X/\mathrm{ker}T \to \mathrm{ran}T$ 表为 $T_0(x + \mathrm{ker}T) = Tx, \forall x$ 满足 $\|T_0\| = \|T\|$.

注: 令 $\pi : X \to X/\mathrm{ker}T$ 为典范投影及 $\iota : \mathrm{ran}T \hookrightarrow Y$ 是自然嵌入. 这样我们分解 $T = \iota \circ T_0 \circ \pi$. 这是对有界映射的分解，被称为 T 的典范分解. 注意到 $(T_0)^{-1}$ 可能不是有界的；例如，T_0 不一定是一个赋范线性空间上的同构. 比较在有限维空间中的线性算子的秩-零化度定理和群论里的第一同构定理.

证明: $\|T_0\| = \sup\left\{ \frac{\|T_0([x])\|}{\|[x]\|}, [x] \in X/\mathrm{ker}T \setminus \{0\} \right\} = \sup\left\{ \frac{\|Tx\|}{\|x+y\|}, x \in X, y \in \mathrm{ker}T \right\} = \sup\left\{ \frac{\|T(x+y)\|}{\|x+y\|}, x \in X, y \in \mathrm{ker}T \right\} = \sup\left\{ \frac{\|Tx\|}{\|x\|}, x \in X \right\} = \|T\|$.

问题 3.39 考虑 $T : \ell^2 \to \ell^2$ 使得 $T(\{x_j\}) := \{x_j/j\}$. 证明 T 是一个有界线性算子，其值域是 ℓ^2 的一个非闭子空间.

证明: (1)T 是线性算子，这是因为对 x, y，有

$$T(x+y) = T(\{x_j + y_j\}) = \left\{ \frac{x_j + y_j}{j} \right\} = \{x_j/j\} + \{y_j/j\} = Tx + Ty. \tag{3.114}$$

(2)T 是有界算子，这是因为对 x，在 T 的作用之后每一项的绝对值都小于等于原 x 相对应项的绝对值. 因此 $\|Tx\| \leqslant \|x\|$. 这给出了 $\|T\| \leqslant 1$.

(3)T 的值域是 ℓ^2 的一个非闭子空间，首先 T 的值域是一个子空间，这是因为对 x, y, λ, μ，

有 $\lambda T(x)+\mu T(y)=T(\lambda x+\mu y)$，为此 T 的值域对线性组合是封闭的，这给出 T 的值域是 ℓ^2 的一个非闭子空间. 然后证明 T 的值域是非闭的，考虑序列

$$x_j^{(n)}=\begin{cases}1, & j\leqslant n;\\ 0, & j>n\end{cases}\tag{3.115}$$

则 $T(x^{(n)})=\left\{\frac{1}{j}\chi_{j\leqslant n}\right\}$，这给出了一个 T 值域上的序列，并且这个序列在 ℓ^2 上是收敛的. 但是这个序列的极限为 $\left\{\frac{1}{j}\right\}$，显然这个序列并不是 T 的值域.

问题 3.40 令 $K:=\{z\in\mathbb{C}:1\leqslant|z|\leqslant 2\}$ 及 $\Phi:C^0(K)\to\mathbb{C},\Phi(f):=\int_{|z|=3/2}f(z)\mathrm{d}z$.

(1) 证明 Φ 是有界线性泛函；

(2) 因此，或以其他方式，证明 $\mathbb{C}[z]$(复系数多项式空间) 在 $C^0(K;\mathbb{C})$ 上不是稠密的；即，*Stone-Weierstrass* 的直接类似情形在 $\mathbb{C}$ 中不成立.

证明：(1) 对函数 f，满足 $|f|\leqslant\|f\|$，因此

$$|\Phi(f)|=\left|\int_{|z|=3/2}f(z)\mathrm{d}z\right|\leqslant\int_{|z|=3/2}|f(z)\mathrm{d}z|\leqslant\|f\|\int_{|z|=3/2}|\mathrm{d}z|\leqslant 3\pi\|f\|.\tag{3.116}$$

可知 $\|\Phi\|\leqslant 3\pi$. 取函数 $f=e^{-\mathrm{i}\ \arg(z)}$ 可令等号取等，此时有 $\|\Phi\|=3\pi$. 因此 Φ 是有界线性泛函.

(2)$\mathbb{C}[z]$ 在 $C^0(K;\mathbb{C})$ 上不是稠密的，这是因为由 Cauchy 积分公式，对任意多项式 $p\in\mathbb{C}[z]$，有

$$\int_{|z|=3/2}p(z)\mathrm{d}z=0.\tag{3.117}$$

这使得 $\mathbb{C}[z]$ 的所有极限函数都满足以上性质，因此无法逼近 (1) 中所给出的函数.

