Homework Assignment # 3 (due Nov.1)

deadline: 2023.11.01, 9:00 am

1. Let $\mathbf{A} = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ be a real square matrix. Recall that the ℓ_1-norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is given by $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$. Show that the induced ℓ_1-norm of the matrix $\mathbf{A}$

$$\|\mathbf{A}\|_1 = \max_{0\neq \mathbf{x}\in\mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|_1}{\|\mathbf{x}\|_1}$$

can be computed by

$$\|\mathbf{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|.$$

2. (a) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show the equivalence between vector norms and operator norms, i.e.,

$$\|\mathbf{x}\|_1 = \sup_{\mathbf{y}\neq 0} \frac{(\mathbf{x},\mathbf{y})}{\|\mathbf{y}\|_\infty}, \quad \|\mathbf{x}\|_\infty = \sup_{\mathbf{y}\neq 0} \frac{(\mathbf{x},\mathbf{y})}{\|\mathbf{y}\|_1},$$

where $(\cdot,\cdot)$ is the inner product of vectors.

   (b) Let $\mathbf{A}$ be a real matrix. Show that $\|\mathbf{A}\|_1 = \|\mathbf{A}^T\|_\infty$.

3. Let $\mathbf{A}$ be a symmetric positive definite matrix. Given the maximal and minimal eigenvalues of $\mathbf{A}$, find the optimal parameter for the Richardson iteration method.

4. Prove the following theorems:

   (a) Suppose $\mathbf{A}$ is a symmetric matrix with positive diagonal elements. If it is strictly diagonally dominant, or irreducible diagonally dominant, then $\mathbf{A}$ is positive definite.

   (b) If $\mathbf{A} = (a_{i,j})_{n\times n} \in \mathbb{R}^{n\times n}$ is symmetric and positive definite with $a_{i,j} \le 0$, $\forall i \neq j$, then the Jacobi method is convergent.

5. The symmetric Gauss-Seidel method is obtained by combining an iteration of Gauss-Seidel method with an iteration of backward Gauss-Seidel method. Precisely, the k-th iteration of the symmetric Gauss-Seidel method for solving $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ is

$$\mathbf{x}_{k+1/2} = \mathbf{x}_k + (\mathbf{D} - \mathbf{L})^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}_k)$$
$$\mathbf{x}_{k+1} = \mathbf{x}_{k+1/2} + (\mathbf{D} - \mathbf{U})^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}_{k+1/2}),$$

i.e., the iteration method can be written as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{P}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}_k).$$

If $\mathbf{A}$ is a symmetric positive definite matrix:

   (a) Compute the preconditioner $\mathbf{P}$ of the symmetric Gauss-Seidel method, and show that $\mathbf{P}$ is symmetric positive definite.

   (b) Show that the spectral radius of the iteration matrix is less than 1.

6. Consider the linear system

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \end{bmatrix},$$

where, $h = 1/n$, $f_i = (3x_i + x_i^2)e^{x_i}$ and $x_i = i/n$ for $i = 1, 2, \cdots, n$.

(a) Given a real matrix $\mathbf{A}$, show that

$$\rho(\mathbf{A}) = \lim_{k\to\infty} \|\mathbf{A}^k\|^{1/k},$$

where $\rho(\mathbf{A})$ is the spectral radius of $\mathbf{A}$ and $\|\cdot\|$ is any matrix norm.

(b) Define the asymptotic convergence rate of an iteration method as $R_\infty = -\ln\rho(\mathbf{B})$ where $\mathbf{B}$ is the iteration matrix. Let $\mathbf{B}_{Jacobi}$ and $\mathbf{B}_{GS}$ be the iteration matrices of the Jacobi method and the Gauss-Seidel method, respectively. Calculate $\rho(\mathbf{B}_{Jacobi}), \rho(\mathbf{B}_{GS})$ and the corresponding asymptotic convergence rates.

(c) Write a computer program to numerically verify the asymptotic convergence rates.

(d) Write a computer program to solve the linear system with the SOR method. For each $n = 1000, 2000, 4000, 8000$, chose different relaxation parameter $\omega_k = 1 + k/101$ for $k = 1, 2, \cdots, 100$ to collect the required iteration numbers until convergence. Plot your results with x-axis for ω and y-axis for iteration number. Use double precision in your computation and set the relative residual tolerance as $\varepsilon_{rel} = 1 \times 10^{-8}$.