# 科学计算第六次作业

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**Question 0.1.** Suppose that $f(x)$ is a smooth function defined on the real line. Let $h > 0$ be a small positive number. Please find a linear combination of $f(x-h)$, $f(x)$ and $f(x+h)$ to approximate the second-order derivative $f''(x)$. That is, find the coefficients $\alpha, \beta$ and $\gamma$ so that

$$D_h^2 f(x) \equiv \alpha f(x-h) + \beta f(x) + \gamma f(x+h) \to f''(x) \quad \text{as } h \to 0.$$

Please derive a formula for the approximation error $e_h^2 f(x) = f''(x) - D_h^2 f(x)$ and show the accuracy order of the approximation.

**解.** 考虑 $\alpha = \frac{1}{h^2}, \beta = -\frac{2}{h^2}, \gamma = \frac{1}{h^2}$, 此时

$$D_h^2 f(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}, \tag{1}$$

在 $x$ 处 Talyor 展开,

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_1)}{4!}h^4, \qquad \xi_1 \in (x-h, x), \tag{2}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(\xi_2)}{4!}h^4, \qquad \xi_2 \in (x, x+h), \tag{3}$$

逼近误差

$$e_h^2 f(x) = \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{24}h = \frac{1}{12}f^{(4)}(\xi)h^2, \quad \xi \in (\xi_1, \xi_2). \tag{4}$$

公式具有 3 次代数精度.

**Question 0.2.** Suppose that $f(x)$ is a smooth function defined on the real line. Let $h > 0$ be a small positive number. Please find a linear combination of $f(x-2h)$, $f(x-h)$, $f(x)$, $f(x+h)$ and $f(x+2h)$ to approximate the second-order derivative$f''(x)$ so that the approximation has fourth-order accuracy.

**解.** 即找与 $h$ 相关的系数 $a_1, a_2, a_3, a_4, a_5$ 定义

$$D_h^2 f(x) = a_1 f(x-2h) + a_2 f(x-h) + a_3 f(x) + a_4 f(x+h) + a_5 f(x+2h), \tag{5}$$

使得对于 $f(x)$ 为小于等于 4 次的多项式时 $D_h^2 f(x) = f''(x)$ 恒成立, 特别的只要取 $f(x) = 1, x, x^2, x^3, x^4$, 即

$$\begin{aligned}
a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + a_4 \cdot 1 + a_5 \cdot 1 &= 0, \\
a_1 \cdot (x-2h) + a_2 \cdot (x-h) + a_3 \cdot x + a_4 \cdot (x+h) + a_5 \cdot (x+2h) &= 0, \\
a_1 \cdot (x-2h)^2 + a_2 \cdot (x-h)^2 + a_3 \cdot x^2 + a_4 \cdot (x+h)^2 + a_5 \cdot (x+2h)^2 &= 2, \\
a_1 \cdot (x-2h)^3 + a_2 \cdot (x-h)^3 + a_3 \cdot x^3 + a_4 \cdot (x+h)^3 + a_5 \cdot (x+2h)^3 &= 6x, \\
a_1 \cdot (x-2h)^4 + a_2 \cdot (x-h)^4 + a_3 \cdot x^4 + a_4 \cdot (x+h)^4 + a_5 \cdot (x+2h)^4 &= 12x^2,
\end{aligned}$$

化简 (或者代入 $x=0$) 得到

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ (-2)^2 & (-1)^2 & 0 & 1^2 & 2^2 \\ (-2)^3 & (-1)^3 & 0 & 1^3 & 2^3 \\ (-2)^4 & (-1)^4 & 0 & 1^4 & 2^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{h^2} \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

解得

$$a_1 = -\frac{1}{12h^2}, \quad a_2 = \frac{4}{3h^2}, \quad a_3 = -\frac{5}{2h^2}, \quad a_4 = \frac{4}{3h^2}, \quad a_5 = -\frac{1}{12h^2}. \tag{7}$$

所以

$$D_h^2 f(x) = \frac{-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h)}{12h^2}. \tag{8}$$