问题 3.41 令 $X=C^0([a,b];\mathbb{R})$. 定义 $T:X\to X$ 为 $(Tf)(t):=\int_a^t f(s)\mathrm{d}s(t\in[a,b])$.

(1) 验证 $T\in\mathcal{B}(X)$，并计算 $\|T\|$.

(2) 求 $k_2:\triangle_{a,b}\equiv\{(s,t)\in\mathbb{R}^2:a\leqslant s\leqslant t\leqslant b\}\to\mathbb{R}$ 使得 $(T^2f)(t)=\int_a^t k_2(s,t)f(s)\mathrm{d}s$. 计算 $\|T^2\|$.

(3) 更一般地，对 $n\geqslant 2$ 定出 $k_n:\triangle_{a,b}\equiv\{(s,t)\in\mathbb{R}^2:a\leqslant s\leqslant t\leqslant b\}\to\mathbb{R}$ 使得 $(T^nf)(t)=\int_a^t k_n(s,t)f(s)\mathrm{d}s$. 计算 $\|T^n\|$.

解：由于空间为 $X \in C^0$，因此所有关于 f 的范数均在上确界范数的意义下讨论.

(1) 首先对于 $f \in X$ 满足 $\|f\| = 1$，有 $|f| \leqslant 1$. 此时 $|(Tf)(t)| = \left|\int_a^t f(s)\mathrm{d}s\right| \leqslant \int_a^t \mathrm{d}s = t-a$，于是可知 $\|Tf\| \leqslant b-a$. 当 $f \equiv 1$ 时等号可以取等，此时可以得到 $\|T\| = b-a$.

(2) 经过积分交换顺序可得

$$\int_a^t k_2(s,t)f(s)\mathrm{d}s = \int_a^t \int_a^s f(r)\mathrm{d}r\mathrm{d}s = \int_a^t \int_r^t f(r)\mathrm{d}s\mathrm{d}r = \int_a^t (t-r)f(r)\mathrm{d}r. \tag{3.118}$$

因此 $k_2(s,t) = t-s$. 此时 $\left|(T^2f)(t)\right| = \left|\int_a^t (t-s)f(s)\mathrm{d}s\right| \leqslant \int_a^t (t-s)\mathrm{d}s = \dfrac{(t-a)^2}{2}$，于是可知 $\|T^2f\| \leqslant \dfrac{(t-a)^2}{2}$，于是可知 $\|T^2f\| \leqslant b-a$. 当 $f \equiv 1$ 时等号可以取等，此时可以得到 $\|T^2\| = \dfrac{(b-a)^2}{2}$.

(3) 我们归纳地证明 $k_n(s,t) = \dfrac{(t-s)^{n-1}}{(n-1)!}$. 当 $n=2$ 时已知成立. 并且，

$$\int_a^t k_{n+1}(s,t)f(s)\mathrm{d}s = \int_a^t \int_a^s k_n(r,s)f(r)\mathrm{d}r\mathrm{d}s = \int_a^t \int_r^t \frac{(s-r)^{n-1}}{(n-1)!}f(r)\mathrm{d}s\mathrm{d}r = \int_a^t \frac{(t-r)^n}{n!}f(r)\mathrm{d}r. \tag{3.119}$$

由此可由 $k_n(s,t) = \dfrac{(t-s)^{n-1}}{(n-1)!}$ 推出 $k_{n+1}(s,t) = \dfrac{(t-s)^n}{n!}$. 此时 $|(T^nf)(t)| = \left|\int_a^t \dfrac{(t-s)^{n-1}}{(n-1)!}f(s)\mathrm{d}s\right| \leqslant \int_a^t \dfrac{(t-s)^{n-1}}{(n-1)!}\mathrm{d}s = \dfrac{(t-a)^n}{n!}$，于是可知 $\|T^nf\| \leqslant \dfrac{(t-a)^n}{n!}$，于是可知 $\|T^nf\| \leqslant \dfrac{(t-a)^n}{n!}$. 当 $f \equiv 1$ 时等号可以取等，此时可以得到 $\|T^n\| = \dfrac{(b-a)^n}{n!}$.

4 Appendix: My Comments

[22]

4.1 Comments on Measure Theory

4.1.1 不同种类测度的定义和关系

测度是一个函数, 它对一个给定集合的某些子集指定一个数, 这个数可以比作大小、体积、概率等等. 传统的积分是在区间上进行的，人们希望把积分推广到任意的集合上，就发展出测度的概念.

