Homework Assignment # 7 (due Dec.27)

deadline: 2023.12.27, 9:00 am

1. Let $\theta = 1/4$ and $T = 16$. Please write C/C++ computer programs to solve the following initial value problem of ordinary differential equation

$$\begin{cases} \dfrac{du(t)}{dt} &= u(1-u)(u-\theta) \qquad \text{for } 0 < t < T \\ u(0) &= \dfrac{3}{10}. \end{cases}$$

with the forward Euler, backward Euler and trapezoidal methods, respectively. Let $\Delta t = T/N > 0$ be the timestep and $\{t_n = n\Delta t\}_{n=0}^{N}$ be the discrete times. Let u_n be the discrete approximation to the value of the exact solution $u(t)$ at time $t = t_n$. Let $w_{\Delta t}$ be the value of the numerical solution u_n at the discrete time $t_n = 1$. For each method, run your program with different time steps $\Delta t's$ to generate a table as the following one.

Δt	1/5	1/10	1/20	1/40
$w_{\Delta t}$				
$w_{\Delta t} - w_{2\Delta t}$				

Verify the accuracy order of the numerical methods by checking the relation of the data $w_{\Delta t} - w_{2\Delta t}$ with the timestep Δt in each table. Please generate a t_n v.s. u_n plot of the numerical solution for each method with $\Delta t = 1/40$.

2. Solve the initial value problem

$$\begin{cases} u'(t) &= u(t) - \dfrac{2\,t}{u(t)} \qquad \text{for } 0 < t < 4 \\ u(0) &= 1 \end{cases}$$

by the classic four-stage Runge-Kutta method with different timestep $\Delta t \in \{1/5, 1/10, 1/20, 1/40\}$. The exact solution to the initial value problem reads $u(t) = \sqrt{1+2\,t}$. Let $e(\Delta t) = u(t_n) - u_n$ be the solution error at time $t_n = 1$. Generate a table as the following one.

Δt	1/5	1/10	1/20	1/40
$e(\Delta t)$				
$e(2\Delta t)/e(\Delta t)$				

3. Solve the initial value problem

$$\begin{cases} u'(t) &= -10\,u(t) + 9\,e^{-t}, \qquad \text{for } 0 < t < 1, \\ u(0) &= 1, \end{cases}$$

whose exact solution reads $u(t) = e^{-t}$, by any time integration method of your own choice with the timestep size $\Delta t = 1/10$, $1/20$ or $1/40$. Validate your code and explain on what you observe.

4. Please write C/C++ computer programs to solve the pendulum equation

$$\theta''(t) + 16\,\sin(\theta(t)) = 0, \qquad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0$$

for $t \in [0,\, 4]$ by the forward Euler, backward Euler and trapezoidal methods as well as a higher order method of your own choice. Denote by θ_n the finite difference approximation of $\theta(t_n)$ at the discrete time $t_n = n\Delta t$. For each integration method, run your computer program with the timestep $\Delta t = 0.05$ and generate a t_n versus θ_n plot of the numerical solution.

5. Suppose that $f(v)$ is a smooth function of $v \in \mathbb{R}$. Let us consider numerically solving the ordinary differential equation

$$u'(t) = f(u(t)),$$

subject to some initial condition. Let $\Delta t > 0$ be the timestep and $\{t_n = n\Delta t\}$ be the discrete times. Assume u_n is a finite difference approximation to the value of the exact solution $u(t)$ at time $t = t_n$. Show that the global solution error $e_n = u(t_n) - u_n$ by each method below has second order accuracy.

(a) the **(explicit) midpoint method** takes the form

$$u_{n+1} = u_n + \Delta t\, f\big(u_n + \frac{\Delta t}{2} f(u_n)\big) \qquad \text{for } n = 0,\, 1,\, 2,\, \cdots .$$

(b) the **implicit midpoint method** takes the form

$$u_{n+1} = u_n + \Delta t\, f\big(\frac{1}{2}(u_n + u_{n+1})\big) \qquad \text{for } n = 0,\, 1,\, 2,\, \cdots .$$

(c) the **modified Euler method** takes the form

$$u_{n+1} = u_n + \frac{\Delta t}{2}\big[f(u_n) + f\big(u_n + \Delta t\, f(u_n)\big)\big] \qquad \text{for } n = 0,\, 1,\, 2,\, \cdots .$$

6. Let $t_n = n\Delta t$ for $n = 0, 1, 2, \cdots$. Show that the approximate solution $u_n \approx u(t_n)$ generated by the Runge-Kutta method below

$$\begin{cases} K_0 = f(u_n), \\ K_1 = f(u_n + \gamma\, \Delta t\, K_0), \\ K_2 = f(u_n + (1-\gamma)\, \Delta t\, K_1), \\ u_{n+1} = u_n + \dfrac{\Delta t}{2}\big(K_1 + K_2\big), \end{cases}$$

for the ODE,

$$\frac{du(t)}{dt} = f(u(t)), \qquad t > 0,$$

has second-order accuracy with any parameter $\gamma \in (0, \frac{1}{2})$, i.e.,

$$e_n = u(t_n) - u_n = O(\Delta t^2),$$

provided that $u_0 = u(0)$ and the slope function $f(u)$ is sufficiently smooth.