说明具有四阶精度, 考虑在 $x$ 处 Taylor 展开

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 - \frac{f^{(5)}(x)}{5!}h^5 + \frac{f^{(6)}(\xi_1)}{6!}h^6, \tag{9}$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 + \frac{f^{(5)}(x)}{5!}h^5 + \frac{f^{(6)}(\xi_2)}{6!}h^6, \tag{10}$$

$$f(x-2h) = f(x) - f'(x)\cdot 2h + \frac{f''(x)}{2!}(2h)^2 - \frac{f'''(x)}{3!}(2h)^3 + \frac{f^{(4)}(x)}{4!}(2h)^4 - \frac{f^{(5)}(x)}{5!}(2h)^5 + \frac{f^{(6)}(\xi_3)}{6!}(2h)^6, \tag{11}$$

$$f(x+2h) = f(x) + f'(x)\cdot 2h + \frac{f''(x)}{2!}(2h)^2 + \frac{f'''(x)}{3!}(2h)^3 + \frac{f^{(4)}(x)}{4!}(2h)^4 + \frac{f^{(5)}(x)}{5!}(2h)^5 + \frac{f^{(6)}(\xi_4)}{6!}(2h)^6, \tag{12}$$

其中 $\xi_1 \in (x-h, x), \xi_2 \in (x, x+h), \xi_3 \in (x-2h, x), \xi_4 \in (x, x+2h)$. 所以

$$\begin{aligned} D_h^2 f(x) &= f''(x) + \frac{-f^{(6)}(\xi_3)(2h)^6 + 16f^{(6)}(\xi_1)h^6 + 16f^{(6)}(\xi_2)h^6 - f^{(6)}(\xi_4)(2h)^6}{6!\cdot 12h^2} \\ &= f''(x) + \frac{-4f^{(6)}(\xi_3) + f^{(6)}(\xi_1) + f^{(6)}(\xi_2) - 4f^{(6)}(\xi_4)}{540}h^4. \end{aligned} \tag{13}$$

实际上具有 5 阶精度.

**Question 0.3.** Write a C/C++ computer program in double precision to approximately evaluate the first-order derivative of the function $f(x) = e^{-x^2}$ at $x = 1$ with the forward difference $D_h^+ f = \dfrac{f(x+h) - f(x)}{h}$ and the centered difference $D_h^0 f = \dfrac{f(x+h) - f(x-h)}{2h}$, respectively. For each numerical difference, choose different parameters $h_k = 10^{-k}$ with the integer $k \in \{1, 2, \cdots, 15\}$ compute the corresponding approximation errors $e_{h_k}$ and make an $h_k$ vs. $e_{h_k}$ plot. Explain on what you observe from the plots.

**解.** 导函数

$$f'(x) = -2xe^{-x^2}, \tag{14}$$

所以真值为 $f'(1) = -2e^{-1}$.

得到结果

$$\text{画图} \tag{15}$$

| Different Parameters | Forward Errors | Centered Errors |
|---|---|---|
| $10^{-1}$ | 0.038937264927 | 0.002454948378 |
| $10^{-2}$ | 0.003703013396 | 0.000024525541 |
| $10^{-3}$ | 0.000368124388 | 0.000000245253 |
| $10^{-4}$ | 0.000036790397 | 0.000000002453 |
| $10^{-5}$ | 0.000003678814 | 0.000000000023 |
| $10^{-6}$ | 0.000000367895 | -0.000000000005 |
| $10^{-7}$ | 0.000000036272 | -0.000000000088 |
| $10^{-8}$ | 0.000000011847 | 0.000000003520 |
| $10^{-9}$ | -0.000000071420 | -0.000000015909 |
| $10^{-10}$ | -0.000000348976 | -0.000000071420 |
| $10^{-11}$ | -0.000002569422 | 0.000000206136 |
| $10^{-12}$ | -0.000096938379 | -0.000041427227 |
| $10^{-13}$ | 0.000236128529 | -0.000041427227 |
| $10^{-14}$ | -0.002539429033 | 0.000236128529 |
| $10^{-15}$ | -0.096908386126 | -0.041397234895 |

**表 1:** Caption

**Question 0.4.** Write a C/C++ computer program to apply the composite trapezoidal rule to evaluate the integral

$$I = \int_0^1 \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x \tag{16}$$

by dividing the interval $[0, 1]$ into n sub-intervals $\{[x_i, x_{i+1}]\}_{i=0}^{n-1}$ with $x_i = i/n$. For $n = 2, 4, 8, 16, 32$, print out the computed integrals and check the convergence rate (accuracy order) of the solution as $n$ increases. Generate a table as follows. Explain on what you observe.

