Homework Assignment # 4 (due Nov.20)

deadline: 2023.11.20, 9:00 am

In this assignment, for each problem which is solved by a variational iterative method, please initialize the solution with zero ($\mathbf{x}_0 = 0$) and stop the iteration when the maximum norm of the residual $\mathbf{r}_k = \mathbf{b} - \mathbf{A}\mathbf{x}_k$ relative to that of the initial residual $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0 = \mathbf{b}$ is less than the tolerance $\varepsilon = 10^{-8}$.

1. Let $\{\mathbf{r}_i\}_{i=0}^k$ be the residual vectors and $\{\mathbf{d}_i\}_{i=1}^k$ be the search directions in the conjugate gradient method for the symmetric and positive definite matrix $\mathbf{A}$. Show that

$$\begin{aligned} \mathbf{d}_j^{\mathrm{T}}\mathbf{A}\mathbf{d}_k &= 0 \qquad \text{for } j < k, \\ \mathbf{r}_j^{\mathrm{T}}\mathbf{r}_k &= 0 \qquad \text{for } j < k, \\ \mathbf{d}_j^{\mathrm{T}}\mathbf{r}_k &= 0 \qquad \text{for } j \le k. \end{aligned}$$

   The notation for the vectors follows our classnote. *The* $\mathbf{A}$*-orthogonality of the search directions indicates the conjugate gradient method will never repeat its searching for the global minimizer of the quadratic functional over the same direction more than once, which is quite different from the steepest descent method.*

2. Let us consider finding the minimizer of the quadratic functional

$$\varphi(x, y, z) = x^2 + \frac{3}{2}\, y^2 + z^2 - xy - 2yz - 3x - 3y + 4z$$

   with the conjugate gradient method. Let $(x_0, y_0, z_0)^{\mathrm{T}} = (1, 1, 1)^{\mathrm{T}}$ be the initial guess. Please compute by hand the first two approximate solutions $(x_1, y_1, z_1)^{\mathrm{T}}$ and $(x_2, y_2, z_2)^{\mathrm{T}}$.

3. Let

$$\mathbf{A} = \frac{1}{(n+1)^3}\begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \ddots & 6 & 3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}_{n\times n}.$$

   Let $\mathbf{f} = (f_1, f_2, \cdots, f_n)^{\mathrm{T}} \in \mathbb{R}^n$ be a known vector, whose i^{th} entry is given by

$$f_i = x_i\,(1 - x_i)\, e^{x_i}$$

   with $x_i = \dfrac{i}{n+1}$. Please write computer programs to solve the linear system

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$

   for the unknown vector $\mathbf{u} \in \mathbb{R}^n$ with the steepest descent (SD) method and the conjugate gradient (CG) method, respectively. Report the iteration numbers used by each iterative method for $n = 9, 19, 39, 79, 159, 319$. What is the relation between the iteration number and the integer $(n+1)$?

n	9	19	39	79	159	319
SD						
CG						

4. The matrix
$$\begin{pmatrix} -2 & 4 & 2 \\ 3 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$$
has eigenvalue $\lambda_1 = 4$ with the associated eigenvector $\mathbf{x} = (1,1,1)^{\mathrm{T}}$. Construct a Householder matrix $\mathbf{H}$ by hand such that
$$\mathbf{HAH} = \begin{pmatrix} 4 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$
and further determine the remaining eigenvalues.

5. Please reduce the following matrix to a right triangular matrix by using Givens' rotations. Print out the right triangular matrix.
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & & \\ 2 & 2 & 3 & \\ & 3 & 3 & 4 \\ & & 4 & 4 \end{bmatrix}$$

6. Make the QR decomposition by hand or computer for the matrix below by the Gram-Schmidt orthogonalization method,
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

7. Use the Householder matrix to make the QR decomposition by hand or computer for the matrix below
$$(a).\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -4 & 5 \end{bmatrix} \qquad (b).\ \begin{bmatrix} 1 & 3 & -2 \\ -1 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$