ANALYSIS • MATH6105 • HOMEWORK #5
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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Question 1. Prove the minimality of the positive measures $\nu^{\pm}$ obtained by the Jordan decomposition theorem. More precisely, suppose that ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$. Show that $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. (Recall that $\nu^{\pm}$ are the unique positive measures such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.)

Question 2. Assume that $\nu \ll \mu$, where ν is a signed measure and μ is a positive measure on $(X, \mathcal{F})$. Let $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$ be the Radon–Nikodym derivative. Describe the Hahn decompositions of ν (with respect to μ) and the measures ν^+, ν^-, and $|\nu|$ in terms of f and μ.

Question 3. Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{I}} \subset L^1(\mu)$ is said to by *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f_\alpha \,\mathrm{d}\mu\right| < \epsilon$ for all $\alpha \in \mathcal{I}$ whenever $\mu(E) < \delta$. Prove that if a sequence $\{f_n\} \subset L^1(\mu)$ converges to some $f \in L^1(\mu)$ in the L^1-norm, then $\{f_n\}$ is uniformly integrable.

Question 4. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $\mathcal{G} \subset \mathcal{F}$ by a σ-subalgebra. Set $\nu := \mu\big|_{\mathcal{G}}$. Prove that for each $f \in L^1(X, \mathcal{F}, \mu)$ there exists $g \in L^1(X, \mathcal{G}, \nu)$ such that

$$\int_E f \,\mathrm{d}\mu = \int_E g \,\mathrm{d}\nu \qquad \text{for all } E \in \mathcal{G}.$$

Moreover, g is unique in the ν-a.e. sense.

Definition 0.1. *We call g the conditional expectation of f with respect to $\mathcal{G}$ and write $g = \mathbb{E}[f|\mathcal{G}]$.*

In the above setting, suppose in addition that $\mathcal{H}$ is a further σ-subalgebra of $\mathcal{G}$. Prove the *towering property* of conditional expectations:

$$\mathbb{E}\big[\mathbb{E}[f|\mathcal{G}]\big|\mathcal{H}\big] = \mathbb{E}[f|\mathcal{H}] \qquad \mu\big|_{\mathcal{H}}\text{-a.e..}$$

Question 5. Find $f \in L^1(\mathbb{R}^d)$ such that there are $C, R > 0$ satisfying

$$\mathcal{M}f(x) \geq C|x|^{-d} \qquad \text{for all } |x| > R.$$

where $\mathcal{M}f$ is the Hardy–Littlewood maximal function of f. Prove that there exists $C' > 0$ such that for $\alpha > 0$ small, it holds that

$$\mathcal{L}^d\{\mathcal{M}f > \alpha\} \geq \frac{C'}{\alpha}.$$

This shows the sharpness of the estimate in the Hardy–Littlewood maximal theorem.

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