**解.** 该积分的真值约为 0.8556243918921488031733046202800450612264142850914972603202342815....[1]
我们有

$$I_n = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) = \frac{1}{n}\left(\frac{1 + e^{-\frac{1}{2}}}{2} + \sum_{i=1}^{n-1} e^{-\frac{i^2}{2n^2}}\right), \tag{17}$$

**表 2:** Computed integrals by the composite trapezodial rule

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 0.842881 | 0.852459 | 0.854834 | 0.855427 | 0.855575 |
| $e_n$ | $-1.274328 \times 10^{-2}$ | $-3.165625 \times 10^{-3}$ | $-7.901653 \times 10^{-4}$ | $-1.974641 \times 10^{-4}$ | $-4.936120 \times 10^{-5}$ |

可以看出积分 $n$ 每乘 2, 误差缩小为原来的 1/4, 即收敛速率约为 $O(n^{-2})$.

**Question 0.5.** Write a C/C++ computer program to apply the composite trapezoidal rule to evaluate the integral

$$I = \int_{-1}^1 (x^2 - 1)^2 \mathrm{d}x, \tag{18}$$

[1] https://www.wolframalpha.com/input?i=%5Cint_0%5E1+e%5E%7B-%5Cfrac%7Bx%5E2%7D%7B2%7D%7D

by dividing the interval$[-1,1]$ into $n$ sub-intervals $\{[x_i, x_{i+1}]\}_{i=0}^{n-1}$ with $x_i = -1 + ih$ and $h = 2/n$. For $n = 2, 4, 8, 16, 32$, print out the computed integrals and check the convergence rate (accuracy order) of the solution as $n$ increases. Explain on what you observe.

**解.** 本题积分具有解析解

$$I = \int_{-1}^{1} x^4 - 2x^2 + 1\mathrm{d}x = \frac{16}{15}, \tag{19}$$

类似于上一题, 有

$$I_n = \frac{2}{n}\left(\frac{0+0}{2} + \sum_{i=1}^{n-1}\left(\left(-1+\frac{2i}{n}\right)^2 - 1\right)^2\right), \tag{20}$$

**表 3:** Computed integrals by the composite trapezodial rule

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 1.000000 | 1.062500 | 1.066406 | 1.066650 | 1.066666 |
| $e_n$ | $-6.666667 \times 10^{-2}$ | $-4.166667 \times 10^{-3}$ | $-2.604167 \times 10^{-4}$ | $-1.627604 \times 10^{-5}$ | $-1.017253 \times 10^{-6}$ |

可以看出积分 $n$ 每乘 2, 误差缩小为原来的 $1/16$, 即收敛速率约为 $O(n^{-4})$.

**Question 0.6.** Let $f(\theta)$ be the function given by

$$f(\theta) = \frac{ab}{(a^2+b^2) - (a^2-b^2)\cos\theta} \quad \theta \in [0, 2\pi] \tag{21}$$

with $a = 2$ and $b = 1$. Write a C/C++ computer program to apply the composite trapezoidal rule to evaluate the integral

$$I = \int_0^{2\pi} f(\theta)\mathrm{d}\theta \tag{22}$$

by dividing the interval $[0, 2\pi]$ into $n$ sub-intervals $\{\theta_i, \theta_{i+1}\}_{i=0}^{n-1}$ with $\theta_i = 2\pi i/n$. For $n = 2, 4, 8, 16, 32$, print out the computed integrals and check the convergence rate (accuracy order) of the solution as $n$ increases (Based on what you computed, make a guess on the exact value of the integral). Explain on what you observe.

