Chapter 3. Direct Methods for Linear Systems.

Section 1. Gauss Elimination Method.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \to (2) - (1)\times \tfrac{1}{2}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3/2 \end{pmatrix} \to \begin{array}{l} \tfrac{3}{2}y = \tfrac{3}{2} \Rightarrow y = 1 \\ 2x + y = 3 \end{array}$$

$$\Rightarrow 2x = 3 - y = 2 \Rightarrow x = 1$$

$$A = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^n$$

Linear System: $Ax = b$ (1). $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

Assume A is invertible

Idea: eliminate the entries in the lower part of the matrix.

For $k = 1, 2, \cdots n-1$ do (as follows)

For $i = k+1, k+2, \cdots n$ do $\quad$ let $l_{ik} = a_{ik}/a_{kk}$

For $j = k, k+1, \cdots n$ do $\quad b_i^{(k+1)} \leftarrow b_i^{(k)} - l_{ik} b_k^{(k)}$

$$a_{ij}^{(k+1)} \leftarrow a_{ij}^{(k)} - l_{ik} a_{kj}^{(k)}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ & & a_{33}^{(3)} \cdots & a_{3n}^{(3)} \\ & & \ddots & \\ & & & a_{nn}^{(n)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_n^{(n)} \end{pmatrix} \quad (2)$$

Assumption: The diagonal entries are always non-zero in the elimination process.

$$a_{kk}^{(k)} \neq 0.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (\times)$$

The upper/right triangular system can be solved by the backward substitution method.

For $k = n, n-1, \cdots 1$. do

$$a_{kk}^{(k)} x_k = b_k^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)} x_j$$

$$x_k = \left(b_k^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)} x_j\right) / a_{kk}^{(k)} \quad (*).$$

algorithm complexity.

$$O(n^3) + O(n^2).$$

section 2. LU-decomposition Method.

Decompose A into the product of a lower triangular and an upper triangular matrix.

Denote the lower and upper triangular matrix by L and U, respectively.

$$A = LU \quad \text{with} \quad L = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \cdots & \cdots & \cdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & \cdots & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}$$

l_{ik}

$$Ax = b \Rightarrow LUx = b.$$

Let $Ux = y$. $\Rightarrow Ly = b$.

step 1. solve $Ly = b$. by forward substitution.

$$\begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad ✓$$

$O(n^2)$

step 2. solve $Ux = y$ by backward substitution.

$$\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

$O(n^2)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ & & & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{pmatrix}$$

$a_{11} = u_{11} \quad a_{12} = u_{12}. \quad a_{1j} = u_{1j} \quad j = 1, 2 \cdots n$

$l_{i1} u_{11} = a_{i1} \quad i = 2, 3 \cdots n \Rightarrow u_{i1} = a_{i1}/u_{11}$

For $K = 1, 2, \cdots n$.

$u_{kj} = ?$

$$a_{kj} = \sum_{m=1}^{K} l_{km} u_{mj} \qquad j = k, k+1 \cdots n$$

$$= u_{kj} + \sum_{m=1}^{k-1} l_{km} u_{mj}$$

$$u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}$$

$l_{ik} = ?$

$$a_{ik} = \sum_{m=1}^{K} l_{im} u_{mj}, \qquad i = k+1, \cdots n$$

$$l_{ik} \cdot u_{kj} = a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mj}$$