ANALYSIS • MATH6105 • HOMEWORK #7
THE PROBLEM SHEETS ARE NOT TO BE HANDED IN

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Question 1. Let X be a Banach space and Y be a normed vector space. Let $T \in \mathcal{B}(X,Y)$ be such that there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$. Prove that T has closed range.

Question 2. Let $K := \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$ and $\Phi : C^0(K) \to \mathbb{C}$, $\Phi(f) := \int_{\{|z|=3/2\}} f(z)\,\mathrm{d}z$.

(1) Show that Φ is a bounded linear functional.
(2) Hence, or otherwise, prove that $\mathbb{C}[z]$ (the space of complex-coefficient polynomials) is *not* dense in $C^0(K;\mathbb{C})$. That is, the direct analogue of Stone–Weierstrass fails in $\mathbb{C}$.

Question 3. Let X be a normed vector space and let Z be a closed (why is it necessary?) subspace of X. Define the quotient norm on X/Z by $\|x+Z\| := \mathrm{dist}(x,Z)$. Prove that it indeed defines a norm. Moreover, if X is a Banach space, then so is X/Z under the quotient norm.

Hint: You may possibly find Question 6 helpful.

Question 4. Let X, Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. Prove that the operator $T_0 : X/\ker T \to \mathrm{ran}T$ given by $T_0(x + \ker T) = Tx$ for every $x \in X$ satisfies $\|T_0\| = \|T\|$.

Remark. Let $\pi : X \to X/\ker T$ be the canonical projection and $\iota : \mathrm{ran}T \hookrightarrow Y$ be the natural inclusion. Then we can factorise $T = \iota \circ T_0 \circ \pi$. This is a factorisation over bounded maps; it is known as the *canonical factorisation* of T. Note that $(T_0)^{-1}$ may fail to be bounded; i.e., T_0 is not necessarily an NVS-isomorphism. Compare with the rank-nullity theorem in (finite-dimensional) linear algebra and the first isomorphism theorem in group theory.

Question 5. Let X be an infinite-dimensional Banach space. Using Baire Category Theorem, show that X cannot have countable Hamel bases.

Question 6. Let X be a normed vector space. A series $\sum x_n$ is said to be absolutely convergent if $\sum_1^\infty \|x_n\| < \infty$. Prove that X is a Banach space if and only if every absolutely convergent series in X also converges in X.

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