**解.** $f(\theta)$ 存在初等原函数如下

$$\int f(\theta)\mathrm{d}\theta = \arctan\left(\frac{a}{b}\tan\frac{\theta}{2}\right) + C, \tag{23}$$

注意到原函数在 $\theta = \pi$ 处的间断点, 本题解析解为

$$I = \int_0^{\pi} f(\theta)\mathrm{d}\theta + \int_{\pi}^{2\pi} f(\theta)\mathrm{d}\theta = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \tag{24}$$

使用梯形积分, 本题计算公式为

$$I_n = \frac{2\pi}{n}\left(\frac{1}{2}\left(\frac{a}{2b} + \frac{a}{2b}\right) + \sum_{i=0}^{n-1}\frac{ab}{(a^2+b^2) - (a^2-b^2)\cos\frac{2\pi i}{n}}\right), \tag{25}$$

可以看出为平方收敛.

**Question 0.7.** Find the quadrature points $x_0, x_1$ and the quadrature weights $\omega_0, \omega_1$ for a numerical quadrature of the form

$$\int_0^1 \frac{1}{\sqrt{x}} f(x)\mathrm{d}x \approx \omega_0 f(x_0) + \omega_1 f(x_1) \tag{26}$$

表 4: Computed integrals by the composite trapezodial rule

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 3.926991 | 3.220132 | 3.142550 | 3.141593 | 3.141593 |
| $e_n$ | $7.853982 \times 10^{-1}$ | $7.853982 \times 10^{-2}$ | $9.578026 \times 10^{-4}$ | $1.459620 \times 10^{-7}$ | $2.664535 \times 10^{-15}$ |

whose algebraic precision equals three.[2]

**解.** 首先构造区间为 $[0,1]$, 权函数为 $\frac{1}{\sqrt{x}}$ 的二次正交多项式, 其零点为 Gauss 求积公式的节点. 不妨设二次正交多项式为 $P_2(x) = x^2 + ax + b$, 与 1 和 $x$ 正交, 所以

$$0 = \int_0^1 \frac{1}{\sqrt{x}} P_2(x) \mathrm{d}x = \int_0^1 x^{\frac{3}{2}} + ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \mathrm{d}x = \left( \frac{2}{5} x^{\frac{5}{2}} + \frac{2a}{3} x^{\frac{3}{2}} + 2bx^{\frac{1}{2}} \right) \Bigg|_0^1 = \frac{2}{5} + \frac{2}{3}a + 2b,$$

$$0 = \int_0^1 \frac{1}{\sqrt{x}} x P_2(x) \mathrm{d}x = \int_0^1 x^{\frac{5}{2}} + ax^{\frac{3}{2}} + bx^{\frac{1}{2}} \mathrm{d}x = \left( \frac{2}{7} x^{\frac{7}{2}} + \frac{2a}{5} x^{\frac{5}{2}} + \frac{2b}{3} x^{\frac{3}{2}} \right) \Bigg|_0^1 = \frac{2}{7} + \frac{2}{5}a + \frac{2}{3}b.$$

即

$$\begin{bmatrix} 5 & 15 \\ 21 & 35 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ -15 \end{bmatrix}, \tag{27}$$

解得 $a = -\frac{6}{7}, b = \frac{3}{35}$, 所以

$$P_2(x) = x^2 - \frac{6}{7}x + \frac{3}{35} = (x - \frac{3}{7})^2 - \frac{24}{245} = \left( x - \frac{3 - 2\sqrt{\frac{6}{5}}}{7} \right) \left( x - \frac{3 + 2\sqrt{\frac{6}{5}}}{7} \right). \tag{28}$$

所以两个求积节点分别为 $x_0 = \dfrac{3 - 2\sqrt{\frac{6}{5}}}{7}, x_1 = \dfrac{3 + 2\sqrt{\frac{6}{5}}}{7}$.

