

**ANALYSIS • MATH6105 • HOMEWORK #6**  
**THE PROBLEM SHEETS ARE NOT TO BE HANDED IN**

SIRAN LI

**Question 1.** Let  $X$  be a topological space. Prove that for any subset  $E \subset X$ , we have

$$X \setminus \text{int}(E) = \overline{X \setminus E} \quad \text{and} \quad X \setminus \overline{E} = \text{int}(X \setminus E).$$

Suppose furthermore that  $X$  is a metric space. Prove that any closed subset of  $X$  is a  $G_\delta$ -set.

**Question 2.** Prove the following variant of the Baire category theorem:

Any locally compact Hausdorff topological space is not a countable union of nowhere dense closed subsets.

**Question 3.** We call  $\mathcal{C} \subset \mathbb{R}$  a **Cantor set** if it is nonempty, compact, perfect (*i.e.*, contains no isolated points), and totally disconnected (*i.e.*, contains no intervals). Prove that every dense  $G_\delta$ -set in  $[0, 1]$  contains a Cantor set.

**Remark.** Cantor sets are very good examples for meagre sets. The intuition is that meagre sets are obtained by taking away all possible subintervals from an interval. We shall use the above result to construct a set  $X \subset \mathbb{R}$  such that for any  $a < b$ , neither  $X \cap [a, b]$  nor  $[a, b] \setminus X$  is meagre.

**Question 4.** Prove the following celebrated result due to Banach:

The set of nowhere differentiable continuous functions on  $[0, 1]$  is residual in  $\mathcal{C}([0, 1])$ . That is, a generic continuous function is nowhere differentiable.

**Question 5.** A number  $0 \leq x \leq 1$  is said to be **Liouville** if for any  $n > 0$  there is a rational number  $p/q$  such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Prove that a generic number in  $[0, 1]$  is Liouville.

**Question 6.** Prove the following variant of the uniform boundedness principle:

Let  $X$  be a Banach space and  $Y$  be a normed vector space. Let  $\{T_i\}_{i \in I} \subset \mathcal{B}(X, Y)$ .

Then either one of the following holds:

- $\sup_{i \in I} \|T_i\| < \infty$ ; or
- there is a residual set  $E \subset X$  such that  $\sup_{i \in I} \|T_i x\| = \infty$  for all  $x \in E$ .

SIRAN LI: 800 DONGCHUAN ROAD, #6 NATURAL SCIENCES BUILDING, SCHOOL OF MATHEMATICAL SCIENCES OFFICE 524, SHANGHAI JIAO TONG UNIVERSITY, MINHANG DISTRICT, SHANGHAI, CHINA (200240)

*Email address:* siran.li@sjtu.edu.cn