1. 测度 我们从对体积的朴素理解出发. 想要测量一个形状比较复杂物体的体积，可以想到的办法就是，把这个形状比较复杂的物体分为一些形状比较简单的部分，然后每部分可以分别计算体积, 最后相加可以得到这个物体总的体积. 我们把这样的想法推广到一般的情形，即得到测度的可数可加性，即对于可数多个不交的集合 $A_1, A_2, \cdots$，满足

$$\sum_{n=1}^{\infty} \mu(A_n) = \mu\Big(\sum_{n=1}^{\infty} A_n\Big). \tag{4.1}$$

值得注意的是，测度需要建立在可测空间上，可测空间可以保证对于集合的任意可数的交、并、补运算都是封闭的.

2. 预测度 由于测度这样一个结构讨论起来相对复杂，因此我们不能简单地断定是否能在一个集合和集合上的一个集族建立测度. 为此, 我们考虑将原本朴素的理解通过一定的方式进行延伸，把这种理解推广到一般的情况.

预测度事实上就可以承载我们原本所具有的一种朴素的理解，它不要求集合上的集族具有对可数集合运算封闭的结构. 例如在 $\mathbb{R}$ 上，我们可以比较自然地给出一个开区间的测度：$\mu_0((a,b)) = b - a$，通过这样的定义，我们可以给出 $\mathbb{R}$ 上所有开集的测度 (因为 $\mathbb{R}$ 上的开集都可以写成可数多个不交开区间之并)，我们可以视这样的一种定义为一个 “预测度” 的结构. 这种结构是便于我们直观理解的，即使在这种结构中，开集对可数交、并、补并不封闭 (事实上对于非空、全集的开集，其补集就不是一个开集)，但这种结构已经是一个足够好的框架来承载我们完备化的一个过程. 将集合上的集族对补集和可数交的运算进行封闭化以后，我们将得到包含这个集族的一个 $\sigma-$ 代数.

3. 外测度

[22]这里是废话大王的说废话空间, 这里的很多理解是浅薄、不严谨的, 欢迎大家向我提出意见, 共同完善这一部分的内容.

4. 完备测度

4.1.2 关于不可测集的构造

在课程中李思然老师并未直接给出一个不可测集构造的证明. 我相信大家在本科《实变函数》课程中已经学过相关的内容，但为了完整性以及方便对课堂上对不可测集应用的理解，我们在此仍给出这方面的证明. 本节内容部分来自于 Stein 的 *Real Analysis* 的 24 页至 25 页.

首先我们在 $\mathbb{R}$ 上构建一个等价类，若对两个数 x 和 y，这两个数的差 $x-y\in\mathbb{Q}$，则记 x 和 y 位于一个等价类. 显然这个等价类满足自反性、对称性和传递性. 两个等价类要么不交要么重合. 为此我们可以将 $[0,1]$ 写成以下的等价类的不交并的形式:

$$[0,1]=\bigcup_{\alpha}\mathcal{E}_{\alpha}. \tag{4.2}$$

我们对以上不交并的结构使用选择公理，对每个集合 $\mathcal{E}$ 找出代表元 x_α，并记 $\mathcal{N}=\{x_\alpha\}$. 我们将使用反证法证明 $\mathcal{N}$ 是不可测的.

假设 $\mathcal{N}$ 是可测的. 取序列 $\{r_k\}_{k=1}^{\infty}$ 取遍 $[-1,1]$ 上的所有有理数，并定义

$$\mathcal{N}_k=\mathcal{N}+r_k. \tag{4.3}$$

则 $\mathcal{N}_k$ 满足:

(1)$\mathcal{N}_k$ 是不交的，设 $x\in\mathcal{N}_i\cap\mathcal{N}_j$，则 $x-r_i\in\mathcal{N},x-r_j\in\mathcal{N}$，但 $x-r_i$ 和 $x-r_j$ 之差为 $r_i-r_j\in\mathbb{Q}$，这说明 $x-r_i$ 和 $x-r_j$ 隶属于同一等价类，但是根据选择公理，同一等价类有且仅有一个元素，这表明 $x-r_i=x-r_j$，这表明 $r_i=r_j$，即 $i=j$.

(2)$[0,1]\subset\bigcup\limits_{k=1}^{\infty}\mathcal{N}_k\subset[-1,2]$. 只证明左边, 对任意 $x\in[0,1]$, 一定存在一个代表元 $\tilde{x}\in\mathcal{N}$ 使得 $x\sim\tilde{x}$，于是存在 $r\in\mathbb{Q}\cap[-1,1]$，使得 $x+r=\tilde{x}$，因此 $x\in\bigcup\limits_{k=1}^{\infty}\mathcal{N}_k$.