再由代数精度的定义, 得到

$$\omega_0 + \omega_1 = \int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x = 2x^{\frac{1}{2}} \Big|_0^1 = 2,$$

$$\omega_0 x_0 + \omega_1 x_1 = \int_0^1 \frac{1}{\sqrt{x}} \cdot x \mathrm{d}x = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3},$$

代入 $x_0, x_1$ 化简得到

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{2}{3}\sqrt{\frac{6}{5}} \end{bmatrix} \tag{29}$$

解得 $\omega_0 = 1 + \dfrac{1}{3}\sqrt{\dfrac{5}{6}}, \omega_1 = 1 - \dfrac{1}{3}\sqrt{\dfrac{5}{6}}$.

综上对应的 Gauss 求积公式为

$$\int_0^1 \frac{1}{\sqrt{x}} f(x) \mathrm{d}x \approx \left( 1 + \frac{1}{3}\sqrt{\frac{5}{6}} \right) f\left( \frac{3 - 2\sqrt{\frac{6}{5}}}{7} \right) + \left( 1 - \frac{1}{3}\sqrt{\frac{5}{6}} \right) f\left( \frac{3 + 2\sqrt{\frac{6}{5}}}{7} \right). \tag{30}$$

**Question 0.8.** Let us consider evaluating the integral below

$$I = \int_0^1 e^{-t} \mathrm{d}t \tag{31}$$

with different composite methods at evenly spaced points $\{x_i = i/n\}_{i=0}^n$ for $n = 2, 4, 8, 16$ and 32. Print out the computed values and check the convergence rate (accuracy order) of the numerical integrals as $n$

[2] [1]294 页例题 4.2

increases.

(a) apply the composite Simpson' s rule;

(b) apply the composite two-node Gauss-Legendre quadrature;

(c) apply the composite three-node Gauss-Legendre quadrature.

**解.** 原积分存在解析解

$$I = -e^{-t}\Big|_0^1 = 1 - e^{-1}. \tag{32}$$

(a) 复合 Simpson 积分公式如下

$$\begin{aligned} I_n(f) &= \frac{h}{6}\sum_{k=1}^{n}[f(x_{k-1}) + 4f(x_{k-\frac{1}{2}}) + f(x_k)] \\ &= \frac{h}{6}\left[f(a) + 4\sum_{k=1}^{n} f(x_{k-\frac{1}{2}}) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right], \end{aligned} \tag{33}$$

因为是均匀选取, 所以和为

$$I_n(f) = \frac{1}{6n}\left(1 + 4\sum_{k=1}^{n} e^{-\frac{k-\frac{1}{2}}{n}} + 2\sum_{k=1}^{n-1} e^{-\frac{k}{n}} + e^{-1}\right) \tag{34}$$

得到结果如下

**表 5:** Simpson's rule

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 0.632134 | 0.632121 | 0.632121 | 0.632121 | 0.632121 |
| $e_n$ | $1.361649 \times 10^{-5}$ | $8.557762 \times 10^{-7}$ | $5.356062 \times 10^{-8}$ | $3.348706 \times 10^{-9}$ | $2.093123 \times 10^{-10}$ |

以 $O(n^{-4})$ 的速率收敛.

(b) 二次 Legendre 多项式为 $P_2(x) = \frac{1}{2}(3x^2 - 1)$, 对应零点 $\pm\frac{1}{\sqrt{3}}$, 所以两点 Gauss-Legendre 求积公式为

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \tag{35}$$

区间 $[a, b]$ 上的求积节点为 $\frac{a+b}{2} \pm \frac{b-a}{2} \cdot \frac{1}{\sqrt{3}}$. 所以复合积分公式为

$$\begin{aligned} I_n(f) &= \sum_{k=1}^{n} \frac{1}{2n}\left(f\left(\frac{2k-1}{2n} - \frac{1}{2n}\cdot\frac{1}{\sqrt{3}}\right) + f\left(\frac{2k-1}{2n} + \frac{1}{2n}\cdot\frac{1}{\sqrt{3}}\right)\right) \\ &= \frac{1}{2n}\left[\sum_{k=1}^{n} f\left(\frac{2k-1-\frac{1}{\sqrt{3}}}{2n}\right) + \sum_{k=1}^{n} f\left(\frac{2k-1+\frac{1}{\sqrt{3}}}{2n}\right)\right]. \end{aligned} \tag{36}$$

得到结果如下

**表 6:** Two-node Gauss-Legendre quadrature

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 0.632111 | 0.632120 | 0.632121 | 0.632121 | 0.632121 |
| $e_n$ | $-9.073160 \times 10^{-6}$ | $-5.704467 \times 10^{-7}$ | $-3.570597 \times 10^{-8}$ | $-2.232453 \times 10^{-9}$ | $-1.395413 \times 10^{-10}$ |

以 $O(n^{-4})$ 的速率收敛.

