ANALYSIS • MATH6105 • HOMEWORK #2
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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Question 1 (Differentiability under the integral sign). Let $f : (X, \mathcal{F}, \mu) \times [a,b] \to \mathbb{C}$ be a function such that $f(\bullet, t)$ is integrable for each fixed $t \in [a,b]$. Write $F(t) = \int_X f(x,t)\,\mathrm{d}\mu(x)$. Suppose that $\frac{\partial f}{\partial t}$ exists and there exists some $g \in L^1(X, \mathcal{F}, \mu)$ such that

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x) \qquad \text{for every } (x,t) \in X \times [a,b].$$

Then F is differentiable with $F'(t) = \int_X (\partial f / \partial t)(x,t)\,\mathrm{d}\mu(x)$.

Using the above, or otherwise, show that

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n! \qquad \text{and} \qquad \int_{\mathbb{R}} x^{2n} e^{-x^2}\,\mathrm{d}x = \frac{(2n)!\sqrt{\pi}}{4^n n!}.$$

Hint: Consider $\int_0^\infty e^{-tx}\,\mathrm{d}x$ and $\int_{\mathbb{R}} e^{-tx^2}\,\mathrm{d}x$.

Question 2. Let f_n, g_n, f, g be integrable functions such that $f_n \to f$, $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$. Deduce that $\int f_n \to \int f$.

Question 3. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Find $\sum_1^\infty \int_0^\infty |f_n(x)|\,\mathrm{d}x$, $\sum_1^\infty \int_0^\infty f_n(x)\,\mathrm{d}x$, and $\int_0^\infty \sum_1^\infty f_n(x)\,\mathrm{d}x$.

Question 4. Find $\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k\,\mathrm{d}x$.

Question 5. Prove that $\lim_{n\to\infty} \int_0^{n^2} n \left(\sin \frac{x}{n}\right) e^{-x^2}\,\mathrm{d}x = \frac{1}{2}$.

Question 6. Show that for $n \geq 2$ and $0 \leq y \leq n$, it holds that

$$\left(1 - \frac{y}{n}\right)^{-n} \geq \left(1 - \frac{y}{n+1}\right)^{-(n+1)}.$$

Hence, or otherwise, show that for $\alpha > 0$ and $\beta > -1$,

$$\lim_{n\to\infty} n^\alpha \int_0^1 x^{\alpha-1} e^{-n\beta x} (1-x)^n\,\mathrm{d}x = (\beta+1)^{-\alpha}\Gamma(\alpha),$$

where Γ is the Gamma function.

Question 7. Let $\alpha > -1$. Show that $f(x) := x^\alpha \log x$ is integrable on $]0,1[$, and that

$$\int_0^1 f(x)\,\mathrm{d}x = -(1+\alpha)^{-2}.$$

Then, deduce for $\beta > -1$ that $g(x) = x^\beta (1-x)^{-1} \log x$ is integrable over $]0,1[$, and that

$$\int_0^1 g(x)\,\mathrm{d}x = -\sum_{n=1}^\infty (n+\beta)^{-2}.$$

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