由于可测集满足单调性和可数可加性，因此我们有

$$1\leqslant\mu\Big(\bigcup_{k=1}^{\infty}\mathcal{N}_k\Big)\leqslant\sum_{k=1}^{\infty}\mu(\mathcal{N}_k)\leqslant 3 \tag{4.4}$$

考虑到 $\mathcal{N}_k$ 是 $\mathcal{N}$ 的一个平移，因此

$$1 \leqslant \sum_{k=1}^{\infty} \mu(\mathcal{N}) \leqslant 3 \tag{4.5}$$

而这将使得 $\mu(\mathcal{N})$ 无法取值，此时导出矛盾.

不可测集的构造打破了我们对实数集结构的美好幻想，其构造的简易性启发我们，如果一个集族中不含不可测集，那么这个集族一定会具有一个相对复杂的结构 (譬如 σ 代数的结构) 或是相对苛刻的条件.

在课堂中借助不可测集论证了 Vitali 定理：如果一个集合 A 的子集都是 Lebesgue 可测的，那么这个集合 A 本身的测度将等于 0，这个表述正好是完备测度的反向表示. 我们考虑把不可测集的构造应用到这里，考虑集合 A 与 $\mathcal{N}$ 的交集，从感觉上来讲，与不可测集做交集并不是什么好事，对于一般的 A 来说，$A\cap\mathcal{N}$ 基本被注定是不可测的，但是由于题目条件的限制，$A\cap\mathcal{N}$ 是可测的. 我们借助这个机会，并结合前述对不可测集的论证为我们带来的灵感，将 A 写成 A 与不可测集交集的不交并，这时在测度上会给到 A 的一个限制. 在这样的思路下，结论中对集合 A 在测度上的限制似乎已经不言而喻.

4.1.3 关于 Lusin 定理的论证

一个通俗易懂但不太严谨的 Lusin 定理的理解：

论证过程总共干了三件事. 第一件事是 that is 那一行是一个简单函数版本的 Lusin，然后打算用这个小的 Lusin 提升为真正的 Lusin，思路是用简单函数逼近可测函数，那么不断逼近这个过程里，在简单版本的 Lusin 里面是要扔掉一个测度为 ε 的集合的，所以我们的第二件事要说明在不断逼近的过程中扔出去的集合的测度一定要有所限制。限制方法就是 $\varepsilon/2^k$. 第三件事是这个逼近是点态逼近，不能保证连续性，保证连续性的逼近是一致逼近，所以我们要用 Egoroff 把点态逼近变成一致逼近.

综合地说，先有一个简单版本的 Lusin，然后从控制去掉集合的测度和改变收敛强度两个方面把它提升成真正的 Lusin.

4.1.4 关于 Sorgenfrey 拓扑不是内正则的

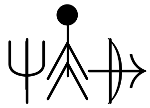
Sorgenfrey 拓扑里的所有紧集 K 都是可数集，这是因为对任意的 $x\in K$，都存在有理数 $q(x)$，使得 $[q(x),x[$ 与 K 的交集仅为单点集 $\{x\}$. 这时让 x 取遍 K 中的所有点，那么对任意两个 $x_i,x_j\in K, x_i\neq x_j$，有 $[q(x_i),xi[\cap[q(x_j,x_j)[=\varnothing$. 这给出了一个 K 中的点到 $\mathbb{Q}$ 的一个一一映射，由此可知 K 是可数集. 因此这样的集合的测度等于 0，无法通过上确界

的方式去逼近一个开集，这给出了 Sorgenfrey 拓扑不是内正则的.

4.1.5 关于 Riesz 表示定理

Riesz 表示定理中关于测度 $\mu(V) := \sup\{\Lambda f : f \prec V\}$ 的理解:

设三维空间上的线性函数 $f : \mathbb{R}^3 \to \mathbb{R}, \boldsymbol{x} = (x, y, z) \mapsto f(x, y, z)$，必然存在三维向量 $\boldsymbol{n} = (a, b, c)$，使得 $f(x, y, z) = ax + by + cz = \boldsymbol{n} \cdot \boldsymbol{x}$. 这里向量 $\boldsymbol{n}$ 是线性函数 f 的梯度，是平面 $f(x, y, z) = 0$ 的法方向，是 f 在 (x, y, z) 处 (移动单位距离下的) 最快上升方向. 回到一般情况下的定义，$f \prec V$ 可以理解为对移动距离的限制，而 $\sup\{\Lambda f\}$ 是在寻找最快上升方向的操作.

My Notes 

- 个人网站：https://www.cnblogs.com/xty0609/[23]
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[23]拖更大王就是我，我就是拖更大王