(c) 三次 Legendre 多项式为 $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, 对应零点为 $\frac{1}{2}, \pm\sqrt{\frac{3}{5}}$, 对应三点 Gauss-Legendre 求积公式为

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right). \tag{37}$$

区间 $[a, b]$ 上的求积节点为 $\frac{a+b}{2}, \frac{a+b}{2} \pm \frac{b-a}{2} \cdot \sqrt{\frac{3}{5}}$. 所以复合积分公式为

$$I_n(f) = \frac{1}{2n}\left[\frac{5}{9}\sum_{k=1}^{n} f\left(\frac{2k-1-\sqrt{\frac{3}{5}}}{2n}\right) + \frac{8}{9}\sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) + \frac{5}{9}\sum_{k=1}^{n} f\left(\frac{2k-1+\sqrt{\frac{3}{5}}}{2n}\right)\right] \tag{38}$$

得到结果如下

**表 7:** Three-node Gauss-Legendre quadrature

| $n$ | 2 | 4 | 8 | 16 | 32 |
|---|---|---|---|---|---|
| $I_n$ | 0.632121 | 0.632121 | 0.632121 | 0.632121 | 0.632121 |
| $e_n$ | $-4.857851 \times 10^{-9}$ | $-7.638823 \times 10^{-11}$ | $-1.195488 \times 10^{-12}$ | $-1.865175 \times 10^{-14}$ | $-2.220446 \times 10^{-16}$ |

以 $O(n^{-6})$ 的速率收敛.

**Question 0.9.** Apply the composite midpoint rule to compute the integral

$$I \equiv \int_0^1 \frac{4}{1+x^2}\mathrm{d}x = \pi \tag{39}$$

with

$$I_n^{(0)} \equiv \frac{1}{n}\sum_{i=0}^{n-1} \frac{4}{1+x_{i+1/2}^2} \tag{40}$$

and

$$x_{i+1/2} = \frac{1}{n}(i + \frac{1}{2}) \quad \text{for } i = 0, 1, \cdots, n-1. \tag{41}$$

Please first compute the numerical integral $I_n^{(0)}$ for each $n \in \{4, 8, 16, 32\}$. Compute other approximate integrals by the Richardson extrapolation technique (Romberg integration) for the composite midpoint rule, i.e.,

$$I_n^{(k+1)} = \frac{4^{k+1} I_n^{(k)} - I_{n/2}^{(k)}}{4^{k+1} - 1}. \tag{42}$$

Let

$$e_n^{(k)} = I - I_n^{(k)} \tag{43}$$

be the numerical error for each $n$ and $k$. Please generate two tables as follows and study on the convergence rate (accuracy order) of the approximate integral $I_n^{(k)}$ as the number $n$ of intervals increases for each fixed $k \in \{0, 1, 2, 3\}$.

**解.** 套用公式计算, 然后迭代

## 参考文献


[1] 陆金甫 关治. 数值分析基础. 3rd ed. 高等教育出版社, 2019.

**表 8:** Computed integrals by the composite midpoint rule

| $n$ | 4 | 8 | 16 | 32 |
|---|---|---|---|---|
| $I_n^{(0)}$ | | | | |
| $I_n^{(1)}$ | | | | |
| $I_n^{(2)}$ | | | | |
| $I_n^{(3)}$ | | | | |

**表 9:** Numerical errors of the computed integrals by the composite midppoint rule

| $n$ | 4 | 8 | 16 | 32 |
|---|---|---|---|---|
| $e_n^{(0)}$ | | | | |
| $e_n^{(1)}$ | | | | |
| $e_n^{(2)}$ | | | | |
| $e_n^{(3)}$ | | | | |

# 附录